"Surgery Obstructions of Fibre Bundles"
by
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Introduction

Chern, Hirzebruch and Serre [6] proved that the signature of the total space of a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ in which the fundamental group $\pi_1(B)$ acts trivially on $H^*(F;\mathbf{R})$ is the product of the signatures of the fibre and base

$$\mathrm{sign}(E) \;=\; \mathrm{sign}(F)\,\mathrm{sign}(B) \in \mathbf{Z}\ ,$$

with sign $= 0$ for manifolds of dimension $\not\equiv 0(\bmod 4)$. Kodaira [13], Atiyah [3] and Hirzebruch [12] constructed various examples of fibre bundles in which $\pi_1(B)$ acts non-trivially on $H^*(F;\mathbf{R})$ and the signature is not multiplicative. Moreover, in the case where both B and F are even-dimensional the Hirzebruch signature theorem

$$\mathrm{sign}(E) \;=\; \langle \mathcal{L}(E), [E]\rangle \in \mathbf{Z}$$

and the Atiyah-Singer index theorem were used by Atiyah [3] to obtain a characteristic class formula for the signature of E involving a contribution from the action of $\pi_1(B)$ on $H^*(F;\mathbf{R})$. The flat $(-1)^k$-symmetric bundle Γ over B with fibres $H^k(F_x;\mathbf{R})$ ($x \in B, k = \dim(F)/2$), has a real (resp. complex) K-theory signature $[\Gamma]_{\mathbf{K}} \in KO(B)$ for $k \equiv 0(\bmod 2)$ (resp. $KU(B)$ for $k \equiv 1(\bmod 2)$) and the twisted signature theorem is

$$\mathrm{sign}(E) \;=\; \langle \mathrm{ch}([\Gamma]_{\mathbf{K}}) \cup \widetilde{\mathcal{L}}(B), [B]\rangle \in \mathbf{Z}\ .$$

Lusztig [19] and Meyer [20] extended this expression to the Γ-twisted signature $\mathrm{sign}(B,\Gamma) \in \mathbf{Z}$ for any sheaf Γ of $(-1)^k$-symmetric forms over an even-dimensional manifold B.

In this paper we apply the algebraic surgery transfer of Lück and Ranicki [18] for a fibration $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a d-dimensional Poincaré complex

$$p^* \;:\; L_n(\mathbf{Z}\pi_1(B)) \;\longrightarrow\; L_{n+d}(\mathbf{Z}\pi_1(E))$$

to a further investigation of the behaviour of the Wall surgery obstruction and the Mishchenko symmetric signature in fibrations. These invariants are generalizations of the equivariant signature, and the characteristic classes have to be replaced by more general L-theory invariants. Instead of dealing with the action of $\pi_1(B)$ on $H^*(F;\mathbf{R})$ we consider the chain homotopy action of $\pi_1(B)$ on the chain complex $C(F)$ induced by the homotopy action of $\pi_1(B)$ on F given by the fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$.

Generalizing the work of Dress [8] and Yoshida [39] for finite groups we study the equivariant L-groups $L^d(\pi, \mathbf{Z})$ of a group π. For $d = 2k$ (resp. $2k+1$) $L^d(\pi, \mathbf{Z})$ is the Witt group of nonsingular π-equivariant $(-1)^k$-symmetric forms (resp. linking forms) on finitely generated free (resp. finite) abelian groups. The equivariant Witt groups $L^{2k}(\pi, \mathbf{Z})$ are well-known, and their applications to group actions, knot theory and the surgery obstruction groups of finite groups have motivated extensive computations, both for finite groups (Alexander, Conner and Hamrick [1]) and infinite groups (Neumann [25]). In the main body of the text we shall take full account of the various orientations: in the introduction we suppose for simplicity that all manifolds and Poincaré complexes are orientable.

The tensor product over $\mathbf{Z}$ with the diagonal π-action induces a pairing

$$\otimes \ : \ L^d(\pi, \mathbf{Z}) \otimes L_n(\mathbf{Z}\pi) \longrightarrow L_{n+d}(\mathbf{Z}\pi) \ .$$

A fibration $F \longrightarrow E \xrightarrow{p} B$ with d-dimensional Poincaré fibre F has a $\pi_1(B)$-equivariant symmetric signature invariant

$$\sigma^*(F, \omega) \in L^d(\pi_1(B), \mathbf{Z}) \ ,$$

given for $d = 2k$ (resp. $2k+1$) by the Witt class of the nonsingular $\pi_1(B)$-equivariant intersection form on $H^k(F)$ (resp. linking form on the torsion subgroup of $H^{k+1}(F)$). It depends only on the fibre F and the fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$. Our main result expresses the composition

$$p_* \circ p^* \ : \ L_n(\mathbf{Z}\pi_1(B)) \ \xrightarrow{p^*} \ L_{n+d}(\mathbf{Z}\pi_1(E)) \ \xrightarrow{p_*} \ L_{n+d}(\mathbf{Z}\pi_1(B))$$

as product with $\sigma^*(F, \omega)$.

Theorem 2.7 (up-down formula) For a fibration $F \longrightarrow E \xrightarrow{p} B$ with F a d-dimensional geometric Poincaré complex we have:

$$p_* \circ p^* \ = \ \sigma^*(F, \omega) \otimes - \ : \ L_n(\mathbf{Z}\pi_1(B)) \ \longrightarrow \ L_{n+d}(\mathbf{Z}\pi_1(B)) \ .$$

■

In the appendix to this paper the algebraic L-theory assembly map of Ranicki [32] is used to relate this expression for $p_* \circ p^*$ to the characteristic class formula for the signature of the total space of a fibre bundle.

Although we shall be mainly concerned with fibrations of connected spaces, it should be noted that the up-down formula 2.7 also applies to finite covers, with the fibre F a genuinely finite 0-dimensional Poincaré complex and $\pi_1(E) \to \pi_1(B)$ the inclusion of a subgroup of finite index. As in the general case, the element $\sigma^*(F,\omega) \in L^0(\pi_1(B),\mathbf{Z})$ is represented by the $\pi_1(B)$-action on the nonsingular symmetric form $(H^0(F),\oplus 1)$ of the fibre. In fact, the up-down formula in this case has already been obtained in Hambleton, Ranicki and Taylor [11].

The up-down formula simplifies considerably for fibrations which are orientable, i.e. with trivial fibre transport $\omega : \pi_1(B) \longrightarrow [F,F]$, generalizing the multiplicativity of the signature in this case. Namely, the $\{1\}$-equivariant Witt group $L^d(\{1\},\mathbf{Z})$ is just the simply-connected symmetric L-group $L^d(\mathbf{Z})$ of Ranicki [29]

$$L^d(\{1\},\mathbf{Z}) \;=\; L^d(\mathbf{Z}) \;=\; \begin{cases} \mathbf{Z} \ \text{(signature)} & \text{if } d \equiv 0 (\text{mod } 4) \\ \mathbf{Z}_2 \ \text{(deRham invariant)} & \text{if } d \equiv 1 (\text{mod } 4) \\ 0 & \text{otherwise} . \end{cases}$$

For an orientable fibration $\sigma^*(F,\omega) \in L^d(\pi_1(B),\mathbf{Z})$ is the image of the simply-connected symmetric signature $\sigma^*(F) \in L^d(\mathbf{Z})$. Combining 2.7 with the factorization

$$\otimes \;:\; L^d(\mathbf{Z}) \otimes L_n(\mathbf{Z}\pi) \;\longrightarrow\; L^d(\pi,\mathbf{Z}) \otimes L_n(\mathbf{Z}\pi) \longrightarrow L_{n+d}(\mathbf{Z}\pi)$$

gives:

Corollary 2.10 For an orientable fibration $F \longrightarrow E \xrightarrow{p} B$ with F a d-dimensional geometric Poincaré complex

$$p_* \circ p^* \;=\; \sigma^*(F) \otimes - \;:\; L_n(\mathbf{Z}\pi_1(B)) \;\longrightarrow\; L_{n+d}(\mathbf{Z}\pi_1(B))$$

is multiplication by the simply-connected symmetric signature $\sigma^*(F) \in L^d(\mathbf{Z})$.

∎

The significance of the algebraic surgery transfer is due to the fact that it does describe the geometric surgery transfer which is defined as follows. Let $F \longrightarrow E \xrightarrow{p} B$ be a smooth fibre bundle of connected smooth compact manifolds such that F is closed. Let d and n be the dimensions of F and B. An element λ in $L_n(\mathbf{Z}\pi_1(B))$ is represented by a normal map $f : M \longrightarrow N$ together with a reference map $g : N \longrightarrow B$ such that f induces a homotopy equivalence on the boundaries. (We suppress bundle data). The pull back construction yields a surgery problem $\overline{f} : \overline{M} \longrightarrow \overline{N}$ with a reference map $\overline{g} : \overline{N} \longrightarrow E$. Its class in $L_{d+n}(\mathbf{Z}\pi_1(E))$ is defined to be the image of λ under the geometric surgery transfer of Quinn [27]

$$p^! : L_n(\mathbf{Z}\pi_1(B)) \longrightarrow L_{n+d}(\mathbf{Z}\pi_1(E))$$

The significance of the geometric surgery transfer is obvious from its definition. It gives the possibility of proving the vanishing of surgery obstructions. The strategy consists of two steps. Firstly, show that the target of a given surgery problem is the total space of an appropriate fibre bundle such that the given surgery problem is the pull back of a surgery problem for the base space. Secondly, show that the surgery transfer vanishes. Hence vanishing theorems are of particular importance. We derive from the up-down formula:

Corollary 6.1 Let G be a compact connected d-dimensional Lie group which is not a torus. Let $G \longrightarrow E \xrightarrow{p} B$ be a G-principal bundle. Then

$$p^* = p^! : L_n(\mathbf{Z}\pi_1(B)) \longrightarrow L_{n+d}(\mathbf{Z}\pi_1(E))$$

is trivial. ■

Surgery transfers appear naturally in the study of group actions, namely, as surgery transfers of the normal sphere bundles of the fixed point sets. In Lück-Madsen [16] and [17] a spectral sequence is constructed which converges to the equivariant surgery obstruction group and whose E^2-term consists of algebraic L-groups. Its differentials are given by surgery transfers of the normal sphere bundles of the fixed point sets. For transformation groups of odd order the spectral sequence collapses. This is not true in the even order case, as already shown by explicit computations for $\mathbf{Z}/2$. Similar spectral sequences occur in the isovariant transverse linear setting of Browder and Quinn [5]. Using these spectral sequences and periodicity results Dovermann and Schultz compare the equivariant and isovariant transverse linear setting in [7].

We shall also give some computations for rational coefficients $\mathbf{Q}$ instead of integer coefficients $\mathbf{Z}$ in section 4. The corresponding L-theory change of ring homomorphism is an isomorphism if one inverts 2, so that rational computations still give good information about integral ones.

A fibration is called untwisted if the pointed fibre transport $\omega^+ : \pi_1(E) \longrightarrow [F, F]^+$ is trivial. We give further vanishing results for this class of fibrations in section 5. This includes orientable fibrations with H-spaces as fibre. The class of untwisted fibrations seems to be the largest class of fibrations where little information about the fibre suffices to prove vanishing results for the surgery transfer.

We are mainly dealing with L-groups of finitely generated free modules, or in other words, with L^h. In section 7 we briefly state the necessary modifications which have to be made when dealing with "decorated" L-groups like L^p or L^s.

We have tried to postpone the main technical parts to the last four sections. In particular it is not necessary to know the construction of the algebraic surgery transfer before section 8, as it only appears in the rather technical proof of the up-down formula 2.7 in sections 9 and 10. We recall which algebraic data are needed to define the algebraic surgery transfer and how the pointed fibre transport provides these data in section 1. Commencing with section 8 we assume that the reader is somewhat familiar with the approach to L-theory using Poincaré chain complexes as developed in Ranicki [30].

We make some comments on the proof of the up-down formula. The proof presented here is fairly straightforward. This has the advantage that one does avoid some machinery and has not to introduce new notions and the disadvantage that the conceptual ideas are somewhat hidden. The motivation for the proof in the L-theory case comes from the proof in the K-theory case as developed in Lück [15]. The relevant products fit together in the following commutative diagram for $i = 0, 1$:

$$\begin{array}{ccc}
K_0^c(\mathbf{Z}\pi_1(B) - \mathbf{Z}\pi_1(E)) \ \otimes \ K_i(\mathbf{Z}\pi_1(B)) & \xrightarrow{\otimes} & K_i(\mathbf{Z}\pi_1(E)) \\
i \otimes id \uparrow & & \uparrow id \\
K_0^c(\Delta - \pi_1(E), \mathbf{Z}) \ \otimes \ K_i(\mathbf{Z}\pi_1(B)) & \xrightarrow{\otimes_t} & K_i(\mathbf{Z}\pi_1(E)) \\
p_* \otimes id \downarrow & & \downarrow p_* \\
K_0^c(\{1\} - \pi_1(B), \mathbf{Z}) \otimes K_i(\mathbf{Z}\pi_1(B)) & \xrightarrow{\otimes_t} & K_i(\mathbf{Z}\pi_1(B)) \\
e \otimes id \downarrow & & \downarrow id \\
\mathrm{Sw}(\pi_1(B), \mathbf{Z}) \otimes K_i(\mathbf{Z}\pi_1(B)) & \xrightarrow{\otimes_{\mathbf{Z}}} & K_i(\mathbf{Z}\pi_1(B))
\end{array}$$

such that up to isomorphism

$$K_0^c(\mathbf{Z}\pi_1(B) - \mathbf{Z}\pi_1(B)) \ = \ K_0^c(\{1\} - \pi_1(B), \mathbf{Z}) \ = \ \mathrm{Sw}(\pi_1(B), \mathbf{Z}) \ .$$

The group $K_0^c(\mathbf{Z}\pi_1(B) - \mathbf{Z}\pi_1(E))$ is the Grothendieck group of chain homotopy representations of $\mathbf{Z}\pi_1(B)$ in $\mathbf{Z}\pi_1(E)$ and $K_0^c(\Delta - \pi_1(E), \mathbf{Z})$ is the Grothendieck group of $\mathbf{Z}\Delta$-chain complexes with a $\pi_1(E)$-twist, where Δ is the kernel of $\pi_1(p) : \pi_1(E) \longrightarrow \pi_1(B)$. $\mathrm{Sw}(\pi_1(B), \mathbf{Z})$ is the Grothendieck-Swan group of $\mathbf{Z}\pi_1(B)$-modules which are finitely generated and free as abelian groups. The pairing $\otimes_{\mathbf{Z}}$ is given by the tensor product over $\mathbf{Z}$ together with the diagonal action. Now the fibre together with the fibre transport define an element $[F, \omega]_1$ in $K_0^c(\mathbf{Z}\pi_1(B) - \mathbf{Z}\pi_1(E))$, and the K-theory transfer p^* is given by the evaluation of the top pairing on $[F, \omega]_1$. It turns out that $[F, \omega]_1$ is the image of an element $[F, \omega]_2 \in K_0^c(\Delta - \pi_1(E), \mathbf{Z})$ under the canonical map i. Let $[F, \omega] \in K_0^c(\{1\} - \pi_1(B), \mathbf{Z})$ be the image of $[F, \omega]_2$ under the map p_* induced by $\pi_1(p)$. As the diagram above commutes $p_* \circ p^*$ agrees with $[F, \omega] \otimes_t ?$. An element in $K_0^c(\{1\} - \pi_1(B), \mathbf{Z})$ is given by a chain complex C of finitely generated free $\mathbf{Z}$-modules, with a homotopy $\pi_1(B)$-action. The homology groups $H_i(C)$ are $\mathbf{Z}\pi_1(B)$-modules which are finitely generated as abelian groups. Using finitely generated free $\mathbf{Z}$-resolutions one obtains elements $[H_i(C)]$ in $\mathrm{Sw}(\pi_1(B), \mathbf{Z})$. The isomorphism $e : K_0^c(\{1\} - \pi_1(B), \mathbf{Z}) \longrightarrow \mathrm{Sw}(\pi_1(B), \mathbf{Z})$ sends the class of C to the K-theory Euler characteristic $\sum_i(-1)^i \cdot [H_i(C)]$. Hence $p_* \circ p^*$ is given by the evaluation of the bottom pairing $\otimes_{\mathbf{Z}}$ on the element $\sum_i(-1)^i \cdot [H_i(F)]$ in $\mathrm{Sw}(\pi_1(B), \mathbf{Z})$. The up-down formula 2.7 is just the L-theory version of this K-theory formula. The equivariant Witt group $L^d(\pi_1(B), \mathbf{Z})$ plays the role of $\mathrm{Sw}(\pi_1(B), \mathbf{Z})$, and the $\pi_1(B)$-equivariant signature $\sigma^*(F, \omega)$ corresponds to $\sum_i(-1)^i \cdot [H_i(F)]$.

We shall construct the L-theory analogue of this diagram in a subsequent paper, obtaining a Witt group for the transfer p^* rather than just $p_* \circ p^*$. This is not a trivial problem, since it involves higher coherences of the homotopy action of $\pi_1(B)$ on the fibre. One does not see this difficulty in homology, since any homotopy action on a space induces an honest action in homology. By sticking to the middle dimensions our proof of the up-down formula is essentially on the homology level, and so avoids higher coherences.

After this paper was written we received the preprint Yan [38], which obtains the up-down formula 2.7 in the special case $\pi_1(F) = \{1\}$, $d = 2k$.

The paper is organized as follows :

1. Algebraic Surgery Transfer

We explain which algebraic data are needed to define the algebraic surgery transfer and how these data can be obtained from the pointed fibre transport.

A ring with involution is an associative ring R with unit 1 together with a function

$^{-}: R \longrightarrow R$ satisfying $\overline{\overline{a}} = a$, $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\overline{a}$ and $\overline{1} = 1$ for $a, b \in R$. Given a group π together with a homomorphism $w : \pi \longrightarrow \{\pm 1\}$, the w-twisted involution on $\mathbf{Z}\pi$ is given by $\overline{g} = w(g) \cdot g^{-1}$. Let R and S be rings with involution. Let C be a d-dimensional finitely generated free S-chain complex. Let C^{d-*} be the S-chain complex $\hom_S(C_{d-*}, S)$, where the involution on S is used to define a left S-structure on $\hom_S(C_{d-*}, S)$. We shall identify $\left(C^{d-*}\right)^{d-*}$ and C. Consider an S-chain equivalence $\alpha : C^{d-*} \longrightarrow C$ such that α^{d-*} and α are chain homotopy equivalent. Then the set $[C, C]_S$ of chain homotopy classes of self chain maps of C becomes a ring with involution $[f] \mapsto [\alpha \circ f^{d-*}\alpha^{-1}]$. Denote by $[C, C]_S^{op}$ the opposite ring. Let $\mathbf{D}_d(S)$ be the additive category with involution having d-dimensional finitely generated free S-chain complexes as objects and homotopy classes of chain maps as morphisms. If $U : R \longrightarrow [C, C]_S^{op}$ is a homomorphism of rings with involution, we call (C, α, U) a symmetric representation of R into $\mathbf{D}_d(S)$. The surgery transfer

1.1 $(C, \alpha, U)^* : L_n(R) \longrightarrow L_{n+d}(S)$

associated to (C, α, U) is defined in Lück-Ranicki [18] and will be reviewed in section 8.

A d-dimensional finite Poincaré complex $F = (F, [F], w(F))$ consists of a finite CW complex F, an orientation homomorphism $w(F) : \pi_1(F) \longrightarrow \{\pm 1\}$ and a fundamental class $[F] \in H^d(F, {}^{w(F)}\mathbf{Z})$ such that $\cap [F] : C^{d-*}(\widetilde{F}) \longrightarrow C_*(\widetilde{F})$ is a $\mathbf{Z}\pi_1(F)$-chain homotopy equivalence. Here we use the $w(F)$-twisted involution to define a left $\mathbf{Z}\pi_1(F)$-structure on $C^{d-*}(\widetilde{F})$ and ${}^{w(F)}\mathbf{Z}$ is the $\mathbf{Z}\pi_1(F)$-module given by $\mathbf{Z}$ and the homomorphism $w(F)$. Any closed manifold is a Poincaré complex. Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of (well-pointed) connected spaces such that F is a d-dimensional Poincaré complex. We do not assume that B and E are manifolds. Let $\widehat{F} \longrightarrow F$ be the pull back of the universal covering of E with the inclusion of the fibre. Denote by $[F, F]$ (resp. $[F, F]^+$) the monoid of (pointed) homotopy classes of (pointed) self-maps of F. Let $[\widehat{F}, \widehat{F}]_{\pi_1(E)}$ be the monoid of $\pi_1(E)$-homotopy classes of $\pi_1(E)$-self-maps of $\widehat{F}$. The elementary construction of the monoid homomorphisms

1.2

τ	: $\pi_1(F)$	$\longrightarrow$	$[F, F]^+$	(operation of the fundamental group)
ω^+	: $\pi_1(E)$	$\longrightarrow$	$[F, F]^+$	(pointed fibre transport)
ω	: $\pi_1(B)$	$\longrightarrow$	$[F, F]$	(fibre transport)
$\widehat{\omega}$	: $\pi_1(B)$	$\longrightarrow$	$[\widehat{F}, \widehat{F}]_{\pi_1(E)}$	(equivariant fibre transport)

based on the homotopy lifting property can be found in Lück [14], section 6.

Let $H^d(\widetilde{F})$ be $H_0(\hom_{\mathbf{Z}\pi_1(F)}(C_{d-*}(\widetilde{F}), \mathbf{Z}\pi_1(F)))$ where $\hom_{\mathbf{Z}\pi_1(F)}(C_{d-*}(\widetilde{F}), \mathbf{Z}\pi_1(F))$ has the untwisted $\mathbf{Z}\pi_1(F)$-structure $(w\phi)(x) = \phi(x)w^{-1}$. Consider a self-map $f : F \longrightarrow F$. Choose a lift $\widetilde{f} : \widetilde{F} \longrightarrow \widetilde{F}$. Let $H^d(\widetilde{f}) : H^d(\widetilde{F}) \longrightarrow H^d(\widetilde{F})$ be given by the $\mathbf{Z}\pi_1(f)^{-1}$-equivariant chain map $\hom_{\mathbf{Z}\pi_1(F)}(C(\widetilde{f}), \mathbf{Z}\pi_1(f)^{-1})$. By Poincaré duality $H^d(\widetilde{F})$ is isomorphic to ${}^{w(F)}\mathbf{Z}$. Hence $w(F)$ depends only on the homotopy type of F and we have $w(F) \circ \pi_1(f) = w(F)$. The homotopy orientation homomorphism $v(F) : [F,F]^+ \longrightarrow \{\pm 1\}$ of F sends $[f]$ to the degree of $H^d(\widetilde{f})$. Define

$$\textbf{1.3} \quad \begin{array}{rcccc} \widehat{v} = v(F) \circ \omega^+ & : & \pi_1(E) & \longrightarrow & \{\pm 1\} \\ \widehat{w} = w \circ \pi_1(p) & : & \pi_1(E) & \longrightarrow & \{\pm 1\} \end{array}$$

Lemma 1.4 *1. The following diagram commutes*

$$\begin{array}{ccccccc} \pi_1(F) & \xrightarrow{\pi_1(i)} & \pi_1(E) & \xrightarrow{\pi_1(p)} & \pi_1(B) & \longrightarrow & \{1\} \\ \downarrow id & & \downarrow \omega^+ & & \downarrow \omega & & \\ \pi_1(F) & \xrightarrow{\tau} & [F,F]^+ & \xrightarrow{pr} & [F,F] & \longrightarrow & \{1\} \end{array}$$

The upper row is an exact sequence of groups. The lower row is exact in the sense that pr is surjective and $pr(f_0) = pr(f_1)$ holds for $f_0, f_1 \in [F,F]^+$, if and only there is $g \in \pi_1(F)$ satisfying $f_0 = \tau(g) f_1$.

