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L^2-TORSION OF HYPERBOLIC MANIFOLDS OF FINITE VOLUME

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Abstract

Suppose $\overline{M}$ is a compact connected odd-dimensional manifold with boundary, whose interior M comes with a complete hyperbolic metric of finite volume. We will show that the L^2-topological torsion of $\overline{M}$ and the L^2-analytic torsion of the Riemannian manifold M are equal. In particular, the L^2-topological torsion of $\overline{M}$ is proportional to the hyperbolic volume of M, with a constant of proportionality which depends only on the dimension and which is known to be nonzero in odd dimensions [HS]. In dimension 3 this proves the conjecture [Lü2, Conjecture 2.3] or [LLü, Conjecture 7.7] which gives a complete calculation of the L^2-topological torsion of compact L^2-acyclic 3-manifolds which admit a geometric JSJT-decomposition.

In an appendix we give a counterexample to an extension of the Cheeger-Müller theorem to manifolds with boundary: if the metric is not a product near the boundary, in general analytic and topological torsion are not equal, even if the Euler characteristic of the boundary vanishes.

0 Introduction

In this paper we study L^2-analytic torsion of a compact connected manifold $\overline{M}$ with boundary such that the interior M comes with a complete hyperbolic metric of finite volume.

Notation 0.1. Let m be the dimension of M. From hyperbolic geometry we know [BP, Chapter D3] that M can be written as

$$M = M_0 \cup_{\partial M_0} E_0 \,,$$

where M_0 is a compact manifold with boundary and E_0 is a finite disjoint union of hyperbolic ends $[0,\infty) \times F_j$. Here each F_j is a closed flat manifold and the metric on the end is the warped product

$$du^2 + e^{-2u} dx^2$$

with dx^2 the metric of F_j. Of course, we can make the ends smaller and also write

$$M = M_R \cup_{\partial M_R} E_R \qquad R \geq 0 \,,$$

where E_R is the subset of E_0 consisting of the components $[R,\infty)\times F_j$. We define

$$T_R = M_{R+1}\cap E_R \qquad R\geq 0\,.$$

Denote by $\widetilde{M_R}, \widetilde{E_R}, \dots$ the inverse images of $M_R, E_R, \dots$ under the universal covering map $\widetilde{M}\to M$. Correspondingly, for the universal covering we have

$$\widetilde{M} = \widetilde{M}_R \cup_{\partial \widetilde{M}_R} \widetilde{E_R}\,.$$

For $\widetilde{E_R}$ we get and fix coordinates such that each component is

$$[R,\infty)\times\mathbb{R}^{m-1} \quad \text{with warped product metric } du^2+e^{-2u}dx^2\,,$$

where dx^2 is the Euclidean metric on $\mathbb{R}^n$. □

Each M_R is a compact connected Riemannian manifold with boundary which is L^2-acyclic (see Corollary 6.5). Its absolute L^2-analytic torsion $T^{(2)}_{\rm an}(M_R)$ is defined. The manifold M has no boundary and finite volume and its L^2-analytic torsion is defined although it is not compact (Remark 1.9). We will recall the notion of L^2-analytic torsion in section 1. The main result of this paper is

Theorem 0.2. *Let $\overline{M}$ be a compact connected manifold with boundary such that the interior M comes with a complete hyperbolic metric of finite volume. Suppose that the dimension m of M is odd. Then we get, if R tends to infinity*

$$\lim_{R\to\infty} T^{(2)}_{\rm an}(M_R) = T^{(2)}_{\rm an}(M)\,.$$ □

Remark 0.3. For compact Riemannian manifolds with boundary and with a product metric near the boundary, analytic torsion and topological torsion differ by $(\ln 2)/2$ times the Euler characteristic of the boundary. This is a result of Lück in the classical situation [Lü1] and of Burghelea et al. [BuFK] for the L^2-version. In particular, analytic torsion does not depend on the metric, as long as it is a product near the boundary and the manifold is acyclic or L^2-acyclic, respectively.

In Appendix A we give examples which show that this is not longer the case for arbitrary metrics. This answers an old question of Cheeger's [C, p. 281]. Of course it also implies that the above extension of the Cheeger-Müller theorem to manifolds with boundary is not true in general.

This requires additional care in the chopping and exhausting process described above.

We will explain the strategy of proof for 0.2 in section 1. Next we discuss consequences of Theorem 0.2 and put it into context with known results.

We will explain in section 1 that the comparison theorem for L^2-analytic and -topological torsion for manifolds with and without boundary of Burghelea, Friedlander, Kappeler and McDonald [BuFK], [BuFKM] now implies

Theorem 0.4. *Let $\overline{M}$ be a compact connected manifold with boundary such that the interior M comes with a complete hyperbolic metric of finite volume. Then*

$$T^{(2)}_{\mathrm{top}}(\overline{M}) = T^{(2)}_{\mathrm{an}}(M)\,. \qquad \square$$

The computations of Lott [L, Proposition 16] and Mathai [M, Corollary 6.7] for closed hyperbolic manifolds extend directly to hyperbolic manifolds without boundary and with finite volume since $\mathbb{H}^m$ is homogeneous. (Notice that we use for the analytic torsion the convention in Lott which is twice the logarithm of the one of Mathai.) Hence Theorem 0.4 implies

Theorem 0.5. *Let $\overline{M}$ be a compact connected manifold with boundary such that the interior M comes with a complete hyperbolic metric of finite volume. Then there is a dimension constant C_m such that*

$$T^{(2)}_{\mathrm{an}}(M) = C_m \cdot \mathrm{vol}(M)\,.$$

Moreover, C_m is zero, if m is even, and $C_3{=}-1/3\pi$ and $(-1)^m C_{2m+1}{>}0$. $\square$

Remark 0.6. The last statement is a result of Hess [HS] and answers the question of Lott [L] whether C_{2m+1} is always nonzero.

For closed manifolds, the proportionality follows directly from the computations of Lott and Mathai and the comparison theorem for L^2-analytical and topological torsion of Burghelea et.al. [BuFKM] (as is also observed there).

Now Theorem 0.4 and Theorem 0.5 together with [Lü2, Theorem 2.1] imply

Theorem 0.7. *Let N be a compact connected orientable irreducible 3-manifold with infinite fundamental group possessing a geometric JSJT-decomposition such that the boundary of N is empty or a disjoint union of incompressible tori. Then all L^2-Betti numbers of N vanish and all Novikov-Shubin invariants are positive. Moreover, we get*

$$T^{(2)}_{\mathrm{top}}(N) = \frac{-1}{3\pi} \cdot \sum_{i=1}^{r} \mathrm{vol}(N_i)\,,$$

where $N_1, \ldots, N_r$ are the hyperbolic pieces of finite volume in the JSJT-decomposition of N. In particular $T^{(2)}_{\mathrm{top}}(N)$ is 0 if and only if N is a graph-manifold, i.e. there are no hyperbolic pieces of finite volume in the JSJT-decomposition. $\square$

Theorem 0.7 has been conjectured in [Lü2, Conjecture 2.3] and [LLü, Conjecture 7.7], where also the relevant notions are explained. The JSJT-decomposition of an irreducible connected orientable compact 3-manifold N with infinite fundamental group is the decomposition of Jaco-Shalen and Johannson by a minimal family of pairwise non-isotopic incompressible not boundary-parallel embedded 2-tori into Seifert pieces and atoroidal pieces. If these atoroidal pieces are hyperbolic, then the JSJT-decomposition is called geometric. Thurston's Geometrization Conjecture says that these atoroidal pieces are always hyperbolic. This conjecture is known to be true if N is Haken, for instance if N has boundary or its first Betti number is positive.

In this context we mention the combinatorial formula for the topological torsion of a 3-manifold N as in Theorem 0.7 which computes this torsion just from a presentation of its fundamental group without using any information about N itself [Lü2, Theorem 2.4]. Theorem 0.7 also implies that for such 3-manifolds the L^2-torsion and the simplicial volume of Gromov agree up to a non-zero multiplicative constant [Lü2, section 2].

The paper is organized as follows:

1 Review of L^2-analytic Torsion and Strategy of Proof

In this section we recall the definition of L^2-analytic torsion (compare [NSh1,2], [L], [M]), discuss the strategy and set the stage for the proof of Theorem 0.2, and prove Theorem 0.4.

Definition 1.1. Let N be a connected compact m-dimensional Riemannian manifold. Let $\Delta_p[\widetilde{N}]$ be the Laplacian on p-forms on the universal covering $\widetilde{N}$ considered as an unbounded self-adjoint operator on L^2 (with absolute boundary conditions, if the boundary is non-empty). Then

$e^{-t\Delta_p[\widetilde{N}]}$ is defined for every $t > 0$ and has a smooth kernel $e^{-t\Delta_p[\widetilde{N}]}(x, y)$. (This kernel is invariant under the diagonal $\Gamma := \pi_1(N)$-operation on $\widetilde{N} \times \widetilde{N}$ given by deck transformations). Let $\mathcal{F}$ be a fundamental domain for the Γ-action. Now define the *normalized trace*

$$\operatorname{Tr}_\Gamma e^{-t\Delta_p[\widetilde{N}]} := \int_{\mathcal{F}} \operatorname{tr}\left(e^{-t\Delta_p[\widetilde{N}]}(x, x)\right) dx \,,$$

where tr is the ordinary trace of an endomorphism of a finite-dimensional vector space. □

DEFINITION 1.2. In the situation of Definition 1.1 we denote by $\Delta_p^{\perp}[\widetilde{N}]$ the operator from the orthogonal complement of the kernel of $\Delta_p[\widetilde{N}]$ to itself which is obtained from $\Delta_p[\widetilde{N}]$ by restriction. We call N of *determinant-class* if for all $p \geq 0$

$$\int_1^\infty \operatorname{Tr}_\Gamma e^{-t\Delta_p^{\perp}[\widetilde{N}]} \, \frac{dt}{t} < \infty \,.$$ □

The condition of determinant-class will be needed to define L^2-analytic torsion and was introduced in [BuFKM, page 754]. If all the Novikov-Shubin invariants of N are positive then N is of determinant-class. There is the conjecture that the Novikov-Shubin-invariants of a compact manifold are always positive [NSh1] or [LLü, Conjecture 7.2]. This has been verified for compact 3-manifolds whose prime factors with infinite fundamental groups admit a geometric JSJT-decomposition [LLü, Theorem 0.1] and for hyperbolic manifolds with or without boundary of finite volume in [L, section VII]. If the fundamental group is residually finite or amenable and N is compact, a proof that the manifold is of determinant-class is given in [BuFK, Theorem A in Appendix A] and [DM1, Theorem 0.2] using [Lü3, section 3]. For generalizations compare Schick [S2].

LEMMA 1.3. *In the situation of Definition 1.1 the normalized trace* $\operatorname{Tr}_\Gamma e^{-t\Delta_p[\widetilde{N}]}$ *has for each* $k \geq 0$ *an asymptotic expansion for* $t \to 0$

$$\operatorname{Tr}_\Gamma e^{-t\Delta_p[\widetilde{N}]} = \sum_{i=0}^{k} t^{-(m-i)/2}(a_i + b_i) + O(t^{(k-m+1)/2}) \,.$$

Moreover, we have

$$a_i = \int_{\mathcal{F}} \alpha_i(x) dx \,;$$

$$b_i = \int_{\mathcal{F} \cap \widetilde{\partial N}} \beta_i(x) dx \,,$$

where $\mathcal{F}$ *is a fundamental domain for the* Γ*-action on* $\widetilde{N}$*, such that* $\mathcal{F} \cap \widetilde{N}$ *is a fundamental domain for the* Γ*-action on the preimage* $\widetilde{\partial N}$ *of* ∂N *under* *the universal covering* $\widetilde{N} \to N$, $\alpha_i(x)$ *is a density on* $\widetilde{N}$ *given locally in terms of the metric and* $\beta_i(x)$ *is a density on* $\widetilde{\partial N}$, *which can be computed locally out of the germ of the metric at the boundary* $\widetilde{\partial N}$. □

Proof. We begin with extending the result of Lott [L, Lemma 4]) to manifolds with boundary. Namely we want to prove the existence of constants $C_1, C_2 > 0$ independent of $x \in \widetilde{N}$ and $t \in (0, \infty)$ such that for the covering projection $q : \widetilde{N} \to N$ we get

$$\left| e^{-t\Delta_p[\widetilde{N}]}(x,x) - e^{-t\Delta_p[N]}(q(x),q(x)) \right| \leq C_1 \cdot e^{-1/C_2 t} \quad \forall x \in \widetilde{N},\ t \in (0,\infty)\,. \tag{1.4}$$

Fix a number $K > 0$ such that the restriction of $q : \widetilde{N} \to N$ to any ball $B_{2K} \subset N$ of radius $2K$ is a trivial covering. Consider $x \in \widetilde{N}$. Choose a connected neighbourhood $V \subset N$ of $q(x)$ such that $B_K(x) \subset V \subset B_{2K}(x)$ and V carries the structure of a Riemannian manifold for which the inclusion of V into N is a smooth map respecting the Riemannian metrics. We can find $\widetilde{V} \subset \widetilde{N}$ such that $x \in \widetilde{V}$ and $\widetilde{V}$ carries the structure of a Riemannian manifold for which the inclusion of $\widetilde{V}$ into $\widetilde{N}$ is a smooth map respecting the Riemannian metrics and q restricted to $\widetilde{V}$ induces an isometric diffeomorphism from $\widetilde{V}$ onto V. Since Theorem 2.26 applies to $\widetilde{V} \subset \widetilde{N}$ and $V \subset N$, we obtain constants $C_1, C_2 > 0$ independent of $x \in \widetilde{N}$ and $t \in (0, \infty)$ satisfying

$$\left| e^{-t\Delta_p[\widetilde{N}]}(x,x) - e^{-t\Delta_p[\widetilde{V}]}(x,x) \right| \leq C_1/2 \cdot e^{-1/C_2 t} \quad \forall t \in (0,\infty)\,; \tag{1.5}$$

$$e^{-t\Delta_p[\widetilde{V}]}(x,x) - e^{-t\Delta_p[V]}\big(q(x),q(x)\big) = 0 \quad \forall t \in (0,\infty)\,; \tag{1.6}$$

$$\left| e^{-t\Delta_p[N]}(q(x),q(x)) - e^{-t\Delta_p[V]}(q(x),q(x)) \right| \leq C_1/2 \cdot e^{-1/C_2 t} \quad \forall t \in (0,\infty)\,. \tag{1.7}$$

Now (1.4) follows from (1.5), (1.6) and (1.7).

For each $k \geq 0$ there is an asymptotic expansion for $t \to 0$

$$\begin{aligned} \operatorname{Tr} e^{-t\Delta_p[N]} &:= \int_N \operatorname{tr}_{\mathbb{R}} \left(e^{-t\Delta_p[N]}(x,x) \right) dx \\ &= \sum_{i=0}^{k} t^{-(m-i)/2} \big(a_i[N] + b_i[N] \big) + O\big(t^{(k-m+1)/2} \big) \end{aligned}$$

with

$$\begin{aligned} a_i[N] &= \int_N \alpha_i[N](x)dx\,; \\ b_i[N] &= \int_{\partial N} \beta_i[N](x)dx\,, \end{aligned}$$

where $\alpha_i(x)[N]$ is a locally in terms of the metric given density on N and $\beta_i(x)$ is density on ∂N, which can be computed locally out of the germ of the metric at the boundary ∂N. This is a classical result of Seeley [Se] and Greiner [Gr]. If we define the desired densities by $\alpha_i := \alpha_i[N] \circ q$ and $\beta_i := \beta_i[N] \circ q|_{\partial \widetilde{N}}$ the claim follows because $\operatorname{vol}(N) \cdot C_1 \cdot e^{-1/C_2 t}$ is $O(t^{(N-m+1)/2})$. □

Now we recall the definition of L^2-analytic torsion (here $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$).

Definition 1.8. Let $\widetilde{N} \to N$ be the universal covering of a compact connected Riemannian manifold N of dimension m. Suppose that N is of determinant-class. Define the *L^2-analytic torsion* of N by

$$T_{\mathrm{an}}^{(2)}(N) := \sum_{p \geq 0} (-1)^p \cdot p \cdot \left(\frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \cdot \operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{N}]} \, dt \Big|_{s=0} + \int_1^\infty \operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{N}]} \frac{dt}{t} \right).$$

Here the first integral is a priori only defined for $\Re(s) > m/2$ but Lemma 1.3 ensures that it has a meromorphic extension to the complex plane with no pole in $s = 0$. The second integral converges because of the assumption that N is of determinant-class.

Remark 1.9. Notice that Definition 1.1, Definition 1.2, Definition 1.8 and Lemma 2.36 carry over to the case where N is a not necessarily compact hyperbolic complete Riemannian manifold with finite volume. This follows from the fact that $\mathbb{H}^m$ is homogeneous. □

The proof of Theorem 0.2 now splits into two separate parts: we study the large time summand using algebraically minded functional analysis, namely, we refine the methods of Lott-Lück [LLü]. The small time behaviour is handled analytically via careful study of the heat kernels. Here the main ingredient is the "principle of not seeing the boundary" due to Kac for functions, and Dodziuk-Mathai [DM2] for forms. We give a short proof, using only unit propagation speed, which was suggested to us by Ulrich Bunke.

Finally we explain how Theorem 0.4 follows from Theorem 0.2. We use Notation 0.1. We want to apply the following result of Burghelea, Friedlander and Kappeler [BuFK, Theorem 3.1].

Theorem 1.10. *Let N be a compact Riemannian manifold of determinant-class such that the Riemannian metric is a product near the boundary.*

Then

$$T_{\mathrm{an}}^{(2)}(N) = T_{\mathrm{top}}^{(2)}(N) + \tfrac{\ln(2)}{2} \cdot \chi(\partial N)\,. \qquad \square$$

Therefore we equip M_R with a new metric which is a product near the boundary.

DEFINITION 1.11. Let M'_R ($R \geq 1$) denote the same manifold as M_R but now equipped with a new Riemannian metric which is equal to the old one outside $T_{R-1} \subset M_R$ and on T_{R-1} given by

$$g_R = du^2 + e^{-2(\phi(R-u)u+(1-\phi(R-u))R)}dx^2\,,$$

where $\phi : [0,\infty] \to [0,1]$ is a smooth function which is identically zero on $[0,\epsilon]$ and identically one on $[1-\epsilon,\infty)$ for some $\epsilon > 0$. Similarly, let E'_R be E_R with the new Riemannian metric $du^2 + e^{-2(\phi(u-R)R+(1-\phi(u-R))u)}dx^2$. $\square$

LEMMA 1.12. *For arbitrary $r, R > 0$, there is an isometric diffeomorphism*

$$\tau_R : \widetilde{E_0} \longrightarrow \widetilde{E_R}\,,$$

which induces by restriction isometric diffeomorphisms $\widetilde{E_r} \to \widetilde{E_{r+R}}$, $\widetilde{T_r} \to \widetilde{T_{r+R}}$, $\widetilde{E'_r} \to \widetilde{E'_{r+R}}$ and $\widetilde{T'_r} \to \widetilde{T'_{r+R}}$.

Proof. Define $\tau_R : [0,\infty)\times\mathbb{R}^{m-1}\to[R,\infty)\times\mathbb{R}^{m-1}$ by $\tau_R(u,x)=(u+R, e^R x)$. $\square$

LEMMA 1.13. *For m odd*

$$\lim_{R\to\infty} T_{\mathrm{an}}^{(2)}(M_R) - T_{\mathrm{an}}^{(2)}(M'_R) = 0\,; \tag{1.14}$$

$$T_{\mathrm{top}}^{(2)}(M'_R) - T_{\mathrm{top}}^{(2)}(\overline{M}) = 0 \qquad \text{for } R \geq 1\,. \tag{1.15}$$

Proof. Observe that for m odd M is L^2-acyclic by [D]). The same holds for M_R by Corollary 6.5 or [CGro, Theorem 1.1]. Let $\{g_u \mid u \in [0,1]\}$ be the obvious family of Riemannian metrics on M_R joining the hyperbolic metric on M restricted to M_R with the metric g_R on M_R introduced in Definition 1.11. Then we get from Corollary 7.13

$$\tfrac{d}{du}\big|_{u=0} T_{\mathrm{an}}^{(2)}(M_R, g_u) = \sum_p (-1)^p d_p[R,u]\,,$$

where $d_p[R,u]$ is an integral of a density $D_p[R,u]$ on ∂M_R which is given locally in terms of the germ of the family of metrics g_u on ∂M_R. If we pull back the density $D_p[R,u]$ to a density $\widetilde{D_p[R,u]}$ on $\widetilde{\partial M_R}$ we can rewrite $d_p[R,u]$ as an integral over the density $\widetilde{D_p[R,u]}$ over $\mathcal{F} \cap \widetilde{\partial M_R}$ for a fundamental domain $\mathcal{F}$. Notice that $\widetilde{D_p[R,u]}$ can be computed locally in terms of the germ of the family of lifted metrics $\widetilde{g_u}$ on $\widetilde{\partial M_R}$. We can write $\widetilde{D_p[R,u]}$ as $f_p[R,u] \cdot dvol_{\widetilde{\partial M_R}}$ for a function $f_p[R,u]$ on $\widetilde{\partial M_R}$. Because the isometries in Lemma 1.12 respect the families of metrics, the functions $f_p[R,u]$

are uniformly bounded in R and u. However, the volume of the domain of integration $\mathcal{F} \cap \widetilde{\partial M_R}$ tends to zero if R tends to ∞. Hence 1.14 follows.

