L^2-Invariants and Their Applications to Geometry, Group Theory and Spectral Theory

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0 Introduction

During the last decades mathematics has developed at an incredible speed and a large amount of new information and results have been accumulated. Therefore mathematics faces the problem that it breaks up into different areas which may not communicate among one another. Fortunately recent developements go in the opposite direction. In particular interactions of different fields have turned out to be very fruitful and lead to new ideas and innovations. The key observation is that some of the techniques developed in one specific field can be exported to other areas and be successfully used to solve problems there. This is only possible if the techniques are so well examined and documented that they are quickly accessible to advanced mathematicians, who are not experts at the particular field, and can be understood by graduate students within a reasonable period of time.

In this article we will present an example of such a new and successful development, namely L^2-invariants. They are modelled on classical notions like homology, Betti numbers or Reidemeister torsion for compact spaces and extend these to non-compact spaces with appropriate group actions. In order to convince the reader about the high potential of L^2-methods, we will present some applications of L^2-invariants to problems about groups, manifolds and K-theory. These problems will have a priori nothing to do with L^2-invariants but their solution will heavily rely on L^2-methods. The study of L^2-invariants requires input from and is linked to topology, geometry, global analysis, operator theory and K-theory and is of interest for representatives of these fields. A lot of work about L^2-invariants has successfully been done but also a lot of very interesting problems are still open. They will create stimulating and highly advanced activities in 2001 and beyond. The challenges are unlimited.

1 Some Theorems

We state some theorems which seem to have nothing to do with L^2-invariants but – as we will see – whose proofs use L^2-methods. The selection below consists of some easy to formulate examples and is not meant to represent the most important results about L^2-invariants, there are plenty of other very interesting and important theorems about them. For simplicity we will often not state the most general formulations. The results below are taken from Cheeger-Gromov [4], Dodziuk [5], Gromov [11], Lück [19],[20] and Cochran-Orr-Teichner [3].

Theorem 1. *Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite CW-model for its classifying space BG. Then its Euler characteristic vanishes, i.e.*

$$\chi(G) := \chi(BG) = 0.$$

Theorem 2. *Let M be a closed hyperbolic manifold of dimension $2n$. Then*

$$(-1)^n \cdot \chi(M) > 0.$$

Theorem 3. *Let M be a closed Kähler manifold of (real) dimension $2n$. Suppose that M is homotopy equivalent to a closed Riemannian manifold with negative sectional curvature. Then*

$$(-1)^n \cdot \chi(M) > 0.$$

Theorem 4. *Let $1 \to H \to G \to K \to 1$ be an extension of infinite groups such that H is finitely generated, G is finitely presented and K contains an element of infinite order. Then*

1. *The deficiency of G satisfies $\operatorname{def}(G) \leq 1$;*
2. *If M is a closed connected oriented 4-manifold with $\pi_1(M) \cong G$, then we get for its signature $\operatorname{sign}(M)$ and its Euler characteristic $\chi(M)$*

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Theorem 5. *Let G be a group and $\mathbb{C}G$ be its complex group ring. Let $G_0(\mathbb{C}G)$ be the Grothendieck group of finitely generated $\mathbb{C}G$-modules. Then*

1. *If G is amenable, the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ of $\mathbb{C}G$ itself is an element of infinite order;*
2. *If G contains the free group $\mathbb{Z} * \mathbb{Z}$ of rank two, then $[\mathbb{C}G] = 0$ in $G_0(\mathbb{C}G)$.*

Theorem 6. *There are non-slice knots in 3-space whose Casson-Gordon invariants are all trivial.*

Here are some explanations. A group G is called *amenable* if there is a G-invariant linear operator $\mu : l^\infty(G, \mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ which satisfies $\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\}$ for all $f \in l^\infty(G, \mathbb{R})$. If a group G contains $\mathbb{Z} * \mathbb{Z}$ as subgroup, it is not amenable. The converse does not hold but (at the time of writing) there is no finitely presented counterexample. Any abelian or finite group is amenable.

The deficiency of a finitely presented group G is the maximum over all differences $g - r$ for all presentations $\langle s_1, s_2, \dots s_g \mid R_1, R_2, \dots R_r \rangle$ of G. It plays an important role in group theory and low dimensional topology. For its computation it is important to have upper bounds. The problem is that the deficiency of a finitely presented group is not always given by the "obvious" presentation. The deficiency of $*_{i=1}^{g}\mathbb{Z}$, $\mathbb{Z}/n$ and $\mathbb{Z}/n \times \mathbb{Z}/n$ is g, 0 and -1 and in these cases they are given by the "obvious" presentations $\langle s_1, s_2 \dots s_g \mid \emptyset \rangle$, $\langle s \mid s^n = 1 \rangle$ and $\langle s, t \mid s^n = t^n = [s,t] = 1 \rangle$. However, the "obvious" presentation of $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ is

$$\langle s_1, t_1, s_2, t_2 \mid s_1^2 = t_1^2 = [s_1, t_1] = s_2^3 = t_2^3 = [s_2, t_2] \rangle$$

but its deficiency is -1 and not -2.

