The Stable Cannon Conjecture

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Poincaré duality groups

Definition (Finite Poincaré complex)

A (connected) finite n-dimensional CW-complex X is a finite n-dimensional Poincaré complex if there is $[X] \in H_n(X; \mathbb{Z}^w)$ such that the induced $\mathbb{Z}\pi$-chain map

$$- \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence.

Theorem (Closed manifolds are Poincaré complexes)

A closed n-dimensional manifold M is a finite n-dimensional Poincaré complex with $w = w_1(X)$.

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A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP.
- $$H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

Theorem (Wall)

If G is a d-dimensional Poincaré duality group for $d \geq 3$ and $\widetilde{K}_0(\mathbb{Z}G) = 0$, then there is a model for BG which is a finite Poincaré complex of dimension d.

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Corollary

If M is a closed aspherical manifold of dimension d, then $\pi_1(X)$ is a d-dimensional Poincaré duality group.

Theorem (Eckmann-Müller, Linnell)

Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.

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A hyperbolic group G is a finitely generated group such that for one (and hence all) choice of symmetric finite set of generators the Cayley graph with the associated word metric is a hyperbolic geodesic metric space.

- A geodesic metric space is called hyperbolic if geodesic triangles are thin in comparison with geodesic triangles in $\mathbb{R}^2$.
- The property hyperbolic is a quasi-isometry invariant.
- Every hyperbolic group is finitely presented and has a finite model for its classifying space of proper actions.
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- A random finitely presented group is hyperbolic.
- One can assign to hyperbolic group a topological space called boundary ∂G such that for any geodesic hyperbolic metric space X on which G acts properly and cocompactly by isometries there is a compactification $\overline{X} = X \amalg \partial G$ such that ∂G is a Z-set in $\overline{X}$. This applies in particular to the Cayley graph.
- Notice that ∂G is independent of X.
- In general ∂G is totally disconnected, in other words, looks like a Cantor set.

Theorem (Hadamard)

If M is a closed smooth Riemannian manifold of dimension n, whose section curvature is negative, then $\pi = \pi_1(M)$ is a torsionfree hyperbolic group with $\partial\pi = S^{n-1}$.

- Actually $\widetilde{M}$ is a geodesic metric space on which π acts freely, properly and cocompactly by isometries.
- There is a diffeomorphism $\widetilde{M} \xrightarrow{\cong} \mathbb{R}^n$ and $\partial\pi$ is the sphere S^{n-1} at infinity.

The main conjectures

Conjecture (Gromov)

Let G be a torsionfree hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

Gromov's Conjecture is true for $n \geq 6$.

Conjecture (Wall)

Every Poincaré duality group is the fundamental group of an aspherical closed manifold.

Conjecture (Cannon's Conjecture in the torsionfree case)

A torsionfree hyperbolic group G has S^2 as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

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Theorem (Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)

A Poincaré duality group G of dimension 3 is the fundamental group of an aspherical closed 3-manifold if and only if it is quasi-isometric to the fundamental group of an aspherical closed 3-manifold.

- A closed 3-manifold is a Seifert manifold if it admits a finite covering $\overline{M} \to M$ such that there exists a S^1-principal bundle $S^1 \to \overline{M} \to S$ for some closed orientable surface S.

Theorem (Bowditch)

If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.

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Theorem (Bestvina)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to S^2.

Theorem (Bestvina-Mess)

Let G be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold M. Then M is hyperbolic and G satisfies the Cannon Conjecture.

- In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group G, whose boundary is S^2, that it is quasiisometric to the fundamental group of some aspherical closed 3-manifold.

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Theorem

Let G be the fundamental group of an aspherical oriented closed 3-manifold. Then G satisfies:

- *G is residually finite and Hopfian;*
- *All its L^2-Betti numbers $b_n^{(2)}(G)$ vanish;*
- *Its deficiency is 0. In particular it possesses a presentation with the same number of generators and relations;*
- *Suppose that M is hyperbolic. Then G is virtually compact special and linear over $\mathbb{Z}$. It contains a subgroup of finite index G' which can be written as an extension $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$ for some closed orientable surface S.*

- Recall that any finitely presented groups occurs as the fundamental group of a closed d-dimensional smooth manifold for every $d \geq 4$.

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A torsionfree hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary ∂G and S^{n-1} have the same Čech cohomology.

Theorem

If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to S^n.

