On the Farrell-Jones Conjecture and its applications

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Outline

- Motivation and statement of the Farrell-Jones Conjecture
- Some prominent conjectures
- The status of the Farrell-Jones Conjecture
- Open problems
- We have prepared more slides than we will probably show.

Algebraic K-and L-theory

Definition (Projective class group $K_0(R)$)

Define the projective class group of a ring R

$$K_0(R)$$

to be the abelian group defined by:

Generators: Isomorphism classes $[P]$ of finitely generated projective R-modules P.

Relations: We get $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules.
- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R) := GL(R)/[GL(R), GL(R)].$$

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\}.$$

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

- Bass (1968) and Quillen (1973) have defined $K_n(R)$ for all $n \in \mathbb{Z}$.
- K-theory has some basic feature such as compatibility with products, Morita equivalence, Bass-Heller-Swan decomposition and localization sequences.
- L-groups $L_n(R)$ are defined in a similar way but now for quadratic forms.
- In contrast to K-theory the L-groups are four-periodic, i.e., $L_n(R) = L_{n+4}(R)$
- In general algebraic K-and L-theory are very hard to compute but of high significance.

Motivation and statement of the Farrell-Jones Conjecture

- Let $\mathcal{H}_*$ be a (generalized) homology theory. It satisfies:
- Suspension

$$\mathcal{H}_n(B\mathbb{Z}) = \mathcal{H}_n(S^1) \cong \mathcal{H}_n(\text{pt}) \oplus \mathcal{H}_{n-1}(\text{pt}) = \mathcal{H}_n(B\{1\}) \oplus \mathcal{H}_{n-1}(B\{1\}).$$

- Mayer-Vietoris-sequence
  If $G = G_1 *_{G_0} G_2$, then we get a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{H}_n(BG_0) \to \mathcal{H}_n(BG_1) \oplus \mathcal{H}_n(BG_2) \to \mathcal{H}_n(BG) \\ \to \mathcal{H}_{n-1}(BG_0) \to \mathcal{H}_{n-1}(BG_1) \oplus \mathcal{H}_{n-1}(BG_2) \to \cdots \end{aligned}$$

- Let R be a regular ring. Then:
- Bass-Heller-Swan (1964) have shown:

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) = K_n(R[\{1\}]) \oplus K_{n-1}(R[\{1\}]).$$

- If $G = G_1 *_{G_0} G_2$ and G_0, G_1 and G_2 are torsionfree and belong to a certain class CL, then Waldhausen (1978) has established the exact sequence

$$\cdots \to K_n(R[G_0]) \to K_n(R[G_1]) \oplus K_n(R[G_2]) \to K_n(R[G]) \\ \to K_{n-1}(R[G_0]) \to K_{n-1}(R[G_1]) \oplus K_{n-1}(R[G_2]) \to \cdots$$

- This raises the question: Is there a generalized homology theory $\mathcal{H}_*$ satisfying

$$\mathcal{H}_n(BG) \cong K_n(RG)$$

  for all torsionfree groups G and $n \in \mathbb{Z}$, where R is a fixed regular ring?
- If yes, we must have for all $n \in \mathbb{Z}$

$$\mathcal{H}_n(\mathrm{pt}) = K_n(R).$$

- Hence our candidate for $\mathcal{H}_*$ is $H_*(-;\mathbf{K}_R)$, the generalized homology theory associated to the (non-connective) K-theory spectrum $\mathbf{K}_R$ of the ring R.

Conjecture (K-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The version of the Farrell-Jones Conjecture above is not true for finite groups unless the group is trivial.
- For instance we get for a finite group G and $R = \mathbb{C}$:

$$\begin{aligned} K_0(\mathbb{C}G) &= R_{\mathbb{C}}(G); \\ H_0(BG; \mathbf{K}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q} &= \mathbb{Q}. \end{aligned}$$

- Also the condition regular is needed in general.
- Namely, we have

$$\begin{aligned} K_n(R[\mathbb{Z}]) &= K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R); \\ H_n(B\mathbb{Z}; \mathbf{K}_R) &= K_n(R) \oplus K_{n-1}(R). \end{aligned}$$

Conjecture (Farrell-Jones Conjecture)

The Farrell-Jones Conjecture for K-theory or L-theory respectively with coefficients in R predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

or

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

respectively is bijective for all $n \in \mathbb{Z}$.

- There is a more complicated version of the Farrell-Jones Conjecture which may be true for all groups G and rings R and makes also sense for twisted group rings and allows orientation homomorphisms in L-theory.
- In the sequel we will refer to this general version.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(E_{\mathcal{F}in}(G)) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Bost Conjecture)

The Bost Conjecture predicts that the assembly map

$$K_n^G(E_{\mathcal{F}in}(G)) \to K_n(l^1(G))$$

is bijective for all $n \in \mathbb{Z}$.