The group $G_0(\mathbb{C}G)$ is the abelian group defined by generators and relations as follows. Generators are isomorphism classes of finitely generated $\mathbb{C}G$-modules. For any exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of finitely generated $\mathbb{C}G$-modules one has the relation $[M_0] - [M_1] + [M_2] = 0$. It should not be confused with the projective class group $K_0(\mathbb{C}G)$ which is defined analogously for finitely generated projective $\mathbb{C}G$-modules. Hardly anything is known about $G_0(\mathbb{C}G)$ for infinite groups G at the time of writing.

A knot in the 3-sphere is *slice* if there exists a locally flat topological embedding of the 2-disk into D^4 whose restriction to the boundary is the given knot. For a long time Casson-Gordon invariants have been the only known obstructions for a knot to be slice. Cochran, Orr and Teichner give in [3] new obstructions for a knot to be slice using L^2-signatures. Thus they can construct an explicit knot, which is not slice but whose Casson-Gordon invariants are all trivial, as stated in Theorem 6.

2 L^2-Betti Numbers

In this section we give the basic definitions and properties of L^2-Betti numbers. Let G be a group. Let $l^2(G)$ be the Hilbert space of formal sums $\sum_{g \in G} \lambda_g \cdot g$ with complex coefficients λ_g such that $\sum_{g \in G} |\lambda_g|^2 < \infty$. The *group von Neumann algebra* $\mathcal{N}(G)$ is the C^*-algebra $B(l^2(G))^G$ of bounded G-operators $l^2(G) \to l^2(G)$. The *von Neumann trace* $\mathrm{tr} : \mathcal{N}(G) \to \mathbb{C}$ sends f to $\langle f(e), e \rangle_{l^2(G)}$, where $e \in G$ is the unit element. It extends to (n,n)-matrices over $\mathcal{N}(G)$ by taking the sum of the traces of the diagonal entries. A *finitely generated Hilbert $\mathcal{N}(G)$-module* V is a Hilbert space V with isometric linear G-action such that there exists a G-projection $p : l^2(G)^n \to l^2(G)^n$ for some natural number n with the property that $\mathrm{im}(p)$ is isometrically linearly G-isomorphic to V. Notice that the projection is not part of the structure, only its existence is required. Define the *von Neumann dimension* $\dim(V) \in [0, \infty)$ of V to be the trace of such a projection p. This is independent of the choice of p. Every possible element in $[0, \infty)$ can occur as $\dim(V)$.

Let X be a free finite G-CW-complex, or equivalently, a G-space occurring as the total space of a G-covering $X \to X/G$ with a compact CW-complex as base space. Let $C_*(X)$ be its cellular $\mathbb{Z}G$-chain complex. Define the *L^2-chain complex* $C_*^{(2)}(X)$ by $l^2(G) \otimes_{\mathbb{Z}G} C_*(X)$. Its chain modules are finite sums of copies of $l^2(G)$ and its differentials $c_p^{(2)}$ are bounded G-operators. Define the *L^2-homology* $H_p^{(2)}(X)$ to be the finitely generated Hilbert $\mathcal{N}(G)$-module $\ker(c_p^{(2)})/\overline{\operatorname{im}(c_{p+1}^{(2)})}$. Notice that we divide by the closure of the image of the $(p+1)$-th differential and not by the image itself in order to get a complete space and thus a Hilbert $\mathcal{N}(G)$-module.

Definition 7. Define the *p-th L^2-Betti number* of the finite free G-CW-complex X by

$$b_p^{(2)}(X;\mathcal{N}(G)) \ := \ \dim(H_p^{(2)}(X)).$$

If Y is a compact connected CW-complex with universal covering $\widetilde{Y}$, we abbreviate

$$b_p^{(2)}(\widetilde{Y}) \ := b_p^{(2)}(\widetilde{Y};\mathcal{N}(\pi_1(Y))).$$

Whenever one introduces a new notion, one should try to justify it. We will do this by explaining that the L^2-Betti numbers have nice and useful properties and that we can give direct applications of this notion.