2. *The $\pi_1(E)$-space $\widehat{F}$ can be identified with $\pi_1(E) \times_{\pi_1(F)} \widetilde{F}$. Let $g \in \pi_1(B)$ and $\widehat{g} \in \pi_1(E)$ be any lift of g. Then $\widehat{\omega}(g)$ is given by $r(\widehat{g}) \times_{\pi_1(F)} \omega^+(\widehat{g}^{-1})$ where $r(\widehat{g}) : \pi_1(E) \to \pi_1(E)$ is right multiplication with $\widehat{g}$.*

3. *The composition of $\widehat{v}$ with $\pi_1(i) : \pi_1(F) \longrightarrow \pi_1(E)$ is the first Stiefel-Whitney class $w(F)$. If $F \longrightarrow E \xrightarrow{p} B$ is a smooth fibre bundle of compact manifolds and w is $w(B)$, then $\widehat{w}\widehat{v}$ is $w(E)$.* ■

We get from $\widehat{\omega}$ a ring homomorphism $U : \mathbf{Z}\pi \longrightarrow [C_*(\widehat{F}), C_*(\widehat{F})]^{op}_{\mathbf{Z}\pi_1(E)}$. The Poincaré duality chain equivalence for $C_*(\widetilde{F})$ induces a $\mathbf{Z}\pi_1(E)$-chain equivalence $\alpha : C^{d-*}(\widehat{F}) \longrightarrow C_*(\widehat{F})$. If we equip $\mathbf{Z}\pi_1(B)$ resp. $\mathbf{Z}\pi_1(E)$ with the w-twisted resp. $\widehat{v}\widehat{w}$-twisted involution, $(C(\widehat{F}), \alpha, U)$ is a symmetric representation of $\mathbf{Z}\pi_1(B)$ in $\mathbf{D}_d(\mathbf{Z}\pi_1(E))$. The transfer $(C(\widehat{F}), \alpha, U)^*$ introduced in 1.1 is the algebraic surgery transfer of the fibration

$$\textbf{1.5} \quad p^* : L_n(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L_{n+d}(\mathbf{Z}\pi_1(E), w(E))$$

2. The up-down Formula for $p_* \circ p^*$

In this section we want to analyse the composition $p_* \circ p^*$ in L-theory. Beforehand we review the computation of $p_* \circ p^*$ for the K-theory transfer $p^* : K_n(\mathbf{Z}\pi_1(B)) \to K_n(\mathbf{Z}\pi_1(E))$ for $n = 0, 1$ of a fibration $F \longrightarrow E \xrightarrow{p} B$ of connected spaces with finitely dominated fibre F in order to motivate the following constructions. Recall that F is finitely dominated, if F is up to homotopy the retract of a finite CW complex. For $\pi = \pi_1(B)$ let $\mathrm{Sw}(\pi, \mathbf{Z})$ be the Grothendieck group of $\mathbf{Z}\pi$-modules which are finitely generated and free as $\mathbf{Z}$-modules. This is the abelian group having isomorphism classes of such modules as generators and any short exact (not necessarily split exact) sequence of such modules $0 \to L \to M \to N \to 0$ gives a relation $[L] - [M] + [N] = 0$. The tensor product over $\mathbf{Z}$ with the diagonal π-action induces a pairing

2.1 $\otimes : \mathrm{Sw}(\pi, \mathbf{Z}) \otimes K_n(\mathbf{Z}\pi) \longrightarrow K_n(\mathbf{Z}\pi)$

This pairing was defined and analysed in Swan [34]. Consider a $\mathbf{Z}\pi$-module M which is finitely generated as $\mathbf{Z}$-module. Let $\{0\} \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow \{0\}$ be a 1-dimensional $\mathbf{Z}\pi$-resolution such that G_0 and G_1 are finitely generated and free as $\mathbf{Z}$-module. Define a class $[M] \in \mathrm{Sw}(\pi, \mathbf{Z})$ depending only on the isomorphism type of M by $[G_0] - [G_1]$. The homotopy action of π on the fibre F given by the fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$ defines a $\mathbf{Z}\pi$-structure on $H_i(F, \mathbf{Z})$. We obtain a class

2.2 $[F] = \sum_{i \geq 0} (-1)^i \cdot [H_i(F, \mathbf{Z})] \in \mathrm{Sw}(\pi, \mathbf{Z})$

Then $p_* \circ p^*$ coincides with $[F] \otimes ?$ (see Lück [15], Munkholm [21], Munkholm-Pedersen [22]). The construction in L-theory is similar, but – as is nearly always the case – harder and the odd-dimensional case is more difficult than the even-dimensional.

We first explain the algebraic setup in even dimensions. Let π be a group and $v, w : \pi \longrightarrow \{\pm 1\}$ be group homomorphisms. For an integer k a nonsingular $(-1)^k$-symmetric form over $\mathbf{Z}$ is a finitely generated free $\mathbf{Z}$-module M together with an isomorphism $\psi : M \longrightarrow \hom_{\mathbf{Z}}(M, \mathbf{Z})$ such that $(-1)^k \cdot \hom_{\mathbf{Z}}(\psi, id) : M \longrightarrow \hom_{\mathbf{Z}}(M, \mathbf{Z})$ agrees with ψ. Here and elsewhere we identify M with $\hom_{\mathbf{Z}}(\hom_{\mathbf{Z}}(M, \mathbf{Z}), \mathbf{Z})$ using the natural isomorphism sending m to $f \mapsto f(m)$. Note that ψ is the same as a nonsingular $(-1)^k$-symmetric bilinear pairing $\overline{\psi} : M \otimes M \longrightarrow \mathbf{Z}$. If M additionally carries a left $\mathbf{Z}\pi$-structure such that $\overline{\psi}(gx, gy) = v(g) \cdot \overline{\psi}(x, y)$ holds for all $x, y \in M$ and $g \in \pi$, we call (M, ψ) a nonsingular (π, v)-equivariant $(-1)^k$-symmetric form over $\mathbf{Z}$. The last condition about $\overline{\psi}$ is equivalent to the assumption that $\psi : M \longrightarrow \hom_{\mathbf{Z}}(M, \mathbf{Z})$ is a $\mathbf{Z}\pi$-map, if π acts on $\hom_{\mathbf{Z}}(M, \mathbf{Z})$ from the left by $(gf)(x) = v(g) \cdot f(g^{-1}x)$ for $f : M \longrightarrow \mathbf{Z}$, $g \in \pi$ and $x \in M$.

Let (M,ψ) be a nonsingular (π,v)-equivariant $(-1)^k$-symmetric form over $\mathbf{Z}$. It is hyperbolic if there is a $\mathbf{Z}\pi$-module $L \subset M$ such that $M \longrightarrow \hom_{\mathbf{Z}}(L,\mathbf{Z})$ sending m to $l \mapsto \psi(m)(l)$ is an epimorphism with L as kernel. Note that L is finitely generated and free as a $\mathbf{Z}$-module. We call two (π,v)-equivariant $(-1)^k$-symmetric forms over $\mathbf{Z}$ stably isomorphic if they become isomorphic after adding hyperbolic ones. Let $L^{2k}(\pi,\mathbf{Z},v)$ be the Witt group of stable isomorphism classes of (π,v)-equivariant $(-1)^k$-symmetric forms over $\mathbf{Z}$. Addition is given by direct sums and the inverse of (M,ψ) is represented by $(M,-\psi)$. The tensor product over $\mathbf{Z}$ together with the diagonal π-action defines a pairing

2.3 $\otimes : L^{2k}(\pi,\mathbf{Z},v) \otimes L_n(\mathbf{Z}\pi,w) \longrightarrow L_{n+2k}(\mathbf{Z}\pi,vw)$

This is essentially the pairing of Fröhlich and McEvett [9], Thomas [35] and Dress [8]. It plays as important a role in induction theory for L-theory as Swan's pairing 2.1 does for K-theory.

Next we deal with the odd-dimensional case. A nonsingular $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$ consists of a finitely generated torsion $\mathbf{Z}$-module M together with an isomorphism $\psi : M \longrightarrow \hom_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ such that $\hom(\psi,id) : M \longrightarrow \hom_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ is $(-1)^{k+1}\cdot\psi$, where we identify M with $\hom_{\mathbf{Z}}(\hom_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z}),\mathbf{Q}/\mathbf{Z})$. One may interpret ψ as a nonsingular $(-1)^{k+1}$-symmetric bilinear pairing $\overline{\psi} : M \otimes M \longrightarrow \mathbf{Q}/\mathbf{Z}$. If M additionally carries a left $\mathbf{Z}\pi$-structure such that $\overline{\psi}(gx,gy) = v(g)\cdot\overline{\psi}(x,y)$ holds for all $x,y \in M$ and $g \in \pi$, we call (M,ψ) a nonsingular (π,v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$. The last condition about $\overline{\psi}$ is equivalent to the assumption that $\psi : M \longrightarrow \hom_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ is a $\mathbf{Z}\pi$-map, if π acts on $\hom_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ from the left by $(gf)(x) = v(g)\cdot f(g^{-1}x)$ for $g \in \pi$ and $x \in M$. A nonsingular (π,v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$ is hyperbolic if there is a $\mathbf{Z}\pi$-submodule $L \subset M$ such that $M \longrightarrow \hom_{\mathbf{Z}}(L,\mathbf{Q}/\mathbf{Z})$ sending m to $l \mapsto \psi(m)(l)$ is an epimorphism with L as kernel. Note that L is a finitely generated torsion $\mathbf{Z}$-module. Let (M,ψ) be a $\mathbf{Q}$-nonsingular (π,v)-equivariant $(-1)^{k+1}$-symmetric form over $\mathbf{Z}$, where $\mathbf{Q}$-nonsingular means that $\psi : M \longrightarrow \hom_{\mathbf{Z}}(M,\mathbf{Z})$ is rationally an isomorphism or, equivalently, has finitely generated torsion $\mathbf{Z}$-modules as kernel and cokernel. Its boundary $(\partial M,\partial\psi)$ is the nonsingular (π,v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$ given by $\partial M = \mathrm{cok}(\psi)$ and

$$\partial\psi : \mathrm{cok}(\psi) \longrightarrow \hom_{\mathbf{Z}}(\mathrm{cok}(\psi),\mathbf{Q}/\mathbf{Z}) \qquad\qquad x \mapsto \left(y \mapsto \frac{x(z)}{s}\right)$$

where $x,y \in \hom_{\mathbf{Z}}(M,\mathbf{Z})$, $z \in M$ and $s \in \mathbf{Z}\backslash\{0\}$ satisfying $sy = \psi(z)$. Two (π,v)-equivariant $(-1)^{k+1}$-symmetric linking forms over $\mathbf{Z}$ are called equivalent if they become isomorphic after adding hyperbolic ones and boundaries. Let $L^{2k+1}(\pi,\mathbf{Z},v)$ be the Witt group of equivalence classes of nonsingular (π,v)-equivariant $(-1)^{k+1}$-symmetric linking forms over

$\mathbf{Z}$. Addition is given by direct sums and the inverse of (M, ψ) is represented by $(M, -\psi)$. Next we define a pairing

2.4 $\otimes : L^{2k+1}(\pi, \mathbf{Z}, v) \otimes L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+2k+1}(\mathbf{Z}\pi, vw)$

Let (M, ψ) be a nonsingular (π, v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$. Choose a finitely generated free $\mathbf{Z}$-resolution $\{0\} \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} M \longrightarrow \{0\}$. We use the following isomorphism as an identification

$$\widehat{\epsilon} : H_0(F^{1-*}, \mathbf{Z}) \longrightarrow \hom_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}) \qquad \overline{g} \mapsto \left(m \mapsto \frac{g(sx)}{s} \right)$$

where $g \in \hom_{\mathbf{Z}}(F_1, \mathbf{Z})$, $x \in F_0$, $s \in \mathbf{Z}$ satisfying $\epsilon(x) = m$ and $sx \in F_1$. As $\hom_{\mathbf{Z}}(F_{1-*}, \mathbf{Z})$ and F are free $\mathbf{Z}$-resolutions, there is a $\mathbf{Z}$-chain equivalence $\alpha : \hom_{\mathbf{Z}}(F_{1-*}, \mathbf{Z}) \longrightarrow F_*$ satisfying $H(\alpha) = \psi$ and $\alpha \simeq (-1)^k \cdot \hom_{\mathbf{Z}}(\phi, id)$. Let $id \otimes \alpha : \mathbf{Z}\pi \otimes_{\mathbf{Z}} \hom_{\mathbf{Z}}(F_{1-*}, \mathbf{Z}) \longrightarrow \mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*$ be the induced $\mathbf{Z}\pi$-chain equivalence. The $\mathbf{Z}\pi$-structure on M yields a homomorphism $V : \pi \longrightarrow [F_*, F_*]^{op}_{\mathbf{Z}}$. This gives a homomorphism $U : \mathbf{Z}\pi \rightarrow [\mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*, \mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*]^{op}_{\mathbf{Z}\pi}$ where $U(g)$ sends $h \otimes x$ to $hg \otimes u(g^{-1})(x)$ for $g \in \pi$,$h \in \pi$ and $x \in F_*$. These data fit together giving a symmetric chain representation $(\mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*, \alpha, U)$ of $\mathbf{Z}\pi$ with the w-twisted involution into $\mathbf{D}_1(\mathbf{Z}\pi)$ with the $(-1)^k \cdot vw$-twisted involution. We have introduced in 1.1 the associated transfer homomorphism from $L_n(\mathbf{Z}\pi, w)$ to $L_{n+1}(\mathbf{Z}\pi, (-1)^k \cdot vw) = L_{n+2k+1}(\mathbf{Z}\pi, vw)$. We shall show in section 11 that this transfer depends only on the class of (M, ψ) in $L^{2k+1}(\pi, \mathbf{Z}, v)$. This finishes the definition of the pairing 2.4.

The groups $L^d(\pi, \mathbf{Z}, v)$ fit into a localization exact sequence of the type studied in Chapter 3 of Ranicki [30]

$$\ldots \rightarrow L^d(\pi, \mathbf{Z}, v) \rightarrow L^d(\pi, \mathbf{Q}, v) \rightarrow L^d(\pi, \mathbf{Z}, \mathbf{Q}, v) \rightarrow L^{d-1}(\pi, \mathbf{Z}, v) \rightarrow \ldots,$$

with $L^d(\pi, \mathbf{Z}, \mathbf{Q}, v)$ the Witt groups of (π, v)-equivariant linking forms\formations on finite abelian groups, $L^{2k}(\pi, \mathbf{Q}, v)$ the Witt group of nonsingular (π, v)-equivariant $(-)^k$-symmetric forms on finite-dimensional vector spaces over $\mathbf{Q}$, and $L^{2k+1}(\pi, \mathbf{Q}, v) = 0$.

We consider a fibration $F \longrightarrow E \xrightarrow{p} B$ and assume that F is an orientable finite Poincaré complex . Then $\widehat{v}$ factorizes over $\pi_1(p)$ by lemma 1.4. The induced homomorphism $v : \pi = \pi_1(B) \longrightarrow \{\pm 1\}$ maps g to the degree of the automorphism $H_d(\omega(g))$ of $H_d(F) \cong \mathbf{Z}$. Suppose that the fibre dimension d is $2k$. The intersection form on F

$$H_k(F, \mathbf{Z})/\mathrm{tors}\,(H_k(F, \mathbf{Z})) \otimes H_k(F, \mathbf{Z})/\mathrm{tors}\,(H_k(F, \mathbf{Z})) \longrightarrow \mathbf{Z} \qquad (x, y) \mapsto \langle x \cup y, [F] \rangle$$

is a nonsingular $(-1)^k$-symmetric form over $\mathbf{Z}$. The fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$ induces a left $\mathbf{Z}\pi$-module structure on it. Thus the intersection form becomes actually a nonsingular (π, v)-equivariant $(-1)^k$-symmetric form over $\mathbf{Z}$ and defines an element $\sigma^*(F,\omega)$ in $L^{2k}(\pi, \mathbf{Z}, v)$. Now suppose that the fibre dimension is $2k+1$. The universal coefficient theorem yields an isomorphism

$$\phi : \operatorname{tors} H_k(F, \mathbf{Z}) \longrightarrow \hom_{\mathbf{Z}}(\operatorname{tors} H^{k+1}(F, \mathbf{Z}), \mathbf{Q}/\mathbf{Z}) \qquad \overline{x} \mapsto \left(f \mapsto \frac{f(y)}{s}\right)$$

where $x \in C_k(F, \mathbf{Z})$, $y \in C_{k+1}(F, \mathbf{Z})$, $s \in \mathbf{Z}\backslash\{0\}$, $f \in C^{k+1}(F, \mathbf{Z})$ and $sx = d(y)$. Poincaré duality gives an isomorphism $[F] \cap - : \operatorname{tors} H^{k+1}(F, \mathbf{Z}) \longrightarrow \operatorname{tors} H_k(F, \mathbf{Z})$. The nonsingular $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$ on F

$$\psi : \operatorname{tors} H_k(F, \mathbf{Z}) \longrightarrow \hom_{\mathbf{Z}}(\operatorname{tors} H_k(F, \mathbf{Z}), \mathbf{Q}/\mathbf{Z})$$

is given by $\hom_{\mathbf{Z}}([F] \cap -, id) \circ \phi$. The fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$ induces a left $\mathbf{Z}\pi$-module structure on $\operatorname{tors} H_k(F, \mathbf{Z})$. Thus the linking form becomes actually a nonsingular (π, v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$ and defines an element $\sigma^*(F,\omega)$ in $L^{2k+1}(\pi, \mathbf{Z}, v)$. All in all we have defined an element

2.5 $\sigma^*(F, \omega) \in L^d(\pi, \mathbf{Z}, v)$

depending on the fibre F and the operation of $\pi_1(B)$ on its homology and a pairing

2.6 $\otimes : L^d(\pi, \mathbf{Z}, v) \otimes L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}(\mathbf{Z}\pi, vw)$

Theorem 2.7 (up-down formula) *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected spaces such that F is an orientable finite d-dimensional Poincaré complex. Let $w : \pi \longrightarrow \{\pm 1\}$ be any homomorphism and $v : \pi \longrightarrow \{\pm 1\}$ be defined as above. Then the composition*

$$p_* \circ p^* : L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}(\mathbf{Z}\pi, vw)$$

is given by $\sigma^*(F, \omega) \otimes ?$ ■

The proof of the up-down formula 2.7 is deferred to sections 9 and 10. We consider the special case of an orientable fibration. Recall that the L-groups are 4-periodic, i.e. $L_n(\mathbf{Z}\pi, w) = L_{n+4}(\mathbf{Z}\pi, w)$. Consider the Witt group $L^d(\{1\}, \mathbf{Z}, 1)$ of the trivial group $\{1\}$. There is a pairing

2.8 $\otimes : L^d(\{1\}, \mathbf{Z}, 1) \otimes L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}(\mathbf{Z}\pi, w)$

which is given by the tensor product over $\mathbf{Z}$ for d even and by the tensor product over $\mathbf{Z}$ of a resolution of the linking form for d odd. The pairings 2.8 and 2.6 are compatible with the canonical homomorphism

$$\text{res} : L^d(\{1\}, \mathbf{Z}, 1) \longrightarrow L^d(\pi, \mathbf{Z}, v)$$

induced by restriction with the trivial map $\pi \longrightarrow \{1\}$. If d is 2 or 3 modulo 4, $L^d(\{1\}, \mathbf{Z}, 1)$ is trivial. A nonsingular symmetric form (M, ψ) over $\mathbf{Z}$ induces a nonsingular symmetric form over $\mathbf{R}$ which can be written as a sum of p copies of $(\mathbf{R}, 1)$ and q copies of $(\mathbf{R}, -1)$. The signature of (M, ψ) is defined to be $p - q$. If d is divisible by 4, the signature induces an isomorphism from $L^d(\{1\}, \mathbf{Z}, 1)$ to $\mathbf{Z}$. The pairing 2.8 is given by multiplication with the signature. The deRham invariant of a nonsingular skew-symmetric linking form (M, ψ) over $\mathbf{Z}$ is the reduction mod 2 of the number of summands in the decomposition of the finite abelian group M as a direct sum of cyclic subgroups of type $\mathbf{Z}/p^s$ for p prime and $s \geq 1$. Suppose that d is 1 modulo 4. Then the deRham invariant defines an isomorphism from $L^d(\{1\}, \mathbf{Z}, 1)$ to $\mathbf{Z}/2$ (see Ranicki [30], section 4.3 for more details, but ignore the first of the two definitions on p.418). Define the homomorphism

2.9 $\nu : L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}(\mathbf{Z}\pi, w)$

by product with the generator in $L^d(\{1\}, \mathbf{Z}, 1) \cong \mathbf{Z}/2$ for $d \equiv 1 (\bmod 4)$.

Corollary 2.10 *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected spaces such that F is an orientable finite Poincaré complex. Assume that π acts trivially on $H_k(F, \mathbf{Z})/\mathit{tors}\, H_k(F, \mathbf{Z})$, if the fibre dimension is $d = 2k$, and trivially on $\mathit{tors}\, H_k(F, \mathbf{Z})$, if $d = 2k + 1$. Then :*

1. *If d is 2 or 3 modulo 4, then $p_* \circ p^*$ vanishes.*
2. *If d is divisible by 4, $p_* \circ p^*$ is multiplication by the signature of F.*
3. *Suppose d is 1 modulo 4. Then $p_* \circ p^*$ is given by the map ν defined above, if the deRham invariant of F is non-trivial, and is zero otherwise. In particular $2 \cdot p_* \circ p^*$ is zero.*

Proof : Under the assumptions above F defines element $\sigma^*(F) \in L^d(\{1\}, \mathbf{Z}, 1)$ whose image under res is just $\sigma^*(F, \omega)$. Now the claim follows from the remarks above and naturality. ∎

Example 2.11 Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected spaces with the d-dimensional sphere S^d as fibre. Suppose $d \geq 2$. Then p is the boundary of a D^{d+1}-fibration with $\pi_1(S^d) = \pi_1(D^{d+1})$, and the surgery transfer is always zero. This is not true for the K-theory transfer on K_0 and K_1 (see Anderson [2]). On the other hand, if the fibration p is a product bundle with S^1 as fibre, the K-theory transfer on both K_0 and K_1 is trivial because of the product formulas for the finiteness obstruction and Whitehead torsion, whereas the surgery transfer is injective modulo 2-torsion by the splitting theorems and Rothenberg sequences of Shaneson [33] and Ranicki [28]. See the appendix to Munkholm and Ranicki [24] for the connection between the S^1-bundle transfer in L-theory and the duality in K-theory.

■

Remark 2.12 For finite π the pring 2.4 is already defined in Yoshida [39]. Yoshida needs the finiteness of π as he has to deal with $\mathbf{Z}\pi$-resolutions instead of $\mathbf{Z}$-resolutions with a homotopy π-action. Let N be a closed manifold with π-action. Consider the fibre bundle $p : B \times_\pi N \longrightarrow B$ with fibre N if $\pi = \pi_1(B)$. The application of the up-down formula to p gives the main result in Yoshida [39] without the assumption that π is finite.

