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L^2-Invariants: Theory and Applications to Geometry and K-Theory

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Preface

There is the general principle to consider a classical invariant of a closed Riemannian manifold M and to define its analog for the universal covering $\widetilde{M}$ taking the action of the fundamental group $\pi = \pi_1(M)$ on $\widetilde{M}$ into account. Prominent examples are the Euler characteristic and the signature of M, which lead to Wall's finiteness obstruction and to all kinds of surgery obstructions such as the symmetric signature or higher signatures. The *p-th L^2-Betti number* $b_p^{(2)}(\widetilde{M})$ arises from this principle applied to the p-th Betti number $b_p(M)$. Some effort is necessary to define L^2-Betti numbers in the case where π is infinite. Typical problems for infinite π are that $\widetilde{M}$ is not compact and that the complex group ring $\mathbb{C}\pi$ is a complicated ring, in general not Noetherian. Therefore some new technical input is needed from operator theory, namely, the group von Neumann algebra and its trace. Analytically Atiyah defined L^2-Betti numbers in terms of the heat kernel on $\widetilde{M}$. There also is an equivalent combinatorial approach based on the cellular $\mathbb{C}\pi$-chain complex of $\widetilde{M}$. It is one of the main important and useful features of L^2-invariants that they can be defined both analytically and combinatorially. There are two further types of L^2-invariants. *L^2-torsion* generalizes the classical notion of Reidemeister torsion from finite to infinite π, whereas *Novikov-Shubin invariants* do not have a classical counterpart.

A very intriguing and important property of L^2-invariants is that they have relations to many other fields. From their construction it is clear that they have connections to operator theory, in particular to von Neumann algebras, and to the spectral theory of the Laplacian on $\widetilde{M}$. For instance Atiyah's motivation to consider L^2-Betti numbers was to establish his L^2-index theorem.

More suprising is the appearance of algebraic K-theory. In all examples where L^2-Betti numbers have been computed explicitly, the values turn out to be rational numbers whose denominators are linked to the orders of finite subgroups of π. This is very suprising in view of the actual definition of L^2-Betti numbers. This phenomenon is linked to questions in algebraic K-theory such as whether any finitely generated projective $\mathbb{C}\pi$-module M is obtained by induction from a finitely generated projective $\mathbb{C}H$-module for a finite subgroup $H \subset \pi$. This leads to the version of the so called *Atiyah Conjecture* that the L^2-Betti numbers are always integers if π is torsionfree.

It turns out that this conjecture implies the *Kaplansky Conjecture* that $\mathbb{C}\pi$ contains no non-trivial zero-divisors if π is torsionfree. For many groups π the Kaplansky Conjecture was not known until the Atiyah Conjecture was proved. We will investigate interactions between L^2-invariants and K-theory and applications of them in both directions throughout this book.

Next we explain a connection to geometry. Provided that M is aspherical, all computations lead to the result that $b_p^{(2)}(\widetilde{M}) = 0$ holds for $2p \neq \dim(M)$ and that $b_n^{(2)}(\widetilde{M}) = (-1)^n \cdot \chi(M)$ is true for the Euler characteristic $\chi(M)$ if $\dim(M) = 2n$ is even. In particular $(-1)^n \cdot \chi(M) \geq 0$ in the case $\dim(M) = 2n$, since each L^2-Betti number is larger or equal to zero by definition. This phenomenon seems to be typical and will be investigated in this book. Recall that M is aspherical if it carries a Riemannian metric with non-positive sectional curvature, but that the converse is not true. If $\dim(M) = 2n$ and M carries a Riemannian metric with negative sectional curvature, then all computations yield $b_n^{(2)}(\widetilde{M}) = (-1)^n \cdot \chi(M) > 0$. Hence L^2-Betti numbers are linked to the Hopf Conjecture which predicts $(-1)^n \cdot \chi(M) \geq 0$ if the $2n$-dimensional closed manifold M carries a Riemannian metric with non-positive sectional curvature, and $(-1)^n \cdot \chi(M) > 0$ if M carries a Riemannian metric with negative sectional curvature. Further connections between L^2-invariants and geometry and group theory will be presented in this book.

Why Study L^2-Invariants?

From the author's point of view there are certain criteria which decide whether a topic or an area in modern mathematics is worth studying or worth further development. Among them are the following:

- The topic has relations to other fields. There is a fruitful exchange of results and techniques with other areas which leads to solutions of problems and to innovations in both the topic of focus and other topics;
- There are some hard open problems which are challenging and promising. They create interesting activity and partial solutions and techniques for their proof already have applications to other problems;
- The topic is accessible with a reasonable amount of effort. In particular talented students are able to learn the basics of the topic within an appropriate period of time and while doing so get a broad basic education in mathematics.

The purpose of this book is to convince the reader that L^2-invariants do satisfy these criteria and to give a comprehensible and detailed approach to them which includes the most recent developments.