The L^2-Betti numbers have a lot of the properties we are used to for (classical) Betti numbers. They are G-homotopy invariants in the sense that $b_p^{(2)}(X) = b_p^{(2)}(Y)$ holds, provided that there is a G-homotopy equivalence $X \to Y$. The Euler characteristic of $\chi(G\backslash X)$ can be computed by $\sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(X)$. They satisfy Poincaré duality, Künneth formula and Morse inequalities, just replace in the corresponding formulas for classical Betti numbers the Betti numbers by L^2-Betti numbers and the spaces by their universal coverings. If X is connected, $b_0^{(2)}(X) = |G|^{-1}$. If G is finite, then $b_p^{(2)}(X;\mathcal{N}(G))$ is the same as $|G|^{-1} \cdot b_p(X)$. There is one important difference between L^2-Betti numbers and Betti numbers. Namely, the L^2-Betti numbers are in contrast to the classical ones multiplicative under finite coverings, i.e. for any d-sheeted covering $X \to Y$ we have $b_p^{(2)}(\widetilde{X}) = d \cdot b_p^{(2)}(\widetilde{Y})$. The corresponding statement $b_p(X) = d \cdot b_p(Y)$ for the classical Betti numbers is in general not true as the d-sheeted covering $S^1 \to S^1, \quad z \mapsto z^d$ shows. From multiplicativity we conclude that $b_p^{(2)}(\widetilde{S^1}) = 0$ for all $p \geq 0$.

Example 8. The following example is quite illuminating although it covers only a comparatively trivial case. Namely, if G is the free abelian group $\mathbb{Z}^n$ of rank n, all these notions can be made much more explicit. One can identify $l^2(\mathbb{Z}^n)$ with the Hilbert space $L^2(T^n)$ of measurable L^2-integrable functions from the torus T^n to $\mathbb{C}$ by Fourier transform. The von Neumann algebra $\mathcal{N}(\mathbb{Z}^n)$ becomes the space $L^\infty(T^n)$ of measurable essentially bounded functions $T^n \to \mathbb{C}$. The von Neumann trace tr sends an element $f \in L^\infty(T^n)$ to its integral $\int_{T^n} f dvol$. An idempotent in $\mathcal{N}(\mathbb{Z}^n)$ is given by a characteristic function χ_M of a measurable subset $M \subset T^n$. The von Neumann dimension of the associated Hilbert $\mathcal{N}(\mathbb{Z})$-submodule $\{f \in L^2(T^n) \mid \chi_M \cdot f = f\}$ of $L^2(T^n)$ is the volume of M. Let $\overline{X} \to X$ be a $\mathbb{Z}^n$-covering of a finite CW-complex X. Denote by F the quotient field of $\mathbb{C}[\mathbb{Z}^n]$. Then the L^2-Betti number $b_p(\overline{X}, \mathcal{N}(\mathbb{Z}^n))$ coincides with the dimension of the F-vector space $F \otimes_{\mathbb{C}[\mathbb{Z}^n]} H_p(\overline{X}, \mathbb{C})$, where $H_p(\overline{X}, \mathbb{C})$ is the singular homology of X with coefficients in $\mathbb{C}$ and the linear $\mathbb{Z}^n$-action coming from the $\mathbb{Z}^n$-action on $\overline{X}$. We will prove an analogous statement for amenable groups G in Corollary 12. If G is not amenable one cannot read off $b_p(\overline{X}, \mathcal{N}(G))$ from the $\mathbb{C}G$-module $H_p(\overline{X}, \mathbb{C})$ in general.

There is a L^2-analogue of the Hodge-deRham Theorem. Let $\overline{M} \to M$ be a G-covering of a closed Riemannian manifold M. Denote by $\mathcal{H}^p_{(2)}(\overline{M})$ the space of L^2-integrable harmonic forms on $\overline{M}$, i.e. smooth p-forms ω on $\overline{M}$ such that ω lies in the kernel of the Laplacian Δ_p and $\int_{\overline{M}} \omega \wedge * \omega < \infty$. Then there is an isometric linear G-isomorphism

$$\mathcal{H}^p_{(2)}(\overline{M}) \xrightarrow{\cong} H_p^{(2)}(\overline{M}; \mathcal{N}(G)).$$

A consequence of this result is that the definition presented here agrees with the original analytic definition of L^2-Betti numbers in terms of the large time behaviour of the trace of the heat kernel $e^{-t\Delta_p}(x,y)$ on $\overline{M}$, which was given by Atiyah in connection with his L^2-index theorem [1]. Namely, for a fundamental domain F of the G-action on $\overline{M}$ and $\mathrm{tr}(e^{-t\Delta_p}(x,x))$ the trace of the endomorphism $e^{-t\Delta_p}(x,x)$ of a finite-dimensional real vector space, Atiyah puts

$$b_p^{(2)}(\overline{M}; \mathcal{N}(G)) \ := \ \lim_{t \to \infty} \int_F \mathrm{tr}(e^{-t\Delta_p}(x,x)) dx.$$

Now we can outline a proof of Theorem 2. The universal covering $\widetilde{M}$ is the hyperbolic space $\mathbb{H}^{2n}$ and a direct calculation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^{2n})$ is zero for $p \neq n$ and different from zero for $p = n$. Since the von Neumann dimension is faithful, we conclude $b_p^{(2)}(\widetilde{M}) = 0$ for $p \neq n$ and $b_n^{(2)}(\widetilde{M}) > 0$. This implies $(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) > 0$.