■

3. Symmetric Signature

If one is willing to invert 2, one can compute surgery obstructions as the difference of symmetric signatures. Let B be a finite n-dimensional Poincaré complex. In Ranicki [30], section 1.2 the symmetric L-group $L^n(\mathbf{Z}\pi_1(B), w(B))$ and the symmetric signature of B

3.1 $\sigma^*(B) \in L^n(\mathbf{Z}\pi_1(B), w(B))$

are defined. The symmetrization map $(1+T) : L_n(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L^n(\mathbf{Z}\pi_1(B), w(B))$ is an isomorphism modulo 8-torsion and sends the surgery obstruction of a normal map $f : M \longrightarrow B$ with B as target to the difference $\sigma^*(M) - \sigma^*(B)$ of the symmetric signatures, if B and M are closed. Product with the generator $E_8 \in L_0(\mathbf{Z}) = \mathbf{Z}$ defines a map $E_8 : L^n(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L_n(\mathbf{Z}\pi_1(B), w(B))$ such that $E_8 \circ (1+T) = 8 \cdot \mathrm{id} : L^n(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L^n(\mathbf{Z}\pi_1(B), w(B))$. Note that our construction of a transfer map on the quadratic L-groups need not extend to the symmetric L-groups (see Lück-Ranicki [18], appendix 2). In even dimensions there is also a pairing on the symmetric L-groups given by the tensor product over $\mathbf{Z}$ and the diagonal operation which is compatible with the symmetrization map :

3.2 $\otimes : L^{2k}(\pi, \mathbf{Z}, v) \otimes L^n(\mathbf{Z}\pi, w) \longrightarrow L^{n+2k}(\mathbf{Z}\pi, vw)$

The details of this pairing are just as in the quadratic case, and are therefore omitted.

Theorem 3.3 *Let $F \longrightarrow E \xrightarrow{p} B$ be a smooth fibre bundle of connected closed manifolds. If the fibre dimension d is odd we have:*

$$128 \cdot p_*\sigma^*(E) = 0 \in L^{n+d}(\mathbf{Z}\pi, vw) \ .$$

If d is even we have:

$$8 \cdot p_*\sigma^*(E) = 8 \cdot \sigma^*(F, \omega) \otimes \sigma^*(B) \in L^{n+d}(\mathbf{Z}\pi, vw)$$

Proof : Suppose that d is even. Let $f : M \longrightarrow N$ be a surgery problem of simply-connected $4l$-dimensional manifolds whose surgery obstruction is the generator in $L_{4l}(\mathbf{Z}) \cong \mathbf{Z}$. Consider the fibre bundle $p \times id_N : E \times N \longrightarrow B \times N$. The following diagram commutes by the up-down formula 2.7, where $\pi = \pi_1(B)$, $\Gamma = \pi_1(E)$ and $w = w(B)$.

$$\begin{array}{ccccc}
L_{n+4l}(\mathbf{Z}\pi_1(B \times N), w(B \times N)) & \longrightarrow & L_n(\mathbf{Z}\pi, w) & \longrightarrow & L^n(\mathbf{Z}\pi, w) \\
\downarrow (p \times id_N)^* & & \downarrow p^* & & \\
L_{n+4l+d}(\mathbf{Z}\pi_1(E \times N), w(E \times N)) & \longrightarrow & L_{m+d}(\mathbf{Z}\Gamma, \widehat{w}\widehat{v}) & & \downarrow \sigma^*(F, \omega) \otimes ? \\
\downarrow (p \times id_N)_* & & \downarrow p_* & & \\
L_{n+4l+d}(\mathbf{Z}\pi_1(B \times N), w(B \times N)) & \longrightarrow & L_{n+d}(\mathbf{Z}\pi, w) & \longrightarrow & L^{n+d}(\mathbf{Z}\pi, vw)
\end{array}$$

The left horizontal arrows are given by induction with the projection to the first factor and the two right horizontal maps are symmetrization homomorphisms. The surgery obstruction of the normal map $id_B \times f : B \times M \longrightarrow B \times N$ lies in the left upper corner. Its image under the clockwise composition in the right lower corner is $8 \cdot \sigma^*(F, \omega) \otimes \sigma^*(B)$ because the product formula for surgery obstructions (see Ranicki [30], section 1.9) and the fact that the symmetrization map from $L_4(\mathbf{Z}, 1)$ to $L^4(\mathbf{Z}, 1)$ sends the generator to the symmetric form of signature 8. For the same reason its image under the anti-clockwise composition is $8 \cdot p_*\sigma^*(E)$.

If d is odd, $p_* \circ p^* : L_n(\mathbf{Q}\pi, w) \longrightarrow L_{n+d}(\mathbf{Q}\pi, vw)$ is zero. This follows from theorem 2.7, since a nonsingular linking form vanishes rationally or from the proof of theorem 4.3. Now a similar argument as above shows that the image of $8 \cdot \sigma^*(E)$ under the change of coefficients homomorphism $L^{n+d}(\mathbf{Z}\pi, vw) \longrightarrow L^{n+d}(\mathbf{Q}\pi, vw)$ is zero. The kernel of this map is 16-torsion. ■

The signature of a nonsingular symmetric form over $\mathbf{Z}$ induces an isomorphism from $L^{4k}(\mathbf{Z},1)$ to $\mathbf{Z}$. Its composition with the induction homomorphism associated to the trivial map $pr : \pi_1(B) \longrightarrow \{1\}$ sends $\sigma^*(B)$ to the signature $\operatorname{sign}(B)$. Note for the trivial group that symmetric L-group $L^d(\mathbf{Z},1)$ can be identified with Witt group $L^d(\{1\},\mathbf{Z},1)$ and the corresponding pairings 2.6 and 3.2 agree. We get from an argument similar to the one used in the proof of theorem 3.3 and corollary 2.10

Corollary 3.4 *Let $F \longrightarrow E \xrightarrow{p} B$ be an smooth fibre bundle of connected closed manifolds. Suppose that p is orientable, i.e. $\omega : \pi_1(B) \longrightarrow [F,F]$ is trivial. Then :*

1. *If the dimension $d \equiv 1 (\operatorname{mod} 4)$, then :*

$$16 \cdot p_*\sigma^*(E) = 0 \in L^{n+d}(\mathbf{Z}\pi, vw) \ .$$

   If the dimension $d \equiv 2,3 (\operatorname{mod} 4)$, then :

$$8 \cdot p_*\sigma^*(E) = 0 \in L^{n+d}(\mathbf{Z}\pi, vw) \ .$$

2. *If the dimension of d is divisible by 4, then :*

$$8 \cdot p_*\sigma^*(E) = 8 \cdot sign(F) \cdot \sigma^*(B) \in L^{n+d}(\mathbf{Z}\pi, vw)$$

3. *Suppose that F, B and E are orientable. If their dimensions are all divisible by 4, then :*

$$sign(E) = sign(F) \cdot sign(B) \in \mathbf{Z} \ .$$

   Otherwise $sign(E) = 0$ ■.

The last statement in corollary 3.4 is a result due to Chern, Hirzebruch and Serre [6]. The examples of Kodaira [13], Atiyah [3], Hirzebruch [12] of fibre bundles in which the signature is not multiplicative show that some condition such as orientability is necessary for the signature to be multiplicative.

Let d be $2k$ and π be $\pi_1(B)$ and assume that $n+d$ is divisible by 4. The structure $\psi : H_k(F,\mathbf{Q}) \longrightarrow \hom_{\mathbf{Q}}(H_k(F,\mathbf{Q}),\mathbf{Q})$ of a nonsingular $(\pi,1)$-equivariant $(-1)^k$-symmetric form over $\mathbf{Q}$ on $H_k(F,\mathbf{Q})$ is induced by the fibre transport, Poincaré duality and the universal coefficient theorem. Let $\sigma^*(B)\cap ? : \hom_{\mathbf{Q}\pi}(C_{n-*}(\widetilde{B},\mathbf{Q}),\mathbf{Q}\pi) \longrightarrow C_*(\widetilde{B})$ be the Poincaré $\mathbf{Q}\pi$-chain equivalence. The following composition is a $\mathbf{Q}\pi$-chain equivalence

$$\hom_{\mathbf{Q}}(C_{n-*}(\widetilde{B},\mathbf{Q}) \otimes_{\mathbf{Q}\pi} H_k(F,\mathbf{Q}),\mathbf{Q}) \xrightarrow{\hom(id\otimes\psi,id)}$$

$$\hom_{\mathbf{Q}}(C_{n-*}(\widetilde{B},\mathbf{Q}) \otimes_{\mathbf{Q}\pi} \hom_{\mathbf{Q}}(H_k(F,\mathbf{Q}),\mathbf{Q}),\mathbf{Q})$$
$$\xrightarrow{\rho} \hom_{\mathbf{Q}\pi}(C_{n-*}(\widetilde{B},\mathbf{Q}),\mathbf{Q}\pi) \otimes_{\mathbf{Q}\pi} H_k(F,\mathbf{Q}) \xrightarrow{\sigma^*(B)\otimes id \cap ?} C_*(\widetilde{B},\mathbf{Q}) \otimes_{\mathbf{Q}\pi} H_k(F,\mathbf{Q})$$

where ρ is the canonical isomorphism. The composition of the inverse of the isomorphism on $H_{n/2}$ induced by the chain map above with the canonical isomorphism from $\hom_{\mathbf{Q}}(H_{n/2}(C_*(\widetilde{B}) \otimes_{\mathbf{Q}\pi} H_k(F,\mathbf{Q}),\mathbf{Q})$ to $\hom_{\mathbf{Q}}(H_{n/2}(B,H_k(F,\mathbf{Q})),\mathbf{Q})$ determines the structure $\phi : H_{n/2}(B,H(F,\mathbf{Q})) \longrightarrow \hom_{\mathbf{Q}}(H_{n/2}(B,H_k(F,\mathbf{Q})),\mathbf{Q})$ of a nonsingular symmetric form over $\mathbf{Q}$. We derive from theorem 3.3:

Corollary 3.5 *Let $F \longrightarrow E \xrightarrow{p} B$ be a smooth fibration of connected closed manifolds such that $d+n$ is divisible by 4. Then the signature $sign(E)$ is zero if d is odd. Suppose that d is $2k$. Then we have :*

$$sign(E) = sign \circ pr_*(\sigma^*(F,\omega) \otimes \sigma^*(B)) = sign(H_{n/2}(B,H_k(F,\mathbf{Q})),\phi)$$

where $pr : \pi \longrightarrow \{1\}$ is the trivial map. ■

Corollary 3.5 was first proved by Meyer [20], Satz I 2.2.

See the appendix for a discussion of the relation between our results and the characteristic class formulae relating the non-multiplicativity of the signature in a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ to the action of $\pi_1(B)$ on $H^*(F;\mathbf{R})$.

4. Rational Computations

If one allows rational coefficients instead of integral ones, computations simplify. The algebraic transfer of a fibration is also defined for rational coefficients and is compatible with the change of coefficients homomorphisms. The change of coefficients homomorphism $L_n(\mathbf{Z}\pi,w) \longrightarrow L_n(\mathbf{Q}\pi,w)$ is an isomorphism if one inverts 2 (see Ranicki [30], page 376). Also the difference between symmetric and quadratic L-theory and the difference between any type of decorated L-groups like L^h, L^s and L^p vanishes, when inverting 2. Hence we obtain a vanishing result for the integral surgery transfer (for any decoration), when 2 is inverted, if we prove the vanishing of the rational algebraic surgery transfer $p^* : L^p_n(\mathbf{Q}\pi_1(B),w) \longrightarrow L^p_{n+d}(\mathbf{Q}\pi_1(E),\widehat{v}\widehat{w})$ for the projective L-groups. Recall that L^p means that we allow finitely generated projective modules instead of finitely generated free ones.

Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected spaces such that F is a finitely dominated Poincaré complex. Finitely dominated means that F is up to homotopy a retract of a finite CW complex. We abbreviate $\pi = \pi_1(B)$, $\Gamma = \pi_1(E)$, and write the kernel of $\pi_1(p) : \Gamma \longrightarrow \pi$ by Δ. Let $w : \pi \longrightarrow \{\pm 1\}$ be any homomorphism. Associated with these data is the rational surgery transfer $p^* : L_n^p(\mathbf{Q}\pi, w) \longrightarrow L_{n+d}^p(\mathbf{Q}\Gamma, \widehat{v}\widehat{w})$. Now suppose that Δ is finite. Then restriction with the epimorphism $\pi_1(p)$ defines a homomorphism

4.1 $\mathrm{res} : L_n^p(\mathbf{Q}\pi, w) \longrightarrow L_n^p(\mathbf{Q}\Gamma, \widehat{w})$

where $\widehat{w}$ was defined to be $w \circ \pi_1(p)$ in 1.3. Let $L^{2k}(\Gamma, \mathbf{Q}, v)$ be the Witt group of stable isomorphism classes of (Γ, v)-equivariant $(-1)^k$-symmetric forms over $\mathbf{Q}$, where $\widehat{v}$ was introduced in 1.3. The tensor product over $\mathbf{Q}$ with the diagonal γ-action gives a pairing

4.2 $\otimes : L^{2k}(\Gamma, \mathbf{Q}, v) \otimes L_n(\mathbf{Q}\Gamma, w) \longrightarrow L_{n+2k}(\mathbf{Q}\Gamma, vw)$

Let $\overline{F}$ be the covering of F associated to $\pi_1(F) \longrightarrow \Delta$. The pointed fibre transport and the intersection pairing induce the structure of a nonsingular $(\Gamma, \widehat{v})$-equivariant $(-1)^k$-symmetric form over $\mathbf{Q}$ on $H_k(F, \mathbf{Q})$, We get a class $\sigma^*(\overline{F}, \omega^+)$ in $L^{2k}(\Gamma, \mathbf{Q}, v)$.

Theorem 4.3 *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected spaces such that F is a finitely dominated Poincaré complex and the kernel Δ of $\pi_1(p)$ is finite. Then*

$$p^* : L_n^p(\mathbf{Q}\pi, w) \longrightarrow L_{n+d}^p(\mathbf{Q}\Gamma, \widehat{v}\widehat{w})$$

is zero if the fibre dimension d is odd, and is the composition $\left(\sigma^*(\overline{F}, \omega^+) \otimes ?\right) \circ res$, if d is $2k$.

Proof : The rational surgery transfer is given by a symmetric representation of $\mathbf{Q}\pi$ in $\mathbf{D}_d(\mathbf{Q}\Gamma)$ (see 1.5 and lemma 1.4)

$$U : \mathbf{Q}\pi \longrightarrow [\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} C(\overline{F}, \mathbf{Q}), \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} C(\overline{F}, \mathbf{Q})]_{\mathbf{Q}\Gamma}^{op}$$

Regard the rational homology $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$ as a d-dimensional $\mathbf{Q}\Gamma$-chain complex using the trivial differential. There is a $\mathbf{Q}\Delta$-chain equivalence $i : \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q}) \longrightarrow C(\overline{F}, \mathbf{Q})$ uniquely determined up to homotopy by the property that it induces the identity on homology. The $\mathbf{Q}\Delta$-chain isomorphism $\alpha : \hom_{\mathbf{Q}}(H_{d-*}(\overline{F}, \mathbf{Q}), \mathbf{Q}) \longrightarrow \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$ is given by Poincaré duality and the natural isomorphism $\hom_{\mathbf{Q}}(H_{d-*}(\overline{F}, \mathbf{Q}), \mathbf{Q}) \longrightarrow H^{d-*}(\overline{F}, \mathbf{Q})$. The pointed fibre transport induces a $\mathbf{Q}\Gamma$-structure on $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$. We put the diagonal $\mathbf{Q}\Gamma$-structure on $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$. Because α extends to a $\mathbf{Q}\Gamma$-chain isomorphism β

from $\hom_{\mathbf{Q}\Gamma}(\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q}), \mathbf{Q}\Gamma)$ to $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$, we obtain a symmetric representation of $\mathbf{Q}\pi$ in $\mathbf{D}_d(\mathbf{Q}\Gamma)$

$$V : \mathbf{Q}\pi \longrightarrow [\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q}), \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})]^{op}_{\mathbf{Q}\Gamma}$$

where $V(g)$ sends $h \otimes x$ to $hg \otimes x$. Define a $\mathbf{Q}\Gamma$-chain map from $\mathbf{Q}\pi \otimes_{\mathbf{Q}} H_*(\overline{F}, \mathbf{Q})$ to $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q})$ by mapping $g \otimes x$ to $\widetilde{g} \otimes \widetilde{g}^{-1}x$ for any lift $\widetilde{g} \in \Gamma$ of $g \in \pi$. Composing this with $\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} i$ defines a $\mathbf{Q}\Gamma$-chain equivalence

$$j : \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q}) \longrightarrow \mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} C(\overline{F}, \mathbf{Q})$$

One easily checks that j is an isomorphism of symmetric representations of $\mathbf{Q}\pi$ in $\mathbf{D}_d(\mathbf{Q}\Gamma)$. Hence the surgery transfer associated with $(\mathbf{Q}\Gamma \otimes_{\mathbf{Q}\Delta} H(\overline{F}, \mathbf{Q}), \beta, V)$ is the rational surgery transfer p^*. We get from the definitions that p^* is the composition

$$L_n(\mathbf{Q}\pi, w) \xrightarrow{\text{res}} L_n(\mathbf{Q}\Gamma, \widehat{w}) \xrightarrow{\otimes[H(\overline{F}, \mathbf{Q}), \alpha]} L_{n+d}(\mathbf{Q}\Gamma, \widehat{v}\widehat{w})$$

where the last map is given by the tensor product over $\mathbf{Q}$ and the diagonal Γ-action. It remains to show that the second map is zero, if d is odd, and $\sigma^*(\overline{F}, \omega^+)\otimes ?$ if d is even.

Let D be the $(k-1)$- resp. k-dimensional $\mathbf{Q}\Gamma$-chain complex obtained by truncating $H(\overline{F}, \mathbf{Q})$ if d is $2k$ resp. $2k+1$. Let $p : H(\overline{F}, \mathbf{Q}) \longrightarrow D$ be the canonical projection. Choose for $\lambda \in L_n(\mathbf{Q}\Gamma, \widehat{w})$ a n-dimensional quadratic Poincaré $\mathbf{Q}\Gamma$-chain complex $(C, \{\psi_s\})$ representing λ. The image of λ under $[H(\overline{F}, \mathbf{Q}), \alpha]\otimes ?$ is represented by a $(n+d)$-dimensional quadratic Poincaré $\mathbf{Q}\Gamma$-chain complex whose underlying $\mathbf{Q}\Gamma$-chain complex is $C \otimes_{\mathbf{Q}} H(\overline{F}, \mathbf{Q})$. Now we can do algebraic surgery on $id \otimes p : C \otimes_{\mathbf{Q}} H(\overline{F}, \mathbf{Q}) \longrightarrow C \otimes D$. The result is contractible if d is odd, and homotopy equivalent to the obvious representative of $\sigma^*(\overline{F}, \omega^+) \otimes \lambda$ if d is even. ■

5. Untwisted Fibrations

A fibration $F \longrightarrow E \xrightarrow{p} B$ is called untwisted, if the pointed fibre transport $\omega^+ : \pi_1(E) \longrightarrow [F, F,]^+$ is trivial, and orientable if the fibre transport $\omega : \pi_1(B) \longrightarrow [F, F]$ is trivial. Let $G_1(F)$ be the kernel of $\tau : \pi_1(F) \longrightarrow [F, F]^+$ as defined by Gottlieb [10]. A fibration is untwisted if and only if it is orientable and $G_1(F) = \pi_1(F)$ (see lemma 1.4). Because $G_1(F) = \pi_1(F)$ holds for H-spaces, an orientable fibration with H-space as fibre is untwisted. Untwisted fibrations seem to be the largest class of fibrations where one can hope to prove the vanishing of the surgery transfer when one has only little information about the fibre.

Let $F \longrightarrow E \xrightarrow{p} B$ be an untwisted fibration of connected spaces such that the fibre F is a d-dimensional finite Poincaré complex. We abbreviate $\pi = \pi_1(B)$ and $\Gamma = \pi_1(E)$ and write Δ for the kernel of $\pi_1(p)$. Consider a homomorphism $w : \pi = \pi_1(B) \longrightarrow \{\pm\}$. Put $\widehat{w}$ to be $w \circ \pi_1(p)$. Since p is untwisted, Δ is central. The tensor product over $\mathbf{Z}\Delta$ induces a pairing

5.1 $\otimes_{\mathbf{Z}\Delta} : L^d(\mathbf{Z}\Delta, 1) \otimes L_n(\mathbf{Z}\Gamma, \widehat{w}) \longrightarrow L_{d+n}(\mathbf{Z}\Gamma, \widehat{w})$

If Δ is trivial, the pairing 5.1 above reduces to the pairing 2.8. Let $\sigma^*(\overline{F}) \in L^d(\mathbf{Z}\Delta, 1)$ be the symmetric signature of the covering $\overline{F}$ of F associated with $\pi_1(F) \longrightarrow \Delta$. The following theorem is proven for the K-theory transfer in Lück [15] page 165. The L-theoretic version is proven similarly.

Theorem 5.2 (down-up formula for untwisted fibrations) *If the conditions above are satisfied, the composition*

$$p^* \circ p_* : L_n(\mathbf{Z}\Gamma, \widehat{w}) \longrightarrow L_{n+d}(\mathbf{Z}\Gamma, \widehat{w})$$

is given by $\sigma^*(\overline{F}) \otimes_{\mathbf{Z}\Delta} ?$. ■

Let F be a connected CW complex such that $\pi_1(F) = \mathbf{Z} \times G$ and $\mathbf{Z} \subset G_1(F)$. Let $q : F' \longrightarrow F$ be the infinite cyclic covering associated with the projection $\pi_1(F) \longrightarrow G$. Choose a representative $i : S^1 \longrightarrow F$ of the generator of $\mathbf{Z}$. Then $q \vee i$ extends to a homotopy equivalence $f : F' \times S^1 \longrightarrow F$ by Lück [15] page 154. We identify F and $F' \times S^1$. If F is a d-dimensional finitely dominated Poincaré complex, then F' is a $(d-1)$-dimensional finitely dominated Poincaré complex.

Let $F \longrightarrow E \xrightarrow{p} B$ be an untwisted fibration of connected spaces such that F is a d-dimensional finitely dominated Poincaré complex. We have the central extension with a free abelian kernel $\Delta' \longrightarrow \Gamma' \longrightarrow \pi$ if Δ' is $\Delta/\mathrm{tors}\,(\Delta)$ and Γ' is $\Gamma/\mathrm{tors}\,(\Delta)$. Let r be the rank of Δ'. Associated to any such extension is a surgery transfer

5.3 $\mathrm{trf} : L_n^\epsilon(R\pi, w) \longrightarrow L_{n+r}^\epsilon(R\Gamma', \widehat{w})$

where ϵ is one of the decorations p, h or s and $\mathbf{Z} \subset R \subset \mathbf{Q}$. It is the surgery transfer associated to the following symmetric representatio of $R\pi$ in $\mathbf{D}_r(R\Gamma')$. Choose an identification $\Delta' = \mathbf{Z}^r$. Let C be the symmetric Poincaré $\mathbf{Z}\Delta'$-chain complex of the universal covering of the r-dimensional torus T^r. The symmetric representation is given by

5.4 $id \otimes_{\mathbf{Z}\Delta'} \alpha : R\Gamma' \otimes_{\mathbf{Z}\Delta'} C^{d-*} \longrightarrow R\Gamma' \otimes_{\mathbf{Z}\Delta'} C$,
$U : R\pi \longrightarrow [R\Gamma \otimes_{\mathbf{Z}\Delta'} C, R\Gamma \otimes_{\mathbf{Z}\Delta'} C]^{op}_{\mathbf{Z}\Gamma'}$

where we have identified $\hom_{R\Gamma'}(R\Gamma' \otimes_{\mathbf{Z}\Delta'} C, R\Gamma')$ with $R\Gamma' \otimes_{\mathbf{Z}\Delta'} C^{d-*}$ and U sends $g \in \pi$ to $h \otimes x \mapsto h\widetilde{g} \otimes x$ for any lift $\widetilde{g} \in \Gamma'$ of g. Let trf be the associated surgery transfer 1.1. Assume that $|\mathrm{tors}\,(\Delta)|$ is invertible in R. Then restriction with the epimorphism $\Gamma \longrightarrow \Gamma'$ induces a homomorphism $\mathrm{res} : L_{n+r}(R\Gamma') \longrightarrow L_{n+r}(R\Gamma)$. Let $\mathrm{sign}(\overline{F}) \in \mathbf{Z}$ be the signature of the covering $\overline{F}$ of F associated with $\pi_1(F) \longrightarrow \Delta$. This is well-defined since $\widehat{F}$ is a finitely dominated Poincaré complex. Let $\nu : L_n(R\pi, w) \longrightarrow L_{n+1}(R\pi, w)$ be the homomorphism defined in 2.9, but now with coefficients in R.

