Introduction to the Farrell-Jones Conjecture

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Group rings

- Let R be a ring, i.e., an associative ring with unit. Let G be a group.

Definition (Group ring RG)

The *group ring RG* is the R-algebra whose underlying R-module is the free R-module generated by the set G and whose multiplication comes from the multiplication in G.

- Elements in RG are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many elements r_g are different from 0.
- The multiplication is given by

$$\left(\sum_{g \in G} r_g \cdot g\right) \cdot \left(\sum_{h \in G} s_h \cdot h\right) := \sum_{k \in G} \left(\sum_{g,h \in G, gh = k} r_g s_h\right) \cdot k.$$

- Let M be a R-module. Suppose that G acts on M by R-linear maps. Then these data determine the structure of an RG-module on M.
  The converse is also true.
- A $R[\mathbb{Z}]$-module M is the same as a R-module together with a R-automorphism of M.
- $R[\mathbb{Z}]$ agrees with the ring $R[z, z^{-1}]$ of finite Laurent series.

- Let X be a path connected CW-complex with fundamental group π.
- Let $p\colon \widetilde{X} \to X$ be its universal covering. Then π acts freely on $\widetilde{X}$ by deck transformations.
- $\widetilde{X}$ inherits from X a CW-structure. The π-action permutes the cells.
- Let $C_*(\widetilde{X})$ be its cellular $\mathbb{Z}$-chain complex. By the induced π-action it becomes a $\mathbb{Z}\pi$-chain complex which consists of free $\mathbb{Z}\pi$-modules.
- Many constructions of invariants for $C_*(X)$ can be generalized to much more refined invariants using $C_*(\widetilde{X})$ with its $\mathbb{Z}\pi$-structure.
- Take $X = S^1$. Then $C_*(\widetilde{X})$ looks like

$$\cdots \to 0 \to \mathbb{Z}[\mathbb{Z}|] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}|],$$

where $t \in \mathbb{Z}$ is a generator.

- Group rings are in general very complicated.
- If R is Noetherian and G is virtually poly-cyclic, then RG is Noetherian.
- There is the conjecture that the converse is true. In particular RG is in general not Noetherian.
- Suppose that $g \in G$ has finite order, let us say n. Put $N = \sum_{i=1}^{n} g^i$. Then
$$N \cdot N = n \cdot N.$$
Hence RG contains a zero-divisor since
$$(n \cdot 1 - N) \cdot N = n \cdot N - N \cdot N = 0.$$
If n is invertible in R, then RG contains an idempotent, namely $\frac{N}{n}$.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G that any idempotent of RG belongs to R.

Conjecture (Unit Conjecture)

Let G be a torsionfree group. Then every unit in RG is of the form $r \cdot g$ for some unit $r \in R$ and some $g \in G$.

- It is an easy exercise to verify the conjectures above for $G = \mathbb{Z}$ by looking at coefficients with the lowest and highest degree.

The projective class group

Definition (Projective R-module)

An R-module P is called *projective* if it satisfies one of the following equivalent conditions:

- P is a direct summand in a free R-module;
- The following lifting problem has always a solution

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & \nwarrow^{\overline{f}} & \uparrow f & & \\ & & P & & \end{array}$$

- If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

- Over a field or, more generally, over a principal ideal domain, every projective module is free.
- If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree.
- For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.
- Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Example (Representations of finite groups)

Let F be a field of characteristic p for p a prime number or 0. Let G be a finite group.

Then F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G. It is a free FG-module only if G is trivial.