For (1.15) we use the fact that the topological L^2-torsion is a homotopy invariant for residually finite fundamental groups [Lü3, Theorem 0.5]. The isometries of the hyperbolic space $\mathbb{H}^m$ can be considered as a subgroup of $Gl(m+1,\mathbb{R})$ [BP, Theorem A.2.4]. Every finitely generated group which possesses a faithful representation in some $Gl(n,K)$ for any field K is residually finite [W, Theorem 4.2]. We have seen that every manifold which possesses a complete hyperbolic metric with finite volume is homotopy equivalent to a finite CW-complex. It follows that the fundamental group of such a manifolds is residually finite. □

Finally we can explain how Theorem 0.4 follows from Theorem 0.2. We begin with the case where $m = \dim(M)$ is odd. Since ∂M is a union of flat manifolds $\chi(\partial M)$ is trivial. Now one just applies Theorem 1.10 for $N = M'_R$ and uses Lemma 1.13. Suppose that m is even. Then the usual proof of Poincaré duality for analytic torsion [RSi, Theorem 2.3] extends directly to L^2-analytic torsion of M (see [L, Proposition 16]) and shows $T^{(2)}_{\mathrm{an}}(M) = 0$. From [Lü2, Theorem 1.6, Theorem 1.7 and Theorem 1.11] we conclude $T^{(2)}_{\mathrm{top}}(\overline{M}) = 0$. This finishes the proof that Theorem 0.2 implies Theorem 0.4.

The rest of this paper is devoted to the proof of Theorem 0.2 (and also of Theorem 2.26 and Corollary 7.13 which we have already used above).

2 Analysis of the Heat Kernel

2.1 Standard Sobolev estimates

Definition 2.1. Let N be a Riemannian manifold with boundary $x \in \partial N$. We define the *boundary exponential map*

$$\exp^{\partial}_x : [0,\infty) \times T_x\partial N \to N : (u,y) \mapsto \nu\big(u, \exp^{\partial N}_x(y)\big)\,.$$

Here $\exp^{\partial N}$ is the exponential map of the boundary with its induced Riemannian metric and ν denotes the geodesic flow of the inward unit normal field.

If we identify $T^{\partial N}_x$ with $\mathbb{R}^{m-1}$ via an orthonormal frame and restrict the boundary exponential map to a subset where it is a diffeomorphism onto its image, we get so called *boundary normal coordinates.*

For an interior point the coordinates induced from the exponential map are called *Gaussian coordinates.* The term *normal coordinates* is used to denote Gaussian coordinates as well as boundary normal coordinates. □

DEFINITION 2.2. A form ω on a Riemannian manifold with boundary fulfills *relative boundary conditions for* Δ^m, $m \geq 1$, if

$$i^*(\Delta^j \omega) = 0 \quad \text{and} \quad i^*(\Delta^j \delta\omega) = 0 \qquad \text{for } 0 \leq j \leq m-1\,.$$

It fulfills *absolute boundary conditions for* Δ^m if

$$i^*(*\Delta^j \omega) = 0 \quad \text{and} \quad i^*(*\Delta^j d\omega) = 0 \qquad \text{for } 0 \leq j \leq m-1\,.$$

Here and in the sequel, $i : \partial M \to M$ always denotes the inclusion of the boundary. □

Remember that both boundary value problems are elliptic. The next result is a standard elliptic estimate (compare f.i. [CGroT, 1.24] if $\partial M = \emptyset$). Since we do not know an explicit reference for manifolds with boundary we include a proof here (compare also [T, Ch. 5, Sec. 9]).

NOTATION 2.3. For real numbers $a, b, c > 0$ we abbreviate

$$a \overset{c}{\leq} b \qquad \text{for} \quad a \leq cb\,.$$

Theorem 2.4. *Let M be a Riemannian manifold with boundary of dimension m, $x_0 \in M$ and r sufficiently small so that $B_r(x_0) \subset M$ is diffeomorphic in normal coordinates to $B_r(x) \subset \mathbb{R}^m_{\geq 0}$ for some $x \in \mathbb{R}^m_{\geq 0}$, where $\mathbb{R}^m_{\geq 0}$ denotes the half space. (The location of x depends on the question whether $B_r(x_0)$ meets the boundary of M).*

Then we can find $C > 0$ so that for all $\omega \in C^\infty$ which fulfill either absolute or relative boundary conditions

$$\left|\omega(x_0)\right|^2 \leq C \sum_{i=0}^{m} |\Delta^i \omega|^2_{L^2(B_r(x_0))}\,.$$

The constant C is a smooth function of the coefficients of the Riemannian metric, its inverse and their derivatives in normal coordinates of $B_r(x_0)$. Moreover, it depends on r (it becomes larger if r becomes smaller).

Proof. We start with the following formula:

LEMMA 2.5. *For every k-form f which fulfills either absolute or relative boundary conditions for Δ and for every function ϕ*

$$\begin{aligned}\left|d(\phi f)\right|^2_{L^2(M)} &+ \left|\delta(\phi f)\right|^2_{L^2(M)} \\ &= (\Delta f, \phi^2 f)_{L^2(M)} + |f \wedge d\phi|^2_{L^2(M)} + |{*f} \wedge d\phi|^2_{L^2(M)}\,.\end{aligned}$$

Proof.

$$\begin{aligned}\big(d(\phi f), d(\phi f)\big) &= \big(d\phi \wedge f, d(\phi f)\big) + \big(\phi df, d(\phi f)\big) \\ &= (d\phi \wedge f, d\phi \wedge f) + (d\phi \wedge f, \phi df) + \big(df, \phi d(\phi f)\big) \\ &= (d\phi\wedge f, d\phi\wedge f) + (d\phi\wedge f, \phi df) + \big(df, d(\phi^2 f) - d\phi\wedge(\phi f)\big) \\ &= |d\phi \wedge f|^2 + (\delta df, \phi^2 f)\,. \end{aligned} \tag{2.6}$$

For the last equation we have used the fact that the boundary contribution of the integration by parts vanishes if either $i^*f=0$ or $i^*(*df)=0$ and that the first equation holds if f fulfills relative boundary conditions, and the second equation holds if f satisfies absolute boundary conditions.

$$\begin{aligned}
\big(\delta(\phi f),\delta(\phi f)\big) &= \big(d*(\phi f), d*(\phi f)\big)\\
&= \big(d(\phi * f), d(\phi * f)\big)\\
&= \big|d\phi\wedge(*f)\big|^2 + \big(\delta d(*f), \phi^2 * f\big)\\
&= \big|d\phi\wedge(*f)\big|^2 + \big((-1)^{mk+m+1} d*d*f, \phi^2 f\big)\\
&= \big|d\phi\wedge(*f)\big|^2 + (d\delta f, \phi^2 f)\,. \qquad (2.7)
\end{aligned}$$

We get the third equation above by applying (2.6). Now add (2.6) and (2.7). □

The next lemma will be needed several times. In the sequel $|\cdot|^2_{H^k(\mathbb{R}^m_{\geq 0})}$ denotes the Sobolev norm on $\mathbb{R}^m_{\geq 0}$ with the standard Euclidean metric, and a function f with compact support in an open subset $V\subset\mathbb{R}^m_{\geq 0}$ is extended to $\mathbb{R}^m_{\geq 0}$ by zero on the complement of V. Moreover, Δ stands for the Laplacian on the Riemannian manifold M.

LEMMA 2.8. *In the situation of Theorem 2.4 let ϕ,ψ be functions with $\operatorname{supp}\phi\subset\operatorname{supp}\psi\subset B_r(x_0)$ and $\psi\equiv 1$ on $\operatorname{supp}\phi$. We use normal coordinates to identify $B(r,x_0)$ with a subset of $\mathbb{R}^m_{\geq 0}$. Then for $k,l\geq 0$ and every form ω which fulfills either absolute or relative boundary conditions for Δ^l*

$$|\phi\omega|^2_{H^{k+2l}(\mathbb{R}^m_{\geq 0})} \overset{C}{\leq} |\phi\omega|^2_{H^k(\mathbb{R}^m_{\geq 0})} + |\phi\Delta^l\omega|^2_{H^k(\mathbb{R}^m_{\geq 0})} + |\psi\omega|^2_{H^{k+2l-1}(\mathbb{R}^m_{\geq 0})}\,. \qquad (2.9)$$

The constant C smoothly depends on the coefficients of the metric, its inverse and their derivatives in normal coordinates in $B(r,x_0)$. In addition, it depends on ϕ.

Proof. We prove only the case of relative boundary conditions.

$$\begin{aligned}
&|\phi\omega|^2_{H^{2l+k}(\mathbb{R}^m_{\geq 0})} \overset{C_1}{\leq} \big|\Delta^l(\phi\omega)\big|^2_{H^k(\mathbb{R}^m_{\geq 0})} + |\phi\omega|^2_{H^k(\mathbb{R}^m_{\geq 0})}\\
&+\sum_{j=0}^{l-1}\big|i^*(\Delta^j(\phi\omega))\big|^2_{H^{2l+k-2j-1/2}(\mathbb{R}^{m-1})}+\big|i^*(\Delta^j\delta(\phi\omega))\big|^2_{H^{2l+k-2j-3/2}(\mathbb{R}^{m-1})} \qquad (2.10)\\
&\overset{2}{\leq} |\phi\omega|^2_{H^k(\mathbb{R}^m_{\geq 0})} + |\phi\Delta^l\omega|^2_{H^k(\mathbb{R}^m_{\geq 0})} + \big|[\Delta^l,\phi]\omega\big|^2_{H^k(\mathbb{R}^m_{\geq 0})}\\
&+\sum_{j=0}^{l-1}\big|\phi\,\underbrace{i^*(\Delta^j\omega)}_{=0}\big|^2_{H^{2l+k-2j-1/2}(\mathbb{R}^{m-1})} + \big|i^*([\Delta^j,\phi]\omega)\big|^2_{H^{2l+k-2j-1/2}(\mathbb{R}^{m-1})}
\end{aligned}$$

$$+\big|\phi\,\underbrace{i^*(\Delta^j\delta\omega)}_{=0}\big|^2_{H^{2l+k-2j-3/2}(\mathbb{R}^{m-1})}+\big|i^*([\Delta^j\delta,\phi]\omega)\big|^2_{H^{2l+k-2j-3/2}(\mathbb{R}^{m-1})} \tag{2.11}$$

$$\overset{C_2}{\le}|\phi\omega|^2_{H^k(\mathbb{R}^m_{\ge 0})}+|\phi\Delta^l\omega|_{H^k(\mathbb{R}^m_{\ge 0})}+|\psi\omega|^2_{H^{2l+k-1}(\mathbb{R}^m_{\ge 0})}\,. \tag{2.12}$$

For (2.10) we use the fact that Δ with relative boundary conditions is elliptic. Then, the same is true for its l^{th} power and therefore we can estimate the Sobolev norm as indicated (compare for instance [S1, 4.15]). Note that the constant C_1 (smoothly) depends on the coefficients of the boundary value problem, which are determined by the Riemannian metric in $B(r,x_0)$ and its derivatives (again in a smooth way). Therefore, C_1 smoothly depends on the metric as desired.

For (2.11) we use the fact that ω fulfills relative boundary conditions and we denote by $[\Delta^j,\phi]$ the commutator of Δ^j and the operator given by multiplication with ϕ.

In (2.12) we use the fact that the commutator $[\Delta^l,\phi]$ is a differential operator of order $2l-1$. Similarly, the commutators $[\Delta^j,\phi]$ have order $2j-1$ and $[\Delta^j\delta,\phi]$ have order $2j$. Also, i^* is the composition of the trace map $H^s(\mathbb{R}^m_{\ge 0})\to H^{s-1/2}(\mathbb{R}^{m-1})$ for $s>1/2$ which simply restricts to the boundary but leaves the bundle unchanged, and the operator of order 0 (bounded on H^s for all s) which projects onto the "tangential" components of forms. Therefore the statement follows. Notice that we can replace ω by $\psi\omega$ above in the last summand as $i^*([\Delta^j,\phi]\omega)=i^*([\Delta^j,\phi]\psi\omega)$. It remains to check what the constant C_2 depends on, i.e. which norm the operators mentioned above have. They all are differential operators and their norm depends on the coefficients. These coefficients are determined by the coefficients of Δ and δ and i^* and their derivatives, but now also by ϕ and its derivatives. So, all together, we have the (smooth) dependence on ϕ and the Riemannian metric as desired. $\square$

Inductive application of the following lemma will prove Theorem 2.4.

LEMMA 2.13. *Adopt the situation of Lemma 2.8 and assume that ϕ is a cutoff function, i.e. $\phi: M\to[0,1]$. Then for every $k\ge 0$ and every ω which fulfills either absolute or relative boundary conditions we get*

$$|\phi\omega|^2_{H^{2k}(\mathbb{R}^m_{\ge 0})}\overset{C}{\le}|\Delta^k\omega|^2_{L^2(B(r,x_0))}+|\Delta^{k-1}\omega|^2_{L^2(B(r,x_0))}+|\Delta\omega|^2_{L^2(B(r,x_0))}$$
$$+|\omega|^2_{L^2(B(r,x_0))}+|\psi\omega|^2_{H^{2k-2}(\mathbb{R}^m_{\ge 0})}\,. \tag{2.14}$$

On $L^2(B(r,x_0))$ we use the norm induced from the Riemannian metric on M. C smoothly depends on the coefficients of the Riemannian met-

ric tensor and its inverse in normal coordinates and on their derivatives. Also it smoothly depends on ϕ and ψ. If $k=1$ we can replace the norm $|\cdot|_{H^{2k-2}(\mathbb{R}^m_{\geq 0})}$ by $|\cdot|_{L^2(B(r,x_0))}$ and the statement remains true.

Proof. We proof only the case of relative boundary conditions.

Since we identified $B(r,x_0)$ with a subset of $\mathbb{R}^m$ we have two norms on $L^2(B(r,x_0))$: the one used in the statement of Lemma 2.13 coming from the Riemannian metric on M and the one coming from Euclidean $\mathbb{R}^m$. We find a constant C depending smoothly on the Riemannian metric tensor and its inverse so that

$$|\cdot|_{L^2(\mathbb{R}^m)} \overset{1/C}{\leq} |\cdot|_{L^2(B(r,x_0))} \overset{C}{\leq} |\cdot|_{L^2(\mathbb{R}^m)} . \tag{2.15}$$

In particular the last statement in Theorem 2.13 for $k=1$ follows from the rest and (2.15).

Choose a cutoff function $\phi_1 : M \to [0,1]$ with $\phi_1 \equiv 1$ on $\operatorname{supp}\phi$ and $\psi \equiv 1$ on $\operatorname{supp}\phi_1$.

$$|\phi\omega|^2_{H^{2k}(\mathbb{R}^m_{\geq 0})} \overset{C_1}{\leq} |\phi\omega|^2_{L^2(B(r,x_0))} + |\phi\Delta^k\omega|^2_{L^2(B(r,x_0))} + |\phi_1\omega|^2_{H^{2k-1}(\mathbb{R}^m_{\geq 0})} \tag{2.16}$$

$$\begin{aligned} &\overset{C_2}{\leq} |\omega|^2_{L^2(B(r,x_0))} + |\Delta^k\omega|^2_{L^2(B(r,x_0))} + |\phi_1\omega|^2_{H^1(\mathbb{R}^m_{\geq 0})} \\ &\qquad + |\phi_1\Delta^{k-1}\omega|^2_{H^1(\mathbb{R}^m_{\geq 0})} + |\psi\omega|^2_{H^{k-2}(\mathbb{R}^m_{\geq 0})} . \end{aligned} \tag{2.17}$$

Inequality (2.16) follows from (2.9) with $k=0$ and l replaced by k, and (2.15). The constant C_1 depends only on ϕ and the Riemannian metric in $B(r,x_0)$. We conclude (2.17) from $0 \leq \phi \leq 1$ and (2.9) now with $k=1$ and $l=k-1$. We clearly can choose ϕ_1 depending smoothly on ϕ and ψ so that C_2 depends smoothly on ϕ, the coefficients of the Riemannian metric tensor and its inverse in normal coordinates and on their derivatives.

It remains to estimate the two H^1-summands appearing in (2.17).

$$\begin{aligned} |\phi_1\omega|^2_{H^1(\mathbb{R}^m_{\geq 0})} &\overset{C_3}{\leq} |\phi_1\omega|^2_{L^2(B(r,x_0))} + \big|d(\phi_1\omega)\big|^2_{L^2(B(r,x_0))} + \big|\delta(\phi_1\omega)\big|^2_{L^2(B(r,x_0))} \\ &\qquad + \big|\phi_1 \underbrace{i^*\omega}_{=0}\big|^2_{H^{1/2}(\mathbb{R}^{m-1})} \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\leq |\omega|^2_{L^2(B(r,x_0))} + (\Delta\omega, \phi_1^2\omega)_{L^2(M)} + |\omega \wedge d\phi_1|^2_{L^2(M)} \\ &\qquad + |{*\omega} \wedge d\phi_1|^2_{L^2(M)} \end{aligned} \tag{2.19}$$

$$\overset{C_4}{\leq} |\omega|^2_{L^2(B(r,x_0))} + |\Delta\omega|_{L^2(B(r,x_0))} \cdot |\phi_1^2\omega|_{L^2(B(r,x_0))} \tag{2.20}$$

$$\leq |\omega|^2_{L^2(B(r,x_0))} + |\Delta\omega|^2_{L^2(B(r,x_0))} + |\phi_1^2\omega|^2_{L^2(B(r,x_0))} \tag{2.21}$$

$$\leq 2 \cdot |\omega|^2_{L^2(B(r,x_0))} + |\Delta\omega|^2_{L^2(B(r,x_0))} \,. \tag{2.22}$$

We get (2.18) since $(d, \delta; i^*)$ is an elliptic boundary value problem and ω fulfills relative boundary conditions. Obviously, C_3 is determined in the same way as above. We conclude (2.19) from Lemma 2.5 and $0 \leq \phi_1 \leq 1$. Equation (2.20) follows from the Cauchy-Schwarz inequality. The constant C_4 in (2.20) involves $\sup_{x\in B(r,x_0)}|d\phi_1(x)|$. Equation (2.21) follows from $ab \leq a^2 + b^2$ for arbitrary real numbers a and b. We get (2.22) from $0 \leq \phi_1 \leq 1$. An identical argument shows for the second H^1-summand

$$|\phi_1\Delta^{k-1}\omega|^2_{H^1(\mathbb{R}^m_{\geq 0})} \overset{c_5}{\leq} 2 \cdot |\Delta^{k-1}\omega|^2_{L^2(B(r,x_0))} + |\Delta^k\omega|^2_{L^2(B(r,x_0))} \,. \tag{2.23}$$

Now Lemma 2.13 follows from (2.16) to (2.23). □

Finally we give the proof of Theorem 2.4. Choose a cutoff function $\phi_1 : M \to [0,1]$ with $\operatorname{supp}\phi_1 \subset B(r,x_0)$ and so that $\phi_1 \equiv 1$ in a neighbourhood of x_0. By Sobolev's lemma

$$\big|\omega(x_0)\big|^2 = \big|\phi_1\omega(x_0)\big|^2 \overset{C_{2m}}{\leq} |\phi_1\omega|^2_{H^{2m}(\mathbb{R}^m_{\geq 0})} \,.$$

This is a computation entirely in $\mathbb{R}^m_{\geq 0}$, therefore C_{2m} depends only on the dimension. Theorem 2.4 follows now by an inductive application of Lemma 2.13. To do this, we have to choose a sequence of cutoff functions $\phi_i : M \to [0,1]$ with $\operatorname{supp}\phi_i \subset B(r,x_0)$ for all i and so that $\phi_{i+1} \equiv 1$ on $\operatorname{supp}\phi_i$. Clearly, we can construct this sequence depending only on r. Since the derivatives of these functions become larger if r becomes smaller, the constant C of Theorem 2.4 has to become larger, too. □

Comparison of heat kernels. In this section we use unit propagation speed (as in [CGroT]) to prove the "principle of not feeling the boundary" of M. Kac. Similar results have been proved by Dodziuk-Mathai [DM2] with a more complicated argument, involving finite propagation speed and Duhamel's principle. Moreover, their method does not yield the statement in the generality we prove (and need) it. The method we use was suggested to us by Ulrich Bunke during the meeting on Dirac operators 1997 at the Banach Center in Warsaw.