Theorem 5.5 *We get for the surgery transfer*

$$p^* : L^p_n(R\pi, w) \longrightarrow L^p_{n+d}(R\Gamma, \widehat{v}\widehat{w})$$

with coefficients in R under the conditions and in the notation above :

1. *p^* is zero if one of the following conditions is satisfied:*
   - *(a) The kernel of $\pi_1(F) \longrightarrow \pi_1(E)$ is infinite*
   - *(b) $d - r \equiv 2, 3 \pmod 4$*
   - *(c) $d - r \equiv 1 \pmod 4$ and $1/2 \in R$*
   - *(d) $d - r \equiv 1 \pmod 4$ and the deRham invariant of $\overline{F}$ is zero.*
2. *If $\pi_1(F) \longrightarrow \pi_1(E)$ has finite kernel and $d - r \equiv 0 \pmod 4$ then p^* is $\mathit{sign}(\overline{F}) \cdot \mathit{res} \circ \mathit{trf}$.*
3. *Suppose that $\pi_1(F) \longrightarrow \pi_1(E)$ has finite kernel, $1/2 \notin R$ and $d - r \equiv 1 \pmod 4$ and the deRham invariant of $\overline{F}$ is not zero. Then p^* is $\nu \circ \mathit{res} \circ \mathit{trf}$.*
4. *If $\mathrm{tors}\,\Delta$ is trivial the items above hold also for $p^* : L^\epsilon_n(\mathbf{Z}\pi, w) \longrightarrow L^\epsilon_{n+d}(\mathbf{Z}\Gamma, \widehat{v}\widehat{w})$ if ϵ is h,p or s and res is the identity.*

Proof : Consider the symmetric representation of π in $\mathbf{D}_d(R\Gamma)$ which is obtained by viewing the symmetric representation of π in $\mathbf{D}_r(\mathbf{Z}\Gamma')$ in 5.4 as a symmetric representation from π into $\mathbf{D}_r(\mathbf{Z}\Gamma)$ by restriction with $\Gamma \longrightarrow \Gamma'$ and tensoring it with $C(\overline{F}, R)$. Because $\left(R\Gamma' \otimes_{R\Delta'} C(\widetilde{T^r}, R)\right) \otimes_R C(\overline{F}, R)$ and $R\Gamma \otimes_{R\pi_1(\widetilde{F})} C(\widetilde{F}, R)$ are $R\Gamma$-chain equivalent, one obtains an isomorphism from this symmetric representation to the one describing p^*. This implies that the algebraic transfer $p^* : L^p_n(R\pi, w) \longrightarrow L^p_{n+d}(R\Gamma, \widehat{v}\widehat{w})$ is the composition

$$L^p_n(R\pi, w) \xrightarrow{\mathrm{trf}} L^p_{n+r}(R\Gamma', \widehat{w}) \xrightarrow{\mathrm{res}} L^p_{n+r}(R\Gamma, \widehat{w}) \xrightarrow{\sigma^*(\overline{F}) \otimes_R ?} L^{n+d}_p(R\Gamma, \widehat{w})$$

where $\sigma^*(\overline{F}) \in L^{d-r}(R,1)$ is the symmetric signature of $\overline{F}$ and $\otimes_R$ is the analogue of the pairing 2.8.

If $d-r \equiv 2,3 \pmod 4$, then $L^{d-r}(\mathbf{Z},1)$ is trivial. The class $\sigma^*(\overline{F}) \in L^{d-r}(\mathbf{Z},1)$ is detected by the signature if $d-r \equiv 0 \pmod 4$, and by the deRham invariant if $d-r \equiv 1 \pmod 4$. Suppose that the kernel of $\pi_1(F) \longrightarrow \pi_1(E)$ is infinite. As $G_1(\overline{F}) = \pi_1(\overline{F})$ and the rank of $\pi_1(\overline{F})$ and the kernel above agree, $\overline{F}$ is homotopy equivalent to $S^1 \times Y$ for some Y. As $S^1 \times Y$ bounds $D^2 \times Y$, the class $\sigma^*(\overline{F}) \in L^{d-r}(\mathbf{Z},1)$ is trivial. If $d-r \equiv 1 \pmod 4$ and $1/2 \in R$, $\sigma^*(\overline{F}) \in L^{d-r}(R,1)$ is trivial. This shows item 1.), 2.) and 3.). One obtains 4.) similarly, using the fact that $\overline{F}$ is a finite simple Poincaré complex, as finiteness obstructions and Whitehead torsions are always trivial over the principal domain $\mathbf{Z}$. ■

Recall from example 2.11 that the surgery transfer of the product bundle with a torus as fibre is injective modulo 2-torsion so that the vanishing results above are the best we can expect.

6. G-Principal Bundles

Corollary 6.1 *Let G be a compact connected Lie group and $G \longrightarrow E \xrightarrow{p} B$ be a G-principal bundle.*

1. *If G is not a torus, then*

$$p^* : L_n(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}(\mathbf{Z}\Gamma, \widehat{v}\widehat{w})$$

is trivial.

2. *Suppose that G is not a torus or that $\pi_1(G) \longrightarrow \pi_1(E)$ is not injective. Then*

$$p^* : L^p_n(\mathbf{Q}\pi_1(B), w) \longrightarrow L^p_{n+d}(\mathbf{Q}\pi_1(E), \widehat{v}\widehat{w})$$

is trivial.

Proof : 1.) Since G is non-abelian, G contains a non-abelian 3-dimensional Lie subgroup H. Let S^1 be the maximal torus in H. We can write the given G-principal bundle as the composition

$$p : E \xrightarrow{p_1} E/S^1 \xrightarrow{p_2} E/H \xrightarrow{p_3} B$$

It is obvious from the definition of the geometric transfer that p^* is the composition $p_1^* \circ p_2^* \circ p_3^*$. Hence it suffices to show that p_2^* is zero. As the fibre of p_2 is S^2, the claim follows from the up-down formula 2.7, or else by 2.11.

2.) follows from theorem 5.5 ■

Recall from example 2.11 that the transfer of the trivial bundles with fibre S^1 is non-trivial so that the vanishing results above are optimal.

7. Change of K-theory

So far we have mainly dealt with L^h in the integral and L^p in the rational case. We make some comments what happens for arbitrary decorations. Consider a fibration $F \longrightarrow E \xrightarrow{p} B$ of connected spaces such that F is a finitely dominated d-dimensional Poincaré complex. Let $w : \pi = \pi_1(B) \longrightarrow \{\pm 1\}$ be a homomorphism. Let X resp. Y be a subgroup of $K_i(\mathbf{Z}\pi_1(B))$ resp. $K_i(\mathbf{Z}\pi_1(E))$ containing the image of $K_i(\mathbf{Z})$ and closed under the w-twisted resp. $\widehat{v}\widehat{w}$-twisted involution (given by taking dual modules and maps) for $i = 0, 1$. Then we have the decorated L-group $L_n^X(\mathbf{Z}\pi_1(B), w)$ resp. $L_n^Y(\mathbf{Z}\pi_1(E), \widehat{v}\widehat{w})$. In order to get a symmetric chain representation from the fibration and hence a well-defined algebraic transfer, we have to specify a base point $b \in B$, a CW complex G and a homotopy equivalence $f : G \longrightarrow F_b$. If i is 0, we assume that the image of the finiteness obstruction $o(G)$ of G under $(j \circ f)_* : K_0(\mathbf{Z}\pi_1(G)) \longrightarrow K_0(\mathbf{Z}\pi_1(E))$ for the inclusion $j : F_b \longrightarrow E$ lies in Y. If i is 1, we assume that G is a finite Poincaré complex and the image of the Whitehead torsion $\tau([G] \cap -)$ of the Poincaré chain equivalence $[G] \cap - : C(\widetilde{G})^{d-*} \longrightarrow C(\widetilde{G})$ under $(j \circ f)_* : \mathrm{Wh}(\pi_1(G)) \longrightarrow \mathrm{Wh}(\pi_1(E))$ lies in Y. Moreover, we assume that the K-theory transfer $p^* : K_i(\mathbf{Z}\pi_1(B)) \longrightarrow K_i(\mathbf{Z}\pi_1(E))$ maps X to Y. Under these assumptions there is is an algebraic surgery transfer

$$p^* : L_n^X(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L_{n+d}^Y(\mathbf{Z}\pi_1(E), \widehat{v}\widehat{w}) \ .$$

It is easy to check in the case $i = 0$ that $(j \circ f)_*(o(G))$ and the surgery transfer p^* do not depend on the choice of b, G and f. This is also true for L^h. In the simple case this choice can matter. If p is a smooth bundle of compact manifolds and $f : G \longrightarrow F_b$ is given by a triangulation, then these choices do not matter and $o(G)$ and $\tau([G] \cap -)$ are trivial.

Let Y' be the image of Y under $p_* : K_i(\mathbf{Z}\pi_1(E)) \longrightarrow K_i(\mathbf{Z}\pi)$ for $\pi = \pi_1(B)$. Then the composition

$$p_* \circ p^* : L_n^X(\mathbf{Z}\pi, w) \longrightarrow L_{n+d}^{Y'}(\mathbf{Z}\pi, vw)$$

is defined. Let Z be an abelian subgroup of $\mathrm{Sw}(\pi, \mathbf{Z})$ closed under the v-twisted involution. Suppose that the pairing 2.1 sends $X \otimes Z$ to Y'. Let $L_Z^d(\pi, \mathbf{Z}, v)$ be the decorated analogue of $L^d(\pi, \mathbf{Z}, v)$, in which it is demanded additionally that the classes of all the $\mathbf{Z}\pi$-modules lie in $Z \subseteq \mathrm{Sw}(\pi, \mathbf{Z})$. We make the following assumption :

7.1

$\sum_{i=0}^{k-1} (-1)^i \cdot [H_i(F)]$ and $\sum_{i=0}^{d} (-1)^i \cdot [H_i(F)]$ lie in $Z \subset \mathrm{Sw}(\pi, \mathbf{Z})$ if $d = 2k$

$[H_k(F)/\mathrm{tors}\, H_k(F)] + \sum_{i=0}^{k-1} (-1)^i \cdot [H_i(F)]$ and $\sum_{i=0}^{d} (-1)^i \cdot [H_i(F)]$
lie in $Z \subset \mathrm{Sw}(\pi, \mathbf{Z})$ if $d = 2k+1$

We shall explain at the end of section 10 that the definitions of 2.5 and 2.6 carry over so that one gets a class

$$\sigma^*(F, \omega) \in L^d_Z(\pi, \mathbf{Z}, v)$$

and a pairing

$$\otimes : L^d_Z(\pi, \mathbf{Z}, v) \otimes L^X_n(\mathbf{Z}\pi, w) \longrightarrow L^{Y'}_{n+d}(\mathbf{Z}\pi, vw)$$

and the up-down formula 2.7 is verified, i.e. $p_* \circ p^*$ is given by $\sigma^*(F, \omega) \otimes ?$.

Let G be a connected compact Lie group and $G \longrightarrow E \xrightarrow{p} B$ be a G-principal bundle. Suppose that G is not a torus. Let $p_1 \circ p_2 \circ p_3$ be the decomposition of p appearing in the proof of theorem 6.1. Then the K-theory transfer p^* is $2 \cdot p_1^* \circ p_3^*$ since the fibre S^2 of p_2 is simply connected and has Euler characteristic 2. Suppose that Y contains not only the image of X under the K-theory transfer p^*, but also the image of X under $p_1^* \circ p_3^*$. Then the argument in theorem 6.1 shows that the surgery transfer $p^* : L^X_n(\mathbf{Z}\pi_1(B), w(B)) \longrightarrow L^Y_{n+d}(\mathbf{Z}\pi_1(E), \widehat{v}\widehat{w})$ vanishes.

8. Review of Construction of Algebraic Surgery Transfer

Let R and S be rings with involution and (C, α, U) be a symmetric representation of R into $\mathbf{D}_d(S)$ as introduced in section 2. In this section we recall the construction of the algebraic surgery transfer $(C, \alpha, U)^* : L_n(R) \longrightarrow L_{n+d}(S)$ as defined in Lück-Ranicki [18].

An involution on an additive category $\mathcal{A}$ is a contravariant functor $* : \mathcal{A} \longrightarrow \mathcal{A}$ together with a natural equivalence $e : id_{\mathcal{A}} \longrightarrow * \circ *$ such that $e(M)^*$ and $(e(M)^{-1})^*$ agree for all objects M in $\mathcal{A}$. An example is the additive category $\mathbf{B}(R)$ of finitely generated based free R-modules with the involution sending a based R-module M to the dual R-modul $\hom_R(M, R)$ with the dual base. The additive category $\mathbf{D}_d(S)$ of finitely generated free d-dimensional S-chain complexes with homotopy classes of chain maps as morphisms possesses the involution sending C to the dual chain complex C^{d-*}. A symmetric representation (C, α, U) of R in $\mathbf{D}_d(S)$ extends in a unique way to a functor of additive categories with involutions, also denoted by $(C, \alpha, U) : \mathbf{B}(R) \longrightarrow \mathbf{D}_d(S)$.

The definition of the quadratic L-groups in terms of forms and formations as well in terms of quadratic Poincaré complexes for modules over a ring with involution carries over directly to additive categories with involution (see Ranicki [31]). In even dimensions an element in $L_{2m}(\mathbf{D}_d(S))$ is represented by a nonsingular $(-1)^m$-quadratic form $(C,[\psi])$, i.e., an object C together with a homotopy class $[\psi]$ of chain maps $\psi : C^{d-*} \longrightarrow C$ such that $\psi + (-1)^m \cdot \psi^{d-*}$ is a chain homotopy equivalence $C^{d-*} \longrightarrow C$. In odd dimensions an element in $L_{2m+1}(\mathbf{D}_d(S))$ is representated by a nonsingular $(-1)^m$-quadratic formation $(C, D, [\mu], [\gamma])$, i.e. objects C and D in $\mathbf{D}_d(S)$ and homotopy classes of chain maps $[\mu]$ and $[\gamma]$ for $\mu : C \longrightarrow D$ and $\gamma : D^{d-*} \longrightarrow C$ such that there exists a chain map $\theta : D^{d-*} \longrightarrow D$ and a chain homotopy $\chi : \theta - (-1)^m \cdot \overline{\theta} \simeq \mu \circ \gamma$ with the property that $\mathrm{cone}(-\mu^{d-*}) \longrightarrow \mathrm{cone}(\mu)_*$ given by

$$\begin{pmatrix} 0 & \gamma \\ (-1)^m \gamma^{d-*} & \chi + (-1)^m \cdot \chi^{d-*} \end{pmatrix} : C^{d-r} \oplus D^{d+1-r} \longrightarrow C_{r-1} \oplus D_r$$

is a chain equivalence. Note that χ and θ are required to exist but are not part of the structure. This definition of form resp. formation corresponds the notion of a 0- resp. 1-dimensional $(-1)^m$-quadratic Poincaré complex in the category $\mathbf{D}_d(S)$. We put $L_{n+4l}(\mathbf{D}_d(S))$ to be $L_n(\mathbf{D}_d(S))$ for $0 \leq n \leq 3$ and $l \geq 0$.

The symmetric representation (C, α, U) extends uniquely to a functor of additive categories with involution, also denoted by $(C, \alpha, U) : \mathbf{B}(R) \longrightarrow \mathbf{D}_d(S)$. A functor of additive categories with involutions induces a map on the (quadratic) L-groups. Hence we obtain a homomorphism :

8.1 $(C, \alpha, U) \otimes ? : L_n(R) \longrightarrow L_n(\mathbf{D}_d(S))$

The main ingredient in the construction of the surgery transfer is the generalized Morita homomorphism

8.2 $\mu : L_n(\mathbf{D}_d(S)) \longrightarrow L_{d+n}(S)$

defined as follows. We begin with the case where n is even, $n = 2m$. Represent the element $\lambda \in L_n(\mathbf{D}_d(S))$ by a nonsingular $(-1)^m$-quadratic form $(C, [\psi])$. Choose a representative ψ of $[\psi]$. Define a d-dimensional $(-1)^m$-quadratic structure $\{\psi_s\}$ on C by $\psi_0 = \psi$ and $\psi_s = 0$ for $s \geq 1$. Then $(C, \{\psi_s\})$ is a d-dimensional $(-1)^m$-quadratic Poincaré complex in the sense of Ranicki [30], section 1.2. Its class in $L_{n+d}(S) = L_d(S, (-1)^m)$ is defined to be $\mu(\lambda)$.

Suppose n is odd, $n = 2m+1$. Represent $\lambda \in L_n(S)$ by a nonsingular $(-1)^m$-quadratic formation $(C, D, [\mu], [\gamma])$. Then $\mu(\lambda) \in L_{d+1}(S, (-1)^m)$ is represented by the following $(d+1)$-dimensional $(-1)^m$-quadratic Poincaré complex. The underlying $(d+1)$-dimensional S-chain complex is the mapping cone of $\mu : C \longrightarrow D$ given by:

$$\mathbf{8.3}\quad \begin{pmatrix} -c & 0 \\ \mu & d \end{pmatrix} : \mathrm{cone}(\mu)_r = C_{r-1} \oplus D_r \longrightarrow \mathrm{cone}(\mu)_{-1} = C_{r-2} \oplus D_{r-1}$$

The $(-1)^m$-quadratic structure is defined by:

8.4

$$\psi_0 = \begin{pmatrix} 0 & \gamma \\ 0 & \chi \end{pmatrix} : \mathrm{cone}(-\mu^{d-*})_r = C^{d-r} \oplus D^{d+1-r} \longrightarrow \mathrm{cone}(\mu)_r = C_{r-1} \oplus D_r$$

$$\psi_1 = \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix} : \mathrm{cone}(-\mu^{d-*})_r = C^{d-r} \oplus D^{d+1-r} \longrightarrow \mathrm{cone}(\mu)_{r-1} = C_{r-2} \oplus D_{r-1}$$

$\psi_s = 0$ for $s \geq 2$

Define the algebraic surgery transfer associated with the symmetric representation (C, α, U) of R into $\mathbf{D}_d(S)$

$$(C, \alpha, U)^* : L_n(R) \longrightarrow L_{n+d}(S)$$

to be the composition $\mu \circ\ ((C, \alpha, U)\otimes ?)$ of the homomorphisms defined in 8.2 and 8.1.

We make some remarks on sign conventions as they differ from those used in Lück-Ranicki [18]. The mapping cone of a chain map was defined in 8.3. This convention has the property that the cellular chain complex of the (geometric) mapping cone of a cellular map is the (algebraic) mapping cone of the induced chain map. Define the suspension of a chain map C to be the mapping cone of $C \longrightarrow \{0\}$. If the following square

$$\begin{array}{ccc} C & \xrightarrow{u} & C' \\ f \downarrow & & \downarrow f' \\ D & \xrightarrow{v} & D' \end{array}$$

commutes up to the homotopy $h : v \circ f \simeq f' \circ u$ we obtain a chain map

$$\mathbf{8.5}\quad \begin{pmatrix} u & 0 \\ h & v \end{pmatrix} : \mathrm{cone}(f) \longrightarrow \mathrm{cone}(f')$$

The suspension ΣC is the mapping cone of $C \longrightarrow \{0\}$. Let C^{-*} be the chain complex given by $(C^{-*})_r = (C_{-r})^*$ and $(c^{-*})_r = (c_{-r+1})^*$. Let C^{d-*} be $\Sigma^d C^{-*}$. Define the dual chain map $f^{d-*}: C^{d-*} \longrightarrow D^{d-*}$ by $(f^{d-*})_r = (f_{d-r})^*$. One easily checks that with these sign conventions $\mathrm{cone}(f)^{d-*} = \mathrm{cone}((-1)^{d+1} \cdot f^{d-*})$.

Note that the $(-1)^m$-quadratic structure 8.4 is not directly a $(-1)^m$-quadratic structure in the sense of Ranicki [30], section 1.2. It is a $(-1)^m$-quadratic structure in the following sense. A $(-1)^m$-quadratic structure on the mapping cone of μ is a collection $\{\psi_s\}$ of maps $\psi_s : \mathrm{cone}(-\mu^{d-*}) \longrightarrow \mathrm{cone}(\mu)$ of graded modules of degree $-s$ such that the following relations hold for $s \geq 0$

$$c\psi_s - (-1)^s \cdot \psi_s c^{d-*} = \psi_{s+1} - (-1)^m \cdot \psi_{s+1}^{d-*}$$

where c is the differential of $\mathrm{cone}(\mu)$ and c^{d-*} the differential of $\mathrm{cone}(-\mu^{d-*})$. It is a Poincaré structure if the chain map $\psi_0 + (-1)^m \psi^{d-*} : \mathrm{cone}(-\mu^{d-*}) \longrightarrow \mathrm{cone}(\mu)$ is a chain homotopy equivalence. Using the identification $\mathrm{cone}(f)^{d-*} = \mathrm{cone}((-1)^{d+1} \cdot f^{d-*})$ and the right signs one can associate to such a $(-1)^m$-quadratic Poincaré structure a $(-1)^m$-quadratic Poincaré structure in the sense of Ranicki [30], section 1.2 (cf. Lück-Ranicki [18] proposition 3.2). The necessary signs are cumbersome so that we stick to the definition above and suppress this step in the sequel. This simplifies the notation in the following sections considerably.

9. Proof of the up-down Formula in Even Base Dimensions

This section prepares the proof of the up-down formula 2.7, and completes it for even base dimensions $n = 2m$.