A User's Guide

We have tried to write this book in a way which enables the reader to pick out his favourite topic and to find the result she or he is interested in quickly and without being forced to go through other material. The various chapters are kept as independent of one another as possible. In the introduction of each chapter we state what input is needed from the previous chapters, which is in most cases not much, and how to browse through the chapter itself. It may also be worthwhile to go through the last section "Miscellaneous" in each chapter which contains some additional information. In general a first impression can be gained by just reading through the definitions and theorems themselves. Of course one can also read the book linearly.

Each chapter includes exercises. Some of them are easy, but some of them are rather difficult. Hints to their solutions can be found in Chapter 16. The exercises contain interesting additional material which could not be presented in detail in the text. The text contains some (mini) surveys about input from related material such as amenable groups, the Bass Conjecture, deficiency of groups, Isomorphism Conjectures in K-theory, 3-manifolds, Ore localization, residually finite groups, simplicial volume and bounded cohomology, symmetric spaces, unbounded operators, and von Neumann regular rings, which may be useful by themselves. (They are listed in the index under "survey". One can also find a list of all conjectures, questions and main theorems in the index.)

If one wants to run a seminar on the book, one should begin with Sections 1.1 and 1.2. Then one can continue depending on the own interest. For instance if one is algebraically oriented and not interested in the analysis, one may directly pass to Chapter 6, whereas an analyst may be interested in the rest of Chapter 1 and then pass to Chapter 2. Chapters 9, 10, 11, 12, 13 and 14 are independent of one another. One may directly approach these chapters and come back to the previous material when it is cited there.

We require that the reader is familiar with basic notions in topology (CW-complexes, chain complexes, homology, manifolds, differential forms, coverings), functional analysis (Hilbert spaces, bounded operators), differential geometry (Riemannian metric, sectional curvature) and algebra (groups, modules, elementary homological algebra).

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Münster, February 2002 *Wolfgang Lück*

Contents

0. Introduction

0.1 What are L^2-Invariants?

There is the classical notion of the p-th Betti number $b_p(X)$ of a finite CW-complex X, for instance a closed manifold, which is the dimension of the complex vector space $H_p(X;\mathbb{C})$. Consider a G-covering $p\colon \overline{X} \to X$. If G is infinite, the p-th Betti number of $\overline{X}$ may be infinite and hence useless. Using some input from functional analysis involving Hilbert spaces, group von Neumann algebras and traces one can define the *p-th L^2-Betti number* $b_p^{(2)}(\overline{X};\mathcal{N}(G))$ of the total space $\overline{X}$ as the non-negative real number given by the von Neumann dimension of the (reduced) L^2-homology of $\overline{X}$. (Often we briefly write $b_p^{(2)}(\overline{X})$ if G is clear from the context.) If G is finite, $b_p^{(2)}(\overline{X}) = |G|^{-1} \cdot b_p(X)$ and we get nothing new. But L^2-Betti numbers carry new information and have interesting applications in the case where G is infinite. In general $b_p^{(2)}(\overline{X})$ of the total space $\overline{X}$ and $b_p(X)$ of the base space X have no relations except for the Euler-Poincaré formula, namely,

$$\chi(X) \;=\; \textstyle\sum_{p\geq 0}(-1)^p \cdot b_p(X) \;=\; \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\overline{X}), \qquad (0.1)$$

where $\chi(X)$ is the Euler characteristic of X (see Section 0.6).

The notion of the classical Reidemeister torsion of $\overline{X}$ for finite groups G will be generalized to the notion of *L^2-torsion* $\rho^{(2)}(\overline{X}) \in \mathbb{R}$ in the case that G is infinite.

There is a third class of L^2-invariants, the *Novikov-Shubin invariants* $\alpha_p(\overline{X})$, which carry no information if G is finite.

All these types of L^2-invariants on the one hand have analytic definitions in terms of the heat kernel on $\overline{X}$, but on the other hand can be defined combinatorially in terms of the cellular $\mathbb{C}G$-chain complex of $\overline{X}$. These two approaches are equivalent. In the analytic context X must be a compact Riemannian manifold. For the combinatorial definition of L^2-Betti numbers and Novikov-Shubin invariants it suffices to require that the base space X is of finite type, i.e. each skeleton of X is finite, but X may be infinite-dimensional.

0.2 Some Applications of L^2-Invariants

In order to convince the reader about the potential of L^2-invariants we state some results which seem to have nothing to do with L^2-invariants but whose proofs — as we will see — use L^2-methods. The selection below consists of some easy to formulate examples and is not meant to represent the most important results about L^2-invariants. There are plenty of other very interesting and important theorems about L^2-invariants, a lot of which will be presented in this book. For simplicity we often will not state the most general formulations in this introduction. All notions appearing in the list of theorems below will be explained in the relevant chapters. The results below are due to Chang-Weinberger, Cheeger-Gromov, Cochran-Orr-Teichner, Dodziuk, Gaboriau, Gromov and Lück.

