# ALGEBRAIC TOPOLOGY I + II (WS 24/25 AND SS 25)

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Abstract. This manuscript is based on the lecture courses *Algebraic Topology I* from the winter term 24/25 and *Algebraic Topology II* from the summer term 25.

## 1. Introduction

This manuscript is based on the lecture courses *Algebraic Topology I* from the winter term 24/25 and *Algebraic Topology II* from the summer term 25.

### 1.1. Prerequisites.

- Topological spaces;
- CW-complexes;
- Coverings;
- Chain complexes and modules over a ring;
- Singular and cellular (co-)homology;
- Basics about smooth manifolds;
- Basics about bundles and vector bundles;

## Contents


1. Introduction 1
1.1. Prerequisites 1
2. Basic definitions and properties of homotopy groups 3
2.1. Review of the fundamental group 3
2.2. Basic definitions and the group structure on homotopy groups 5
2.3. Functorial properties of homotopy groups 7
2.4. Homotopy groups and coverings 8
2.5. The long exact sequence of a pair and a triple 9
2.6. Connectivity 12
2.7. Homotopy groups and colimits 13
3. Hopf's Degree Theorem 13
3.1. Some basics about differential topology and the mapping degree 13
3.2. The proof of Hopf's Degree Theorem 14
3.3. The homotopy groups of the $n$-sphere in the degree $\leq n$ 19
4. The Cellular Approximation Theorem 20
5. The Whitehead Theorem 23
6. CW-Approximation 28
7. The category of compactly generated spaces 32
7.1. Open subsets 32
7.2. The retraction functor $k$ 33
7.3. Mapping spaces, product spaces, and subspaces 33
7.4. Basic features of the category of compactly generated spaces 33
8. Cofibrations 34


*Date*: WS 24/25.

8.1. Basics about cofibrations 34
8.2. Cofibrations and NDR-pairs 38
8.3. Relative $CW$-complexes are cofibrations 40
8.4. Well-pointed spaces 40
8.5. Comparing pointed homotopy and homotopy 41
8.6. The Homotopy Theorem for pushouts and cofibrations 42
8.7. (Pointed) cylinders, cones and suspensions 45
8.8. Turning a map into a cofibration 47
8.9. The Cofiber Sequence 48
8.10. Group structures on the Puppe Sequence 50
9. Fibrations 50
9.1. Basics about fibrations 50
9.2. Turning a map into a fibration 52
9.3. Homotopy Theorem for pullbacks and fibrations 53
9.4. The fiber transport 54
9.5. Homotopy equivalences and fibrations 54
9.6. The Fiber Sequence 55
9.7. Group structures on pointed sets associated to the Fiber Sequence 57
9.8. The adjunction between suspension and loop spaces 58
9.9. Locally trivial bundles are fibrations 58
9.10. Duality between cofibrations and fibrations 59
10. The long exact homotopy sequence associated to a fibration 59
10.1. The homotopy sequence 59
10.2. The Hopf fibration 61
10.3. Homotopy groups of loop spaces 61
10.4. Homotopy groups of classifying spaces $BG$ 62
10.5. On the homotopy groups of some classical Lie groups 62
11. The Excision Theorem of Blakers-Massey 63
11.1. The statement of the Excision Theorem of Blakers-Massey 63
11.2. The proof of the Excision Theorem of Blakers-Massey 64
11.3. The Excision Theorem for $n = 0$ 69
11.4. Some applications of the Excision Theorem of Blakers-Massey 70
11.5. The Freudenthal Suspension Theorem 73
11.6. Stable homotopy groups 75
12. The Hurewicz Theorem 76
12.1. The Hurewicz homomorphism 76
12.2. The Hurewicz Theorem 77
12.3. The relative Hurewicz Theorem 79
12.4. Applications of the Hurewicz Theorem 80
13. Moore spaces 81
14. Eilenberg-MacLane spaces 85
15. Postnikov towers 88
16. Spectra 89
16.1. Basics about spectra 89
16.2. Homology and cohomology theories for pointed spaces and pairs 91
16.3. The homology and cohomology theory assigned to a spectrum 94
16.4. Brown's Representation Theorem 98
16.5. Basics about vector bundles 99
16.6. Thom spaces and Thom spectra 101
16.7. Topological $K$-theory 102
16.8. Outlook 103
17. The Pontrjagin-Thom Construction 104

17.1. $\xi$-bordism 104
17.2. The Pontrjagin-Thom construction of $\xi$-bordism 105
17.3. The Pontrjagin-Thom construction and bordism for stable systems of bundles 108
17.4. Unoriented bordism 109
17.5. The unoriented bordism ring 112
17.6. Conventions about orientations 113
17.7. Oriented bordism 114
17.8. The oriented bordism ring 114
17.9. Framed bordism 115
References 118

## 2. Basic definitions and properties of homotopy groups

2.1. **Review of the fundamental group.** We briefly recall the notion and the basic properties of the *fundamental group* $\pi_1(X,x)$ of a pointed space $(X,x)$

Let $X = (X,x)$ be a *pointed space*, i.e., a topological space $X$ with an explicit choice of a so called *base point* $x \in X$. Denote by $I$ the unit interval $[0,1]$. A *loop at $x$ in $X$* is a map of pairs $w\colon (I,\partial I) \to (X,\{x\})$. Elements in $\pi_1(X,x)$ are homotopy classes of loops at $x$ in $X$. Note that this means that two loops $w,w'\colon (I,\partial I) \to (X,\{x\})$ are homotopic if there is a homotopy $h\colon I \times I \to X$ such that $h(s,0) = w(s)$, $h(s,1) = w'(s)$, and $h(0,t) = h(1,t) = x$ hold for all $s,t \in I$. Given two loops $v,w$ at $x$ in $X$, we get a new loop $v * w$ by putting

$$v * w(s) = \begin{cases} v(2s) & \text{if } s \in [0,1/2]; \\ w(2s-1) & \text{if } s \in [1/2,1]. \end{cases}$$

The group structure on $\pi_1(X,x)$ is given by the formula $[v] \cdot [w] = [v * w]$. The unit element is given by the constant loop $c_x\colon (I,\partial I) \to (X,\{x\})$ sending $s \in I$ to $x$ and the inverse of $[w]$ is given by $[w^-]$ for $w^-\colon (I,\partial I) \to (X,\{x\}),\ s \mapsto w(1-s)$.

Here are some basic properties of the fundamental group:

- A pointed map $f\colon (X,x) \to (Y,y)$ induces a group homomorphism
$$\pi_1(f,x)\colon \pi_1(X,x) \to \pi_1(Y,y), \quad [w] \mapsto [f \circ w]$$
which depends only on the pointed homotopy class of $f$;
- We get a functor from the category of pointed spaces to the category of groups;
- Given pointed spaces $(X_i,x_i)$ for $i = 0,1$, we get from the two projections $\mathrm{pr}_i\colon (X_0 \times X_1,(x_0,x_1)) \to (X_i,x_i)$ for $i = 0,1$ an isomorphism
$$\pi_1(\mathrm{pr}_0,(x_0,x_1)) \times \pi_1(\mathrm{pr}_1,(x_0,x_1))\colon \pi_1(X_0 \times X_1,(x_0,x_1)) \xrightarrow{\cong} \pi_1(X_0,x_0) \times \pi_1(X_1,x_1);$$
- Let $p\colon X \to Y$ be a covering. Choose $x \in X$ and put $y = p(x)$. Then the induced map $\pi_1(p,x)\colon \pi_1(X,x) \to \pi_1(Y,y)$ is injective.

  If $p$ is actually a $G$-covering for the group $G$ and $X$ is path connected, then we obtain an exact sequence of groups
$$1 \to \pi_1(X,x) \xrightarrow{\pi_1(p,x)} \pi_1(Y,y) \xrightarrow{\partial} G \to 1;$$
- The mapping degree induces an isomorphism $\pi_1(S^1) \xrightarrow{\cong} \mathbb{Z}$;

- There is a *Seifert-van Kampen Theorem*. It allows to read off a presentation of the fundamental group from the 2-skeleton $X_2$ and implies that the inclusion $X_2 \to X$ induces an isomorphism $\pi_1(X_2, x) \to \pi_1(X, x)$ for any choice of base point $x \in X$. In particular $\pi_1(X, x)$ vanishes if $X$ is a $CW$-complex which has no 1-cells. Moreover, $\pi_1(\bigvee_{i=1}^r S^1, x)$ is the free group of rank $r$. So in general $\pi_1(X)$ is not abelian. Actually any group occurs as $\pi_1(X, x)$ for a 2-dimensional path connected $CW$-complex $X$;
- We get a functor $T_1$ from the fundamental groupoid $\Pi(X)$ to the category of groups by sending an object in $\Pi(X)$ which is a point $x \in X$ to $\pi_1(X, x)$. A morphism $[u]\colon x \to y$ in $\Pi(X)$ is a homotopy class $[u]$ relative endpoints of paths $u\colon I \to X$ from $x$ to $y$. It is send to the group homomorphism $T_1([u])\colon \pi_1(X, x) \to \pi_1(X, y)$ mapping $[w]$ to $[u^- * w * u]$. Recall that the composite of the morphism $[u]\colon x \to y$ and $[v]\colon y \to z$ in $\Pi(X)$ is given by $[v] \circ [u] = [u * v]$. One easily checks $T_1([v] \circ [u]) = T_1([v]) \circ T_1([u])$. Recall that there is a canonical isomorphism of $\pi_1(X, x)$ to the opposite of the group $\operatorname{aut}_{\Pi(X)}(x)$;
- Consider two maps $f_0, f_1\colon X \to Y$. Let $h\colon X \times I \to Y$ be a homotopy between $f_0$ and $f_1$. Choose a base point $x$ and put $y_i = f_i(x)$ for $i = 0, 1$. Let $u\colon I \to Y$ be the path from $y_0$ to $y_1$ given by $u(t) = h(x, t)$. We obtain a group isomorphism $T_1([u])\colon \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(Y, y_1)$ and the following diagram of groups commutes

(2.1)
$$\begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \nearrow^{\pi_1(f_0, x_0)} & \\ \pi_1(X, x) & & \cong \downarrow T_1([u]) \\ & \searrow_{\pi_1(f_1, x_1)} & \\ & & \pi_1(Y, y_1). \end{array}$$

Now consider a *pointed pair* $(X, A, x)$, i.e., a topological pair $(X, A)$ together with a choice of a base point $x \in A$. Define the set $\pi_1(X, A, x)$ as the set of homotopy classes relative $\{0\}$ of maps of pairs $w\colon (I, \partial I) \to (X, A)$ satisfying $w(0) = x$, or, equivalently, of homotopy classes of maps of triads $(I; \{0\}, \{1\}) \to (X, \{x\}, A)$. Note that $w(1)$ is not necessarily equal to $x$ and is only required to lie in $A$. If $A = \{x\}$, then $\pi_1(X, A, x)$ agrees with $\pi_1(X, x)$. In general there is no group structure on $\pi_1(X, A, x)$.

Define $\pi_0(X)$ as the *set of path components of* $X$. Note that this is the same as the homotopy classes of maps $\{\bullet\} \to X$. If $(X, x)$ is pointed map, we sometimes write $\pi_0(X, x)$ instead of $\pi_0(X)$ to indicate that the set $\pi_0(X)$ has a preferred base point, namely the path component containing $x$.

Next we construct the (in some sense exact) sequence

(2.2)
$$\pi_1(A, x) \xrightarrow{\pi_1(i,x)} \pi_1(X, x) \xrightarrow{\pi_1(j,x)} \pi_1(X, A, x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X, A) \to \{*\}.$$

The map $\pi_1(i, x)$ is the group homomorphism given by the inclusion $i\colon (A, x) \to (X, x)$. The map of sets $\pi_1(j, x)\colon \pi_1(X, x) \to \pi_1(X, A, x)$ is the obvious map given by forgetting that $w(1) = x$ holds in connection with $\pi_1(X, x)$. The map $\partial_1$ sends $[w]$ represented by $w\colon (I, \partial I) \to (X, A)$ to the path component of $A$ containing $w(1)$. The map of sets $\pi_0(i)$ sends the path component $C$ of $A$ to the path component $D$ of $X$ containing $i(C)$. The pointed set $\pi_0(X, A)$ is the quotient of the set $\pi_0(X)$

by collapsing the image of $\pi_0(i)\colon \pi_0(A) \to \pi_0(X)$ to one element and $\pi_0(j)$ is the obvious projection.

This sequence is exact in the following sense. The image of $\pi_1(i,x)$ is the preimage under $\pi_1(j,x)$ of the element in $\pi_1(X,A,x)$ given by the constant map $c_x\colon I \to X$. The image of $\pi_1(j,x)$ is the preimage under $\partial_1$ of the path component of $A$ containing $x$. The image of $\partial_1$ is the preimage under $\pi_0(i)$ of the path component of $X$ containing $x$. The image of $\pi_0(i)$ is the preimage under $\pi_0(j)$ of the preferred base point in $\pi_0(X,A)$. The map $\pi_0(j)$ is surjective.

2.2. **Basic definitions and the group structure on homotopy groups.** Next we want to generalize the notion of the fundamental group to the notion of the homotopy group in degree $n$ for all integers $n \geq 1$. The basic idea is to replace $I = [0,1]$ and $\partial I = \{0,1\}$ by the *n-dimensional cube*

$$I^n = \prod_{i=1}^{n} [0,1] = \{(s_1, s_2, \dots, s_n) \mid s_i \in [0,1]\}$$

where we define

$$\partial I^n = \{(s_1, s_2, \dots, s_n) \mid s_i \in I, \exists i \in \{1,2,\dots,n\} \text{ with } s_i \in \{0,1\}\}.$$

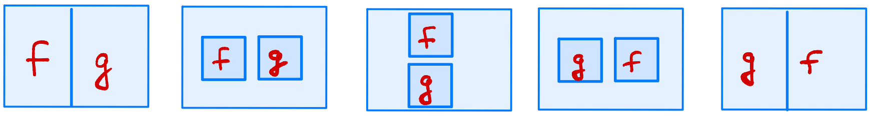
Given a pointed space $X$, we define the set $\pi_n(X,x)$ to be the set of homotopy classes $[f]$ of maps of pairs $f\colon (I^n, \partial I^n) \to (X, \{x\})$. Given two elements $[f]$ and $[g]$, we define their product $[f] \cdot [g]$ by the homotopy class of the map of pairs $f * g\colon (I^n, \partial I^n) \to (X, \{x\})$ defined by

$$f * g(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{if } s_1 \in [0,1/2]; \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{if } s_1 \in [1/2,1]. \end{cases} \tag{2.3}$$

The unit is given by the homotopy class $[c_x]$ of the constant map $c_x\colon (I^n, \partial I^n) \to (X, \{x\})$. The inverse of $[f]$ is the class $[f^-]$ for the map $f^-\colon (I^n, \partial I^n) \to (X, \{x\})$ sending $(s_1, s_2, \dots, s_n)$ to $(1 - s_1, s_2, \dots, s_n)$. The proof that this defines a group $\pi_n(X,x)$ called *n-homotopy group* of the pointed space $(X,x)$ is the essentially the same as the one for $\pi_1(X)$. The construction above for $n = 1$ agrees with the definition of $\pi_1(X,x)$ presented in Subsection 2.1. If we define $I^0$ to be $\{\bullet\}$ and $\partial I^0 = \emptyset$, the definition of the set $\pi_0(X,x)$ above agrees with the definition of $\pi_0(X)$ as the set of path components of $X$. Recall that $\pi_0(X)$ has no group structure in general and the $\pi_1(X,x)$ is not necessarily commutative. However, the following lemma is true.

**Lemma 2.4.** *The group $\pi_n(X,x)$ is abelian for $n \geq 2$.*

*Proof.* The basic observation is that in the cube $I^n$ for $n \geq 2$ there is enough room to show $[f] \cdot [g] = [g] \cdot [f]$. The desired homotopy is indicated for $n = 2$ by the following sequence of pictures:

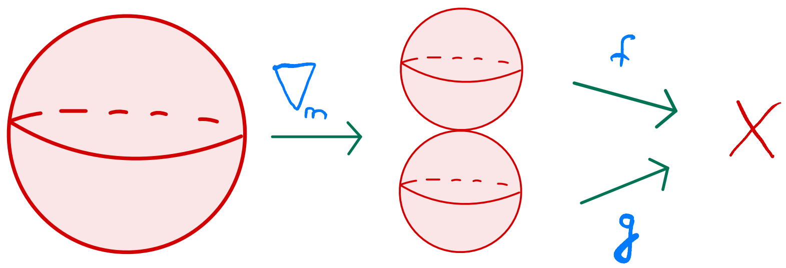



The homotopy begins by shrinking the domains of $f$ and $g$ to smaller subcubes of $I^n$, where the region outside these subcubes is mapped to the basepoint. After this has been done, there is room to slide the two subcubes around anywhere in $I^n$ as long as they stay disjoint. Hence for $n \geq 2$ they can be slided past each other, interchanging their positions. Then to finish the homotopy, the domains of $f$ and $g$ can be enlarged back to their original size. The whole process can actually be done using just the coordinates $s_1$ and $s_2$, keeping the other coordinates fixed. $\square$

Any map of pairs $f\colon (I^n, \partial I^n) \to (X, \{x\})$ factorizes in a unique way over the projection $\mathrm{pr}\colon I^n \to I^n/\partial I^n$ to a pointed map $\overline{f}\colon (I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x)$. Obviously this is compatible with the notion of a homotopy of maps of pairs $(I^n, \partial I^n) \to (X, \{x\})$ and of a pointed homotopy of pointed maps $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x)$. There is an obvious homeomorphism of pairs $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (S^n, \{s\})$ for the fixed base point $s = (1, 0, \dots, 0) \in S^n$. Hence we can interprete an element in $\pi_n(X, x)$ as a pointed homotopy of pointed maps $(S^n, s) \to (X, x)$. The multiplication in this picture is given as follows. Consider pointed maps $f_i\colon (S^n, s) \to (X, x)$ for $i = 0, 1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X, x)$. They define a pointed map $f_0 \vee f_1\colon (S^n \vee S^n, s) \to (X, x)$. Let

$$\nabla_n\colon S^n \to S^n \vee S^n \tag{2.5}$$

be the so-called *pinching map* which is obtained by collapsing the equator $S^{n-1} \subseteq S^n$ given by $\{(x_0, x_1, \dots, x_n) \in S^n \mid x_n = 0\}$ to a point. Then $[f_0]\cdot[f_1]$ is represented by the composite $f_0 \vee f_0 \circ \nabla_n$, as illustrated in the following picture

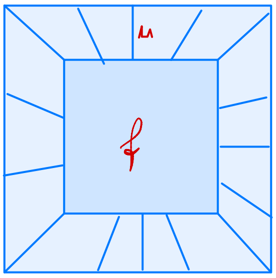



The interpretation in terms of pointed maps $(S^n, s) \to (X, x)$ is useful for some theoretical considerations and in connection with $CW$-complexes, whereas the picture in terms of maps of pairs $(I^n, \partial I^n) \to (X, \{x\})$ is better suited for some constructions and proofs, e.g., the proof of Lemma 2.4.

2.3. **Functorial properties of homotopy groups.** Obviously a map of pointed spaces $f\colon (X, x) \to (Y, y)$ induces a group homomorphism $\pi_n(f, x)\colon \pi_n(X, x) \to \pi_n(Y, y)$ for $n \geq 1$ by composition. We get a functor from the category of pointed spaces to the category of abelian groups by sending $(X, x)$ to $\pi_n(X, x)$ for $n \geq 2$, whereas for $n = 1$ we get a functor from the category of pointed spaces to the category of groups by sending $(X, x)$ to $\pi_1(X, x)$ for $n = 1$. We get a functor from the category of topological spaces to sets by sending $X$ to $\pi_0(X)$.

Obviously $\pi_n(f, x)$ depends only on the pointed homotopy class of $f$ and $\pi_0(f)$ depends only on the homotopy class of $f$.

Next we construct for every $n \geq 2$ a functor $T_n$ from $\Pi(X)$ to the category of abelian groups. It sends an object in $\Pi(X)$, which is a point $x$ in $X$, to the abelian group $\pi_n(X, x)$. Consider a morphism $[u]\colon x \to y$ in $\Pi(X)$ represented by a path $u$ in $X$ from $x$ to $y$. It is sent to the homomorphism of abelian groups $T_n([u])\colon \pi_n(X, x) \to \pi_n(X, y)$ defined as follows. Consider $[f] \in \pi_n(X, x)$ represented by the map $f\colon (I^n, \partial I^n) \to (X, \{x\})$. Consider a new map $uf\colon (I^n, \partial I^n) \to (X, \{x\})$ by shrinking the domain of $f$ to a smaller concentric cube in $I^n$, then inserting the path $u$ on each radial segment in the shell between this smaller cube and $\partial I^n$, as indicated in the picture below




We leave it to the reader to figure out the elementary proof that this definition is independent of all the choices and indeed yields a functor $T_n$ from $\Pi(X)$ to the category of abelian groups.

Recall that there is a canonical isomorphism of $\pi_1(X,x)$ to the opposite of the group $\operatorname{aut}_{\Pi(X)}(x)$. Hence we obtain from the functor $T_n$ above the structure of a $\mathbb{Z}[\pi_1(X,x)]$-module on $\pi_n(X,x)$ for $n \geq 2$. Recall that for $n = 1$ the functor $T_1$ is actually given by conjugation.

Consider two maps $f_0, f_1\colon X \to Y$. Let $h\colon X \times I \to Y$ be a homotopy between $f_0$ and $f_1$. Choose a base point $x$ and put $y_i = f_i(x)$ for $i = 0,1$. Let $u\colon I \to Y$ be the path from $y_0$ to $y_1$ given by $u(t) = h(x,t)$. For $n \geq 2$ we obtain an isomorphism of abelian groups $T_n([u])\colon \pi_n(Y,y_0) \xrightarrow{\cong} \pi_n(Y,y_1)$ and the following diagram of abelian groups commutes

(2.6)
$$\begin{array}{ccc} & & \pi_n(Y,y_0) \\ & \overset{\pi_n(f_0,x_0)}{\nearrow} & \\ \pi_n(X,x) & & \cong \big\downarrow T_n([u]) \\ & \underset{\pi_n(f_1,x_1)}{\searrow} & \\ & & \pi_n(Y,y_1). \end{array}$$

A consequence of (2.1) and (2.6) is that a homotopy equivalence $f\colon X \to Y$ induces for every $x \in X$ and $n \geq 1$ a bijection $\pi_n(f,x)\colon \pi_1(X,x) \xrightarrow{\cong} \pi_n(Y,f(x))$. Moreover, for a path connected space $X$ the isomorphism class of $\pi_n(X,x)$ is independent of the choice of $x \in X$. Therefore we sometimes write $\pi_n(X)$ instead of $\pi_n(X,x)$

Given pointed spaces $(X_i,x_i)$ for $i = 0,1$, we get from the two projections $\mathrm{pr}_i\colon (X_0 \times X_1,(x_0,x_1)) \to (X_i,x_i)$ for $i = 0,1$ a group isomorphism

$$\pi_n(\mathrm{pr}_0,(x_0,x_1)) \times \pi_n(\mathrm{pr}_1,(x_0,x_1))\colon \pi_n(X_0 \times X_1,(x_0,x_1)) \xrightarrow{\cong} \pi_n(X_0,x_0) \times \pi_n(X_1,x_1)$$

for every $n \geq 1$.

2.4. **Homotopy groups and coverings.**

**Theorem 2.7** (Homotopy groups and covering)**.** *Let $p\colon X \to Y$ be a covering. Choose a base point $x \in X$ and put $y = p(x)$. Then for $n \geq 2$ the map induced by $p$*

$$\pi_n(p,x)\colon \pi_n(X,x) \to \pi_n(Y,y)$$

*is bijective.*

*Proof.* Consider a map $f\colon S^n \to Y$ sending the base point $s$ to $y$. Since $n \geq 2$ holds by assumption, $S^n$ is simply connected. Hence the image of $\pi_1(f,x)$ is contained in the image $\pi_1(p,x)$. A standard theorem about coverings and liftings implies that we can find a lift $\widetilde{f}\colon (S^n,s) \to (X,x)$ of $f$, i.e., a pointed map $\widetilde{f}$ satisfying $p \circ \widetilde{f} = f$. This shows that $\pi_n(p,x)$ is surjective for $n \geq 2$.

Injectivity follows from the standard theorem about lifting homotopies along coverings, the argument is the same as for the injectivity of $\pi_1(p,x)$. This standard theorem says that for a map $u\colon Z \to X$ and a homotopy $h\colon Z \times I \to Y$ with $h_0 = p \circ u$ we can find precisely one homotopy $\widetilde{h}\colon Z \times I \to X$ with $p \circ \widetilde{h} = h$ and $\widetilde{h}_0 = u$. □

Theorem 2.7 implies for a connected $CW$-complex $X$ that for the universal covering $p\colon \widetilde{X} \to X$ and any choice of base points $\widetilde{x} \in \widetilde{X}$ and $x \in X$ with $p(\widetilde{x}) = x$ the map $\pi_n(p,\widetilde{x})\colon \pi_n(\widetilde{X},\widetilde{x}) \to \pi_n(X,x)$ is bijective for $n \geq 2$. If we additionally assume that $\widetilde{X}$ is contractible, we get $\pi_n(X,x) = 0$ for $n \geq 2$. In particular we get for any base point $s \in S^1$ and $n \geq 1$

$$\pi_n(S^1,s) \cong \begin{cases} \mathbb{Z} & \text{if } n = 1; \\ \{1\} & \text{if } n \geq 2, \end{cases} \tag{2.8}$$

since the universal covering of $S^1$ is given by the map $\mathbb{R} \to S^1$ sending $t \in \mathbb{R}$ to $\exp(2\pi i t)$.

2.5. **The long exact sequence of a pair and a triple.** Consider a pointed pair $(X,A,x)$, i.e., a pair of topological spaces $(X,A)$ together with a base point $x \in A$. We can consider $I^{n-1}$ as the subspace of $I^n$ given by those points $(s_1, s_2, \ldots, s_n)$ satisfying $s_n = 0$. Let $J_{n-1}$ be the subspace of $\partial I^n$ which is the closure of $\partial I^n \setminus I^{n-1}$ in $\partial I^n$. Explicitly we get

$$\begin{aligned} J_{n-1} = (\partial I^n \setminus I^{n-1}) \cup \partial I^{n-1} = \{&(s_1, s_2, \ldots, s_n) \in I^n \\ &\mid (\exists i \in \{1,2,\ldots,(n-1)\} \text{ with } s_i \in \{0,1\}) \text{ or } \ (s_n = 1)\}. \end{aligned}$$

Obviously $I_{n-1} \cup J_{n-1} = \partial I^n$ and $I_{n-1} \cap J_{n-1} = \partial I^{n-1}$. For $n \geq 1$ we define the set $\pi_n(X,A,x)$ as the set homotopy classes $[f]$ of maps of triples $f\colon (I^n, \partial I^n, J_{n-1}) \to (X, A, \{x\})$. For $n \geq 2$, this becomes a group by defining $[f_0]\cdot[f_1]$ by the class $[f_0 * f_1]$ for the maps of triples $f_0 * f_1\colon (I^n, \partial I^n, J_{n-1}) \to (X, A, \{x\})$ defined in (2.3). There is no reasonable group structure on $\pi_1(X,A,x)$. It is not hard to check that this group structure on $\pi_n(X,A,x)$ for $n \geq 2$ is well-defined and that the following result is true.

**Lemma 2.9.** *The group $\pi_n(X,A,x)$ is abelian for $n \geq 3$.*

Note that there is an obvious identification of $\pi_n(X,\{x\},x)$ defined above and of $\pi_n(X,x)$ defined in Subsection 2.2.

Obviously we obtain a functor from the category of pointed pairs to the category of groups by $\pi_2(X,A,x)$ and a functor from the category of pointed pairs to the category of abelian groups by $\pi_n(X,A,x)$ for $n \geq 3$. If two maps $f_0, f_1\colon (X,A,x) \to (Y,B,y)$ of pointed pairs are homotopic as maps of pointed pairs, then $\pi_n(f_0,x) = \pi_n(f_1,x)$ holds for $n \geq 1$. Given a pair $(X,A)$, one can define a functor $T_n$ from the fundamental groupoid $\Pi(A)$ of $A$ to the category of groups or abelian groups by assigning to a point $x \in A$ the homotopy group $\pi_2(X,A,x)$ or $\pi_n(X,A,x)$ for $n \geq 3$, the construction appearing in Subsection 2.3 for a space $X$ carries directly over. In particular $\pi_n(X,A,x)$ inherits the structure of a $\mathbb{Z}[\pi_1(A,x)]$-module for $n \geq 3$.

A map of triples $f\colon (I^n, \partial I^n, J_{n-1}) \to (X, A, \{x\})$ factorizes uniquely through the projection $\mathrm{pr}\colon (I^n, \partial I^n, J_{n-1}) \to (I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1})$ to a map of pointed pairs $(I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1}) \to (X,A,x)$. There is a homeomorphism $(I^n/J_{n-1}, \partial I^n/J_{n-1}, \{J_{n-1}/J_{n-1}\}) \xrightarrow{\cong} (D^n, S^{n-1}, \{s\})$ of triples. Hence one can define $\pi_n(X,A,x)$ also the set of homotopy classes of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \to (X,A,x)$. The multiplication in this picture is given as follows. Consider pointed maps of pointed pairs $f_i\colon (D^n, S^{n-1}, s) \to (X,A,x)$ for $i = 0,1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X,A,x)$. They define a pointed map of pointed pairs $f_0 \vee f_1\colon (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s) \to (X,A,x)$. Let

$$\nabla'_n\colon D^n \to D^n \vee D^n \tag{2.10}$$

be the so-called *pinching map* which is obtained by collapsing $D^{n-1} \subseteq D^n$ given by $\{(x_1,\ldots,x_n) \in D^n \mid x_n = 0\}$ to a point. Note that $\nabla'_n$ is a map of pointed pairs $(D^n, S^{n-1}, s) \to (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s)$ and its restriction to $(S^{n-1}, s)$ is the pinching map defined in (2.5). Then $[f_0]\cdot[f_1]$ is represented by the composite $f_0 \vee f_0 \circ \nabla'_n$.

Define for $n \geq 2$ a group homomorphism $\partial_n\colon \pi_n(X,A,x) \to \pi_1(A,x)$ by sending the class $[f]$ of the map of pointed pairs $f\colon (D^n, S^{n-1}, s) \to (X,A,s)$ to the pointed homotopy class of maps of pointed spaces obtained by restriction to $(S^{n-1}, s)$. Let $i\colon A \to X$ and $j\colon X \to (X,A)$ be the canonical inclusions.

**Theorem 2.11.** *We obtain a long exact sequence of groups infinite to the left*

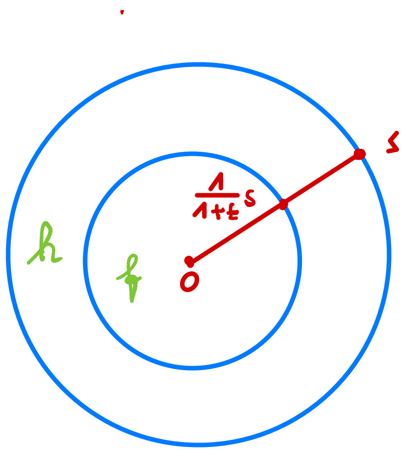
$$\cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A,x) \xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X,x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X,A,x)$$
$$\xrightarrow{\partial_{n+1}} \pi_n(A,x) \xrightarrow{\pi_n(i,x)} \pi_n(X,x) \xrightarrow{\pi_n(j,x)} \cdots$$
$$\cdots \xrightarrow{\pi_2(j,x)} \pi_2(X,A,x) \xrightarrow{\partial_2} \pi_1(A,x) \xrightarrow{\pi_1(i,x)} \pi_1(X,x).$$

*Proof.* We only show exactness at $\pi_n(X,A,x)$, the proofs at the other places are analogous. Consider a pointed map $f\colon (S^n,s) \to (X,x)$. The image of the class $[f]$ under the composite $\pi_n(X,x) \xrightarrow{\pi_n(j,x)} \pi_n(X,A,x) \xrightarrow{\partial_n} \pi_{n-1}(A,x)$ is by construction represented by the constant map $c_x\colon S^{n-1} \to A$ and hence zero. This shows $\mathrm{im}(\pi_n(j,x)) \subseteq \ker(\partial_n)$. It remains to prove $\ker(\partial_n) \subseteq \mathrm{im}(\pi_n(j,x))$.