The next definition extends the notion of a Riemannian manifold of bounded geometry to manifolds with boundary (compare Schick [S1, chapter 3], where these manifolds are discussed in greater detail).

Definition 2.24. A Riemannian manifold (N, g) (possibly with boundary) is called a manifold of *bounded geometry* if constants C_k; $k \in \mathbb{N}$ and $R_I, R_C > 0$ exist, so that the following holds:

1. The geodesic flow of the unit inward normal field induces a diffeomorphism of $[0,2R_C)\times\partial N$ onto its image $C(\partial N)$, the geodesic collar. Let $\pi: C(\partial N)\to\partial N$ be the corresponding projection;
2. $\forall x\in N$ with $d(x,\partial N)>R_C/2$ the exponential map $T_xN\to N$ is a diffeomorphism on $B_{R_I}(0)$;
3. $\forall x\in N$ with $d(x,\partial N)<R_C$ we have boundary normal coordinates defined on $[0,R_C)\times(B(0,R_C)\subset T^{\partial N}_{\pi(x)})$;
4. For every $k\in\mathbb{N}$ and every $x\in\mathbb{N}$ the derivatives up to order k of the Riemannian metric tensor g_{ij} and its inverse g^{ij} in Gaussian coordinates (if $d(x,\partial N)>R_C/2$) or in normal boundary coordinates (if $d(x,\partial N)<R_C$) resp., are bounded by C_k. □

Example 2.25. Any covering of a compact manifold with the induced metric is a manifold of bounded geometry. □

Theorem 2.26. *Let N be a Riemannian manifold possibly with boundary which is of bounded geometry. Let $V\subset N$ be a closed subset which carries the structure of a Riemannian manifold of the same dimension as N such that the inclusion of V into N is a smooth map respecting the Riemannian metrics. (We make no assumptions about the boundaries of N and V and how they intersect.) For fixed $p\geq 0$ let $\Delta[V]$ and $\Delta[N]$ be the Laplacians on p-forms on V and N, considered as unbounded operators with either absolute boundary conditions or with relative boundary conditions (see Definition 2.2). Let $\Delta[V]^ke^{-t\Delta[V]}(x,y)$ and $\Delta[N]^ke^{-t\Delta[N]}(x,y)$ be the corresponding smooth integral kernels. Let k be a non-negative integer.*

Then there is a monotone decreasing function $C_k(K)$ from $(0,\infty)$ to $(0,\infty)$ which depends only on the geometry of N (but not on V,x,y,t) and a constant C_2 depending only on the dimension of N such that for all $K>0$ and $x,y\in V$ with $d_V(x):=d(x,N-V)\geq K$, $d_V(y)\geq K$ and all $t>0$:

$$\left|\Delta[V]^ke^{-t\Delta[V]}(x,y)-\Delta[N]^ke^{-t\Delta[N]}(x,y)\right|\leq C_k(K)e^{-\frac{(d_V(x)^2+d_V(y)^2+d(x,y)^2)}{C_2t}}.$$

Proof. In the sequel Δ stands for both $\Delta[N]$ and $\Delta[V]$. The Fourier transform of an L^1-function $f:\mathbb{R}\to\mathbb{C}$ is defined by $\widehat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}f(s)e^{-is\xi}ds$. Because of the well-known rules

$$\widehat{e^{-s^2/2}}=e^{-\xi^2/2};$$
$$\widehat{\frac{d^mf}{ds^m}}=i^m\xi^m\cdot\widehat{f};$$
$$\lambda\cdot\widehat{g}(\lambda\xi)=\widehat{f}(\xi)\qquad\text{for } g(s):=f(\lambda s),$$

we conclude for $\xi \in \mathbb{R}$

$$\frac{(-1)^m}{\sqrt{\pi t}} \cdot \int_0^\infty \left(\frac{d^{2m}}{ds^{2m}} e^{-s^2/4t} \right) \cos(s\xi) ds = \frac{(-1)^m}{\sqrt{2t}} \cdot \Re\left(\left(\widehat{\frac{d^{2m}}{ds^{2m}} e^{-s^2/4t}} \right)(\xi) \right) = \xi^{2m} e^{-t\xi^2} .$$

By the spectral theorem applied to $\sqrt{\Delta}$ we get for non-negative integers l, m and k

$$\Delta^m \Delta^l \Delta^k e^{-t\Delta} = \frac{(-1)^{l+m+k}}{\sqrt{\pi t}} \cdot \int_0^\infty \frac{d^{2(m+l+k)}}{ds^{2(m+l+k)}} e^{-s^2/4t} \cos(s\sqrt{\Delta}) ds . \tag{2.27}$$

Choose $x_0, y_0 \in V$ with $d_V(x_0), d_V(y_0) \geq K$. Notice for the sequel that the ball with radius R less or equal to K around y_0 in N and the one in V agree and will be denoted by $B_R(y_0)$. Moreover, the intersection of $B_R(y_0)$ with ∂N and with ∂V agree. For a smooth p-form u with $\operatorname{supp}(u) \subset B_{K/4}(y_0)$ which satisfies the absolute or relative boundary conditions and hence lies in the domain for both $\Delta[V]$ and $\Delta[N]$, we consider now the function f on V given by

$$f := \left(\Delta[N]^k e^{-t\Delta[N]} - \Delta[V]^k e^{-t\Delta[V]} \right) u .$$

We conclude from (2.27)

$$\Delta^m \Delta^l f = \int_0^\infty t^{-2(m+l+k)-1/2} P_{m,l,k}(s, \sqrt{t}) e^{-s^2/4t} \big(\cos(s\sqrt{\Delta[N]}) - \cos(s\sqrt{\Delta[V]}) \big) u \, ds , \tag{2.28}$$

where $P_{m,l,k}$ is a universal polynomial with real coefficients. Next we show

$$\cos\left(s\sqrt{\Delta[N]}\right) u - \cos\left(s\sqrt{\Delta[V]}\right) u = 0 \qquad \text{on } B_{K/4}(x_0) \quad \text{for } s \leq \max\left\{ d_V(y_0)/2, d(x_0, y_0)/2 \right\} . \tag{2.29}$$

Note that $\cos(s\sqrt{\Delta})u$ fulfills the wave equation with initial data u. By unit propagation speed for the wave equation on manifolds with boundary [T, 6.1]), $\operatorname{supp}(\cos(s\sqrt{\Delta})u) \subset B_{d_V(y_0)}(y_0) \subset V$ for $s \leq d_V(y_0)/2 < d_V(y_0) - K/4$. Moreover, because of $\operatorname{supp} u \subset B_{K/4}(y_0)$ and the uniqueness of solutions of the wave equation we get on V

$$\cos\left(s\sqrt{\Delta[N]}\right) u = \cos\left(s\sqrt{\Delta[V]}\right) u \quad \text{for} \quad s \leq d_V(y_0)/2 . \tag{2.30}$$

If $d(x_0, y_0) \geq d_V(y_0) \geq K$, we know from unit propagation speed that

$$\operatorname{supp}\left(\cos(s\sqrt{\Delta})u \right) \cap B_{K/4}(x_0) = \emptyset \text{ for } s \leq d(x_0, y_0)/2 < d(x_0, y_0) - K/2 , \tag{2.31}$$

since $\operatorname{supp} u \subset B_{K/4}(y_0)$. Now (2.29) follows from (2.30) and (2.31). Since $|\cos(s\xi)| \leq 1$ and hence $|\cos(s\sqrt{\Delta})u|_{L^2} \leq |u|_{L^2}$ we conclude from (2.28) and (2.29)

$$
\begin{aligned}
&|\Delta^m\Delta^l f|_{L^2(B_{K/4}(x_0))}\\
&\le 2\biggl(\int_{\max\{d_V(y_0)/2,d(x_0,y_0)/2\}}^{\infty} t^{-(2(m+l+k)+1/2)}P_{m,l,k}(s,\sqrt{t})e^{-s^2/4t}ds\biggr)|u|_{L^2}\\
&\qquad\qquad\le C_{m,l,k}e^{-\max\{d_V(y_0)/2,d(x_0,y_0)/2\}^2/C_{m,l}t}|u|_{L^2}
\end{aligned}
$$

by an elementary estimate of the integral. Since N is of bounded geometry and the heat kernel fulfills absolute boundary conditions, the elliptic estimates of Theorem 2.4 yield pointwise bounds

$$
\bigl|\Delta^l f(x_0)\bigr| \le D_l(K)\cdot e^{-\max\{d_V(y_0)/2,d(x_0,y_0)/2\}^2/E_l t}|u|_{L^2(B_{K/4}(x_o))}\,, \quad (2.32)
$$

where $D_l(K)$ is a monotone decreasing function in $K>0$ which is given in universal expressions involving the norm of curvature and a bounded number of its derivatives on $B_{K/4}(x_0)$ and is independent of x_0, y_0 and t, and $E_l>0$ depends only on the dimension of N.

Now $\Delta^l f(x_0)=\int \Delta_y^l(\Delta^k e^{-t\Delta[N]}(x_0,y)-\Delta^k e^{-t\Delta[V]}(x_0,y))u(y)\,dy$. Since the estimates (2.32) hold for a dense subset $\{u\}\subset L^2(B_{K/4}(y_0))$ we conclude

$$
\begin{aligned}
&\bigl|\Delta_y^l(\Delta^k e^{-t\Delta[N]}(x_0,y)-\Delta^k e^{-t\Delta[V]}(x_0,y))\bigr|_{L^2(B_{K/4}(y_0))}\\
&\qquad\qquad\le D_{l,k}(K)\cdot e^{-\max\{d_V(y_0)/2,d(x_0,y_0)/2\}^2/E_l t}\,.
\end{aligned}
$$

The very same reasoning as above yields pointwise bounds

$$
\begin{aligned}
&\bigl|(\Delta^k e^{-t\Delta[V]}(x_0,y_0)-\Delta^k e^{-t\Delta[N]}(x_0,y_0)\bigr|\\
&\qquad\qquad\le C_k(K)e^{-\max\{d_V(y_0)/2,d(x_0,y_0)/2\}^2/C_2 t}\,, \quad (2.33)
\end{aligned}
$$

where $C_k(K)>0$ is a monotone decreasing function in $K>0$ which is independent of x_0,y_0 and t, and $C_2>0$ depends only on the dimension of N. Since the heat kernel is symmetric we also have

$$
\begin{aligned}
&\bigl|(\Delta^k e^{-t\Delta[V]}(x_0,y_0)-\Delta^k e^{-t\Delta[N]}(x_0,y_0)\bigr|\\
&\qquad\qquad\le C_k(K)e^{-\max\{d_V(x_0)/2,d(x_0,y_0)/2\}^2/C_2 t}\,. \quad (2.34)
\end{aligned}
$$

Theorem 2.26 follows from (2.33) and (2.34). □

Theorem 2.35. *Let N be a Riemannian manifold possibly with boundary which is of bounded geometry. For $t_0>0$ we find $c(t_0)$, depending on the bounds of the metric tensor and its inverse and finitely many of their derivatives in normal coordinates (but not on $x,y\in N$) so that*

$$
\bigl|e^{-t\Delta[V]}(x,y)\bigr|\le c(t_0) \qquad \forall t\ge t_0\,.
$$

Proof. The norm of the bounded operator $\Delta^m\Delta^l e^{-t\Delta}$ is, by the spectral theorem, bounded by $\sup_{x\ge 0}x^{m+l}e^{-tx}$. For $t\ge t_0$ this is bounded by some constant C_{t_0}. This L^2-bound can be converted to a pointwise bound exactly in the same way as above. Here, via the Sobolev estimates of Theorem 2.4, the constant depends on the geometry of N. $\square$

Convergence of the small t part of determinants. In this section, we study the small t summands in the Definition 1.8 of $T_{\mathrm{an}}^{(2)}(N)$. We use Lemma 1.3 to rewrite the first integral involving small t in Definition 1.8 in a form which does not involve meromorphic extension. Namely, for

$$d_p^{sm} := \int_0^1 \left(\mathrm{Tr}_\Gamma\, e^{-t\Delta_p^\perp[\widetilde{N}]} - \sum_{i=0}^m t^{-(m-i)/2}(a_i+b_i) \right) \frac{dt}{t} + \sum_{i=0}^m c(i,m)(a_i+b_i)\,;$$

$$c(i,m) := -\frac{m-i}{2} \qquad \text{for } i \neq m;$$

$$c(m,m) := -\tfrac{d\Gamma}{ds}\Big|_{s=1}\,,$$

we want to show

LEMMA 2.36.

$$\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\cdot \mathrm{Tr}_\Gamma\, e^{-t\Delta_p^\perp[\widetilde{N}]}dt\Big|_{s=0} = d_p^{sm}\,.$$

Proof. If $h(s)$ is a holomorphic function defined in an open neighbourhood of $s=0$ then one easily checks using $\Gamma(s+1) = s\cdot\Gamma(s)$ and $\Gamma(1)=1$

$$\tfrac{d}{ds}\tfrac{1}{\Gamma(s)}\cdot h(s)\big|_{s=0} = h(0)\,;$$

$$\tfrac{d}{ds}\tfrac{1}{\Gamma(s)}\cdot \tfrac{1}{s}\big|_{s=0} = -\tfrac{d\Gamma}{ds}\big|_{s=1}\,.$$

Notice that the function $\int_0^1 t^{s-1}\cdot(\mathrm{Tr}_\Gamma\, e^{-t\Delta_p^\perp[\widetilde{N}]} - \sum_{i=0}^m t^{-(m-i)/2}(a_i+b_i))dt$ is holomorphic for $\Re(s) > -1/2$. Hence the claim follows from the following computation

$$\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\cdot t^{-(m-i)/2}dt\Big|_{s=0} = \frac{d}{ds}\frac{1}{\Gamma(s)}\cdot\frac{1}{s-(m-i)/2}\Big|_{s=0}. \qquad \square$$

PROPOSITION 2.37. *In the situation of Theorem 0.2 and with Notation 0.1 we get*

$$\lim_{R\to\infty} d_p^{sm}[\widetilde{M_R}] = d_p^{sm}[\widetilde{M}]\,.$$

Proof. First we choose the fundamental domain $\mathcal{F}\subset\widetilde{M}$ for the $\Gamma=\pi_1(X)$-action on $\widetilde{M}$ such that

$$\mathcal{F}_R := \widetilde{M_R}\cap\mathcal{F}\,;$$

$$\partial\mathcal{F}_R := \partial\widetilde{M_R}\cap\mathcal{F}\,;$$

are fundamental domains for the induced Γ-action on $\widetilde{M_R}$ and $\widetilde{\partial M_R}$ and under the identifications of Notation 0.1 we get

$$\mathcal{F}_R - \mathcal{F}_S = [S,R] \times \coprod_j \mathcal{G}_j\,; \tag{2.38}$$

$$\partial\mathcal{F}_R = \{R\} \times \coprod_j \mathcal{G}_j\,, \tag{2.39}$$

where $\mathcal{G}_j \subset \mathbb{R}^{m-1}$ is a fundamental domain for the universal covering $\mathbb{R}^{m-1} \to F_j$ of the flat closed $m-1$-dimensional manifold sitting at the j-th component of E_0.

We will only consider $R > 2$. We have to estimate $|d_p^{sm}[\widetilde{M}] - d_p^{sm}[\widetilde{M_R}]|$. Recall from Definition 1.1 and Lemma 1.3 that

$$\begin{aligned} \operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{M_R}]} &= \int_{\mathcal{F}_R} \operatorname{tr}_\mathbb{R} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x)\,dx\,; \\ a_i[\widetilde{M_R}] &= \int_{\mathcal{F}_R} \alpha_i[\widetilde{M_R}](x)\,dx\,; \\ b_i[\widetilde{M_R}] &= \int_{\partial\mathcal{F}_\mathcal{R}} \beta_i[\widetilde{M_R}](x)\,dx \end{aligned} \tag{2.40}$$

and similarly for $\widetilde{M}$. The functions $\alpha_i[\widetilde{M_R}](x)$ and $\beta_i[\widetilde{M_R}](x)$ are determined by the geometry of $\widetilde{M_R}$ in a neighbourhood of x. In particular, $\alpha_i[\widetilde{M_R}]$ does not depend on R, and coincides with $\alpha_i[\widetilde{M}]$. Moreover, $\operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x) - \sum_{i=0}^m t^{-m/2+i/2}\alpha_i[\widetilde{M_R}](x)$ is $O(t^{1/2})$ uniformly in x for $x \in \widetilde{M_{R-1}} \subset \widetilde{M_R}$ by Theorem 2.1 applied to $N = \widetilde{M}$, $V = \widetilde{M_R}$ and $K = 1$ since $\alpha_i[\widetilde{M_R}](x) = \alpha_i[\widetilde{M}](x)$. The function $\operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M}]}(x,x) - \sum_{i=0}^m t^{-m/2+i/2}\alpha_i[\widetilde{M}](x)$ is $O(t^{1/2})$ uniformly in x since the isometry group of $\widetilde{M} = \mathbb{H}^m$ is transitive. This shows that the following splitting makes sense, i.e. the integrals do converge and the obvious interchange of integration are allowed. Namely, we write $d_p^{sm}[\widetilde{M}] - d_p^{sm}[\widetilde{M_R}]$ as a sum with the following summands:

$$\begin{aligned} s_1 := \int_{\mathcal{F}_{R/2}} \int_0^1 \Big(&\operatorname{tr} e^{-t\Delta_p^\perp(\widetilde{M})}(x,x) - \operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x) \\ &- \sum_i t^{-m/2+i/2}\big(\underbrace{\alpha_i[\widetilde{M}](x) - \alpha_i[\widetilde{M_R}](x)}_{=0}\big)\Big)\,\tfrac{dt}{t}\,dx\,; \end{aligned}$$

$$s_2 := \int_{\mathcal{F}_{R-1}-\mathcal{F}_{R/2}} \int_0^1 \Big(\operatorname{tr} e^{-t\Delta_p^\perp(\widetilde{M})}(x,x) - \operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x)$$

$$- \sum_i t^{-m/2+i/2} \Big(\underbrace{\alpha_i[\widetilde{M}](x) - \alpha_i[\widetilde{M_R}](x)}_{=0} \Big) \Big) \tfrac{dt}{t}\, dx\,;$$

$$s_3 := \int_{\mathcal{F}-\mathcal{F}_{R-1}} \int_0^1 \Big(\operatorname{tr} e^{-t\Delta_p^\perp(\widetilde{M})}(x,x) - \sum_i t^{-m/2+i/2} \alpha_i[\widetilde{M}](x) \Big) \tfrac{dt}{t}\, dx\,;$$

$$s_4 := \int_0^1 \Bigg(\int_{\mathcal{F}_R-\mathcal{F}_{R-1}} \operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x)dx$$
$$- \sum_i t^{-m/2+i/2} \Bigg(\int_{\mathcal{F}_R-\mathcal{F}_{R-1}} \alpha_i[\widetilde{M_R}](x)dx + \int_{\partial\mathcal{F}_{\mathcal{R}}} \beta_i[\widetilde{M_R}](x')\, dx' \Bigg) \Bigg) \tfrac{dt}{t}\,;$$

$$s_5 := \sum_{i=0}^m c(i,m) \int_{\mathcal{F}_R} \underbrace{\alpha_i[\widetilde{M}](x) - \alpha_i[\widetilde{M_R}](x)}_{=0} dx = 0\,;$$

$$s_6 := \sum_{i=0}^m c(i,m) \int_{\mathcal{F}-\mathcal{F}_R} \alpha_i[\widetilde{M}](x)dx\,;$$

$$s_7 := \sum_{i=0}^m c(i,m) \int_{\partial\mathcal{F}_{\mathcal{R}}} \beta_i[\widetilde{M_R}](x')dx'\,.$$