Some prominent conjectures

- Construction of idempotents in *RG*
  Suppose that $g \in G$ has finite order $|g|$. Put $N = \sum_{i=1}^{|g|} g^i$. Then

$$N \cdot N = |g| \cdot N.$$

If $|g|$ is invertible in R and different from 1, then RG contains a non-trivial idempotent, namely $\frac{N}{|g|}$.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and a field F that 0 *and* 1 *are the only idempotents in FG.*

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

- If $f\colon M \to N$ is a homotopy equivalence of closed aspherical manifolds, then the Novikov Conjecture predicts

$$f_*\mathcal{L}(M) = \mathcal{L}(N).$$

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Wall and Farrell-Jones.

Conjecture (Gromov)

If G is a torsionfree hyperbolic group whose boundary is a standard sphere, then there is a closed aspherical manifold M with $G = \pi_1(M)$.

- There are further interesting prominent conjectures by Bass and by Moody and conjectures about L^2-invariants and about Poincaré duality groups, which we do not state.
- One of the basic features of the Farrell-Jones Conjecture is that it implies all the conjectures mentioned above, where in some cases one has to assume dim ≥ 5.
- The Farrell-Jones Conjecture is the basic ingredient for computations of $K_n(RG)$ and $L_n(RG)$.

Status of the Farrell-Jones Conjectures

Theorem (Main Theorem (Bartels-Echterhoff-Farrell-Lück-Wegner Reich (2008-2012)))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures (in its most general form) holds. It has the following properties:

- *Hyperbolic groups belong to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*

Theorem (continued)

- CAT(0)*-groups belong to* $\mathcal{FJ}$*;*
- *Virtually poly-cyclic groups belong to* $\mathcal{FJ}$*;*
- *Cocompact lattices in almost connected Lie groups belong to* $\mathcal{FJ}$*;*
- *All* 3*-manifold groups belong to* $\mathcal{FJ}$*.*

- The groups above are certainly wild in Bridson's universe of groups.
- Examples are:
  - Lacunary hyperbolic groups in the sense of Olshanskii-Osin-Sapir (2009);
  - Tarski monsters, i.e., groups which are not virtually cyclic and whose proper subgroups are ofm order p for a fiyed orime p;
  - Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2006)).

- Arzhantseva-Delzant (2008) have constructed groups with expanders as colimits of hyperbolic groups.
- These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis (2002).
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture, the Bost Conjecture and hence also the other conjectures such as the conjectures due to Borel and Novikov mentioned above.
- In particular the results of Higson-Lafforgue-Skandalis (2002) show that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$ is not bijective.

- Davis- Januszkiewics (1991) have constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

Open problems

- What are candidates for groups or closed aspherical manifolds for which the conjectures due to Farrell-Jones, Novikov or Borel may be false?
- There are still many interesting groups for which the Farrell-Jones Conjecture is open.
- Examples are:
  - Solvable groups;
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - Out(F_n);
  - Thompson groups.

- Can methods of proof for the Baum-Connes or the Farrell-Jones Conjecture be transferred from one setting to the other?

Comparing the Farrell-Jones and the Baum-Connes Conjecture

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_{\mathbb{Z}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{Z}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_{\mathbb{R}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{R}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_{C_r^*(?;\mathbb{R})}^p)[1/2] & \xrightarrow{\cong} & L_n^p(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{K}_{\mathbb{R}}^{\text{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{K}_{\mathbb{C}}^{\text{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{C}))[1/2]
\end{array}$$

Examples of Computations

Theorem (Torsionfree hyperbolic groups)

If G is a torsionfree hyperbolic group, then we get isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C),C\subseteq G,C\neq 1\\ C \text{ maximal cyclic}}} \left(\mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)\right) \right) \xrightarrow{\cong} K_n(RG),$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG).$$

- The Baum-Connes Conjecture and the Bost Conjecture are also known to be true for hyperbolic groups and reduce therefore for obvious reasons for a torsionfree hyperbolic group to

$$K_n(BG) \cong K_n(C_r^*(G)) \cong K_n(l^1(G)).$$

Comparing algebraic and topological K-theory

Theorem (L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

About the proof

- One needs to interpret the assembly map which is easiest described in terms of homotopy theory as a forget control homomorphism.
- Then the task is to show how to get control.
- This is achieved for hyperbolic and CAT(0)-groups by constructing flow spaces which mimic the geodesic flow on a Riemannian manifold with negative or non-positive sectional curvature.
- The proof of the inheritance results is of homotopy theoretic nature.
- Poly-cyclic groups are handled by transfer methods.