Theorem 0.2 (see Theorem 1.35 (2) **and Corollary 6.75).** *Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite CW-model for its classifying space BG. Then its Euler characteristic vanishes, i.e.*

$$\chi(G) := \chi(BG) = 0.$$

Theorem 0.3 (see Theorem 1.62 and Theorem 11.6). *Let M be a closed manifold of even dimension $2m$. Suppose that M is hyperbolic, or more generally, that its sectional curvature satisfies $-1 \leq \sec(M) < -(1-\frac{1}{m})^2$ .Then*

$$(-1)^m \cdot \chi(M) > 0.$$

Theorem 0.4 (see Theorem 11.14 and Theorem 11.15). *Let M be a closed Kähler manifold of (real) dimension $2m$. Suppose that M is homotopy equivalent to a closed Riemannian manifold with negative sectional curvature. Then*

$$(-1)^m \cdot \chi(M) > 0.$$

Moreover, M is a projective algebraic variety and is Moishezon and Hodge.

Theorem 0.5 (see Theorem 7.25). *Let $1 \to H \to G \to K \to 1$ be an extension of infinite groups such that H is finitely generated and G is finitely presented. Then*

(1) The deficiency of G satisfies $\operatorname{def}(G) \leq 1$;
(2) If M is a closed connected oriented 4-manifold with $\pi_1(M) \cong G$, then we get for its signature $\operatorname{sign}(M)$ and its Euler characteristic $\chi(M)$

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Theorem 0.6 (see Theorem 9.38). *Let $i\colon H \to G$ be the inclusion of a normal finite subgroup H into an arbitrary group G. Then the maps coming from i and the conjugation action of G on H*

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) &\to \mathrm{Wh}(G); \\ \mathrm{Wh}(H)^G &\to \mathrm{Wh}(G) \end{aligned}$$

have finite kernel, where Wh *denotes the Whitehead group.*

Theorem 0.7 (see Theorem 9.66). *Let G be a group and $\mathbb{C}G$ be its complex group ring. Let $G_0(\mathbb{C}G)$ be the Grothendieck group of finitely generated (not necessarily projective) $\mathbb{C}G$-modules. Then*

(1) If G is amenable, the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ is an element of infinite order;
(2) If G contains the free group $\mathbb{Z} * \mathbb{Z}$ of rank two, then $[\mathbb{C}G] = 0$ in $G_0(\mathbb{C}G)$.

Theorem 0.8 (see Section 15.4). *There are non-slice knots in 3-space whose Casson-Gordon invariants are all trivial.*

Theorem 0.9 (see Section 7.5). *There are finitely generated groups which are quasi-isometric but not measurably equivalent.*

Theorem 0.10 (see Section 15.1). *Let M^{4k+3} be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then there are infinitely many smooth manifolds which are homotopy equivalent to M (and even simply and tangentially homotopy equivalent to M) but not homeomorphic to M.*

0.3 Some Open Problems Concerning L^2-Invariants

The following conjectures will be treated in detail in Section 2.5 and Chapters 10, 11, 12, 13 and 14. They have created a lot of activity. This book contains proofs of these conjectures in special cases which rely on general methods and give some structural insight or consist of explicit computations. Recall that a free G-CW-complex X is the same as the total space of a G-covering $X \to G\backslash X$ with a CW-complex $G\backslash X$ as base space, and that X is called finite or of finite type if the CW-complex $G\backslash X$ is finite or of finite type.

Conjecture 0.11 (Strong Atiyah Conjecture). *Let X be a free G-CW-complex of finite type. Denote by $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $|H|^{-1}$, where H runs through the finite subgroups of G. Then we get for the L^2-Betti numbers of X*

$$b_p^{(2)}(X) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

In Subsection 10.1.4 we will explain that there are counterexamples to the strong Atiyah Conjecture 0.11 due to Grigorchuk and Żuk, but no counterexample is known to the author at the time of writing if one replaces $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ by $\mathbb{Q}$ or if one assumes that there is an upper bound for the orders of finite subgroups of G. The author is not aware of a counterexample to the following conjectures at the time of writing.

Conjecture 0.12. (Positivity and rationality of Novikov-Shubin invariants). *Let X be a free G-CW-complex of finite type. Then its Novikov-Shubin invariants $\alpha_p(X)$ are positive rational numbers unless they are ∞ or ∞^+.*

Conjecture 0.13 (Singer Conjecture). *Let M be an aspherical closed manifold. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ satisfy*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \qquad \textit{if } 2p \neq \dim(M)$$

and $(-1)^m \cdot \chi(M) \geq 0$ if $\dim(M) = 2m$ is even.

Let M be a closed connected Riemannian manifold with negative sectional curvature. Then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2p \neq \dim(M); \\ > 0 & \textit{if } 2p = \dim(M), \end{cases}$$

and $(-1)^m \cdot \chi(M) > 0$ if $\dim(M) = 2m$ is even.