Next we indicate the proof of Theorem 4. The hard part which we will not explain is to show that $b_1^{(2)}(G) := b_1^{(2)}(EG; \mathcal{N}(G))$ vanishes under the assumptions of Theorem 4. Notice that BG has finite 2-skeleton so that the definition of $b_1^{(2)}(G)$ makes sense (see also Definition 10). We have to show for any presentation $\langle s_1, s_2, \dots s_g \mid R_1, R_2 \dots R_r \rangle$ of G that $g - r \leq 1$. Let X be the finite 2-dimensional CW-complex associated to this presentation. It has 1 cell of dimension zero, g cells of dimension one and r cells of dimension two. Since the classifying map $f : X \to BG$ is 2-connected, we conclude $b_p^{(2)}(\widetilde{X}) = b_p^{(2)}(G) = 0$ for $p = 0, 1$. This implies

$$g - r = 1 - \chi(X) = 1 - b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}) \leq 1.$$

If M is an oriented closed 4-manifold with $\pi_1(M) \cong G$, we get $b_p^{(2)}(\widetilde{M}) = b_p^{(2)}(G) = 0$ for $p \leq 1$. Poincaré duality implies $\chi(M) = b_2^{(2)}(\widetilde{M})$. By the L^2-index theorem of Atiyah [1], $\operatorname{sign}(M) = \dim(H_p^{(2)}(\widetilde{M})_+) - \dim(H_p^{(2)}(\widetilde{M})_-)$ for some subspaces $H_p^{(2)}(\widetilde{M})_\pm$ of $H_p^{(2)}(\widetilde{M})$. This implies

$$|\operatorname{sign}(M)| \leq \dim(H_p^{(2)}(\widetilde{M})) = \chi(M).$$

One may ask whether the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ are linked to the ordinary Betti numbers $b_p(X)$ for a finite CW-complex X. Except for the equality $\sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\widetilde{X}) = \sum_{p\geq 0}(-1)^p \cdot b_p(X) = \chi(X)$ there seems to be no relations. There are examples of l-dimensional finite CW-complexes X for $l \geq 2$ such that $b_p^{(2)}(\widetilde{X})$ is any given non-negative rational number for $1 \leq p \leq l-1$ and $b_p(X) = 0$ for $1 \leq p \leq l-1$, or on the other hand such that $b_p(X)$ is any given non-negative integer for $1 \leq p \leq l-1$ and $b_p^{(2)}(\widetilde{X}) = 0$ for $p \leq l-1$. There is however an asymptotic relation. Namely, let X be a finite CW-complex such that there is a nested sequence $\pi_1(X) = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ of normal subgroups Γ_n of $\pi_1(X)$ of finite index $[\pi_1(X) : \Gamma_n]$ with $\cap_{n\geq 0}\Gamma_n = \{1\}$. Let $X_n \to X$ be the covering associated to $\Gamma_n \subset \pi_1(M)$. Then [18]

$$b_p^{(2)}(\widetilde{X}) \;=\; \lim_{n\to\infty} \frac{b_p(X_n)}{[\pi_1(X) : \Gamma_n]}.$$

3 An Algebraic Approach

In this section we develop a more algebraic approach to L^2-Betti numbers following [20] (see also [8]), where we forget the topology on $\mathcal{N}(G)$ and consider its ring structure only. This algebraic approach will give us more flexibility. Thus we will be able to extend this notion to more general G-spaces. This will be the basic ingredient for further applications.

Let R be an associative ring with unit. Let K be a R-submodule of an R-module M. Define the *closure of K in M* to be the R-submodule of M

$$\overline{K} := \{x \in M \mid f(x) = 0 \text{ for all } f \in \hom_R(M,R) \text{ with } K \subset \ker(f)\}.$$

For a finitely generated R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\mathbf{T}M := \overline{\{0\}} = \{x \in M \mid f(x) = 0 \text{ for all } f \in \hom_R(M,R)\};$$
$$\mathbf{P}M := M/\mathbf{T}M.$$

If P is a finitely generated projective $\mathcal{N}(G)$-module, there is an idempotent $A \in M_n(\mathcal{N}(G))$ such that $\operatorname{im}(A)$ is $\mathcal{N}(G)$-isomorphic to P. Define the *von Neumann dimension* of P by the von Neumann trace of A. This number $\dim(P) \in [0,\infty)$ depends only on the isomorphism class of P and not on the choice of A.