Consider a map of pointed pairs $f\colon (D^n, S^{n-1}, s) \to (X,A,x)$ such that $[f]$ lies in the kernel of $\partial_n\colon \pi_n(X,A,x) \to \pi_{n-1}(A,x)$. Then the map of pointed spaces $f|_{S^{n-1}}\colon (S^{n-1}, s) \to (A,x)$ is nullhomotopic as pointed map of pointed spaces. Choose such a nullhomotopy $h\colon S^{n-1} \times I \to A$ with $h_0 = f|_{S^{n-1}}$ and $h_1 = c_x$ for the constant function. Note that $h(s,t) = x$ holds for $t \in I$. Define a homotopy $k\colon D^n \times I \to X$ as follows:

$$k(z,t) = \begin{cases} f((t+1)z) & \text{if } ||z|| \leq \frac{1}{1+t}; \\ h\big(\frac{z}{||z||}, 2||z|| - \frac{2}{1+t}\big) & \text{if } ||z|| \geq \frac{1}{1+t}. \end{cases}$$

Roughly speaking, $k_t$ is given on the disk $\frac{1}{1+t} \cdot D^n$ of radius $\frac{1}{1+t}$ by $f$ with an appropriate scaling of $z$ and on the anulus between $\frac{1}{1+t} \cdot S^{n-1}$ and $S^{n-1}$ by the restriction of the homotopy $h$ to $S^1 \times [2 - 2/(1+t), 1]$




We have $k(z,0) = f(z)$ for $z \in D^n$, $k(s,t) = x$ for $t \in I$, $k(z,t) \in A$ for $z \in S^{n-1}$ and $t \in I$, and $k(z,1) = x$ for $z \in S^{n-1}$. Hence $k$ is a homotopy of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \to (X, A, x)$ between $k_0 = f$ and $k_1$. Therefore $[f] = [k_1]$ holds in $\pi_n(X,A,x)$. Since $k_1(z) = x$ holds for $z \in S^{n-1}$, the class $[k_1]$ lies in the image of $\pi_n(j,x)\colon \pi_n(X,x) \to \pi_n(X,A,x)$. Hence we get $\operatorname{im}(\pi_n(j,x)) = \ker(\partial_n)$. □

**Remark 2.12.** Let $G$ be any group. Then we can find a path connected pointed 2-dimensional $CW$-complex $(A,x)$ with $\pi_1(A,x) \cong G$. Let $X$ be the cone over $A$. Then we obtain a path connected pointed 3-dimensional $CW$-complex $(X,A,x)$ such that $\pi_2(X,A,x) \cong \pi_1(A,x) \cong G$ holds by Theorem 2.11.

**Remark 2.13.** One can combine the exact sequences appearing in Theorem 2.2 and Theorem 2.11 to an exact sequence

$$(2.14)\quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A,x) \xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X,x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X,A,x) \xrightarrow{\partial_{n+1}} \pi_n(A,x) \xrightarrow{\pi_n(i,x)} \pi_n(X,x) \xrightarrow{\pi_n(j,x)} \cdots \xrightarrow{\pi_2(j,x)} \pi_2(X,A,x) \xrightarrow{\partial_2} \pi_1(A,x) \xrightarrow{\pi_1(i,x)} \pi_1(X,x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X,A) \to \{*\}$$

which is compatible with the group structures as long as these exist.

It is not hard to check that one obtains for a triple $(X,B,A)$ and a base point $x \in A$ an exact sequence of the shape

$$(2.15)\quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(B,A,x) \xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X,A,x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X,B,x) \xrightarrow{\partial_{n+1}} \pi_n(B,A,x) \xrightarrow{\pi_n(i,x)} \pi_n(X,A,x) \xrightarrow{\pi_n(j,x)} \cdots \xrightarrow{\pi_2(j,x)} \pi_2(X,B,x) \xrightarrow{\partial_2} \pi_1(B,A,x) \xrightarrow{\pi_1(i,x)} \pi_1(X,A,x) \xrightarrow{\pi_1(j,x)} \pi_1(X,B,x) \xrightarrow{\partial_1} \pi_0(B,A) \xrightarrow{\pi_0(i)} \pi_0(X,A) \xrightarrow{\pi_0(j)} \pi_0(X,B) \to \{*\}$$

which is compatible with the group structures as long as these exist.

**Remark 2.16** (Long exact homotopy sequence of a pointed map)**.** Let $f\colon (X,x) \to (Y,y)$ be a map of pointed spaces. Denote by $\operatorname{cyl}(f)$ its mapping cylinder. Note that we obtain a pointed pair $(\mathrm{cyl}(f), X, x)$. The canoncial projection $\mathrm{cyl}(f) \to Y$ is a homotopy equivalence and satisfies $\mathrm{pr}(x) = y$. Hence it induces an isomorphism of groups $\pi_n(\mathrm{pr}, x)\colon \pi_n(\mathrm{cyl}(f), x) \xrightarrow{\cong} \pi_n(Y, y)$ for $n \geq 1$ and a bijection $\pi_0(\mathrm{cyl}(f)) \xrightarrow{\cong} \pi_0(Y)$. Define $\pi_n(f, x) = \pi_n(\mathrm{cyl}(f), X, x)$ for $n \geq 1$. Let $\pi_0(f)$ be the quotient of $\pi_0(Y)$ obtained by collapsing the image of $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$. Then the long exact sequence of the pointed pair $(\mathrm{cyl}(f), X, x)$ of (2.14) yields the long exact homotopy sequence of the map $f$

$$(2.17) \quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(f,x)} \pi_{n+1}(Y, y) \to \pi_{n+1}(f, x) \xrightarrow{\partial_{n+1}} \pi_n(X, x) \xrightarrow{\pi_n(f,x)} \pi_n(Y, y) \to \cdots \to \pi_2(f, x) \xrightarrow{\partial_2} \pi_1(X, x) \xrightarrow{\pi_1(f,x)} \pi_1(Y, y) \xrightarrow{\partial_1} \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y) \to \pi_0(f) \to \{1\}.$$

Note that $\pi_n(f, x)$ can have two different meanings in the notation above.

## 2.6. Connectivity.

**Definition 2.18** (Connectivity)**.** A space $X$ is called 0-*connected* if $\pi_0(X)$ consists of one point, or, equivalently, $X$ is path connected. It is called $n$-*connected* for $n \geq 1$ if $X$ is path connected and $\pi_k(X, x)$ is trivial for every base point $x$ and $1 \leq k \leq n$. It is called $\infty$-*connected* or *weakly contractible* if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x$ and $k \geq 1$.

A map $f\colon X \to X$ is called 0-*connected* if the induced map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective. It is called $n$-*connected* for $n \geq 1$, if the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is bijective and for every base point $x$ the map $\pi_k(f, x)\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is bijective for $1 \leq k < n$ and surjective for $k = n$. It is called $\infty$-*connected* or a *weak homotopy equivalence* if the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is bijective and for every base point $x$ and $k \geq 1$ the map $\pi_k(f, x)\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is bijective. Note that $f$ is $n$-*connected* if and only if $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective and the group $\pi_k(f, x)$ defined in Remark 2.16 is trivial for $1 \leq k \leq n$.

A pair $(X, A)$ is called $n$-*connected* for $n \geq 0$ or $n = \infty$, if the inclusion $i\colon A \to X$ is $n$-connected. This is equivalent to the condition that $\pi_0(X, A)$ and $\pi_k(X, A, x)$ for $1 \leq k \leq n$ are trivial.

**Remark 2.19.** One easily checks that the following assertions are equivalent for a pointed space $(X, x)$ and $n \geq 1$:

- $\pi_n(X, x)$ is trivial for any base point $x \in X$;
- Every map $S^n \to X$ is nullhomotopic;
- Every map $S^n \to X$ extends to a map $D^{n+1} \to X$.

This implies that the following assertions are equivalent for a space $X$ and $n \geq 0$ or $n = \infty$:

- $X$ is $n$-connected;
- Given any $k$ with $0 \leq k \leq n$, every map $S^k \to X$ is nullhomotopic;
- Given any $k$ with $0 \leq k \leq n$, every map $S^k \to X$ extends to a map $D^{k+1} \to X$.

Moreover, the following assertions are equivalent for a pair $(X, A)$ and $n \geq 0$ or $n = \infty$:

- $(X, A)$ is $n$-connected;
- Given any $k$ with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic relative $S^{k-1}$ to a map $D^k \to A$;
- Given any $k$ with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic through such maps to a map $D^k \to A$;
- Given any $k$ with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic through such maps to a constant map $D^k \to A$.

### 2.7. **Homotopy groups and colimits.**

**Theorem 2.20** (Homotopy groups and colimits)**.** *Let $X$ be a topological Hausdorff space with a sequence of closed subspaces $X_0 \subset X_1 \subseteq \cdots \subseteq X$ such that $X$ is the union of the $X_i$-s and carries the colimit topology.*

*Then for every $x_0 \in X$ and $n \geq 1$ the canonical group homomorphism induced by the inclusions $j_k \colon X_k \to X$*

$$\operatorname{colim}_{k\to\infty} \pi_n(j_k, x_0) \colon \operatorname{colim}_{k\to\infty} \pi_n(X_k, x_0) \to \pi_n(X, x_0)$$

*is bijective. Also the map of sets*

$$\operatorname{colim}_{k\to\infty} \pi_0(j_k) \colon \operatorname{colim}_{k\to\infty} \pi_0(X_k) \to \pi_0(X, x_0)$$

*is bijective.*

*Proof.* We first prove that for any compact subset $C \subseteq X$ there exists a natural number $k$ with $C \subseteq X_k$. Suppose that for every $k \geq 0$ we have $C \nsubseteq X_k$. Then we can choose a sequence of $x_0, x_1, x_2, \ldots$ in $C$ and a strictly monotone increasing function $j \colon \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ with $x_i \in X_{j(i)} \setminus X_{j(i-1)}$ for $i = 1, 2, \ldots$. Put $S = \{x_0, x_1, x_2, \ldots\}$. Obviously $S$ is infinite. Let $T \subseteq S$ be any subset. Note that the intersection $T \cap X_k$ is finite and hence a closed subset of $X_k$ for $k = 0, 1, 2, \ldots$. Since $X$ carries the colimit topology, $T$ is closed in $X$. Hence $S$ is a discrete subset of $X$. As $C$ is compact and $S$ is a closed subset of $C$, the set $S$ is compact. As $S$ is a discrete and compact set, it must be finite, a contradiction.

We only treat the case $n \geq 1$, the case $n = 0$ is analogous. Consider an element $[f] \in \pi_n(X, x_0)$ represented by a pointed map $f \colon (S^n, s) \to (X, x_0)$. Then image of $f$ lies already in $X_i$ for some $i \geq 0$. Hence $[f]$ lies in the image of the map $\pi_n(X_i, x_0) \to \pi_n(X, x_0)$ induced by the inclusion $X_i \to X$. This implies that $[f]$ lies in the image of $\operatorname{colim}_{k\to\infty} \pi_n(j_k, x_0) \colon \operatorname{colim}_{k\to\infty} \pi_n(X_k, x_0) \to \pi_n(X, x_0)$. Hence this map is surjective. To prove injectivity, we consider an element $[g]$ in its kernel. There exists $i \geq 0$ and an element $[g'] \in \pi_n(X_i, x_0)$ such that the structure map $\pi_n(X_i, x_0) \to \operatorname{colim}_{j\to\infty} \pi_n(X_j, x_0)$ sends $[g']$ to $[g]$. The element $[g']$ lies in the kernel of the map $\pi_n(X_i, x_0) \to \pi_n(X, x_0)$ induced by the inclusion $X_i \to X$. If $g' \colon (S^n, s) \to (X_i, x_0)$ is a representative of $[g']$, there is a nullhomotopy $h \colon S^n \times I \to X$ for it. The image of $h$ lies already in $X_j$ for some $j$ with $i \leq j$. Hence the image of $[g']$ under the map $\pi_n(X_i, x_0) \to \pi_n(X_j, x_0)$ induced by the inclusion $X_i \to X_j$ is trivial. This implies that $[g]$ is trivial. □

## 3. Hopf's Degree Theorem

In this section we give the proof of the following theorem.

**Theorem 3.1** (Hopf's Degree Theorem)**.** *Let $M$ be a connected oriented closed smooth manifold of dimension $n \geq 1$. Then the degree defines a bijection*

$$\deg \colon [M, S^n] \to \mathbb{Z}.$$

### 3.1. **Some basics about differential topology and the mapping degree.**

Its proof needs some preparation. We recall some basic facts about differential topology and the mapping degree.

- Let $M$ and $N$ be smooth manifolds. Then a (continuous) map $f \colon M \to N$ is homotopic to a smooth map. If two smooth maps $M \to N$ are homotopic, then one can find a smooth homotopy between them.
- Let $M$ and $N$ be smooth manifolds and $L \subseteq N \setminus \partial N$ be a smooth submanifold without boundary. Then any smooth map $f \colon M \to N$ with $f(\partial M) \cap L = \emptyset$ is smoothly homotopic relativ $\partial M$ to a map $g \colon M \to N$ which is *transversal to* $L$ at every $x \in M$, i.e., we have either $f(x) \notin L$ or we

have $f(x) \in L$ and $T_xf(T_xM) + T_{f(x)}L = T_{f(x)}N$. If $\dim(M) + \dim(L) < \dim(N)$ holds, then $f$ is transversal to $L$ if and only if $f(M) \cap L = \emptyset$.

- If $L = \{y\}$ for $y \in N \setminus \partial N$, then we say that $y$ is a *regular value* of $f$ if $f$ is transversal to $\{y\}$.
- Every smooth map $f\colon M \to N$ has a regular value $y \in N \setminus \partial N$. Actually the points in $N \setminus \partial N$ for which $y$ is not a regular value has measure zero in $N$ by the Theorem of Sard.

  If $y \in N \setminus \partial N$ is a regular value of $f$, $M$ is compact, and $\dim(M) = \dim(N)$, then $f^{-1}(y)$ is finite and for every $x \in f^{-1}(y)$ the differential induces an isomorphism $T_xf\colon T_xM \to T_yN$.
- Let $f\colon M \to N$ be a map of connected oriented compact smooth oriented manifolds of dimension $n$ such that $f(\partial M) \subseteq \partial N$ holds. Let $y \in N \setminus \partial N$ be any regular value. For $x \in f^{-1}(y) \subseteq M \setminus \partial M$ the orientations on $M$ and $N$ yield orientations on the finite dimensional vector spaces $T_xM$ and $T_yN$. Define $\epsilon(x) \in \{\pm 1\}$ to be 1 if $T_xf\colon T_xM \xrightarrow{\cong} T_yN$ respects these orientations and to be $-1$ otherwise.

  Recall degree of $f$ is the natural number for which $H_n(f)\colon H_n(M, \partial M) \to H_n(N, \partial N)$ sends $[M, \partial M]$ to $\deg(f) \cdot [N, \partial N]$. We get

$$\deg(f) = \sum_{x \in f^{-1}(y)} \epsilon(x). \tag{3.2}$$

  This formula is well-known for $\partial M = \partial N = \emptyset$. The proof in this case extends directly to the more general case above. Or one considers the map of closed oriented manifolds $f \cup_{\partial f} f\colon M \cup_{\partial M} M \to N \cup_{\partial N} N$ for $\partial f\colon \partial M \to \partial N$ given by $f|_{\partial M}$.
- Let $M$ be a smooth Riemannian manifold and $x \in M \setminus \partial M$. Then there is an $\epsilon > 0$, an open subset $U$ of $M$ containing $x$, and a diffeomorphism called *exponential map*

$$\exp_x\colon D^\circ_\epsilon T_xM := \{v \in T_xM \mid ||x|| < \epsilon\} \to U \tag{3.3}$$

  such that the differential $T_0 \exp_x\colon T_0(T_xM) \to T_xM$ of $\exp_x$ at $0 \in T_xM$ becomes the identity under the canoncial identification $T_0(T_xM) = T_xM$.

3.2. **The proof of Hopf's Degree Theorem.** We prove Hopf's Degree Theorem 3.1 by induction over the dimension $n = \dim(M)$. If $n = 1$, then $M$ is diffeomorphic to $S^1$ and elementary covering theory shows that the degree induces a bijection $\deg\colon [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. The induction step from $(n-1)$ to $n \geq 2$ is done as follows.

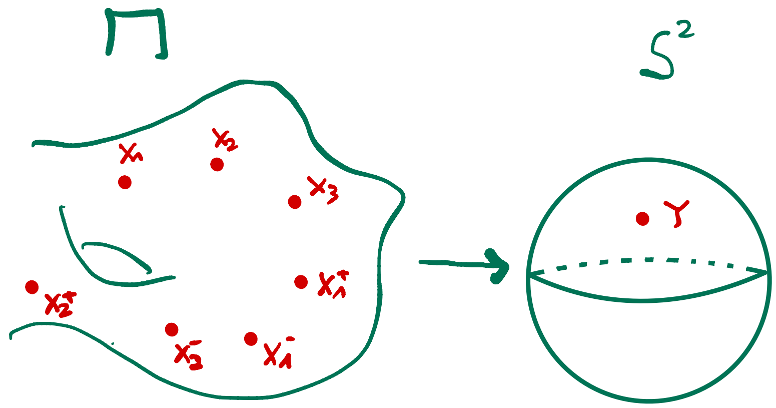
Fix $x \in M$ and an embedding $i\colon D^n \hookrightarrow M$ such that $i(0) = x$ holds and $T_0i\colon T_0D^n \xrightarrow{\cong} T_xM$ is compatible with the orientations coming from the standard orientation on $D^n$ and the given orientation on $M$. Define the collaps map $c\colon M \to D^n/S^{n-1} \cong S^n$ by sending $i(x)$ for $x \in D^n$ to the element given by $x$ in $D^n/S^{n-1}$ and every point $y \in M \setminus i(D^n)$ to the point $S^{n-1}/S^{n-1}$ in $D^n/S^{n-1}$. We conclude from (3.2) applied to the regular value $z \in D^n/S^{n-1} = S^n$ given by $0 \in D^n$ of $c$ that $\deg(c) = 1$. Given any $d \in \mathbb{Z}$, there exists a selfmap $u_d\colon S^n \to S^n$ with $\deg(u_d) = d$. It can be constructed as the $(n-1)$-fold suspension of the map $S^1 \to S^1$ sending $z$ to $z^d$. Then $\deg(u_d \circ c) = d$. This shows that $\deg\colon [M, S^n] \to \mathbb{Z}$ is surjective.

In order to show that $\deg\colon [M, S^n] \to \mathbb{Z}$ is injective, we have to show that two smooth maps $f, g\colon M \to S^n$ with $\deg(f) = \deg(g)$ are homotopic. Since there is diffeomorphism $u\colon S^n \to S^n$ with degree $-1$ and $\deg(u \circ f) = -\deg(f)$, we can assume in the sequel that $d = \deg(f) = \deg(g)$ satisfies $d \geq 0$.

We can change $f$ and $g$ up to homotopy and find $y \in S^n$ such that both $f$ and $g$ are smooth and have $y$ as regular value. Then we can write

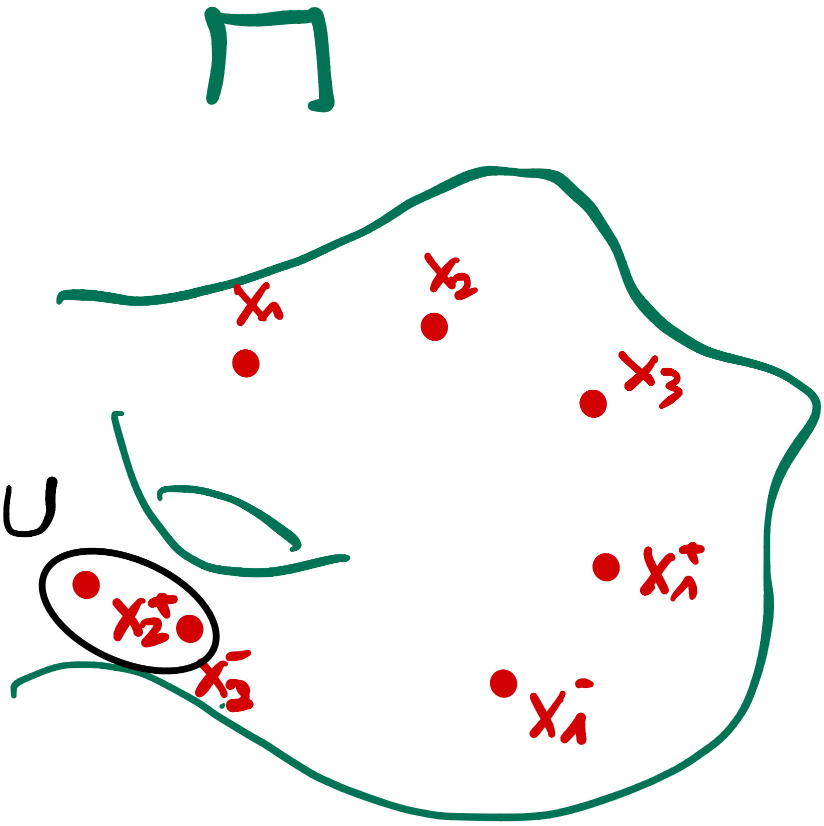
$$f^{-1}(y) = \{x_1, x_2, \ldots, x_d\} \amalg \{x_1^+, x_1^-, \ldots, x_m^+, x_m^-\}$$

for some $m \geq 0$ such that $\epsilon(x_i) = 1$ for $i = 1, 2, \ldots, d$ and $\epsilon(x_j^\pm) = \pm 1$ holds for $j = 1, 2, \ldots, m$.



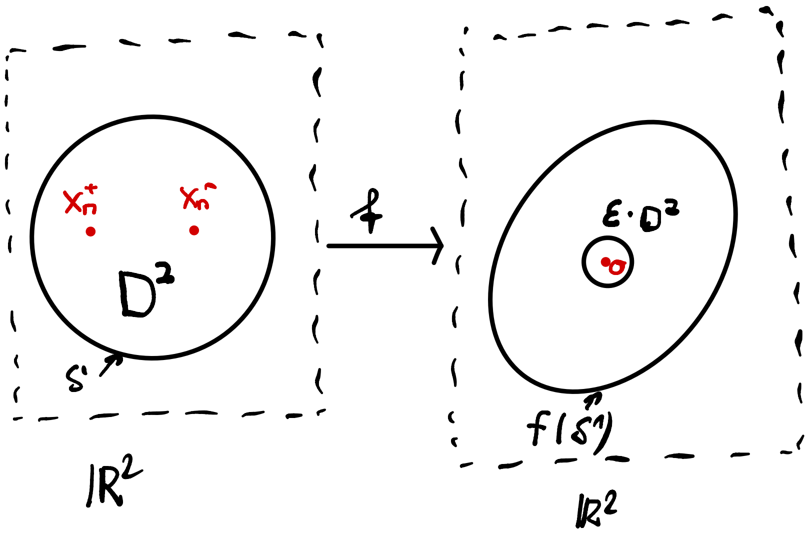

We next describe a procedure how to change $f$ up to homotopy so that $m = 0$, or, equivalently $f^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$ holds. This will be done by an inductive procedure where we change $f$ up to homotopy such that $m \geq 1$ becomes $(m-1)$, in other words, we get rid of the points $x_m^+$ and $x_m^-$.

Choose an embedded arc in $M$ joining $x_m^+$ and $x_m^-$ that does not meet any of the other points in $f^{-1}(y)$. Let $U$ be an open neighbourhood of $x_m^-$ that is diffeomorphic to $\mathbb{R}^n$. Now perform a local homotopy of $f$ along this arc to move $x_m^-$ so close to $x_m^+$ such that $x_m^-$ lies in $U$.

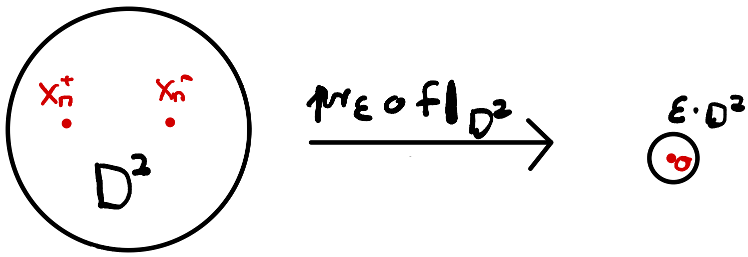


Hence it suffices to prove the following: Given a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f$ is transversal to $0 \in \mathbb{R}^n$, the preimage $f^{-1}(0)$ consists of precisely two points $x_0$ and $x_1$ belonging to the interior of the disk $D^n \subseteq \mathbb{R}^n$, the differential $T_{x_0}f\colon T_{x_0}\mathbb{R}^n \to T_0\mathbb{R}^n$ is bijective and reverses the standard orientations, and the differential $T_{x_1}f\colon T_{x_1}\mathbb{R}^n \to T_0\mathbb{R}^n$ is bijective and preserves the standard orientations, then we can change $f$ up to homotopy relative $\mathbb{R}^n \setminus D^n$ so that $f^{-1}(0)$ is empty.

Choose $\epsilon > 0$ so small that the image of $S^{n-1} \subseteq \mathbb{R}^n$ under $f$ does not meet the interior of $\epsilon \cdot D^n$.




Let $\mathrm{pr}_\epsilon \colon \mathbb{R}^n \to \epsilon \cdot D_n$ be the retraction that sends $x \in \mathbb{R}^n$ to $\frac{\epsilon}{||x||} \cdot x$ if $||x|| \geq \epsilon$, and to $x$ if $||x|| \leq \epsilon$. Then $\mathrm{pr}_\epsilon \circ f$ induces a map of compact oriented manifolds $(D^n, S^{n-1}) \to (\epsilon \cdot D^n, \epsilon \cdot S^{n-1})$. By inspecting the preimage of $0 \in \epsilon \cdot D^n$ we conclude from (3.2) that its degree is zero.




Since the following diagram commutes and the vertical maps given by boundary homomorphisms of pairs are isomorphism of infinite cyclic groups respecting the fundamental classes

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow{H_n(f)} & H_n(\epsilon \cdot D^n, \epsilon \cdot S^{n-1}) \\ \cong \downarrow & & \downarrow \cong \\ H_n(S^{n-1}) & \xrightarrow[H_n(f|_{S^{n-1}})]{} & H_n(\epsilon \cdot S^{n-1}) \end{array}$$

the induced map $(\mathrm{pr}_\epsilon \circ f)|_{S^{n-1}} \colon S^{n-1} \to \epsilon \cdot S^{n-1}$ has degree zero and hence is nullhomotopic by the induction hypothesis. This implies that the map $f_0 \colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ induced by $f$ is nullhomotopic and hence extends to a map $f_1 \colon D^n \to \mathbb{R}^n \setminus \{0\}$. Let $f' \colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$ be the map whose restriction to $D^n$ is $f_1$ and whose restriction to $\mathbb{R}^n \setminus D^n$ agrees with the restriction of $f$ to $\mathbb{R}^n \setminus D^n$. We obtain a homotopy $h \colon f \simeq f'$ of maps $\mathbb{R}^n \to \mathbb{R}^n$ by $h(x,t) = t \cdot f'(x) + (1-t) \cdot f$ that is stationary outside the interior of $D^n$. Since the image of $f'$ does not contain zero, the claim follows.

This argument applies also to $g$. If $d = 0$, then $\mathrm{im}(f)$ and $\mathrm{im}(g)$ are contained in the contractible subspace $S^n \setminus \{y\}$ of $S^n$ and hence $f$ and $g$ are homotopic. It remains to consider the case $d \geq 1$. Then we can find finite subsets $\{x_1, x_2, \ldots, x_d\}$ and $\{x_1', x_2', \ldots, x_d'\}$ of $M$ such that $f^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$ and $g^{-1}(y) = \{x_1', x_2', \ldots, x_d'\}$ holds and the differentials $T_{x_i} f \colon T_{x_i} M \to T_y S^n$ and $T_{x_i'} g \colon T_{x_i'} M \to T_y S^n$ are orientation preserving isomorphisms for $i = 1, 2, \ldots, n$. Now we can construct a diffeomorphism $a \colon M \to M$ which is homotopic to the identity and satisfies $w(x_i) = x_i'$ for $i = 1, 2, \ldots, d$. Then $g$ and $g' = g \circ a$ are homotopic, $f^{-1}(y) = g'^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$ and the differentials $T_{x_i} f \colon T_{x_i} M \to T_y S^n$ and $T_{x_i} g' \colon T_{x_i} M \to T_y S^n$ are orientation preserving isomorphisms for $i = 1, 2, \ldots, d$. It remains to show that $f$ and $g'$ are homotopic.

For this purpose we need the following construction. Let $u_0, u_1 \colon \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ be linear $\mathbb{R}$-isomorphisms which are orientation preserving. Then we can find a homotopy $h \colon \mathbb{R}^n \times I \to \mathbb{R}^n$ such that $h_0 = u_0$ and $h_1 = u_1$ holds and $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ is a orientation preserving automorphism for $t \in I$. This follows from the fact that $\{A \in GL_n(\mathbb{R}) \mid \det(A) > 0\}$ is path connected for $n \geq 1$. Define the homotopy

$$H \colon \mathbb{R}^n \times I \to \mathbb{R}^n, \quad (v,t) \mapsto \begin{cases} h_t(v) & \text{if } ||v|| \leq 1; \\ h_{(2-||v||)\cdot t}(v) & \text{if } 1 \leq ||v|| \leq 2; \\ u_0(v) & \text{if } ||v|| \geq 2. \end{cases}$$

Then we have

$$\begin{aligned} H_t^{-1}(0) &= 0 \quad \text{for } t \in I; \\ H_0 &= u_0; \\ H_t(v) &= u_0(v) \quad \text{for } t \in I \text{ and } ||v|| \geq 2; \\ H_1(v) &= \begin{cases} u_1(v) & \text{if } ||v|| \leq 1; \\ h_{(2-||v||)}(v) & \text{if } 1 \leq ||v|| \leq 2; \\ u_0(v) & \text{if } ||v|| \geq 2. \end{cases} \end{aligned}$$

So $H$ is a homotopy between $H_0 = u_0$ and $H_1$ which is stationary on $\{v \in V \mid ||v|| \geq 2\}$ and satisfies $H_t^{-1}(0) = 0$ for $t \in I$ and $H_1(v) = u_1(v)$ for $||v|| \leq 1$.

Using this construction and the exponential map (3.3), we can change $g'$ by a homotopy to a map $g'' \colon M \to S^n$, such that for $i = 1, 2, \ldots, d$ there are disjoint embedded disks $D_i^n \subseteq M$ such that $0 \in D_i^n$ corresponds to $x_i$, $f|_{D_i^n} = g''|_{D_i^n}$ holds and we have $f^{-1}(y) = (g'')^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$. Let $X$ be the complement in $M$ of the disjoint union $\coprod_{i=1}^d D_i^n \setminus \partial S_i^{n-1}$. This is a manifold with boundary

$\partial X = \coprod_{i=1}^{d} S_i^{n-1}$ such that $f(X)$ and $g''(X)$ are contained in $S^n \setminus \{y\}$ and $f|_{\partial X} = g''|_{\partial X}$ holds. As $S^n \setminus \{y\}$ is contractible, the maps $f|_X$ and $g''|_X$ from $X$ to $S^n$ are homotopic relative $\partial X$. Recall that $f$ and $g''$ agree on $\coprod_{i=1}^{d} D_i^n$. Hence $f$ and $g''$ are homotopic as maps $M \to S^n$. This implies that the maps $f$ and $g$ from $M$ to $S^n$ are homotopic. This finishes the proof of Hopf's Degree Theorem 3.1.

### 3.3. **The homotopy groups of the $n$-sphere in the degree $\leq n$.**

**Theorem 3.4.** *We get for every $n \geq 1$*

$$\pi_k(S^n) \cong \begin{cases} \{0\} & k < n; \\ \mathbb{Z} & k = n. \end{cases}$$

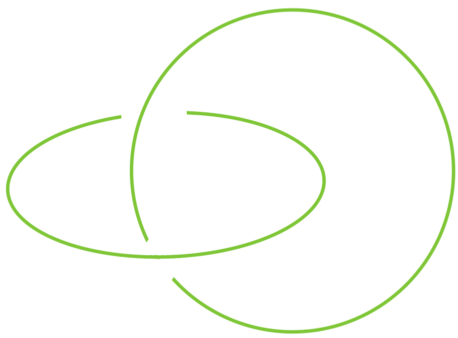
*There is an explicit isomorphism $\mathbb{Z} \xrightarrow{\cong} \pi_n(S^n)$ which sends $1 \in \mathbb{Z}$ to $[\mathrm{id}_{S^n}]$. Its inverse $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ sends $[f]$ to the degree of $f$.*

*Proof.* Suppose $k < n$. Let $f\colon S^k \to S^n$ be any map. Since we can change any map $f\colon S^k \to S^n$ up to homotopy into a smooth map transversal to $y \in S^n$, we can change $f$ by a homotopy to map $S^n \to S^n \setminus \{y\}$. As $S^n \setminus \{y\}$ is contractible, $f$ is nullhomotopic. This implies $\pi_k(S^n, s) = \{0\}$ for every $s \in S$.