Conjecture 0.14 (L^2-torsion for aspherical manifolds). *If M is an aspherical closed manifold of odd dimension $2m+1$, then the L^2-torsion of its universal covering satisfies*

$$(-1)^m \cdot \rho^{(2)}(\widetilde{M}) \;\geq\; 0.$$

If M is a closed connected Riemannian manifold of odd dimension $2m+1$ with negative sectional curvature, then

$$(-1)^m \cdot \rho^{(2)}(\widetilde{M}) \;>\; 0.$$

If M is an aspherical closed manifold whose fundamental group contains an amenable infinite normal subgroup, then

$$\rho^{(2)}(\widetilde{M}) \;=\; 0.$$

Conjecture 0.15 (Zero-in-the-spectrum Conjecture). *Let $\widetilde{M}$ be the universal covering of an aspherical closed Riemannian manifold M. Then for some $p \geq 0$ zero is in the spectrum of the minimal closure*

$$(\Delta_p)_{\min}\colon \operatorname{dom}\left((\Delta_p)_{\min}\right) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Conjecture 0.16 (Approximation Conjecture). *Let G be a group. Let $\{G_i \mid i \in I\}$ be an inverse system of normal subgroups of G directed by inclusion over the directed set I. Suppose that $\cap_{i\in I} G_i = \{1\}$. Let X be a free G-CW-complex of finite type. Then $G_i\backslash X$ is a free G/G_i-CW-complex of finite type and*

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; \lim_{i\in I} b_p^{(2)}(G_i\backslash X;\mathcal{N}(G/G_i)).$$

Conjecture 0.17 (Simplicial volume and L^2-invariants). *Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then all the L^2-Betti numbers and the L^2-torsion of the universal covering $\widetilde{M}$ vanish, i.e.*

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad \textit{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

0.4 L^2-Invariants and Heat Kernels

The *p-th L^2-Betti number* $b_p^{(2)}(\overline{M})$ of a G-covering $p\colon \overline{M} \to M$ of a closed Riemannian manifold M was first defined by Atiyah [9, page 71] in connection with his L^2-index theorem. By means of a Laplace transform, Atiyah's original definition agrees with the one given by the non-negative real number

$$b_p^{(2)}(\overline{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\, d\mathrm{vol}\,. \tag{0.18}$$

Here $\mathcal{F}$ is a fundamental domain for the G-action on $\overline{M}$ and $e^{-t\Delta_p}(x,y)$ is the heat kernel on p-forms on $\overline{M}$. The p-th L^2-Betti number $b_p^{(2)}(\overline{M})$ measures the size of the kernel of the Laplacian acting on smooth p-forms on $\overline{M}$. If G is trivial, then $b_p^{(2)}(\overline{M})$ is the same as the ordinary Betti number $b_p(M)$ which is the real dimension of the p-th singular cohomology with real coefficients of M. One important consequence of the L^2-index theorem is the Euler-Poincaré formula (0.1) (see Theorem 1.35 (2)).

The *p-th Novikov-Shubin invariant* $\alpha_p^{\Delta}(\overline{M})$ measures how fast the expression $\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\, d\mathrm{vol}$ approaches its limit $b_p^{(2)}(\overline{M})$ for $t \to \infty$ (see (0.18)). The larger $\alpha_p^{\Delta}(\overline{M})$ is, the "thinner" is the spectrum of the p-th Laplacian on $\overline{M}$ at zero.

Notice that the L^2-Betti numbers and the Novikov-Shubin invariants are invariants of the large time asymptotics of the heat kernel and hence invariants of the global geometry, in contrast to invariants of the small time asymptotics, such as indices of operators, which are of local nature. For instance the Novikov-Shubin invariant associated to the Laplacian acting on

0-forms of the universal covering of a closed Riemannian manifold M is determined by group theoretic properties of the fundamental group $\pi_1(M)$ such as its growth rate or the question whether it is amenable (see Theorem 2.55 (5)).

In view of the definitions of the L^2-Betti numbers and Novikov-Shubin invariants, the strong Atiyah Conjecture 0.11 and the Conjecture 0.12 about the positivity and rationality of Novikov-Shubin invariants are very surprising. Some explanation for the strong Atiyah Conjecture 0.11 comes from connections with algebraic K-theory, whereas the only evidence for the Conjecture 0.12 about the positivity and rationality of Novikov-Shubin invariants is based on computations, and no conceptual reasons are known.

The third important L^2-invariant is the L^2-*torsion* $\rho^{(2)}(\overline{M})$ which was introduced by Carey-Mathai, Lott, Lück-Rothenberg, Mathai and Novikov-Shubin. It is only defined under a certain technical assumption, namely, that $\overline{M}$ is of determinant class. This condition is conjecturally always satisfied and we will suppress it in this discussion. If all L^2-Betti numbers of $\overline{M}$ vanish, the L^2-torsion $\rho^{(2)}(\overline{M})$ is independent of the Riemannian metric and depends only on the simple homotopy type. Actually, there is the conjecture that it depends only on the homotopy type (see Conjecture 3.94). Its analytic definition is complicated.

This analytic approach via the heat kernel is important in the following situations. One can compute the L^2-Betti numbers of the universal covering $\widetilde{M}$ of a closed Riemannian manifold M if M is hyperbolic (see Theorem 1.62), or, more generally, satisfies certain pinching conditions (see Theorem 11.4, Theorem 11.5 and Theorem 11.6). There are explicit computations of the L^2-Betti numbers, the Novikov-Shubin invariants and the L^2-torsion of the universal covering of a closed manifold M if M is a locally symmetric space (see Theorem 5.12 and Section 5.4). The proof of the Proportionality Principle 3.183 relies on the analytic description. The proofs of these facts do not have combinatorial counterparts.