Let C be a finitely generated free $\mathbf{Z}$-chain complex. A π-twist on C is a choice of $\mathbf{Z}$-chain maps $C(g) : C \longrightarrow C$ for $g \in \pi$ such that $C(1) = id$ and $C(g) \circ C(h) \simeq C(gh)$ holds for $g, h \in \pi$. In particular, we obtain a homomorphism $\pi \longrightarrow [C, C]_{\mathbf{Z}}$ by sending g to $[C(g)]$ and a $\mathbf{Z}\pi$-structure on the homology $H(C)$. A map $f : (C, \{C(g)\}) \longrightarrow (D, \{D(g)\})$ of chain complexes with a π-twist is a chain map $f : C \longrightarrow D$ satisfying $f \circ C(g) \simeq D(g) \circ f$ for all $g \in \pi$. Let $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^b$ be a $\mathbf{Z}\pi$-homomorphism where $\mathbf{Z}\pi^a$ is the direct sum of a copies of $\mathbf{Z}\pi$. Equip $\mathbf{Z}\pi \otimes C$ with the $\mathbf{Z}\pi$-structure given by $u \cdot (v \otimes x) = uv \otimes x$. In this and in the next section all tensor products are understood to be over the integers $\mathbf{Z}$. We define a $\mathbf{Z}\pi$-chain map

9.1 $\mu \otimes_t C : \mathbf{Z}\pi^a \otimes C \longrightarrow \mathbf{Z}\pi^b \otimes C$

as follows. As μ can be viewed as a matrix of homomorphism $\mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$, it suffices to specify $\mu \otimes_t C$ in the case $a = b = 1$. Let $\mu(1) \in \mathbf{Z}\pi$ be $\sum_{g \in \pi} \lambda_g \cdot g$. Then $\mu \otimes_t C$ sends $u \otimes x$ to $\sum_{g \in \pi} \lambda_g \cdot ug \otimes C(g^{-1})(x)$. One should not confuse $\mu \otimes_t C$ with $\mu \otimes C$ which sends $g \otimes x$ to $\mu(g) \otimes x$. Let $\rho_2 : \mathbf{Z}\pi^a \otimes H(C) \longrightarrow H(\mathbf{Z}\pi^a \otimes C)$ be the canonical map. This is an isomorphism as $\mathbf{Z}\pi$ is $\mathbf{Z}$-free. Define $\rho_1 : \mathbf{Z}\pi^a \otimes H(C) \longrightarrow \mathbf{Z}\pi^a \otimes H(C)$ by sending $g \otimes x$ to $g \otimes g^{-1}x$. We obtain a $\mathbf{Z}\pi$-isomorphism

9.2 $\rho = \rho_2 \circ \rho_1 : \mathbf{Z}\pi^a \otimes H(C) \longrightarrow H(\mathbf{Z}\pi^a \otimes C)$

if π acts diagonally on the source. We shall use ρ as an identification. One easily checks:

Lemma 9.3 *Let $\mu_1 : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^b$ and $\mu_2 : \mathbf{Z}\pi^b \longrightarrow \mathbf{Z}\pi^c$ be $\mathbf{Z}\pi$-homomorphisms. Consider a map of chain complexes with a π-twist $f : (C, \{C(g)\}) \longrightarrow (D, \{D(g)\})$. Then :*

1. $(\mu_2 \otimes_t C) \circ (\mu_1 \otimes_t C) \simeq_{\mathbf{Z}\pi} (\mu_2 \circ \mu_1) \otimes_t C$
2. $(\mu_1 \otimes_t D) \circ (\mathbf{Z}\pi^a \otimes f) \simeq_{\mathbf{Z}\pi} (\mathbf{Z}\pi^b \otimes f) \circ (\mu_1 \otimes_t C)$
3. $H(\mu_1 \otimes_t C) \circ \rho = \rho \circ (\mu \otimes H(C))$
4. $H(\mathbf{Z}\pi^a \otimes f) \circ \rho = \rho \circ (\mathbf{Z}\pi^a \otimes H(f))$ ■

Our main example is the cellular $\mathbf{Z}$-chain complex $C = C(F)$ of the fibre together with a choice $C(g)$ of representatives of the chain homotopy class of chain maps $C \longrightarrow C$ induced by the fibre transport $\omega(g)$. Let $\alpha : C^{d-*} \longrightarrow C$ be a representative of the homotopy class of chain maps given by the Poincaré chain equivalence $\cap[F] : C^{d-*} \longrightarrow C$. Recall that the homomorphism $v : \pi \longrightarrow \{\pm 1\}$ sends g to the degree of the automorphism $H(\omega(g))$ of $H_d(F) \cong \mathbf{Z}$. Equip C^{d-*} with the π-twist $C^{d-*}(g) = v(g) \cdot C(g)^{d-*}$. Then α is a map of chain complexes with a twist.

Recall that $\mathbf{Z}\pi^* = \hom_{\mathbf{Z}\pi}(\mathbf{Z}\pi, \mathbf{Z}\pi)$ carries the $\mathbf{Z}\pi$-structure with respect to the w-twisted involution $^- : \mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$. We shall use the following $\mathbf{Z}\pi$-isomorphism as an identification

9.4 $ev : \mathbf{Z}\pi^* \longrightarrow \mathbf{Z}\pi \qquad f \mapsto \overline{f(1)}$

Recall that $(\mathbf{Z}\pi \otimes C)^{d-*}$ is equipped with the $\mathbf{Z}\pi$-structure with respect to the vw-twisted involution $^- : \mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$. We shall use the following $\mathbf{Z}\pi$-isomorphism as an identification

9.5 $\mathbf{Z}\pi \otimes C^{d-*} \longrightarrow (\mathbf{Z}\pi \otimes C)^{d-*}$

sending $u \otimes f$ to $v \otimes x \mapsto \overline{vf(x)\overline{u}}$. Given a $\mathbf{Z}\pi$-map $\mu : \mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$, let $\overline{\mu} : \mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$ be the $\mathbf{Z}\pi$-map satisfying $\overline{\mu}(1) = \overline{\mu(1)}$ if $^-$ is the w-twisted involution. We have $\overline{\mu} = \mu^*$ under the identification 9.4. This extends obviously to $\mathbf{Z}\pi$-maps $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^b$. One easily checks that the following data define a symmetric representation $(\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)$ of $\mathbf{Z}\pi$ with the w-twisted involution into $\mathbf{D}_d(\mathbf{Z}\pi)$ with the vw-twisted involution:

9.6 $\mathbf{Z}\pi \otimes \alpha : \mathbf{Z}\pi \otimes C^{d-*} \longrightarrow \mathbf{Z}\pi \otimes C$

$$U : \mathbf{Z}\pi \longrightarrow [\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes C]^{op}_{\mathbf{Z}\pi} \qquad\qquad g \mapsto [r(g) \otimes_t C]$$

where $r(g) : \mathbf{Z}\pi \longrightarrow \mathbf{Z}\pi$ maps u to ug. One derives directly from lemma 1.4 and the definitions that the homomorphisms $p_* \circ p^*$ and $(\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)^*$ from $L_n(\mathbf{Z}\pi, w)$ to $L_{n+d}(\mathbf{Z}\pi, vw)$ agree.

The strategy of the proof of the up-down formula is described as follows. Given $\lambda \in L_n(\mathbf{Z}\pi, w)$, choose an appropriate $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex of dimension d resp. $d+1$ representing $p_* \circ p^*(\lambda) \in L_{d+n}(\mathbf{Z}\pi, w)$ if $n = 2m$ resp $n = 2m+1$. Do algebraic surgery on this chain complex and show that the result is homotopy equivalent to a $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex which represents $\sigma^*(F, \omega) \otimes \lambda$. As algebraic surgery and homotopy equivalence do not change the class of a $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex in the L-group, the up-down formula follows. To do the algebraic surgery and find the right homotopy equivalence, we need some preliminaries. Namely, we are going to construct the following data if $d = 2k$ respectively $d = 2k+1$:

9.7
1. A finitely generated free $\mathbf{Z}$-chain complex D of dimension k. If d is $2k$, the differential d_k is injective.
2. A chain epimorphism $p : C \longrightarrow D$ which is k-connected. If d is $2k+1$, then $H_k(p)$ has $\operatorname{tors} H_k(C)$ as kernel.
3. Chain maps $D(g) : D \longrightarrow D$ for $g \in \pi$ such that $p \circ C(g)$ and $D(g) \circ p$ agree. Let $\operatorname{cone}(p)(g) : \operatorname{cone}(p) \longrightarrow \operatorname{cone}(p)$ be the chain map given by $C(g)_{*-1} \oplus D(g)_*$. Then $\{D(g)\}$ and $\{\operatorname{cone}(p)(g)\}$ define π-twists.
4. A finitely generated free chain complex E. If d is $2k$, then E is concentrated in dimension k and E_k is $H_k(C)/\operatorname{tors} H_k(C)$. If d is $2k+1$, then E is concentrated in dimensions $k+1$ and k, the differential e_{k+1} is injective and E is a resolution of $\operatorname{tors} H_k(C)$.
5. A chain map $q : C \longrightarrow E$ such that $H_k(q) : H_k(C) \longrightarrow H_k(E) = H_k(C)/\operatorname{tors} H_k(C)$ is the canonical projection if $d = 2k$ and $H_k(q) : H_k(C) \longrightarrow H_k(E) = \operatorname{tors} H_k(C)$ induces the identity on $\operatorname{tors} H_k(C)$ if $d = 2k+1$.

6. A π-twist $E(g)$ on E. Let $\eta : \Sigma^{-1}\text{cone}(p) \longrightarrow E$ be the composition of $q : C \longrightarrow D$ with the canonical projection $\Sigma^{-1}\text{cone}(p) \longrightarrow C$. Then η is a map of chain complexes with π-twist.

7. We get chain maps $\xi : D^{d-*} \longrightarrow \Sigma^{-1}\text{cone}(p)$ by $(\alpha p^{d-*}, 0)$ and $\nu : \text{cone}(\xi) \longrightarrow E$ by $(0, \eta)^{tr}$.

8. There is for all i a short exact sequence

$$0 \longrightarrow H_i(D^{d-*}) \xrightarrow{H_i(\xi)} H_i(\Sigma^{-1}\text{cone}(p)) \xrightarrow{H_i(\eta)} H_i(E) \longrightarrow 0 \quad .$$

This implies that ν is a homotopy equivalence.

Roughly speaking, the data about D and p will be used for the algebraic surgery and the data about E and q for the homotopy equivalence of quadratic Poincaré complexes. We construct this data in the case $d = 2k$ first.

Let D be

$$\ldots \longrightarrow \{0\} \longrightarrow \text{im}(c_k) \hookrightarrow C_{k-1} \longrightarrow C_{k-2} \longrightarrow \ldots \longrightarrow C_0$$

Define $p : C \longrightarrow D$ by $p_i = 0$ for $i > k$, $p_k = c_k$ and $p_i = id$ for $i < k$. For $g, h \in \pi$ choose a homotopy $H(g,h) : C(g) \circ C(h) \simeq C(gh)$. There are maps of graded modules $D(g) : D_* \longrightarrow D_*$ of degree zero and $K(g,h) : D_* \longrightarrow D_{*+1}$ of degree with the property that $p \circ C(g) = D(g) \circ p$ and $K(g,h) \circ p = p \circ H(g,h)$ hold. As p is surjective, $D(g)$ is a chain map and $K(g,h)$ a homotopy between $D(g) \circ D(h)$ and $D(gh)$. Obviously $D(1)$ and $C(1)$ are the identity. Now one easily checks that items 1.), 2.) and 3.) in the list 9.7 are satisfied.

Define E as required in item 4.) of 9.7. Since $\text{im}(c_k) \subset C_{k-1}$ is free and the sequence $0 \longrightarrow \ker(c_k) \longrightarrow C_k \longrightarrow \text{im}(c_k) \longrightarrow 0$ is exact, there is a $\mathbf{Z}$-map $r : C_k \longrightarrow \ker(c_k)$ whose restriction to $\ker(c_k)$ is the identity. Let $\text{pr} : \ker(c_k) \longrightarrow H_k(C)/\text{tors}\, H_k(C)$ be the canonical projection. Define a chain map $q : C \longrightarrow E$ by $q_k = \text{pr} \circ r$. Obviously item 5.) in 9.7 holds. Define the π-twist on E by the $\mathbf{Z}\pi$ structure on $E_k = H_k(C)/\text{tors}\, H_k(C)$. Then $q : C \longrightarrow E$ is a map of chain complexes with π-twist, as it induces a $\mathbf{Z}\pi$-map on homology. This proves item 6.) of 9.7.

The compositions $p \circ \alpha \circ p^{d-*}$ is zero because of item 1.) in 9.7 since $D_i^{d-*} = \{0\}$ for $i < k$, $D_i = \{0\}$ for $i > k$ and d_k is injective. Hence $\xi : D^{d-*} \longrightarrow \text{cone}(p)$ given by $(\alpha \circ p^{d-*}, 0)^{tr}$ is a chain map. Items 1.) and 2.) of 9.7 imply that $H_i(\Sigma^{-1}\text{cone}(p))$ is $\{0\}$ for $i \leq k-1$ and the canonical map $H_k(\text{cone}(p)) \longrightarrow H_k(C)$ is injective with $\text{tors}\, H_k(C)$ as image. Therefore $H_k(\eta \circ \xi)$ is zero. Since E is concentrated in one dimension $\eta \circ \xi$ is zero and $\nu : \text{cone}(\xi) \longrightarrow E$ given by $(0, \eta)^{tr}$ is a chain map. This proves item 7.)

Next we prove the exactness of the sequence in item 8.). Note that this together with the long homology sequence of ξ implies that ν is a homology equivalence, and hence a homotopy equivalence. Since $\Sigma^{-1}\mathrm{cone}(p)$ is $(k-1)$-connected, we can choose a chain complex X homotopy equivalent to $\Sigma^{-1}\mathrm{cone}(p)$ such that $X_i = \{0\}$ for $i \le k-1$. The mapping cone of $p^{d-*} : D^{d-*} \longrightarrow C^{d-*}$ is $\left(\Sigma^{-1}\mathrm{cone}((-1)^{d+1} \cdot p)\right)^{d-*} \simeq X^{d-*}$. Obviously $H_i(X^{d-*}) = \{0\}$ for $i \ge k+1$ and $H_k(X^{d-*}) \subset (X^{d-*})_k$ is free. We derive from the long homology sequence of p^{d-*} that $H_i(p^{d-*})$ is bijective for $i \ge k+1$ and the following sequence is exact

$$0 \longrightarrow H_k(D^{d-*}) \xrightarrow{H_k(p^{d-*})} H_k(C^{d-*}) \longrightarrow H_k(X^{d-*}) \longrightarrow 0$$

Since $H_k(D^{d-*}) \otimes \mathbf{Q}$ is isomorphic to $H_k(D) \otimes \mathbf{Q}$ and $H_k(D) = \{0\}$ by item 2.) in 9.7, the module $H_k(D^{d-*})$ is torsion. This implies that $H_k(p^{d-*}) : H_k(D^{d-*}) \longrightarrow H_k(C^{d-*})$ is injective and has $\mathrm{tors}\, H_k(C^{d-*})$ as image. Since $H_i(\alpha) : H_i(C^{d-*}) \longrightarrow H_i(C)$ is bijective, $H_i(\xi) : H_i(D^{d-*}) \longrightarrow H_i(\mathrm{cone}(p))$ is bijective for $i \ge k$. Now item 8.) in 9.7 follows. This finishes the verification and construction of the data 9.7 in the even-dimensional case $d = 2k$.

Next we treat the case $d = 2k+1$. Let K be the kernel of the canonical projection $\ker(c_k) \longrightarrow H_k(C)/\mathrm{tors}\, H_k(C)$. As its image is free K is a direct summand in $\ker(c_k)$ and hence in C_k. In particular K and C_k/K are free and we can choose a retraction $r : C_k \longrightarrow K$. Let D be

$$\ldots \longrightarrow \{0\} \longrightarrow C_k/K \xrightarrow{c_k} C_{k-1} \longrightarrow C_{k-2} \longrightarrow \ldots \longrightarrow C_0$$

and $p : C \longrightarrow D$ be the obvious projection. One verifies items 1.) 2.) and 3.) of 9.7 as done in the case $d = 2k$ above.

Choose E as required in item 4.) of 9.7. Let $\epsilon : E_k \longrightarrow H_k(E) = \mathrm{tors}\, H_k(C)$ be the canonical projection. Let $\mathrm{pr} : K \longrightarrow \mathrm{tors}\, H_k(C)$ be the canonical epimorphism. Choose $\overline{\mathrm{pr}} : K \longrightarrow E_k$ satisfying $\epsilon \circ \overline{\mathrm{pr}} = \mathrm{pr}$. Put $q_k : C_k \longrightarrow E_k$ to be $\overline{\mathrm{pr}} \circ r$ where $r : C_k \longrightarrow K$ is a retraction. Because $\epsilon \circ q_k \circ c_{k+1}$ vanishes, we can choose $q_{k+1} : C_{k+1} \longrightarrow E_{k+1}$ such that $e_{k+1} \circ q_{k+1} = q_k \circ c_{k+1}$ holds. Hence we obtain a chain map $q : C \longrightarrow E$ satisfying item 5.) of 9.7.

Since E is a resolution of the $\mathbf{Z}\pi$-module $\mathrm{tors}\, H_k(C)$, we may choose chain maps $E(g) : E \longrightarrow E$ inducing multiplication with g on homology for each $g \in \pi$. As $\Sigma^{-1}\mathrm{cone}(p)$ is $(k-1)$-connected and $H_k(\eta) : H_k(\Sigma^{-1}\mathrm{cone}(p)) \longrightarrow H_k(E)$ is a $\mathbf{Z}\pi$-homomorphism, η is a map of chain complexes with a π-twist. This proves item 6.) of 9.7.

The compositions $p \circ \alpha \circ p^{d-*}$ and $q \circ \alpha \circ p^{d-*}$ are zero for dimension reasons. Hence item 7.) of 9.7 is true.

It remains to verify item 8.). Because of item 1.) and 2.) of 9.7 it suffices to show that $H_i(p^{d-*}) : H_i(D^{d-*}) \longrightarrow H_i(C^{d-*})$ is bijective for $i \geq k+1$. The mapping cone of p^{d-*} is $\left(\Sigma^{-1}\mathrm{cone}((-1)^{d+1} \cdot p)\right)^{d-*}$). Since $H_i(\Sigma^{-1}\mathrm{cone}(p))$ is zero for $i \leq k-1$ and is torsion for $i = k$, the universal coefficient theorem shows that $H_i(\Sigma^{-1}\mathrm{cone}(p)^{d-*}) = \{0\}$ for $i \geq k+1$. One may derive this also by an argument as in the case $d = 2k$ above using a chain complex X which is homotopy equivalent to $\Sigma^{-1}\mathrm{cone}(p)$ and satisfies $X_i = \{0\}$ for $i \leq k-1$. This finishes the construction and verification of the data 9.7.

Now we are ready to prove the up-down formula 2.7 in even base dimensions $n = 2m$. Represent $\lambda \in L_n(\mathbf{Z}\pi, w)$ by a nonsingular $(-1)^m$-quadratic form $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^a$ over $\mathbf{Z}\pi$ with respect to the w-twisted involution, i.e. a $\mathbf{Z}\pi$-map μ such that $\mu + (-1)^m\overline{\mu}$ is an isomorphism. We obtain a $(-1)^m$-quadratic structure $\{\psi_s\}$ on $\mathbf{Z}\pi^a \otimes C$ if we define $\psi_0 = (\mu \otimes_t C) \circ (\mathbf{Z}\pi^a \otimes \alpha) : \mathbf{Z}\pi^a \otimes C^{d-*} \longrightarrow \mathbf{Z}\pi^a \otimes C$ and $\psi_s = 0$ for $s \geq 1$. The class of the d-dimensional $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex $(\mathbf{Z}\pi^a \otimes C, \{\psi_s\})$ in $L_{d+n}(\mathbf{Z}\pi, vw)$ is $p_* \circ p^*(\lambda)$.

The composition $(\mathbf{Z}\pi^a \otimes p) \circ (\psi + (-1)^m \cdot \psi^{d-*}) \circ (\mathbf{Z}\pi^a \otimes p^{d-*})$ is zero by item 1.) of 9.7. Hence $\{\psi_s\} \in Q_d(\mathbf{Z}\pi^a \otimes C)$ can be extended to $\{(\partial\psi, \psi)_s\} \in Q_{d+1}(\mathbf{Z}\pi^a \otimes p)$ by $\partial\psi = 0$. Denote by $(\overline{C}, \{\overline{\psi}\})$ the result under surgery on $(\mathbf{Z}\pi^a \otimes p, \{(\partial\psi, \psi)_s\})$ as defined in Ranicki [30], section 1.5. We obtain a chain map

$$\widetilde{\xi} : \mathbf{Z}\pi^a \otimes D^{d-*} \longrightarrow \Sigma^{-1}\mathrm{cone}(\mathbf{Z}\pi^a \otimes p) = \mathbf{Z}\pi^a \otimes \Sigma^{-1}\mathrm{cone}(p)$$

by $\left((\psi_0 + (-1)^m \cdot \psi_0^{d-*}) \circ (\mathbf{Z}\pi^a \otimes p^{d-*}), 0\right)^{tr}$. By definition $\overline{C}$ is the mapping cone of $\widetilde{\xi}$. Let

$$\widetilde{\eta} : \mathbf{Z}\pi^a \otimes \Sigma^{-1}\mathrm{cone}(p) \longrightarrow \mathbf{Z}\pi^a \otimes E$$

be $\mathbf{Z}\pi^a \otimes \eta$. One computes $H_k(\widetilde{\eta} \circ \widetilde{\xi}) = \mathbf{Z}\pi^a \otimes H_k(\eta \circ \xi)$. using the identification 9.2 and lemma 9.3. Since $H_k(\eta \circ \xi)$ vanishes by item 8.) of 9.7 and E is concentrated in one dimension by item 4.) of 9.7, $\widetilde{\eta} \circ \widetilde{\xi}$ is zero. Hence we obtain a chain map

$$\widetilde{\nu} : \overline{C} \longrightarrow \mathbf{Z}\pi^a \otimes E$$

by $(0, \widetilde{\eta})$. If we choose a $\mathbf{Z}$-base for C and D, we get an induced $\mathbf{Z}\pi$-base for the source and target of $\widetilde{\nu}$ and the Whitehead torsion $\tau(\widetilde{\nu}) \in \widetilde{K}_1(\mathbf{Z}\pi)$ of $\widetilde{\nu}$ is defined. It is independent of the choice of the $\mathbf{Z}$-base above, since we work in the reduced K_1-group. Recall from Lück [15], section 5 that associated to the chain complex with π-twist $(D^{d-*}, \{D^{d-*}(g)\})$ there is a K-theory transfer $(D^{d-*}, \{D^{d-*}(g)\})^* : \widetilde{K}_1(\mathbf{Z}\pi) \longrightarrow \widetilde{K}_1(\mathbf{Z}\pi)$ (and also for $K_0(\mathbf{Z}\pi)$).