The degree defines a bijection $\deg\colon [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$ because of Hopf's Degree Theorem 3.1 for $n \geq 1$. By inspecting the proof of surjectivity of this map we see that the forgetful map $\pi_n(S^n, s) \to [S^n, S^n]$ is surjective. We conclude from (2.1) and (2.6) that the forgetful map $\pi_n(S^n, s) \to [S^n, S^n]$ is injective. □

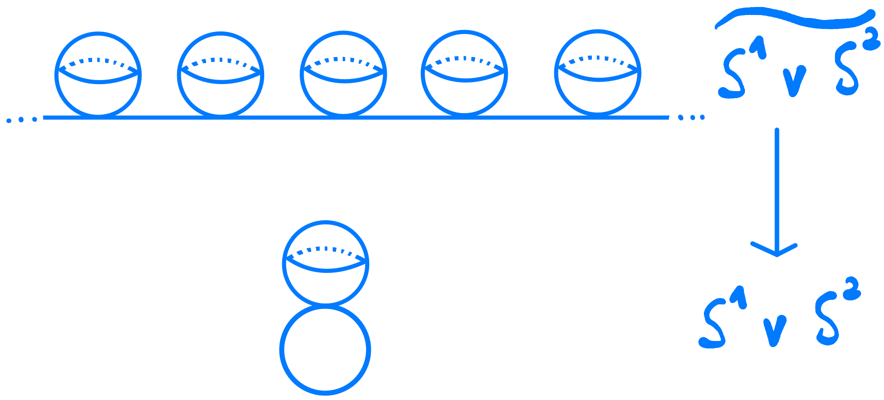
**Example 3.5** (The Hopf map and $\pi_3(S^2)$)**.** One may think that $\pi_k(S^n, s)$ vanishes for $k > n$ as $H_k(S^n)$ vanishes for $k > n$. But this is not true as the following example due to Hopf shows. We can think of $S^3$ as the subset of $\mathbb{C}^2$ given by $\{(z_1, z_2) \mid z_1\overline{z_1} + z_2\overline{z_2} = 1\}$. We get an $S^1$-action on $S^3$ by $z \cdot (z_1, z_2) = (zz_1zz_2)$. This action is free and the quotient space $S^3/S^1$ is homeomorphic to $S^2$. Thus we get a projection $p\colon S^3 \to S^2$. We will later show that $\pi_3(S^2) \cong \mathbb{Z}$ holds with the class $[p]$ of $p$ as generator, see Theorem 10.5.

One indication that $[p]$ is not zero in $\pi_3(S^2)$ is the observation that the preimages of the north and the south pole of $S^2$ are two embedded $S^1$-s in $S^3$ which are linked.



**Example 3.6** ($\pi_n(S^1 \vee S^n)$ is not finitely generated.)**.**

Consider $X = S^1 \vee S^n$ for $n \geq 2$. Its universal covering $\widetilde{X}$ is obtained from $\mathbb{R}$ by glueing to each element in $\mathbb{Z}$ a copy of $S^n$ along the base point.




The map $\widetilde{X} \to \bigvee_{i\in\mathbb{Z}} S^n$ given by collapsing $\mathbb{R}$ to point turns out to be a point homotopy equivalence. This can be seen by a direct inspection or follows from Lemm 8.25 and Theorem 8.28. Hence we conclude $\pi_n(X) \cong \pi_n(\widetilde{X}) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^n)$ from Theorem 2.7. For each $k \in \mathbb{Z}$ we have the pointed inclusion $j_k \colon S^n \to \bigvee_{i\in\mathbb{Z}} S^n$ of the $k$-th summand and the pointed projection $\mathrm{pr}_k \colon \bigvee_{i\in\mathbb{Z}} S^n \to S^n$ onto the $k$-th summand. Obviously $\mathrm{pr}_k \circ j_k$ is the identity and $\mathrm{pr}_k \circ j_l$ is the constant map for $k \neq l$. Hence the map $\bigoplus_{i\in\mathbb{Z}} \pi_n(j_i) \colon \bigoplus_{i\in\mathbb{Z}} \pi_n(S^n) \to \pi_n(\bigvee_{i\in\mathbb{Z}} S^n)$ is injective. As $\pi_n(S^n) \cong \mathbb{Z}$, the abelian group $\pi_n(S^1 \vee S^n)$ is not finitely generated.

Actually, we know that $\pi_n(S^1 \cup S^n)$ is a $\mathbb{Z}[\pi_1(S^1)]$-module and it will turn out that it is $\mathbb{Z}[\pi_1(S^1)]$-isomorphic to $\mathbb{Z}[\pi_1(S^1)]$.

**Remark 3.7** (Outlook about $\pi_k(S^n)$ for $k > n$)**.** It is an open (and extremely hard) problem to compute $\pi_k(S^n, s)$ for $2 \leq n < k$ in general. There is not even a formula known which might give the answer. There is no obvious pattern in the computations, one has carried out so far. At least one knows that $\pi_k(S^n)$ is finite for $k > n$ except for $\pi_{4i-1}(S^{2i})$ for $i \geq 1$ which is a direct sum of a copy of $\mathbb{Z}$ and some finite abelian group.

## 4. The Cellular Approximation Theorem

In this section we want to sketch the proof of the following theorem.

**Theorem 4.1** (Cellular Approximation Theorem)**.** *Let $(X, A)$ be a CW-pair and $Y$ be a CW-complex. Let $f \colon X \to Y$ be a map whose restriction $f|_A \colon A \to Y$ to $A$ is cellular. Then $f$ is homotopic relative $A$ to a cellular map $X \to Y$.*

By a colimit argument one can reduce the proof of the Cellular Approximation Theorem 4.1 to the proof of following lemma.

**Lemma 4.2.** *Consider any $k \in \{0, 1, 2, \ldots\}$. Let $f \colon X \to Y$ be a map of CW-complexes. Suppose that $f(X_{k-1}) \subseteq Y_{k-1}$ holds.*

*Then we can change $f$ up to homotopy relative $X_{k-1}$ such that $f(X_k) \subseteq Y_k$ holds.*

In order to arrange that $f(X_k) \subseteq Y_k$ holds, we must achieve for every closed $k$-dimensional cell $e$ of $X$ by a homotopy of $f|_e$ relative $\partial e$ that $e$ does not meet any cell of $Y$ of dimension $> k$. Note that each compact subset of $Y$ meets only finitely many cells. Hence for a closed cell of $e$ of $X$ of dimension $k$ there are only finitely many closed cells $e_1, e_2, \ldots, e_m$ of $Y$ satisfying $f(e) \cap e_i \neq \emptyset$. Choose $\{i \in 1, 2, \ldots, m\}$ such the the dimension of $e_i$ is greater than $\dim(e)$. If such an $i$ does not exists, we are already done for $e$. If such $i$ exists, we can arrange that $\dim(e_i) \geq \dim(e_j)$ holds for all $j \in \{1, 2, \ldots, m\}$ and we have to change $f|_e$ up to homotopy relative $\partial e$ such that $f(e)$ meets only the cells $e_1, e_2, \ldots, e_{i-1}, e_{i+1}, e_m$ of $Y$. Therefore it suffices to show the following lemma.

**Lemma 4.3.** *Consider $0 \leq k < l$. Let $(W, V)$ be pair for which there exists a pushout*

$$\begin{array}{ccc} S^{l-1} & \xrightarrow{q} & V \\ \downarrow & & \downarrow \\ D^l & \xrightarrow{Q} & W. \end{array}$$

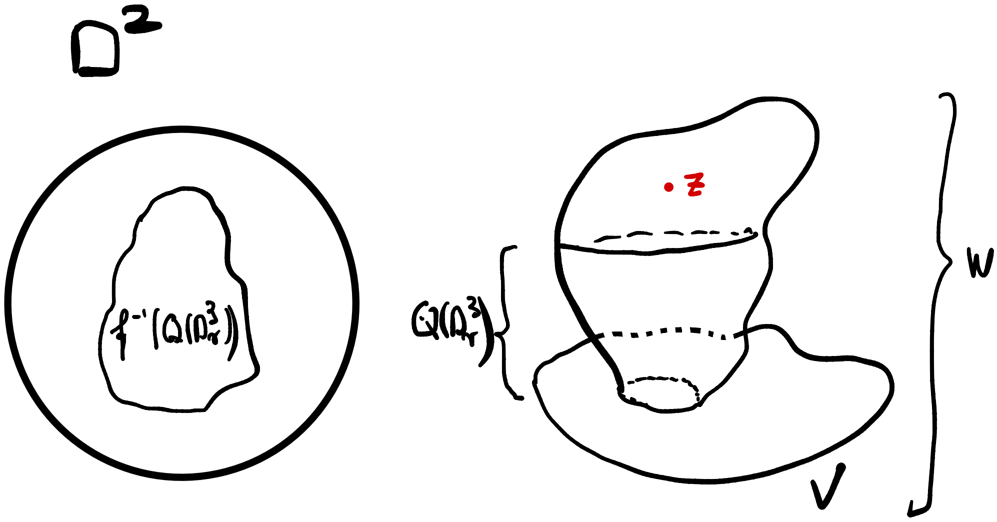
*Consider any map $f \colon (D^k, S^{k-1}) \to (W, V)$.*

*Then $f$ is homotopic relative $S^{k-1}$ to a map $D^k \to V$.*

Consider any point $z \in W \setminus V$. Then $(W \setminus \{z\}, V)$ is a strong deformation retraction, i..e, there exists a homotopy $h \colon W \setminus \{z\} \times I \to W \setminus \{z\}$ such that $h(y, 0) = y$ and $h(y, 1) \in V$ hold for $y \in W \setminus \{z\}$ and $h(y, t) = y$ holds for $y \in V$ and $t \in I$. Hence Lemma 4.3 follows from the next lemma.

**Lemma 4.4.** *Consider the situation of Lemma 4.3. Then there exists $z \in W \setminus V$ such that $f$ is homotopic relative $S^{k-1}$ to a map $D^k \to W \setminus \{z\}$.*

*Sketch of proof.* Fix $r \in (0, 1)$. Let $D^l_r \subseteq D^l$ be the open ball of radius $r$, i.e, $\{x \in D^l \mid ||x|| < r\}$. If $D^k \setminus f^{-1}(Q(D^l_r)) = \emptyset$, we are obviously done. Hence we can assume withtout loss of generality that $D^k \setminus f^{-1}(Q(D^l_r))$ is non-empty. Then one can arrange by an improved version of the Whitney Approximation Theorem that $f$ is homotopy relative to $D^k \setminus f^{-1}(Q(D^l_r))$ to a map $g \colon (D^l, S^{l-1}) \to (W, V)$ such that the map induced by $g$ from the open subset $f^{-1}(Q(D^l_r))$ of $D^k$ to the open subset $Q(D^L_r)$ of $W$, which we can be equipped with the structure of a smooth manifold diffeomorphic to $D^L_r$, is smooth.




Since by Sard's Theorem this smooth map $g$ has a regular value $z$ and $k < l$, we get $g(D^k) \subseteq W \setminus \{z\}$. □

This finishes the sketch of the proof of the Cellular Approximation Theorem 4.1.

**Corollary 4.5.** *Consider $n \geq 0$. Let $(X, A)$ be a CW-pair such that all cells in $X \setminus A$ have dimension $> n$. Then $(X, A)$ is $n$-connected. In particular $(X, X_n)$ is $n$-connected for a CW-complex $X$.*

*Proof.* We only deal with the case, where $A$ is non-empty. The proof for $A = \emptyset$ follows from the one, where $A = \{x\}$ for any zero-cell $\{x\} \in X$, since $X$ is the disjoint union of its path components and every path component contains a zero-cell.

First we show that $\pi_0(f) \colon \pi_0(A) \to \pi_0(X)$ is surjective for $n = 0$ and bijective for $n \geq 1$. Surjectivity follows from Cellular Approximation Theorem 4.1 applied to any map $\{\bullet\} \to X$ using the fact that $X_0 = A$ holds. Note for the sequel that any path component of a $CW$-complex must contain a zero-cell. By the Cellular Approximation Theorem 4.1 any path in $X$ connecting two zero-cells in $A$ is homotopic relative endpoints to a path in $A$ as $X_1 = A$ holds if $n \geq 1$. This shows the bijectivity of $\pi_0(f)$ if $n \geq 1$.

It remains to show that $\pi_i(X, A, a) = \{1\}$ holds for any base point $a \in A$ and $i \in \{1, 2, \ldots, n\}$ Since any path component of $A$ contains a zero-cell, diagrams (2.1) and (2.6) imply that we can assume without loss of generality that $a$ is a zero-cell of $A$. Consider an element $[f] \in \pi_i(X, A, a)$ given by a map of triples $f \colon (D^i, S^{i-1}, \{s\}) \to (X, A, \{a\})$. Equip $S^{i-1}$ with the $CW$-structure consisting of precisely two cells, namely one 0-cell $\{s\}$ given by the base point $s$ and one $(i-1)$-cell. By the Cellular Approximation Theorem 4.1 the map $f|_{S^{i-1}} \colon S^{i-1} \to A$ is relative $\{s\}$ homotopic to cellular map. One easily checks that this implies that $f \colon (D^i, S^{i-1}, \{s\}) \to (X, A, \{a\})$ is homotopic as a map of triples to a map $f'$ such that $f'|_{S^{i-1}} \colon S^{i-1} \to A$ is cellular. (This is a standard cofibration argument as we will see later, or done by direct inspection.) By the Cellular Approximation Theorem 4.1 the map $f'$ is homotopic relative $S^{i-1}$ to map $f'' \colon (D^i, S^{i-1}) \to (X_i, A)$.

As $X_i = A$ holds and hence $\pi_1(X_i, X_i, a)$ is trivial by the long exact sequence of the pointed pair $(X_i, X_i, a)$, see Theorem 2.11, we conclude $[f] = [f'] = [f''] = 1$ in $\pi_i(X, A, a)$. □

## 5. The Whitehead Theorem

In this section we want to prove the following theorem.

**Theorem 5.1** (Whitehead Theorem)**.** *Let $f\colon Y \to Z$ be a map.*

*(i) Consider any $n \in \{0, 1, 2, \dots\}$. Then the following assertions are equivalent:*
  - *(a) The map induced by composition with $f$*
  $$f_*\colon [X, Y] \to [X, Z], \quad [g] \mapsto [f \circ g]$$
  *is bijective for every CW-complex $X$ of dimension $\dim(X) < n$ and is surjective for every CW-complex $X$ of dimension $\dim(X) = n$;*
  - *(b) The map $f\colon Y \to Z$ is $n$-connected;*

*(ii) The following assertions are equivalent:*
  - *(a) The map induced by composition with $f$*
  $$f_*\colon [X, Y] \to [X, Z], \quad [g] \mapsto [f \circ g];$$
  *is bijective for every CW-complex $X$;*
  - *(b) The map $f\colon Y \to Z$ is a weak homotopy equivalence.*

Its proof needs some preparations.

**Lemma 5.2.** *Let $Y$ be a space which is $n$-connected for some $n \in \{0, 1, 2, \dots\} \amalg \{\infty\}$. Let $(X, A)$ be a relative CW-complex whose relative dimension $\dim(X, A)$ is less or equal to $n$.*

*Then any map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.*

*Proof.* We construct for $k = -1, 0, 1, 2, \dots$ with $k \le n$ maps $f_k\colon X_k \to Y$ such that $f_{-1}\colon X_{-1} = A \to Y$ is the given map $f$ and we have $f_k|_{X_{k-1}} = f_{k-1}$ for $k \ge 0$. Then Lemma 5.2 is a consequence of the following argument. If $n < \infty$, then we can take $F = f_n$. If $n = \infty$, we define $F = \operatorname{colim}_{k\to\infty} f_k$ having in mind that by the definition of a $CW$-pair we have $X = \operatorname{colim}_{k\to\infty} X_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to $k$ is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i\in I} S^{k-1} & \xrightarrow{\coprod_{i\in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^k & \xrightarrow{\coprod_{i\in I} Q_i} & X_k. \end{array}$$

We conclude from the pushout property that we can construct $f_k$ from $f_{k-1}$ if for any $i \in I$ we can extend the composite $f_{k-1} \circ q_i\colon S^{k-1} \to Y$ to a map $D^k \to Y$. This can be done as $Y$ is by assumption $k$-connected. □

**Lemma 5.3.** *Let $(Y, B)$ be a pair which is $n$-connected for some $n \in \{0, 1, 2, \dots\} \amalg \{\infty\}$. Let $(X, A)$ be a relative CW-complex whose relative dimension $\dim(X, A)$ is less or equal to $n$.*

*Then any map $f\colon (X, A) \to (Y, B)$ is homotopic relative $A$ to a map $f\colon (X, A) \to (Y, B)$ with $g(X) \subseteq B$.*

*Proof.* We construct for $k = -1, 0, 1, 2, \dots$ with $k \le n$ a map

$$h_k\colon X_k \times I \cup_{X_k \times \{0\}} X \times \{0\} \to Y$$

such that the following conditions are satisfied:

- $h_{-1}\colon A \times I \cup_{A\times\{0\}} \cup X \times \{0\} \to X$ sends $(a,t)$ to $f(a)$ for $(a,t) \in A \times I$ and $(x,0)$ to $f(x)$ for $x \in X$.
- We have $h_k(x,0) = f(x)$ for $x \in X$;
- We have $h_k(x,1) \in B$ for every $x \in X_k$
- For $0 \le k \le n$ we have $h_k|_{X_{k-1}\times I} = h_{k-1}|_{X_{k-1}\times I}$.

Then Lemma 5.3 is a consequence of the following argument. If $n < \infty$, then $h = h_n$ is the desired homotopy relative $A$ from $f$ to a map with image in $B$. Suppose $n = \infty$. Since $X = \operatorname{colim}_{k\to\infty} X_k$, we get $X \times I = \operatorname{colim}_{k\to\infty}(X_k \times I)$ and we obtain the desired homotopy $h$ by $\operatorname{colim}_{k\to\infty} h_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to $k$ is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i\in I} S^{k-1} & \xrightarrow{\coprod_{i\in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^k & \xrightarrow{\coprod_{i\in I} Q_i} & X_k. \end{array}$$

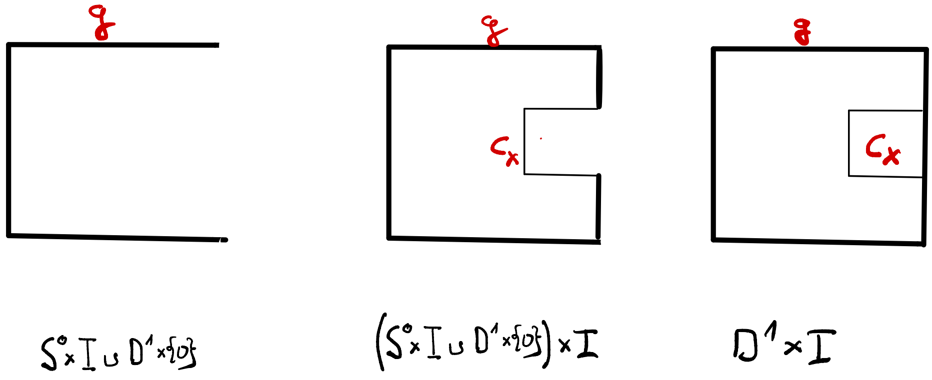
Then we obtain a pushout

$$\begin{array}{ccc} \coprod_{i\in I} S^{k-1} \times I \cup_{S^{k-1}\times\{0\}} D^k \times \{0\} & \xrightarrow{\coprod_{i\in I} q_i'} & X_{k-1} \times I \cup_{X_{k-1}\times\{0\}} X \times \{0\} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^k \times I & \xrightarrow{\coprod_{i\in I} Q_i'} & X_k \times I \cup_{X_k\times\{0\}} X \times \{0\} \end{array}$$

where $q_i'$ is given by $q_i \times \mathrm{id}_I \cup_{q_i\times \mathrm{id}_{\{0\}}} Q_i \times \mathrm{id}_{\{0\}}$. We conclude from the pushout property that it suffices to construct for every $i \in I$ an extension of the map

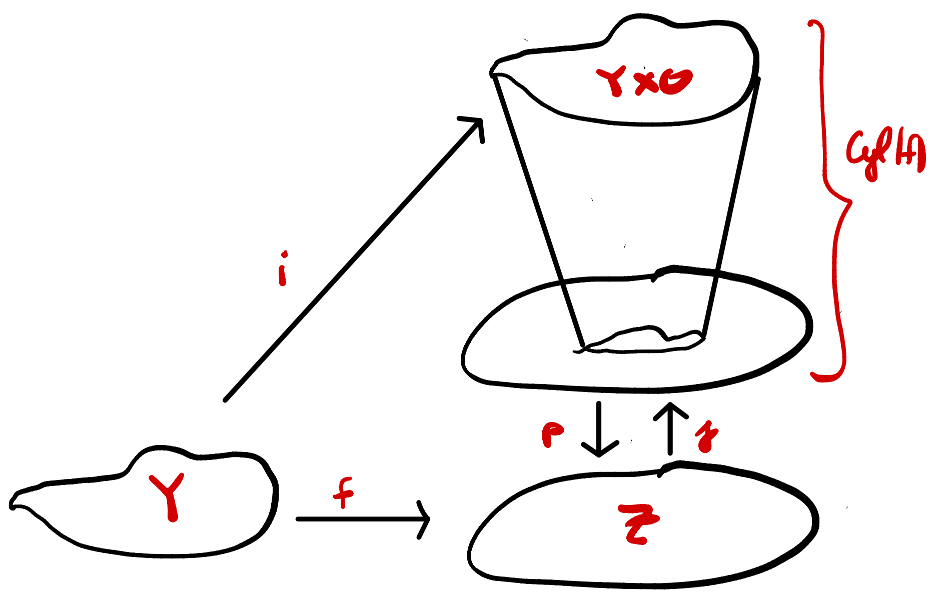
$$u = h_{k-1} \circ q_i'\colon S^{k-1} \times I \cup_{S^{k-1}\times\{0\}} D^k \times \{0\} \to Y$$

to a map $U\colon D^k \times I \to Y$ such that $g(D^k \times \{1\}) \subseteq B$ holds. Up to homeomorphism the pair $(S^{k-1} \times I \cup_{S^{k-1}\times\{0\}} D^k \times \{0\}, S^{k-1} \times \{1\})$ can be identified with $(D^k, S^{k-1})$. So we can think of $u$ as a map of triples $(D^k, S^{k-1}, \{s\}) \to (Y, B, \{x\})$ for $x = u(s)$. Hence it defines a element in $\pi_k(Y, B, x)$. As $\pi_k(Y, B, x)$ is by assumption trivial, there is a homotopy of maps of triples $(S^{k-1} \times I \cup_{S^{k-1}\times\{0\}} D^k \times \{0\}, S^{k-1} \times \{1\}, \{(s,1)\}) \to (Y, B, \{x\})$ from $u$ to the constant map $c_x$. Obviously the latter map extends to the constant map $c_x\colon D^k \times I, D^k \times \{1\}, \{(s,1\}) \to (Y, B, \{x\})$. Hence we can extend $u$ to a map $U\colon (D^k \times I, D^k \times \{1\}, \{s\}) \to (Y, B, \{x\})$.




This finishes the proof of Lemma 5.3. □

*Proof of the Whitehead Theorem 5.1.* Let $\mathrm{cyl}(f)$ be the mapping cylinder of $f$. Let $i\colon X \to \mathrm{cyl}(f)$ and $j\colon Y \to \mathrm{cyl}(f)$ be the canonical inclusions and $p\colon \mathrm{cyl}(f) \to Y$ be the canonical projection. Then $p \circ i = f$, $p \circ j = \mathrm{id}_Y$, and $j \circ p \simeq \mathrm{id}_{\mathrm{cyl}(f)}$. Hence we can assume without loss of generality that $f\colon Y \to Z$ is an inclusion of pairs, otherwise replace the given $f\colon Y \to Z$ by $i\colon Y \to \mathrm{cyl}(f)$.



$\implies$ (i)a and (ii)b $\implies$ (ii)b

The surjectivity of $f_*\colon [X,Y] \to [X,Z]$ follows for $\dim(X) \le n$ directly from Lemma 5.3 applied to a map $g\colon (X,\emptyset) \to (Z,Y)$. Finally we prove the injectivity of $f_*$ under the assumption that either $n = \infty$ or $\dim(X) < n < \infty$ holds. Consider $g_0, g_1\colon X \to Y$ and a homotopy $h\colon f \circ g_0 \simeq f \circ g_1$ of maps from $X$ to $Z$. We obtain a map of pairs $(h, g_0 \amalg g_1)\colon (X \times I, X \times \{0,1\}) \to (Z,Y)$. This map is homotopic relative $X \times \{0,1\}$ to a map $k\colon X \times I \to Y$ by Lemma 5.3 since $\dim(X \times I) \le n$ holds. Obviously $k$ is a homotopy of maps $X \to Y$ between $g_0$ and $g_1$.

(i)a $\implies$ (i)b and (ii)a $\implies$ (ii)b The map $\pi_0(f)\colon \pi_0(Y) \to \pi_0(Z)$ can be identified with the map $f_*\colon [\{\bullet\},Y] \to [\{\bullet\},Z]$. Hence the claim is true for $n = 0$. So it suffices to treat the case $n \in \mathbb{Z}^{\ge 1} \amalg \{\infty\}$. Then $\pi_0(f)$ is bijective. It remains to show for any $y \in Y$ that $\pi_k(f,y)\colon \pi_k(Y,y) \to \pi_k(Z,f(y))$ is bijective for $1 \le k < n$ and surjective for $1 \le k \le n$.

We begin with surjectivity for $1 \le k \le n$. Choose an index set $I$ and a map $v\colon (S,s) \to (Z,f(y))$ for $S = \bigvee_{i \in I} S^k$ equipped with the obvious base point $s$ such that $\pi_k(v,s)\colon \pi_k(S,s) \to \pi_k(Z,f(y))$ is surjective. Then we can find by assumption a map $u\colon S \to Y$ such that $f \circ u$ is homotopic to $v$. For an appropriate path $w\colon [0,1] \to Z$ from $u(s)$ to $z$, we obtain a commutative diagram

$$\begin{array}{ccc} \pi_k(Y,u(s)) & \xrightarrow{\pi_k(f,u(s))} & \pi_k(Z, f\circ u(s)) \\ {\scriptstyle \pi_k(u,s)}\uparrow & & \downarrow {\scriptstyle t_{[f\circ w]}}\ \cong \\ \pi_k(S,s) & \xrightarrow[\pi_k(v,s)]{} & \pi_k(Z,z). \end{array}$$

Next we show injectivity for $1 \le k < n$. Choose an index set $I$ and a map $u\colon (S,s) \to (Z,f(y))$ for $S = \bigvee_{i \in I} S^k$ equipped with the obvious base point $s$ such that the sequence $\pi_k(S,s) \xrightarrow{\pi_k(u,s)} \pi_k(Y,y) \xrightarrow{\pi_k(f,y)} \pi_k(Z,f(y))$ is exact. The the composite $f \circ u\colon S \to Z$ is nullhomotopic. Since $S$ has dimension $\le (n-1)$, the map $f_*\colon [S,Y] \to [S,Z]$ is bijective by assumption. Hence $u$ is nullhomotopic. This implies that there is a path $w\colon [0,1] \to Y$ from $y$ to some point $y'$ such that the composite $\pi_k(u,s)\colon \pi_k(S,s) \to \pi_k(Y,y)$ with the isomorphism $t_{[w]}\colon \pi_k(Y,y) \xrightarrow{\cong} \pi_k(Y,y')$ is trivial. Hence $\pi_k(u,s)\colon \pi_k(S,s) \to \pi_k(Y,y)$ is trivial. This implies that the kernel of $\pi_k(f,y)$ is trivial and hence that $\pi_k(f,y)$ is injective.

This finishes the proof of the Whitehead Theorem 5.1. □

**Corollary 5.4.** *Let $f\colon X \to Y$ be a map of CW-complexes. Then $f$ is a homotopy equivalence if and only $f$ is a weak homotopy equivalence.*

*Proof.* We conclude from the diagrams (2.1) and (2.6) that $f$ is a weak homotopy equivalence if it is a homotopy equivalence. Suppose that $f$ is a weak homotopy equivalence. Theorem 5.1 (ii) implies that $f_*\colon [Y,X] \to [Y,Y]$ is bijective. Let $a\colon Y \to X$ be map with $f_*([a]) = [f \circ a] = \mathrm{id}_Y$. Then $a$ is a weak homotopy equivalence. Theorem 5.1 (ii) again implies that $a_*\colon [X,Y] \to [X,X]$ is bijective. So we can choose a map $b\colon X \to Y$ with $[a \circ b] = [\mathrm{id}_X]$. This implies $b \simeq f \circ a \circ b \simeq f$. Hence $a$ is a homotopy inverse of $f$ and in particular $f$ is a homotopy equivalence. □

**Example 5.5** ($S^\infty$)**.** Define the real vector space $\mathbb{R}^\infty := \bigoplus_{i=1}^\infty \mathbb{R}$. It inherits a norm by

$$||(x_1,x_2,x_3,\ldots)|| = \sqrt{\sum_{i=1}^{\infty} x_i^2}.$$

In particular $\mathbb{R}^\infty$ inherits a metric and the structure of a topological space. We can identify the topological space $\mathbb{R}^n$ with the subspace consisting of points $(x_1, x_2, \ldots)$ for which $x_i = 0$ for $i > n$ holds. Let $S^\infty \subseteq \mathbb{R}^\infty$ be the subspace consisting of points $z$ satisfying $||z|| = 1$. Then $S^n$ can be identified with $S^\infty \cap \mathbb{R}^{n+1}$ for $n \geq 0$. Moreover, we get:

(i) We have the nested sequence $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^\infty$ such that $S^\infty$ is the unions of the $S^n$-s. The colimit topology with respect to this filtration is not the subspace topology $S^\infty \subseteq \mathbb{R}^\infty$;

(ii) $S^\infty$ equipped with the colimit topology carries a $CW$-structure with $S^n$ as $n$-skeleton;

(iii) $S^\infty$ equipped with the subspace topology does not carry the structure of a $CW$-complex;

(iv) $S^\infty$ equipped with the subspace topology is contractible;

(v) $S^\infty$ equipped with the colimit topology is contractible;

(vi) Consider the identity $S^\infty \to S^\infty$, where we equip the domain with the colimit topology and the codomain with the subspace topology. Then this map is bijective and continuous and is a homotopy equivalence but is not a homeomorphism.

For $n \geq 1$ consider the element $a_n$ in $S^\infty$ whose $i$-th entry is $\sqrt{1 - n^{-1}}$ for $i = 1$, $n^{-1}$ for $i = 2, \ldots, n+1$, and is 0 for $i \geq (n+2)$. Let $A = \{a_n \mid n \geq 1\}$. Since the intersection of $A$ with $S^n$ is finite for $n \geq 1$, it is a closed subspace of $S^\infty$ with respect to the colimit topology. Since $(1, 0, 0, \ldots)$ does not belong to $A$ and $\lim_{n\to\infty} a_n = (1, 0, 0 \ldots)$ holds with respect to the metric above, $A$ is not closed with respect to the subspace topology. This finishes the proof of assertion (i).

We leave the obvious proof of the assertion (ii) is left to the reader.

Assertion (iii) is proved as follows. Suppose that $S^\infty$ with the subspace topology has a $CW$-structure. Since then $S^\infty$ is a metrizable $CW$-complex, it must be locally compact by [6, Theorem B on page 81]. This implies there is an $\epsilon > 0$ such that that the intersection of $S^\infty$ with the closed ball of radius $\epsilon$ around $(1, 0, 0, \ldots)$ is compact. Hence we can find $\delta > 0$ such that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sqrt{1-\delta} \cdot e_1 + \sqrt{\delta} \cdot e_n$ with $e_i$ the $i$-th element of the standard base belongs to the intersection of $S^\infty$ with the closed ball of radius $\epsilon$ around $(1, 0, 0, \ldots)$. Hence it has a subsequence which is a Cauchy sequence. Since this is not the case, we get a contradiction.

Next we prove assertion (iv). Let $s\colon S^\infty \to S^\infty$ be the shift map sending $(x_1, x_2, x_3, \ldots)$ to $(0, x_1, x_2, x_3, \ldots)$ Define

$$h\colon S^\infty \times I \to S^\infty, \quad x \mapsto \frac{t \cdot s(x) + (1-t) \cdot x}{||t \cdot s(x)x + (1-t) \cdot x||}.$$

This is a homotopy between $\mathrm{id}_{S^\infty}$ and $s$. Now consider the homotopy

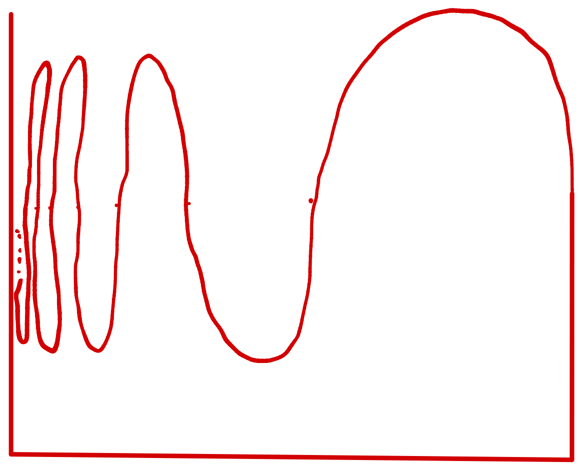
$$k\colon S^\infty \times I \to S^\infty, \quad x \mapsto \frac{(1-t) \cdot s(x) + t \cdot e_1}{||(1-t) \cdot s(x) + t \cdot e_1||}$$

for $e_1 = (1, 0, 0, \ldots)$. Then $k$ is a homotopy between $s$ and the constant map $S^\infty \to S^\infty$ with value $e_1$. Hence $S^\infty$ with the subspace topology is contractible.