We study each of these summands individually. For s_1 and s_2 we use Theorem 2.26 applied to $N = \widetilde{M}$, $V = \widetilde{M_R}$ and $K = 1$. Note that $d(\widetilde{M_a}, \widetilde{M} - \widetilde{M_b}) = b - a$ for $b/2 \le a \le b$. This implies for appropriate constants C_1 and C_2 independent of R:

$$|s_1| \le \operatorname{vol}(M_{R/2}) \int_0^1 C_1 e^{-R^2/4C_2 t} \frac{dt}{t} \le 4 \operatorname{vol}(M) C_1 C_2 R^{-2} e^{-R^2/4C_2}\,;$$

$$\begin{aligned} |s_2| &\le \operatorname{vol}(M_{R-1} - M_{R/2}) \int_0^1 C_1 e^{-1/C_2 t} \frac{dt}{t} \\ &\le \operatorname{vol}(M - M_{R/2}) C_1 C_2 = \operatorname{vol}(E_{R/2}) C_1 C_2\,. \end{aligned} \tag{2.41}$$

Therefore, both terms tend to zero for $R \to \infty$. For s_3 and s_6 observe that $\widetilde{M} = \mathbb{H}^m$ has transitive isometry group. It follows that

$$I_3 := \int_0^1 \Big(\operatorname{tr} e^{-t\Delta_p^\perp(\widetilde{M})}(x,x) - \sum_i t^{-m/2+i/2} \alpha_i[\widetilde{M}](x) \Big) \tfrac{dt}{t}$$

is a constant independent of x and the same holds for

$$I_6 := \sum_{i=0}^m c(i,m) \alpha_i[\widetilde{M}](x)\,.$$

Therefore

$$|s_3| \le |I_3| \cdot \operatorname{vol}(M - M_{R-1}) = |I_3| \cdot \operatorname{vol}(E_{R-1}) \xrightarrow{R\to\infty} 0\,;$$

$$|s_6| \le |I_6| \cdot \operatorname{vol}(M - M_R) = |I_6| \cdot \operatorname{vol}(E_R) \xrightarrow{R\to\infty} 0\,.$$

For arbitrary $R, S \geq 0$ and $x' \in \partial\widetilde{M}_R$ and $y' \in \partial\widetilde{M}_S$ we find neighbourhoods which are isometric. It follows that

$$I_7 := \sum_{i=0}^{m} c(i,m)\beta_i[\widetilde{M_R}](x')$$

does not depend on x' neither on R. Therefore

$$|s_7| \leq |I_7| \operatorname{vol}(\partial M_R) \xrightarrow{R\to\infty} 0 .$$

Since s_5 is zero it remains to treat s_4. We will treat only the case where $M_R - M_{R-2}$ has only one component, otherwise one applies the following argument to each component separately. Define

$$\mathbb{H}^m_R \ := \ (-\infty, R] \times \mathbb{R}^{m-1}$$

with the warped product metric as in Notation 0.1. We split s_4 into three summands

$$s_{41} := \int_0^1 \int_{\mathcal{F}_R-\mathcal{F}_{R-1}} \big(\operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x) - \operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}-\widetilde{M_{R-2}}]}(x,x)\big) dx \tfrac{dt}{t}\,;$$

$$s_{42} := \int_0^1 \int_{\mathcal{F}_R-\mathcal{F}_{R-1}} \big(\operatorname{tr} e^{-t\Delta_p^\perp[\widetilde{M_R}-\widetilde{M_{R-2}}]}(x,x) - \operatorname{tr} e^{-t\Delta_p^\perp[\mathbb{H}^m_R]}(x,x)\big) dx \tfrac{dt}{t}\,;$$

$$\begin{aligned} s_{43} := \int_0^1 \int_{\mathcal{F}_R-\mathcal{F}_{R-1}} & \operatorname{tr} e^{-t\Delta_p^\perp[\mathbb{H}^m_R]}(x,x)\, dx \\ & - \Big(\sum_i t^{-m/2+i/2} \int_{\mathcal{F}_R-\mathcal{F}_{R-1}} \alpha_i[\widetilde{M_R}](x)\, dx + \int_{\partial\mathcal{F}_\mathcal{R}} \beta_i[\widetilde{M_R}](x')\, dx'\Big) \tfrac{dt}{t} . \end{aligned}$$

For s_{41} we want to apply Theorem 2.26 for $V = \widetilde{M_R} - \widetilde{M_{R-2}}$, $N = \widetilde{M_R}$ and $K = 1$. Since $\widetilde{M_R}$ has constant sectional curvature -1 for all R and a neighbourhood of $\widetilde{M_R}$ is isometric to a neighbourhood of $\widetilde{M_0}$, the constants appearing in Theorem 2.26 can be chosen independently of R.

If $x \in \widetilde{M_R} - \widetilde{M}_{R-1}$ then $d_{\widetilde{M_R}-\widetilde{M_{R-2}}}(x) \geq 1$ and Theorem 2.26 yields

$$\big|e^{-t\Delta_p^\perp[\widetilde{M_R}]}(x,x) - e^{-t\Delta_p^\perp[\widetilde{M_R}-\widetilde{M_{R-2}}]}(x,x)\big| \leq D_1 \cdot e^{-D_2/t}$$

for constants D_1 and D_2 independent of x, t and R. Since the volume of $\widetilde{M_R} - \widetilde{M_{R-1}}$ tends to zero if R goes to ∞ the same is true for s_{41}. The same argument when replacing $\widetilde{M_R}$ by $\mathbb{H}^m_R$ yields that s_{42} tends to zero if R goes to ∞.

Recall that $\alpha[\widetilde{M_R}](x,x)$ and $\beta[\widetilde{M_R}](x,x)$ are determined by the geometry of $\widetilde{M_R}$ in a neighborhood of x. Hence we conclude

$$s_{43} := \int_0^1 \int_{\mathcal{F}_R - \mathcal{F}_{R-1}} \operatorname{tr} e^{-t\Delta_p^\perp[\mathbb{H}^m_R]}(x,x)dx$$
$$- \Big(\sum_i t^{-m/2+i/2} \int_{\mathcal{F}_R - \mathcal{F}_{R-1}} \alpha_i[\mathbb{H}^m_R](x)\,dx + \int_{\partial\mathcal{F}_\mathcal{R}} \beta_i[\mathbb{H}^m_R](x')dx' \Big) \tfrac{dt}{t} \,.$$

In the sequel we use the identifications (2.38) and (2.39). Notice that each isometry $i : \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ induces an isometry $\mathrm{id} \times i : \mathbb{H}^m_R \to \mathbb{H}^m_R$ and that $\mathbb{H}^m_R$ is isometric to $\mathbb{H}^0_m$ by the same argument as in the proof of Lemma 1.12. This implies that

$$\beta_i[\mathbb{H}^m_R](R,y) = f_i\,;$$
$$\alpha_i[\mathbb{H}^m_R](u,y) = g_i(u+1-R)\,;$$
$$\operatorname{tr} e^{-t\Delta_p^\perp[\mathbb{H}^m_R]}\big((u,y),(u,y)\big) = h(t,u+1-R)\,;$$

holds for all $u \in (-\infty, R]$, $y \in \mathbb{R}^{m-1}$ and an appropriate number f_i, appropriate functions $g_i(u)$ and an appropriate function $h(t,u)$, which are all independent of y or R. Since the volume of $\{u\} \times \mathcal{G}$ in $\{u\} \times \mathbb{R}^{m-1}$ is $e^{-(m-1)u}$-times the volume of $\mathcal{G} \subset \mathbb{R}^{m-1}$, we get

$$s_{43} = e^{-(m-1)R} \cdot \mathrm{vol}(\mathcal{G}) \cdot \int_0^1 \int_0^1 \Big(h(t,u)e^{-2u+2}du$$
$$- \sum_i t^{-\frac{m+i}{2}} \cdot \big(g_i(u)e^{-2u+2} - f_i\big)\Big) du \tfrac{dt}{t} \,.$$

Hence s_{43} tends to zero if R goes to ∞. This finishes the proof of Proposition 2.37. □

3 The Large t Summand of the Torsion Corresponds to Small Eigenvalues

Here, we will show that it suffices to control uniformly the small eigenvalues, to get convergence of $\int_1^\infty t^{-1} \operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{M_R}]}\,dt$. For this section let p be a fixed non-negative integer. We start with some notation.

Definition 3.1. Let N be a compact Riemannian manifold possibly with boundary. Let $E^p_\lambda[\widetilde{N}]$ be the right-continuous spectral family of $\Delta_p^\perp[\widetilde{N}]$, i.e. $\Delta_p[\widetilde{N}]$ restricted to the complement of its kernel. Define

$$F\big(\Delta_p^\perp[\widetilde{N}], \lambda\big) = \operatorname{Tr}_\Gamma E^p_\lambda[\widetilde{N}]\,. \tag{3.2}$$

Abbreviate $F(\lambda) = F(\Delta_p^\perp[\widetilde{N}], \lambda)$ and $E_\lambda = E^p_\lambda[\widetilde{N}]$. □

Definition 3.2 is of course consistent with the definition (see Definition 4.1 and Lemma 5.8) of spectral density function we will use later.

LEMMA 3.3. *For every $\lambda < \infty$ we have $F(\lambda) < \infty$.*

Proof. Remember that $e^{-\Delta_p[\tilde{N}]}$ and therefore also $e^{-\Delta_p^{\perp}[\tilde{N}]}$ are of Γ-trace class. It follows that the operator $\chi_{[0,\lambda]}(\Delta_p^{\perp})e^{-\Delta_p^{\perp}}$ has the same property, since the characteristic function $\chi_{[0,\lambda]}(t)$ of the interval $[0,\lambda]$ is clearly bounded and the trace class operators form an ideal in the algebra of bounded operators. Then $E(\lambda) = \chi_{[0,\lambda]}(\Delta_p^{\perp})e^{\Delta_p^{\perp}}\chi_{[0,\lambda]}(\Delta_p^{\perp})e^{-\Delta_p^{\perp}}$ is also of Γ-trace class because $\chi_{[0,\lambda]}(t)e^t$ is a bounded function, too. This concludes the proof. $\square$

Now observe that for $t \geq 1$

$$\begin{aligned} t^{-1}\operatorname{Tr}_\Gamma e^{-t\Delta_p^{\perp}[\widetilde{N}]} &= t^{-1}\int_0^\epsilon e^{-t\lambda}dF(\lambda) + t^{-1}\operatorname{Tr}_\Gamma \int_\epsilon^\infty e^{-t\lambda}dE_\lambda \\ &\overset{F(0)=0}{=} -t^{-1}\int_0^\epsilon (-t)e^{-t\lambda}F(\lambda)d\lambda + t^{-1}e^{-t\epsilon}F(\epsilon) \\ &\qquad\qquad + t^{-1}e^{-t\epsilon}\operatorname{Tr}_\Gamma \int_\epsilon^\infty e^{-t(\lambda-\epsilon)}dE_\lambda \\ &\overset{t\geq 1}{\leq} \int_0^\epsilon e^{-t\lambda}F(\lambda)d\lambda + \frac{e^{-t\epsilon}}{t}F(\epsilon) + \frac{e^{-t\epsilon}}{t}\operatorname{Tr}_\Gamma \int_\epsilon^\infty e^{-(\lambda-\epsilon)}dE_\lambda \\ &\leq \int_0^\epsilon e^{-t\lambda}F(\lambda)d\lambda + \frac{e^{-t\epsilon}}{t}F(\epsilon) + \frac{e^{-t\epsilon}}{t}e^\epsilon \operatorname{Tr}_\Gamma \int_0^\infty e^{-\lambda}dE_\lambda \\ &= \int_0^\epsilon e^{-t\lambda}F(\lambda)d\lambda + \frac{e^{-t\epsilon}}{t}F(\epsilon) + \frac{e^{-t\epsilon}}{t}e^\epsilon \operatorname{Tr}_\Gamma e^{-\Delta_p^{\perp}[\widetilde{N}]}\,. \end{aligned} \tag{3.4}$$

Therefore the next step in the proof of Theorem 0.2 is the following:

PROPOSITION 3.5. *For every $t \geq 1$ we have*

$$\lim_{R\to\infty}\operatorname{Tr}_\Gamma e^{-t\Delta_p^{\perp}[\widetilde{M_R}]} = \operatorname{Tr}_\Gamma e^{-t\Delta_p^{\perp}[\widetilde{M}]}\,.$$

Proof. By Theorem 2.35 and the local isometries of Lemma 1.12 there is $c = c(1)$ independent of R so that

$$\left|e^{-t\Delta_p[\widetilde{M_R}]}(x,x)\right| \leq c \qquad \forall t \geq 1\ \forall x \in \widetilde{M_R}\,.$$

Since $\widetilde{M} = \mathbb{H}^m$ is homogeneous we can choose c so that this inequality also holds for M in place of M_R. Notice that m is odd by assumption. Hence $\Delta_p^{\perp}[\widetilde{M}] = \Delta_p[\widetilde{M}]$ by [D]. We conclude $\Delta_p^{\perp}[\widetilde{M_R}] = \Delta_p[\widetilde{M_R}]$ from

Corollary 6.5 or [CGro, Theorem 1.1]. Given $t \geq 1$, we get for $R \geq 2$

$$\begin{aligned}
&|\operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{M_R}]} - \operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\widetilde{M}]}| \\
&\quad = \left| \int_{\mathcal{F}_R} \operatorname{tr} e^{-\Delta_p[\widetilde{M_R}]}(x,x)dx - \int_{\mathcal{F}} \operatorname{tr} e^{-\Delta_p(\widetilde{M}]}(x,x)dx \right| \\
&\quad \leq \int_{\mathcal{F}_{R/2}} \left| \operatorname{tr} e^{-t\Delta_p[\widetilde{M_R}]}(x,x) - \operatorname{tr} e^{-t\Delta_p(\widetilde{M}]}(x,x) \right| dx \\
&\qquad + \int_{\mathcal{F}_R - \mathcal{F}_{R/2}} \operatorname{tr} e^{-t\Delta_p[\widetilde{M_R}]}(x,x)dx + \int_{\mathcal{F} - \mathcal{F}_{R/2}} \operatorname{tr} e^{-t\Delta_p[\widetilde{M}])}(x,x)dx \\
&\quad \leq \operatorname{vol}(M_{R/2}) C_1 e^{-R^2/4C_2 t} + c \operatorname{vol}(M_R - M_{R/2}) + c \operatorname{vol}(E_{R/2}) \\
&\quad \leq \operatorname{vol}(M) C_1 e^{-R^2/4C_2 t} + 2c \operatorname{vol}(E_{R/2}) \xrightarrow{R \to \infty} 0 \,.
\end{aligned}$$

The existence of $C_1, C_2 > 0$ follows from Theorem 2.6 applied to $N = \widetilde{M}$, $V = \widetilde{M_R}$, $K = R/2$. Note that $d(\tilde{M}_{R/2}, \tilde{M} - \tilde{M}_R) = R/2$. This finishes the proof of Proposition 3.5. □

Corollary 3.6. *If we find $\epsilon > 0$ and a function $G(\lambda)$ so that for every $R \geq 0$*

$$F\big(\Delta_p^\perp[\widetilde{M_R}], \lambda\big) \leq G(\lambda) \qquad \forall \lambda \leq \epsilon$$

with

$$\int_1^\infty \bigg(\int_0^\epsilon e^{-t\lambda} G(\lambda) d\lambda \bigg) dt < \infty \,,$$

then the large time summand in the analytic L^2-torsion of M_R converges to the corresponding summand for M

$$\textstyle\int_1^\infty t^{-1} \operatorname{Tr}_\Gamma e^{-t\Delta^\perp(\widetilde{M_R})} dt \xrightarrow{R \to \infty} \int_1^\infty t^{-1} \operatorname{Tr}_\Gamma e^{-t\Delta^\perp(\widetilde{M})} dt \,.$$

Proof. By Proposition 3.5 we have pointwise convergence of the integrand. We want to apply the Theorem of Dominated Convergence. Inequality (3.4) shows that the assumption just guarantees the existence of a dominating integrable function because $\operatorname{Tr}_\Gamma e^{-\Delta^\perp(\widetilde{M_R})}$ is bounded by Proposition 3.5 and $\int_1^\infty e^{-t\epsilon}/t\, dt < \infty$. □

4 Spectral Density Functions

In this section we have to go through some of the proofs of [LLü, Section 1 and Section 2] since we need the results there in a more precise form later. For this section, let $\mathcal{A}$ be a von Neumann algebra with finite trace Tr. Our main example will be $\mathcal{A} = \mathcal{N}(\Gamma)$, the von Neumann algebra of a discrete group. In this section, all morphisms will be Hilbert-$\mathcal{A}$-module morphisms, i.e. bounded $\mathcal{A}$-equivariant operators, unless explicitly specified differently.

Spectral density functions.

DEFINITION 4.1. Suppose $f : U \to V$ is a Hilbert-$\mathcal{A}$-module morphism. Define its *spectral density function*

$$F(f,\lambda) := \operatorname{Tr} E_{\lambda^2}(f^*f) \qquad \lambda \geq 0\,.$$

Here $E_\lambda(f^*f)$ is the right-continuous spectral family of the positive self adjoint operator f^*f. Note that $F(f,\lambda)$ is monotone increasing. We say f is *left Fredholm* if $F(f,\lambda) < \infty$ for some $\lambda > 0$. If $F(f,0) < \infty$ set

$$\overline{F}(f,\lambda) := F(f,\lambda) - F(f,0).$$

LEMMA 4.2. *Let $f : U \to V$ and $g : V \to W$ be given. Let $i : V \to V'$ be injective with closed range and $p : U \to U'$ surjective with $\dim_{\mathcal{A}}(\ker(p)) < \infty$. Then*

1) $F(f,\lambda) \leq F(gf, \|g\|\lambda) \quad \forall\lambda$;
2) $F(g,\lambda) \leq F(gf, \|f\|\lambda)$, *if f is left Fredholm with dense image;*
3) $F(gf,\lambda) \leq F(g,\lambda^{1-r}) + F(f,\lambda^r) \quad \forall 0 < r < 1$;
4) $F(if,\lambda) \leq F(f, \|i^{-1}\|\lambda)$, *where $i^{-1} : i(V) \to V$ is bounded by the Open Mapping Theorem;*
5) $F(f,\lambda) \leq F(fp, \|p\|\lambda)$;
6) $F(f^*f, \sqrt{\lambda}) = F(f,\lambda)$.

Proof. 1)–3) are proven in [LLü, Lemma 1.6] and imply 4) and 5). 6) is a direct consequence of the definition. □

LEMMA 4.3. *Adopt the situation of Lemma 4.2 and suppose that the kernels of all morphisms in question are finite $\mathcal{A}$-dimensional. Then*

1) $\overline{F}(f,\lambda) \leq \overline{F}(gf, \|g\|\lambda)$, *if* $\ker g \cap \operatorname{im} f = \{0\}$;
2) $\overline{F}(g,\lambda) \leq \overline{F}(gf, \|f\|\lambda)$, *if f is left Fredholm with dense image;*
3) $\overline{F}(gf,\lambda) \leq \overline{F}(g,\lambda^{1-r}) + \overline{F}(f,\lambda^r)$ *for all* $0 < r < 1$, *if* $\ker g \subset \overline{\operatorname{im} f}$;
4) $\overline{F}(if,\lambda) \leq \overline{F}(f, \|i^{-1}\|\lambda)$;
5) $\overline{F}(fp,\lambda) \leq \overline{F}(f, \|p^{-1}\|\lambda)$, *for* $p^{-1} : U \to (\ker p)^\perp$;
6) $\overline{F}(f,\lambda) \leq \overline{F}(fp, \|p\|\lambda) + \dim_{\mathcal{A}} \ker p$.