Definition (Projective class group $K_0(R)$)

Let R be an (associative) ring (with unit).
Define its *projective class group*

$$K_0(R)$$

to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.
- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.
- Induction

  Let $f \colon R \to S$ be a ring homomorphism. Given an R-module M, let f_*M be the S-module $S \otimes_R M$. We obtain a homomorphism of abelian groups

$$f_* \colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P].$$

- Compatibility with products

  The two projections from $R \times S$ to R and S induce isomorphisms

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

- Morita equivalence

  Let R be a ring and $M_n(R)$ be the ring of (n,n)-matrices over R.
  We can consider R^n as a $M_n(R)$-R-bimodule and as a R-$M_n(R)$-bimodule.
  Tensoring with these yields mutually inverse isomorphisms

$$\begin{array}{lclcl} K_0(R) & \xrightarrow{\cong} & K_0(M_n(R)), & [P] & \mapsto & [{}_{M_n(R)}R^n{}_R \otimes_R P]; \\ K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R), & [Q] & \mapsto & [{}_R R^n{}_{M_n(R)} \otimes_{M_n(R)} Q]. \end{array}$$

Example (Principal ideal domains)

If R is a principal ideal domain. Let F be its quotient field. Then we obtain mutually inverse isomorphisms

$$\begin{array}{rclcrcl} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & & n & \mapsto & [R^n]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example (Representation ring)

Let G be a finite group and let F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$. Taking the character of a representation yields an isomorphism

$$R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} = K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \operatorname{class}(G, \mathbb{C}),$$

where $\operatorname{class}(G; \mathbb{C})$ is the complex vector space of class functions $G \to \mathbb{C}$, i.e., functions, which are constant on conjugacy classes.

Example (Dedekind domains)

- Let R be a Dedekind domain, for instance the ring of integers in an algebraic number field.
- Call two ideals I and J in R equivalent if there exists non-zero elements r and s in R with $rI = sJ$. The ideal class group C(R) is the abelian group of equivalence classes of ideals under multiplication of ideals.
- Then we obtain an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

- The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p.

Theorem (Swan (1960))

If G is finite, then $\widetilde{K}_0(\mathbb{Z}G)$ is finite.

- Topological K-theory

  Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X. This is the zero-th term of a generalized cohomology theory $K^*(X)$ called topological K-theory. It is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = \{0\}$.
- Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Theorem (Swan (1962))

There is an isomorphism

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Wall's finiteness obstruction

Definition (Finitely dominated)

A CW-complex X is called *finitely dominated* if there exists a finite (= compact) CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

- A finite CW-complex is finitely dominated.
- A closed smooth manifold is a finite CW-complex.

Problem

Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

Definition (Wall's finiteness obstruction)

A finitely dominated *CW*-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its *finiteness obstruction* as follows.

- Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$.
- Let $C_*(\widetilde{X})$ be the cellular chain complex. It is a free $\mathbb{Z}\pi$-chain complex.
- Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$.
- Define
$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

- This is a kind of Euler characteristic but now counting modules themselves and not their rank.

Theorem (Wall (1965))

A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ *vanishes.*

- A finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex since $\widetilde{K}_0(\mathbb{Z}) = \{0\}$.
- Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Theorem (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$)

The following statements are equivalent for a finitely presented group G:

- *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*
- $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$.

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the *K_1-group of a ring R*

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- This is the same as $GL(R)/[GL(R), GL(R)]$.
- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

which in general is not bijective.
- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of the universal determinant for R.

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$*.*

Proof.

- The ring $\mathbb{Z}$ possesses an Euclidean algorithm.
- Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1).
- This implies that any element in $K_1(\mathbb{Z})$ is represented by ± 1.

□

- Let G be a finite group. The in contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable.

Whitehead torsion

Definition (*h*-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann, mid 60-s)

Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorpic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

□

- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$.
  The Alexander trick does not work smoothly.
  Indeed, there exists so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- Given a finitely presented group G, an element $\xi \in \mathrm{Wh}(G)$ and a closed manifold M of dimension $n \geq 5$ with $G \cong \pi_1(M)$, there exists an h-cobordism W over M with $\tau(W, M) = \xi$.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Negative K-theory

Definition (Bass-Nil-groups)

Define for $n = 0, 1$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for K_1 (1964))

There is an isomorphism, natural in R,

$$K_0(R) \oplus K_1(R) \oplus \mathrm{NK}_1(R) \oplus \mathrm{NK}_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]) = K_1(R[\mathbb{Z}]).$$

Definition (Negative K-theory)

Define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t,t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for negative K-theory)

For $n \leq 1$ there is an isomorphism, natural in R,

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition (Regular ring)

A ring R is called *regular* if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

- Principal ideal domains are regular. In particular $\mathbb{Z}$ and any field are regular.
- If R is regular, then $R[t]$ and $R[t, t^{-1}] = R[\mathbb{Z}]$ are regular.
- If R is regular, then RG in general is not Noetherian or regular.