Assertion (v) follows from Theorem 2.20, Theorem 3.4, and Corollary 5.4 using assertion (ii). Alternatively, the proof for assertion (iv) does carry over to assertion (v).

Assertion (vi) is a direct consequence of the other assertions.

**Example 5.6** (Warsaw circle)**.** Let $W$ be the *Warsaw circle*, i.e., the compact subsets of $\mathbb{R}^2$ given by the union of $\{(x, \sin(2\pi/x)) \mid x \in (0, 1]\}$, $\{(1, y) \mid y \in [-2, 0]\}$, $\{(x, -2) \mid x \in [0, 1]\}$ and $\{(0, y) \mid y \in [-2, 1]\}$.



Then the projection $p\colon W \to \{\bullet\}$ is a weak homotopy equivalence but not a homotopy equivalence. In particular $W$ is a compact space which is not homotopy equivalent to a $CW$-complex.

**Remark 5.7** (Whitehead Theorem for pairs)**.** There is the following version of the Whitehead Theorem 5.1 (ii) for pairs. Let $(F, f)\colon (Y, B) \to (Z, C)$ be a map of pairs. Then the following assertions are equivalent:

(i) The maps $F\colon Y \to Z$ and $f\colon B \to C$ are weak homotopy equivalences;
(ii) For every $CW$-pair $(X, A)$ the maps of the homotopy classes of pairs induced by composition with $(F, f)$

$$(F, f)_*\colon [(X, A), (Y, B)] \to [(X, A), (Z, C)], \quad [(G, g)] \mapsto [(F \circ G, g \circ f)]$$

is bijective.

## 6. CW-Approximation

**Definition 6.1** ($n$-coconnected maps)**.** A map $f\colon X \to Y$ is called *$n$-coconnected* for $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$, if for any base point $x \in X$ the map $\pi_i(f, x)\colon \pi_i(X, x) \to \pi_i(Y, f(x))$ is injective if $i = n$, and is bijective if $i > n$.

Consider a natural number $n$ and a map $f\colon X \to Y$. Then $f$ is a weak homotopy equivalence if and only if it is both $n$-connected and $n$-coconnected.

**Definition 6.2** ($n$-$CW$-model for a pair)**.** Consider a topological pair $(Y, A)$ such that $A$ is a $CW$-complex and $n \in \mathbb{Z}^{\geq 0}$. (The subcomplex $A$ may be empty.) An *$n$-$CW$-model* for $(Y, A)$ consists of an $n$-connected pair of $CW$-complexes $(Z, A)$ together with an $n$-coconnected map $f\colon Z \to Y$ satisfying $f|_A = \mathrm{id}_A$.

**Theorem 6.3** ($n$-$CW$-models)**.** *Consider a topological pair $(Y, A)$ such that $A$ is a $CW$-complex and $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Then there exists an $n$-$CW$-model*

$$(f, \mathrm{id}_A)\colon (Z, A) \to (Y, A)$$

*such that $Z \setminus A$ contains no cells of dimension $\leq n$.*

*Proof.* We construct a sequence of nested spaces $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \cdots$ and maps $f_i\colon Z_i \to Y$ for $i \geq n$ such that the following holds:

- $Z_n = A$ and $f_n = \mathrm{id}_A$;
- $f_i|_{Z_{i-1}} = f_{i-1}$ for $i = (n+1), (n+2), \ldots$;
- There exists for $i \geq n$ a pushout of the shape

$$\begin{array}{ccc} \coprod_{j \in J_i} S^i & \xrightarrow{\coprod_{j_i \in J_i} q_i^j} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{j \in J_i} D^{i+1} & \xrightarrow{\coprod_{j \in J_i} Q_i^j} & Z_{i+1} \end{array}$$

  such that the image of each map $q_j$ does not meet any closed cell in $A$ of dimension $> i$;
- For any base point $z \in Z_i$ the map $\pi_j(f_i, z)$ is injective for $j = n$, bijective for $n < j \leq i$, and surjective for $j = i$.

Before we explain the construction of these data, we explain how we get the desired $n$-$CW$-model from it. Namely, we define $Z = \operatorname{colim}_{i\to\infty} Z_i$ and $f = \operatorname{colim}_{i\to\infty} f_i \colon Z \to X$. Then $(Z, A)$ is a $CW$-pair and the $i$-skeleton $Z_i$ of $Z$ is the complement of the union of the open cells of dimension $> i$ of $A$ in $Z_i$. In particular $Z \setminus Z_i$ contains no cells of dimension $\leq i$. Since $Z \setminus A$ contains no $k$-cells for $0 \leq k \leq n$, the pair $(Z, A)$ is $n$-connected by Corollary 4.5. We conclude from Corollary 4.5 again that the map $\pi_m(Z_i, z_i) \to \pi_m(Z, z_i)$ induced by the inclusion $Z_i \to Z$ is bijective for $m < i$ and surjective for $m = i$ for any $i \geq n$ and $z_i \in Z_i$. Hence the map $f$ is $n$-coconnected by Theorem 2.20.

Finally we carry out the construction of the sequence $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \cdots$ and the sequence of maps $f_i \colon Z_i \to Y$. The induction beginning is obvious, take $Z_n = A$ and $f_n = \mathrm{id}_A$. The induction step how to construct $Z_{i+1}$ and $f_{i+1}$, when $Z_i$ and $f_i$ have already been established, is done as follows. For each path component $C$ of $A$ choose a zero-cell $x_C$ in $A$ which is contained in $C$. Then for every element $u$ in the kernel of the map $\pi_i(f_i, x_C) \colon \pi_i(Z_i, x_C) \to \pi_i(Y, x_C)$ choose a pointed map $q_{C,u} \colon (S^i, s) \to (Z_i, x_C)$ with $u = [q_{C,u}]$. The define $Z'_{i+1}$ as the pushout

$$\begin{array}{ccc} \coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} S^i & \xrightarrow{\coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} q_{C,u}} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{\substack{C \in \pi_0(A) \\ u \in \ker(\pi_i(f_i, x_C))}} D^{i+1} & \longrightarrow & Z'_{i+1}. \end{array}$$

Since each $[q_{C,u}]$ lies in the kernel of $\pi_i(f_i, x_C)$, each map $f_i \circ q_{C,u} \colon S^i \to Y$ can be extended to a map $\overline{q_{C,u}} \colon D^{i+1} \to Y$. By the Cellular Approximation Theorem 4.1 we can additionally arrange that the image of each map $\overline{q_{C,u}}$ has trivial intersection with any open cell of $A$ of dimension $i > i + 1$.

The collection of these extensions yield a map $f'_{i+1} \colon Z'_{i+1} \to Y$ by the pushout property. We have for $j \leq (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z_i, x_C) & \xrightarrow{\pi_j(f_i, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow_{\pi_j(f'_{i+1}, x_C)} & \\ \pi_j(Z'_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z_i \to Z_{i+1}$. The vertical arrow is bijective for $j \leq (i-1)$ and surjective for $j = i$ by Corollary 4.5. Hence

$\pi_j(f'_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \leq (i-1)$, as $\pi_j(f_i, x_C)$ has these properties by the induction hypothesis. Consider an element $v$ in the kernel of $\pi_i(f'_{i+1}, x_C)$. Choose $u \in \pi_i(Z_i, x_C)$ whose image under the vertical arrow is $v$. Then $u$ lies in the kernel of $\pi_i(f_i, x_C)$. By construction $u$ lies in the kernel of the vertical arrow. Hence $v$ is trivial. Therefore $\pi_i(f'_{i+1}, x_C)$ is injective. As $\pi_i(f_i, x_C)$ is surjective by the induction hypothesis, $\pi_i(f'_{i+1}, x_C)$ is surjective. This implies that $\pi_j(f'_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \leq i$ for all $C \in \pi_0(A)$.

Now consider any $C \in \pi_0(A)$ and any element $[w_C] \in \pi_{i+1}(Y, x_C)$. Choose a map $w_C\colon (S^{i+1}, s) \to (Y, x_C)$ representing $[w_C]$. Define the desired space $Z_{i+1}$ and the desired map $f_{i+1}\colon Z_{i+1} \to Y$ by

$$\begin{aligned} Z_{i+1} &= Z'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} S^{i+1}, \\ f_{i+1} &= f'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} w_C. \end{aligned}$$

We have for $j \leq (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z'_{i+1}, x_C) & \xrightarrow{\pi_j(f'_{i+1}, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow_{\pi_j(f_{i+1}, x_C)} & \\ \pi_j(Z_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z'_{i+1} \to Z_{i+1}$. The left vertical arrow is bijective for $j < i$ and surjective for $j = i$ by Corollary 4.5. It is also injective for $j = i$, since the inclusion $Z'_{i+1} \to Z_{i+1}$ has an obvious retraction $Z_{i+1} \to Z'_{i+1}$. Hence the left vertical arrow is bijective for $j \leq i$. This implies that $\pi_j(f_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \leq i$ for all $C \in \pi_0(A)$. Moreover, by construction any element $[w_C]$ is in the image of $\pi_j(f_{i+1}, x_C)$. Hence $\pi_j(f_{i+1}, x_C)$ is surjective for all $C \in \pi_0(A)$. Since $\pi_0(A) \to \pi_0(Z_{i+1})$ is surjective, we conclude from the diagrams (2.1) and (2.6) that for any base point $z \in Z_{i+1}$ the map $\pi_j(f_{i+1}, z)$ is injective for $i = n$, bijective for $n < j \leq i$, and surjective for $j = (i+1)$.

This finishes the proof of Theorem 6.3. □

**Remark 6.4.** One can think of the $n$-$CW$-model $f\colon (Z, A) \to (Y, A)$ as a sort of homotopy theoretic hybrid of $A$ and $Y$. If $n = 0$ and $Y$ is path connected, then the hybrid looks like $Y$ in the sense that $f$ is a weak homotopy equivalence. As $n$ increases, the hybrid looks more and more like $A$, and less and less like $Y$. If we take $n = \infty$, then the inclusion $A \to Z$ is a weak homotopy equivalence and can actually be realized by $Z = A$ and $\mathrm{id}_A$.

More precisely, if $k\colon A \to Z$ and $l\colon A \to Y$ are the inclusions and $a \in A$ is a base point, we get a factorization

$$\pi_i(l, a)\colon \pi_i(A, a) \xrightarrow{\pi_i(k,a)} \pi_i(Z, a) \xrightarrow{\pi_i(f,a)} \pi_i(Y, a)$$

such that the following holds:

- If $i < n$, then the first map $\pi_i(k, a)$ is bijective;
- If $i = n$, then the first map $\pi_i(k, a)$ is surjective and the second map $\pi_i(f, a)$ is injective;
- If $i > n$, then the second map $\pi_i(f, a)$ is bijective.

**Corollary 6.5.** *Consider a $CW$-pair $(X, A)$ and $n \in \mathbb{N}$. Then the following assertions are equivalent:*

*(i) There is a $CW$-pair $(Z, A)$ such that $(X, A)$ and $(Z, A)$ are homotopy equivalent relative $A$ and $Z \setminus A$ contains no cells of dimension $\leq n$;*
*(ii) The pair $(X, A)$ is $n$-connected.*

*Proof.* (i) $\implies$ (ii) This follows from Corollary 4.5.

(ii) $\implies$ (i) We obtain from Theorem 6.3 an $n$-model $(f, \mathrm{id}_A) \colon (Z, A) \to (X, A)$ such that $Z \setminus A$ contains no cells of dimension $\leq n$. Since $(Z, A)$ and $(X, A)$ are $n$-connected and $f$ is $n$-coconnected, $f \colon Z \to X$ is a weak homotopy equivalence inducing the identity on $A$. A version of the Whitehead Theorem 5.1 (ii) relative $A$ implies that $(X, A)$ and $(Z, A)$ are homotopy equivalent relative $A$. $\square$

In particular any path connected $CW$-complex is homotopy equivalent to a $CW$-complex $Z$ having precisely one 0-cell.

**Example 6.6.** Let $X$ be path connected $CW$-complex. We conclude from Theorem 2.7 that a 1-connected $CW$-model for $X = (X, \emptyset)$ is given by the universal covering $\widetilde{X} \to X$.

For this section the case $n = 0$ is important which we treat next.

**Definition 6.7.** Consider a space $Y$. A *$CW$-approximation* $(X, f)$ of $Y$ is a $CW$-complex $X$ together with a weak homotopy equivalence $f \colon X \to Y$.

**Theorem 6.8** (Existence and uniqueness of $CW$-approximations)**.** *Let $Y$ be a topological space. Then:*
*(i) There exists a $CW$-approximation $(X, f)$ of $Y$;*
*(ii) Let $(X, f)$ and $(X', f')$ be two $CW$-approximations of $Y$. Then there exists a homotopy equivalence $g \colon X \to X'$ for which the following diagram commutes up to homotopy*

$$\begin{array}{ccc} X & \xrightarrow{\;\;g\;\;} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y. & \end{array}$$

*The homotopy equivalence $g$ is up to homotopy uniquely determined by the property $f' \circ g \simeq f$.*

*Proof.* (i) Consider a path component $C$ of $Y$. From Theorem 6.3 applied to the pair $(C, \emptyset)$ and $n = 0$ we obtain a $CW$-complex $X_C$ and weak homotopy equivalence $f_C \colon X_C \to C$. Then we get from $X = \coprod_{C \in \pi_0(C)} X_C$ and $f = \coprod_{C \in \pi_0(C)} f_C$ a $CW$-approximation of $Y$.

(ii) We conclude from the Whitehead Theorem 5.1 (ii) that there exists a map $g \colon X \to X'$ which is uniquely determined up to homotopy by the property $f' \circ g \simeq f$. The map $g$ is a weak homotopy equivalence and hence a homotopy equivalence by Corollary 5.4. $\square$

**Remark 6.9.** One may think of Theorem 6.8 as the topological analogue of the fact that any positive $R$-chain complex $C_*$ possesses a *projective $R$-resolution* $f_* \colon P_* \to C_*$, i.e., a projective positive $R$-chain complex $P_*$ together with an $R$-chain map $f_* \colon P_* \to C_*$ inducing an isomorphism on all homology modules, and that for two projective resolutions $(P_*, f_*)$ and $(P'_*, f'_*)$ of $C_*$ there is a $R$-chain homotopy equivalence $g_* \colon P_* \to P'_*$ which is a up to $R$-chain homotopy uniquely determined by the property $f'_* \circ g_* \simeq f_*$.

**Theorem 6.10.** *Let $f \colon X \to Y$ be a weak homotopy equivalence of spaces. Then the induced map on singular homology $H_n(f) \colon H_n(X) \to H_n(Y)$ is bijective for all $n \geq 0$.*

*Proof.* See [24, Theorem 9.5.3 on page 237]. □

**Remark 6.11** ($CW$-approximations for pairs)**.** Consider a pair $(Y, B)$. Choose a $CW$-approximation $u\colon A \to B$ for $B$. Let $\operatorname{cyl}(u)$ be the mapping cylinder of $u$. It contains the $CW$-complex $A$ as subspace. Let $g\colon (X, A) \to (\operatorname{cyl}(f), A)$ be a 0-$CW$-model which exists by Theorem 6.3. Thus we obtain a pair of $CW$-complexes $(X, A)$ together with a weak homotopy equivalence $g\colon X \to Y$ satisfying $g|_A = \operatorname{id}_A$. Let $p\colon \operatorname{cyl}(f) \to Y$ be the projection which is a homotopy equivalence and satisfies $p|_A = u$. Let $f\colon X \to Y$ be the composite $p \circ v$. Then $f\colon X \to Y$ and $f|_A = u\colon A \to B$ are weak homotopy equivalences. So we obtain a $CW$-approximation $(f, u)\colon (X, A) \to (Y, B)$ for pairs.

A relative version of the Whitehead Theorem 5.1 (ii), see Remark 5.7, shows that for two such $CW$-approximations $f\colon (X, A) \to (Y, B)$ and $f'\colon (X', A') \to (Y, B)$ there is a homotopy equivalence of pairs $g\colon (X, A) \to (X', A')$ which is up to homotopy uniquely determined by the property that $f$ and $f' \circ g$ are homotopic as maps of pairs $(X, A) \to (Y, B)$.

## 7. The category of compactly generated spaces

We briefly recall some basics about compactly generated spaces. More information and proofs can be found in [18]. A topological space $X$ is *compactly generated* if it is a Hausdorff space and a set $A \subseteq X$ is closed if and only if for any compact subset $C \subset X$ the intersection $C \cap A$ is a closed subspace of $C$.

Every locally compact space, and every space satisfying the first axiom of countability, e.g., a metrizable space, is compactly generated. If $p\colon X \to Y$ is an identification of topological spaces and $X$ is compactly generated and $Y$ is Hausdorff, then $Y$ is compactly generated. A closed subset of a compactly generated space is again compactly generated. For open subsets one has to be careful as it is explained in Subsection 7.1.

### 7.1. Open subsets.

Recall that a topological space $B$ is called *regular* if for any point $x \in X$ and closed set $A \subseteq X$ there exists open subsets $U$ and $V$ with $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. A Hausdorff space is called *locally compact* if every $x \in X$ possesses a compact neighborhood. Equivalently, for every $x \in X$ and open neighborhood $U$ there exists an open neighborhood $V$ of $x$ such that the closure of $V$ in $X$ is compact and contained in $U$, see [16, Lemma 8.2 in Section 3-8 on page 185].

**Definition 7.1** (Quasi-regular open set and regular space)**.** An open subset $U \subseteq B$ is called *quasi-regular* if for any $x \in X$ there exists an open neighborhood $V_x$ whose closure in $B$ is contained in $U$.

**Lemma 7.2.**

- *(i) Let $B$ be a compactly generated Hausdorff space. A quasi-regular open subset $U \subseteq B$ equipped with the subspace topology is compactly generated;*
- *(ii) Let $f\colon X \to Y$ be a (continuous) map between (not necessarily compactly generated) spaces. If $V \subseteq Y$ is a quasi-regular open subset, then $f^{-1}(V) \subseteq X$ is a quasi-regular open subset;*
- *(iii) The intersection of finitely many quasi-regular open subsets is again a quasi-regular open subset;*
- *(iv) A space is regular if and only if every open subset is quasi-regular;*
- *(v) Any locally compact Hausdorff space, any metrizable space, and every $CW$-complex are regular;*
- *(vi) Every open subset of a $CW$-complex is quasi-regular and, equipped with the subspace topology, compactly generated.*

*Proof.* (i) See [18, page 135].

(ii) Consider a point $x \in f^{-1}(V)$. Choose an open set $W$ of $Y$ such that $f(x) \in W$ and the closure of $W$ in $B$ is contained in $V$. Then $f^{-1}(W)$ is an open subset of $X$ which contains $x$ and whose closure in $X$ is contained in $f^{-1}(V)$.

(iii) Let $U_1$, $U_2$, ..., $U_r$ be quasi-regular open subsets. Consider $x \in U := \bigcap_{i=1}^r U_i$. Choose for every $i = 1, 2 \ldots, r$ an open subset $V_i$ with $x \in V_i$ such that the closure $\overline{V_i}$ of $V_i$ in $B$ is contained in $U_i$. Put $V := \bigcap_{i=1}^r V_i$. Then $x \in V$ and $\overline{V} \subseteq \cap_{i=1}^r \overline{V_i} \subseteq U$. Hence $U$ is a quasi-regular open subset.

(iv) See [16, Lemma 2.1 in Section 4-2 on page 196].

(v) This is obvious for locally compact spaces. Metrizable spaces are treated in [16, Theorem 2.3 in Section 4-2 on page 198]. Every $CW$-complex is paracompact, see [15], and hence in particular regular, see [16, Theorem 4.1 in Section 6-4 on page 255].

(vi) This follows from assertions (i), (iv), and (v). □

7.2. **The retraction functor $k$.** There is a construction which assigns to a topological Hausdorff space $X$ a new topological space $k(X)$ such that $X$ and $k(X)$ have the same underlying sets, $k(X)$ is compactly generated, $X$ and $k(X)$ have the same compact subsets, the identity $k(X) \to X$ is continuous and is a homeomorphism if and only if $X$ is compactly generated. Namely, define the new topology on $k(X)$ by declaring a subset $A \subseteq X$ to be closed if and only if for every compact subset of $X$ the intersection $A \cap C$ is a closed subset of $C$.

7.3. **Mapping spaces, product spaces, and subspaces.** Given two compactly generated spaces $X$ and $Y$, denote by $\operatorname{map}(X,Y)_{k.o.}$ the set of maps $X \to Y$ with the compact-open-topology, i.e., a subbasis for the compact-open-topology is given by the sets $W(C,U) = \{f \colon X \to Y \mid f(C) \subseteq U\}$, where $C$ runs through the compact subsets of $X$ and $U$ runs though the open subsets of $Y$. Note that $\operatorname{map}(X,Y)_{k.o.}$ is not compactly generated in general. We denote by $\operatorname{map}(X,Y)$ the topological space given by $k(\operatorname{map}(X,Y)_{k.o.})$. Sometimes we abbreviate $\operatorname{map}(X,Y)$ by $Y^X$ and denote for a map $f \colon Y \to Z$ the induced map $\operatorname{map}(\operatorname{id}_X, f) \colon \operatorname{map}(X,Y) \to \operatorname{map}(X,Z)$, $g \mapsto f \circ g$ by $f^X \colon Y^X \to Z^X$. If $X$ and $Y$ are compactly generated spaces, then $X \times Y$ stands for $k(X \times_p X)$, where $X \times_p Y$ is the topological space with respect to the "classical" product topology.

If $A \subseteq X$ is a subset of a compactly generated space, the subspace topology means that we take $k(A_{st})$ for $A_{st}$ the topology space given by the "classical" subspace topology on $A$.

Roughly speaking, all the usual constructions of topologies are made compactly generated by passing from $Y$ to $k(Y)$ in order to stay within the category of compactly generated spaces.

7.4. **Basic features of the category of compactly generated spaces.** The category of compactly generated spaces has the following convenient features:

- A map $f \colon X \to Y$ of compactly generated spaces is continuous if and only if its restriction $f|_C \colon C \to Y$ to any compact subset $C \subseteq X$ is continuous;
- If $X$, $Y$, and $Z$ are compactly generated spaces, then the obvious maps

$$\begin{aligned} \operatorname{map}(X, \operatorname{map}(Y,Z)) &\xrightarrow{\cong} \operatorname{map}(X \times Y, Z); \\ \operatorname{map}(X, Y \times Z) &\xrightarrow{\cong} \operatorname{map}(X,Y) \times \operatorname{map}(X,Z), \end{aligned}$$

  are homeomorphisms and the map given by composition

$$\operatorname{map}(X,Y) \times \operatorname{map}(Y;Z) \to \operatorname{map}(X,Z)$$

is continuous;
- The product of two identifications is again an identification;
- If $X$ is locally compact and $Y$ compactly generated, then $X \times Y$ and $X \times_p Y$ are the same topological spaces;
- Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of inclusions of compactly generated spaces such that $X_i$ is a closed subspace of $X_{i+1}$ for $i = 0, 1, 2, \ldots$.

  Then the colimit $\operatorname{colim}_{i\to\infty} X_i$ exists in the category of compactly generated Hausdorff spaces. Moreover, if $Y$ is a compactly generated space, then $\operatorname{colim}_{i\to\infty}(X_i \times Y)$ exists in the category of compactly generated spaces and the canonical map

$$\operatorname{colim}_{i\to\infty}(X_i \times Y) \xrightarrow{\cong} \big(\operatorname{colim}_{i\to\infty} X_i\big) \times Y$$

  is a homeomorphism;
- In the category of compactly generated spaces the pushout of a diagram $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ exists if $f_1$ or $f_2$ is the inclusion of a closed subspace;
- Given a Hausdorff space $Y$, the canonical map $k(Y) \to Y$ is a weak homotopy equivalence and induces an isomorphism on singular homology.
- Given a pushout in the category of compactly generated spaces, its product with a compactly generated space is again a pushout in the category of compactly generated spaces.
- The product of two $CW$-complexes is again a $CW$-complex;

**Remark 7.3** (Compactly generated weak Hausdorff spaces)**.** There is also the category of compactly generated weak Hausdorff spaces, see [19]. The main advantage in contrast to the category of compactly generated Hausdorff spaces, see [18], is that in the category of compactly generated weak Hausdorff spaces colimits for small diagrams, for instance pushouts or filtered colimits, always exist, see [19, Corollary 2.23]. In the category of compactly generated spaces one can define the pushout of a diagram $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ only if for the pushout in the classical setting

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \overline{f_2}} \\ X_2 & \xrightarrow{\overline{f_1}} & X \end{array}$$

the space $X$ is Hausdorff, since the retraction functor $k$ digests only Hausdorff spaces. Note that $X$ is Hausdorff if $f_1$ or $f_2$ is an inclusion of a closed subspace. Therefore in the case treated in the manuscript this condition is always satisfies and the pushout exists in the category of compactly generated Hausdorff spaces.

The same discussion applies to the colimit $\operatorname{colim}_{i\to\infty} X_i$ of a sequence of inclusions of compactly generated spaces of $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$.

For simplicity we will discuss these issues not anymore and will work in the category of compactly generated Hausdorff spaces throughout this manuscript.

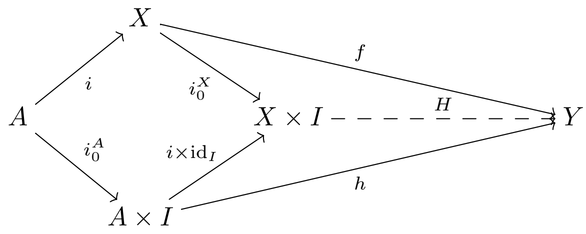
## 8. Cofibrations

### 8.1. Basics about cofibrations.

**Definition 8.1** (Homotopy extension property)**.** A map $i\colon A \to X$ has the *homotopy extension property* (HEP) for the space $Y$, if for any map $f\colon X \to Y$ and any homotopy $h\colon A \times I \to Y$ with $h_0 = f \circ i$, there exists a homotopy $H\colon X \times I \to Y$ with $H_0 = f$ and $H \circ (i \times \mathrm{id}_I) = h$.

In other words, HEP for the space $Y$ means that the extension problem indicated by the following diagram has a solution $H$ for every map $f\colon X \to Y$ and homotopy $h\colon A \times I \to Y$ satisfying $h(a, 0) = f(a)$ for every $a \in A$

(8.2)




where $i_0^A(a) = (a, 0)$ for $a \in A$ and $i_0^X(x) = (x, 0)$ for $x \in X$.

Equivalently, one may describe the homotopy extension property by the following diagram

(8.3)
$$\begin{array}{ccc} A & \xrightarrow{\overline{h}} & \operatorname{map}(I, Y) \\ {\scriptstyle i}\downarrow & \overset{\overline{H}}{\nearrow} & \downarrow{\scriptstyle e_Y^0} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

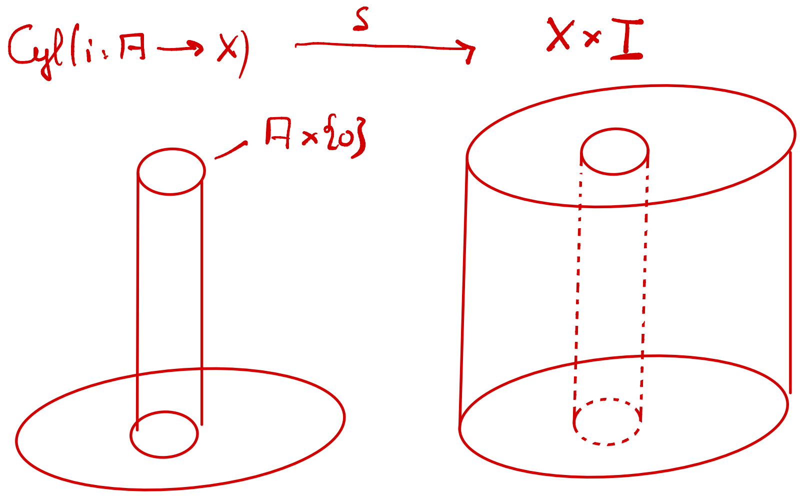
where $e_Y^0$ is given by evaluation at 0 and $\overline{h}$ corresponds to $h$ under the adjunction homeomorphism $\operatorname{map}(A \times I, Y) \xrightarrow{\cong} \operatorname{map}(A, \operatorname{map}(I, Y))$, and analogously for $H$ and $\overline{H}$.

**Definition 8.4** (Cofibration)**.** A map $i\colon A \to X$ is called a *cofibration* if it has the homotopy extension property for every space $Y$.

Recall that the mapping cylinder of a map $i\colon A \to X$ is defined by the pushout

(8.5)
$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle i_0^A}\downarrow & & \downarrow{\scriptstyle k} \\ A \times I & \xrightarrow[l]{} & \operatorname{cyl}(i) \end{array}$$

and there is a canonical map $s\colon \operatorname{cyl}(i) \to X \times I$ defined as the pushout of the inclusion $i_0^X\colon X \to X \times I$ and the map $i \times \mathrm{id}_I\colon A \times I \to X \times I$.




**Proposition 8.6.** *The following assertions are equivalent for a map* $i\colon A \to X$*:*

(i) *The map* $i\colon A \to X$ *is a cofibration;*
(ii) *The map* $i\colon A \to X$ *has the homotopy extension property for the mapping cylinder* $\operatorname{cyl}(i)$*;*
(iii) *The canonical map* $s\colon \operatorname{cyl}(i) \to X \times I$ *has a retraction* $r\colon X \times I \to \operatorname{cyl}(i)$*.*

*Proof.* (i) $\implies$ (ii) This is obvious.

(ii) $\implies$ (iii) If we apply the homotopy lifting property to the map $k\colon X \to \operatorname{cyl}(i)$ and the homotopy $l\colon A \times I \to \operatorname{cyl}(f)$, we obtain a map $r\colon X \times I \to \operatorname{cyl}(i)$ such that $r \circ i_0^X = k$ and $r \circ (i \times \operatorname{id}_I) = l$ hold. Since we have $r \circ s \circ k = r \circ i_0^X = k$ and $r \circ s \circ l = r \circ (i \times \operatorname{id}_I) = l$, we conclude from the pushout property that $r \circ s = \operatorname{id}_{\operatorname{cyl}(i)}$ holds.

(iii) $\implies$ (i) Consider any map $f\colon X \to Y$ and any homotopy $h\colon A \times I \to Y$ satisfying $h_0 = f|_A$. We obtain from the pushout property a map $a\colon \operatorname{cyl}(i) \to Y$ such that $a \circ k = f$ and $a \circ l = h$ hold. Now define $H\colon X \times I \to Y$ to be $a \circ r$. Then $H \circ (i \times \operatorname{id}_I) = a \circ r \circ (i \times \operatorname{id}_I) = a \circ r \circ s \circ l = a \circ l = h$ and $H \circ i_0^X = a \circ r \circ i_0^X = a \circ r \circ s \circ k = a \circ k = f$ hold. Therefore $i$ has the homotopy lifting property for every space $Y$ and hence is a cofibration. □

**Remark 8.7** (Cofibrations are closed embeddings)**.** Note that Proposition 8.6 implies that a cofibration $i\colon A \to X$ is a *closed embedding*, i.e., its image $i(A)$ is a closed subspace of $X$ and that $i$ induces a homeomorphism $A \xrightarrow{\cong} i(A)$. Namely, the the composite $j\colon A \xrightarrow{i_1^A} A \times I \xrightarrow{l} \operatorname{cyl}(i)$ is a closed embedding and $j$ can be written as the composite $A \xrightarrow{i} X \xrightarrow{i_1^X} X \times I \xrightarrow{r} \operatorname{cyl}(i)$ because of $r \circ i_1^X \circ i = r \circ (i \times \operatorname{id}_I) \circ i_1^A = r \circ s \circ l \circ i_1^A = l \circ i_1^A = j$. Now use the fact that a map $u$ is a closed embedding if the composite $v \circ u$ of it with some other map $v$ is a closed embedding.