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The main results

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

satisfying

1. *The space BG is a finite 3-dimensional CW-complex;*
2. *The map $H_n(f, \mathbb{Z}) \colon H_n(M; \mathbb{Z}) \xrightarrow{\cong} H_n(BG; \mathbb{Z})$ is bijective for all $n \geq 0$;*
3. *The simple algebraic surgery obstruction $\sigma(f, \overline{f}) \in L_3^s(\mathbb{Z}G)$ vanishes.*

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Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Stable Cannon Conjecture)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2.

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

such that the map f is a simple homotopy equivalence.

Theorem (Stable Cannon Conjecture, continued)

Moreover:

Let $\widehat{M} \to M$ *be the G-covering associated to the composite of the isomorphism* $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ *with the projection* $G \times \pi_1(N) \to G$. *Suppose additionally that N is aspherical and* $\dim(N) \geq 3$.

Then $\widehat{M}$ *is homeomorphic to* $\mathbb{R}^3 \times N$. *Moreover, there is a compact topological manifold* $\overline{\widehat{M}}$ *whose interior is homeomorphic to* $\widehat{M}$ *and for which there exists a homeomorphism of pairs*

$$(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N).$$

- The last two theorems follow from the Cannon Conjecture.
- By the product formula for surgery theory the second last theorem implies the last theorem.
- The manifold M appearing in the last theorem is unique up to homeomorphism by the Borel Conjecture, provided that $\pi_1(N)$ satisfies the Farrell-Jones Conjecture.
- If we take $N = T^k$ for some $k \geq 2$, then the Cannon Conjecture is equivalent to the statement that this M is homeomorphic to $M' \times T^k$ for some closed 3-manifold M'.

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The existence of a normal map

Theorem (Existence of a normal map)

Let X be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle ξ over BG and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

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Proof.

- Stable vector bundles over X are classified by the first and second Stiefel-Whitney class $w_1(\xi)$ and $w_2(\xi)$ in $H^*(X;\mathbb{Z}/2)$.
- Let ξ be a k-dimensional vector bundle over X such that $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(\xi) \cup w_1(\xi)$ holds.
- A spectral sequence argument applied to $\Omega_3(X, w_1(X))$ shows that there is a closed 3-manifold M together with a map $f\colon M \to X$ of degree one such that $f^* w_1(X) = w_1(M)$.
- Then $w_1(f^*\xi) = w_1(M)$ and the Wu formula implies $w_2(M) = w_1(f^*\xi) \cup w_1(f^*\xi)$.
- Hence $f^*\xi$ is stably isomorphic to the stable tangent bundle of M and we get the desired normal map.

□

The total surgery obstruction

- Consider an aspherical finite n-dimensional Poincaré complex X such that $G = \pi_1(X)$ is a Farrell-Jones group, i.e., satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive categories, and $\mathcal{N}(X)$ is non-empty. (For simplicity we assume $w_1(X) = 0$ in the sequel.)

- We have to find one normal map of degree one

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- Recall that the simple surgery obstruction defines a map

$$\sigma^s \colon \mathcal{N}(X) \to L_n^s(\mathbb{Z}G).$$

- Fix a normal map $(f_0, \overline{f_0})$.

- Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}(X) & \xrightarrow{\sigma^s(-,-)-\sigma^s(f_0,\overline{f_0})} & L_n^s(\mathbb{Z}G) \\ s_0 \downarrow \cong & & \cong \uparrow \operatorname{asmb}_n^s(X) \\ H_n(X;\mathbf{L}_{\mathbb{Z}}^s\langle 1\rangle) & \xrightarrow[H_n^G(\operatorname{id}_X;\mathbf{i})]{} & H_n(X;\mathbf{L}_{\mathbb{Z}}^s) \end{array}$$

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of $(f, \overline{f})$ to the difference $\sigma^s(f, \overline{f}) - \sigma^s(f, \overline{f_0})$ of simple surgery obstructions.

- An easy spectral sequence argument yields a short exact sequence

$$0 \to H_n(X; \mathbf{L}^s_{\mathbb{Z}}\langle 1 \rangle) \xrightarrow{H_n(\mathrm{id}_X; \mathbf{i})} H_n(X; \mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

- Consider the composite

$$\mu^s_n(X) \colon \mathcal{N}(X) \xrightarrow{\sigma^s} L^s_n(\mathbb{Z}G, w) \xrightarrow{\mathrm{asmb}^s_n(X)^{-1}} H_n(X; \mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

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- We conclude that there is precisely one element, called the total surgery obstruction,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element $[(f, \overline{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$.