The group von Neumann algebra $\mathcal{N}(G)$ has one very important and useful property as a ring, it is a semi-hereditary ring, i.e. any finitely generated submodule of a projective module is projective. This has the consequence that for a submodule $K \subset M$ of a finitely generated $\mathcal{N}(G)$-module M the quotient $M/\overline{K}$ is finitely generated and projective and $\overline{K}$ is a direct summand in M. In particular we conclude for a finitely generated $\mathcal{N}(G)$-module M that $\mathbf{P}M$ is finitely generated projective and

$$M \cong \mathbf{P}M \oplus \mathbf{T}M. \tag{1}$$

Theorem 9. *There is precisely one function which associates to an arbitrary $\mathcal{N}(G)$-module M an element $\dim(M) \in [0, \infty]$ and has the following properties.*

1. *Continuity*
   If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(G)$-module M, then
   $$\dim(K) \;=\; \dim(\overline{K});$$
2. *Cofinality*
   Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \cup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then
   $$\dim(M) \;=\; \sup\{\dim(M_i) \mid i \in I\};$$
3. *Additivity*
   If $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$ is an exact sequence of $\mathcal{N}(G)$-modules, then
   $$\dim(M_1) \;=\; \dim(M_0) + \dim(M_2),$$
   where $r + s$ for $r, s \in [0, \infty]$ is the ordinary sum of two real numbers, if both r and s are not ∞, and is ∞ otherwise;
4. *Extension Property*
   If M is finitely generated projective, then $\dim(M)$ agrees with the previous notion.

Notice that there are some similarities between the ring $\mathbb{Z}$ of the integers and the ring $\mathcal{N}(G)$. If $R = \mathbb{Z}$ and M is a finitely generated abelian group, then $\mathbf{T}M$ is just its torsion submodule in the ordinary sense. The splitting (1) in the special case $R = \mathbb{Z}$ is the splitting of a finitely generated abelian group as the direct sum of its torsion subgroup and a finitely generated free abelian group. The von Neumann dimension of a finitely generated $\mathcal{N}(G)$-module M with $\mathbf{T}M = M$ is zero in analogy to the fact that the rank of a finite abelian group is zero. If one replaces in the statements of Theorem 9 $\mathcal{N}(G)$ by $\mathbb{Z}$ and requires in the Extension Property that $\dim(M)$ for a finitely generated abelian group is the usual rank, then all statements remain true and $\dim(M)$ becomes the dimension of the rational vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$. However, there are two important differences. A finite von Neumann algebra is in general not Noetherian and hence harder to study than the Noetherian ring $\mathbb{Z}$. On the other hand the dimension of a finitely generated projective $\mathcal{N}(G)$-module can be an arbitrary small positive real number, and hence the dimension of a countable direct sum of non-trivial finitely generated projective $\mathcal{N}(G)$-modules can be a finite number. This can never happen over $\mathbb{Z}$.

Now consider a topological space X with a G-action. Denote by $H_*(X;\mathcal{N}(G))$ the singular homology of X with coefficients in $\mathcal{N}(G)$, i.e. the homology of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)$.

Definition 10. Define the *p-th L^2-Betti number* of the G-space X by

$$b_p^{(2)}(X;\mathcal{N}(G)) \ := \ \dim(H_*(X;\mathcal{N}(G))) \ \in [0,\infty].$$

Define the *p-th L^2-Betti number* of a group G by

$$b_p^{(2)}(G) \ := \ b_p^{(2)}(EG;\mathcal{N}(G)) \ \in [0,\infty].$$

If X is a finite free G-CW-complex, Definition 7 and Definition 10 agree. If furthermore X is the total space of a G-covering $\overline{M} \to M$ of a closed Riemannian manifold M, then these two definitions agree with Atiyah's heat kernel definition $\lim_{t\to\infty} \int_F \operatorname{tr}(e^{-t\Delta_p}(x,x))dx$.

The following theorem is the main ingredient in some of the applications.

Theorem 11. *Let G be an amenable group. Then $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$ in the sense that for any $\mathbb{C}G$-module M we have*

$$\dim(\operatorname{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G),M)) \ = \ 0 \qquad \textit{for } p \geq 1.$$

The von Neummann algebra $\mathcal{N}(G)$ is flat over $\mathbb{C}G$ if G is virtually cyclic. Conjecturally virtually cyclic groups are the only groups with this property. If $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, then $H_p(X;\mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathbb{C}G} H_p(X)$ because the corresponding universal coefficient spectral sequence collapses. If G is amenable, the associated universal spectral sequence does not collapse on the nose but from the dimension point of view by Theorem 9 and Theorem 11. Therefore we get

Corollary 12. *Let G be amenable and X a G-space. Then*

$$b_p^{(2)}(X;\mathcal{N}(G)) \ = \ \dim(\mathcal{N}(G) \otimes_{\mathbb{Z}G} H_p(X;\mathbb{Z})).$$

Now we can give the proof of Theorem 1. If G itself is infinite amenable, then $b_0^{(2)}(G) = 0$ since G is infinite, and $b_p^{(2)}(G) = 0$ for $p \geq 1$ by Corollary 12 since $H_p(EG) = 0$ for $p \geq 1$. If G contains a normal infinite subgroup H, then all L^2-Betti numbers of H vanish. There is a Serre spectral sequence associated to the fibration $BH \to BG \to B(G/H)$ converging to $H_{p+q}(EG;\mathcal{N}(G))$. The dimension of $E^2_{p,q}$ vanishes since all L^2-Betti numbers of H vanish, and the claim follows. Notice that with our purely algebraic approach classical machinery like homological algebra and spectral sequences applies directly.