**Lemma 8.8.**

(i) *If* $i\colon A \to X$ *is a cofibration, then there exists a retraction* $r\colon X \times I \to X \times \{0\} \cup_{A \times \{0\}} A \times I$*;*

(ii) *If there exists a retraction* $r\colon X \times I \to X \times \{0\} \cup_{A\times\{0\}} A \times I$ *and the inclusion* $A \to X$ *is a closed embedding, then the inclusion* $i\colon A \to X$ *is a cofibration.*

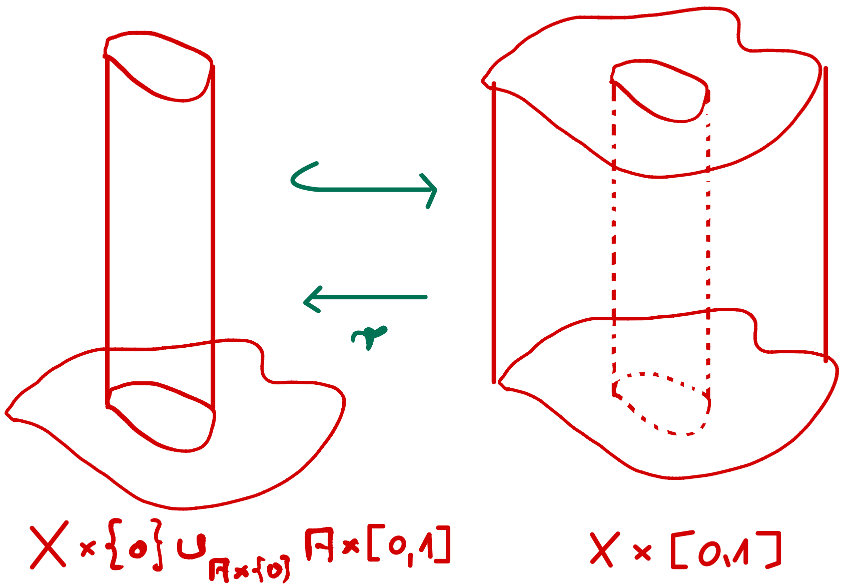
*Proof.* (i) Put $Y = X \times \{0\} \cup_{A\times\{0\}} A \times I$. Define $f\colon X \to Y$ by $f(x) = (x, 0)$ and $h\colon A \times I \to Y$ by $h(a,t) = (a,t)$. From the homotopy extension property applied to $Y$, $f$, and $h$ we obtain the desired retraction $r = H\colon X \times I \to X \times \{0\} \cup_{A\times\{0\}} A \times I$.

(ii) Note that $X \times \{0\} \cup_{A\times\{0\}} A \times I \subseteq X \times I$ is to be understood to be equipped with the subspace topology. Since the inclusion $A \to X$ is a closed embedding, we get with this topology a pushout

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \times \{0\} \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & X \times \{0\} \cup_{A\times\{0\}} A \times I \end{array}$$

where all maps are inclusions.

Consider $f\colon X \to Y$ and $h\colon A \times I \to Y$ with $f \circ i_0^A = h_0$. Consider the map $g := f \cup h\colon X \times \{0\} \cup_{A\times\{0\}} A \times I \to Y$. The desired homotopy $H\colon X \times I \to Y$ is then given by $g \circ r$. □




**Lemma 8.9.** *Consider a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\bar{f}]{} & Y \end{array}$$

*such that* $i\colon A \to X$ *is a cofibration.*

*Then* $\bar{i}\colon B \to Y$ *is a cofibration.*

*Proof.* Suppose that $h\colon B \times I \to Z$ and $\varphi\colon Y \to Z$ with $h_0 = \varphi \circ \bar{i}$ are given. Then we get a homotopy $h' = h \circ (f \times \mathrm{id}_I)$ and a map $\varphi' = \varphi \circ \bar{f}\colon X \to Z$ satisfying

$h_0' = \varphi' \circ i$. Since $i$ is a cofibration, we get a homotopy $H' \colon X \times I \to Z$ satisfying $H_0' = \varphi'$ and $H' \circ (i \times \mathrm{id}_I) = h'$. We have the pushout

$$\begin{array}{ccc} A \times I & \xrightarrow{f \times \mathrm{id}_I} & B \times I \\ {\scriptstyle i \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle \overline{i} \times \mathrm{id}_I} \\ X \times I & \xrightarrow[\overline{f} \times \mathrm{id}_I]{} & Y \times I. \end{array}$$

Hence $H'$ and $h$ define a map $H \colon Y \times \mathrm{id}_I \to Z$ which is uniquely determined by $H \circ (\overline{f} \times \mathrm{id}_I) = H'$ and $H \circ (\overline{i} \times \mathrm{id}_I) = h$. We get $H_0 = \varphi$, since $H_0$ and $\varphi$ have the same composite with $\overline{f}$ and $\overline{i}$. $\square$

**Lemma 8.10.** *Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of cofibrations. Let $X$ be its colimit$_{i \to \infty} X_i$.*

*Then the canonical map $X_0 \to X$ is a cofibration.*

*Proof.* Because of Remark 8.7 we can assume without loss of generality that $X_i$ is a closed subspace of both $X_{i+1}$ and $X$ for $i = 0, 1, 2, \ldots$ and $X = \bigcup_{i=0}^{\infty} X_i$. Because of Lemma 8.8 (ii) it suffices to construct a retraction $R \colon X \times I \to X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I$. Since $X \times I = \mathrm{colim}_{i \to \infty} X \times \{0\} \cup_{X_i \times \{0\}} (X_i \times I)$ holds, it suffices to construct a sequence of maps

$$r_i \colon X \times \{0\} \cup_{X_i \times \{0\}} X_i \times I \to X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I$$

for $i = 0, 1, 2, \ldots$ such that $r_i|_{X \times \{0\} \cup_{X_{i-1} \times \{0\}} X_{i-1} \times I} = r_{i-1}$ holds for $i = 1, 2, 3, \ldots$ and $r_0 = \mathrm{id}_{X \times \{0\} \cup_{X_0 \times \{0\}} X_0 \times I}$ holds.

We construct the desired retractions $r_i$ by induction over $i = 0, 1, 2, \ldots$. The induction beginning is obvious. The induction step from $(i-1)$ to $i \geq 1$ is done as follows. Since $X_{i-1} \to X_i$ is a cofibration, there exists a retraction $r_i' \colon X_i \times I \to X_i \times \{0\} \cup_{X_i \times \{0\}} X_{i-1} \times I$ by Lemma 8.8 (ii). It extends to a retraction

$$r_i' \colon X \times \{0\} \cup_{X_i \times \{0\}} X_i \times I \to X \times \{0\} \cup_{X_{i-1} \times \{0\}} X_{i-1} \times I$$

by $\mathrm{id}_{X \times \{0\}} \cup r_i'$. Now define $r_i$ to be the composite $r_{i-1} \circ r_i''$. $\square$

### 8.2. **Cofibrations and NDR-pairs.**

**Definition 8.11** (NDR-pair)**.** We call a pair $(X, A)$ an NDR-*pair* or *neighborhood deformation retract*, if there are maps $h \colon X \times I \to X$ and $v \colon X \to I$ satisfying:

- $h(a, t) = a$ for $a \in A$ and $t \in I$;
- $h(x, 0) = x$ for $x \in X$;
- $v^{-1}(0) = A$;
- $h(x, t) \in A$ for $x \in X$ and $t \in I$ with $v(x) < t$.

**Lemma 8.12.** *Let $(X, A)$ be a pair. Let $i \colon A \to X$ be an inclusion. Then the following assertions are equivalent:*

(i) *The map $i \colon A \to X$ is a cofibration;*
(ii) *The pair $(X, A)$ is an NDR-pair.*

*Proof.* (i) $\implies$ (ii) We get from Remark 8.7 that $A \subseteq X$ is closed and from Lemma 8.8 (i) a retraction $r \colon X \times I \to X \times \{0\} \cup_{A \times \{0\}} A \times I$. Define

$$h \colon X \times I \to X, \quad x \mapsto \mathrm{pr}_X \circ r(x),$$

and

$$v \colon X \to I, \quad x \mapsto \sup\{|t - \mathrm{pr}_I \circ r(x, t)| \mid t \in I\},$$

where $\mathrm{pr}_X \colon X \times I \to X$ and $\mathrm{pr}_I \colon X \times I \to I$ are the canonical projections. One easily checks that $h$ and $v$ satisfy the conditions appearing in Definition 8.11.

(ii) $\implies$ (i) Given the maps $h$ and $v$, we can define a retraction $r\colon X \times I \to X \times \{0\} \cup_{A \times \{0\}} A \times I$ by

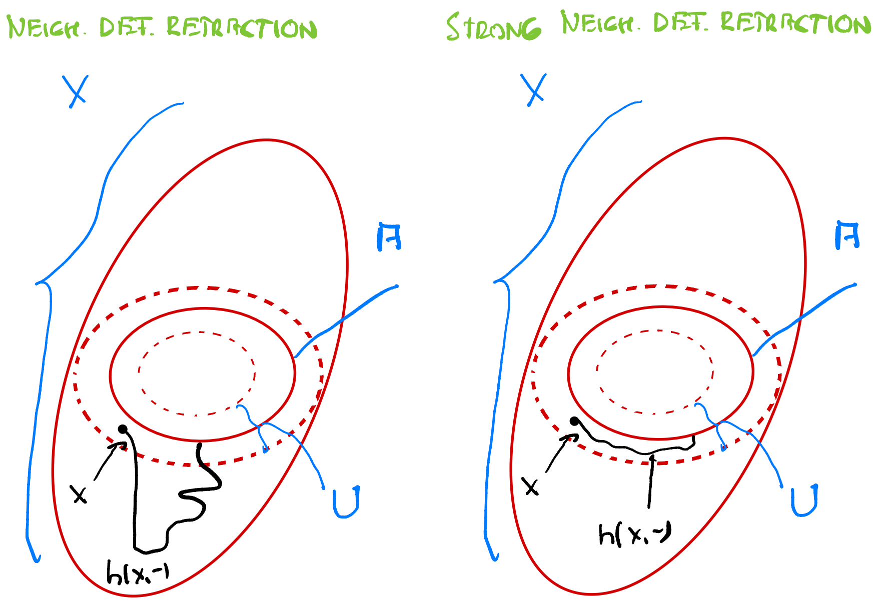
$$r(x,t) = \begin{cases} (h(x,t),0) & \text{if } t \le v(x); \\ (h(x,t), t - v(x)) & \text{if } t \ge v(x). \end{cases}$$

Since $A = v^{-1}(0)$ holds, $A \subseteq X$ is closed. Moreover, $A$ is a $G_\delta$-subset of $X$, i.e., $A$ is the intersection of countably many open subsets of $X$. Lemma 8.8 (ii) implies that $i\colon A \to X$ is a cofibration. □

**Definition 8.13** (Strong neighborhood deformation retraction)**.** Consider a pair $(X, A)$. We call $A$ a *strong neighborhood deformation retraction* of $X$, if $A \subseteq X$ is closed, there is an open neighborhood $U$ of $A$ in $X$ such that the inclusion $i\colon A \to U$ has retraction $r\colon U \to A$, and there exists a homotopy relative $A$ between $\mathrm{id}_U$ and $i \circ r$, or, equivalently, $A \subseteq X$ is closed, there is an open neighborhood $U$ of $A$ in $X$ and a homotopy $h\colon U \times I \to U$ such that $h(u,0) = u$ and $h(u,1) \in A$ holds for $u \in U$ and we have $h(a,t) = a$ for $a \in A$ and $t \in I$.

**Definition 8.14** (Neighborhood deformation retraction)**.** Consider a pair $(X, A)$ We call $A$ a *neighborhood deformation retraction* of $X$, if $A \subseteq X$ is closed, there is an open neighborhood $U$ of $A$ in $X$ and a homotopy $h\colon U \times I \to X$ such that $h(u,0) = u$ and $h(u,1) \in A$ hold for $u \in U$ and we have $h(a,t) = a$ for $a \in A$ and $t \in I$.

**Remark 8.15** (Strong neighborhood deformation retraction versus neighborhood deformation retraction)**.** The difference between Definition 8.13 and Definition 8.14 is that in Definition 8.13 the target of $h$ is $U$, whereas in Definition 8.14 the target of $h$ is $X$. Hence a strong neighborhood deformation retraction is a neighborhood deformation retraction. The converse is not true in general.




**Remark 8.16** (NDR-pairs versus neighborhood deformation retractions)**.** Let $(X, A)$ be an NDR-pair in the sense of Definition 8.11 which is equivalent to $i\colon A \to X$ being a cofibration by Lemma 8.12. Then it is a neighborhood deformation retraction in the sense of Definition 8.14. Namely, given $h\colon X \times I \to X$ and $v\colon X \to I$

as in Definition 8.11, we get by $U = v^{-1}([0,1))$ and $h|_{U\times I}\colon U \times I \to X$ the data required in Definition 8.14. The converse is nor true in general.

Now suppose that $(X, A)$ is neighborhood deformation retraction in the sense of Definition 8.14 and assume additionally that there is a map $w\colon X \to I$ satisfying $w^{-1}(0) = A$ and $U = w^{-1}([0,1))$. The latter additional condition is known to be automatically satisfied if $X$ is a perfectly normal space, i.e., a metric space or a $CW$-complex, and $A \subseteq X$ is closed, or if $X$ is a normal space and $A \subseteq X$ is a closed $G_\delta$-subset of $X$. Then we obtain a retraction $r\colon X \times I \to X \times \{0\} \cup_{A\times\{0\}} A \times I$ by

$$r(x,t) = \begin{cases} (x,t) & \text{if } x \in w^{-1}(0); \\ (h(x, t/2w(x)), 0) & \text{if } x \in w^{-1}((0,1/2]), t \le 2w(x); \\ (h(x,1), t - 2w(x)) & \text{if } x \in w^{-1}((0,1/2]), 2w(x) \le t \le 1; \\ (h(x, 2t(1-w(x))), 0) & \text{if } x \in w^{-1}([1/2,1)); \\ (x,0) & x \in w^{-1}(1). \end{cases}$$

Hence $i\colon A \to X$ is a cofibration by Lemma 8.8 (ii) which is equivalent to $(X, A)$ being an NDR-pair by Lemma 8.12.

8.3. **Relative $CW$-complexes are cofibrations.**

**Theorem 8.17** (Relative $CW$-complexes are cofibrations)**.** *Let $(X, A)$ be a relative $CW$-complex. Then the inclusion $i\colon A \to X$ is a cofibration.*

*Proof.* Because of Lemma 8.10 it suffices to prove that the inclusion $X_i \to X_{i+1}$ is a cofibration for $i = 0, 1, 2, \ldots$. Choose a pushout

$$\begin{array}{ccc} \coprod_{j\in J_i} S^i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ \coprod_{j\in J_i} D^{i+1} & \longrightarrow & X_{i+1}. \end{array}$$

By Lemma 8.9 it suffices to show that the left vertical arrow in the diagram above is a cofibration. This follows from the fact that the inclusion $S^i \to D^{i+1}$ is a cofibration which is a consequence of Lemma 8.8 (ii). □

One can actually show the following stronger result which we state without giving the proof which follows essentially from the fact that $(D^{i+1}, S^i)$ is a strong neighborhood deformation retraction and is similar to the one of Theorem 8.17.

**Theorem 8.18** ($CW$-complexes and strong neighborhood deformation retraction)**.** *Let $(B, A)$ be a strong neighborhood deformation retraction. Let $(X, B)$ be a relative $CW$-complex. Then the pair $(X, A)$ is a strong neighborhood deformation retraction.*

We omit the proof of the following result whose prove is similar to the one of Theorem 8.18.

**Theorem 8.19** ($CW$-complexes are locally contractible)**.** *Every $CW$-complex $X$ is locally contractible, i.e., for every point $x \in X$ and every open neighborhood $V$ of $x$ in $X$ there exists an open neighborhood $U$ of $x$ in $X$ such that $\overline{U} \subseteq V$ holds and $U$ and $\overline{U}$ are contractible.*

8.4. **Well-pointed spaces.**

**Definition 8.20** (Well-pointed space)**.** A well-pointed space $(X, x)$ is a pointed space such that the inclusion of the base point $\{x\} \to X$ is a cofibration.

**Lemma 8.21.** *Let $\{(X_i, x_i) \mid i \in I\}$ be a collection of well-pointed spaces. Then $\bigvee_{i\in I}(X_i, x_i)$ with the canonical base point is well-pointed.*

*Proof.* We have the pushout

$$\begin{array}{ccc} \coprod_{i\in I}\{x_i\} & \longrightarrow & \{\bullet\} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i\in I} X_i & \longrightarrow & \bigvee_{i\in I}(X_i, x_i) \end{array}$$

where the left vertical arrow is the obvious inclusion and a cofibration. Now apply Lemma 8.9. □

8.5. **Comparing pointed homotopy and homotopy.** Consider a well-pointed space $(X, x)$ and space $Y$. Next we define a covariant functor

$$\gamma = \gamma_{(X,x),Y} \colon \Pi(Y) \to \mathsf{Sets} \tag{8.22}$$

from the fundamental groupoid $\Pi(Y)$ to the category $\mathsf{Sets}$ of sets. It sends the element $y \in Y$ to the set $[(X,x),(Y,y)]^0$ of pointed homotopy classes of pointed maps $(X,x) \to (Y,y)$. Consider a morphism $[w] \colon y_0 \to y_1$ in $\Pi(Y)$ represented by path $w \colon I \to Y$ with $w(0) = y_0$ and $w(1) = y_1$ and an element $[f] \in [(X,x),(Y,y_0)]^0$ represented by a pointed map $f \colon (X,x) \to (Y,y_0)$. Since the inclusion $\{x\} \to X$ is a cofibration, we can find a homotopy $h \colon X \times I \to Y$ such that $h_0 = f_0$ and $h(x,t) = w(t)$ holds. Now we define $\gamma([w])([f]) = [h_1]$. We omit the proof, which is essentially based on the fact that the inclusion $\{x\} \times I \to X \times I$ is a cofibration, that this definitions makes sense and yields the functor $\gamma$ announced in (8.22).

If we fix a point $y \in Y$, we get using the identification of $\pi_1(Y,y)$ with $\operatorname{aut}_{\Pi(Y)}(y)$ an operation

$$\pi_1(Y,y) \times [(X,x),(Y,y)]^0 \to [(X,x),(Y,y)]^0. \tag{8.23}$$

One easily checks that the forgetful map $[(X,x),(Y,y)]^0 \to [X,Y]$ induces a bijection

$$\pi_1(Y,y)\backslash[(X,x),(Y,y)]^0 \xrightarrow{\cong} [X,Y]. \tag{8.24}$$

Note that for a simply connected pointed space $(Y,y)$, the bijection (8.24) reduces to a bijection $[(X,x),(Y,y)]^0 \xrightarrow{\cong} [X,Y]$.

If we take $(X,x) = (S^n, s)$, the operation (8.23) yields an operation of $\pi_1(Y,y)$ on $\pi_n(Y,y)$. If $n = 1$, this is the conjugation action, where $[w]$ acts on $[u] \in \pi_1(Y,y)$ by $[u] \mapsto [w]\cdot[u]\cdot[w]^{-1}$. If $n \geq 2$, then $\pi_n(Y,y)$ is abelian and the $\pi_1(Y,y)$-action is by automorphism of abelian groups. Hence we get a left $\mathbb{Z}[\pi_1(Y,y)]$-module structure on $\pi_n(Y,y)$ for $n \geq 2$.

Suppose that $Y$ is path connected and has a universal covering $p \colon \widetilde{Y} \to Y$. Choose $\widetilde{y} \in \widetilde{Y}$ with $p(\widetilde{y}) = y$. Recall that $\widetilde{Y}$ comes with a $\pi_1(Y,y)$-action. Fix a natural number $n \geq 2$. We get for $[u] \in \pi_1(Y,y)$ a homeomorphism $l_{[u]} \colon \widetilde{Y} \xrightarrow{\cong} \widetilde{Y}$ by left multiplication with $[u]$. Choose a path $v \colon I \to \widetilde{Y}$ from $[u]\cdot\widetilde{y}$ to $\widetilde{y}$. Then we get an isomorphism $T_n([v]) \colon \pi_n(\widetilde{Y}, [u]\cdot\widetilde{y}) \xrightarrow{\cong} \pi_n(\widetilde{Y},\widetilde{y})$ from (2.6), which is independent of the choice of $v$ as $\widetilde{Y}$ is simply connected. Now we define a left $\pi_1(Y,y)$-action on $\pi_n(\widetilde{Y},\widetilde{y})$ by letting $[u] \in \pi_1(Y,y)$ act on $\pi_n(\widetilde{Y},\widetilde{y})$ by the composite $\pi_n(\widetilde{Y},\widetilde{y}) \xrightarrow{\pi_n(l_{[u]},\widetilde{y})} \pi_n(\widetilde{Y},[u]\cdot\widetilde{y}) \xrightarrow{T_n([v])} \pi_n(\widetilde{Y},\widetilde{y})$. One easily checks that this defines a left $\mathbb{Z}[\pi_1(Y,y)]$-module structure on the abelian group $\pi_n(\widetilde{Y},\widetilde{y})$. Recall the isomorphism $\pi_n(p,\widetilde{y}) \colon \pi_n(\widetilde{Y},\widetilde{y}) \xrightarrow{\cong} \pi_n(Y,y)$ from Theorem 2.7. One easily checks that it is compatible with the left $\mathbb{Z}[\pi_1(Y,y)]$-module structures on $\pi_n(\widetilde{Y},\widetilde{y})$ and $\pi_n(Y,y)$ constructed above.

**Lemma 8.25.** *Let $f \colon (X,x) \to (Y,y)$ be a pointed map of well-pointed spaces. Suppose that $f$ is a homotopy equivalence (after forgetting the base points).*

*Then $f$ is a pointed homotopy equivalence.*

*Proof.* Choose a homotopy inverse $g' \colon Y \to X$ of $f$. Because of (8.24) we can change $g'$ up to homotopy such that $g'(y) = x$ holds. The map $f_* \colon [(Y,y),(X,x)]^0 \to [(Y,y),(Y,y)]^0$ sends $[g']$ to the element $[f \circ g']$ which is mapped under the projection $[(Y,y),(Y,y)]^0 \to [Y,Y]$ to $[\mathrm{id}_Y]$. Because of (8.24) there is a element $v \in \pi_1(Y,y)$ satisfying $[\mathrm{id}_Y] = v \cdot [f \circ g']$ in $[(Y,y),(Y,y)]^0$. Since $\pi_1(f,x) \colon \pi_1(X,x) \to \pi_1(Y,y)$ is bijective, we can find $u \in \pi_1(X,x)$ with $\pi_1(f,x)(u) = v$. Let $g \colon (Y,y) \to (X,x)$ be a pointed map satisfying $[g] = u \cdot [g']$ in $[(Y,y),(X,x)]^0$. Then we get in $[(Y,y),(Y,y)]^0$

$$[f \circ g] = f_*([g]) = f_*(u \cdot [g']) = v \cdot f_*([g']) = v \cdot [f \circ g'] = [\mathrm{id}_Y].$$

Hence $g$ is a pointed homotopy right inverse of $f$. The same argument applied to $g$ shows that $[g]$ has a pointed homotopy right inverse. This implies that $f$ is a pointed homotopy equivalence. □

8.6. **The Homotopy Theorem for pushouts and cofibrations.** Given a space $B$, let $\mathsf{Top}^B$ be the category of topological spaces under $B$. Objects are maps $u \colon B \to X$. A morphism from $u \colon B \to X$ to $v \colon B \to Y$ is a map $f \colon X \to Y$ satisfying $f \circ u = v$. We call two such morphism $f_0, f_1 \colon u \to v$ homotopic if they are homotopic through morphisms in $\mathsf{Top}^B$, i.e., there exists a homotopy $h \colon X \times I \to Y$ such that $h_0 = f_0$ and $h_1 = f_1$ holds and we have $h_t \circ u = v$ for every $t \in I$. Let $\mathrm{h}\text{-}\mathsf{Top}^B$ be the associated homotopy category, i.e., the set of objects of $\mathrm{h}\text{-}\mathsf{Top}^B$ and $\mathsf{Top}^B$ agree and a morphism from $u$ to $v$ in $\mathrm{h}\text{-}\mathsf{Top}^B$ is a homotopy class of morphisms from $u$ to $v$ in $\mathsf{Top}^B$.

Let $\mathsf{Cof}^B$ and $\mathrm{h}\text{-}\mathsf{Cof}^B$ respectively be the full subcategory of $\mathsf{Top}^B$ and $\mathrm{h}\text{-}\mathsf{Top}^B$ respectively consisting of those objects $i \colon B \to X$ for which $i$ is a cofibration.

Given two spaces $A$ and $B$, define $\Pi(A,B)$ to be the following category. Objects are maps $f \colon A \to B$. A morphism from $f_0$ to $f_1$ is a homotopy class $[h]$ relative $A \times \{0,1\}$ of maps $h \colon A \times I \to B$ with $h_0 = f_0$ and $h_1 = f_1$. Note that $h$ itself is a homotopy between $f_0$ and $f_1$ and $[h]$ is the homotopy class of such homotopies represented by $h$. If $A = \{\bullet\}$, then $\Pi(\{\bullet\}, B)$ is the fundamental groupoid $\Pi(B)$ of $B$. Note that $\Pi(A,B)$ is a groupoid. Given a cofibration $i \colon A \to X$, we next sketch the construction of a contravariant functor

$$\beta_i \colon \Pi(A,B) \to \mathrm{h}\text{-}\mathsf{Cof}^B. \tag{8.26}$$

An object $f \colon A \to B$ is sent to the cofibration $\gamma_i(f) \colon B \to Y_f$ given by the following pushout and Lemma 8.9

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \gamma_i(f)} \\ X & \xrightarrow[\beta_i(f)]{} & Y_f. \end{array}$$

Now consider two maps $f_0, f_1 \colon A \to B$ together with a homotopy $h \colon A \times I \to B$ with $h_0 = f_0$ and $h_1 = f_1$. As $i$ is a cofibration, there exists a homotopy $H \colon X \times I \to Y_{f_0}$ with $H_0 = \beta_i(f_0)$ and $H \circ (i \times \mathrm{id}_I) = \gamma_i(f_0) \circ h$. Since we have the pushout

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \gamma_i(f_1)} \\ X & \xrightarrow[\beta_i(f_1)]{} & Y_{f_1} \end{array}$$

there is precisely one map $u \colon Y_{f_1} \to Y_{f_0}$ uniquely determined by the property that $u \circ \beta_i(f_1) = H_1$ and $u \circ \gamma_i(f_1) = \gamma_i(f_0)$ hold. Obviously $u$ is a morphism from

$\gamma_i(f_1)\colon B \to Y_{f_1}$ to $\gamma_i(f_0)\colon B \to Y_{f_0}$ in $\mathtt{Cof}^B$. Thanks to $H$, we have

$$u \circ \beta_i(f_1) \simeq \beta_i(f_0). \tag{8.27}$$

We omit the proof that $[u]$ depends only on $[h]$ which can be found in [24, Proposition 5.2.1 on page 107]. So we can define $\beta_i([h]) = [u]$. We also omit the proof that $\beta_i$ is a contravariant functor. Note that $\beta_i([u])$ is represented by a homotopy equivalence as $\pi(A, B)$ is a groupoid.

**Theorem 8.28** (Homotopy Theorem for pushouts and cofibrations)**.** *Consider a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \overline{i}} \\ X & \xrightarrow[\overline{f}]{} & Y \end{array}$$

*such that $i\colon A \to X$ is a cofibration and $f$ is a homotopy equivalence.*

*Then $\overline{i}\colon B \to Y$ is a cofibration and $\overline{f}$ is a homotopy equivalence.*

*Proof.* The map $\overline{i}$ is a cofibration by Lemma 8.9. Let $g\colon B \to A$ be a homotopy inverse of $f$. Consider the pushout

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ {\scriptstyle \overline{i}}\downarrow & & \downarrow{\scriptstyle \overline{\overline{i}}} \\ Y & \xrightarrow[\overline{g}]{} & Z. \end{array}$$

The map $\overline{\overline{i}}$ is a cofibration by Lemma 8.9. Since $g \circ f \simeq \mathrm{id}_A$, we get from the contravariant functor $\beta_i\colon \Pi(A, A) \to \mathrm{h}\text{-}\mathtt{Cof}^A$ of (8.26) and from (8.27) a homotopy equivalence $u\colon Z \to X$ such that $u \circ \overline{g} \circ \overline{f} \simeq \mathrm{id}_X$ holds. Hence $\overline{f}$ has a left homotopy inverse. Interchanging the role of $f$ and $g$ shows that $\overline{f}$ has a right homotopy inverse. Hence $\overline{f}$ is a homotopy equivalence. □

Theorem 8.28 can easily be extended to the following theorem.

**Theorem 8.29** (Homotopy Theorem for maps between pushouts)**.** *Let the following two diagrams be pushouts*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ Y_2 & \xrightarrow[l_2]{} & Y \end{array}$$

*where the left vertical arrows $i_2$ and $k_2$ are cofibrations. Let $f_i\colon X_i \to Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f\colon X \to Y$ the map induced by $f_0$, $f_1$, and $f_2$ and the pushout property.*

*Then $f$ is a homotopy equivalence.*

**Remark 8.30.** The condition that the maps $i_2$ and $k_2$ are cofibrations appearing in Theorem 8.29 is necessary as the following examples shows.

We take as pushouts

$$\begin{array}{ccc} S^n & \xrightarrow{i_1} & D^{n+1} \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ D^{n+1} & \xrightarrow[j_2]{} & S^{n+1} \end{array} \qquad\qquad \begin{array}{ccc} S^n & \xrightarrow{k_1} & \{\bullet\} \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ \{\bullet\} & \xrightarrow[l_2]{} & \{\bullet\} \end{array}$$

and define $f_0 = \mathrm{id}_{S^n}$ and $f_1$, $f_2$, and $f$ to be the the projections.

**Example 8.31.** Let $B$ be the compact subset of $\mathbb{R}^2$ given by

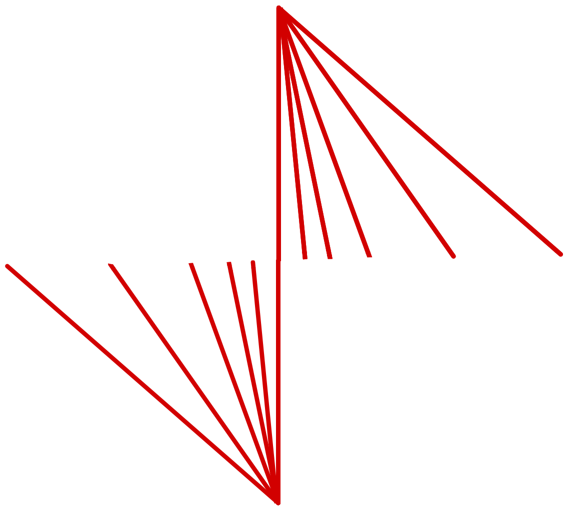
$$B = \{(1/n, 0) \in \mathbb{R}^2 \mid n \in \mathbb{Z}^{\geq 1}\} \amalg \{(0,0)\}.$$

Let $C$ be the cone over $B$ with cone point $(0,1)$ in $\mathbb{R}^2$, i.e.,

$$C = \{x \in \mathbb{R}^2 \mid \exists t \in I \text{ and } b \in B \text{ satisfying } x = t \cdot b + (1-t) \cdot (0,1)\}.$$

Define

$$A = \{x \in \mathbb{R}^2 \mid x \in C \text{ or } -x \in C\}.$$



Then we have:

(i) The inclusion $\{(0,1)\} \to C$ is a cofibration;
(ii) $C$ is contractible;
(iii) $A$ is not contractible;
(iv) The inclusion $\{(0,0)\} \to A$ is not a cofibration;
(v) The inclusion $\{(0,0)\} \to C$ is not a cofibration;
(vi) $C$ is a not $CW$-complex;
(vii) $A$ is not a $CW$-complex.

Since we have the pushout whose left vertical arrow is the obvious inclusion and a cofibration

$$\begin{array}{ccc} B \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ B \times I & \longrightarrow & C \end{array}$$

assertions (i) and (ii) follow from Theorem 8.28.