Theorem (Total surgery obstruction)

- *There exists a normal map of degree one $(f, \overline{f})$ with target X and vanishing simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G)$ if and only if $s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$ vanishes.*
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ANR-homology manifolds

Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a closed ANR-homology manifold.

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- Every closed topological manifold is a closed ANR-homology manifold.
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5. Then M is a topological manifold if and only if $\iota(M) = 1$.*
- *The Quinn obstruction and the total surgery obstruction are related for an aspherical closed ANR-homology manifold M of dimension ≥ 5 by*

$$\iota(M) = 8 \cdot s(X) + 1.$$

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Proof of the Theorem about the vanishing of the surgery obstruction

Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex X that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$8 \cdot s(X \times Y) + 1 = (8 \cdot s(X) + 1) \cdot (8 \cdot s(y) + 1).$$

- This implies

$$s(X \times T^3) = s(X).$$

- Hence it suffices to show that $s(X \times T^3)$ vanishes.

Proof of the Theorem about the vanishing of the surgery obstruction

Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex X that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$8 \cdot s(X \times Y) + 1 = (8 \cdot s(X) + 1) \cdot (8 \cdot s(y) + 1).$$

- This implies

$$s(X \times T^3) = s(X).$$

- Hence it suffices to show that $s(X \times T^3)$ vanishes.

Proof (continued).

- There exists an aspherical closed ANR-homology manifold M and a homotopy equivalence $f\colon M \to X \times T^3$.
- There is a Z-compactification $\overline{\widetilde{X}}$ of $\widetilde{X}$ by the boundary $\partial G = S^2$.
- One constructs an appropriate Z-compactification $\overline{\widetilde{M}}$ of $\widetilde{M}$ so that we get a ANR-homology manifold $\overline{\widetilde{M}}$ whose boundary is a topological manifold and whose interior is $\widetilde{M}$.
- By adding a collar to $\overline{\widetilde{M}}$ one obtains a ANR-homology manifold Y which contains $\widetilde{M}$ as an open subset and contains an open subset U which is homeomorphic to $\mathbb{R}^6$.

□

Proof (continued).

- Hence we get

$$8s(X \times T^3) + 1 = 8s(M) + 1 = i(M) = i(\widetilde{M})$$
$$= i(Y) = i(U) = i(\mathbb{R}^6) = 1.$$

- This implies $s(X \times T^3) = 0$ and hence $s(X) = 0$.

□

Appendix: Pulling back the boundary

Notation

Let $(\overline{Y}, Y)$ be a topological pair. Put $\partial Y := \overline{Y} \setminus Y$. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Pulling back the boundary is a construction of a topological pair $(\overline{X}, X)$ and a continuous map $\overline{f}\colon \overline{X} \to X$

- It has the desired universal universal property which we will not state here.
- Its basic properties are:

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Let $(\overline{Y}, Y)$ be a topological pair. Put $\partial Y := \overline{Y} \setminus Y$. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Pulling back the boundary is a construction of a topological pair $(\overline{X}, X)$ and a continuous map $\overline{f}\colon \overline{X} \to X$

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- Its basic properties are:

Lemma

- *$Y \subseteq \overline{Y}$ is dense and the closure of the image of f in $\overline{Y}$ contains ∂Y, then $X \subseteq \overline{X}$ is dense;*
- *$\overline{Y}$ is compact, $Y \subseteq \overline{Y}$ is open and $f\colon X \to Y$ is proper. Then $\overline{X}$ is compact;*
- *We have for the topological dimension of $\overline{X}$*

$$\dim(\overline{X}) \leq \dim(X) + \dim(\overline{Y}) + 1;$$

- *The induced map $\overline{f}$ induces a homeomorphism $\partial f\colon \partial X \to \partial Y$.*

Definition (Z-set)

A closed subset Z of a compact *ANR* X is called a Z-set if for every open subset U of X the inclusion $U \setminus (U \cap Z) \to U$ is a homotopy equivalence.

- The boundary of a manifold is a Z-set in the manifold.

Lemma

Consider a pair $(\overline{Y}, Y)$ of spaces such that $\overline{Y}$ is a ANR and ∂Y is a Z-set in $\overline{Y}$. Consider a homotopy equivalence $f \colon X \to Y$ which is continuously controlled. Let $(\overline{f}, f) \colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary along f.

Then $\overline{X}$ is an ANR and $\partial X \subseteq \overline{X}$ is a Z-set.

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