Proof. This follows from Lemma 4.2. In assertion 2) use [LLü, Lemma 1.4] to conclude $F(gf,0) \leq F(g,0)$. In assertion 3) use the easy argument in the proof of [LLü, Lemma 1.11.3] to conclude $F(gf,0) = F(g,0) + F(f,0)$. □

LEMMA 4.4. *If $\dim_{\mathcal{A}} \ker\phi < \infty$ and $\dim_{\mathcal{A}} \ker\phi^* < \infty$, then*

$$\overline{F}(\phi,\lambda) = \overline{F}(\phi^*,\lambda)\,.$$

This also holds if ϕ is an unbounded $\mathcal{A}$-operator.

Proof. The proof in [LLü, Lemma 1.12.6] literally holds for unbounded operators, too. □

LEMMA 4.5. *Let $\phi : U_1 \to V_1$, $\gamma : U_2 \to V_1$ and $\xi : U_2 \to V_2$ be morphisms of Hilbert-$\mathcal{A}$-modules. Then*

1) $F\big(\left(\begin{smallmatrix}\phi & 0\\ 0 & \xi\end{smallmatrix}\right), \lambda\big) = F(\phi, \lambda) + F(\xi, \lambda)$;
2) *Suppose ϕ is invertible. Then*
$$F\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}, \lambda\right) \le F\big(\phi, (4+2\|\gamma\|\|\phi^{-1}\|)\lambda\big) + F\big(\xi, (4+2\|\gamma\|\|\phi^{-1}\|)\lambda\big);$$
3) $F\big(\left(\begin{smallmatrix}\phi & \gamma\\ 0 & \xi\end{smallmatrix}\right), \lambda\big) \le F(\phi, \lambda^r) + F(\xi, (4+2\|\gamma\|)\lambda^{1-r})$ *holds for $0 < r < 1$, provided that $\lambda < (4+2\|\gamma\|)^{1/(r-1)}$ is true;*
4) $F(\phi, \lambda) \le F\big(\left(\begin{smallmatrix}\phi & \gamma\\ 0 & \xi\end{smallmatrix}\right), 2(1+\|\gamma\|+\|\xi\|)\lambda\big)$;
5) *If ϕ has dense image and ϕ or $\left(\begin{smallmatrix}\phi & \gamma\\ 0 & \xi\end{smallmatrix}\right)$ are left Fredholm then for $\lambda < 1$ we have*
$$F(\xi, \lambda) \le F\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}, 2\big(1+\|\gamma\|+\|\phi\|\big)\lambda\right).$$

Proof. We will use the elementary fact
$$\left\|\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}\right\| \le 2\big(\|\phi\| + \|\xi\| + \|\gamma\|\big)$$
and the decompositions
$$\begin{pmatrix}\phi & 0\\ 0 & \xi\end{pmatrix} = \begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}\begin{pmatrix}1 & -\phi^{-1}\gamma\\ 0 & 1\end{pmatrix}; \tag{4.6}$$
$$\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix} = \begin{pmatrix}1 & \gamma\\ 0 & \xi\end{pmatrix}\begin{pmatrix}\phi & 0\\ 0 & 1\end{pmatrix}; \tag{4.7}$$
$$\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \xi\end{pmatrix}\begin{pmatrix}\phi & \gamma\\ 0 & 1\end{pmatrix}. \tag{4.8}$$

1) is obvious.
2) Apply Lemma 4.2.5 to (4.6) and use assertion 1).
3) Apply Lemma 4.2.3 to (4.7) and use assertions 1) and 2).
4) Apply Lemma 4.2.1 to (4.7).
5) If $\left(\begin{smallmatrix}\phi & \gamma\\ 0 & \xi\end{smallmatrix}\right)$ is left Fredholm, then ϕ is left Fredholm by [LLü, Lemma 1.11.1]. Suppose that ϕ has dense image and is left Fredholm. Then $\left(\begin{smallmatrix}\phi & \gamma\\ 0 & 1\end{smallmatrix}\right)$ has dense image and is left Fredholm by [LLü, Lemma 1.12.3]. Apply Lemma 4.3.2 to (4.8) and use assertion 1). □

LEMMA 4.9. *Adopt the situation of Lemma 4.5. Suppose all relevant kernels have finite $\mathcal{A}$-dimension. Then*

1) $\overline{F}\big(\left(\begin{smallmatrix}\phi & 0\\ 0 & \xi\end{smallmatrix}\right), \lambda\big) = \overline{F}(\phi, \lambda) + \overline{F}(\xi, \lambda)$;
2) *Suppose ϕ is invertible. Then*
$$\overline{F}\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}, \lambda\right) \le \overline{F}\big(\phi, (4+2\|\gamma\|\|\phi^{-1}\|)\lambda\big) + \overline{F}\big(\xi, (4+2\|\gamma\|\|\phi^{-1}\|)\lambda\big);$$

3) *Suppose ξ is injective or ϕ has dense image and is left Fredholm. Then for $0 < r < 1$ and $\lambda < (4+2\|\gamma\|)^{1/(r-1)}$ we have*
$$\overline{F}\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix},\lambda\right) \leq \overline{F}(\phi,\lambda^r) + \overline{F}\big(\xi,(4+2\|\gamma\|)\lambda^{1-r}\big)\,.$$
4) *If ξ is injective then*
$$\overline{F}(\phi,\lambda) \leq \overline{F}\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}, 2\big(1+\|\gamma\|+\|\xi\|\big)\lambda\right)\,;$$
5) *If ϕ has dense image and ϕ or $\left(\begin{smallmatrix}\phi & \gamma\\ 0 & \xi\end{smallmatrix}\right)$ are left Fredholm, then*
$$\overline{F}(\xi,\lambda) \leq \overline{F}\left(\begin{pmatrix}\phi & \gamma\\ 0 & \xi\end{pmatrix}, 2\big(1+\|\gamma\|+\|\phi\|\big)\lambda\right) + \dim_{\mathcal{A}}\ker\phi\,.$$

Proof. This follows from Lemma 4.5 as Lemma 4.3 follows from Lemma 4.2. □

Next we extend the notion of $F(f,\lambda)$ and $\bar{F}(f,\lambda)$ from Definition 4.1 for morphisms to chain complexes.

Definition 4.10. Let $0 \to C^0 \xrightarrow{c^0} C^1 \xrightarrow{c^1} \cdots$ be a cochain complex of Hilbert-$\mathcal{A}$-modules. Define the *spectral density function*
$$F^p(C^*,\lambda) := F\big(c^p| : \operatorname{im}(c^{p-1})^\perp \to C^{p+1},\lambda\big)\,.$$
If $F^p(C^*,0) < \infty$ set
$$\overline{F}^p(C^*,\lambda) := F^p(C^*,\lambda) - F^p(C^*,0)\,.$$
We call C^* *Fredholm* at p if $F^p(C^*,\lambda) < \infty$ for some $\lambda > 0$. □

Short exact sequences of cochain complexes. In this subsection we will express for a short exact sequence of Hilbert $\mathcal{A}$-cochain complexes $0 \to C^* \to D^* \to E^* \to 0$ the spectral density function of D^* in terms of the spectral density functions of C^*, E^* and the long weakly exact homology sequence.

Theorem 4.11. *Let $0 \to C^* \xrightarrow{j} D^* \xrightarrow{q} E^* \to 0$ be an exact sequence of Hilbert cochain complexes as above. Suppose C^* and E^* are Fredholm at p. Then D^* is Fredholm at p, δ_p is left Fredholm and*
$\overline{F}_p(D^*,\lambda) \leq \overline{F}_p(E^*,c_E\cdot\lambda^{1/2}) + \overline{F}(\delta^p,c_\delta\cdot\lambda^{1/4}) + \overline{F}_p(C^*,c_C\cdot\lambda^{1/4})$ *for* $0\leq\lambda< c_1$.
Here $\delta^p : H^p(E^*) \to H^{p+1}(C^*)$ is the connecting homomorphism in the long weakly exact L^2-cohomology sequence ([CGro, Theorem 2.1], [LLü, Theorem 2.2]) and c_E, c_C, c_δ, c_1 are explicitly determined in terms of the norms of d^p, j^p, q^p and their inverses. We use the convention that $H^p(E^*)$ and $H^p(C^*)$ are subquotients of the corresponding cochain complexes with the induced norm. Explicitly:
$$c_E = \big(4+2\|d^p\|\big)\|q_{p+1}\|\|q_p^{-1}\|\,;$$

$$
\begin{aligned}
c_C &= \|j_{p+1}^{-1}\|^{1/2}\|j_p\|\,;\\
c_\delta &= \|j_{p+1}^{-1}\|^{1/2}\bigl(4+2\|j_{p+1}^{-1}\|\|d^p\|\bigr)\|q_p^{-1}\|\,;\\
c_1 &= \min\bigl\{(4+2\|d^p\|)^{-1/2},(4+2\|j_{p+1}^{-1}\|\|d^p\|)^{-1/2}\bigr\}\,.
\end{aligned}
$$

Here, the inverse of an operator with closed image always means the obvious operator from the image to the orthogonal complement of the kernel.

Proof. For the proof, we repeat the proof of [LLü, Theorem 2.3] (where chain and not cochain complexes are treated) and take care not only of the Novikov-Shubin invariants but of all of the spectral density functions. Lott-Lück [LLü, page 28] construct a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\overline{q_p} & \xrightarrow{i} & D^p/\overline{\mathrm{im}(d_{p-1})} & \xrightarrow{\overline{q_p}} & E^p/\ker(e^p) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\partial_p} & & \downarrow{\scriptstyle\overline{d^p}} & & \downarrow{\scriptstyle\overline{e^p}} & & \\
0 & \longrightarrow & C^{p+1} & \xrightarrow{j_{p+1}} & D^{p+1} & \xrightarrow{q_{p+1}} & E^{p+1} & \longrightarrow & 0
\end{array}
$$

and show that ∂_p is Fredholm. The diagram yields a splitting

$$
\begin{aligned}
D^p/\overline{\mathrm{im}(d_{p-1})} = \ker\overline{q_p}\oplus\ker\overline{q_p}^{\perp} &\xrightarrow{\overline{d^p}=\begin{pmatrix} j_{p+1}\partial_p & \gamma\\ 0 & q_{p+1}^{-1}\overline{e^p}\,\overline{q_p}\end{pmatrix}} j(C^{p+1})\oplus\ker q_{p+1}^{\perp}\\
&= D^{p+1}\,.
\end{aligned}
$$

Because $\|\gamma\|\le\|\overline{d^p}\|=\|d^p\|$ and $q_{p+1}^{-1}\overline{e^p}\,\overline{q_p}$ is injective, Lemma 4.9.3 implies

$$
\begin{aligned}
\overline{F}_p(D^*,\lambda) &= \overline{F}(\overline{d^p},\lambda)\\
&\le \overline{F}(j_{p+1}\partial_p,\lambda^{1/2})+\overline{F}\bigl(q_{p+1}^{-1}\overline{e^p}\,\overline{q_p},(4+2\|d^p\|)\lambda^{1/2}\bigr)\\
&\quad\ \forall\lambda<(4+2\|d^p\|)^{-2}\,.
\end{aligned}
$$

Since $\|\overline{q_p}^{-1}\|\le\|q_p^{-1}\|$ we conclude from Lemma 4.3 4.) and 5.) that for all $\lambda<(4+2\|d^p\|)^{-2}$

$$
\begin{aligned}
\overline{F}_p(D^*,\lambda) &\le \overline{F}\bigl(\partial_p,\|j_{p+1}^{-1}\|\lambda^{1/2}\bigr)+\overline{F}\bigl(\overline{e^p},(4+2\|d^p\|)\|q_{p+1}\|\|\overline{q_p}^{-1}\|\lambda^{1/2}\bigr)\\
&\le \overline{F}\bigl(\partial_p,\|j_{p+1}^{-1}\|\lambda^{1/2}\bigr)+\overline{F}_p\bigl(E^*,(4+2\|d^p\|)\|q_{p+1}\|\|q_p^{-1}\|\lambda^{1/2}\bigr)\,.
\end{aligned} \tag{4.12}
$$

Now we have to examine ∂_p further. Its range actually lies in $\ker(c_{p+1})$ [LLü, page 28] and Lemma 4.3.4 applied to $\ker(\overline{q_p})\xrightarrow{\partial_p}\ker(c_{p+1})\overset{i}{\hookrightarrow}C^{p+1}$ implies

$$\overline{F}(\partial_p,\lambda)=\overline{F}(i\circ\partial_p,\lambda)\,. \tag{4.13}$$

Lott-Lück [LLü, page 29] construct the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\widehat{q_p}) & \xrightarrow{i_1} & \ker(\overline{q_p}) & \xrightarrow{\widehat{q_p}} & H_p(E) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \overline{\partial_p}} & & \big\downarrow{\scriptstyle \partial_p} & & \big\downarrow{\scriptstyle \delta_p} & & \\
0 & \longrightarrow & \overline{\operatorname{im}(c^p)} & \xrightarrow{i_2} & \ker(c_{p+1}) & \xrightarrow{pr} & H_{p+1}(C) & \longrightarrow & 0
\end{array}$$

and prove that the induced operator $\overline{\partial_p}$ is Fredholm and has dense image. We get the splitting

$$\begin{aligned}
&\ker(\overline{q_p}) = \\
&\ker(\widehat{q_p}) \oplus \ker(\widehat{q_p})^{\perp} \xrightarrow{\partial_p = \begin{pmatrix} \overline{\partial_p} & \gamma' \\ 0 & pr^{-1}\delta_p\widehat{q_p} \end{pmatrix}} \overline{\operatorname{im}(c^p)} \oplus \big(\operatorname{im}(c^p)^{\perp} \cap \ker(c_{p+1})\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \ker(c_{p+1})\,.
\end{aligned}$$

Because of $\|\gamma'\| \le \|\partial_p\|$ Lemma 4.9.3 implies

$$\overline{F}(\partial_p,\lambda) \le \overline{F}(\overline{\partial_p},\lambda^{1/2}) + \overline{F}\big(pr^{-1}\delta_p\widehat{q_p},(4+2\|\partial_p\|)\lambda^{1/2}\big) \text{ for } \lambda < \big(4+2\|\partial_p\|\big)^{-2}.$$

Note that $\widehat{q_p}^{-1} = q_p^{-1} \circ (pr : \ker e^p \to H^p(E))^{-1}$, therefore $\|\widehat{q_p}^{-1}\| \le \|q_p^{-1}\|$. Moreover, $\partial_p = j_{p+1}^{-1} \circ \overline{d^p}$, hence $\|\partial_p\| \le \|j_{p+1}^{-1}\| \cdot \|\overline{d^p}\| = \|j_{p+1}^{-1}\| \cdot \|d^p\|$. Since a non-trivial projection and its inverse always have norm 1, we conclude from Lemma 4.3 4) and 5) that for all $\lambda < (4+2\|j_{p+1}^{-1}\|\|d^p\|)^{-2}$

$$\overline{F}(\partial_p,\lambda) \le \overline{F}(\overline{\partial_p},\lambda^{1/2}) + \overline{F}\big(\delta_p,\|q_p^{-1}\|(4+2\|j_{p+1}^{-1}\|\|d^p\|)\lambda^{1/2}\big)\,. \qquad (4.14)$$

It remains to identify $\overline{\partial_p}$. Lott-Lück [LLü, page 29] define maps so that the following diagram is commutative:

$$\begin{array}{ccccc}
C^p & = & C^p & \xrightarrow{\widetilde{\overline{j_p}}} & \ker\widehat{q_p} \\
\big\downarrow{\scriptstyle j_p} & & \big\downarrow{\scriptstyle \overline{j_p}} & & \big\downarrow{\scriptstyle incl} \\
D^p & \longrightarrow & D^p/\overline{\operatorname{im}(d_{p-1})} & \xrightarrow{\mathrm{id}} & D^p/\overline{\operatorname{im}(d_{p-1})}
\end{array}$$

and show that $c^p = \overline{\partial_p} \circ \widetilde{\overline{j_p}}$ and that $\widetilde{\overline{j_p}}$ has dense image and is left Fredholm. Because of $\|\widetilde{\overline{j_p}}\| = \|\overline{j_p}\| \le \|j_p\|$ Lemma 4.3.2 implies

$$\overline{F}(\overline{\partial_p},\lambda) \le \overline{F}_p\big(C^*,\|j_p\|\lambda\big)\,. \qquad (4.15)$$

Now Theorem 4.11 follows from (4.12), (4.13), (4.14) and (4.15). □

5 Sobolev and L^2-complexes

In this section we show how the study of the spectral density function of the Laplacian with absolute boundary conditions, considered as an unbounded operator on L^2, can be translated to the study of the spectral density functions of Sobolev de Rham complexes without any boundary conditions.

As intermediate steps we study an L^2-de Rham complex with absolute boundary conditions, then a Sobolev complex with absolute boundary conditions, and in a last step the Sobolev complex without boundary conditions. We need the last one, because only here, an exact Mayer-Vietoris sequence can be obtained. Efremov [E] and Lott-Lück [LLü] use the same reductions. We repeat and refine their arguments, because we need more precise information on the spectral density functions than they do.

Definition 5.1. Let N be a complete m-dimensional Riemannian manifold. If N has no boundary, define for each integer $p \geq 0$ and each real number $s \geq 0$ the *Sobolev norm* $|\ |_{H^s}$ on the space $C_0^\infty(\Lambda^p(N))$ of smooth p-forms with compact support by (compare [T, page 363])

$$|\omega|_{H^s} := \left|(1+\Delta_p)^{s/4}\omega\right|_{L^2}.$$

If N has boundary, define for integers $p, s \geq 0$ the *Sobolev norm* $|\ |_{H^s}$ on the space $C_0^\infty(\Lambda^p(N))$ inductively by

$$|\omega|_{H^0} = |\omega|_{L^2};$$
$$|\omega|^2_{H^{s+1}} = |\omega|^2_{H^s} + |d^p\omega|^2_{H^s} + |\delta^p\omega|^2_{H^s} + \left|i^*(*\omega)\right|^2_{H^{s+1/2}},$$

where $i: \partial N \to N$ is the inclusion of the boundary and $\delta^p = (-1)^{p(m-p)} * d^{m-p} *$ is the adjoint of d^p with respect to the L^2-inner product. The *Sobolev space* $H^s(\Lambda^p(N))$ is the completion of $C_0^\infty(\Lambda^p(N))$ with respect to $|\,|_{H^s}$. □

Next we introduce the various relevant Sobolev and L^2-chain complexes.

Definition 5.2. Let N be a complete m-dimensional Riemannian manifold. Define its *Sobolev cochain complex* $D_p^*[N]$ which is concentrated in dimensions $p-1$, p and $p+1$ by

$$\ldots \to 0 \to H^2\big(\Lambda^{p-1}(N)\big) \xrightarrow{d^{p-1}} H^1\big(\Lambda^p(N)\big) \xrightarrow{d^p} L^2\big(\Lambda^{p+1}(N)\big) \to 0 \to \ldots.$$

Define the *Sobolev cochain complex with absolute boundary conditions* $D^*_{p,abs}[N]$ which is again concentrated in dimensions $p-1$, p, $p+1$ by

$$\ldots \to 0 \to H^2_{abs}\big(\Lambda^{p-1}(N)\big) \xrightarrow{d} H^1_{abs}\big(\Lambda^p(N)\big) \xrightarrow{d} L^2\big(\Lambda^{p+1}(N)\big) \to 0 \to \ldots,$$

where

$$H^1_{abs}\big(\Lambda^{p-1}(N)\big) := \big\{\omega \in H^1(\Lambda^{p-1}(N)) \bigm| i^*(*\omega) = 0\big\} \subset H^1\big(\Lambda^{p-1}(N)\big);$$
$$H^2_{abs}\big(\Lambda^{p-1}(N)\big) := \big\{\omega \in H^2(\Lambda^{p-1}(N)) \bigm| i^*(*\omega) = 0 = i^*(*d\omega)\big\} \subset H^2\big(\Lambda^{p-1}(N)\big).$$

The inclusions induce a cochain map and we will use on $D^*_{p,abs}[N]$ the norm induced from $D^*_p[N]$.

Define the *L^2-cochain complex* $L^*_p[N]$ which is again concentrated in dimensions $p-1$, p, $p+1$ by

$$\ldots \to 0 \to L^2\big(\Lambda^{p-1}(N)\big) \xrightarrow{d^{p-1}} L^2\big(\Lambda^p(N)\big) \xrightarrow{d^p} L^2\big(\Lambda^{p+1}(N)\big) \to 0 \to \ldots ,$$

where d^p is the closure of the operator with domain $C_0^\infty(\Lambda^p N)$. □

Notice that the differentials in $D^*_p[N]$ and $D^*_{p,abs}[N]$ are bounded operators, what is not true for $L^*_p[N]$.