Lemma 9.8 *$\widetilde{\nu} : \overline{C} \longrightarrow \mathbf{Z}\pi^a \otimes E$ is a homotopy equivalence. Its Whitehead torsion is the image under $(D^{d-*}, \{D^{d-*}(g)\})^*$ of the class in $\widetilde{K}_1(\mathbf{Z}\pi)$ represented by $\mu + (-1)^m \cdot \overline{\mu}$.*

Proof : There is a homotopy H from $\widetilde{\xi}$ to $(\mathbf{Z}\pi^a \otimes \xi) \circ (\mu \otimes_t D^{d-*} + (-1)^m \cdot \overline{\mu} \otimes_t D^{d-*})$ because of lemma 9.3. We obtain a chain map

$$\nu' = \begin{pmatrix} \mu \otimes_t D^{d-*} + (-1)^m \cdot \overline{\mu} \otimes_t D^{d-*} & 0 \\ H & id \end{pmatrix} : \overline{C} = \operatorname{cone}(\widetilde{\xi}) \longrightarrow \operatorname{cone}(\mathbf{Z}\pi^a \otimes \xi)$$

As $\mu \otimes_t D^{d-*} + (-1)^m \cdot \overline{\mu} \otimes_t D^{d-*}$ is a homotopy equivalence, ν' is a homotopy equivalence. The Whitehead torsion of ν' is the Whitehead torsion of $\mu \otimes_t D^{d-*} + (-1)^m \cdot \overline{\mu} \otimes_t D^{d-*}$ which is by definition the image under $(D^{d-*}, \{D^{d-*}(g)\})^*$ of the class in $\widetilde{K_1}(\mathbf{Z}\pi)$ represented by $\mu + (-1)^m \cdot \overline{\mu}$. One can identify the target of ν' with $\mathbf{Z}\pi^a \otimes \operatorname{cone}(\xi)$. As ν is by item 8.) of 9.7 a homotopy equivalence $\mathbf{Z}\pi^a \otimes \nu : \mathbf{Z}\pi^a \otimes \operatorname{cone}(\xi) \longrightarrow \mathbf{Z}\pi^a \otimes E$ is a simple homotopy equivalence. Its composition with ν' is $\widetilde{\nu}$ and the claim follows. ■

Let $\{\phi_s\}$ be the quadratic structure on $\mathbf{Z}\pi^a \otimes E$ obtained by pulling back the quadratic structure $\{\overline{\psi}_s\}$ by $\widetilde{\nu}$, i.e., $\phi_s = \widetilde{\nu} \circ \overline{\psi}_s \circ \widetilde{\nu}^{d-*}$. As $\widetilde{\nu}$ is $(0, q, 0) : D^{d-*-1} \oplus C_* \oplus D_{*+1} \longrightarrow E_*$ one easily checks from the definition of $\overline{\psi}$ in Ranicki [30], section 1.5 that ϕ_s is zero for $s \geq 1$ and ϕ_0 is $q \circ (\mu \otimes_t C) \circ (\mathbf{Z}\pi^a \otimes \alpha) \circ \mathbf{Z}\pi^a \otimes q^{d-*}$. Hence

$$H_i(\phi_0) : H_i(\mathbf{Z}\pi^a \otimes E^{d-*}) \longrightarrow H_i(\mathbf{Z}\pi^a \otimes E)$$

can be identified using 9.2, lemma 9.3 and item 4.) of 9.7 with

$$\mu \otimes \phi_F : \mathbf{Z}\pi^a \otimes \hom_{\mathbf{Z}}(H_k(F)/\operatorname{tors} H_k(F), \mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes H_k(F)/\operatorname{tors} H_k(F)$$

if $d = 2k$ and ϕ_F is the intersection form on F and with

$$\mu \otimes \phi_F : \mathbf{Z}\pi^a \otimes \hom_{\mathbf{Z}}(\operatorname{tors} H_k(F), \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes \operatorname{tors} H_k(F)$$

if $d = 2k+1$ and ϕ_F is the linking form. The class of the d-dimensional $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex $(\mathbf{Z}\pi^a \otimes E, \{\phi_s\})$ in $L_{n+d}(\mathbf{Z}\pi, vw) = L_d(\mathbf{Z}\pi, (-1)^m \cdot vw)$ has been shown to be $\sigma^*(F, \omega) \otimes \lambda$. This finishes the proof of the up-down formula 2.7 in the case of even base dimension $n = 2m$ for L^h.

Finally, we make some remarks how this extends to the intermediate L-groups in case of K_1-decorations. The case K_0 is completely analogous. Given a finitely generated free $\mathbf{Z}$-chain complex with a π-twist $D, \{D(g)\}$, let $s(D) \in \operatorname{Sw}(\pi, \mathbf{Z})$ be the class $\sum_{i \geq 0} (-1)^i \cdot [H_i(D)]$. The assumptions 7.1 just say that $s(D)$ and $s(C)$ lie in $Z \subset \operatorname{Sw}(\pi, \mathbf{Z})$ where C and D are the chain complexes with a π-twist defined in 9.7. The long homology sequence of ξ becomes an exact $\mathbf{Z}\pi$-sequence if one identifies $H_i(\operatorname{cone}(\xi))$ with $H_i(E)$ by $H_i(\nu)$. The long exact sequence of p is also compatible with the $\mathbf{Z}\pi$-structures. This implies $s(E) = s(C) - s(D) + s(D^{d-*})$. By the universal coefficient theorem $(-1)^d \cdot s(D^{d-*})$ is the image of $s(D)$ under the v-twisted involution. Thus $s(D^{d-*})$ and $[\operatorname{tors} H_k(F)] = (-1)^k \cdot s(E)$ lie in $Z \subset \operatorname{Sw}(\pi, \mathbf{Z})$ and $\sigma^*(F, \omega)$ defines an element in $L^d_Z(\pi, \mathbf{Z}, v)$. The K-theoretic transfer homomorphism $(D^{d-*}, \{D^{d-*}(g)\})^* : \widetilde{K}_1(\mathbf{Z}\pi) \longrightarrow \widetilde{K}_1(\mathbf{Z}\pi)$ sends X to Y' since it is given by the pairing 2.1 and $s(D) \in \mathrm{Sw}(\pi, \mathbf{Z})$. Now one easily checks that the proof above for the up-down formula for L^h goes through for the intermediate L-groups.

10. Proof of the up-down Formula in Odd Base Dimensions

Next, we finish the proof of the up-down formula 2.7 in the case where the base dimension is odd, $n = 2m+1$. We give two different proofs. The first one is based on the Shaneson splitting and is comparatively short but does not carry over to the intermediate L-groups. The second one is a blown up version of the proof in the even-dimensional case and holds also for the intermediate L-groups. We restrict ourselves to the proof for L^h.

Let $i : L^h_n(\mathbf{Z}\pi, w) \longrightarrow L^s_{n+1}(\mathbf{Z}[\mathbf{Z} \times \pi], w)$ be the homomorphism given by the tensor-product with the symmetric $\mathbf{Z}[\mathbf{Z}]$-chain complex of the universal covering of S^1. Since the Euler characteristic of S^1 is zero the image lies in the L^s-groups. This map is a split injection by the results of Shaneson [33] and Ranicki [28]. Let $(\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)$ be the symmetric representation 9.6 whose surgery transfer $(\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)^*$ is just $p_* \circ p^*$. The Whitehead torsion of $\mathbf{Z}\pi^a \otimes \alpha : \mathbf{Z}\pi^a \otimes C^{d-*} \longrightarrow \mathbf{Z}\pi^a \otimes C$ is zero. Hence $(\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)$ induces transfer maps on both L^h and L^s. One easily verifies that the following square commutes

$$\begin{array}{ccc} L^h_n(\mathbf{Z}\pi, w) & \xrightarrow{i} & L^s_{n+1}(\mathbf{Z}[\pi \times \mathbf{Z}], w) \\ (\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)^* \downarrow & & \downarrow (\mathbf{Z}\pi \otimes C, \mathbf{Z}\pi \otimes \alpha, U)^* \\ L^h_{n+d}(\mathbf{Z}\pi, vw) & \xrightarrow{i} & L^s_{n+d+1}(\mathbf{Z}[\pi \times \mathbf{Z}], vw) \end{array}$$

One also checks easily that this square remains comutative if we substitute the vertical arrows by $\sigma^*(F, \omega)\otimes ?$. The up-down formula for L^h in the odd-dimensional case $n = 2m+1$ now follows from the up-down formula for L^s in the even case $n = 2m+2$ and injectivity of i. A similar argument applies to L^p instead of L^h.

Next we give a different proof which is much more complicated but carries over to the intermediate L-groups as well. Recall that a nonsingular $(-1)^m$-quadratic formation $(\mathbf{Z}\pi^a, \mathbf{Z}\pi^b, \mu, \gamma)$ consists of $\mathbf{Z}\pi$-homomorphisms $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^b$ and $\gamma : \mathbf{Z}\pi^b \longrightarrow \mathbf{Z}\pi^a$ such that there is a $\mathbf{Z}\pi$-map $\theta : \mathbf{Z}\pi^b \longrightarrow \mathbf{Z}\pi^b$ with the properties that $\theta - (-1)^m \cdot \overline{\theta} = \mu \circ \gamma$ holds and the following square is cartesian:

10.1

$$\begin{array}{ccc}
\mathbf{Z}\pi^b & \xrightarrow{\gamma} & \mathbf{Z}\pi^a \\
-\overline{\mu} \downarrow & & \downarrow \mu \\
\mathbf{Z}\pi^a & \xrightarrow{\overline{\gamma}} & \mathbf{Z}\pi^b
\end{array}$$

Consider an element $\lambda \in L_{2m+1}(\mathbf{Z}\pi, vw)$. Represent λ by a nonsingular $(-1)^m$-quadratic formation $(\mathbf{Z}\pi^a, \mathbf{Z}\pi^b, \mu, \gamma)$. Choose a $\mathbf{Z}\pi$-chain map $\theta : D^{d-*} \longrightarrow D$ and a $\mathbf{Z}\pi$-homotopy χ from $\theta \otimes_t C - (-1)^m \cdot \overline{\theta} \otimes_t C$ to $(\mu \otimes_t C) \circ (\gamma \otimes_t C) \circ (\mathbf{Z}\pi^b \otimes \alpha)$. Equip $\mathrm{cone}(\mu \otimes_t C)$ with the $(-1)^m$-quadratic structure $\{\psi_s\}$ defined accordingly to 8.4 as follows

10.2

$$\psi_0 = \begin{pmatrix} 0 & \gamma \otimes_t C \circ \mathbf{Z}\pi^b \otimes \alpha \\ 0 & \chi \end{pmatrix} : \mathrm{cone}(\mu \otimes_t C)^{d+1-*} \longrightarrow \mathrm{cone}(\mu \otimes_t C)_*$$

$$\psi_1 = \begin{pmatrix} 0 & 0 \\ 0 & \theta \otimes_t C \circ \mathbf{Z}\pi^b \otimes \alpha \end{pmatrix} : \mathrm{cone}(\mu \otimes_t C)^{d+1-*} \longrightarrow \mathrm{cone}(\mu \otimes_t C)_{*-1}$$

$\psi_s = 0$ for $s \geq 2$

Then the class of the $(-1)^m$-quadratic Poincaré $\mathbf{Z}\pi$-chain complex $(\mathrm{cone}(\mu \otimes_t C), \{\psi_s\})$ of dimension $d+1$ in $L_{n+d}(\mathbf{Z}\pi, vw) = L_{d+1}(S, (-1)^m \cdot vw)$ is $p_* \circ p^*(\lambda)$.

The following square commutes (strictly, not only up to homotopy) by item 3.) in 9.7

10.3

$$\begin{array}{ccc}
\mathbf{Z}\pi^a \otimes C & \xrightarrow{\mathbf{Z}\pi^a \otimes p} & \mathbf{Z}\pi^a \otimes D \\
\mu \otimes_t C \Big\downarrow & & \Big\downarrow \mu \otimes_t D \\
\mathbf{Z}\pi^b \otimes C & \xrightarrow{\mathbf{Z}\pi^b \otimes p} & \mathbf{Z}\pi^b \otimes D
\end{array}$$

We get a chain map $\widehat{p} : \mathrm{cone}(\mu \otimes_t C) \longrightarrow \mathrm{cone}(\mu \otimes_t D)$ by

$$\begin{pmatrix} \mathbf{Z}\pi^a \otimes p_{*-1} & 0 \\ 0 & \mathbf{Z}\pi^b \otimes p_* \end{pmatrix} : \mathbf{Z}\pi^a \otimes C_{*-1} \oplus \mathbf{Z}\pi^b \otimes C_* \longrightarrow \mathbf{Z}\pi^a \otimes D_{*-1} \oplus \mathbf{Z}\pi^b \otimes D_*$$

By item 1.) in 9.7 $\widehat{p}\circ\psi_s\circ\widehat{p}^{d-*}$ is zero for all $s\geq 0$. Hence $\{\psi_s\}\in Q_{d+1}(\mathrm{cone}(\mu\otimes_t C))$ extends to $\{(\partial\psi,\psi)_s\}\in Q_{d+2}(\widehat{p})$ if we put $\partial\psi$ to be zero. Let $(\overline{C},\{\overline{\psi_s}\})$ be the result of algebraic surgery on $(\widehat{p},\{(\partial\psi,\psi)_s\})$. Because $\widehat{p}\circ(\psi_0+(-1)^m\psi_0^{d-*})\circ\widehat{p}^{d+1-*}$ is zero by item 1.) in 9.7, we obtain a chain map

$$\widetilde{\xi}:\mathrm{cone}(\overline{\mu}\otimes_t D^{d-*})\longrightarrow\Sigma^{-1}\mathrm{cone}(\widehat{p})$$

by $\left((\psi_0+(-1)^m\psi_0^{d-*})\circ\widehat{p}^{d+1-*},0\right)^{tr}$. By definition $\mathrm{cone}(\widetilde{\xi})$ is $\overline{C}$. The following diagram commutes up to homotopy by item 6.) in 9.7 and lemma 9.3.

$$\begin{array}{ccc}
\mathbf{Z}\pi^a\otimes\Sigma^{-1}\mathrm{cone}(p) & \xrightarrow{\mathbf{Z}\pi^a\otimes\eta} & \mathbf{Z}\pi^a\otimes E\\
\Big\downarrow{\scriptstyle \mu\otimes_t\Sigma^{-1}\mathrm{cone}(p)} & & \Big\downarrow{\scriptstyle \mu\otimes_t E}\\
\mathbf{Z}\pi^b\otimes\Sigma^{-1}\mathrm{cone}(p) & \xrightarrow{\mathbf{Z}\pi^b\otimes\eta} & \mathbf{Z}\pi^b\otimes E
\end{array}$$

Choose a corresponding homotopy H. We get a $\mathbf{Z}\pi$-chain map

$$\widetilde{\eta}=\begin{pmatrix}\mathbf{Z}\pi^a\otimes\eta & 0\\ H & \mathbf{Z}\pi^b\otimes\eta\end{pmatrix}:\Sigma^{-1}\mathrm{cone}(\widehat{p})=\mathrm{cone}(\mu\otimes_t\Sigma^{-1}\mathrm{cone}(p))\longrightarrow\mathrm{cone}(\mu\otimes_t E)$$

We derive from item 1.) of 9.7 that $\widetilde{\eta}\circ\widetilde{\xi}$ is zero. Hence we obtain a chain map

$$\widetilde{\nu}:\overline{C}=\mathrm{cone}(\widetilde{\xi})\longrightarrow\mathrm{cone}(\mu\otimes_t E)$$

by $(\widetilde{\eta},0)^{tr}$.

Lemma 10.4 *$\widetilde{\nu}:\overline{C}\longrightarrow$ cone$(\mu\otimes_t E)$ is a homotopy equivalence. Its Whitehead torsion is the image under the K-theory transfer $(D^{d-*},\{D^{d-*}(g)\})^*$ of the Whitehead torsion of the nonsingular $(-1)^m$-quadratic formation $(\mathbf{Z}\pi^a,\mathbf{Z}\pi^b,\mu,\gamma)$.*

Proof : We use the following abbreviations :

$$f_1=(\gamma\otimes_t C)\circ(\mathbf{Z}\pi^b\otimes\alpha)\circ(\mathbf{Z}\pi^b\otimes p^{d-*}):\mathbf{Z}\pi^b\otimes D^{d-*}\longrightarrow\mathbf{Z}\pi^a\otimes C$$
$$f_2=(-1)^m\cdot\ (\mathbf{Z}\pi^b\otimes\alpha^{d-*})\circ(\overline{\gamma}\otimes_t C^{d-*})\circ(\mathbf{Z}\pi^a\otimes p^{d-*}):\mathbf{Z}\pi^a\otimes D^{d-*}\longrightarrow\mathbf{Z}\pi^b\otimes C$$

$$g_1=(\mathbf{Z}\pi^a\otimes\alpha)\circ(\mathbf{Z}\pi^a\otimes p^{d-*}):\mathbf{Z}\pi^a\otimes D^{d-*}\longrightarrow\mathbf{Z}\pi^a\otimes C$$
$$g_2=(-1)^m\cdot\ (\mathbf{Z}\pi^b\otimes\alpha)\circ(\mathbf{Z}\pi^b\otimes p^{d-*}):\mathbf{Z}\pi^b\otimes D^{d-*}\longrightarrow\mathbf{Z}\pi^b\otimes C$$

Then $\widetilde{\xi} : \mathrm{cone}(\overline{\mu} \otimes_t D^{d-*}) \longrightarrow \Sigma^{-1}\mathrm{cone}(\widehat{p})$ is given by

$$\begin{pmatrix} f_1 & 0 \\ * & f_2 \\ 0 & 0 \end{pmatrix} : (\mathbf{Z}\pi^b \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^b \otimes D^{d-*}) \to (\mathbf{Z}\pi^a \otimes C_{*-1}) \oplus (\mathbf{Z}\pi^b \otimes C_*) \oplus \Sigma^{-1}\mathrm{cone}(\mu \otimes_t D)_*$$

As the square 10.1 is cartesian, we can find a cartesian square

10.5
$$\begin{array}{ccc} \mathbf{Z}\pi^a & \xrightarrow{\delta} & \mathbf{Z}\pi^b \\ \mu \downarrow & & \downarrow -\overline{\mu} \\ \mathbf{Z}\pi^b & \xrightarrow{\epsilon} & \mathbf{Z}\pi^a \end{array}$$

and a $\mathbf{Z}\pi$-map $\Omega : \mathbf{Z}\pi^b \longrightarrow \mathbf{Z}\pi^a$ satisfying

10.6
$$\begin{aligned} \mu \circ \Omega &= id - \overline{\gamma} \circ \epsilon \\ \Omega \circ \mu &= id - \gamma \circ \delta \end{aligned}$$

If one regards the square 10.1 as a chain homotopy equivalence from the first to the second column, the square 10.5 describes a chain homotopy inverse and Ω a chain homotopy. Define a chain map $\kappa : \mathrm{cone}(\mu \otimes_t D^{d-*}) \longrightarrow \mathrm{cone}(\overline{\mu} \otimes_t D^{d-*})$ by

$$\begin{pmatrix} \delta \otimes_t D^{d-(*-1)} & 0 \\ * & \epsilon \otimes_t D^{d-*} \end{pmatrix} : (\mathbf{Z}\pi^a \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^b \otimes D^{d-*}) \longrightarrow$$
$$(\mathbf{Z}\pi^b \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^a \otimes D^{d-*})$$

where $*$ denotes any homotopy from $(\overline{\mu} \otimes_t D^{d-*}) \circ (\delta \otimes_t D^{d-*})$ to $(\epsilon \otimes_t D^{d-*}) \circ (\mu \otimes_t D^{d-*})$. Such a homotopy $*$ exists as the square 10.5 is commutative. Because of the relations 10.6 and lemma 9.3 we can choose homotopies

$$h_1 : f_1 \circ (\delta \otimes_t D^{d-*}) + g_1 \circ (\Omega \otimes_t D^{d-*}) \circ (\mu \otimes_t D^{d-*}) \simeq g_1$$
$$h_2 : f_2 \circ (\epsilon \otimes_t D^{d-*}) + g_2 \circ (\mu \otimes_t D^{d-*}) \circ (\Omega \otimes_t D^{d-*}) \simeq g_2$$

Define a map of degree one $H : \mathrm{cone}(\mu \otimes_t D^{d-*}) \longrightarrow \Sigma^{-1}\mathrm{cone}(\widehat{p})$ by

$$\begin{pmatrix} h_1 & g_1 \circ (\Omega \otimes_t D^{d-*}) \\ 0 & -h_2 \\ 0 & 0 \end{pmatrix} : (\mathbf{Z}\pi^a \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^b \otimes D^{d-*}) \longrightarrow$$

$$(\mathbf{Z}\pi^a \otimes C_{*-1}) \oplus (\mathbf{Z}\pi^b \otimes C_*) \oplus (\Sigma^{-1}\mathrm{cone}(\mu \otimes_t C)_*)$$

One checks using item 1.) and 2.) of 9.7 that the compositions of $\mathbf{Z}\pi^a \otimes p$ and $\mathbf{Z}\pi^a \otimes q$ with h_1 and H as well as the compositions of $\mathbf{Z}\pi^b \otimes p$ and $\mathbf{Z}\pi^b \otimes q$ with h_2 are zero. Now one easily verifies that H is a homotopy between $\widetilde{\xi} \circ \widetilde{\kappa}$ and a chain map $\widetilde{g} : \mathrm{cone}(\mu \otimes_t D^{d-*}) \longrightarrow \Sigma^{-1}\mathrm{cone}(\widehat{p})$ of the following shape

$$\begin{pmatrix} g_1 & 0 \\ * & g_2 \\ 0 & 0 \end{pmatrix} : (\mathbf{Z}\pi^a \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^b \otimes D^{d-*})$$
$$\longrightarrow (\mathbf{Z}\pi^a \otimes C_{*-1}) \oplus (\mathbf{Z}\pi^b \otimes C_*) \oplus (\Sigma^{-1}\mathrm{cone}(\mu \otimes_t D)_*) \ .$$

We have constructed a diagram which commutes up to the homotopy H.

$$\begin{array}{ccc} \mathrm{cone}(\mu \otimes_t D^{d-*}) & \xrightarrow{\widetilde{g}} & \Sigma^{-1}\mathrm{cone}(\widehat{p}) \\ \widetilde{\kappa} \downarrow & & \downarrow id \\ \mathrm{cone}(\overline{\mu} \otimes_t D^{d-*}) & \xrightarrow{\widetilde{\xi}} & \Sigma^{-1}\mathrm{cone}(\widehat{p}) \end{array}$$

Let $\nu' : \mathrm{cone}(\widetilde{g}) \longrightarrow \mathrm{cone}(\widetilde{\xi}) = \overline{C}$ be the induced chain map (see 8.5). It is a homotopy equivalence as the square 10.5 is cartesian and therefore $\widetilde{\kappa}$ is a homotopy equivalence. The Whitehead torsion of ν' is the Whitehead torsion of $\widetilde{\kappa}$ which can be identified with the negative of the image of the Whitehead torsion of the cartesian square 10.1 under the K-theory transfer $(D^{d-*}, \{D^{d-*}(g)\})^*$. The mapping cone of $\widetilde{g}$ can be identified with the mapping cone of the following chain map $\widehat{g} : \mathbf{Z}\pi^a \otimes \mathrm{cone}(\xi) \longrightarrow \mathbf{Z}\pi^b \otimes \mathrm{cone}(\xi)$ given by

$$\begin{pmatrix} \mu \otimes_t D^{d-*} & 0 & 0 \\ * & \mu \otimes_t C & 0 \\ 0 & 0 & \mu \otimes_t D \end{pmatrix} : (\mathbf{Z}\pi^a \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^a \otimes C_*) \oplus (\mathbf{Z}\pi^a \otimes D_{*+1})$$
$$\longrightarrow (\mathbf{Z}\pi^b \otimes D^{d-(*-1)}) \oplus (\mathbf{Z}\pi^b \otimes C_*) \oplus \mathbf{Z}\pi^b \otimes D_{*+1}$$

The composition of $\widetilde{\nu}$ with ν' is given by the following up to homotopy commutative square together with an appropiate choice of homotopy (see 8.5).