Next we give the proof of Theorem 5. Suppose that G is amenable. By Theorem 11 the following map is well-defined because it is compatible with the relations in $G_0(\mathbb{C}G)$

$$\dim : G_0(\mathbb{C}G) \to \mathbb{R} \qquad [M] \mapsto \dim(\mathcal{N}(G) \otimes_{\mathbb{C}G} M).$$

It sends $[\mathbb{C}G]$ to 1 and hence $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ has infinite order. Now suppose that $\mathbb{Z} * \mathbb{Z} \subset G$. The inclusion i induces by induction a homomorphism $i_* : G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}]) \to G_0(\mathbb{C}G)$ which sends $[\mathbb{C}[\mathbb{Z} * \mathbb{Z}]]$ to $[\mathbb{C}G]$. The cellular chain complex of the universal covering of $S^1 \vee S^1$ yields a short exact $\mathbb{C}[\mathbb{Z}*\mathbb{Z}]$-sequence $0 \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}] \oplus \mathbb{C}[\mathbb{Z}*\mathbb{Z}] \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}] \to \mathbb{C} \to 0$. This shows $[\mathbb{C}[\mathbb{Z} * \mathbb{Z}]] = -[\mathbb{C}] \in G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$. Choose an epimorphism $f : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$. It induces by restriction a homomorphism $f^* : G_0(\mathbb{C}\mathbb{Z}) \to G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$ which sends $[\mathbb{C}] \in G_0(\mathbb{C}\mathbb{Z})$ to $[\mathbb{C}] \in G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$. The cellular chain complex of the universal covering of S^1 yields a short exact $\mathbb{C}\mathbb{Z}$-sequence $0 \to \mathbb{C}\mathbb{Z} \to \mathbb{C}\mathbb{Z} \to \mathbb{C} \to 0$. This shows $[\mathbb{C}] = 0 \in G_0(\mathbb{C}\mathbb{Z})$. We conclude $[\mathbb{C}G] = 0 \in G_0(\mathbb{C}G)$.

4 Novikov-Shubin Invariants and L^2-Torsion

There are further L^2-invariants which are analytically defined in terms of the heat kernel on the universal covering $\widetilde{M}$ of a closed Riemannian manifold M, namely the Novikov-Shubin invariants $\alpha_p(\widetilde{M})$ and the L^2-torsion $\rho^{(2)}(\widetilde{M})$ (see [15], [22], and [23]). Novikov-Shubin invariants measure how fast $\int_F \operatorname{tr}(e^{-t\Delta_p}(x,x))dx$ approaches its limit $b_p^{(2)}(\overline{M}, \mathcal{N}(G)))$ for $t \to \infty$, or equivalently, the difference between the L^2-homology $H_p^{(2)}(\overline{M}, \mathcal{N}(G)) := \ker(c_p^{(2)})/\overline{\operatorname{im}(c_{p+1}^{(2)})}$ and its unreduced version $\ker(c_p^{(2)})/\operatorname{im}(c_{p+1}^{(2)})$. The definition of L^2-torsion is modelled upon the classical notion of Reidemeister torsion. Novikov-Shubin invariants and L^2-torsion have topological counterparts in terms of the combinatorial Laplace operator on the cellular chain complex, which are known to coincide with their analytical versions [2], [7]. The proof in [2] consists of a deep analysis of the Witten deformation of the L^2-deRham complex of the universal covering. Novikov-Shubin invariants are homotopy invariants. The L^2-torsion is a simple homotopy invariant, provided that all L^2-Betti numbers vanish. There is the conjecture that the L^2-torsion is even a homotopy invariant if all L^2-Betti numbers vanish. This conjecture is equivalent to the K-theoretic statement that the homomorphism induced by the Fuglede-Kadison determinant $\operatorname{Wh}(\pi_1(M)) \to \mathbb{R}$ is trivial. Thus one gets nice connections between heat kernels and geometry. The L^2-torsion $\rho(\widetilde{M})$ of a closed hyperbolic odd dimensional manifold is known to be proportional to the volume. This reproves at least in the odd dimensional case the well-known statement that the volume of a closed hyperbolic manifold depends only on its fundamental group. The L^2-torsion $\rho^{(2)}(\widetilde{M})$ of an irreducible compact orientable 3-manifold M with incompressible torus boundary and infinite fundamental group is up to a constant the sum of the volumes of its hyperbolic pieces in the Jaco-Shalen-Johannson splitting along incompressible tori, provided that all non-Seifert pieces are hyperbolic as predicted by Thurston's Geometrization Conjecture. It can be read off from a presentation of the fundamental group $\pi_1(M)$ without knowing M itself. If a closed aspherical manifold M carries a non-trivial S^1-action, then all L^2-Betti numbers $b_p(\widetilde{M})$ and the L^2-torsion $\rho^{(2)}(\widetilde{M})$ vanish. Thus we can conclude using L^2-invariants the well-known statement that a closed hyperbolic manifold cannot carry a non-trivial S^1-action. The question is still open whether it may admit an S^1-foliation.