Theorem (Bass-Heller-Swan decomposition for regular rings)

Suppose that R is regular. Then

$$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \leq -1; \\ \mathrm{NK}_n(R) &= 0 \quad \text{for } n \leq 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition reduces for $n \leq 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

- There are also higher algebraic K-groups $K_n(R)$ for $n \geq 2$ due to Quillen (1973). They are defined as homotopy groups of certain spaces or spectra.
- Most of the well known features of $K_0(R)$ and $K_1(R)$ extend to both negative and higher algebraic K-theory. For instance the Bass-Heller-Swan decomposition holds also for higher algebraic K-theory.

- Notice the following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$, which look similar:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

- If G and K are groups, then we have the following formulas, which look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G * K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G * K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

Definition (Spectrum)

A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n) \colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n) \colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \; = \; \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.

- Given two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$, their one-point-union and their smash product are defined to be the pointed spaces

$$\begin{aligned} X \vee Y &:= \{(x, y_0) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\} \subseteq X \times Y; \\ X \wedge Y &:= (X \times Y)/(X \vee Y). \end{aligned}$$

  We have $S^{n+1} \cong S^n \wedge S^1$.
- The sphere spectrum $\mathbf{S}$ has as n-th space S^n and as n-th structure map the homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
- Let X be a pointed space. Its suspension spectrum $\Sigma^\infty X$ is given by the sequence of spaces $\{X \wedge S^n \mid n \geq 0\}$ with the homeomorphism $(X \wedge S^n) \wedge S^1 \cong X \wedge S^{n+1}$ as structure maps. We have $\mathbf{S} = \Sigma^\infty S^0$.

Definition (Ω-spectrum)

Given a spectrum $\mathbf{E}$, we can consider instead of the structure map $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$ its adjoint

$$\sigma'(n)\colon E(n) \to \Omega E(n+1) = \operatorname{map}(S^1, E(n+1)).$$

We call $\mathbf{E}$ an *Ω-spectrum* if each map $\sigma'(n)$ is a weak homotopy equivalence.

Definition (Homotopy groups of a spectrum)

Given a spectrum **E**, define for $n \in \mathbb{Z}$ its *n-th homotopy group*

$$\pi_n(\mathbf{E}) := \operatorname{colim}_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$ with k-th structure map

$$\pi_{k+n}(E(k)) \xrightarrow{\sigma'(k)} \pi_{k+n}(\Omega E(k+1)) = \pi_{k+n+1}(E(k+1)).$$

- Homotopy groups of spectra are always abelian (in contrast to the fundamental group of a space).
- Notice that a spectrum can have in contrast to a space non-trivial negative homotopy groups.
- If **E** is an Ω-spectrum, then $\pi_n(\mathbf{E}) = \pi_n(E(0))$ for all $n \geq 0$.

- Eilenberg-MacLane spectrum

  Let A be an abelian group. The n-th Eilenberg-MacLane space $EM(A, n)$ associated to A for $n \geq 0$ is a CW-complex with $\pi_m(EM(A, n)) = A$ for $m = n$ and $\pi_m(EM(A, n)) = \{0\}$ for $m \neq n$. The associated Eilenberg-MacLane spectrum $\mathbf{H}(A)$ has as n-th space $EM(A, n)$ and as n-th structure map a homotopy equivalence $EM(A, n) \to \Omega EM(A, n+1)$.