Sobolev complexes with and without boundary conditions.

Arbitrary manifolds. Let N be a compact Riemannian manifold possibly with boundary. The inclusion induces a cochain map $i^* : D^*_{p,abs}[\widetilde{N}] \to D^*_p[\widetilde{N}]$ (see Definition 5.2). Given a geodesic collar of width $w > 0$, Lott-Lück [LLü, Definition 5.4 and Lemma 5.5] define maps

$$K^p_s : H^s\big(\Lambda^p(\widetilde{N})\big) \to H^{s+1}\big(\Lambda^{p-1}(\widetilde{N})\big) \quad s = 0,1,2\,,$$

which induce maps

$$K^p_{s,abs} : H^s_{abs}\big(\Lambda^p(\widetilde{N})\big) \to H^{s+1}_{abs}\big(\Lambda^{p-1}(\widetilde{N})\big) \qquad s = 0,1$$

and show that they have the following properties:

LEMMA 5.3. 1) *The maps K^p_s are bounded;*

2) *$K^p_s\omega$ depends only on the restriction of ω to the interior of the collar of width w, and $\operatorname{supp} K^p_s\omega$ is contained in this collar;*

3) *The maps*

$$j^{p-1}_p := 1 - d^{p-2}K^{p-1}_2 - K^{p-1}_1 d^{p-1}\,;$$
$$j^p_p := 1 - d^{p-1}K^p_1 - K^{p+1}_0 d^p\,;$$
$$j^{p+1}_p := 1 - d^p K^{p+1}_0\,;$$

constitute a chain map

$$j^*_p : D^*_p[\widetilde{N}] \to D^*_{p,abs}[\widetilde{N}]\,;$$

4) *The inclusion i^* and the map j^* induce (inverse) homotopy equivalences $\overline{i}^*_p : \overline{D}^*_p \to \overline{D}^*_{abs,p}$ and $\overline{j}^*_p : \overline{D}^*_{abs,p} \to \overline{D}^*_p$ between the reduced complexes*

$$\overline{D}^*_p := \ldots \to 0 \to D^p_p[\widetilde{N}]/\operatorname{clos}(\operatorname{im} d^{p-1}) \to D^{p+1}_p[\widetilde{N}] \to 0 \to \ldots$$
$$\overline{D}^*_{p,abs} := \ldots \to 0 \to D^p_{abs,p}[\widetilde{N}]/\operatorname{clos}(\operatorname{im} d^{p-1}) \to D^{p+1}_{p,abs}[\widetilde{N}] \to 0 \to \ldots$$

The corresponding homotopies to the identity are induced from $K^*_* \circ i^*$ and $i^* \circ K^*_*$. □

Application to hyperbolic manifolds. We use these results to compare the spectral density function of $D_p^*[\widetilde{M_R}]$ and $D_{p,abs}^*[\widetilde{M_R}]$. Note that for $R \geq 1$, $\widetilde{M_R}$ has a geodesic collar of width $1/3$, and all these collars are isometric to the one of $\widetilde{M_1}$. In particular we get

$$\big\|K_s^p[\widetilde{M_R}]\big\| \leq C \quad s = 0,1,2\,, \quad p \geq 0\,; \tag{5.4}$$

$$\big\|j_p^q[\widetilde{M_R}]\big\| \leq C \quad p \geq 0\,, \quad q = p-1, p, p+1\,; \tag{5.5}$$

with a constant C not depending on R since the maps K_s^p involve only the collar of the boundary. Now we can use the following theorem of Gromov-Shubin [GroSh, Proposition 4.1].

Theorem 5.6. *Let C^* and D^* be cochain complexes of Hilbert $\mathcal{N}(\Gamma)$-modules with not necessarily bounded differential, $f^* : C^* \to D^*$ and $g^* : D^* \to C^*$ bounded homotopy equivalences and $T^* : C^* \to C^{*-1}$ a homotopy between g^*f^* and* id. *Then for the spectral density functions the following holds*

$$F_p(C^*, \lambda) \leq F_p\big(D^*, \|f_{p+1}\|^2\|g_p\|^2\lambda\big) \quad \forall \lambda < (2\|T_{p+1}\|)^{-2}\,. \qquad \square$$

Proposition 5.7. *We find constants $C_1, C_2 > 0$, independent of R, so that*

$$F_p\big(D_{p,abs}^*[\widetilde{M_R}], C_1^{-1}\lambda\big) \leq F_p\big(D_p^*[\widetilde{M_R}], \lambda\big) \leq F_p\big(D_{p,abs}^*[\widetilde{M_R}], C_1\lambda\big) \quad \forall \lambda \leq C_2\,;$$

$$F_p\big(D_{p,abs}^*[\widetilde{T_R}], C_1^{-1}\lambda\big) \leq F_p\big(D_p^*[\widetilde{T_R}], \lambda\big) \leq F_p\big(D_{p,abs}^*[\widetilde{T_R}], C_1\lambda\big) \qquad \forall \lambda \leq C_2\,.$$

Proof. For $\widetilde{M_R}$ this is a direct consequence of Theorem 5.6 applied to the homotopy equivalence in Lemma 5.3.4, the estimates (5.4) and (5.5) and of the fact, that the p-spectral density function of $D_p^*[\widetilde{N}]$ and $D_{p,abs}^*[\widetilde{N}]$, respectively, coincide by definition with the on of $\overline{D}_p^*[\widetilde{N}]$ and $\overline{D}_{p,abs}^*[\widetilde{N}]$, respectively. The case of $\widetilde{T_R}$ is completely analogous. $\square$

L^2-complexes and Sobolev complexes with boundary conditions. To compare spectral density functions of Sobolev complexes and L^2-de Rham complexes, we need the following formula for these functions:

Lemma 5.8. *Suppose (E^*, d^*) is a Hilbert-$\mathcal{A}$ cochain complex. Here, we use the broader definition of Gromov-Shubin [GroSh, section 4], where d^* are closed, but not necessarily bounded operators. Then*

$$F_p(E^*, \lambda) = \sup_{L \in S_\lambda^p} \dim_\Gamma L\,,$$

where S_λ^p is the set of all closed Γ-invariant subspaces L of $\ker(d^p)^\perp \cap \mathcal{D}(d^p) \subset E^p$ so that

$$|d^p x| \leq \lambda \cdot |x| \qquad \forall x \in L\,.$$

Proof. The proof in [LLü, Lemma 1.5], where the proposition is stated only for bounded d^*, works without modifications also for unbounded operators. $\square$

To apply this theorem it is necessary to compute the orthogonal complement of the kernel of the differential for the considered complexes.

LEMMA 5.9. *Let N be a compact Riemannian manifold possibly with boundary. Then*

$$\ker\big(d^p : H^1_{abs}(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N}))\big)^{\perp_{H^1}} = H^1_{abs}\big(\Lambda^p(\widetilde{N})\big) \cap \overline{\delta C^\infty_{abs}\big(\Lambda^{p+1}(\widetilde{N})\big)}^{L^2},$$

where $C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N})) := \{\omega \in C^\infty_0(\Lambda^{p+1}(\widetilde{N})) \mid i^*(*\omega)=0\}$ *with* $i : \partial\widetilde{N} \hookrightarrow \widetilde{N}$ *the inclusion.*

Proof. $H^1_{abs}(\Lambda^p(\widetilde{N})) \cap \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}^{L^2} \subset \ker(d^p : H^1_{abs}(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N})))^\perp$:

Let $x \in H^1_{abs} \cap \overline{\delta C^\infty_{abs}}^{L^2}$ with $x = \lim_{L^2} \delta x_n$, $y \in H^1_{abs}$ with $dy = 0$. Then

$$(x,y)_{H^1} = (x,y)_{L^2} + (dx, \underbrace{dy}_{=0})_{L^2} + (\delta x, \delta y)_{L^2} + \big(i^*(*x), \underbrace{i^*(*y)}_{=0}\big)_{H^{1/2}}$$

$$(x,y)_{L^2} = \lim(\delta x_n, y)_{L^2} = \lim(x_n, dy)_{L^2} \pm \int_{\partial\tilde{N}} y \wedge *x_n = 0$$

If $z \in C^\infty_0(\tilde{N} - \partial\tilde{N})$ then

$$(\delta x, z) = (x, dz) + \int_{\partial\tilde{N}} z \wedge *x = \lim(\delta x_n, dy) = \lim(\underbrace{\delta\delta}_{=0} x_n, z) = 0$$

The set of these z is dense in L^2 hence $\delta x = 0$.

It remains to show the opposite inclusion. Put

$$\mathcal{H}^p = \big\{x \in H^\infty(\Lambda^p(\widetilde{N})) \mid dx = 0 = \delta x \text{ and } i^*(*x) = 0\big\}.$$

(Because of elliptic regularity, we could replace H^∞ by $C^\infty \cap L^2$.) There is the orthogonal decomposition [S1, Theorem 5.10] or [BuFK]:

$$L^2\big(\Lambda^p(\widetilde{N})\big) = \mathcal{H}^p \oplus \overline{dC^\infty_0(\Lambda^{p-1}(\widetilde{N}))} \oplus \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}. \quad (5.10)$$

Hence it suffices to show that $y \in \ker(d^p : H^1_{abs}(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N})))^{\perp_{H^1}}$ is orthogonal to both $\mathcal{H}^p$ and $\overline{dC^\infty_0(\Lambda^{p-1}(\widetilde{N}))}$ in $L^2(\Lambda^p(\widetilde{N}))$. The first claim follows from

$$0 = (x,y)_{H^1} = (x,y)_{L^2} \quad \forall x \in \mathcal{H}^p,$$

and for the second claim it suffices to prove $\delta y = 0$. We have

$$(dx, y)_{H^1} = 0 \quad \forall x \in H^\infty\big(\Lambda^{p-1}(\widetilde{N})\big) \text{ with } i^*(*dx) = 0.$$

Since $i^*(*y) = 0$ this implies

$$\begin{aligned} 0 &= (dx, y)_{L^2} + (\delta dx, \delta y)_{L^2} + (\underbrace{dd\delta x}_{=0}, y)_{L^2} \\ &= (x, \delta y)_{L^2} + (\delta dx, \delta y)_{L^2} + (d\delta x, \delta y)_{L^2} \\ &= \big((1 + \Delta_{p-1})x, \delta y\big)_{L^2} . \end{aligned}$$

Now

$$\begin{aligned} &(1 + \Delta_{p-1}) : H^\infty_{abs}\big(\Lambda^{p-1}(\widetilde{N})\big) \\ &\qquad = \big\{x \in H^\infty(\Lambda^{p-1}(\widetilde{N}));\ i^*(*x) = 0 = i^*(*dx)\big\} \to L^2\big(\Lambda^{p-1}(\widetilde{N})\big) \end{aligned}$$

has dense image (compare [S1, Theorem 5.19]) and therefore $\delta y = 0$. This finishes the proof of Lemma 5.9. □

Lemma 5.11. *Let N be a compact Riemannian manifold possibly with boundary and $L^*_p[\widetilde{N}]$ be the cochain complex introduced in Definition 5.2. Then*

$$\begin{aligned} &\mathcal{D}(d) \cap \ker\big(d^p : L^2(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N}))\big)^{\perp_{L^2}} \\ &\qquad\qquad = H^1_{abs}\big(\Lambda^p(\widetilde{N})\big) \cap \overline{\delta C^\infty_{abs}\big(\Lambda^{p+1}(\widetilde{N})\big)}^{L^2} \end{aligned}$$

with C^∞_{abs} as in Lemma 5.9.

Proof. First remember that $d + \delta : C^\infty(\Lambda^\bullet \widetilde{N}) \to C^\infty(\Lambda^\bullet(\widetilde{N}))$ with either absolute (i.e. $i^*(*\omega) = 0$) or relative (i.e. $i^*(\omega) = 0$) boundary conditions are formally self adjoint elliptic boundary value problems. We will establish that $H^1_{abs}(\Lambda^p(\widetilde{N})) \cap \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}^{L^2}$ is perpendicular to $\ker(d^p)$, and that any form which is perpendicular to $\ker(d^p)$ and is contained in the domain of d lies in $H^1_{abs}(\Lambda^p(\widetilde{N})) \cap \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}$.

For the first statement take $x \in C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))$ and $y \in \ker d^p$. In particular, a sequence $y_n \in C^\infty_0(\Lambda^{p+1}(\widetilde{N}))$ exists with $y_n \xrightarrow{L^2} y$ and $dy_n \xrightarrow{L^2} 0$. Then

$$(y, \delta x)_{L^2} = \lim (y_n, \delta x)_{L^2} \overset{i^*(*x)=0}{=} \lim (dy_n, x)_{L^2} = 0 .$$

Therefore, $H^1_{abs}(\Lambda^p(\widetilde{N})) \cap \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}^{L^2}$ and $\ker(d^p)$ are perpendicular. For the second statement, suppose $x \in \mathcal{D}(d)$ is perpendicular to $\ker d^p$. Choose $x_n \in C^\infty_0(\Lambda^p(\widetilde{N}))$ with $x_n \xrightarrow{L^2} x$ and $dx_n \to dx$. For every $y \in C^\infty_{abs}(\Lambda^\bullet(\widetilde{N}))$ we have

$$\begin{aligned} \big((d + \delta)y, x\big)_{L^2} &\overset{dy \in \ker d}{=} (\delta y, x)_{L^2} = \lim_{n\to\infty} (\delta y, x_n)_{L^2} \\ &\overset{i^*(*y)=0}{=} \lim_{n\to\infty} (y, dx_n)_{L^2} = (y, dx)_{L^2} \end{aligned}$$

Adjoint elliptic regularity [S1, Lemma 4.19, Corollary 4.22] shows that $x \in H^1_{loc}(\Lambda^p(\widetilde{N}))$. Then $(\delta x, y)_{L^2} = (x, dy) = 0$ holds since $\forall y \in C_0^\infty(\tilde{N} - \partial\tilde{N})$ $dy \in \ker d$. We conclude $\delta x = 0$. This means that $\forall y \in C_0^\infty(\tilde{N})$

$$0 \overset{dy\in\ker d}{=} (dy, x) = (y, \underbrace{\delta x}_{=0}) \pm \int_{\partial\tilde{N}} y \wedge *x\,.$$

Since $\{i^*(y)\}$ is dense in $L^2(\partial\tilde{N})$, $i^*(*x) = 0$, i.e. $x \in H^1_{abs}(\Lambda^p(\widetilde{N}))$. The L^2-splitting (5.10) implies $x \in \overline{\delta C^\infty_{abs}}$. $\square$

Now we can compare the spectral density functions of L_p^* and $D^*_{abs,p}$.

Proposition 5.12. *Let N be any compact manifold with boundary, $\Gamma := \pi_1(N)$. Then*

$$F_p\big(L_p^*[\widetilde{N}], \lambda\big) \le F_p\big(D^*_{abs,p}[\widetilde{N}], \lambda\big) \qquad \forall\lambda\,;$$
$$F_p\big(D^*_{abs,p}[\widetilde{N}], \lambda\big) \le F_p\big(L_p^*[\widetilde{N}], \sqrt{2}\lambda\big) \qquad \forall\lambda \le 1/\sqrt{2}\,.$$

Proof. We will use Lemma 5.8.

We start with the first inequality. Let $L \subset \ker(d^p : L^2(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N})))^{\perp_{L^2}} \cap \mathcal{D}(d^p)$ be a closed Γ-invariant subspace with $|dx|_{L^2} \le \lambda|x|_{L^2}$ for all $x \in L$. Hence we get $|dx|_{L^2} \le \lambda|x|_{H^1}$ for all $x \in L$. Moreover, $L \subset \ker(d^p : H^1(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N})))^{\perp_{H^1}}$ since the two orthogonal complements are equal by Lemma 5.11 and Lemma 5.9. L is closed also with respect to the H^1-topology because this is finer than the L^2-topology.

By Lemma 5.8, $F_p(L_p^*(\widetilde{N}), \lambda) = \sup_L \dim_\Gamma L$, where the supremum is over all such L. We have just seen that for the computation of $F_p(D^*_{abs}(\widetilde{N}),\lambda)$ we have to take the supremum over a larger set. This implies the first inequality. It remains to prove the second.

Let $\lambda \le 1/\sqrt{2}$ and let $L \subset H^1_{abs}(\Lambda^p(\widetilde{N}))$ be a closed Γ-invariant subspace with $L \perp_{H^1} \ker(d^p : H^1_{abs}(\Lambda^p(\widetilde{N})) \to L^2(\Lambda^{p+1}(\widetilde{N})))$ and $|dx|_{L^2} \le \lambda|x|_{H^1}$ for all $x \in L$. Then Lemma 5.11 implies for all $x \in L$ that

$$|x|^2_{H^1} = |x|^2_{L^2} + |dx|^2_{L^2}. \tag{5.13}$$

$$\Longrightarrow \quad |x|_{L^2} \le |x|_{H^1} \le (1-\lambda^2)^{-1/2} \cdot |x|_{L^2}\,. \tag{5.14}$$

Equation (5.14) says that on L the L^2-norm and the H^1-norm are equivalent, so that L is a closed subspace of $L^2(\Lambda^p(\widetilde{N}))$. We conclude from (5.13)

$$|dx|^2_{L^2} \le \lambda^2|x|^2_{H^1} \le \lambda^2|x|^2_{L^2} + \lambda^2|dx|^2_{L^2} \le \lambda^2|x|^2_{L^2} + \tfrac{1}{2}|dx|^2_{L^2}$$
$$\Longrightarrow |dx|_{L^2} \le \sqrt{2}\lambda|x|_{L^2}$$

Now the second inequality follows as above. $\square$

L^2-complexes and the Laplacian. Let N be a compact manifold with boundary as above. In the last paragraph, we studied the unbounded operator d on $\ker(d)^\perp \subset L^2(\widetilde{N})$ with domain $H^1_{abs} \cap \overline{\delta C^\infty_{abs}}$. Obviously, this coincides with the unbounded operator $d+\delta$ with domain H^1_{abs}, restricted to $\overline{\delta C^\infty_{abs}}$. This boundary value problem is elliptic, hence $(d+\delta)$ with domain H^1_{abs} is self adjoint in the Hilbert space sense ([S1, Theorem 4.25]). The adjoint of d restricted to $\overline{\delta C^\infty_{abs}}$ is therefore $d+\delta$ composed with projection onto $\overline{\delta C^\infty_{abs}}$. The square of $d+\delta$ is just

$$\Delta = (d+\delta)^2 \quad \text{with domain } H^2_{abs} = \left\{\omega \in H^2;\ i^*(*\omega) = 0 = i^*(*d\omega)\right\}.$$

This is exactly the operator we have to study. We want to compare its spectral density function with the one of $L^*_p(\widetilde{N})$. Recall that $\Delta_p^\perp$ is the operator from the orthogonal complement of the kernel of Δ_p to itself obtained by restriction. Note that the splitting (5.10) of L^2 induces a splitting of the Laplacian:

$$\Delta_p^\perp[\widetilde{N}] = \delta^{p+1}d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}} \oplus d^{p-1}\delta^p|_{\overline{dC^\infty_0(\Lambda^{p-1}(\widetilde{N}))}}. \tag{5.15}$$

LEMMA 5.16. *We have*

$$(\delta^{p+1}d^p)|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}} = (d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}})^*(d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}});$$
$$(d^{p-1}\delta^p)|_{\overline{dC^\infty_0(\Lambda^{p-1}(\widetilde{N}))}} = (d^{p-1}|_{\overline{\delta C^\infty_{abs}(\Lambda^{p-1}(\widetilde{N}))}})(d^{p-1}|_{\overline{\delta C^\infty_{abs}(\Lambda^{p-1}(\widetilde{N}))}})^*.$$

Here $d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}}$ *is the unbounded operator on the subspace* $\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}$ *of* $L^2(\Lambda^p(\widetilde{N}))$ *with range* $\overline{dC^\infty_0(\Lambda^p(\widetilde{N}))}$ *and with domain* $H^1_{abs}(\Lambda^p(\widetilde{N})) \cap \overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}$.