5 Some Open Conjectures

The following conjectures are at the time of writing still open. The first one was raised as a question by Atiyah [1].

Conjecture 13 (Atiyah Conjecture). *A finitely generated group G satisfies the Atiyah Conjecture if the following equivalent statements are true, where $\mathbb{Z}[\mathcal{FIN}^{-1}]$ is the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting all the orders of finite subgroups of G.*

1. *For any G-covering $\overline{M} \to M$ of a closed Riemannian manifold M and $p \geq 0$ we have*
$$\lim_{t\to\infty} \int_F \operatorname{tr}(e^{-t\Delta_p}(x,x))dx \ \in \mathbb{Z}[\mathcal{FIN}^{-1}];$$
2. *For any G-covering $\overline{X} \to X$ of a compact CW-complex X we have*
$$b_p^{(2)}(\overline{X}; \mathcal{N}(G)) \in \mathbb{Z}[\mathcal{FIN}^{-1}];$$
3. *Let $A \in M(m,n,\mathbb{Z}G)$ be an (m,n)-matrix with coefficients in $\mathbb{Z}G$. Denote by $R_A : l^2(G)^m \to l^2(G)^n$ the induced bounded G-operator. Then*
$$\dim(\ker(R_A)) \in \mathbb{Z}[\mathcal{FIN}^{-1}];$$
4. *Let M be a finitely presented $\mathbb{Z}G$-module. Then*
$$\dim(\mathcal{N}(G) \otimes_{\mathbb{Z}G} M) \in \mathbb{Z}[\mathcal{FIN}^{-1}].$$

Notice that the statements (3) and (4) make sense for any group G. They are true for a group G if and only if they are true for any finitely generated subgroup of G. The Atiyah Conjecture implies the following classical conjecture

Conjecture 14 (Kaplansky Conjecture). *A group G is torsionfree if and only if $\mathbb{Q}G$ has no non-trivial zero-divisors.*

If G contains an element g of finite order $n > 1$, then $N := \frac{1}{n} \cdot \sum_{i=1}^{n} g^i$ is a non-trivial zero-divisor because of $N \cdot (1 - N) = 0$. Suppose that G is torsionfree and $x \in \mathbb{Q}G$ is a non-trivial zero-divisor. By multiplying x with an appropriate integer we can achieve $x \in \mathbb{Z}G$. The G-operator $r_x : l^2(G) \to l^2(G)$ given by right multiplication with x has a non-trivial kernel. Since by the Atiyah Conjecture the dimension of the kernel is an integer and the kernel is a closed subspace of $l^2(G)$, the kernel must have dimension 1 and hence be equal to $l^2(G)$. Hence $x = 0$, a contradiction. For most of the groups, for which the Kaplansky Conjecture is known, the method of proof was to attack and solve the Atiyah-Conjecture. There are exceptions. For instance the Kaplansky Conjecture has been proven by for congruence subgroups Γ_p for which the Atiyah-Conjecture is not known to be true.

We recall that the class of *elementary amenable* groups is defined as the smallest class of groups which contains all finite and all abelian groups, and is closed under taking subgroups, forming factor groups, group extensions and upwards directed unions. Any elementary amenable group is amenable, but the converse is in general not true.

Theorem 15 (Linnell [14]). *The Atiyah Conjecture is true for G if G occurs in an extension $1 \to F \to G \to A \to 1$ for a free group F and an elementary amenable group A, provided that there is a bound on the order of the finite subgroups of G.*

The proof uses ingredients from ring theory, K-theory and operator theory. It is an example, where one has to use unexpected methods from different areas to prove a purely algebraic statement such as the Kaplansky Conjecture. Linnell's work shows that the Atiyah Conjecture is linked to the Isomorphism Conjecture of Farrell and Jones in algebraic K-theory. It says for $K_0(\mathbb{C}G)$ that the natural map

$$\operatorname{colim}_{H \subset G, |H| < \infty} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

is bijective, where the colimit is taken with respect to inclusion and conjugation. An analytic approach to the Atiyah Conjecture is not known. Actually Linnell has proven the Atiyah Conjecture for a bigger class of groups than stated in Theorem 15. Recently the class of groups, for which the Atiyah Conjecture is known, has been considerably enlarged by Schick [24].