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{Z}\pi^a \otimes \mathrm{cone}(\xi) \arrow[r, "\mathbf{Z}\pi^a \otimes \nu"] \arrow[d, "\widehat{g}"'] & \mathbf{Z}\pi^a \otimes E \arrow[d, "\mu \otimes_t E"] \\
\mathbf{Z}\pi^b \otimes \mathrm{cone}(\xi) \arrow[r, "\mathbf{Z}\pi^b \otimes \nu"] & \mathbf{Z}\pi^b \otimes E
\end{tikzcd}$$

Since ν is a homotopy equivalence by item 8.) of 9.7 $\mathbf{Z}\pi^a \otimes \nu$ and $\mathbf{Z}\pi^b \otimes \nu$ and hence $\widetilde{\nu} \circ \nu'$ are $\mathbf{Z}\pi$-homotopy equivalences with trivial Whitehead torsion. This finishes the proof of lemma 10.4. ■

Equip $\mathrm{cone}(\mu \otimes_t E)$ with the quadratic structure $\{\phi_s\} = \{\nu \circ \overline{\psi}_s \circ \nu^{d+1-*}\}$. One computes directly from 10.2 that ϕ is given by :

$$\phi_0 = \begin{pmatrix} 0 & (\mathbf{Z}\pi^a \otimes q) \circ (\gamma \otimes_t C) \circ (\mathbf{Z}\pi^b \otimes \alpha) \circ (\mathbf{Z}\pi^b \otimes q^{d-*}) \\ 0 & (\mathbf{Z}\pi^b \otimes q) \circ \chi \circ (b \otimes_t q^{d-*}) \end{pmatrix}$$

$$\phi_1 = \begin{pmatrix} 0 & 0 \\ 0 & (\mathbf{Z}\pi^b \otimes q) \circ (\mathbf{Z}\pi^b \otimes \theta) \circ (\mathbf{Z}\pi^b \otimes \alpha) \circ (\mathbf{Z}\pi^b \otimes q^{d-*}) \end{pmatrix}$$

$\phi_s = 0$ for $s \geq 2$

We can identify

$$H_k((\mathbf{Z}\pi^a \otimes q) \circ (\gamma \otimes_t C) \circ (\mathbf{Z}\pi^b \otimes \alpha) \circ (\mathbf{Z}\pi^b \otimes p^{d-*})) : H_k(\mathbf{Z}\pi^b \otimes E) \longrightarrow H_k(\mathbf{Z}\pi^a \otimes E)$$

using the identification 9.2, lemma 9.3 and and item 4.) of 9.7 with

$$\gamma \otimes \phi_F : \mathbf{Z}\pi^b \otimes \hom_{\mathbf{Z}}(H_k(F)/\mathrm{tors}\, H_k(F), \mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes H_k(F)/\mathrm{tors}\, H_k(F)$$

if $d = 2k$ and ϕ_F is the intersection form on F, and with

$$\gamma \otimes \phi_F : \mathbf{Z}\pi^a \otimes \hom_{\mathbf{Z}}(\mathrm{tors}\, H_k(F), \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes \mathrm{tors}\, H_k(F)$$

if $d = 2k + 1$ and ϕ_F is the linking form. Analogously we can identify

$$H_k(\mu \otimes_t E) : H_k(\mathbf{Z}\pi^a \otimes E) \longrightarrow H_k(\mathbf{Z}\pi^b \otimes E)$$

with

$$\mu \otimes id : \mathbf{Z}\pi^a \otimes \hom_{\mathbf{Z}}(H_k(F)/\mathrm{tors}\, H_k(F), \mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes H_k(F)/\mathrm{tors}\, H_k(F)$$

if $d = 2k$, and with

$$\mu \otimes id : \mathbf{Z}\pi^a \otimes \hom_{\mathbf{Z}}(\mathrm{tors}\, H_k(F), \mathbf{Q}/\mathbf{Z}) \longrightarrow \mathbf{Z}\pi^a \otimes \mathrm{tors}\, H_k(F)$$

if $d = 2k+1$. Now it follows from the definitions that $(E, \{\phi_s\})$ represents $\sigma^*(F,\omega) \otimes \lambda$ in $L_{d+n}(\mathbf{Z}\pi, vw)$. This finishes the proof of the up-down formula 2.7.

11. The Pairing 2.4 is Well-Defined

This section is devoted to the proof that the pairing 2.4 is well-defined. We use the notation of section 3. Let (M, ψ) be a nonsingular (π, v)-equivariant $(-1)^{k+1}$-symmetric linking form over $\mathbf{Z}$. We have to show that the transfer $(\mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*, id \otimes \alpha, U)^*$ is trivial if (M, ψ) is hyperbolic or a boundary. The strategy of the proof is in all cases the following. We represent $\lambda \in L_n(\mathbf{Z}\pi, w)$ by a nonsingular quadratic form or formation and look at a Poincaré complex $(C, \{\psi_s\})$ representing the image of λ under $(\mathbf{Z}\pi \otimes_{\mathbf{Z}} F_*, id \otimes \alpha, U)^*$. We give a chain map $\widehat{p} : C \longrightarrow D$. Recall that the Q-groups are homology groups of certain chain complexes (see [30], section 1.1). We have the class $\widehat{p}_*(\{\psi_s\}) \in Q_{n+d}(D)$ given by the cycle $\{\widehat{p} \circ \psi_s \circ \widehat{p}^{n+d-*}\}$. We shall specify a chain $\{\partial\psi_s\}$ whose image under the differential is just $\{\widehat{p} \circ \psi_s \circ \widehat{p}^{n+d-*}\}$. Hence $\widehat{p}_*(\{\psi_s\}) \in Q_{n+d}(D)$ is zero. This guarantees that we can do surgery on $\widehat{p}$. We leave to the reader the easy verification that the result under surgery is a contractible Poincaré complex. Then the claim follows.

Assume that (M, ψ) is the boundary of (N, ϕ). Choose F to be $N \longrightarrow \hom_{\mathbf{Z}}(N, \mathbf{Z})$ and the π-twist to be the $\mathbf{Z}\pi$-structure. Moreover, the Poincaré duality map $\alpha : F^{1-*} \longrightarrow F$ can be choosen to be the $\mathbf{Z}\pi$-chain map which is $(-1)^k \cdot id$ in dimension 1 and id in dimension 0. Let $1(N)$ be the chain complex concentrated in dimension 1 and having N as chain module there. It inherits a π-twist from the $\mathbf{Z}\pi$-structure on N. Let $p : F \longrightarrow 1(N)$ be given by $p_1 = id$.

If $n = 2m$ and the nonsingular $(-1)^m$-quadratic form $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^a$ represents λ, we choose $\hat{p}$ to be $\mathbf{Z}\pi^a \otimes p : C = \mathbf{Z}\pi^a \otimes F \longrightarrow D = \mathbf{Z}\pi^a \otimes 1(N)$. Then $\{\widehat{p} \circ \psi_s \circ \widehat{p}^{n+d_*}\}$ is zero for dimension reasons. So we can choose $\{\partial\psi_s\}$ to be zero.

If $n = 2m+1$ and λ is represented by the nonsingular $(-1)^m$-quadratic formation $(\mathbf{Z}\pi^a, \mathbf{Z}\pi^b, \mu, \gamma)$ we define the chain map $\widehat{p} : C = \operatorname{cone}(\mu \otimes_t F) \longrightarrow \operatorname{cone}(\mu \otimes_t 1(N))$ by $(\mathbf{Z}\pi^a \otimes p) \oplus (\mathbf{Z}\pi^b \otimes p)$. This is possible, as p is a $\mathbf{Z}\pi$-map and so $(\mu \otimes_t 1(N)) \circ (\mathbf{Z}\pi^a \otimes p)$ agrees with $(\mathbf{Z}\pi^b \otimes p) \circ (\mu \otimes_t F)$. Moreover, we can choose the homotopy χ appearing in the definition 8.4 of ψ_0 of C to be zero . Now one easily checks again that $\{\widehat{p} \circ \psi_s \circ \widehat{p}^{n+d_*}\}$ is zero.

Now assume that (M, ψ) is hyperbolic. Let

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{q} \hom_{\mathbf{Z}}(L, \mathbf{Q}/\mathbf{Z}) \longrightarrow 0$$

be the corresponding sequence. If $F_1 \longrightarrow F_0 \xrightarrow{\epsilon} M$ is the $\mathbf{Z}$-resolution of M, let G_0 be F_0, $\epsilon' : G_0 \longrightarrow \hom_{\mathbf{Z}}(L, \mathbf{Q}/\mathbf{Z})$ be the composition $q \circ \epsilon$ and G_1 be the kernel of ϵ'. There is precisely one chain map $p : F \longrightarrow G$ satisfying $p_0 = id$. If $\{F(g)\}$ is the π-twist on F, there is precisely one π-twist on G such that $G(g) \circ p$ agrees with $p \circ F(g)$ for all $g \in \pi$.

If $n = 2m$ and the nonsingular $(-1)^m$-quadratic form $\mu : \mathbf{Z}\pi^a \longrightarrow \mathbf{Z}\pi^a$ represents λ, we choose $\hat{p}$ to be $\mathbf{Z}\pi^a \otimes p : C = \mathbf{Z}\pi^a \otimes F \longrightarrow D = \mathbf{Z}\pi^a \otimes G$. Then the chain map $\{\widehat{p} \circ \psi_0 \circ \widehat{p}^{d-*}\}$ induces $(\gamma \otimes H(D)) \circ (\mathbf{Z}\pi^a \otimes H(p \circ \alpha \circ p^{d-*}))$ on homology using the identification 9.2 by lemma 9.3. Because G^{d-*} and G are resolutions and $H(p \circ \alpha \circ p^{d-*})$ is zero, we can choose a nullhomotopy $\partial\psi_0$ for $\{\widehat{p} \circ \psi_0 \circ \widehat{p}^{d-*}\}$. Put $\partial\psi_s$ to be zero for $s \geq 1$.

If $n = 2m + 1$ and λ is represented by the nonsingular $(-1)^m$-quadratic formation $(\mathbf{Z}\pi^a, \mathbf{Z}\pi^b, \mu, \gamma)$, we define the chain map $\widehat{p} : C = \mathrm{cone}(\mu \otimes_t F) \longrightarrow \mathrm{cone}(\mu \otimes_t G)$ by $(\mathbf{Z}\pi^a \otimes p) \oplus (\mathbf{Z}\pi^b \otimes p)$. Choose nullhomotopies

$$H_0 : (\mathbf{Z}\pi^a \otimes p) \circ (\gamma \otimes_t F) \circ (\mathbf{Z}\pi^b \otimes \alpha) \circ (\mathbf{Z}\pi^b \otimes p^{d-*}) \simeq 0$$
$$H_1 : (\mathbf{Z}\pi^b \otimes p) \circ (\theta \otimes_t F) \circ (\mathbf{Z}\pi^b \otimes \alpha) \circ (\mathbf{Z}\pi^b \otimes p^{d-*}) \simeq 0$$

Let $\partial\psi_s$ be given by

$$\partial\psi_0 = \begin{pmatrix} 0 & -H_0 \\ 0 & 0 \end{pmatrix} : \mathbf{Z}\pi^a \otimes G^{1-*} \oplus \mathbf{Z}\pi^b \otimes G^{1-(*-1)} \longrightarrow \mathbf{Z}\pi^a \otimes G_* \oplus \mathbf{Z}\pi^b \otimes G_{*+1}$$
$$\partial\psi_1 = \begin{pmatrix} 0 & 0 \\ 0 & -H_1 \end{pmatrix} : \mathbf{Z}\pi^a \otimes G^{1-*} \oplus \mathbf{Z}\pi^b \otimes G^{1-(*-1)} \longrightarrow \mathbf{Z}\pi^a \otimes G_{*-1} \oplus \mathbf{Z}\pi^b \otimes G_*$$
$$\partial\psi_s = 0 \text{ for } s \geq 2$$

If we alter the cycle $\{\widehat{p} \circ \psi_s \circ \widehat{p}^{n+d-*}\}$. by the boundary given by $\{\partial\psi_s\}$, we obtain a new cycle $\{\psi'_s\}$ which represents the same element as $\{\psi_s\}$ in the Q-group and is of the following shape

$$\psi'_0 = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} : \mathbf{Z}\pi^a \otimes G^{1-*} \oplus \mathbf{Z}\pi^b \otimes G^{1-(*-1)} \longrightarrow \mathbf{Z}\pi^a \otimes G_{*-1} \oplus \mathbf{Z}\pi^b \otimes G_*$$
$$\psi'_s = 0 \text{ for } s \geq 1$$

Since $\{\psi'_s\}$ is a cycle, x is actually a chain map $x : \mathbf{Z}\pi^a \otimes G^{1-*} \longrightarrow \Sigma^{-1}\mathbf{Z}\pi^a \otimes G_*$. As the first differential of G is injective, x must be zero. Hence $\{\psi'_s\}$ is zero.

Appendix: Characteristic Class Formulae

We shall now use the algebraic L-theory assembly map of Ranicki [32] to relate the expression of 2.7

$$p_* \circ p^* \;=\; \sigma^*(F,p) \otimes - \;:\; L_n(\mathbf{Z}[\pi_1(B)]) \longrightarrow L_{n+d}(\mathbf{Z}[\pi_1(B)])$$

(where $\sigma^*(F,p) \equiv \sigma^*(F,\omega)$) for a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ of manifolds with the characteristic class formulae for the signature of E.

In the first instance we recall the results of Atiyah [3], Lusztig [19] and Meyer [20] expressing the twisted signature in terms of characteristic classes, by means of the Atiyah-Singer index theorem.

Atiyah [3] considered a differentiable fibre bundle $F \longrightarrow E \xrightarrow{p} B$ of oriented manifolds with $\dim(F) = 2k$. The action of $\pi_1(B)$ on $H^k(F;\mathbf{R})$ determines a flat vector bundle Γ over B with the fibres $H^k(F_x;\mathbf{R})$ ($x \in B$) nonsingular $(-)^k$-symmetric forms over $\mathbf{R}$. For $k \equiv 0 (\bmod 2)$ the bundle splits as $\Gamma = \Gamma^+ \oplus \Gamma^-$ with the form positive/negative definite on $\Gamma^\pm$. For $k \equiv 1 (\bmod 2)$ the Hodge $*$-operator defines a complex structure on Γ, so that there is defined a complex conjugate bundle Γ^*. The topological K-theory signature of Γ is defined by

$$[\Gamma]_{\mathbf{K}} \;=\; \begin{cases} \Gamma^+ - \Gamma^- \in KO(B) & \text{if } k \equiv 0 (\bmod 2) \\ \Gamma^* - \Gamma \in KU(B) & \text{if } k \equiv 1 (\bmod 2) \ . \end{cases}$$

The twisted signature formula of [3] in the case $\dim(B) = 2j$, $\dim(E) = 2(j+k) \equiv 0 (\bmod 4)$ is

$$\mathrm{sign}(E) \;=\; \langle \mathrm{ch}([\Gamma]_{\mathbf{K}}) \cup \widetilde{\mathcal{L}}(B), [B]_{\mathbf{Q}} \rangle \in \mathbf{Z}$$

with ch the Chern character, $[B]_{\mathbf{Q}} \in H_{2j}(B;\mathbf{Q})$ the fundamental class and $\widetilde{\mathcal{L}}$ the modification of the Hirzebruch $\mathcal{L}$-genus defined by

$$\widetilde{\mathcal{L}}(B) \;=\; \prod_{i=1}^{j} \frac{x_i}{\tanh x_i/2} \in H^{4*}(B;\mathbf{Q})$$

with $x_1, x_2, \ldots, x_j$ notional elements of degree 2 such that the ith Pontrjagin class $p_i(\tau_B) \in H^{4i}(B;\mathbf{Q})$ of the tangent bundle τ_B is the ith elementary symmetric function in $x_1^2, x_2^2, \ldots, x_j^2$. The tangent bundle of E is the Whitney sum $\tau_E = i_*\tau_F \oplus p^*\tau_B$ of the pushforward $i_*\tau_F$ along the fibre inclusion $i : F \to E$ of τ_F and the pullback $p^*\tau_B$ along the projection p of τ_B, with

$$p_*(\widetilde{\mathcal{L}}(i_*\tau_F) \cap [E]_{\mathbf{Q}}) \;=\; \mathrm{ch}([\Gamma]_{\mathbf{K}}) \cap [B]_{\mathbf{Q}} \in H_{4*}(B;\mathbf{Q}) \ .$$

Lusztig [19] considered a flat complex vector bundle Γ over an oriented differentiable manifold B with $\dim(B) = 2j$, such that the fibres Γ_x ($x \in B$) are nonsingular hermitian forms over $\mathbf{C}$. The twisted signature $\mathrm{sign}(B,\Gamma) \in \mathbf{Z}$ is defined to be the signature of the nonsingular hermitian form on $H^k(B;\Gamma)$. The complex K-theory signature of Γ is defined by $[\Gamma]_{\mathbf{K}} = \Gamma^+ - \Gamma^- \in KU(B)$ for any splitting $\Gamma = \Gamma^+ \oplus \Gamma^-$ with the hermitian form positive/negative definite on $\Gamma^{\pm}$. The twisted signature formula of [19] is

$$\mathrm{sign}(B,\Gamma) \;=\; \langle \mathrm{ch}([\Gamma]_{\mathbf{K}}) \cup \widetilde{\mathcal{L}}(B), [B]_{\mathbf{Q}} \rangle \in \mathbf{Z} \;.$$

Meyer [20] considered a locally constant sheaf Γ over an oriented topological manifold B with $\dim(B) = 2j$, such that the stalks Γ_x $(x \in B)$ are nonsingular $(-)^k$-symmetric forms over $\mathbf{R}$. The twisted signature $\mathrm{sign}(B,\Gamma) \in \mathbf{Z}$ is defined to be the signature of the nonsingular symmetric form on $H^k(B;\Gamma)$. The topological K-theory signature of Γ is the topological K-theory signature in the sense of [3] of the flat vector bundle $\tilde{\Gamma}$ over B with fibres $H^k(\Gamma_x;\mathbf{R})$

$$[\Gamma]_{\mathbf{K}} \;=\; [\tilde{\Gamma}]_{\mathbf{K}} \in \begin{cases} KO(B) & \text{if } k \equiv 0 (\mathrm{mod}\, 2) \\ KU(B) & \text{if } k \equiv 1 (\mathrm{mod}\, 2) \;. \end{cases}$$

The twisted signature formula of [20] is

$$\mathrm{sign}(B,\Gamma) \;=\; \langle \widetilde{\mathrm{ch}}([\Gamma]_{\mathbf{K}}) \cup \mathcal{L}(B), [B]_{\mathbf{Q}} \rangle \in \mathbf{Z}$$

with $\widetilde{\mathrm{ch}} = \mathrm{ch} \circ \psi^2$ the modified Chern character obtained by composition with the second Adams operation ψ^2 and $\mathcal{L}$ the original Hirzebruch $\mathcal{L}$-genus defined by

$$\mathcal{L}(B) \;=\; \prod_{i=1}^{j} \frac{x_i}{\tanh x_i} \in H^{4*}(B;\mathbf{Q}) \;.$$

For any complex n-plane bundle α over B with total Chern class

$$c(\alpha) \;=\; \prod_{i=1}^{n} (1 + y_i) \in H^{2*}(B)$$

the Chern characters $\mathrm{ch}(\alpha) = \sum\limits_{i=1}^{n} e^{y_i}$, $\widetilde{\mathrm{ch}}(\alpha) = \sum\limits_{i=1}^{n} e^{2y_i} \in H^{2*}(B;\mathbf{Q})$ are such that

$$(\widetilde{\mathrm{ch}}(\alpha) \cup \mathcal{L}(B))_{2j} \;=\; (\mathrm{ch}(\alpha) \cup \widetilde{\mathcal{L}}(B))_{2j} \in H^{2j}(B;\mathbf{Q}) \;,$$

since for any $i \geq 0$

$$\begin{aligned} \mathcal{L}(B)_{4i} \;&=\; 2^{2i-j} \widetilde{\mathcal{L}}(B)_{4i} \in H^{4i}(B;\mathbf{Q}) \;, \\ \widetilde{\mathrm{ch}}(\alpha)_{2j-4i} \;&=\; 2^{j-2i} \mathrm{ch}(\alpha)_{2j-4i} \in H^{2j-4i}(B;\mathbf{Q}) \;. \end{aligned}$$

Thus the twisted signature can be expressed as

$$\mathrm{sign}(B,\Gamma) \;=\; \langle \widetilde{\mathrm{ch}}([\Gamma]_{\mathbf{K}}) \cup \mathcal{L}(B), [B]_{\mathbf{Q}} \rangle \;=\; \langle \mathrm{ch}([\Gamma]_{\mathbf{K}}) \cup \widetilde{\mathcal{L}}(B), [B]_{\mathbf{Q}} \rangle \in \mathbf{Z} \;.$$

(See 'Mannigfaltigkeiten und Modulformen', Bonn notes on lectures of Hirzebruch, pp. 83-4. We are indebted to Michael Crabb for this reference.)

Next, we describe the algebraic L-theory assembly map of Ranicki [32].

Let $\mathbf{L}^{\cdot} = \{\mathbf{L}^d | d \in \mathbf{Z}\}$, $\mathbf{L}. = \{\mathbf{L}_d | d \in \mathbf{Z}\}$ be the algebraic L-spectra defined in §13 of [32]. $\mathbf{L}^d$ is the Kan Δ-set with n-simplexes the d-dimensional symmetric Poincaré n-simplexes over $\mathbf{Z}$, such that

$$\Omega \mathbf{L}^d \;=\; \mathbf{L}^{d+1} \;\;,\;\; \pi_*(\mathbf{L}^{\cdot}) \;=\; L^*(\mathbf{Z}) \;,$$

and similarly for $\mathbf{L}_d$, $\mathbf{L}.$ in the quadratic case.

An 'n-dimensional symmetric cycle' over a simplicial complex B is an inverse system

$$(C,\phi) \;=\; \{(C[\tau],\phi[\tau]) \,|\, \tau \in B\}$$

of $(n+|\tau|-k)$-dimensional symmetric $(k-|\tau|)$-simplexes over $\mathbf{Z}$, with the support $\{\tau \in B| \, C[\tau] \neq 0\}$ contained in a finite subcomplex $B_0 \subseteq B$ with $(k+2)$ vertices (so that there exists an embedding $B_0 \subseteq \partial\Delta^{k+1}$). The cycle is 'locally Poincaré' if each $(C[\tau],\phi[\tau])$ $(\tau \in B)$ is an $(n+|\tau|-k)$-dimensional symmetric Poincaré $(k-|\tau|)$-simplex over $\mathbf{Z}$. The cycle is 'globally Poincaré' if the assembly n-dimensional symmetric complex over $\mathbf{Z}[\pi_1(B)]$

$$(C[\tilde{B}],\phi[\tilde{B}]) \;=\; \bigcup_{\tilde{\tau}\in\tilde{B}} (C[\tau],\psi[\tau])$$

is Poincaré, with $\tilde{B}$ the universal cover of B. The generalized homology group $H_n(B;\mathbf{L}^{\cdot})$ is identified in §13 of [32] with the cobordism group of locally Poincaré n-dimensional symmetric cycles over B.

The visible symmetric L-group $VL^n(B)$ of a simplicial complex B is the cobordism group of globally Poincaré n-dimensional symmetric cycles over B. (For a classifying space $B = B\pi$ these are the original visible symmetric L-groups $VL^*(\mathbf{Z}[\pi])$ of Weiss [37]). The forgetful maps

$$\begin{array}{l} L_n(\mathbf{Z}[\pi_1(B)]) \;\longrightarrow\; VL^n(B) \;;\; (C,\psi) \;\longrightarrow\; (C,(1+T)\psi) \;, \\ VL^n(B) \;\longrightarrow\; L^n(\mathbf{Z}[\pi_1(B)]) \;;\; (C,\phi) \;\longrightarrow\; (C[\tilde{B}],\phi[\tilde{B}]) \end{array}$$

are isomorphisms modulo 8-torsion. Passing from local to global Poincaré cycles defines assembly maps

$$A \;:\; H_n(B;\mathbf{L}^{\cdot}) \;\longrightarrow\; VL^n(B)$$

for any simplicial complex B. The visible symmetric signature of an n-dimensional geometric Poincaré complex B is an element

$$\sigma^*(B) \;=\; (C,\phi) \in VL^n(B)$$

with $C[\tau] = \mathbf{Z}$ $(\tau \in B)$. The visible symmetric signature of an n-dimensional PL manifold B is the assembly

$$\sigma^*(B) \;=\; A([B]_{\mathbf{L}}) \in VL^n(B)$$

of the canonical $\mathbf{L}^{\cdot}$-orientation

$$[B]_{\mathbf{L}} \;=\; \{(C[\tau],\phi[\tau])|\tau\in B\}\in H_n(B;\mathbf{L}^{\cdot})$$

with $C[\tau]=\mathbf{Z}$.