Proof. We first prove that the Hilbert space adjoint d^* of $d|_{\overline{\delta C^\infty_{abs}}} : \overline{\delta C^\infty_{abs}} \to \overline{dC^\infty_0}$ is δ with domain $\overline{dC^\infty_0} \cap H^1_{abs}$.

If $x \in \mathcal{D}(d^*) \subset \overline{dC^\infty_0} \implies x \in \mathcal{D}(d)$ and $dx = 0$.

Moreover, for arbitrary $y \in H^1_{abs} \cap \overline{\delta C^\infty_{abs}}$ we have $\delta y = 0$. Therefore

$$\big((d+\delta)y, x\big)_{L^2} = (dy, x)_{L^2} = (y, d^*x)_{L^2}.$$

If $y \in H^1_{abs} \cap \overline{dC^\infty_0}$ then

$$\big((d+\delta)y, x\big)_{L^2} = (\delta y, x)_{L^2} = 0$$

because $\delta H^1_{abs} \perp dC^\infty_0$. But also $(y, d^*x)_{L^2} = 0$ because $d^*x \in \overline{\delta C^\infty_{abs}}$ and $dC^\infty_0 \perp \delta C^\infty_{abs}$. Adjoint elliptic regularity [S1, Lemma 4.19] implies now that $x \in H^1_{loc}$. We have to show $x \in H^1_{abs}$, i.e. $dx, \delta x \in L^2$ and $i^*(*x) = 0$. Now

$$dx = 0 \in L^2; \qquad \delta x \overset{dx=0}{=} (d+\delta)x = d^*x \in L^2.$$

$$\left((x, d\delta y) = (d^*x, \delta y) = (\delta x, \delta y) = (x, d\delta y) \pm \int_{\partial\tilde{N}} \delta y \wedge *x \quad \forall y \in C_0^\infty(\tilde{N})\right)$$
$$\Longrightarrow\ i^*(*x) = 0\,.$$

Clearly $\delta d|_{\mathcal{D}(d^*d)} = \Delta = \Delta^\perp$. To prove the lemma it remains to show that the domains coincide, i.e. that

$$\mathcal{D}(\Delta) \cap \overline{\delta C^\infty_{abs}} = \mathcal{D}\big(d^*(d|_{\overline{\delta C^\infty_{abs}}})\big)\,.$$

Now $\mathcal{D}(\Delta) = H^2_{abs} = \{x \in H^2;\ i^*(*x) = 0 = i^*(*dx)\} \subset \mathcal{D}(d^*d)$.

If, on the other hand, $x \in \mathcal{D}(d^*d|)$ then $x \in H^1_{abs} \cap \overline{\delta C^\infty_{abs}}$ and $dx \in H^1_{abs}$. I.e. $(d+\delta)x \in H^1$ because $\delta x = 0$. Since also $i^*(*x) = 0$, by elliptic regularity $x \in H^2$. The boundary conditions are fulfilled therefore $x \in H^2_{abs} = \mathcal{D}(\Delta)$.

The proof for $(d\delta)|_{\overline{dC_0^\infty}}$ is similar. □

Proposition 5.17. *Let N be a compact Riemannian manifold possibly with boundary. Then*

$$F\big(\Delta_p^\perp[\widetilde{N}], \sqrt{\lambda}\big) = F_p\big(L_p^*[\widetilde{N}], \lambda\big) + F_{p-1}\big(L_p^*[\widetilde{N}], \lambda\big)\,.$$

Proof. This follows from (5.15), Lemma 4.9 1), Lemma 5.16, Lemma 4.2 6) and Lemma 4.4 by the following calculation:

$$\begin{aligned} F\big(\Delta_p^\perp[\widetilde{N}],\sqrt{\lambda}\big) &= F(\delta^{p+1}d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}},\sqrt{\lambda}) + F(d^{p-1}\delta^p|_{\overline{dC^\infty_0(\Lambda^{p-1}(\widetilde{N}))}},\sqrt{\lambda}) \\ &= F(d^p|_{\overline{\delta C^\infty_{abs}(\Lambda^{p+1}(\widetilde{N}))}},\lambda) + F(d^{p-1}|_{\overline{\delta C^\infty_{abs}(\Lambda^{p-1}(\widetilde{N}))}},\lambda) \\ &= F_p\big(L_p^*[\widetilde{N}],\lambda\big) + F_{p-1}\big(L^*_{p-1}[\widetilde{N}],\lambda\big)\,. \end{aligned}$$
□

6 Spectral Density Functions for M_R

In this section, we estimate the spectral density function of the Laplacian on $\widetilde{M_R}$ independently of R and finish the proof of the main Theorem 0.2.

The following sequence of Hilbert cochain complexes is exact [LLü, Lemma 5.14]

$$0 \to D_p^*[\widetilde{M}] \xrightarrow{\ j\ } D_p^*[\widetilde{M_R}] \oplus D_p^*[\widetilde{E_{R-1}}] \xrightarrow{\ q\ } D_p^*[\widetilde{T_{R-1}}] \to 0 \qquad (6.1)$$

with $j(\omega) := i^*_{M_R}\omega \oplus i^*_{E_{R-1}}\omega$ and $q(\omega_1 \oplus \omega_2) := i^*_{T_{R-1}}\omega_1 - i^*_{T_{R-1}}\omega_2$, where we use Notation 0.1 and i_X are inclusion maps. We are interested in the spectral density function of $D_p^*[\widetilde{M_R}]$ at $* = p$. We get from Lemma 4.5.1

$$F_p\big(D_p^*[\widetilde{M_R}],\lambda\big) \le F_p\big(D_p^*[\widetilde{M_R}] \oplus D_p^*[\widetilde{E_{R-1}}],\lambda\big)\,. \qquad (6.2)$$

The latter can be estimated in terms of the spectral densities of $D_p^*[\widetilde{M}]$ and $D_p^*[\widetilde{T_{R-1}}]$ by Theorem 4.11. To apply this theorem, we have to check the

Fredholm condition, compute the connecting homomorphism δ^p of the long weakly exact L^2-cohomology sequence of our short exact sequence and the constants appearing in the statement of the Theorem 4.11.

LEMMA 6.3. 1) *We get for all $R \geq 1$ and $\lambda \geq 0$*

$$F\big(\Delta_p^{\perp}[\widetilde{T_{R-1}}], \lambda\big) \leq F\big(\Delta_p^{\perp}[\widetilde{T_0}], \lambda\big)\,; \tag{6.4}$$

2) *There are positive numbers $\alpha > 0$ and $\epsilon > 0$ such that for $\lambda \leq \epsilon$*

$$F\big(\Delta_p^{\perp}[\widetilde{M}], \lambda\big) \leq \lambda^{\alpha}\,;$$
$$F\big(\Delta_p^{\perp}[\widetilde{T_0}], \lambda\big) \leq \lambda^{\alpha}\,;$$

3) *$D_p^*[\widetilde{M}]$ and $D_p^*[\widetilde{T_{R-1}}]$ are Fredholm at $* = p$;*

4) *The p-th L^2-cohomology of $D_p^*[\widetilde{M}]$ and of $D_p^*[\tilde{T}_{R-1}]$ vanishes. In particular the boundary morphism δ^p is trivial.*

Proof. 1) The isometry of Lemma 1.12 between $\widetilde{T_{R-1}}$ and $\widetilde{T_0}$ intertwines $\Delta_p[\widetilde{T_{R-1}}]$ and $\Delta_p[\widetilde{T_0}]$. In particular, spectral projections which enter in the definition of the spectral density function are mapped onto each other. However, this does not imply that the spectral density functions are equal because we have to take regularized dimensions, which involve the action of the fundamental group, and the obtained isometries do not respect the group action. Note that we explicitly get the regularized dimension by integrating the trace of the kernel of the corresponding projection operator over a fundamental domain of the diagonal. The isometry above maps the fundamental domain of $\widetilde{T_{R-1}}$ onto a subset of the fundamental domain in $\widetilde{T_0}$ (after suitable choices).

2) We conclude from [L, Proposition 39, Proposition 46] that closed hyperbolic manifolds and closed manifolds with virtually abelian fundamental groups have positive Novikov-Shubin invariants. Since these are homotopy invariants and agree with their combinatorial counterparts [LLü, Theorem 5.13], [E], [GroSh], the same is true for T_0. Note that $F(\Delta_p^{\perp}(\tilde{M}), \lambda) = f_{p,\mathbb{H}^m}(\lambda) \cdot \mathrm{vol}(M)$ depends only on the homogeneous structure of $\mathbb{H}^m$ and the volume of the quotient. In particular, its behavior near zero (given by $f_{p,\mathbb{H}^m}$ can be computed using a closed quotient.

3) and 4) Because of 2), $\Delta_p^{\perp}[\widetilde{M}]$ and $\Delta_p^{\perp}[\widetilde{T_{R-1}}]$ are left-Fredholm. Moreover, the kernel of $\Delta_p^{\perp}[\widetilde{M}]$ is trivial since $\widetilde{M}$ is $\mathbb{H}^m$ and for odd m there are no harmonic L^2-forms on $\mathbb{H}^m$ [D]. Also, T_{R-1} is homotopy equivalent to a flat manifold which has trivial L^2-cohomology. Since L^2-cohomology is a homotopy invariant of compact manifolds there are no harmonic L^2-forms

on $\widetilde{T_{R-1}}$. Now Proposition 5.7, Proposition 5.12, Proposition 5.17 imply 3) and the vanishing of the L^2-cohomology of $D_p^*[\widetilde{M}]$ and $D_p^*[\widetilde{T_{R-1}}]$. □

COROLLARY 6.5. *M_R is L^2-acyclic for all $R > 0$.*

Proof. The long weakly exact cohomology sequence of (6.1) together with Lemma 6.3.4 implies that the p-th L^2-cohomology of $D_p^*[\tilde{M}_R]$ vanishes. Propositions 5.7, 5.12 and 5.17 show that also the L^2-cohomology as it is usually defined is trivial. □

LEMMA 6.6. *We can choose the constants c_E, c_C and c_1 of Theorem 4.11 applied to the sequence* (6.1) *at $* = p$ independently of R.*

Proof. It suffices to find a constant $C < 0$ independent of R such that for all $R \geq 2$ the following holds

$$\left\| d^p : H^1(\Lambda^p(\widetilde{M_R})) \to L^2(\Lambda^{p+1}(\widetilde{M_R}))\right\| \leq 1\,; \tag{6.7}$$

$$\left\| d^p : H^1(\Lambda^p(\widetilde{E_{R-1}})) \to L^2(\Lambda^{p+1}(\widetilde{E_{R-1}}))\right\| \leq C\,; \tag{6.8}$$

$$\|j_{p+1}\| \leq \sqrt{2}\,; \tag{6.9}$$

$$\|q_{p+1}\| \leq 1\,; \tag{6.10}$$

$$\|j_p\| \leq C\,; \tag{6.11}$$

$$\|q_p\| \leq C\,; \tag{6.12}$$

$$\|j_{p+1}^{-1}\| \leq 1\,; \tag{6.13}$$

$$\|j_p^{-1}\| \leq 1\,; \tag{6.14}$$

$$\|q_p^{-1}\| \leq C\,; \tag{6.15}$$

We get (6.7) directly from the definition of the Sobolev norm 5.1. We conclude (6.8) from the fact that E_{R-1} is isometrically diffeomorphic to E_0 (Lemma 1.12). We obtain (6.9) from

$$|j_{p+1}\omega|_{L^2}^2 = \int_{\widetilde{M_R}} \left|\omega(x)\right|^2 dx + \int_{\widetilde{E_{R-1}}} \left|\omega(x)\right|^2 dx \leq \int_{\widetilde{M}} \left|\omega(x)\right|^2 + \int_{\widetilde{M}} \left|\omega(x)\right|^2 = 2\left|\omega\right|_{L^2}^2 .$$

and similarly for (6.10). For j_p in (6.11) observe

$$\begin{aligned} |\omega|_{\widetilde{M_R}}|_{H^1}^2 + |\omega|_{\widetilde{E_{R-1}}}|_{H^1}^2 &\leq 2\big(|\omega|_{L^2}^2 + |d\omega|_{L^2}^2 + |\delta\omega|_{L^2}^2\big) + \left|i^*(*\omega)\right|_{H^{1/2}(\partial\widetilde{M_R})}^2 \\ &\qquad + \left|i^*(*\omega)\right|_{H^{1/2}(\partial\widetilde{E_{R-1}})}^2 \\ &= 2|\omega|_{H^1}^2 + \left|i^*(*\omega)\right|_{H^{1/2}(\partial\widetilde{M_R})}^2 + \left|i^*(*\omega)\right|_{H^{1/2}(\partial\widetilde{E_{R-1}})}^2 . \end{aligned}$$

Therefore, we only have to deal with restriction to the boundary. Choose a cutoff function $\chi : \mathbb{R} \to [0,1]$ such that, for some $0 < \epsilon < 1$, $\chi(u) = 0$ for

$|u| \geq \epsilon$ and $\chi(u) = 1$ for $|u| \leq \epsilon/2$ holds. Define a function

$$\chi_R : M \longrightarrow [0,1]$$

which vanishes on $\widetilde{M_0}$ and sends $(u,x) \in [0,\infty) \times \mathbb{R}^{m-1} = \widetilde{E_0}$ to $\chi(u-R)$. The support of χ_R lies in the interior of $\widetilde{T_{R-1}} \cup \widetilde{T_R}$. Then

$$\begin{aligned}&\left|i^*(*\omega)\right|^2_{H^{1/2}(\partial\widetilde{M_R})}\\ &= \left|i^*(*\chi_R\omega)\right|^2_{H^{1/2}(\partial\widetilde{M_R})} \leq C^2_{\widetilde{T_{R-1}}\cup\widetilde{T_R}} |\chi_R\omega|^2_{H^1(\widetilde{M})} \leq C^2_{\widetilde{T_{R-1}}\cup\widetilde{T_R}} C^2_{\chi_R} |\omega|^2_{H^1}\,.\end{aligned}$$

The isometries of Lemma 1.12 which map $\widetilde{T_{R-1}} \cup \widetilde{T_R}$ to $\widetilde{T_0} \cup \widetilde{T_1}$ interchange χ_R and χ_1. Because the Sobolev norms are defined locally in terms of the geometry, the existence of these isometries shows that we can choose $C_{\widetilde{T_{R-1}}\cup\widetilde{T_R}} C_{\chi_R}$ independently of R. Since a similar argument applies to $|i^*(*\omega)|^2_{H^{1/2}(\partial\widetilde{E_{R-1}})}$ we get (6.11). The proof of (6.12) is similar.

To show (6.13), for every $\omega_1 \oplus \omega_2 \in L^2(\widetilde{M_R}) \oplus L^2(\widetilde{E_{R-1}})$ with $\omega_1|_{T_{R-1}} = \omega_2|_{T_{R-1}}$ we must find an element $\omega \in L^2(\widetilde{M})$ with $j(\omega) = \omega_1 \oplus \omega_2$ and $|\omega|^2_{L^2} \leq |\omega_1|^2_{L^2} + |\omega_2|^2_{L^2}$. The latter is easy to achieve. Namely, define

$$\omega(x) := \begin{cases} \omega_1(x)\,; & x \in \widetilde{M_R}\,, \\ \omega_2(x)\,; & x \in \widetilde{E_{R-1}}\,. \end{cases}$$

We use the same method to prove (6.14). It remains to prove (6.15).

Choose for $R = 1$ a bounded operator

$$\mathrm{Ex}_0 : H^1\big(\Lambda^p(\widetilde{T_0})\big) \to H^1\big(\Lambda^p(\widetilde{E_0})\big)$$

which satisfies $\mathrm{Ex}_0(\omega)|_{\widetilde{T_0}} = \omega$ for all $\omega \in H^1(\Lambda^p(\widetilde{T_0}))$. For arbitrary R, define the corresponding extension operator

$$\mathrm{Ex}_R : H^1\big(\Lambda^p(\widetilde{T_R})\big) \to H^1\big(\Lambda^p(\widetilde{E_R})\big)$$

using Ex_0 and the isometries of $\widetilde{E_0}$ and $\widetilde{E_{R-1}}$ given in Lemma 1.12. Since the H^1-norm is defined entirely in terms of the Riemannian metric, the norms of all the extension operators E_R are equal. We get for $\omega \in H^1(\Lambda^p(\widetilde{T_0}))$

$$\begin{gathered} q_p\big(0 \oplus \mathrm{Ex}_{R-1}(\omega)\big) = \omega\,; \\ \big|0 \oplus \mathrm{Ex}_{R-1}(\omega)\big|_{H^1} \leq \|\mathrm{Ex}_{R-1}\| |\omega|_{H^1}\,. \end{gathered}$$

This implies $\|q_p^{-1}\| \leq \|\mathrm{Ex}_{R-1}\| = \|\mathrm{Ex}_0\|$. This shows (6.15) and finishes the proof of Lemma 6.6. □

Proposition 6.16. *There is a constant C so that*

$$G(\lambda) = F\big(\Delta_p^\perp[\widetilde{T_0}], C\cdot\lambda^{1/2}\big) + F\big(\Delta_p^\perp[\widetilde{M}], C\cdot\lambda^{1/4}\big)\,.$$

fulfills the assumptions in Corollary 3.6.

Proof. We conclude from Theorem 4.11, equation (6.2), Lemma 6.3 and Lemma 6.6 that there is a constant $C > 0$ independent of R such that for all $R \geq 1$ and $0 \leq \lambda \leq C^{-1}$

$$F_p\big(D_p^*[\widetilde{M_R}], \lambda\big) \leq F_p\big(D_p^*[\widetilde{T_0}], C \cdot \lambda^{1/2}\big) + F_p\big(D_p^*[\widetilde{M}], C \cdot \lambda^{1/4}\big)\,.$$

Now we apply Proposition 5.7, Proposition 5.12, Proposition 5.17. One checks easily that all relevant statements also hold for $\widetilde{M}$ although M itself is not compact because $\widetilde{M}$ is isometric to $\mathbb{H}^m$ which is homogeneous. Hence there is a constant $C > 0$ independent of R such that for all $R \geq 1$ and $0 \leq \lambda \leq C^{-1}$

$$F\big(\Delta_p^\perp[\widetilde{M_R}], \lambda\big) \leq G(\lambda)\,.$$

We conclude from Lemma 6.3.2

$$\begin{aligned}
\int_1^\infty \left(\int_0^\epsilon e^{-t\lambda} G(\lambda) d\lambda\right) dt &\leq \int_1^\infty \left(\int_0^\epsilon e^{-t\lambda} (\lambda^{1/2} + \lambda^{1/4}) d\lambda\right) dt \\
&\leq 2 \cdot \int_1^\infty \left(\int_0^\epsilon e^{-t\lambda} \lambda^{1/4} d\lambda\right) dt \\
&\leq 2 \cdot \int_0^\epsilon \left(\int_1^\infty e^{-t\lambda} dt\right) \lambda^{1/4} d\lambda \\
&\leq 2 \cdot \int_0^\epsilon -e^{-t\lambda} \cdot \lambda^{-3/4} d\lambda \\
&< \infty\,.
\end{aligned}$$

This proves Proposition 6.16. □

This finishes the proof of our main Theorem 0.2 because of Corollary 3.6 and Proposition 6.16 for the large t summand and Lemma 2.36 and Proposition 2.37 for the small t summand.

7 L^2-analytic Torsion and Variation of the Metric

In the next lemma we extend a result of Lott [L, p. 480] to manifolds with boundary.

Lemma 7.1. *Let N be a compact manifold, possibly with boundary. Let $(g_u)_{u\in[0,1]}$ be a continuous family of Riemannian metrics on N. Then*

$$\operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\tilde{N}, g_u]} \xrightarrow{t\to\infty} 0 \quad \text{uniformly in } u \in [0,1]\,.$$

Here, $\Delta^\perp$ is Δ restricted to the orthogonal complement of its kernel.