Next we show assertion (iii). Suppose that $A$ is contractible. Since $A$ is in particular path connected, we can find a map $h\colon A \times I \to A$ with $h(a,0) = (0,0)$ and $h(a,1) = a$ for all $a \in A$. Since any path from $(-1/n, 0)$ to $(0,0)$ in $A$ must go through $(0,-1)$, we can find elements $t_n^- \in I$ with $h(-1/n, t_n^-) = (0,-1)$ for $n \geq 1$. Since $I$ is compact, we can find a strictly monotone increasing function $N\colon \mathbb{N} \to \mathbb{N}$ and an element $t^- \in I$ with $\lim_{n\to\infty} t^-_{N(n)} = t^-$. As $\lim_{n\to\infty} 1/N(n) = 0$ holds, we conclude $h((0,0), t^-) = \lim_{n\to\infty} h(-1/N(n), t^-_{N(n)}) = (0,-1)$. Since $h$

is continuous, we can choose $t^-$ such that $h((0,0),t) = (0,-1)$ for $t \in I$ implies $t^- \le t$. Analogously we construct $t^+ \in I$ such that $h((0,0),t^+) = (0,1)$ holds and $h((0,0),t) = (0,1)$ for $t \in I$ implies $t^+ \le t$. Next we consider only the case $t^+ \le t^-$, the other case is completely analogous. Obviously $t^+ \ne t^-$ holds and hence $t^+ < t^-$, By continuity $\lim_{n\to 0} h(-1/n, t^+) = h(0,t^+) = (0,1)$. Hence there is a natural number $n_0$ such that $h(-1/n,t^+) = (0,u_n)$ with $u_n \ge 0$ holds for $n \ge n_0$. Since any path from $(-1/n,0)$ to $(0,u)$ for $u \ge 0$ in $A$ must go through $(0,-1)$ we can for every $n \ge n_0$ elements $s_n^- \in I$ with $h(-1/n,s_n^-) = (0,-1)$ and $s_n^- \le t^+$. Since $I$ is compact, we can find a strictly monotone increasing function $N' \colon \mathbb{N} \to \mathbb{N}$ and an element $s^- \in I$ with $\lim_{n\to\infty} s^-_{N'(n)} = s^-$. Obviously $s^- \le t^+$. As $\lim_{n\to\infty} 1/N'(n) = 0$ holds, we conclude $(0,-1) = \lim_{n\to\infty} h((1/n,0),s_n^-) = h((0,0),s^-)$. This implies $t^- \le s^-$. Hence we get $t^- \le t^+$, a contradiction.

Suppose that assertions (iv) is not true. As $C$ and hence also $\{-x \in \mathbb{R}^n \mid x \in C\}$ are contractible by assertion (ii), Theorem 8.29 implies that $A$ is contractible. Since we have already proved that $A$ is not contractible, assertion (iv) follows.

Suppose that the inclusion $\{(0,0)\} \to C$ is a cofibration. Then also the inclusion $\{(0,0)\} \to \{-x \in \mathbb{R}^n \mid x \in C\}$ is a cofibration. This implies by Lemma 8.21 that the inclusion $\{(0,0)\} \to A$ is a cofibration. Hence assertion (iv) implies assertion (v).

Since the point $(0,0)$ in $C$ has the property that any neighborhood of it in $C$ which does not contain $(1,0)$ is not contractible, assertion (vi) follows from Theorem 8.19. The proof of assertion (vii) is analogous.

8.7. **(Pointed) cylinders, cones and suspensions.** Consider a space $X$. Recall that its *cylinder* is defined by $X \times I$, its *cone* $\operatorname{cone}(X)$ by the pushout

$$\begin{array}{ccc} X \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \operatorname{cone}(X), \end{array}$$

and its *suspension* by the pushout

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{cone}(X) \\ \downarrow & & \downarrow \\ \operatorname{cone}(X) & \longrightarrow & \Sigma X. \end{array}$$

Equivalently, one can define $\Sigma X$ to be the quotient of $X \times [-1,1]$ under the equivalence relation generated by $(x_0,1) \sim (x_1,1)$ for $x_0, x_1 \in X$ and $(x_0,-1) \sim (x_1,-1)$ for $x_0, x_1 \in X$. There is an obvious pushout

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{cone}(X) \\ \downarrow & & \downarrow \\ \{\bullet\} & \longrightarrow & \Sigma X. \end{array}$$

Consider a pointed space $(X,x)$. Its *reduced mapping cylinder* $\operatorname{cyl}(X,x)$, its *reduced mapping cone* $\operatorname{cone}(X,x)$, and its *reduced suspension* $\Sigma(X,x)$ are the defined by the pushouts

(8.32)
$$\begin{array}{ccc} \{x\} \times I & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \operatorname{cyl}(X,x), \end{array}$$

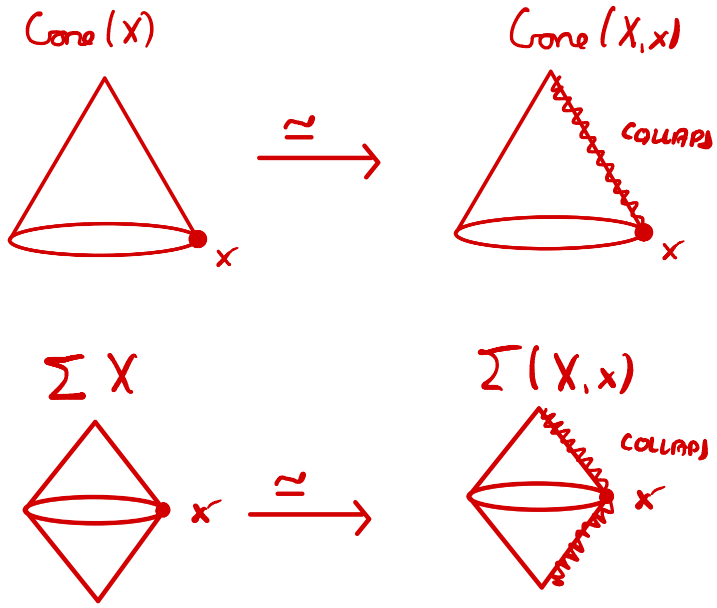
(8.33)
$$\begin{array}{ccc} \{x\} \times I \cup X \times \{1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \operatorname{cone}(X,x), \end{array}$$

and

(8.34)
$$\begin{array}{ccc} \{x\} \times I \cup X \times \{0,1\} & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \Sigma(X,x). \end{array}$$

and come with a preferred base point. Note that $\Sigma(X,x)$ can be identified with the smash product $(S^1,s) \wedge (X,x) = (S^1 \times X)/(S^1 \times x \cup \{s\} \times X)$.

Given a well-pointed pointed space $(X,x)$, the canoncial projections $\operatorname{cyl}(X) \to \operatorname{cyl}(X,x)$, $\operatorname{cone}(X) \to \operatorname{cone}(X,x)$, and $\Sigma X \to \Sigma(X,x)$ are pointed homotopy equivalences by Lemma 8.25, Theorem 8.28, and Theorem 8.29.




There are obvious pushouts

$$\begin{array}{ccc} \{x\} \times I & \longrightarrow & \{\bullet\} \\ \downarrow & & \downarrow \\ \operatorname{cyl}(X,x) & \longrightarrow & \operatorname{cone}(X,x), \end{array}$$

and

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{cone}(X,x) \\ \downarrow & & \downarrow \\ \operatorname{cone}(X,x) & \longrightarrow & \Sigma(X,x). \end{array}$$

Let $f\colon (X,x) \to (Y,y)$ be a pointed map. Its *reduced mapping cone* $\operatorname{cone}(f,x)$ is defined by the pushout

(8.35)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \operatorname{cone}(X,x) & \longrightarrow & \operatorname{cone}(f,x) \end{array}$$

or, equivalently by the pushout

(8.36)
$$\begin{array}{ccc} (\{x\} \times I) \cup (X \times \{0\}) \cup (X \times \{1\}) & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & \operatorname{cone}(f,x) \end{array}$$

where $u$ sends $(x,t)$ for $t \in I$ to $y$, $(z,1)$ to $f(z)$ for $z \in X$ and $(z,0)$ to $y$ for $z \in X$.

Note that $\operatorname{cone}(f,x)$ comes with a preferred base point for which the pushout (8.35) is a diagram of pointed maps of pointed spaces.

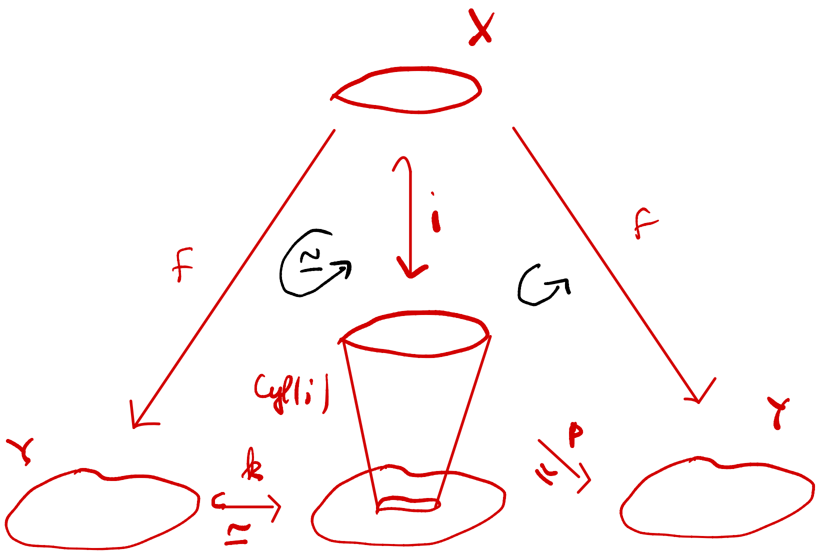
8.8. **Turning a map into a cofibration.** Consider a map $f\colon X \to Y$. Then it can be replaced up to homotopy equivalence by a cofibration. Namely let $i\colon X \to \operatorname{cyl}(f)$ be the canonical inclusion and $p\colon \operatorname{cyl}(i) \to Y$ be the projection. Then we get the factorization

$$f\colon X \xrightarrow{i} \operatorname{cyl}(f) \xrightarrow{p} Y$$

where $i$ is a cofibration and $p$ is a homotopy equivalence. Actually we get a diagram

(8.37)
$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & \downarrow{\scriptstyle i} & {}^{f}\searrow & \\ Y & \xrightarrow[k]{\simeq} & \operatorname{cyl}(f) & \xrightarrow[p]{\simeq} & Y \end{array}$$

where the left triangle commutes up to homotopy, the right triangle commutes, the two horizontal maps $k$ and $p$ are homotopy equivalences which are homotopy inverse to one another, and the vertical arrow $i$ is a cofibration.




8.9. **The Cofiber Sequence.** A pointed map $f\colon (X,x) \to (Y,y)$ induces by composition for every pointed space $(B,b)$ a map

$$f^*\colon [(Y,y),(B,b)]^0 \to [(X,x),(B,b)]^0, \quad [u] \mapsto [u \circ f]$$

which depends only on the pointed homotopy class of $f$. A sequence $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{g} (Z,z)$ of maps of pointed spaces is called *homotopy coexact* if for each pointed space $(B,b)$ the induced sequence of pointed sets

$$[(Z,z),(B,b)]^0 \xrightarrow{g^*} [(Y,y),(B,b)]^0 \xrightarrow{f^*} [(X,y),(B,b)]^0$$

is exact at $[(Y,y),(B,b)]^0$ in the sense that the image of $g^*$ is the preimage of $f^*$ of the base point in $[(X,y),(B,b)]^0$ given by $[c_b]$ for the constant pointed map $c_b\colon (X,x) \to (B,b)$. Note that this implies that $g \circ f$ is pointed homotopy equivalent to the constant map $c_z\colon (X,x) \to (Z,z)$.

**Lemma 8.38.** *Let $f\colon (X,x) \to (Y,y)$ be a pointed map. Let $j\colon (Y,y) \to \operatorname{cone}(f,x)$ be the canonical inclusion which is a map of pointed spaces.*

*Then the sequence*

$$(X,x) \xrightarrow{f} (Y,y) \xrightarrow{j} (\operatorname{cone}(f,x),*)$$

*is homotopy coexact.*

*Proof.* This is a direct consequence of the pushout (8.36) which says that a pointed map $(\operatorname{cone}(f,x),*) \to (B,b)$ is the same as a pointed map $v\colon (Y,y) \to (B,b)$ together with a pointed homotopy $h\colon X \times I \to B$ between the constant map $c_b$ and $v \circ f$. ∎

One can iterate this construction and obtains a homotopy coexact sequence of pointed sets, infinite to the right,

$$X \xrightarrow{f} Y \xrightarrow{j} \operatorname{cone}(f) \xrightarrow{f_2} \operatorname{cone}(j) \xrightarrow{f_3} \operatorname{cone}(f_2) \xrightarrow{f_3} \cdots$$

where we omit the base points from the notation and homotopy coexact means that it is exact as a sequence of pointed sets at $Y$, $\operatorname{cone}(f)$, $\operatorname{cone}(j)$, ....

The further investigation replace the iterated mapping cones with homotopy equivalent spaces which are more appealing, namely iterated suspensions.

Suppose additionally, that the pointed map $f\colon (X,x) \to (Y,y)$ is a cofibration (after forgetting the base points). Note that then we can think of $X$ as a closed subspace of $Y$ and $f$ as the inclusion of $X$ into $Y$, see Remark 8.7. Then we obtain a pointed map $p\colon (\mathrm{cone}(f,x),*) \to (Y/X,*)$ which is homotopy equivalence by Theorem 8.29. We conclude from Lemma 8.25 that $p$ is a pointed homotopy equivalence. Hence the following diagram of pointed sets commutes

(8.39)
$$\begin{array}{ccccc} (X,x) & \xrightarrow{f} & (Y,y) & \xrightarrow{j} & (\mathrm{cone}(f,x),*) \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle p} \\ (X,x) & \xrightarrow{f} & (Y,y) & \xrightarrow{q} & (Y/X,*) \end{array}$$

where $q\colon X \to X/Y$ is the canonical projection and all vertical arrows are pointed homotopy equivalences. Hence the sequence $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{q} (Y/X,*)$ is homotopy coexact.

Note that $j\colon (Y,y) \to \mathrm{cone}(f,x)$ is a cofibration and $\mathrm{cone}(f,x)/Y$ is homeomorphic to $\Sigma(X,x)$ regardless whether $f$ is a cofibration or not. Hence we obtain from (8.39) a commutative diagram of pointed sets

$$\begin{array}{ccccc} (Y,y) & \xrightarrow{j} & (\mathrm{cone}(f,x),*) & \xrightarrow{k} & (\mathrm{cone}(j,*),*) \\ \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{cone}(f,x)}} & & \downarrow{\scriptstyle p'} \\ (Y,y) & \xrightarrow{j} & (\mathrm{cone}(f,x),*) & \xrightarrow{g} & (\Sigma(X,x),*) \end{array}$$

where all vertical arrows are pointed homotopy equivalences and $g = p' \circ k$. Hence the sequence

$$(X,x) \xrightarrow{f} (Y,y) \xrightarrow{j} (\mathrm{cone}(f,x),*) \xrightarrow{g} (\Sigma(X,x),*)$$

is homotopy coexact. Iterating this process leads to the following result. Denote by $\Sigma^n$ the $n$-fold suspension.

**Theorem 8.40** (Cofiber sequence)**.** *Consider a pointed map $f\colon (X,x) \to (Y,y)$. Then we obtain a homotopy coexact sequence, infinite to the right*

(8.41)
$$\begin{aligned} (X,x) &\xrightarrow{f} (Y,y) \xrightarrow{j} (\mathrm{cone}(f,x),*) \xrightarrow{g} (\Sigma(X,x),*) \xrightarrow{\Sigma f} (\Sigma(Y,y),*) \\ &\xrightarrow{\Sigma j} \Sigma(\mathrm{cone}(f,x),*) \xrightarrow{\Sigma g} (\Sigma^2(X,x),*) \xrightarrow{\Sigma^2 f} (\Sigma^2(Y,y),*) \\ &\xrightarrow{\Sigma^2 j} \Sigma^2(\mathrm{cone}(f,x),*) \xrightarrow{\Sigma^2 g} (\Sigma^3(X,x),*) \xrightarrow{\Sigma^3 f} \cdots . \end{aligned}$$

Note that this sequence (8.41) is natural in $f$. Moreover, it yields for every pointed space $(B,b)$ the following exact sequence of pointed sets, which is infinite to the left, natural in both $f$ and $(B,b)$, and sometimes called the *Puppe sequence*:

(8.42)
$$\begin{aligned} \cdots &\xrightarrow{\Sigma^2 g^*} [\Sigma^2 \mathrm{cone}(f),B]^0 \xrightarrow{\Sigma^2 j^*} [\Sigma^2 Y,B]^0 \xrightarrow{\Sigma^2 f^*} [\Sigma^2 X,B]^0 \xrightarrow{\Sigma g^*} [\Sigma\,\mathrm{cone}(f),B]^0 \\ &\xrightarrow{\Sigma j^*} [\Sigma Y,B]^0 \xrightarrow{\Sigma f^*} [\Sigma X,B]^0 \xrightarrow{g^*} [\mathrm{cone}(f),B]^0 \xrightarrow{j^*} [Y,B]^0 \xrightarrow{f^*} [X,B]^0. \end{aligned}$$

Here and also sometimes in the sequel we omit the base points from the notation. Note the obvious fact that the map $f^*\colon [Y,B]^0 \to [X,B]^0$ is not necessarily surjective.

8.10. **Group structures on the Puppe Sequence.** Let $(X, x)$ be a well-pointed space and $(Y, y)$ be a pointed space. We have the pinching map $\nabla_1 \colon S^1 \to S^1 \vee S^1$ of (2.5). It induces a pinching map

$$\nabla_X \colon \Sigma(X, x) = S^1 \wedge X \xrightarrow{\nabla_1 \wedge \mathrm{id}_X} (S^1 \vee S^1) \wedge X = (S^1 \wedge X) \vee (S^1 \wedge X) = \Sigma(X, x) \vee \Sigma(X, x).$$

Now we can define a group structure on $[\Sigma X, Y]^0$ by

$$[\Sigma X, Y]^0 \times [\Sigma X, Y]^0 \to [\Sigma X, Y]^0, \quad [f] \cdot [g] \mapsto [(f \vee g) \circ \nabla_X]. \tag{8.43}$$

Analogously to the proof of Lemma 2.4, one can show that this group structure is abelian on $[\Sigma^n X, Y]^0$ for $n \geq 2$. If we take $(X, x)$ to be $(S^0, s)$, then the groups $[\Sigma^n S^0, (Y, y)]^0 = [S^n, Y]^0$ and $\pi_n(Y, y)$ agree.

The exact Puppe sequence (8.42) appearing in Theorem 8.40 is an exact sequence of groups or abelian groups in the ranges where the group structures are defined on the sets of pointed homotopy classes.

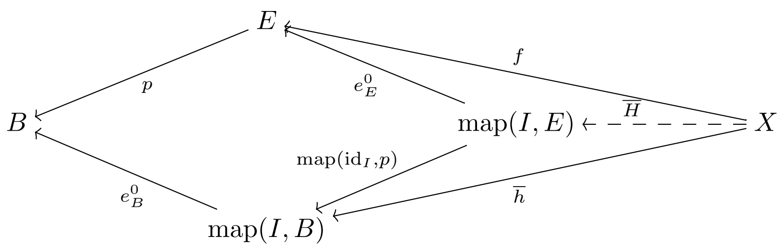
## 9. Fibrations

### 9.1. Basics about fibrations.

**Definition 9.1** (Homotopy lifting property)**.** A map $p \colon E \to B$ has the *homotopy lifting property* (HLP) for the space $X$, if for each homotopy $h \colon X \times I \to B$ and each map $f \colon X \to E$ satisfying $p \circ f = h_0$, there is a homotopy $H \colon X \times I \to E$ with $p \circ H = h$ and $H_0 = f$.

In other words, the HLP for a space $X$ means that the extension problem indicated by the following diagram has a solution $\overline{H}$ for every map $f \colon X \to E$ and homotopy $h \colon X \times I \to Y$ satisfying $p \circ f = h_0$

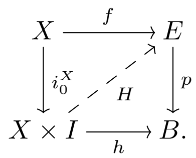
(9.2)




where $e^0_B$ and $e^0_E$ are given by evaluation at 0 and $\overline{h}$ is the adjoint of $h$ under the canonical adjunction homeomorphism $\mathrm{map}(X \times I, B) \xrightarrow{\cong} \mathrm{map}(X, \mathrm{map}(I, X))$, and analogously for $H$ and $\overline{H}$.

Equivalently, one may describe the HLP with by the following diagram

(9.3)




**Definition 9.4** (Fibration)**.** A map $p \colon E \to B$ is called a *fibration* or *Hurewicz fibration* if it has the homotopy lifting property for every space $X$.

A map $p \colon E \to B$ is called a *Serre fibration* if it has the homotopy lifting property for the cube $I^n$ for all $n \geq 1$.

Define for a map $p\colon E \to B$ the space $W(p)$ by the pullback

(9.5)
$$\begin{array}{ccc} W(p) & \xrightarrow{e_p} & E \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{map}(I,B) & \xrightarrow[e_B^0]{} & B. \end{array}$$

Explicitly $W(p) = \{(e,w) \in E \times \operatorname{map}(I,B) \mid p(e) = w(0)\} \subseteq E \times \operatorname{map}(I,B)$ and $e_p$ sends $(e,w)$ to $e$ and $\overline{p}$ sends $(e,w)$ to $w$. Note that we obtain from (9.5) a map

$$r\colon \operatorname{map}(I,E) \to W(p)$$

uniquely determined by the property that $e_p \circ r = e_E^0$ and $\overline{p} \circ r = \operatorname{map}(\mathrm{id}_I, p)$ holds.

If we have base points $e \in E$ and $b \in B$ with $f(e) = b$, then $W(p)$ inherits the base point $* = (e, c_b)$ for the constant map $c_b\colon I \to B$ with image $\{b\}$ and the diagram (9.5) is a diagram of pointed spaces.

**Proposition 9.6.** *The following assertions are equivalent for a map $p\colon E \to B$:*

*(i) $p$ is a fibration;*
*(ii) $p$ has the* HLP *for $W(p)$;*
*(iii) The map $r\colon \operatorname{map}(I,E) \to W(p)$ has a section $s$.*

*Proof.* (i) $\implies$ (ii) This is obvious.

(ii) $\implies$ (iii) If we apply the HLP to the map $e_p\colon W(p) \to E$ and the homotopy $h\colon W(p) \times I \to B$ which corresponds under the adjunction homeomorphism $\operatorname{map}(W(p) \times I, B) \xrightarrow{\cong} \operatorname{map}(W(p), \operatorname{map}(I,B))$ to $\overline{p}$, we get a map $\overline{s}\colon W(p) \times I \to E$. Let $s\colon W(p) \to \operatorname{map}(I,E)$ be the map corresponding to $\overline{s}$ under the adjunction homeomorphism $\operatorname{map}(W(p) \times I, E) \xrightarrow{\cong} \operatorname{map}(W(p), \operatorname{map}(I,E))$. Since the composite of $r \circ s$ and of $\mathrm{id}_{W(p)}$ with both $e_p$ and $\overline{p}$ agree, we get $r \circ s = \mathrm{id}_{W(p)}$.

(iii) $\implies$ (i) Consider a homotopy $h\colon X \times I \to B$ and a map $f\colon X \to E$ satisfying $p \circ f = h_0$. Because of the pullback (9.5) we get from $(f,p)$ a map $u\colon X \to W(p)$. Let $\overline{H}\colon X \to \operatorname{map}(I,E)$ be the composite $s \circ u$. Let $H\colon X \times I \to E$ be the homotopy corresponding to $\overline{H}$ under the adjunction homeomorphism $\operatorname{map}(X \times I, E) \xrightarrow{\cong} \operatorname{map}(X, \operatorname{map}(I,E))$. Then $H$ is a solution to the HLP given by $(h,f)$. This shows that $p$ is a fibration. □

**Proposition 9.7.** *Consider the pullback*

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & Y \\ {\scriptstyle \overline{q}}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[f]{} & C \end{array}$$

*If $q$ is a fibration, then its pullback $\overline{q}$ along $f$ is a fibration.*

*Proof.* Consider a map $u\colon A \to X$ and a homotopy $h\colon A \times I \to B$ such that $h_0 = \overline{q} \circ u$ holds. As $q$ is a fibration, we get from the HLP applied to the map $\overline{f} \circ u\colon A \to Y$ and the homotopy $f \circ h\colon A \times [0,1] \to Y$ a homotopy $\overline{H}\colon A \times I \to Y$ satisfying $q \circ \overline{H} = f \circ h$ and $\overline{H}_0 = \overline{f} \circ u$. Since the diagram above is a pullback, we get a map $H\colon A \times [0,1] \to X$ uniquely determined by $\overline{f} \circ H = \overline{H}$ and $\overline{q} \circ H = h$. Since $H_0$ and $u$ have the the same composite with both $\overline{f}$ and $\overline{q}$, we get $H_0 = f$. Hence $\overline{q}$ has the HLP and therefore is a fibration. □

The elementary proof of the next result can be found in [24, Proposition 5.5.4 on page 116 and Proposition 5.5.5 and 5.5.6 on page 117].

**Proposition 9.8.** *Let $Z$ be a (compactly generated) space. Let $i\colon A \to B$ be a cofibration and $p\colon E \to B$ be a fibration. Then:*

(i) *The induced map*
$$\mathrm{map}(i, \mathrm{id}_Z)\colon \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$$
*is a fibration;*

(ii) *The induced map*
$$\mathrm{map}(\mathrm{id}_Z, p)\colon \mathrm{map}(Z, E) \to \mathrm{map}(Z, B)$$
*is a fibration;*

(iii) *The canonical map* $\mathrm{map}(I, E) \to W(p)$ *sending $v$ to $(v(0), p \circ v)$ is a fibration;*

(iv) *Consider the pullback*
$$\begin{array}{ccc} E_i & \xrightarrow{\overline{i}} & E \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[i]{} & B. \end{array}$$
*Then the upper horizontal arrow $\overline{i}\colon E_i \to E$ is a cofibration.*

The elementary proof the next result can be found in [24, Corollary 5.5.3 on page 116].

**Proposition 9.9** (Improved HLP)**.** *Let $p\colon E \to B$ be a fibration and $i\colon A \to B$ be a cofibration which is the inclusion of a closed subspace $A$ of $B$. Consider a homotopy $h\colon X \times I \to B$ and a map $f\colon A \times I \cup X \times \{0\} \to E$. Let $j\colon A \times I \cup X \times \{0\} \to X \times I$ be the obvious inclusion. Suppose $p \circ f = h \circ j$.*

*Then there exists a homotopy $H\colon X \times I \to E$ satisfying $p \circ H = h$ and $H \circ j = f$, in other words, we can solve the following extension problem*
$$\begin{array}{ccc} A \times I \cup X \times \{0\} & \xrightarrow{f} & E \\ {\scriptstyle j}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[h]{} & B. \end{array}$$

9.2. **Turning a map into a fibration.** Let $f\colon X \to Y$ be a map. Consider the space $W(f)$ defined in 9.5. Then the composite $q_f\colon W(f) \xrightarrow{\overline{f}} \mathrm{map}(I, Y) \xrightarrow{e_Y^1} Y$ is a fibration by the following argument.

Consider a homotopy $h\colon A \times I \to Y$ and a map $u\colon A \to X$ satisfying $f \circ u = h_0$. Since (9.5) is a pullback, there is a homotopy $H\colon A \times I \to W(f)$ which is uniquely determined by the properties that $e_f \circ H$ is the composite $A \times I \xrightarrow{\mathrm{pr}_A} A \xrightarrow{u} X$ for $\mathrm{pr}_A$ the canonical projection and that $\overline{f} \circ H\colon A \times I \to \mathrm{map}(I, B)$ agrees under the adjunction $\mathrm{map}(A \times I, \mathrm{map}(I, Y)) \xrightarrow{\cong} \mathrm{map}(A \times I \times I \to Y)$ with the composite $A \times I \times I \xrightarrow{\mathrm{id}_A \times v} A \times I \xrightarrow{h} Y$ for the map $v\colon I \times I \to I$ sending $(s, t) \to s \cdot t$. Explicitly $H$ sends $(a, t)$ to the pair $(u(a), w)$ where $w\colon I \to Y$ sends $s$ to $h(a, st)$. One easily checks that $H_0 = u$ and $q_f \circ H = h$ holds.

We have the inclusion $i\colon X \to W(f)$ sending $x$ to $(x, c_{f(x)})$. Its composite with the map $e_f\colon W(f) \to X$ appearing in (9.5) is the identity on $X$. Define a homotopy $k\colon W(f) \times [0, 1] \to W(f)$ by sending $((x, w), t)$ to $(x, w_t)$ for the path $w_t\colon I \to Y$ sending $s$ to $w(st)$. Then $k_0 = i \circ e_f$ and $k_1 = \mathrm{id}_{W(p)}$. Hence $e_f$ is a homotopy equivalence with homotopy inverse $i$. Obviously $q_f \circ i = f$ holds. So we get a factorization $f\colon X \xrightarrow{i} W(f) \xrightarrow{q_f} Y$ into a homotopy equivalence $i$ followed by a fibration $q_f$. Actually we obtain a diagram

(9.10)
$$\begin{array}{ccccc} X & \xrightarrow[\simeq]{i} & W(f) & \xrightarrow[\simeq]{e_f} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q_f} & \swarrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

such that the left triangle commutes, the right triangle commutes up to homotopy, the two horizontal arrows are homotopy equivalences and homotopy inverse to one another, and the middle vertical arrow $q_f$ is a fibration. Recall that we have

$$\begin{aligned} W(f) &= \{(x,w) \in X \times \operatorname{map}(I,Y) \mid p(x) = w(0)\}; \\ i(x) &= (x, c_{f(x)}); \\ e_f(x,w) &= x; \\ q_f(x,w) &= w(1). \end{aligned}$$

### 9.3. **Homotopy Theorem for pullbacks and fibrations.**

**Theorem 9.11** (Homotopy Theorem for pullbacks and fibrations)**.** *Consider the pullback*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & E \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B. \end{array}$$

*Suppose that $p$ is a fibration and $f$ is a homotopy equivalence. Then $\overline{p}$ is a fibration and $\overline{f}$ is a homotopy equivalence.*

*Proof.* We have already shown in Proposition 9.7 that $\overline{p}$ is a fibration. The proof that $\overline{f}$ is a homotopy equivalence is omitted and can be found in [24, Proposition 5.5.10 on page 118]. □

Theorem 9.11 can easily be extended to the following theorem.

**Theorem 9.12** (Homotopy Theorem for maps between pullbacks)**.** *Let the following two diagrams be pullbacks*

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X_0 \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{k_1} & Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ Y_2 & \xrightarrow[l_2]{} & Y_0 \end{array}$$

*where the right vertical arrows $j_1$ and $l_1$ are fibrations. Let $f_i\colon X_i \to Y_i$ be homotopy equivalences for $i = 0,1,2$ satisfying $l_1 \circ f_1 = f_0 \circ j_1$ and $l_2 \circ f_2 = f_0 \circ j_2$. Denote by $f\colon X \to Y$ the map induced by $f_0$, $f_1$, and $f_2$ and the pullback property.*

*Then $f$ is a homotopy equivalence.*

**Remark 9.13.** The condition that $j_1$ and $l_1$ are fibrations appearing in Theorem 9.12 is necessary as the following examples shows.

Given a pointed space $(X,x)$, let $P(X,x)$ be the subspace of $\operatorname{map}(I,X)$ consisting of path $w$ with $w(0) = x$ and $\Omega(X,x)$ be the subspace of $\operatorname{map}(I,X)$ consisting of path $w$ with $w(0) = w(1) = x$. Often $\Omega(X,x)$ is called the *loop space* of $X$. One easily checks that $P(X,x)$ is contractible. We take as pullbacks

$$\begin{array}{ccc} \Omega(X,x) & \xrightarrow{i_1} & P(X,x) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \{\bullet\} & \xrightarrow[j_2]{} & X \end{array} \qquad\qquad \begin{array}{ccc} \{\bullet\} & \xrightarrow{k_1} & \{\bullet\} \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ \{\bullet\} & \xrightarrow[l_2]{} & X \end{array}$$

where $j_1$ is by evaluation at 1 and $j_2$, $l_1$, and $l_2$ have as image the base point $x$. Take $f_1\colon P(X,x) \to \{\bullet\}$ to be the projection, $f_2 = \mathrm{id}_{\{\bullet\}}$, and $f_0 = \mathrm{id}_X$. Note that $\Omega(X,x)$ is in general not contractible.

9.4. **The fiber transport.** Let $p\colon E \to B$ be a fibration. Next we construct a functor

$$\tau\colon \Pi(B) \to \text{h-}\mathsf{Top}. \tag{9.14}$$

It sends an object $x$ in the fundamental groupoid to the fiber $F_x := p^{-1}(x)$ of $p$ over $x$. Consider a morphism $[w]\colon x \to y$. Choose a path $w\colon I \to B$ with $w(0) = x$ and $w(1) = y$ representing $w$. Apply HLP to the inclusion $i_x\colon F_x \to E$ and the homotopy $h\colon F_x \times I \xrightarrow{\mathrm{pr}_I} I \xrightarrow{w} B$ for the projection $\mathrm{pr}_I$. This yields a homotopy $H\colon F_x \times I \to E$ with $H_0 = i_x$ and $p \circ H = h$. Then $H_1$ is a map $F_x \to F_y$ and we define $\tau([w]) = [H_1]$. We leave it to the reader to check that $[H_1]$ depends only on $[w]$ and is independent of the choices of $w$ and $H$ and yields a covariant functor.