0.5 L^2-Invariants and Cellular Chain Complexes

One important feature of all these L^2-invariants is that they can also be defined for a G-covering $p\colon \overline{X} \to X$ of a finite CW-complex X in terms of the cellular $\mathbb{Z}G$-chain complex $C_*(\overline{X})$. For L^2-Betti numbers and Novikov-Shubin invariants it suffices to require that X is of finite type. The associated L^2-*chain complex* $C_*^{(2)}(\overline{X})$ is defined by $l^2(G)\otimes_{\mathbb{Z}G}C_*(\overline{X})$. Each chain module $C_*^{(2)}(\overline{X})$ is a Hilbert space with isometric G-action of the special form $l^2(G)^n$, where $l^2(G)^n$ is the n-fold sum of the Hilbert space $l^2(G)$. Each differential $c_p^{(2)}$ is a bounded G-equivariant operator. The *p-th* L^2-*homology* $H_p^{(2)}(\overline{X})$ is defined to be the quotient of the kernel of $c_p^{(2)}$ by the closure of the image of $c_{p+1}^{(2)}$. Dividing out the closure of the image has the effect that $H_p^{(2)}(\overline{X})$ is again a Hilbert space with isometric G-action. It actually comes with the structure of a *finitely generated Hilbert $\mathcal{N}(G)$-module*, where $\mathcal{N}(G)$ denotes the *von Neumann algebra of the group G*. This additional structure allows to define the *von Neumann dimension* of $H_p^{(2)}(\overline{X})$. Dodziuk has shown that this non-negative real number agrees with $b_p^{(2)}(\overline{X})$ as defined in (0.18) (see Theorem 1.59 and (1.60)). One can also read off the Novikov-Shubin invariants and the L^2-torsion from $C_*^{(2)}(\overline{X})$ by results of Efremov (see Theorem 2.68) and Burghelea-Friedlander-Kappeler-McDonald (see Theorem 3.149). The p-th Novikov-Shubin invariant $\alpha_p(\overline{X})$ measures the difference between the image of $c_p^{(2)}$ and the closure of the image of $c_p^{(2)}$.

The point of this cellular description is that it is much easier to handle and calculate than the analytic counterpart. For instance one can show homotopy invariance of L^2-Betti numbers, Novikov-Shubin invariants and L^2-torsion and prove some very useful formulas like sum formulas, product formulas, fibration formulas and so on using the combinatorial approach (see Theorem 1.35, Theorem 2.55, Theorem 3.93, Theorem 3.96 and Theorem 3.100). The combinatorial approach allows to show for an aspherical closed manifold M that all L^2-Betti numbers and the L^2-torsion of its universal covering vanish provided M carries a non-trivial S^1-action (see Theorem 3.105). There exists a combinatorial proof that all L^2-Betti numbers of the universal covering of a mapping torus of a self map of a CW-complex of finite type vanish (see Theorem 1.39). No analytic proofs or no simpler analytic proofs of these results are known to the author. The combination of the analytic and combinatorial methods yields a computation of the L^2-invariants of the universal covering of a compact 3-manifold provided Thurston's Geometrization Conjecture holds for the pieces appearing in the prime decomposition of M (see Theorem 4.1, Theorem 4.2 and Theorem 4.3).

For a kind of algorithmic computation of L^2-invariants based on the combinatorial approach we refer to Theorem 3.172.

The possibility to take both an analytic and a combinatorial point of view is one of the main reasons why L^2-invariants are so powerful.

0.6 L^2-Betti Numbers and Betti Numbers

Let $\widetilde{X} \to X$ be the universal covering of a connected CW-complex X of finite type. Then the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ of $\widetilde{X}$ and the (classical) Betti numbers $b_p(X)$ share some basic properties such as homotopy invariance, the Euler-Poincaré formula, Poincaré duality, Morse inequalities, Künneth formulas and so on, just replace in the corresponding statement for the classical Betti numbers $b_p(X)$ by $b_p^{(2)}(\widetilde{X})$ everywhere (see Theorem 1.35). There is also an L^2-Hodge de Rham Theorem 1.59 which is one important input in the proof of Theorem 0.3.

But there are also differences. One important extra feature of the L^2-Betti numbers is that they are multiplicative under finite coverings in the following sense. If $p\colon Y \to X$ is a finite d-sheeted covering, then $b_p^{(2)}(\widetilde{Y}) = d \cdot b_p^{(2)}(\widetilde{X})$ (see Theorem 1.35 (9)). This implies for instance $b_p^{(2)}(\widetilde{S^1}) = 0$ for all $p \geq 0$ since there is a d-sheeted covering $S^1 \to S^1$ for $d \geq 2$. The corresponding statement is not true for the Betti numbers. This is one reason why L^2-Betti numbers more often tend to be zero than the classical Betti numbers. Often this is the key phenomenon for applications. Another reason for it is the fact that $b_0^{(2)}(\widetilde{X})$ is 0 if $\pi_1(X)$ is infinite and is $|\pi_1(X)|^{-1}$ if $\pi_1(X)$ is finite (see Theorem 1.35 (8)), whereas $b_0(X)$ is always 1.