- Algebraic K-theory spectrum

  For a ring R there is the algebraic K-theory spectrum $\mathbf{K}_R$ with the property

$$\pi_n(\mathbf{K}_R) = K_n(R) \quad \text{for } n \in \mathbb{Z}.$$

- Algebraic L-theory spectrum

  For a ring with involution R there is the algebraic L-theory spectrum $\mathbf{L}_R^{\langle -\infty \rangle}$ with the property

$$\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R) \quad \text{for } n \in \mathbb{Z}.$$

- Topological K-theory spectrum

  By Bott periodicity there is a homotopy equivalence

$$\beta \colon BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z}).$$

  The topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$.
  The structure maps are given in even degrees by the map β and in odd degrees by the identity $\mathrm{id} \colon \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$.

Definition (Homology theory)

Let Λ be a commutative ring, for instance $\mathbb{Z}$ or $\mathbb{Q}$.
A *homology theory* $\mathcal{H}_*$ with values in Λ-modules is a covariant functor from the category of CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance.
- Long exact sequence of a pair.
- Excision.

  If (X, A) is a CW-pair and $f \colon A \to B$ is a cellular map , then

$$\mathcal{H}_n(X, A) \xrightarrow{\cong} \mathcal{H}_n(X \cup_f B, B).$$

Definition (continued)

- Disjoint union axiom.

$$\bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right).$$

Definition (Smash product)

Let **E** be a spectrum and X be a pointed space. Define the smash product $X \wedge \mathbf{E}$ to be the spectrum whose n-th space is $X \wedge E(n)$ and whose n-th structure map is

$$X \wedge E(n) \wedge S^1 \xrightarrow{\mathrm{id}_X \wedge \sigma(n)} X \wedge E(n+1).$$

Theorem (Homology theories and spectra)

Let $\mathbf{E}$ *be a spectrum. Then we obtain a homology theory* $H_*(-;\mathbf{E})$ *by*

$$H_n(X,A;\mathbf{E}) := \pi_n\left((X \cup_A \operatorname{cone}(A)) \wedge \mathbf{E}\right).$$

which satisfies

$$H_n(pt;\mathbf{E}) = \pi_n(\mathbf{E}).$$

Example (Stable homotopy theory)

The homology theory associated to the sphere spectrum **S** is stable homotopy $\pi_*^s(X)$.

The groups $\pi_n^s(\mathrm{pt})$ are finite abelian groups for $n \neq 0$ by a result of Serre (1953).

Their structure is only known for small n.

Example (Singular homology theory with coefficients)

The homology theory associated to the Eilenberg-MacLane spectrum $\mathbf{H}(A)$ is singular homology with coefficients in A.

Example (Topological K-homology)

The homology theory associated to the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ is K-homology $K_*(X)$. We have

$$K_n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

The Isomorphism Conjectures for torsionfree groups

Conjecture (K-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of R;
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$ for $n \in \mathbb{Z}$.
- BG is the classifying space of the group G, i.e., the base space of the universal G-principal G-bundle $G \to EG \to BG$. Equivalently, $BG = EM(G, 1)$. The space BG is unique up to homotopy.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture for torsionfree groups)

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG, where $K_*(-) = H_*(-; \mathbf{K}^{\mathrm{top}})$ for $\mathbf{K}^{\mathrm{top}}$ is the topological K-theory spectrum.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G which is the closure in the norm topology of $\mathbb{C}G$ considered as subalgebra of $\mathcal{B}(l^2(G))$.
- There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G; \mathbb{R})).$$

Consequences of the Isomorphism Conjectures for torsionfree groups

- In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the K-theoretic and L-theoretic respectively Farrell-Jones Conjecture for the coefficient ring R.
- Let $\mathcal{BC}$ be the class of groups which satisfy the Baum-Connes Conjecture.

Lemma

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then:

- $K_n(RG) = 0$ *for* $n \leq -1$*;*
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

Lemma

Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\mathrm{Wh}(G)$ *is trivial.*

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

Proof (continued).

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$.
  We get an exact sequence

$$0 \to H_0(BG; K_1(\mathbb{Z})) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G,G] \to 0.$$

- This implies $\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0$.

□

In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-Lück-Reich(2008))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree;*
- *G is torsionfree and sofic, e.g., residually amenable;*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Proof.