**Proposition 9.15.** *Let $p\colon E \to B$ be a fibration over a path connected space $B$. Then for any two points $x$ and $y$ the fibers $F_x$ and $F_y$ are homotopy equivalent.*

*Proof.* This follows from the functor $\tau$ of (9.14) and the fact that $\Pi(B)$ is a groupoid. ☐

9.5. **Homotopy equivalences and fibrations.**

**Definition 9.16** (Fiber homotopy equivalence)**.** Let $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ be fibrations over $B$.

A *fiber preserving map* $f\colon p_0 \to p_1$ is a map $f\colon E_0 \to E_1$ satisfying $p_1 \circ f = p_0$.

Two such fiber preserving maps $f_0, f_1\colon p_0 \to p_1$ are called *fiber homotopy equivalent* if there is a homotopy $h\colon E_0 \times I \to E_1$ such that $h_0 = f_0$ and $h_1 = f_1$ hold and $h_t\colon E_0 \to E_1$ is a fiber preserving map $h_t\colon p_0 \to p_1$ for each $t \in I$.

A fiber preserving map $f\colon E_0 \to E_1$ is a fiber homotopy equivalence if there is a fiber preserving map $g\colon E_1 \to E_0$ such that $g \circ f$ is fiber homotopy equivalent to $\mathrm{id}_{E_0}$ and $f \circ g$ is fiber homotopy equivalent to $\mathrm{id}_{E_1}$.

**Theorem 9.17** (Characterization of fiber homotopy equivalences)**.** *Let $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ be fibrations over $B$.*

*Then a fiber preserving map $f\colon p_0 \to p_1$ is a fiber homotopy equivalence if and only if the underlying map $f\colon E_0 \to E_1$ is a homotopy equivalence.*

*Proof.* The proof is indicated for instance in [13, Proposition in Section 5 of Chapter 7 on page 52]. ☐

**Theorem 9.18** (Homotopy Covering Theorem)**.** *Let $p\colon E \to B$ be a fibration. Consider two maps $f_0, f_1\colon X \to B$ which are homotopic. Let $p_i\colon E_i \to X$ be the fibration obtained by the pulling back of $p$ to $f_i$ for $i = 0, 1$.*

*Then $p_0$ and $p_1$ are fiber homotopy equivalent.*

*Proof.* See [20, Proposition 15.16 on page 344]. ☐

**Corollary 9.19.** *Let $p\colon E \to B$ be a fibration over a contractible space $B$. Then $p$ is fiber homotopy equivalent to a trivial fibration $B \times F \to B$.*

9.6. **The Fiber Sequence.** A pointed map $f\colon (X,x) \to (Y,y)$ induces by composition for every pointed space $(A,a)$ a map

$$f_*\colon [(A,a),(X,x)]^0 \to [(A,a),(Y,y)]^0, \quad [u] \mapsto [f \circ u]$$

which depends only on the pointed homotopy class of $f$. A sequence $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{g} (Z,z)$ of maps of pointed space is called *homotopy exact* if for each pointed space $(A,a)$ the induced sequence of pointed sets

$$[(A,a),(X,x)]^0 \xrightarrow{f_*} [(A,a),(Y,y)]^0 \xrightarrow{g_*} [(A,a),(Z,z)]^0$$

is exact at $[(A,a),(Y,y)]^0$ in the sense that the image of $f_*$ is the preimage of $g_*$ of the base point in $[(A,a),(Z,z)]$ given by $[c_z]$ for the constant pointed map $c_z\colon (B,b) \to (Z,z)$. Note that this implies that $g \circ f$ is pointed homotopy equivalent to the constant map $c_z\colon (X,x) \to (Z,z)$.

Recall that $P(Y,y)$ is the subspace of $\operatorname{map}(I,Y)$ consisting of path $w$ with $w(0) = y$. It has the constant path $c_y$ as base point. Equivalently, one define $P(Y,y)$ by the pullback

(9.20)
$$\begin{array}{ccc} P(Y,y) & \xrightarrow{\overline{c_y}} & \operatorname{map}(I,Y) \\ \downarrow & & \downarrow{\scriptstyle e_Y^0} \\ \{\bullet\} & \xrightarrow[c_y]{} & Y. \end{array}$$

Define the space $P(f,x)$ by the pullback

(9.21)
$$\begin{array}{ccc} P(f,x) & \xrightarrow{\overline{f}} & P(Y,y) \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle e_Y^1} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

The space $P(f,x)$ inherits from the base points $x \in X$ and $c_y \in P(Y,y)$ a base point $*$ for which the diagram (9.21) becomes a diagram of pointed spaces. Explicitly $P(f,x)$ is the subspace of $X \times \operatorname{map}(I,Y)$ consisting of those pairs $(z,w)$ for which $w(0) = y$ and $w(1) = f(z)$ holds. The map $\overline{f}$ sends $(z,w)$ to $w$ and $p_f$ sends $(z,w)$ to $z$.

**Lemma 9.22.** *Let $f\colon (X,x) \to (Y,y)$ be a pointed map of pointed spaces. Let $p_f\colon P(f,x) \to X$ be the map defined in* (9.21)*. Then*

$$(P(f,x),*) \xrightarrow{p_f} (X,x) \xrightarrow{f} (Y,y)$$

*is homotopy exact.*

*Proof.* This is a direct consequence of the pullback (9.21) and the adjunction $\operatorname{map}(A,\operatorname{map}(I,X)) \xrightarrow{\cong} \operatorname{map}(A \times I, X)$. Namely, they imply that a pointed map $(A,a) \to (P(f,x),*)$ is the same as a pointed map $u\colon (A,a) \to (X,x)$ together with a pointed homotopy $h\colon (A,a) \times I \to (Y,y)$ between the constant map $c_y\colon B \to Y$ and $f \circ u\colon (A,a) \to (Y,y)$. □

One can iterate this process and obtains a homotopy exact sequence, infinite to the left

$$\cdots \xrightarrow{f_5} P(f_3) \xrightarrow{f_4} P(f_2) \xrightarrow{f_3} P(f_1) \xrightarrow{f_2} P(p_f) \xrightarrow{f_1} P(f) \xrightarrow{p_f} X \xrightarrow{f} Y.$$

Here and also sometimes in the sequel we omit the base points from the notation.

The further investigations replace the spaces $P(f_1), P(f_2), \ldots$ by more appealing spaces, namely by iterated loop spaces $\Omega^n(X,x)$. Recall that the loop space $\Omega(X,x)$

is the subspace of $\operatorname{map}(I, X)$ consisting of maps $w \colon I \to X$ with $w(0) = w(1) = x$. It can also be described by the pullback

$$\begin{array}{ccc} \Omega(X,x) & \xrightarrow{i} & P(X,x) \\ \downarrow & & \downarrow{\scriptstyle e^1_X} \\ \{\bullet\} & \xrightarrow[c_x]{} & X \end{array} \tag{9.23}$$

or, equivalently,

$$\begin{array}{ccc} \Omega(X,x) & \xrightarrow{i} & \operatorname{map}(I,X) \\ \downarrow & & \downarrow{\scriptstyle \operatorname{map}(i,\operatorname{id}_X)} \\ \{\bullet\} & \xrightarrow{} & \operatorname{map}(\partial I,X) \end{array} \tag{9.24}$$

where $i \colon \partial I \to I$ is the inclusion and the lower horizontal arrow has the constant map $c_x$ with value $x$ as image.

Suppose additionally that the pointed map $f \colon (X,x) \to (Y,y)$ is a fibration (after forgetting the base points) and that $(X,x)$ and $(Y,y)$ are well-pointed. We have the commutative diagram

$$\begin{array}{ccccc} \{\bullet\} & \xrightarrow{c_y} & Y & \xleftarrow{f} & X \\ \simeq \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \operatorname{id}_Y} & & \downarrow{\scriptstyle \operatorname{id}_X} \\ P(Y,y) & \xrightarrow[e^1_Y]{} & Y & \xleftarrow[f]{} & X \end{array}$$

where $j$ is the map onto the base point $*$ on $P(Y,y)$. The pullback of the upper row is $f^{-1}(y)$, whereas the pullback of the lower row is $P(f,x)$ because of (9.21). All vertical maps are homotopy equivalences. Hence the diagram induces by the Homotopy Theorem 9.11 a homotopy equivalence

$$g \colon f^{-1}(y) \xrightarrow{\simeq} P(f,x).$$

It is a pointed homotopy equivalence by Lemma 8.25, since its domain and codomain are well-pointed. The following diagram of well-pointed spaces commutes

$$\begin{array}{ccccc} (f^{-1}(y),x) & \xrightarrow{i_y} & (X,x) & \xrightarrow{f} & (Y,y) \\ {\scriptstyle g}\downarrow \simeq & & \downarrow{\scriptstyle \operatorname{id}_X} & & \downarrow{\scriptstyle \operatorname{id}_Y} \\ (P(f,y),*) & \xrightarrow[p_f]{} & (X,x) & \xrightarrow[f]{} & (Y,y) \end{array} \tag{9.25}$$

for $i_x$ the inclusion and all vertical arrows are pointed homotopy equivalences. Since the lower row is homotopy exact, the upper row is homotopy exact.

The map $p_f \colon P(f,x) \to X$ is a fibration by Proposition 9.7 applied to the pullback (9.7) since the inclusion $\{y\} \to Y$ is a cofibration and hence $e^1_Y \colon P(Y,y) \to Y$ is a fibration by Proposition 9.8 (i). Then we obtain a pullback

$$\begin{array}{ccc} (\Omega(Y,y),*) & \xrightarrow{i_f} & P(f,x) \\ \downarrow & & \downarrow{\scriptstyle p_f} \\ \{\bullet\} & \xrightarrow[c_x]{} & X \end{array}$$

from the pullbacks (9.21) and (9.23). Explicitly $i_f\colon \Omega(Y,y) \to P(f,x)$ sends $w$ to $(x,w)$ and induces a homeomorphism $i_f\colon \Omega(Y,y) \to p_f^{-1}(x)$. Hence the sequence of pointed spaces

$$\Omega(Y,y) \xrightarrow{i_f} (P(f,x),*) \xrightarrow{p_f} (X,x) \xrightarrow{f} (Y,y)$$

is homotopy exact. Iterating this process yields the following result.

**Theorem 9.26** (Fiber sequence)**.** *Let $f\colon (X,x) \to (Y,y)$ be a map of well-pointed spaces. Then we obtain a homotopy exact sequence, infinite to the left,*

$$(9.27) \quad \cdots \xrightarrow{\Omega^2 p_f} (\Omega^2(X,x),*) \xrightarrow{\Omega^2 f} (\Omega^2(Y,y),*) \xrightarrow{\Omega i_f} (\Omega(P(f,x),*),,*) \xrightarrow{\Omega p_f} (\Omega(X,x),*) \xrightarrow{\Omega f} (\Omega(Y,y),*) \xrightarrow{i_f} (P(f,x),*) \xrightarrow{p_f} (X,x) \xrightarrow{f} (Y,y).$$

Note this sequence (9.27) is natural in $f$ and yields for any pointed space $(B,b)$ the long exact sequence of pointed sets, infinite to the left,

$$(9.28) \quad \cdots \xrightarrow{(\Omega^2 p_f)_*} [B,\Omega^2(X,x)]^0 \xrightarrow{(\Omega^2 f)_*} [B,\Omega^2(Y,y)]^0 \xrightarrow{(\Omega i_f)_*} [B,\Omega P(f,x)]^0 \xrightarrow{(\Omega p_f)_*} [B,\Omega(X,x)]^0 \xrightarrow{(\Omega f)_*} [B,\Omega(Y,y)]^0 \xrightarrow{(i_f)_*} [B,P(f,x)]^0 \xrightarrow{(p_f)_*} [B,X]^0 \xrightarrow{f_*} [B,Y]^0$$

where we have omitted the base points of the pointed spaces involved. Note the obvious fact that the map $f_*\colon [B,X]^0 \to [B,Y]^0$ is not surjective in general.

### 9.7. Group structures on pointed sets associated to the Fiber Sequence.

**Definition 9.29** (Group object in $\mathrm{h}\text{-}\mathsf{Top}^0$)**.** A *group object in* $\mathrm{h}\text{-}\mathsf{Top}^0$ is a pointed space $(X,x)$ together with pointed maps

$$\begin{array}{rcl} m\colon (X\times X,(x,x)) & \to & (X,x); \\ i\colon (X,x) & \to & (X,x), \end{array}$$

satisfying the following conditions:

(i) The two pointed maps $(X,x) \to (X,x)$ sending $y$ to $m(x,y)$ and $m(y,x)$ respectively are pointed homotopic to the identity;
(ii) The two pointed maps $m \circ (\mathrm{id}_X \times m)$ and $m \circ (m \times \mathrm{id}_X)$ from $(X \times X \times X,(x,x,x))$ to $(X,x)$ are pointed homotopic;
(iii) The two pointed maps $m \circ (\mathrm{id}_X \times i)$ and $m \circ (\mathrm{id}_X \times i)$ from $(X\times X,(x,x))$ to $(X,x)$ are pointed homotopic to the constant map $c_x$.

Sometimes group objects in $\mathrm{h}\text{-}\mathsf{Top}^0$ are called *associative H-spaces with inverse.*

**Example 9.30** (Examples for group object in $\mathrm{h}\text{-}\mathsf{Top}^0$)**.** A topological group is obviously an example of a group object in $\mathrm{h}\text{-}\mathsf{Top}^0$. Our main example is the loop space $\Omega(X,x)$ of a well-pointed space $(X,x)$ where $m\colon \Omega(X,x)\times\Omega(X,x) \to \Omega(X,x)$ sends $(v,w)$ to the concatenation $v*w$ and $i\colon \Omega(X,x) \to \Omega(X,x)$ sends $w$ to the inverse path $w^-$.

**Remark 9.31.** Let $(B,b)$ be a pointed space and $(X,x)$ be a group object in $\mathrm{h}\text{-}\mathsf{Top}^0$. Then $[(B,b),(X,x)]$ inherits a group structure by the multiplication given by

$$[(B,b),(X,x)]^0 \times [(B,b),(X,x)]^0 \to [(B,b),(X,x)]^0, \quad ([f],[g]) \mapsto [m\circ(f\times g)].$$

The unit is given by the class $[c_x]$ of the constant map. The inverse of $[f] \in [(B,b),(X,x)]^0$ is given by $[i \circ f]$.

In particular we obtain for a well-pointed space $(X,x)$ a group structure on $[(B,b),(\Omega(X,x),*)]^0$. This group structure on $[(B,b),(\Omega^n(X,x),*)]^0$ is abelian for

$n \geq 2$. The sequence (9.28) is compatible with the group structures as long as they exist.

9.8. **The adjunction between suspension and loop spaces.** Let $(X, x)$ and $(Y, y)$ be well-pointed spaces. Then there is a natural adjunction homeomorphism

$$\mathrm{ad}\colon \mathrm{map}((\Sigma(X,x),*),(Y,y))^0 \xrightarrow{\cong} \mathrm{map}((X,x),(\Omega(Y,y),*))^0 \tag{9.32}$$

between mapping spaces of pointed spaces. It is uniquely determined by the property that it makes the following diagram commutative,

$$\begin{array}{ccc} \mathrm{map}((\Sigma(X,x),*),(Y,y))^0 & \xrightarrow{\mathrm{ad}} & \mathrm{map}((X,x),(\Omega(Y,y),*))^0 \\ \downarrow & & \downarrow \\ \mathrm{map}(X \times I, Y) & \xrightarrow[\cong]{} & \mathrm{map}(X, \mathrm{map}(I,Y)) \end{array}$$

where the lower horizontal arrow is the natural adjunction homeomorphism, the left vertical is the closed embedding coming from the projection $X \times I \to \Sigma(X,x)$ and the right vertical arrow is the closed embedding coming from the canonical inclusion $\Omega(Y,y) \to \mathrm{map}(I,Y)$. By passing to $\pi_0$, we obtain from (9.32) natural adjunction bijection

$$[(\Sigma(X,x),*),(Y,y)]^0 \xrightarrow{\cong} [(X,x),(\Omega(Y,y),*)]^0. \tag{9.33}$$

It is compatible with the group structure on the domain introduced in (8.43) and on the codomain introduced in Remark 9.31.

If we take $(X,x) = (S^n, s)$, we obtain for $n = 0, 1, 2, \ldots$ a natural bijection of groups

$$\pi_{n+1}(Y,y) \xrightarrow{\cong} \pi_n(\Omega(Y,y),*) \tag{9.34}$$

for $n \geq 0$. Iterating this, we get a bijection of groups

$$[(S^0,s),(\Omega^n(X,x),*)]^0 = \pi_0(\Omega^n(X,x)) \xrightarrow{\cong} \pi_n(X,x). \tag{9.35}$$

9.9. **Locally trivial bundles are fibrations.** The proof of the following result can be found in [24, Theorem 13.4.1 on page 32].

**Theorem 9.36** (Being a fibration is a local property)**.** *Let $p\colon E \to B$ be a continuous map and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of $B$. Suppose that $\mathcal{U}$ is numerable, i.e., admits a subordinate partition of unity, and that $p|_{p^{-1}(U_i)}\colon p^{-1}(U_i) \to U_i$ is a fibration for every $i \in I$.*

*Then $p$ is a fibration.*

Recall that a *partition of unity subordinate to $\mathcal{U}$* is a family $\{t_i\colon U_i \to [0,1] \mid i \in I\}$ of functions $t_i$ satisfying:

- The support $\mathrm{supp}(t_i) := \overline{\{b \in B \mid t_i(b) \neq 0\}} \subseteq B$ of $t_i$ is contained in $U_i$ for $i \in I$;
- The family $\{t_i\colon U_i \to [0,1] \mid i \in I\}$ is *locally finite*, i.e., for every $b \in B$ there is an open neighborhood $V$ together with a finite subset $I_0 \subseteq I$ such that $t_i(v) = 0$ holds for all $v \in V$ and $i \in I \setminus I_0$;
- The (finite) sum $\sum_{i \in I} t_j(b)$ is 1 for every $b \in B$.

A space $B$ is called *paracompact* if every open covering $\mathcal{U} = \{U_i \mid i \in I\}$ has a refinement $\mathcal{V} = \{v_j \mid j \in J\}$ which is *locally finite*, i.e., for every $b \in B$ there exists an open neighborhood $W$ of $b$ in $B$ and a finite subset $I_0 \subseteq I$ satisfying $W \cap U_i \implies i \in I_0$. Note that such $\mathcal{V}$ is automatically numerable. Every metric space is paracompact, see [16, Theorem 4.3 on page 256]. Every $CW$-complex is paracompact, see [15] or [7, Theorem 1.3.5]. Theorem 9.36 and the discussion above imply the following result.

**Theorem 9.37** (Locally trivial bundles are fibrations)**.** *Let $p\colon E \to B$ be a locally trivial bundle over a paracompact space, e.g., a principal $G$-bundle for a topological group $G$, a vector bundle, or a covering over a space $B$ which is a CW-complex or a metric space.*

*Then $p$ is a fibration.*

9.10. **Duality between cofibrations and fibrations.** There is a kind of duality between cofibrations and fibrations which we want to discuss next. One has to interchange $X \times I$ and $\mathrm{map}(I, X)$, interchange pushouts with pullbacks, $\Sigma$ with $\Omega$ and invert all arrows. Here is a list of some examples.

- (HEP) and (HLP)

  Consider the diagrams (8.3) and (9.3)

$$\begin{array}{ccc} Y & \xleftarrow{\;f\;} & X \\ {\scriptstyle e_Y^0}\uparrow & \swarrow{\scriptstyle H} & \uparrow{\scriptstyle i} \\ \mathrm{map}(I,Y) & \xleftarrow[\;\overline{h}\;]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\;f\;} & E \\ {\scriptstyle i_0^X}\downarrow & \nearrow{\scriptstyle H} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\;h\;]{} & B; \end{array}$$

- mapping cylinder and W(p)

  Consider the diagrams (8.5) and (9.5)

$$\begin{array}{ccc} \mathrm{cyl}(i) & \xleftarrow{\;k\;} & X \\ {\scriptstyle l}\uparrow & & \uparrow{\scriptstyle i} \\ A \times I & \xleftarrow[\;i_0^A\;]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W(p) & \xrightarrow{\;e_p\;} & E \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{map}(I,B) & \xrightarrow[\;e_B^0\;]{} & B; \end{array}$$

- Turning a maps into cofibration or fibration

  Consider the diagrams (8.37) and (9.10)

$$\begin{array}{ccccc} Y & \underset{\simeq}{\xleftarrow{\;p\;}} & \mathrm{cyl}(f) & \underset{\simeq}{\xleftarrow{\;k\;}} & Y \\ & {\scriptstyle f}\nwarrow & \uparrow{\scriptstyle i} & \nearrow{\scriptstyle f} & \\ & & X & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} X & \underset{\simeq}{\xrightarrow{\;i\;}} & W(f) & \underset{\simeq}{\xrightarrow{\;e_p\;}} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q_f} & \swarrow{\scriptstyle f} & \\ & & Y; & & \end{array}$$

- The Homotopy Theorems 8.28 and 9.11 where the relevant diagrams are

$$\text{the pushout} \quad \begin{array}{ccc} Y & \xleftarrow[\;\overline{f}\;]{} & X \\ {\scriptstyle \overline{i}}\uparrow & & \uparrow{\scriptstyle i} \\ B & \xleftarrow{\;f\;} & A \end{array} \qquad \text{and the pullback} \quad \begin{array}{ccc} \overline{X} & \xrightarrow{\;\overline{f}\;} & E \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\;f\;]{} & B. \end{array}$$

- The Cofiber Sequence appearing in Theorem 8.40 and the Fiber Sequence appearing in Theorem 9.26 as well as the long exact sequences of pointed homotopy classes associated to them, see (8.42) and (9.28).

## 10. The long exact homotopy sequence associated to a fibration

### 10.1. **The homotopy sequence.**

**Theorem 10.1** (The long exact homotopy sequence associated to a fibration)**.** *Let $(E, e)$ and $(B, b)$ be well-pointed spaces. Let $p\colon E \to B$ be a fibration with $p(e) = b$. Put $F_b = p^{-1}(b)$. Denote by $i\colon F_b \to E$ the inclusion.*

*Then we obtain a long exact sequence, infinite to the left*

$$(10.2)\qquad \cdots \xrightarrow{\partial_{n+1}} \pi_n(F_b,e) \xrightarrow{\pi_n(i,e)} \pi_n(E,e) \xrightarrow{\pi_n(p,e)} \pi_n(B,b) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} \pi_1(F_b,e) \xrightarrow{\pi_1(i,e)} \pi_1(E,e) \xrightarrow{\pi_1(p,e)} \pi_1(B,b) \xrightarrow{\partial_1} \pi_0(F_b) \xrightarrow{\pi_0(i)} \pi_0(E) \xrightarrow{\pi_0(p)} \pi_0(B)$$

*with the following properties:*

- *It is an exact sequence of groups in the range until $\pi_1(B,b)$;*
- *It is exact at $\pi_1(B,b)$ in the sense that the image of $\pi_1(p,e)$ is the preimage of the component in $F_b$ containing $e$ under $\partial_1$;*
- *It is exact at $\pi_0(F_b)$ in the sense that the image of $\partial_1$ is the preimage of the component in $E$ containing $e$ under $\pi_0(i)$;*
- *It is exact at $\pi_0(E)$ in the sense that the image of $\pi_0(i)$ is the preimage of the component in $B$ containing $b$ under $\pi_0(p)$;*
- *The boundary operator $\partial_{n+1}\colon \pi_{n+1}(B,b) \to \pi_n(F_b,e)$ is defined as follows. Consider $u \in \pi_{n+1}(B,b)$. Choose a map $h\colon S^n \times [0,1] \to B$ which sends $S^n \times \{0,1\} \cup \{s\} \times I$ to $b$ such that for the pointed standard homeomorphism*

  $$q\colon (S^n \times [0,1]/(S^n \times \{0,1\} \cup \{s\}) \times I, *) \xrightarrow{\cong} (S^{n+1}, b)$$

  *the composite $\overline{h} = h \circ q\colon (S^{n+1}, s) \to (B,b)$ represents $u$. Choose a solution $H$ to the lifting problem*

  $$\begin{array}{ccc} S^n \times \{0\} \cup \{s\} \times I & \xrightarrow{c_e} & E \\ \Big\downarrow i & \overset{H}{\nearrow} & \Big\downarrow p \\ S^n \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

  *which exists by Proposition 9.9. Then $\partial_{n+1}(x)$ is represented by the pointed map $H_1\colon (S^n, s) \to (F_b, e)$.*

*Proof.* This follows from the exact sequence (9.28) applied in the case $(B,b) = (S^0, s)$,

$$\cdots \xrightarrow{(\Omega^2 p_p)_*} [S^0, \Omega^2 E]^0 \xrightarrow{(\Omega^2 p)_*} [S^0, \Omega^2 B]^0 \xrightarrow{(\Omega i_p)_*} [S^0, \Omega P(p,e)]^0 \xrightarrow{(\Omega p_p)_*} [S^0, \Omega E]^0 \xrightarrow{(\Omega p)_*} [S^0, \Omega B]^0 \xrightarrow{(i_p)_*} [S^0, P(p,e)]^0 \xrightarrow{(p_p)_*} [S^0, E]^0 \xrightarrow{p_*} [S^0, B]^0,$$

the isomorphism (9.35)

$$[(S^0, s), (\Omega^n(X,x), *)]^0 = \pi_0(\Omega^n(X,x)) \xrightarrow{\cong} \pi_n(X,x)$$

and the diagram (9.25) which becomes in the situation considered here

$$\begin{array}{ccccc} (F_b, e) & \xrightarrow{i_b} & (X,x) & \xrightarrow{f} & (Y,y) \\ g \Big\downarrow \simeq & & \Big\downarrow \mathrm{id}_X & & \Big\downarrow \mathrm{id}_Y \\ (P(p,e), *) & \xrightarrow[p_p]{} & (X,x) & \xrightarrow[f]{} & (Y,y). \end{array}$$

□

**Remark 10.3** (Serre fibrations and the homotopy sequence)**.** In order to have the long exact homotopy sequence of Theorem 10.1 available, one needs only to know that $p\colon E \to B$ is a Serre fibration, see [24, Theorem 6.3.2 on page 130].The obvious version of Theorem 9.36 holds also for Serre fibrations, see [24, Theorem 6.3.3 on page 130].

10.2. **The Hopf fibration.** Fix $d \in \mathbb{Z}^{\geq 1}$. We can consider $S^1$ as a subgroup of $\mathbb{C} \setminus \{0\}$ with respect to multiplication of complex numbers. In particular $S^1$ acts diagonally on $\mathbb{C}^{d+1}$. Then $S^{2d+1} \subseteq \mathbb{C}^{d+1}$ inherits an $S^1$-action, which is free. Recall that $\mathbb{CP}^d$ is the set of 1-dimensional complex vector spaces of $\mathbb{C}^{d+1}$ and is equipped with the quotient topology with respect to the map $f\colon \mathbb{C}^{d+1} \setminus \{0\} \to \mathbb{CP}^d$ sending $z \in \mathbb{C}^{d+1} \setminus \{0\}$ to the 1-dimensional complex vector space generated by $z$. Consider the map $p\colon S^{2d+1} \to \mathbb{CP}^d$ obtained by restricting $f$ to $S^{2d+1}$. Then $p\colon S^{2d+1} \to \mathbb{CP}^d$ is an identification. Obviously it factorizes over the projection $\mathrm{pr}\colon S^{2d+1} \to S^{2d+1}/S^1$ into a bijective map $u\colon S^{2d+1}/S^1 \xrightarrow{\cong} \mathbb{CP}^d$. Since $p$ and pr are identifications, $u$ is a bijective identification and hence a homeomorphism. Now one easily checks that $p\colon S^{2d+1} \to \mathbb{CP}^d$ is a principal $S^1$-bundle. Theorem 9.36 implies that $p$ is a fibration. From Theorem 10.1 we obtain a long exact sequence of groups

$$\cdots \xrightarrow{\partial_{n+1}} \pi_n(S^1) \to \pi_n(S^{2d+1}) \xrightarrow{\pi_n(p)} \pi_n(\mathbb{CP}^d) \xrightarrow{\partial_n}$$
$$\cdots \xrightarrow{\partial_2} \pi_1(S^1) \to \pi_1(S^{2d+1}) \xrightarrow{\pi_1(p)} \pi_1(\mathbb{CP}^d) \to \{1\}.$$

Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = \{0\}$ for $n \geq 2$ by (2.8), we obtain an isomorphism

$$\pi_n(p)\colon \pi_n(S^{2d+1}) \to \pi_n(\mathbb{CP}^d) \quad \text{for } n \geq 3 \tag{10.4}$$

and an exact sequence of abelian groups

$$\{0\} \to \pi_2(S^{2d+1}) \to \pi_2(\mathbb{CP}^d) \to \pi_1(S^1) \to \pi_1(S^{2d+1}) \to \pi_1(\mathbb{CP}^d) \to \{0\}.$$

Recall that $\pi_m(S^n) \cong \{0\}$ for $m < n$ and $\pi_n(S^n) \cong \mathbb{Z}$ by Theorem 3.4. Hence $\mathbb{CP}^d$ is simply connected and we get an isomorphism

$$\pi_2(\mathbb{CP}^d) \cong \mathbb{Z}.$$

Since $\mathbb{CP}^1$ is homeomorphic to $S^2$, we get from (10.4) the following theorem, which we have already briefly discussed in Example 3.5.

**Theorem 10.5** ($\pi_3(S^2)$ is infinite cyclic)**.** *The abelian group $\pi_3(S^2)$ is an infinite group with $[p]$ for the so called Hopf map $p\colon S^3 \to \mathbb{CP}^1 = S^2$ as generator.*

10.3. **Homotopy groups of loop spaces.**

**Proposition 10.6.** *Let $(E, e)$ and $(B, b)$ be well-pointed spaces. Let $p\colon E \to B$ be a fibration with $p(e) = b$. Put $F_b = p^{-1}(b)$. Suppose that $E$ is weakly contractible. Then we get isomorphisms*

$$\pi_{n+1}(B, b) \xrightarrow{\cong} \pi_n(F_b, e)$$

*for $n \geq 1$.*

*Proof.* This follows directly from Theorem 10.1. □

Suppose that $(X, x)$ is a path connected well-pointed space. Then we have the fibration $p\colon P(X, x) \to X$ whose fiber over $x \in X$ is the loop space $\Omega(X, x)$. As $P(X, x)$ is contractible, we get from Proposition 10.6 for $n \geq 0$ a preferred isomorphism of groups

$$\partial_{n+1}(X, x)\colon \pi_{n+1}(X, x) \xrightarrow{\cong} \pi_n(\Omega(X, x)). \tag{10.7}$$

Note that $\pi_0(\Omega(X, x)) = [(S^0, s), (\Omega(X, x), *)]^0$ has a group structure by Example 9.30 and Remark 9.31. Iterating this, we get for every $n \geq 1$ a group isomorphism $\pi_n(X, x) \cong \pi_0(\Omega^n(X, x))$, as already mentioned in (9.35).

10.4. **Homotopy groups of classifying spaces $BG$.** Let $G$ be a topological group and $p\colon EG \to BG$ be the universal principal $G$-bundle. Recall that it has the property that the pullback construction defines for every $CW$-complex $X$ a bijection

$$[X, BG] \xrightarrow{\cong} \{\text{isomorphism classes of principal } G\text{-bundles over } X\} \tag{10.8}$$

and is up to isomorphism of $G$-bundles uniquely characterized by the property that $EG$ is weakly contractible. Proposition 10.6 implies that $BG$ is path connected and satisfies for $n \geq 1$

$$\pi_n(BG) \cong \pi_{n-1}(G) \tag{10.9}$$

for $e \in G$ the unit element.