If $\pi_1(X)$ is finite, then $b_p^{(2)}(\widetilde{X}) = |\pi_1(X)|^{-1} \cdot b_p(\widetilde{X})$. If $\pi_1(X)$ is infinite, the only general relation between the L^2-Betti numbers of $\widetilde{X}$ and the Betti numbers of X is the Euler-Poincaré formula (0.1). Given an integer $l \geq 1$ and a sequence r_1, r_2, ..., r_l of non-negative rational numbers, we construct in Example 1.38 a group G such that BG is of finite type and

$$\begin{aligned} b_p^{(2)}(G) &:= b_p^{(2)}(EG) = \begin{cases} r_p & \text{for } 1 \leq p \leq l; \\ 0 & \text{for } l+1 \leq p; \end{cases} \\ b_p(G) &:= b_p(BG) \quad = 0 \qquad\qquad \text{for } p \geq 1. \end{aligned}$$

On the other hand we can construct for any sequence n_1, n_2, ... of non-negative integers a CW-complex X of finite type such that $b_p(X) = n_p$ and $b_p^{(2)}(\widetilde{X}) = 0$ hold for $p \geq 1$.

However, there is an asymptotic relation between the L^2-Betti numbers of $\widetilde{X}$ and the Betti numbers of X. Recall that the Betti numbers are not multiplicative. One may try to force multiplicativity of the Betti numbers by stabilizing under finite coverings as follows. Suppose that $\pi_1(X)$ possesses a nested sequence of normal subgroups of finite index

$$\pi_1(X) = G_0 \supset G_1 \supset G_2 \supset G_3 \supset \ldots$$

with $\cap_{i=0}^{\infty} G_i = \{1\}$. Then $G_i\backslash\widetilde{X}$ is a CW-complex of finite type and there is a $[G : G_i]$-sheeted covering $G_i\backslash\widetilde{X} \to X$. One may consider $\lim_{i\to\infty} \frac{b_p(G_i\backslash\widetilde{X})}{[G:G_i]}$. This expression is automatically multiplicative if the limit exists and is independent of the nested sequence. Actually it turns out that this is true and

$$\lim_{i\to\infty} \frac{b_p(G_i\backslash\widetilde{X})}{[G : G_i]} \;=\; b_p^{(2)}(\widetilde{X}).$$

This result is a special case of the Approximation Conjecture 0.16 which will be investigated in Chapter 13.

0.7 L^2-Invariants and Ring-Theory

A more algebraic approach will be presented in Chapter 6. It will enable us to define L^2-Betti numbers for arbitrary G-spaces and in particular for groups without any restrictions on BG. This allows to apply standard techniques of algebraic topology and homological algebra directly to L^2-Betti numbers. The idea is to view the group von Neumann algebra $\mathcal{N}(G)$ just as a ring forgetting the functional analysis and the topology. The von Neumann algebra $\mathcal{N}(G)$ has zero-divisors and is not Noetherian unless G is finite. This makes $\mathcal{N}(G)$ complicated as a ring. But it has one very nice property, it is *semihereditary*, i.e. any finitely generated submodule of a projective module is itself projective (see Theorem 6.5 and Theorem 6.7 (1)). This justifies the slogan that $\mathcal{N}(G)$ behaves like the ring $\mathbb{Z}$ if one ignores the facts that $\mathbb{Z}$ has no zero-divisors and is Noetherian. The main input for the ring-theoretic approach is the construction of a dimension function for arbitrary modules over the group von Neumann algebra $\mathcal{N}(G)$ (Theorem 6.7). It is uniquely characterized by the condition that it satisfies Additivity, Continuity and Cofinality and extends the classical dimension function for finitely generated projective modules which is defined in terms of the von Neumann trace of idempotents in $M_n(\mathcal{N}(G))$. One applies it to the $\mathcal{N}(G)$-modules $H_p(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X))$ for a G-space X and gets an extension of the notion of L^2-Betti numbers to arbitrary G-spaces if one allows the value ∞. The second key result is that for amenable G the von Neumann algebra $\mathcal{N}(G)$ looks like a flat $\mathbb{C}G$-module from the point of view of dimension theory (see Theorem 6.37).

In Chapter 8 we introduce the algebra $\mathcal{U}(G)$ of operators affiliated to the group von Neumann algebra. From an algebraic point of view $\mathcal{U}(G)$ can be described as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative set of non-zero divisors. The main ring theoretic property of $\mathcal{U}(G)$ is that it is *von Neumann regular* (see Theorem 8.22 (3)) which is a stronger property than to be semihereditary. The dimension theory of $\mathcal{N}(G)$ extends to $\mathcal{U}(G)$ (see Theorem 8.29). The relation of $\mathcal{U}(G)$ to $\mathcal{N}(G)$ is analogous to the relation of $\mathbb{Q}$ to $\mathbb{Z}$.