Conjecture 16 (Singer Conjecture). *Let M be a closed aspherical manifold of dimension n. Then*

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad &\text{,if } 2p \neq n; \\ (-1)^m \cdot \chi(M) &\geq 0 \qquad &\text{,if } n = 2m, m \in \mathbb{Z}. \end{aligned}$$

If M carries a metric of negative sectional curvature, then

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad &\text{,if } 2p \neq n; \\ b_p^{(2)}(\widetilde{M}) &> 0 \qquad &\text{,if } 2p = n; \\ (-1)^m \cdot \chi(M) &> 0 \qquad &\text{,if } n = 2m, m \in \mathbb{Z}. \end{aligned}$$

Notice that any closed manifold with non-positive sectional curvature is aspherical, i.e. its universal covering is contractible, by Hadamard's Theorem. If M satisfies the Singer Conjecture, then all L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ are integers as predicted by the Aiyah Conjecture 13 (1) because the only possible non-trivial L^2-Betti number must be up to sign the Euler characteristic and hence an integer. Gromov (see [11] or Theorem 3) proves the Singer Conjecture in the case of negative sectional curvature, provided that M is a Kähler manifold. In dimension 3 the Singer Conjecture for aspherical manifolds is proven by Lott-Lück [17] assuming Thurston's Geometrization Conjecture. The Singer Conjecture is proven for manifolds with pinched negative sectional curvature by Donnelly-Xavier [6] and Jost-Xin [13].

Conjecture 17 (Zero-in-the Spectrum Conjecture). *Let M be a closed aspherical Riemannian manifold. Then the following equivalent statements are true*

1. *The Laplacian $\Delta_p : l^2\Omega^p(\widetilde{M}) \to l^2\Omega^p(\widetilde{M})$ has zero in its spectrum for some $p \geq 0$;*
2. *$H_p(\widetilde{M};\mathcal{N}(\pi_1(M)) := H_p(\mathcal{N}(\pi_1(M)) \otimes_{\mathbb{Z}\pi_1(M)} C_*^{\text{sing}}(\widetilde{M})) \neq 0$ for some $p \geq 0$;*

This conjecture is not true if one drops the condition aspherical as shown by Farber and Weinberger [9]. For more information about this conjecture and for which cases it has been proven we refer to the survey article of Lott [16].

Conjecture 18. *Let M be a closed orientable aspherical manifold whose simplical volume in the sense of Gromov vanishes. Then all its L^2-Betti numbers and its L^2-torsion vanishes.*

The conjecture above for L^2-Betti numbers is due to Gromov [12]. The notion of simplical volume is treated in [10] and defined as follows. Let $C_*^{\text{sing}}(M,\mathbb{R})$ be the singular chain complex of M with coefficients in the real numbers $\mathbb{R}$. An element c in $C_p^{\text{sing}}(M,\mathbb{R})$ is given by a finite $\mathbb{R}$-linear combination $c = \sum_{i=1}^{s} r_i \cdot \sigma_i$ of singular p-simplices σ_i in M. Define the l^1-norm of c by $\|c\|_1 = \sum_{i=1}^{s} |r_i|$. For $\alpha \in H_m(M;\mathbb{R})$ define

$$\|\alpha\|_1 = \inf\left\{\|c\|_1 \;\mid c \in C_m^{\text{sing}}(M;\mathbb{R}) \text{ is a cycle representing } \alpha\right\}.$$

The *simplicial volume* of M is defined by $\|M\| := \|\ [M]\ \|_1$, where $[M]$ is the image of the fundamental class of M under the change of ring homomorphism on singular homology $H_n(M;\mathbb{Z}) \longrightarrow H_n(M;\mathbb{R})$. The simplicial volume does not seem to be related to L^2-invariants from its definition. There is no conceptual idea why Conjecture 18 should be true, there is only some evidence based on calculations. For instance it is true for closed orientable hyperbolic manifolds, for aspherical orientable 3-manifold, provided Thurston's Geometrization Conjecture holds, and for closed orientable aspherical manifolds whose fundamental group is solvable or which carry a non-trivial S^1-action.

For more information about L^2-invariant we refer for instance to [12] and [21]. We hope that we could convince the reader that L^2-invariants represent an accessible modern field, where a lot of further activities will take place in the future. In our view it is a good model how mathematics should evolve in the future, where more and more sophisticated methods and ideas will be required and therefore interaction and exchange of knowledge and techniques will become more and more important.

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