- Let p be an idempotent in FG. We want to show $p \in \{0, 1\}$.
- Denote by $\epsilon \colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module.
  Since $G \in \mathcal{FJ}_K(F)$, we can conclude that

$$i_* \colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is surjective.
  Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

$$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

Proof (continued).

- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain

$$FG^m \cong i_*(P) \oplus FG^n.$$

- Inserting this in the first equation yields

$$(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.$$

- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$.

□

Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

Let G be a torsionfree group with $G \in \mathcal{BC}$. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

Proof.

- We can prove the claim even for $p \in C_r^*(G)$.
- There is a trace map

$$\mathrm{tr}\colon C_r^*(G) \to \mathbb{C}$$

  which sends $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e\rangle_{l^2(G)}$.
- The L^2-index theorem due to Atiyah (1976) shows that the composite

$$K_0(BG) \to K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$

  coincides with

$$K_0(BG) \xrightarrow{K_0(\mathrm{pr})} K_0(\mathrm{pt}) = \mathbb{Z} \xrightarrow{i} \mathbb{C}.$$

Proof (continued).

- Hence $G \in \mathcal{BC}$ implies $\mathrm{tr}(p) \in \mathbb{Z}$.
- Since $\mathrm{tr}(1) = 1$, $\mathrm{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\mathrm{tr}(p) \in \mathbb{R}$ and $0 \leq \mathrm{tr}(p) \leq 1$.
- We conclude $\mathrm{tr}(0) = \mathrm{tr}(p)$ or $\mathrm{tr}(1) = \mathrm{tr}(p)$.
- This implies already $p = 0$ or $p = 1$.

□

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Wall(1969) and Farrell-Jones(1989).
- There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-Lück (2009)).

Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

- The Borel Conjecture in dimension 1 and 2 is obviously true.
- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\text{top}}(M)$ consists of one element.

Theorem (Ranicki (1992))

There is an exact sequence of abelian groups, called algebraic surgery exact sequence, for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$.

The general formulation the Isomorphism Conjectures

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict for a finite group G

$$K_0(BG) \cong K_0(C_r^*(G)) \cong R_{\mathbb{C}}(G).$$

However, $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} K_0(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ and $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ holds if and only if G is trivial.

- If G is torsionfree, then the version of the K-theoretic Farrell-Jones Conjecture predicts

$$\begin{aligned} H_n(B\mathbb{Z};\mathbf{K}_R) = H_n(S^1;\mathbf{K}_R) &= H_n(\mathrm{pt};\mathbf{K}_R) \oplus H_{n-1}(\mathrm{pt};\mathbf{K}_R) \\ &= K_n(R) \oplus K_{n-1}(R) \cong K_n(R\mathbb{Z}). \end{aligned}$$

  In view of the Bass-Heller-Swan decomposition this is only possible if $\mathrm{NK}_n(R)$ vanishes which is true for regular rings R but not for general rings R.
- However, there are more technical versions of the Farrell-Jones Conjecture which make sense for all groups and more general coefficients, where one allows twisted group rings or orientation homomorphisms.
- We present their formulation without giving details.

Definition (Additive category)

An additive category $\mathcal{A}$ is a small category $\mathcal{A}$ such that for two objects A and B the morphism set $\operatorname{mor}_{\mathcal{A}}(A, B)$ has the structure of an abelian group and the direct sum $A \oplus B$ of two objects A and B exists and the obvious compatibility conditions hold.

Example

Examples of additive categories are the category of R-modules and of finitely generated projective R-modules. Further examples are the category of R-chain complexes and the homotopy category of R-chain complexes.