Proof. If $E_\lambda(u)$ is the right continuous spectral family of $\Delta_p^\perp[\tilde{N}, g_u]$ and

$F(\lambda,u) = \operatorname{Tr}_\Gamma E_\lambda(u)$ we have by (3.4) for every $\epsilon > 0$ and $t \geq 1$

$$\operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\tilde{N},g_u]} \leq \int_0^\epsilon te^{-t\lambda}F(\lambda,u)d\lambda + e^{-t\epsilon}F(\epsilon,u) + e^{-t\epsilon+\epsilon}\operatorname{Tr}_\Gamma e^{-\Delta_p^\perp[\tilde{N},u]} . \tag{7.2}$$

By Proposition 5.17 we have

$$F(\lambda,u) = F_p\big(L_p^*[\tilde{N},g_u],\lambda^2\big) + F_{p-1}\big(L_{p-1}^*[\tilde{N},g_u],\lambda^2\big) . \tag{7.3}$$

The complexes L^* are defined in Definition 5.2. The identity map induces a bounded isomorphism between $L_p^*[\tilde{N},g_u]$ and $L_p^*[\tilde{N},g_0]$ with norm $a_p(u)$ (the norm is in general different from 1 because the inner products are different). Denote by $b_p(u)$ the norm of the inverse. Since g_u is continuous $a := \sup_{p=0,\dots,m;\, u\in[0,1]} a_p(u)$ and $b := \sup_{p=0,\dots m;\, u\in[0,1]} b_p(u)$ exist. By Theorem 5.6 we can compare the spectral density functions in the following way for λ (i.e. ϵ) sufficiently small:

$$F_p\big(L_p^*[\tilde{N},g_u],\lambda\big) \leq F_p\big(L_p^*[\tilde{N},g_0],a^2b^2\lambda\big) . \tag{7.4}$$

Now (7.2), (7.3) and (7.4) imply

$$\begin{aligned} &\operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\tilde{N},g_u]} \\ &\quad\leq \int_0^\epsilon te^{-t\lambda}F(ab\lambda,0)d\lambda + e^{-t\epsilon}F(ab\epsilon,0) + e^{-t\epsilon+\epsilon}\operatorname{Tr}_\Gamma e^{-\Delta_p^\perp[\tilde{N},u]} . \end{aligned} \tag{7.5}$$

By its explicit construction, the integral kernel of $e^{-\Delta_p^\perp[\tilde{N},u]}$ is a continuous function of u and therefore $\operatorname{Tr}_\Gamma e^{-\Delta^\perp[\tilde{N},u]}$ is uniformly bounded in u. Moreover,

$$\begin{aligned} &\int_0^\epsilon te^{-t\lambda}F(ab\lambda,0)\;d\lambda + e^{-t\epsilon}F(ab\epsilon,0) \\ &= \int_0^\epsilon e^{-t\lambda}d_\lambda F(ab\lambda) = \int_0^{ab\epsilon} e^{-t\lambda/ab}d_\lambda F(\lambda) \\ &\leq \int_0^\infty e^{-t\lambda/ab}dF(\lambda) = \operatorname{Tr}_\Gamma e^{-(t/ab)}\Delta_p^\perp[\tilde{N},g_0] \xrightarrow{t\to\infty} 0 . \end{aligned}$$

It follows that $\operatorname{Tr}_\Gamma e^{-t\Delta_p^\perp[\tilde{N},g_u]} \xrightarrow{t\to\infty} 0$ uniformly in u. □

Theorem 7.6. *Suppose N^m is a compact manifold, possibly with boundary. Suppose N is of determinant class. Let $(g_u)_{0\leq u\leq 1}$ be a smooth family of Riemannian metrics on N. Let $*_u$ be the Hodge-$*$-operator with respect to the metric g_u. Set $V_p = V_p(g_u) := \big(\frac{d}{du} *_p (g_u)\big) \circ *_p(g_u)^{-1}$.*

The analytic and L^2-analytic torsion are smooth functions of u, and

$$\begin{aligned} &\tfrac{d}{du}\big|_{u=0}\big(T_{\mathrm{an}}^{(2)}(N,g_u) - T_{\mathrm{an}}(N,g_u)\big) \\ &\qquad = \sum_p (-1)^{p+1}\big(\operatorname{Tr}_\Gamma V_p|_{\ker\Delta_p(\tilde{N})} - \operatorname{Tr} V_p|_{\ker\Delta_p(N)}\big) . \end{aligned} \tag{7.7}$$

Proof. Define

$$f(u,\Lambda) := \sum_p (-1)^p p \biggl(\int_0^\Lambda \operatorname{Tr}_\Gamma e^{-T\Delta_p[\tilde N, g_u]} - \operatorname{Tr} e^{-T\Delta_p[N,g_u]} \\ - \theta(T-1)(b_p^{(2)} - b_p)\tfrac{dT}{T} - \gamma(b_p^{(2)} - b_p)dT \biggr)\,, \quad (7.8)$$

where γ is the Euler–Mascheroni constant, θ the Heaviside step function and b_p are the ordinary Betti numbers. Lott proves the theorem for $\partial N = \emptyset$ [L, Proposition 7]. His proof applies to our situation because of the following remarks:

1. The asymptotic expansions for $\operatorname{Tr}_\Gamma e^{-t\Delta_p[\tilde N, g_u]}$ and $\operatorname{Tr} e^{-t\Delta_p[N,g_u]}$ are equal by (1.4) and therefore Lott's Corollary 5 with proof generalizes to our situation. This implies
$$f(u,\Lambda) \xrightarrow{\Lambda\to\infty} T_{\mathrm{an}}^{(2)}(N,g_u) - T_{\mathrm{an}}(N,g_u)\,. \quad (7.9)$$
2. Set $B(u,T) := \sum_p (-1)^p(\operatorname{Tr}_\Gamma e^{-T\Delta_p(\tilde N, g_u)} - \operatorname{Tr} e^{-T\Delta_p(N,g_u)})$. Then
$$\tfrac{dB}{du} = \sum_p (-1)^{p+1} T \tfrac{d}{dT}(\operatorname{Tr}_\Gamma V e^{-T\Delta_p(\tilde N)} - \operatorname{Tr} V e^{-T\Delta_p(N)})\,. \quad (7.10)$$
Note that Duhamel's principle works also for the manifold N with boundary, so that Lott's proof applies. The only crucial point is the question whether $\exp(-T\Delta_p[\tilde N, g_u])$ is differentiable with respect to u. This follows from the explicit construction of this operator (f.i. in Ray/Singer [RSi, 5.4]) which works also on $\tilde N$ because of bounded geometry.
3. The principle of not feeling the boundary in Theorem 2.26 implies that
$$\tfrac{d}{dT}(\operatorname{Tr}_\Gamma V e^{-T\Delta_p[\tilde N, g_u]} - \operatorname{Tr} V e^{-T\Delta_p[N,g_u]}) \\ = -\operatorname{Tr}_\Gamma V\Delta_p[\tilde N] e^{-T\Delta_p[\tilde N]} + \operatorname{Tr} V\Delta_p[N]e^{-T\Delta_p[N]}$$
is bounded for $0 \le T \le \Lambda < \infty$ in the same way as it implies Lemma 1.3 because V is a local operator. As g_u is smooth in u, the expression is uniformly bounded for $0 \le T \le \Lambda$ and $0 \le u \le 1$. Therefore we can integrate (7.10) to get
$$\tfrac{d}{du} f(u,\Lambda) = \sum_p (-1)^{p+1}(\operatorname{Tr}_\Gamma V_p e^{-\Lambda\Delta_p[\tilde N, g_u]} - \operatorname{Tr} V_p e^{-\Lambda\Delta_p[N,g_u]})\,.$$
and hence
$$f(u,\Lambda) = f(0,\Lambda) \\ + \sum_p (-1)^{p+1} \int_0^u \operatorname{Tr}_\Gamma V_p e^{-\Lambda\Delta_p[\tilde N, g_v]} - \operatorname{Tr} V_p e^{-\Lambda\Delta_p[N,g_v]} dv\,. \quad (7.11)$$

4. Since N is of determinant class, $\mathrm{Tr}_\Gamma\, e^{-t\Delta^\perp[\tilde N, g_u]} \xrightarrow{t\to\infty} 0$ uniformly in $u \in [0,1]$ by Lemma 7.1. Here $\Delta^\perp$ is Δ restricted to the orthogonal complement of its kernel. Because $e^{-t\Delta} = e^{-t\Delta^\perp} + \mathrm{pr}_{\ker\Delta}$ and $V(g_u)$ is a local operator which is uniformly bounded in u we can interchange integral and limit in (7.11) and conclude with (7.9)
$$T^{(2)}_{\mathrm{an}}(N, g_u) - T_{\mathrm{an}}(N, g_u) = T^{(2)}_{\mathrm{an}}(N, g_0) - T_{\mathrm{an}}(N, g_0) +$$
$$\sum_p (-1)^{p+1} \int_0^u \mathrm{Tr}_\Gamma\, V_p|_{\ker \Delta_p[\tilde N, g_v]} - \mathrm{Tr}\, V_p|_{\ker\Delta_p[N, g_v]} dv\,.$$

The last equation implies (7.7). □

Definition 7.12. The trace of $e^{-t\Delta_p(N,g_0)}V_p$ has an asymptotic expansion for $t \to 0$ because V_p is local. Let d_p be the coefficient of t^0 of the boundary contribution to this asymptotic expansion. This is an integral over a density on ∂N which is given locally in terms of the germ of the family of metrics g_u on ∂N (compare Cheeger [C, p. 278]).

Corollary 7.13. *In the situation of Theorem 7.6 and with Definition 7.12*
$$\frac{d}{du}\Big|_{u=0} T^{(2)}_{\mathrm{an}}(N, g_u) = \sum_p (-1)^p (d_p - \mathrm{Tr}_\Gamma\, V_p|_{\ker\Delta_p(\tilde N)})\,.$$

Proof. This follows from Theorem 7.6 and the computation of $\frac{d}{du}\big|_{u=0} T_{\mathrm{an}}(N, g_u)$ by Cheeger [C, 3.27] (note that Cheeger's α is just $-V$). □

A Examples for Nontrivial Metric Anomaly

In Corollary 7.13, we extended Cheeger's computation of the deviation of torsion under variation of the Riemannian metric on a compact manifold with boundary from classical analytical torsion to L^2-analytic torsion. Here, we will show that the abstract correction term (in the acyclic case)
$$\sum_p (-1)^p d_p$$
can be nonzero. Note that the formula relating analytic and topological torsion implies that the correction term is zero as long as the metric has a product structure near the boundary. Our examples show that this formula (i.e. the extension of the Cheeger-Müller theorem to manifolds with boundary) is not true in general (Corollary A.5). We use absolute boundary conditions for the examples (as we have done in the main text), but one can produce counterexamples with relative boundary conditions exactly in the same way.

Branson/Gilkey [BrG] explicitly compute the first few coefficients of the asymptotic expansion of the heat operator for manifolds with boundary and local boundary conditions. We will use these results to give two and three dimensional examples with nontrivial metric anomaly. Using a product formula this yields examples also in arbitrary dimensions > 1.

The examples

Dimension = 2. Here, we work with $S^1 \times [0,3]$. On $S^1 \times [0,1)$, we choose the following family of metrics:

$$g_u = f(x,u)(dx^2 + dy^2) \quad \text{with} \quad (y,x) \in S^1 \times [0,1)\,.$$

Here $S^1 = \mathbb{R}/\mathbb{Z}$ with the induced metric. We choose $(1, dx, dy, dx \wedge dy)$ as basis for the exterior algebra. Then $|dx| = f^{-1/2} = |dy|$ and $(dx, dy) = 0$. Consequently $|dx \wedge dy| = f^{-1}$. For the Hodge-$*$-operator we get the following

$$*1 = f dx \wedge dy * dx = dy * dy = -dx * (dx \wedge dy) = f^{-1}\,.$$

Consequently (remember that $V_p = \partial * / \partial u \cdot *^{-1}$)

$$V_0 = -f' f^{-1}\,, \qquad V_1 = 0\,, \qquad V_2 = f' f^{-1}\,.$$

Here $f' = \partial f / \partial u$. We extend these metrics to smooth metrics on $S^1 \times [0,3]$, which are constant (in u) product metrics on $S^1 \times (2,3]$. The specific choice does not affect the boundary terms we want to compute.

Now we specify $f(x,u)$: choose $f(x,u) = (1 + ux)$.

Proposition A.1. *For the family of metrics g_u on $S^1 \times [0,3]$, the boundary contribution of the metric anomaly is nonzero:*

$$d_0 - d_1 + d_2 \neq 0\,.$$

Proof. We have to evaluate the corresponding expressions in [BrG, 7.2]. d_p is the coefficient of t^0, which in dimension two is the second nontrivial coefficient. Therefore

$$d_p = (24\pi)^{-1}\left(\int_N (6V_p E + F_p \tau) + \int_{\partial N} \big(2V_p L_{aa} + 3\psi_p (V_p)_{;N} + 12 V_p S\big) \right).$$

The integral over N does not matter, because we already know that there is no interior contribution to the metric anomaly (in the acyclic case), so the alternating sum of these summands has to vanish and we do not have to specify the terms in the integrand.

Since $f' f^{-1} = 0$ at the boundary, $V_p = 0$ at the boundary, and the boundary contribution in d_p reduces to

$$(8\pi)^{-1} \int_{\partial N} \big(\psi_p (V_p)_{;N}\big)\,.$$

Here $(V_p)_{;N}$ denotes covariant differentiation in normal direction to the boundary. Since $\partial/\partial x$ is the inward pointing unit normal to the boundary: $\phi_{;N} = \partial\phi/\partial x$. In particular

$$(V_0)_{;N} = -(1+ux)^{-1} + ux(1+ux)^{-2} = -(V_2)_{;N} \quad \text{and} \quad (V_1)_{;N} = 0\,.$$

ψ_p is the dimension of the subspace which fulfills Neumann boundary conditions minus the dimension of the subspace with Dirichlet boundary conditions, i.e. $\psi_0 = 1$, where Neumann boundary conditions are in effect, and $\psi_2 = -1$, because on Λ^2 we have to impose Dirichlet boundary conditions.

Taking everything together yields (since $x = 0$ at the boundary)

$$d_0 - d_1 + d_2 = -(4\pi)^{-1} \int_{S^1} 1\, dy = -(4\pi)^{-1} \neq 0\,. \qquad \square$$

Dimension = 3. To produce a three dimensional example, we make a similar Ansatz on $[0,3]\times T^2$:

$$g_u = f(u,x)^2(dx^2 + dy^2 + dz^2) \quad \text{with} \quad f(u,x) = (1 + x + ux)\,.$$

Here $(x,y,z) \in [0,1) \times \mathbb{R}^2/\mathbb{Z}^2$, and we give $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ the quotient metric. The boundary is given by $x = 0$. Again we extend these metrics, so that $(2,3]\times T^2$ gets the standard product metric.

Computations as above give

$$\begin{aligned}
*1 &= f^3 dx \wedge dy \wedge dz \\
*dx &= f\, dy \wedge dz & *dy &= f\, dz \wedge dx & *dz &= f\, dx \wedge dy \\
*(dy \wedge dz) &= f^{-1} dx & *(dz \wedge dx) &= f^{-1} dy & *(dx \wedge dy) &= f^{-1} dz \\
*(dx \wedge dy \wedge dz) &= f^{-3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
V_0 &= -3f'f^{-1} & V_1 &= -f'f^{-1} \\
V_2 &= f'f^{-1} & V_3 &= 3f'f^{-1} \quad \text{with} \quad f'f^{-1} = x(1+x+ux)^{-1}\,.
\end{aligned} \tag{A.2}$$

In dimension three, the coefficient of t^0 is the third nontrivial coefficient. Branson/Gilkey [BrG, 7.2] compute this coefficient as follows:

$$\begin{aligned}
1536\pi \cdot d_p = \int_{\partial N} & V_p \times T \\
& + (V_p)_{;N}\big((6\psi_N^p + 30\psi_D^p)k + 96 S_p\big) \\
& + 24\psi_p (V_p)_{;;N}\,.
\end{aligned}$$

Here T is a complicated expression in terms of the geometry but $V_p(0) = 0$ due to our choice of f. Since ∂_x is the unit normal vector to the boundary, as above

$$d_p = (256\pi)^{-1} \int_{\partial N} 4\psi_p \underbrace{(V_p)_{;\partial_x\partial_x}}_{\substack{\text{iterated covariant derivative}\\ \text{in normal direction}}} + \tfrac{\partial V_p}{\partial x}\big((\psi_N^p + 5\psi_D^p)k + 16S_p\big)dy\,dz\,. \tag{A.3}$$

Here k is the mean curvature of the boundary (the trace of the second fundamental form). ψ_N^p is the trace of the projection onto the subspace of Λ^p, where Neumann boundary conditions are in effect, ψ_D^p is the trace of the projection onto the subspace with Dirichlet boundary conditions and $\psi = \psi_N - \psi_D$. Explicitly we get:

p	0	1	2	3
ψ_N^p	1	2	1	0
ψ_D^p	0	1	2	1
ψ_p	1	1	-1	-1 .

Together with (A.2) this easily implies that in our example most of the summands cancel. We are left with

$$d_0 - d_1 + d_2 - d_3 = (16\pi)^{-1} \sum_{p=0}^{3} (-1)^p \int_{\partial N} \frac{\partial V_p}{\partial x} S_p\,.$$

It remains to identify S_p. This is the trace of the (local) operator $A := \partial/\partial x - \nabla_N$ restricted to the subspace of Λ^p where we impose Neumann boundary condition, i.e. restriction to all of Λ^0, to the span of dy and dz in Λ^1, to the span of $dy\wedge dz$ in Λ^2, and to zero in Λ^3. Therefore $S_0 = 0 = S_3$. For 3-manifolds, A_1 and A_2 are given in the proof of [BrG, 7.2] (table on the bottom of p. 267). We get: $S_1 = -k = S_2$. In our situation, using the fact that ∂_x is normal to the boundary and has unit length there, that the Levi-Civita connection is compatible with the metric and torsion free

$$\begin{aligned} k &= (\nabla_y\partial_y, \partial_x) + (\nabla_z\partial_z, \partial_x) = (\partial_y, \nabla_y\partial_x) + (\partial_z, \nabla_z\partial_x) \\ &= (\partial_y, \nabla_x\partial_y) + (\partial_z, \nabla_x\partial_z) = \nabla_x(\partial_y, \partial_y)/2 + \nabla_x(\partial_z, \partial_z)/2 \\ &= \partial f^2/\partial x = 2(1 + x + ux)(1 + u)\,. \end{aligned}$$

It follows with (A.2) (remember $x = 0$ on ∂N) that

$$d_0 - d_1 + d_2 - d_3 = -(8\pi)^{-1} \int_{T^2} \frac{\partial f' f^{-1}}{\partial x} k\,dy\,dz = -(4\pi)^{-1}(1 + u) \neq 0\,.$$

Here we used

$$\frac{\partial f' f^{-1}}{\partial x} = (1 + x + ux)^{-1} - x(1 + u)(1 + x + ux)^{-2} \overset{x=0}{=} 1\,.$$

Arbitrary dimensions

PROPOSITION A.4. *For every dimension $m > 1$ we find a compact manifold N with boundary with a family of metrics g_u on N so that the boundary contribution to the metric anomaly is nonzero.*

Proof. For $m = 2$ and $m = 3$, we have shown that we can use $N = S^1 \times [0,3]$ or $N = S^1 \times S^1 \times [0,3]$, respectively. For higher dimensions, use $N \times S^{2k}$ with suitable $k \in \mathbb{N}$ with the corresponding family of product metrics. We use the product formula

$$T_{\mathrm{an}}(N \times S^{2k}) = T_{\mathrm{an}}(N)\chi(S^{2k}) + \chi(N)T_{\mathrm{an}}(S^{2k}) = 2T_{\mathrm{an}}(N)\,.$$

Using an acyclic representation on N we get nontrivial metric anomaly of $N \times S^{2k}$ which is entirely due to the boundary. For L^2-analytic torsion observe that N is L^2-acyclic so that the same argument applies. □

COROLLARY A.5. *For every dimension $m > 1$ we find compact Riemannian manifolds M^m with boundary which are L^2-acyclic, so that the difference between L^2-analytic and L^2-topological torsion is different from $\chi(\partial M)(\ln 2)/2$. Similarly, we find acyclic finite dimensional orthogonal representations of $\pi_1(M)$ so that the same statement holds for the corresponding classical analytic and topological torsion.*

In other words, we get counterexamples for the extension of the Cheeger-Müller theorem (and its L^2-counterpart) to arbitrary compact Riemannian manifolds with boundary.

Proof. The manifolds of Proposition A.4 do the job for metrics g_{u_0} with $u_0 \neq 0$ sufficiently small. This follows from the validity of the Cheeger-Müller theorem for the product metric case g_0 (by [Lü1], [BuFK]), and the anomaly formulas [C, 3.27] or Corollary 7.13, respectively. □

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