From the point of view of representation theory of finite groups the passage from $\mathbb{C}G$ to $\mathcal{N}(G)$ is the natural one for infinite groups. Namely, two finitely generated projective $\mathcal{N}(G)$-modules P and Q are $\mathcal{N}(G)$-isomorphic if and only if their center valued von Neumann dimensions $\dim^u_{\mathcal{N}(G)}(P)$ and $\dim^u_{\mathcal{N}(G)}(Q)$ agree (see Theorem 9.13). If G is finite, this reduces to the well-known theorem that two complex finite-dimensional G-representations are isomorphic if and only if they have the same character.

This algebraic approach may be preferred by algebraists who do not have much background in (functional) analysis.

Linnell's Theorem 10.19 says that the strong Atiyah Conjecture 0.11 is true for a class of groups $\mathcal{C}$ which contains all extensions of free groups with elementary amenable groups as quotients, provided that there is an upper bound on the orders of finite subgroups. Its proof is based on techniques from ring theory, in particular localization techniques, and from K-theory. The following square of inclusions of rings plays an important role as explained below

$$\begin{array}{ccc} \mathbb{C}G & \longrightarrow & \mathcal{N}(G) \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array} \tag{0.19}$$

where $\mathcal{D}(G)$ denotes the *division closure* of $\mathbb{C}G$ in $\mathcal{U}(G)$.

0.8 L^2-Invariants and K-Theory

The strong Atiyah Conjecture 0.11 is related to K-theory in the following way. It is equivalent to the statement that for any finitely presented $\mathbb{C}G$-module M the generalized dimension $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M)$ (see Theorem 6.5 and Theorem 6.7 (1)) of the $\mathcal{N}(G)$-module $\mathcal{N}(G) \otimes_{\mathbb{C}G} M$ takes values in $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ (see Lemma 10.7). Notice that any non-negative real number occurs as $\dim_{\mathcal{N}(G)}(P)$ for a finitely generated projective $\mathcal{N}(G)$-module P, if G contains $\mathbb{Z}$ as subgroup (see Example 1.11, Theorem 6.24 (4) and Theorem 6.29 (2)). So the point is to understand the passage from $\mathbb{C}G$ to $\mathcal{N}(G)$, not only to investigate modules over $\mathcal{N}(G)$.

One may first consider the weaker statement that for any finitely generated *projective* $\mathbb{C}G$-module M the generalized dimension $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M)$ takes values in $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$. This is equivalent to the statement that the composition $K_0(\mathbb{C}G) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathbb{R}$ must have its image in $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$, where i is the change of rings map. This is certainly true for the composition

$$\bigoplus_{\substack{H \subset G \\ |H| < \infty}} K_0(\mathbb{C}H) \xrightarrow{a} K_0(\mathbb{C}G) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathbb{R}$$

where a is the sum of the various change of rings maps. The Isomorphism Conjecture 9.40 for $K_0(\mathbb{C}G)$ implies that a is surjective and hence that the image of $K_0(\mathbb{C}G) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\dim_{\mathcal{N}(G)}} \mathbb{R}$ is contained in $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$.

The proof of Linnell's Theorem 10.19 can be split into two parts, a ring-theoretic one and a K-theoretic one. Namely, one proves that any finitely presented $\mathbb{C}G$-module becomes finitely generated projective over the ring $\mathcal{D}(G)$ (see (0.19)) and that the composition

$$\bigoplus_{\substack{H \subset G \\ |H| < \infty}} K_0(\mathbb{C}H) \xrightarrow{a} K_0(\mathbb{C}G) \xrightarrow{j} K_0(\mathcal{D}(G))$$

for j the change of rings map is surjective (see Section 10.2). Then the claim follows from (0.19) and the facts that the change of rings homomorphism

$K_0(\mathcal{N}(G)) \to K_0(\mathcal{U}(G))$ is bijective (see Theorem 9.20 (1)) and that the dimension function $\dim_{\mathcal{N}(G)}$ for $\mathcal{N}(G)$ extends to a dimension function $\dim_{\mathcal{U}(G)}$ for $\mathcal{U}(G)$ satisfying $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M) = \dim_{\mathcal{N}(G)}(M)$ for any $\mathcal{N}(G)$-module M (see Theorem 8.29).

The extension of the dimension function to arbitrary modules has some applications to G-theory of $\mathbb{C}G$ as already mentioned in Theorem 0.7 (see Subsection 9.5.3). Computations of the middle K-theory and of the L-theory of von Neumann algebras and the associated algebras of affiliated operators are presented in Chapter 9. L^2-methods also lead to results about the Whitehead group $\mathrm{Wh}(G)$ (see Theorem 0.6) and some information about the Bass Conjecture (see Subsection 9.5.2). The question whether the L^2-torsion in the L^2-acyclic case is a homotopy invariant is equivalent to the question whether the map induced by the Fuglede-Kadison determinant $\mathrm{Wh}(G) \to \mathbb{R}$ is trivial (see Conjecture 3.94). This question is related to the Approximation Conjecture 0.16 by the Determinant Conjecture 13.2 (see Lemma 13.6 and Theorem 13.3 (1)). The Approximation Conjecture 0.16 also plays a role in proving that the class of groups for which the strong Atiyah Conjecture 0.11 is true is closed under direct and inverse limits (see Theorem 10.20).