Definition (The K-theoretic Farrell-Jones Conjecture with additive categories as coefficients)

The K-theoretic Farrell-Jones Conjecture for G with additive categories as coefficients says that the projection $\underline{\underline{E}}G \to G/G$ induces for all $n \in \mathbb{Z}$ and all additive categories $\mathcal{A}$ with right G-action an isomorphism

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_{\mathcal{A}}) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right).$$

- IF $\mathcal{A}$ is the additive category of finitely generated projective R-modules equipped with the trivial G-action, then the assembly map above can be identified with the classical one

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

- Roughly speaking, the Farrell-Jones Conjecture predicts how one can compute $K_n(RG)$ if one knows $K_m(RV)$ for all virtually cyclic subgroups V of G and all $m \leq n$ taking their relation coming from inclusions and conjugation into account.
- The advantage of the approach via additive categories is that it includes the case of twisted group rings and more generally of crossed product rings and that many inheritance properties such as the inheritance to subgroups is built in.

- In some sense the coefficient ring R becomes a kind of dummy variable.
- There is also a version for L-theory and a version for topological K-theory of group C^*-algebras which is called the Baum-Connes Conjecture.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0,f_0) = \operatorname{sign}_x(M_1,f_1).$$

- Both the Farrell-Jones Conjecture for L-theory and the Baum-Connes Conjecture imply the Novikov Conjecture.

The status of the Farrell-Jones Conjecture

Theorem (Main Theorem Bartels-Farrell-Lück-Reich (2008/2009))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures hold with coefficients in any additive G-category (with involution). It has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*

Theorem (continued)

- *If we demand for the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all* CAT(0)*-groups;*
- *The last statement is also true all cocompact lattices in almost connected Lie groups.*

- For all applications presented in these talks the version, where we demand for the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, is sufficient.

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems and have some exotic properties which may cause the failure of the Farrell-Jones Conjecture.
- One needs a property which can be used to detect a non-trivial element which is not in the image of the assembly map or is in its kernel.

Computational aspects

Theorem (The algebraic K-theory of torsionfree hyperbolic groups)

Let G be a torsionfree hyperbolic group and let R be a ring (with involution). Then we get an isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C),\, C \subseteq G,\, C \neq 1 \\ C \text{ maximal cyclic}}} \mathrm{NK}_n(R) \right) \xrightarrow{\cong} K_n(RG);$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG);$$

Theorem (Lück (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

Comments on the proof

Here are the basic steps of the proof of the main Theorem.

Step 1: Interpret the assembly map as a forget control map. Then the task is to give a way of gaining control.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}$ holds if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- *Z*-set-condition
  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;
- Long thin coverings
  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists a $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above.
This is often done by constructing a certain flow space and use the flow to let a given not yet perfect covering flow into a good one. The construction of the flow space for CAT(0)-space is one of the main ingredients.

K-theory versus L-theory

- So far the K-theory case has been easier to handle.
- The reason is that at some point a transfer argument comes in. After applying the transfers the element gets controlled on the total space level and then is pushed down to the base space.
- The transfer $p^!$ for a fiber bundle $F\colon E \to B$ has in K-theory the property that $p^! \circ p_*$ is multiplication with the Euler characteristic.
- In most situations F is contractible and hence obviously $p^! \circ p_*$ is the identity what is needed for the proof.
- In the L-theory case $p^! \circ p_*$ is multiplication with the signature. If the fiber is a sphere, then $p^! \circ p_*$ is zero.
- One needs a construction which makes out of a finite CW-complex with Euler characteristic 1 a finite Poincare complex with signature 1 or a chain complex or module analogue.

- Such a construction is given by the multiplicative hyperbolic form.
- Given a finitely projective R-module P over the commutative ring R, define a symmetric bilinear R-form $H_{\otimes}(P)$ by

$$\big(P \otimes P^*\big) \times \big(P \otimes P^*\big) \to R, \quad (p \otimes \alpha, q \otimes \beta) \mapsto \alpha(q) \cdot \beta(p).$$

- If one replaces $\otimes$ by $\oplus$ and $\cdot$ by $+$, this becomes the standard hyperbolic form.
- The multiplicative hyperbolic form induces a ring homomorphism

$$K_0(R) \to L^0(R), \quad [P] \mapsto [H_{\otimes}(P)].$$

- It is an isomorphism for $R = \mathbb{Z}$.

Concluding Remarks