The generalized homology group $H_n(B;\mathbf{L}.)$ is the cobordism group of locally Poincaré n-dimensional quadratic cycles over B. The surgery obstruction group $L_n(\mathbf{Z}[\pi_1(B)])$ is the cobordism group of globally Poincaré n-dimensional quadratic cycles over B, as well as the cobordism group of n-dimensional quadratic Poincaré complexes over $\mathbf{Z}[\pi_1(B)]$. The quadratic L-theory assembly map

$$A \;:\; H_n(B;\mathbf{L}.) \;\longrightarrow\; L_n(\mathbf{Z}[\pi_1(B)])\;;\;(C,\psi)\;\longrightarrow\;(C[\tilde{B}],\psi[\tilde{B}])$$

is defined in §9 of [32] by passing from local to global Poincaré duality. The surgery obstruction of a normal map $(f,b):M\to B$ of closed n-dimensional manifolds is the assembly

$$\sigma_*(f,b) \;=\; A([f,b]_{\mathbf{L}})\in L_n(\mathbf{Z}[\pi_1(B)])$$

of an $\mathbf{L}.$-homology surgery invariant $[f,b]_{\mathbf{L}}\in H_n(B;\mathbf{L}.)$ with symmetrization

$$(1+T)[f,b]_{\mathbf{L}} \;=\; f_*[M]_{\mathbf{L}}-[B]_{\mathbf{L}}\in H_n(B;\mathbf{L}^{\cdot})\;.$$

A 'd-dimensional symmetric Poincaré cocycle' over a simplicial complex B is a directed system $\{(C[\tau],\phi[\tau])\,|\,\tau\in B\}$ of $(d+|\tau|)$-dimensional symmetric $|\tau|$-simplexes over $\mathbf{Z}$, i.e. a Δ-map $B\to\mathbf{L}^d$. The generalized cohomology group $H^{-d}(B;\mathbf{L}^{\cdot})=[B,\mathbf{L}^d]$ is the cobordism group of d-dimensional symmetric Poincaré cocycles over B.

A d-dimensional symmetric Poincaré cocycle (C,ϕ) over a finite simplicial complex B is *homogenous* if the structure maps $C[\sigma]\to C[\tau]$ $(\sigma\leq\tau\in B)$ are chain equivalances, in which case for any 0-simplex $*\in B$ the fundamental group $\pi_1(B)$ acts on the chain homotopy type of the 'fibre' d-dimensional symmetric Poincaré complex $(C[*],\phi[*])$. Let $L^d(B,\mathbf{Z})$ be the cobordism group of homogenous d-dimensional symmetric Poincaré cocycles over B, and define an assembly map

$$A \;:\; L^d(B,\mathbf{Z}) \;\longrightarrow\; L^d(\pi_1(B),\mathbf{Z})\;;\;(C,\phi)\;\longrightarrow\;(C[*],\phi[*])\;.$$

Tensor product over $\mathbf{Z}$ makes $\mathbf{L}^{\cdot}$ into a ring spectrum, and $\mathbf{L}.$ is an $\mathbf{L}^{\cdot}$-module spectrum. The evaluation of the cap product pairing

$$\cap \;:\; H^r(B;\mathbf{L}^{\cdot})\otimes H_n(B;\mathbf{L}^{\cdot}) \;\longrightarrow\; H_{n-r}(B;\mathbf{L}^{\cdot})$$

on the canonical $\mathbf{L}^{\cdot}$-coefficient orientation $[B]_{\mathbf{L}}\in H_n(B;\mathbf{L}^{\cdot})$ of an n-dimensional PL manifold B defines $\mathbf{L}^{\cdot}$-coefficient Poincaré duality isomorphisms

$$-\cap[B]_{\mathbf{L}} \;:\; H^*(B;\mathbf{L}^{\cdot}) \;\longrightarrow\; H_{n-*}(B;\mathbf{L}^{\cdot})\;.$$

For any simplicial complex B use the cap product pairing

$$\cap \ : \ H^{-d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes VL^n(B) \longrightarrow VL^{n+d}(B)$$

and the forgetful map

$$L^d(B,\mathbf{Z}) \longrightarrow H^{-d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \ ; \ (C,\phi) \longrightarrow (C,\phi)$$

to define a product pairing

$$\otimes \ : \ L^d(B,\mathbf{Z}) \otimes VL^n(B) \longrightarrow VL^{n+d}(B)$$

and define similarly the product pairings

$$\begin{aligned} \otimes \ &: \ L^d(B,\mathbf{Z}) \otimes H_n(B;\mathbf{L}^{\boldsymbol{\cdot}}) \longrightarrow H_{n+d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \ , \\ \otimes \ &: \ L^d(B,\mathbf{Z}) \otimes H_n(B;\mathbf{L}_{\boldsymbol{\cdot}}) \longrightarrow H_{n+d}(B;\mathbf{L}_{\boldsymbol{\cdot}}) \ . \end{aligned}$$

The products and assembly maps are related by commutative squares

$$\begin{array}{ccc} L^d(B,\mathbf{Z}) \otimes VL^n(B) & \longrightarrow & VL^{n+d}(B) \\ \downarrow & & \downarrow \\ L^d(\pi_1(B),\mathbf{Z}) \otimes VL^n(\mathbf{Z}[\pi_1(B)]) & \longrightarrow & VL^{n+d}(\mathbf{Z}[\pi_1(B)]) \ , \end{array}$$

$$\begin{array}{ccc} L^d(B,\mathbf{Z}) \otimes H_n(B;\mathbf{L}^{\boldsymbol{\cdot}}) & \longrightarrow & H_{n+d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \\ \downarrow & & \downarrow \\ L^d(B,\mathbf{Z}) \otimes VL^n(B) & \longrightarrow & VL^{n+d}(B) \ , \end{array}$$

$$\begin{array}{ccc} H^{-d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes H_n(B;\mathbf{L}_{\boldsymbol{\cdot}}) & \longrightarrow & H_{n+d}(B;\mathbf{L}_{\boldsymbol{\cdot}}) \\ \downarrow & & \downarrow \\ L^d(\pi_1(B),\mathbf{Z}) \otimes L_n(\mathbf{Z}[\pi_1(B)]) & \longrightarrow & L_{n+d}(\mathbf{Z}[\pi_1(B)]) \ , \end{array}$$

$$\begin{array}{ccc} L^d(B,\mathbf{Z}) \otimes H_n(B;\mathbf{L}_{\boldsymbol{\cdot}}) & \longrightarrow & H_{n+d}(B;\mathbf{L}_{\boldsymbol{\cdot}}) \\ \downarrow & & \downarrow \\ H^{-d}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes L_n(\mathbf{Z}[\pi_1(B)]) & \longrightarrow & L_{n+d}(\mathbf{Z}[\pi_1(B)]) \ . \end{array}$$

A fibration $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a d-dimensional geometric Poincaré complex induces transfer maps in the quadratic and visible symmetric L-groups

$$\begin{aligned} p^* &: L_n(\mathbf{Z}[\pi_1(B)]) \longrightarrow L_{n+d}(\mathbf{Z}[\pi_1(E)]) \ ; \ (C,\psi) \longrightarrow (C^!,\psi^!) \\ p^* &: VL^n(B) \longrightarrow VL^{n+d}(E) \ ; \ (C,\phi) \longrightarrow (C^!,\phi^!) \end{aligned}$$

with $C^![\tau] = C[p\tau]$ $(\tau \in B)$. In general, there is no transfer $p^* : L^n(\mathbf{Z}[\pi_1(B)]) \to L^{n+d}(\mathbf{Z}[\pi_1(E)])$ in the symmetric L-groups (cf. Appendix 2 of [18]).

A PL fibre bundle $F \longrightarrow E \xrightarrow{p} B$ with the fibre a d-dimensional PL manifold F induces transfer maps in $\mathbf{L}^\cdot$-homology

$$\begin{aligned} p^* &: H_n(B;\mathbf{L}^\cdot) \longrightarrow H_{n+d}(E;\mathbf{L}^\cdot) \ , \\ p^* &: H_n(B;\mathbf{L}_\cdot) \longrightarrow H_{n+d}(E;\mathbf{L}_\cdot) \end{aligned}$$

which commute with the assembly maps

$$\begin{array}{ccc} H_n(B;\mathbf{L}^\cdot) & \xrightarrow{A} & VL^n(B) \\ p^* \big\downarrow & & \big\downarrow p^* \\ H_{n+d}(E;\mathbf{L}^\cdot) & \xrightarrow{A} & VL^{n+d}(E) \ , \end{array}$$

$$\begin{array}{ccc} H_n(B;\mathbf{L}_\cdot) & \xrightarrow{A} & L_n(\mathbf{Z}[\pi_1(B)]) \\ p^* \big\downarrow & & \big\downarrow p^* \\ H_{n+d}(E;\mathbf{L}_\cdot) & \xrightarrow{A} & L_{n+d}(\mathbf{Z}[\pi_1(E)]) \ . \end{array}$$

The $\pi_1(B)$-equivariant symmetric signature is the assembly

$$\sigma^*(F,p) = A([F,p]_{\mathbf{L}}) \in L^d(\pi_1(B),\mathbf{Z})$$

of the $\mathbf{L}^\cdot$-coefficient fibre transport

$$(F,p)_{\mathbf{L}} \in L^d(B,\mathbf{Z})$$

represented by the Δ-map

$$(F,p)_{\mathbf{L}} : B \longrightarrow \mathbf{L}^d \ ; \ \tau \longrightarrow (C(p^{-1}(\tau)),\phi_\tau) \ .$$

The proof is by a direct generalization of the expression in §16 of [32] of the visible symmetric signature of F as the assembly of the canonical $\mathbf{L}$-orientation. Let $[F,p]_{\mathbf{L}} \in H^{-d}(B;\mathbf{L}^\cdot)$ be the image of $(F,p)_{\mathbf{L}} \in L^d(B,\mathbf{Z})$, such that

$$\begin{aligned}
p_* \circ p^* &= [F,p]_{\mathbf{L}} \otimes - \ : \ H_n(B;\mathbf{L}^{\cdot}) \longrightarrow H_{n+d}(B;\mathbf{L}^{\cdot}) \ , \\
p_* \circ p^* &= [F,p]_{\mathbf{L}} \otimes - \ : \ VL^n(B) \longrightarrow VL^{n+d}(B) \ , \\
p_* \circ p^* &= [F,p]_{\mathbf{L}} \otimes - \ : \ L_n(\mathbf{Z}[\pi_1(B)]) \longrightarrow L_{n+d}(\mathbf{Z}[\pi_1(B)]) \ .
\end{aligned}$$

This gives an alternative proof of 2.7, at least in outline.

For a PL fibre bundle $F \longrightarrow E \xrightarrow{p} B$ with base an n-dimensional PL manifold B and fibre a d-dimensional PL manifold F the total space is an $(n+d)$-dimensional PL manifold E. The tangent bundle of E is a direct sum $\tau_E = i_*\tau_F \oplus p^*\tau_B$ (as in the differentiable case). Let $(E(i_*\tau_F), S(i_*\tau_F))$ be the $(n+2d)$-dimensional manifold with boundary defined by the total space of the (D^d, S^{d-1})-bundle of the d-plane bundle $i_*\tau_F$ over E. The $\mathbf{L}^{\cdot}$-coefficient Thom class of $i_*\tau_F$ is an element $U_{i_*\tau_F} \in \dot{H}^d(T(i_*\tau_f);\mathbf{L}^{\cdot})$, with $\dot{H}^*$ reduced cohomology and $T(i_*\tau_F) = E(i_*\tau_F)/S(i_*\tau_F)$ the Thom space. The Poincaré duality isomorphism

$$\begin{aligned}
[E(i_*\tau_F)]_{\mathbf{L}} \cap - \ : \ \dot{H}^d(T(i_*\tau_f);\mathbf{L}^{\cdot}) &= H^d(E(i_*\tau_f), S(i_*\tau_F);\mathbf{L}^{\cdot}) \\
&\longrightarrow H_{n+d}(E(i_*\tau_f);\mathbf{L}^{\cdot}) = H_{n+d}(E;\mathbf{L}^{\cdot})
\end{aligned}$$

is such that

$$[E(i_*\tau_F)]_{\mathbf{L}} \cap U_{i_*\tau_F} = [E]_{\mathbf{L}} \in H_{n+d}(E;\mathbf{L}^{\cdot}) \ .$$

The canonical $\mathbf{L}^{\cdot}$-orientation $[E]_{\mathbf{L}}$ of E is the transfer of $[B]_{\mathbf{L}} \in H_n(B;\mathbf{L}^{\cdot})$

$$[E]_{\mathbf{L}} = p^*[B]_{\mathbf{L}} \in H_{n+d}(E;\mathbf{L}^{\cdot}) \ ,$$

and

$$p_*[E]_{\mathbf{L}} = p_*p^*[B]_{\mathbf{L}} = [F,p]_{\mathbf{L}} \cap [B]_{\mathbf{L}} \in H_{n+d}(B;\mathbf{L}^{\cdot})$$

is the Poincaré dual of the fibre transport $[F,p]_{\mathbf{L}} \in H^{-d}(B;\mathbf{L}^{\cdot})$. The transfer map in $\mathbf{L}^{\cdot}$-cohomology

$$p^! \ : \ H^0(E;\mathbf{L}^{\cdot}) \cong H_{n+d}(E;\mathbf{L}^{\cdot}) \xrightarrow{p_*} H_{n+d}(B;\mathbf{L}^{\cdot}) \cong H^{-d}(B;\mathbf{L}^{\cdot})$$

is such that

$$p^!(1) = [F,p]_{\mathbf{L}} \in H^{-d}(B;\mathbf{L}^{\cdot}) \ .$$

This is an L-theoretic analogue of the result of Becker and Schultz ([4], Section 6) that for a differentiable fibre bundle the transfer map $p^! : KO(E) \to KO(B)$ in real K-theory is such that

$$p^!(1) = \sum_i (-1)^i \tilde{B} \times_{\pi_1(B)} H^i(F;\mathbf{R}) \in KO(B)$$

with $\tilde{B}$ the universal cover of B.

Rationally, $\mathbf{L}^{\cdot}$ has the homotopy type of a product of Eilenberg-MacLane spectra

$$\mathbf{L}^{\cdot}\otimes\mathbf{Q} \;=\; \prod_{i=0}^{\infty} K(\mathbf{Q},4i)\ .$$

The canonical $\mathbf{L}^{\cdot}$-orientation $[B]_{\mathbf{L}}\in H_n(B;\mathbf{L}^{\cdot})$ of an oriented n-dimensional manifold B is such that

$$[B]_{\mathbf{L}}\otimes 1 \;=\; \mathcal{L}(B)\cap[B]_{\mathbf{Q}}\in H_n(B;\mathbf{L}^{\cdot})\otimes\mathbf{Q} \;=\; H_{n-4*}(B;\mathbf{Q})$$

with $[B]_{\mathbf{Q}}\in H_n(B;\mathbf{Q})$ the ordinary fundamental class.

A homogenous $2k$-dimensional symmetric Poincaré cocycle (C,ϕ) over a finite simplicial complex B determines a representation of $\pi_1(B)$ on $H^k(C[*];\mathbf{R})$ preserving the nonsingular $(-)^k$-symmetric form over $\mathbf{R}$. Working as in Meyer [20] define a flat vector bundle over B with fibre $H^k(C[*];\mathbf{R})$

$$\Gamma \;=\; \tilde{B}\times_{\pi_1(B)} H^k(C[*];\mathbf{R})$$

equipped with an automorphism $A:\Gamma\to\Gamma$ such that $A^2=(-)^kI$. For $k\equiv 0(\mathrm{mod}\,2)$ the ± 1-eigenspaces of A are the bundles $\Gamma^{\pm}=\tilde{B}\times_{\pi_1(B)}H^k(C[*];\mathbf{R})^{\pm}$ over B with fibres complementary subspaces $H^k(C[*];\mathbf{R})^{\pm}\subseteq H^k(C[*];\mathbf{R})$ where the symmetric form is positive/negative definite, and the real K-theory signature of (C,ϕ) is defined by

$$[C,\phi]_{\mathbf{K}} \;=\; [\Gamma^+]-[\Gamma^-]\in KO(B)\ .$$

For $k\equiv 1(\mathrm{mod}\,2)$ A defines a complex structure on Γ, and the complex K-theory signature of (C,ϕ) is defined by

$$[C,\phi]_{\mathbf{K}} \;=\; [\Gamma^*]-[\Gamma]\in KU(B)$$

with Γ^* the complex conjugate bundle. The topological K-theory signature defines morphisms

$$L^{2k}(B,\mathbf{Z}) \;\to\; K(B)\ ;\ (C,\phi) \;\longrightarrow\; [C,\phi]_{\mathbf{K}}$$

(where $K=KO$ or KU) such that there is defined a commutative diagram

$$\begin{array}{ccc}
L^{2k}(B,\mathbf{Z}) & \longrightarrow & H^{-2k}(B;\mathbf{L}^{\cdot}) \\
\big\downarrow & & \big\downarrow \\
K(B) & \xrightarrow{\widetilde{\mathrm{ch}}} & H^{-2k+2*}(B;\mathbf{Q})\ .
\end{array}$$

Given a PL fibre bundle $F\longrightarrow E\xrightarrow{p} B$ with the fibre a F a $2k$-dimensional manifold let (C,ϕ) be the homogenous $2k$-dimensional symmetric Poincaré cocycle representing the $\mathbf{L}^{\cdot}$-coefficient fibre transport $(F,p)_{\mathbf{L}}\in L^{2k}(B,\mathbf{Z})$ with

$$C[\tau] \;=\; C(p^{-1}(\tau))\ \ (\tau\in B)\ \ ,\ \ H^*(C[*];\mathbf{R}) \;=\; H^*(F;\mathbf{R})\ .$$

The topological K-theory signature of the bundle $\Gamma = \tilde{B} \times_{\pi_1(B)} H^k(F;\mathbf{R})$ over B is denoted by

$$[F,p]_{\mathbf{K}} \ = \ [\Gamma]_{\mathbf{K}} \in K(B) \ .$$

By the above, the rationalization of $[F,p]_{\mathbf{L}} \in H^{-2k}(B;\mathbf{L}^{\boldsymbol{\cdot}})$ is identified with the Chern character of $[F,p]_{\mathbf{K}}$

$$[F,p]_{\mathbf{L}} \otimes 1 \ = \ \widetilde{\mathrm{ch}}([F,p]_{\mathbf{K}}) \in H^{-2k}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes \mathbf{Q} \ = \ H^{-2k+4*}(B;\mathbf{Q}) \ ,$$

and so

$$\begin{aligned} p_* \circ p^* \ &= \ \widetilde{\mathrm{ch}}([F,p]_{\mathbf{K}}) \otimes - \ : \ H_*(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes \mathbf{Q} \ \longrightarrow \ H_{*+2k}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes \mathbf{Q} \ , \\ p_* \circ p^* \ &= \ \widetilde{\mathrm{ch}}([F,p]_{\mathbf{K}}) \otimes - \ : \ VL^*(B) \otimes \mathbf{Q} \ = \ L_*(\mathbf{Z}[\pi_1(B)]) \otimes \mathbf{Q} \\ &\qquad\qquad \longrightarrow \ VL^{*+2k}(B) \otimes \mathbf{Q} \ = \ L_{*+2k}(\mathbf{Z}[\pi_1(B)]) \otimes \mathbf{Q} \ . \end{aligned}$$

Let now $F \longrightarrow E \xrightarrow{p} B$ be a PL fibre bundle with the fibre F a $2k$-dimensional manifold and the base B an n-dimensional manifold, so that the total space E is an $(n+2k)$-manifold. The canonical $\mathbf{L}^{\boldsymbol{\cdot}}$-orientation $[E]_{\mathbf{L}} \in H_{n+2k}(E;\mathbf{L}^{\boldsymbol{\cdot}})$ of E has image $p_*[E]_{\mathbf{L}} \in H_{n+2k}(B;\mathbf{L}^{\boldsymbol{\cdot}})$ with rationalization

$$\begin{aligned} p_*[E]_{\mathbf{L}} \otimes 1 \ &= \ ([F,p]_{\mathbf{L}} \cap [B]_{\mathbf{L}}) \otimes 1 \\ &= \ p_*(\mathcal{L}(E) \cap [E]_{\mathbf{Q}}) \\ &= \ \widetilde{\mathrm{ch}}([F,p]_{\mathbf{K}}) \cap (\mathcal{L}(B) \cap [B]_{\mathbf{Q}}) \in H_{n+2k-*}(B;\mathbf{Q}) \ . \end{aligned}$$

For $n+2k \equiv 0 (\mathrm{mod}\, 4)$ the signature of E is given by

$$\begin{aligned} \mathrm{sign}(E) \ &= \ \langle \mathcal{L}(E), [E]_{\mathbf{Q}} \rangle \\ &= \ \langle \mathcal{L}(E), p^*[B]_{\mathbf{Q}} \rangle \\ &= \ \langle p_*\mathcal{L}(E), [B]_{\mathbf{Q}} \rangle \\ &= \ \langle \widetilde{\mathrm{ch}}([F,p]_{\mathbf{K}}) \cup \mathcal{L}(B), [B]_{\mathbf{Q}} \rangle \\ &= \ \langle \mathrm{ch}([F,p]_{\mathbf{K}}) \cup \widetilde{\mathcal{L}}(B), [B]_{\mathbf{Q}} \rangle \in \mathbf{Z} \end{aligned}$$

as in [3]. The degree 0 component of $\mathrm{ch}([F,p]_{\mathbf{K}}) \in H^{-2k+4*}(B;\mathbf{Q})$ is the rank of the virtual bundle $[F,p]_{\mathbf{K}}$, which is $\mathrm{sign}(F) \in \mathbf{Z} \subset H^0(B;\mathbf{Q}) = \mathbf{Q}$ (to be interpreted as 0 if $k \equiv 1 (\mathrm{mod}\, 2)$). If all the other components are 0 then the signature is multiplicative

$$\mathrm{sign}(E) \ = \ \mathrm{sign}(F)\,\mathrm{sign}(B) \in \mathbf{Z} \ ,$$

and more generally

$$\begin{aligned} p_* \circ p^* \ &= \ \mathrm{sign}(F) \otimes - \ : \ H_*(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes \mathbf{Q} \ \longrightarrow \ H_{*+2k}(B;\mathbf{L}^{\boldsymbol{\cdot}}) \otimes \mathbf{Q} \ , \\ p_* \circ p^* \ &= \ \mathrm{sign}(F) \otimes - \ : \ VL^*(B) \otimes \mathbf{Q} \ = \ L_*(\mathbf{Z}[\pi_1(B)]) \otimes \mathbf{Q} \\ &\qquad\qquad \longrightarrow \ VL^{*+2k}(B) \otimes \mathbf{Q} \ = \ L_{*+2k}(\mathbf{Z}[\pi_1(B)]) \otimes \mathbf{Q} \ . \end{aligned}$$

See Neumann [26] for a class of groups G such that $\mathrm{ch}([F,p]_{\mathbf{K}})$ is concentrated in degree 0 if the action of $\pi_1(B)$ on $H^k(F;\mathbf{C})$ factors as

$$\omega : \pi_1(B) \longrightarrow G \longrightarrow \mathrm{Aut}\, H^k(F;\mathbf{C}) \ .$$

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