0.9 L^2-Invariants and Aspherical Manifolds

Let M be an aspherical closed manifold, for instance a closed Riemannian manifold with non-positive sectional curvature. Then the Singer Conjecture 0.13, Conjecture 0.14 about L^2-torsion for aspherical manifolds and the zero-in-the-spectrum Conjecture 0.15 put some restrictions on the L^2-invariants of its universal covering. There are special cases where these conjectures have been proved by computations. For instance if M is a compact 3-manifold (see Chapter 4), a locally symmetric space (see Corollary 5.16) or carries a Riemannian metric whose sectional curvature satisfies certain pinching conditions (see Theorem 11.4, Theorem 11.5 and Theorem 11.6). They also have been proved under additional assumptions like the existence of a non-trivial S^1-action (see Theorem 3.105), the existence of the structure of a Kähler hyperbolic manifold (see Theorem 11.14) or the existence of a normal infinite (elementary) amenable subgroup of $\pi_1(X)$ (see Theorem 3.113 and Theorem 7.2). But it is still very mysterious why Poincaré duality together with asphericity may have such implications, or what kind of mechanism is responsible for these phenomenons. The status of Conjecture 0.17 about simplicial volume and L^2-invariants is similar. Conjectures 0.13, 0.14, 0.15 and 0.17 become false if one drops the condition that M is aspherical. Without this assumption it is easy to construct counterexamples to all but the zero-in-the-spectrum Conjecture 0.15. Counterexamples in the non-aspherical case to the zero-in-the-spectrum Conjecture 0.15 are presented by Farber-Weinberger [187] (see also [258]). We will deal with them in Section 12.3.

0.10 L^2-Invariants and Groups

L^2-Betti numbers $b_p^{(2)}(G)$ (and also Novikov-Shubin invariants $\alpha_p(G)$) can be defined for arbitrary (discrete) groups if one allows the value ∞. In Chapter 7 the L^2-Betti numbers of groups are investigated and in particular the question when they vanish is studied. The vanishing of all L^2-Betti numbers of G implies the vanishing of the L^2*-Euler characteristic* $\chi^{(2)}(G)$ of G. The notion of L^2-Euler characteristic agrees with the classical notion of Euler characteristic $\chi(BG)$ (or more generally the virtual Euler characteristic) if the latter is defined. Actually Theorem 0.2 is proved by showing that all L^2-Betti numbers of a group G vanish if G contains a normal infinite amenable subgroup. This example shows that it is important to extend the definition of L^2-Betti numbers from those groups for which BG is finite to arbitrary groups even if one may only be interested in groups with finite BG. Namely, if G has a finite model for BG, this does not mean that a normal subgroup $H \subset G$ has a model of finite type for BH. The vanishing of the first L^2-Betti number $b_1^{(2)}(G)$ has consequences for the deficiency of the group. The hard part of the proof of Theorem 0.5 is to show the vanishing of $b_1^{(2)}(G)$, then the claim follows by elementary considerations.

We show in Theorem 7.10 that all L^2-Betti numbers of Thompson's group F vanish. This is a necessary condition for F to be amenable. The group F cannot be elementary amenable and does not contain $\mathbb{Z} * \mathbb{Z}$ as subgroup but (at the time of writing) it is not known whether F is amenable or not.

In Section 7.4 a number $\rho^{(2)}(f) \in \mathbb{R}$ is associated to an automorphism $f\colon G \to G$ of a group G provided that BG has a finite model. One also needs the technical assumption of det $\geq$ 1-class which is conjecturally always true and proved for a large class of groups and will be suppressed in the following discussion. This invariant has nice properties such as the trace property $\rho^{(2)}(g \circ f) = \rho^{(2)}(f \circ g)$ and multiplicativity $\rho^{(2)}(f^n) = n \cdot \rho^{(2)}(f)$ and satisfies a sum formula $\rho^{(2)}(f_1 *_{f_0} f_2) = \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0)$ (see Theorem 7.27). If $f = \pi_1(g)$ for an automorphism $g\colon F \to F$ of a compact orientable 2-dimensional manifold F different from S^2, D^2 and T^2, then $\rho^{(2)}(f)$ is, up to a constant, the sum of the volumes of the hyperbolic pieces appearing in the Jaco-Shalen-Johannson-Thurston decomposition of the mapping torus of g along tori into Seifert pieces and hyperbolic pieces (see Theorem 7.28). If F is closed and g is irreducible, then $\rho^{(2)}(g) = 0$ if and only if g is periodic, and $\rho^{(2)}(g) \neq 0$ if and only if g is pseudo-Anosov.

In Section 7.5 the question is discussed whether or not the L^2-Betti numbers, Novikov-Shubin invariants and the L^2-torsion are quasi-isometry invariants or invariants of the measure equivalence class of a countable group G. Theorem 0.9 is one of the main applications of L^2-Betti numbers to measurable equivalence.