10.5. **On the homotopy groups of some classical Lie groups.** Denote by $\mathbb{F}$ one of the (skew)fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ given by the reals numbers, the complex numbers, or the quaternions. We have the associated orthogonal, unitary, or symplectic groups which are Lie groups:

$$\begin{array}{lllll} \mathrm{O}(n) & = & \mathrm{O}(n,\mathbb{R}) & = & \{A \in \mathrm{M}(n,n,\mathbb{R}) \mid A^tA = I_n\}; \\ \mathrm{SO}(n) & = & \mathrm{SO}(n,\mathbb{R}) & = & \{A \in \mathrm{O}(n) \mid \det(A) = 1\}; \\ \mathrm{U}(n) & = & \mathrm{O}(n,\mathbb{C}) & = & \{A \in \mathrm{M}(n,n,\mathbb{C}) \mid A^*A = I_n\}; \\ \mathrm{SU}(n) & = & \mathrm{SO}(n,\mathbb{C}) & = & \{A \in \mathrm{U}(n) \mid \det(A) = 1\}; \\ \mathrm{Sp}(n) & = & \mathrm{O}(n,\mathbb{H}). & & \end{array}$$

The action of these groups on the unit spheres yield locally trivial fiber bundles and hence by Theorem 9.36 fibrations for $d = \dim_{\mathbb{R}}(\mathbb{F})$

$$\begin{array}{lllll} \mathrm{O}(n,\mathbb{F}) & \xrightarrow{i} & \mathrm{O}(n+1,\mathbb{F}) & \to & S^{d(n+1)-1}; \\ \mathrm{SO}(n,\mathbb{F}) & \xrightarrow{j} & \mathrm{SO}(n+1,\mathbb{F}) & \to & S^{d(n+1)-1}. \end{array}$$

The inclusions $i$ and $j$ come from $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

By passing to colimits we get the topological groups

$$\begin{array}{lll} \mathrm{O}(\infty;\mathbb{F}) & = & \operatorname{colim}_{n\to\infty} \mathrm{O}(n,\mathbb{F}); \\ \mathrm{SO}(\infty;\mathbb{F}) & = & \operatorname{colim}_{n\to\infty} \mathrm{SO}(n,\mathbb{F}). \end{array}$$

Since $S^{d(n+1)-1}$ is $(d(n+1)-2)$-connected by Theorem 3.4, we conclude from Theorem 2.20 and Theorem 10.1

**Proposition 10.10.**

*(i) For $1 \leq n < m$ the inclusions $\mathrm{O}(n,\mathbb{F}) \to \mathrm{O}(m,\mathbb{F})$ and $\mathrm{SO}(n,\mathbb{F}) \to \mathrm{SO}(m,\mathbb{F})$ are $(d(n+1)-2)$-connected;*

*(ii) For $1 \leq n$ the inclusions $\mathrm{O}(n,\mathbb{F}) \to \mathrm{O}(\infty,\mathbb{F})$ and $\mathrm{SO}(n,\mathbb{F}) \to \mathrm{SO}(\infty,\mathbb{F})$ are $(d(n+1)-2)$-connected.*

The associated *Stiefel manifold* of orthogonal $k$-frames in $\mathbb{F}^n$ are defined by

$$\begin{array}{lllll} V_k(\mathbb{R}^n) & = & \mathrm{O}(n,\mathbb{R})/\mathrm{O}(n-k,\mathbb{R}) & \cong & \mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n-k,\mathbb{R}); \\ V_k(\mathbb{C}^n) & = & \mathrm{U}(n,\mathbb{R})/\mathrm{U}(n-k,\mathbb{C}) & \cong & \mathrm{SU}(n,\mathbb{R})/\mathrm{SU}(n-k,\mathbb{C}); \\ V_k(\mathbb{H}^n) & = & \mathrm{Sp}(n)/\mathrm{Sp}(n-k). & & \end{array}$$

We have the fibration $\mathrm{O}(n-k,\mathbb{R}) \to \mathrm{O}(n,\mathbb{R}) \to V_k(\mathbb{R}^n)$ and analogous fibrations for $V_k(\mathbb{C}^n)$ and $V_k(\mathbb{H}^n)$. The next proposition is a direct consequence of Theorem 10.1 and Proposition 10.10.

**Proposition 10.11.** *The space $V_k(\mathbb{F}^n)$ is $(d(n-k+1)-2)$-connected.*

There is a fibration $V_k(\mathbb{F}^n) \to V_{k+1}(\mathbb{F}^{n+1}) \xrightarrow{p} V_1(\mathbb{F}^{n+1})$, where $p$ sends a frame $\{v_1, v_2, \dots, v_{k+1}\}$ to the frame $\{v_{k+1}\}$. The next proposition follows from Theorem 10.1 and Proposition 10.11.

**Proposition 10.12.** *The inclusion $V_k(\mathbb{F}^n) \to V_{k+1}(\mathbb{F}^{n+1})$ is $(d(n+1)-2)$-connected.*

**Proposition 10.13.** *We have*

$$\begin{aligned} \pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) &\cong \mathbb{Z}; \\ \pi_{4(n-k)+3}(V_k(\mathbb{H}^n)) &\cong \mathbb{Z}; \\ \pi_{n-k}(V_k(\mathbb{R}^n)) &\cong \begin{cases} \mathbb{Z} & \text{if } k = 1 \text{ or } (n-k) \text{ even;} \\ \mathbb{Z}/2 & \text{if } k \geq 2 \text{ and } (n-k) \text{ odd.} \end{cases} \end{aligned}$$

*Proof.* The cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{H}$ follows by induction using $V_1(\mathbb{F}^n) \cong S^{dn-1}$ and $\pi_n(S^n) \cong \mathbb{Z}$ and Proposition 10.12. The case $\mathbb{F} = \mathbb{R}$ needs more than we have accumulated so far and can be found in [24, Proposition 6.8.5 on page 148]. □

**Example 10.14.** There are homeomorphisms $\mathrm{SO}(2) \xrightarrow{\cong} S^1$ and $\mathrm{SO}(3) \xrightarrow{\cong} \mathbb{RP}^3$. Since the universal covering of $S^1$ is the principal $\mathbb{Z}$-bundle $\mathbb{R} \to S^1$ and the universal covering of $\mathbb{RP}^3$ is the principal $\mathbb{Z}/2$-bundle $S^3 \to \mathbb{RP}^3$, we conclude from Proposition 10.10

$$\pi_1(\mathrm{SO}(n)) \cong \begin{cases} \mathbb{Z} & n = 2; \\ \mathbb{Z}/2 & n \geq 3, \end{cases}$$

and $\pi_1(\mathrm{SO}) \cong \mathbb{Z}/2$.

We conclude $\pi_2(\mathrm{SO}(3)) \cong \pi_2(\mathbb{RP}^3) \cong \pi_2(S^3) \cong \{0\}$ from Theorem 2.7 and Theorem 3.4. Proposition 10.10 (ii) implies $\pi_2(\mathrm{SO}) = \{0\}$. Actually for every compact Lie group $G$ we have $\pi_2(G, g) = \{0\}$ for any base point $g \in G$, see [4, Proposition 7.5 on page 225].

## 11. The Excision Theorem of Blakers-Massey

11.1. **The statement of the Excision Theorem of Blakers-Massey.** One basic feature of a homology theory is excision. Consider any (generalized) homology theory $\mathcal{H}_*$ with values in $R$-modules for a commutative ring $R$. Consider a $CW$-complex $Y$ with $CW$-subcomplexes $Y_0$, $Y_1$, and $Y_0$ satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$, or a topological space $Y$ with open subspaces $Y_0$, $Y_1$, and $Y_0$ satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$. Then the map induced by the inclusion $(Y_2, Y_0) \to (Y, Y_1)$ induces for all $n \in \mathbb{Z}$ an $R$-isomorphism

$$\mathcal{H}_n(Y_2, Y_0) \xrightarrow{\cong} \mathcal{H}_n(Y, Y_1).$$

This yields a a long exact Mayer-Vietoris sequence of $R$-modules, infinite to both sides,

$$\begin{aligned} \cdots \xrightarrow{\mathcal{H}_{n+1}(j_1)-\mathcal{H}_{n+1}(j_2)} \mathcal{H}_{n+1}(X) \xrightarrow{\partial_{n+1}} \mathcal{H}_n(X_0) \xrightarrow{\mathcal{H}_n(i_1)\oplus\mathcal{H}_n(i_2)} \mathcal{H}_n(X_1) \oplus \mathcal{H}_n(X_2) \\ \xrightarrow{\mathcal{H}_n(j_1)-\mathcal{H}_n(j_2)} \mathcal{H}_n(X) \xrightarrow{\partial_n} \mathcal{H}_{n-1}(X_0) \xrightarrow{\mathcal{H}_{n-1}(i_1)\oplus\mathcal{H}_{n-1}(i_2)} \cdots \end{aligned}$$

where $i_k \colon X_0 \to X_k$ and $j_k \colon X_k \to X$ for $k = 1, 2$ are the inclusions. The corresponding statement is not true for homotopy groups as the following example shows.

**Example 11.1.** Consider the $CW$-complex $Y = S^1 \vee S^n$ with the $CW$-subcomplexes $Y_1 = S^1$, $Y_2 = S^n$, and $Y_0 = \{\bullet\}$ for $n \in \mathbb{Z}^{\geq 2}$. Suppose that $\pi_n(Y_2, X_0) \xrightarrow{\cong} \pi_n(Y, Y_1)$ is an isomorphism. Since $\pi_n(S^n, \{\bullet\})$ is isomorphic to $\pi_n(S^n)$ by the long exact homotopy sequence of the pair $(S^n, \{\bullet\})$, we conclude from Theorem 3.4

that $\pi_n(S^1 \vee S^n, S^1)$ is infinite cyclic. Theorem 3.4 and the exact sequence of abelian groups $\pi_n(S^1) \to \pi_n(S^1 \vee S^n) \to \pi_n(S^1 \vee S^n, S^1)$ coming from the long exact homotopy sequence of the pair $(S^1 \vee S^n, S^1)$, see Theorem 2.11 imply that $\pi_n(S^1 \vee S^n)$ is a subgroup of an infinite cyclic group and hence a finitely generated abelian group. This contradicts Example 3.6. Hence $\pi_n(S^n, \{\bullet\}) \to \pi_n(S^1 \vee S^n, S^1)$ is not bijective.

One of the main results of this course is the next theorem due to Blakers and Massey which shows excision in a very special case for homotopy groups.

**Theorem 11.2** (The Excision Theorem of Blakers-Massey). *Consider $p, q \in \mathbb{Z}^{\geq 1}$. Let $Y$ be a topological space with open subspaces $Y_0$, $Y_1$, and $Y_0$ satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$. Suppose that for any base point $y_0 \in Y_0$ we have*

$$\begin{array}{rcll} \pi_i(Y_1, Y_0, y_0) & = & \{0\} & \text{for } 0 < i < p; \\ \pi_i(Y_2, Y_0, y_0) & = & \{0\} & \text{for } 0 < i < q. \end{array}$$

*Then, for every base point $y_0 \in Y_0$, the map induced by the inclusion $i \colon (Y_2, Y_0) \to (Y_1, Y_0)$*

$$\pi_n(i, y_0) \colon \pi_n(Y_2, Y_0, y_0) \to \pi_n(Y, Y_1, y_0)$$

*is surjective for $1 \leq n = p + q - 2$ and bijective for $1 \leq n \leq p + q - 3$.*

If $p = 1$, then there is no condition on $(Y_1, Y_0)$ in Theorem 11.2. Note that in Theorem 11.2 only the case $n \geq 1$ is treated, we will say something for $n = 0$ in Subsection 11.3.

## 11.2. The proof of the Excision Theorem of Blakers-Massey.

The following rather elementary proof of the Excision Theorem 11.2 of Blakers-Massey is due to Dieter Puppe. The proof needs some preparation.

We begin with introducing some notation.

**Notation 11.3** (Cubes and faces in $\mathbb{R}^n$). A cube in $\mathbb{R}^n$ for $n \geq 1$ is a subset of the form

$$\begin{aligned} W = W(a, \delta, L) := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \\ a_i \leq x_i \leq a_i + \delta \text{ for } i \in L, a_i = x_i \text{ for } i \notin L\} \end{aligned}$$

for $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $\delta > 0$, and a (possibly empty) subset $L \subseteq \{1, 2, \ldots, n\}$. The dimension $\dim(W)$ of $W$ is defined to be $|L|$.

A face of $W'$ of $W$ is a subset of $W$ of the form

$$W' = \{x \in W \mid x_i = a_i \text{ for } i \in L_0, x_j = a_j + \delta \text{ for } j \in L_1\}$$

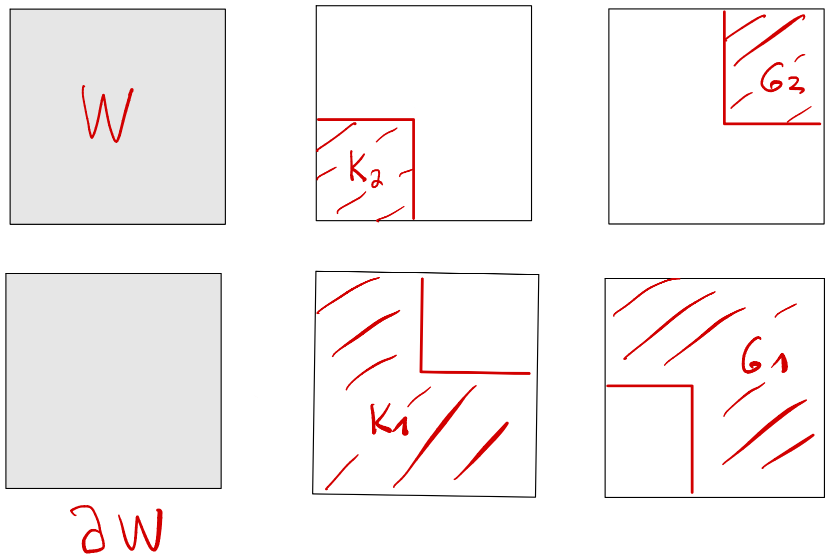
for some (possibly empty) subsets $L_0 \subseteq L$ and $L_1 \subseteq L$. (The subset $W'$ may be empty or equal to $W$.)

Let $\partial W$ be the union of all faces $W'$ of $W$ which are not equal to $W$.

For $1 \leq p \leq n$ we define the following subsets of $W$:

$$\begin{array}{rcl} K_p(W) & = & \{w \in W \mid x_i < a_i + \delta/2 \text{ for at least } p \text{ values } i \in L\}; \\ G_p(W) & = & \{w \in W \mid x_i > a_i + \delta/2 \text{ for at least } p \text{ values } i \in L\}. \end{array}$$

For $p > \dim(W)$ we define $K_p(W)$ and $G_p(W)$ to be the empty sets. Note that $K_p(W)$ and $G_p(W)$ become smaller and smaller as $p$ becomes bigger and $G_p(W) \cap K_q(W) = \emptyset$ if $p + q > \dim(W)$ hold.




Next we prove a technical lemma which will enter in the important Proposition 11.5. It essentially says that a map $W \to Y$, which satisfies a certain condition on the boundary $\partial W$, can be changed up to homotopy relative $\partial W$ such that the resulting map satisfies the analog of this condition on $W$ and not only on $\partial W$.

**Lemma 11.4.** *Consider a pair $(Y, A)$, a cube $W \subseteq \mathbb{R}^n$, and a map $f \colon W \to Y$. Suppose that for $p \le \dim(W)$ we have $f^{-1}(A) \cap W' \subseteq K_p(W')$ for all faces $W' \subseteq \partial W$.*

*Then there exists a map $g \colon W \to Y$ with the following properties:*

*(i) $g$ is homotopic relative $\partial W$ to $f$;*
*(ii) We have $g^{-1}(A) \subseteq K_p(W)$.*

*The same conclusion holds if we replace $K_p(W)$ by $G_p(W)$ in assertion (ii).*

*Proof.* Obviously we can assume without loss of generality that $W$ is the standard cube $I^n = \prod_{i=1}^n [0,1] = W((0,0,\dots,0),1,\{1,2,\dots,n\})$. Let $I_2^n$ be the subcube of $I^n$ given by $\prod_{i=1}^n [0,1/2] = W((0,0,\dots,0),1/2,\{1,2,\dots,n\})$. Put $x_4 = (1/4,1/4,\dots,1/4)$.

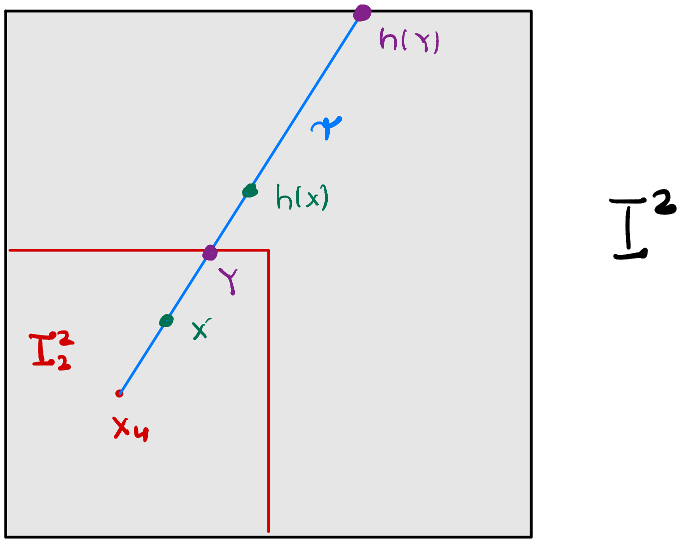
Define a map

$$h \colon I^n \to I^n$$

by expanding $I_2^n$ to $I^n$ by radial projection with center $x_4$. Here is the precise definition of $h$. Let $x \in I^n$ be any point. If $x = x_4$, we define $h(x) = x_4$. Suppose that $x \neq x_4$. Consider the ray

$$r \colon \mathbb{R}^{\ge 0} \to \mathbb{R}^n, \quad t \mapsto x_4 + t \cdot (x - x_4)$$

starting at $x_4$ through $x$. Let $P(x)$ be its intersection point with $\partial I_2^n$ and $Q(x)$ be its intersection point with $\partial I^n$. If $x$ lies in the segment $[P(x), Q(x)]$ of the ray $r$, it is sent to $Q(x)$. Suppose that $x$ lies on the segment $[x_4, P(x)]$. If we write $x = x_4 + t\cdot(P(x) - x_4)$ for some $t \in [0,1]$ then $h(x)$ is defined to be $x_4 + t\cdot(Q(x) - x_4)$. In other words, $h$ sends the segment $[x_4, P(x)]$ affinely to the segment $[x_4, Q(x)]$ and the segment $[P(x), Q(x)]$ to the point $Q(x)$. Obviously $h$ is homotopic relative $\partial W$ to $\mathrm{id}_W$.

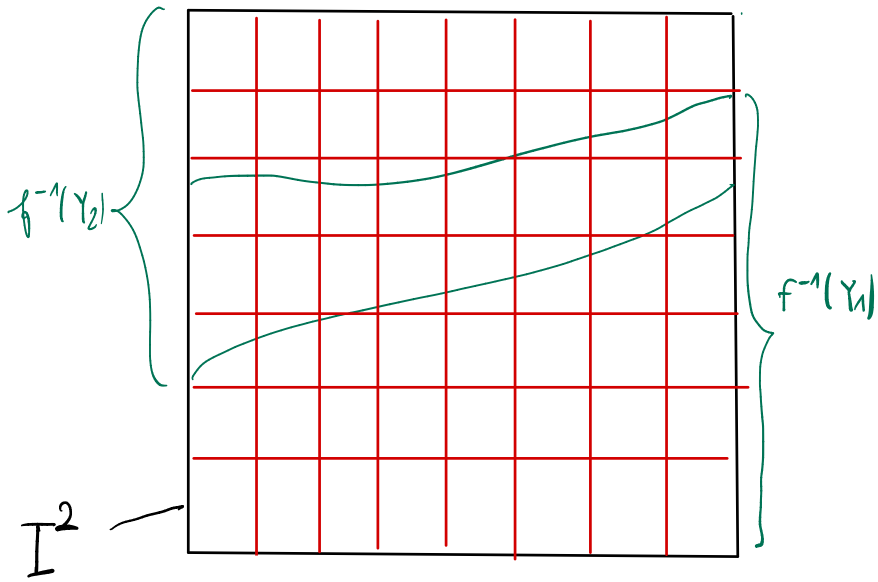



Now we set $g = f \circ h$. Then $g$ is homotopic relative $\partial W$ to $f$. It remains to show $g^{-1}(A) \subseteq K_p(W)$.

Consider $z \in I^n$ with $g(z) \in A$. If $z_i < 1/2$ holds for $i = 1, 2, \ldots, n$, then $z \in K_n(I^n) \subseteq K_p(I^n)$. So it suffices to treat the case, where $z_i \geq 1/2$ holds for at least one $i \in \{1, 2, \ldots, n\}$. Then $h(z) \in \partial I^n$ holds by definition. Choose a face $W' \subseteq \partial I^n$ with $h(z) \in W'$. Since $h(z) \in f^{-1}(A)$ holds, we get $h(z) \in W' \cap f^{-1}(A)$ and hence $h(z) \in K_p(W')$. Hence we have $h(z)_i < 1/2$ for at least $p$ many elements $i \in \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots\}$ with $h(z)_i < 1/2$ we get $h(z)_i = 1/4 + t \cdot (z_i - 1/4)$ with $t \geq 1$ and hence $z_i < 1/2$. This shows $z \in K_p(W)$. This finishes the proof of Lemma 11.4 for $K_p(W)$, the version for $G_p(W)$ is proven analogously. □

The next proposition contains the main technical result needed for the proof of Theorem 11.2.

For the remainder of this subsection let $Y$ be a topological space $Y$ with open subspaces $Y_0$, $Y_1$, and $Y_0$ satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$ and we consider $p, q \in \mathbb{Z}^{\geq 0}$.




**Proposition 11.5.** *Suppose that $(Y_1, Y_0)$ is $p$-connected and $(Y_2, Y_0)$ is $q$-connected. Let $f\colon I^n \to Y$ be a map. Let $\mathcal{W} = \{W\}$ be a subdivision of $I^n$ into cubes $W$ such that either $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$ holds. (It exists as $I^n$ is compact.)*

*Then there exists a homotopy $h\colon I^n \times I \to Y$ with $h_0 = f$ satisfying for every $W \in \mathcal{W}$:*

- *(i) If $f(W) \subseteq Y_j$ holds, then we have $h_t(W) \subseteq Y_j$ for every $t \in I$, where $j \in \{0, 1, 2\}$;*
- *(ii) If $f(W) \subseteq Y_0$ holds, then we have $h_t|_W = f|_W$ for every $t \in I$;*
- *(iii) If $f(W) \subseteq Y_1$ holds, then we have $h_1^{-1}(Y_1 \setminus Y_0) \cap W \subseteq K_{p+1}(W)$;*
- *(iv) If $f(W) \subseteq Y_2$ holds, then we have $h_1^{-1}(Y_2 \setminus Y_0) \cap W \subseteq G_{q+1}(W)$.*

*Proof.* We enlarge the collection of cubes $\mathcal{W}$ such that for every $W \in \mathcal{W}$ all of its faces belong to $\mathcal{W}$. Let $C_k \subseteq I^n$ be the union of all cubes $W \in \mathcal{W}$ with $\dim(W) \leq k$. We construct for $k = 0, 1, 2 \ldots, n$ a homotopy $h[k]\colon C_k \times I \to Y$ satisfying for each cube $W \in \mathcal{W}$ of dimension $\leq k$ the conditions (i), (ii), (iii), and (iv) such that $h[k]|_{C_{k-1} \times I} = h[k-1]$ holds for $k = 1, 2, \ldots, n$. Then the desired homotopy is $h = h[n]$.

Note in the sequel that for a cube $W \in \mathcal{W}$, for which we have $f(W) \subseteq Y_0$ and condition (ii) holds, conditions (iii) and (iv) are automatically satisfied, since then $h_1^{-1}(Y_1 \setminus Y_0)$ and $h_1^{-1}(Y_2 \setminus Y_0)$ are empty. Moreover, if a cube $W \in \mathcal{W}$ satisfies both $f(W) \subseteq Y_1$ and $f(W) \subseteq Y_2$, then we have $f(W) \subseteq Y_0$, and for each cube $W \in \mathcal{W}$ we have $f(W) \subseteq Y_1$ or $f(W) \subseteq Y_2$. So every cube $W \in \mathcal{W}$ satisfies precisely one of the following conditions:

- $f(W) \subseteq Y_0$;
- $f(W) \subseteq Y_2$ and $f(W) \not\subseteq Y_1$;
- $f(W) \subseteq Y_1$ and $f(W) \not\subseteq Y_2$.

We begin with $k = 0$. Consider a cube $W$ in $\mathcal{W}$ of dimension 0. If $W_0 \subseteq Y_0$, define $h[0]_t(W_0) = W_0$ for $t \in I$. This is forced upon us by condition (ii). Suppose $f(W) \subseteq Y_1$ and $f(W) \subsetneq Y_2$ hold. As $(Y_1, Y_0)$ is 0-connected, we can choose a path $w\colon I \to Y_1$ from $f(W)$ to a point $y \in Y_0$. We define $h[0](W, t) = w(t)$ for $t \in I$. Then conditions (i) and (iii) are satisfied for trivial reasons. Analogously one defines $h[0]$ in the case, where $f(W) \subseteq Y_2$ and $f(W) \subsetneq Y_0$ hold. This finishes the construction of $h[0]$. One easily checks that all the conditions (i), (ii), (iii), and (iv) are satisfied for every 0-dimensional cube $W$ by $h[0]$.

Next we deal with the induction step from $(k-1)$ to $k$. Consider a cube of dimension $k$. Then $\partial W = W \cap C_{k-1}$. Since $\partial W \to W$ is a cofibration, we can extend $h[k-1]_{\partial W \times I}$ to $W \times I$ such that conditions (i) and (ii) are satisfied. So we get a homotopy $h[k]' \colon C_k \times I \to Y$ such that conditions (i) and (ii) hold for $h[k]'$ and any $W \in \mathcal{W}$ with $\dim(W) \le k$ and the restriction of $h[k]'$ to $C_{k-1} \times I$ satisfies conditions (i), (ii), (iii), and (iv) for any $W \in \mathcal{W}$ with $\dim(W) = (k-1)$.

The homotopy $h[k]'$ is not yet the desired homotopy $h[k]$. It remains to explain why we can change $h[k]'$ further such that all the conditions (i), (ii), (iii), and (iv) are satisfied for each cube $W$ with $\dim(k) \le k$. For this purpose we consider the map $h[k]'_1 \colon C_k \to Y$ and construct an appropriate homotopy $h[k]'' \colon C_k \times I \to Y$ with $h[k]''_0 = h[k]'_1$ and will get the desired homotopy $h[k] \colon C_k \times I \to Y$ by $h[k]' * h[k]''$.

Consider a cube $W$. We explain how to define $h[k]''|_{W \times [0,1]}$ with $\dim(W) = k$. If $h[k]'_1(W) \subseteq Y_0$, then we define $h[k]''_t|_W = h[k]'_1|_W$ for $t \in [0,1]$. Suppose that $h[k]'_1(W) \subseteq Y_1$ and $h[k]'_1(W) \subsetneq Y_2$ holds. If $\dim(W) \le p$, there exists a homotopy $l$ relative $\partial W$ with $l_0 = h'[k]_1$ and $l_1(W) \subseteq Y_0$, since the pair $(Y_1, Y_0)$ is $p$-connected. Define $h[k]''|_{W \times I}$ by $l$. If $\dim(W) > p$, we use Lemma 11.4 with $f = h[k]'|_W$ to define $h[k]''|_{W \times I}$. We treat the case $h[k]'_1(W) \subseteq Y_2$ and $h[k]'_1(W) \subsetneq Y_1$ analogously. This finishes the construction of $h[k]''$ and hence of the desired homotopy $h[k]$. Note that $h[k]''$ is stationary on $C_{k-1}$. One easily checks that $h(k)$ satisfies conditions (i), (ii), (iii), and (iv) for any $W \in \mathcal{W}$ with $\dim(W) \le k$. Hence the proof of Proposition 11.5 is finished. □

Denote by $F(Y_1, Y, Y_2)$ the subspace of $\operatorname{map}(I, Y)$ given by

$$F(Y_1, Y, Y_2) := \{w \colon I \to Y \mid w(0) \in Y_1, w(1) \in Y_2\}.$$

So we are looking at paths in $Y$ starting somewhere in $Y_1$ and ending somewhere in $Y_2$. Define $F(Y_1, Y_1, Y_0)$ to be the subspace of $\operatorname{map}(I, Y_1)$ given by

$$F(Y_1, Y_1, Y_0) := \{w \colon I \to Y_1 \mid w(1) \in Y_0\}.$$

So here we are looking at paths in $Y_1$ ending somewhere in $Y_0$. Since we can think of $\operatorname{map}(I, Y_1)$ as a subspace of $\operatorname{map}(I, Y)$, we can also think of $F(Y_1, Y_1, Y_0)$ as a subspace of $F(Y_1, Y, Y_2)$.

**Proposition 11.6.** *Suppose that $(Y_1, Y_0)$ is $p$-connected and $(Y_2, Y_0)$ is $q$-connected. Then the inclusion*

$$F(Y_1, Y_1, Y_0) \to F(Y_1, Y, Y_2)$$

*is $(p+q-1)$-connected.*

*Proof.* Consider a map of pairs

$$\varphi \colon (I^n, \partial I^n) \to (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

for any $n \le (p+q-1)$. We have to find a homotopy $h$ with $h_0 = \varphi$ such that the image of $h_1$ is contained in $F(Y_1, Y_1, Y_0)$.

By the adjunction $\operatorname{map}(I^n \times I, Z) \xrightarrow{\cong} \operatorname{map}(I^n, \operatorname{map}(I; Z))$ the map $\varphi$ is the same as a map $\Phi \colon I^n \times I \to Y$ satisfying:

(i) $\Phi(x, 0) \in Y_1$ for $x \in I^n$;
(ii) $\Phi(x, 1) \in Y_2$ for $x \in I^n$;
(iii) $\Phi(x, 1) \in Y_0$ for $x \in \partial I^n$, $t \in I$.

In the sequel we call a map $\Phi \colon I^n \times I \to Y$ satisfying the three conditions above admissible. We have to show that any such admissible map $\Phi$ can be homotoped through admissible maps to an admissible map $\Phi' \colon I^n \times I \to Y$ with the property $\Phi'(I^n \times I) \subseteq Y_1$.

Starting with an admissible map $\Phi\colon I^{n+1} = I^n \times I \to Y$, we apply Proposition 11.5 and obtain a new admissible map $\Psi$. One easily checks that the homotopy coming from Proposition 11.5 is a homotopy through admissible maps.

Consider the projection $\mathrm{pr}\colon I^n \times I \to I^n$. Next we show that the images of $\Psi^{-1}(Y \setminus Y_1)$ and $\Psi^{-1}(Y \setminus Y_2)$ under pr are disjoint. Suppose the contrary. So there are $y \in I^n$, $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ and $z_2 \in \Psi^{-1}(Y \setminus Y_1)$ with $\mathrm{pr}(z_1) = y = \mathrm{pr}(z_2)$. Choose a cube $W \subseteq I^{n+1}$ with $z_1 \in W$. Since $z_1 \in \Psi^{-1}(Y \setminus Y_1)$ holds, we conclude $z_1 \in K_{p+1}(W)$ from condition (iii) appearing in Proposition 11.5. This implies that $y \in K_p(I^n)$ holds. Analogously one shows $y \in G_q(I^n)$, now using condition (iv) appearing in Proposition 11.5. This is a contradiction since $K_p(I^n) \cap G_q(I^n)$ is empty if $n < p + q$ holds.

The intersection of $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\partial I^n$ is empty since $\Psi$ is admissible and hence $\Psi(\partial I^n) \times I \subseteq Y_1$ holds. Hence the closed subsets $\mathrm{pr}(\Psi^{-1}(Y \setminus Y_1))$ and $\partial I^n \cup \mathrm{pr}(\Psi^{-1}(Y \setminus Y_2))$ of $I^n$ are disjoint. Choose a continuous function $\tau\colon I^n \to I$ which assumes the value 0 on $\Psi^{-1}(Y \setminus Y_1)$ and the value 1 on $\partial I^n \cup \mathrm{pr}(\Psi^{-1}(Y \setminus Y_2))$. Then we obtain a homotopy through admissible maps

$$h\colon (I^n \times I) \times I \to Y, \quad ((x,t),s) \mapsto \Psi(x,(1-s)t + st\tau(x))$$

such that $h_0 = \Psi$ and $h_1(I^n \times I) \subseteq Y_1$ holds. This finishes the proof of Proposition 11.6. □

Now we are ready to give the proof of the Excision Theorem 11.2.

*Proof of Theorem 11.2.* We have the path fibration $\mathrm{map}(I,Y) \to Y$ sending $w$ to $w(0)$, see Proposition 9.8 (i). The induced map $p\colon F(Y_1,Y,Y_2) \to Y_1$ sending $w$ to $w(0)$ is a fibration by Proposition 9.7.The fiber over a point $y_1 \in Y_1$ is $F(\{y_0\},Y,Y_2)$. We obtain a commutative diagram of fibrations

$$\begin{array}{ccc}
F(\{y_0\},Y_1,Y_0) & \xrightarrow{i} & F(\{y_0\},Y,Y_2) \\
\downarrow & & \downarrow \\
F(Y_1,Y_1,Y_0) & \xrightarrow{j} & F(Y_1,Y,Y_2) \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\mathrm{id}} & Y_1
\end{array}$$

where $i$ and $j$ are the inclusions. We have already shown that $j$ is $(p+q-1)$-connected by Proposition 11.6. Using the long exact homotopy sequences of the two fibrations above and a kind of Five-Lemma argument shows that $i$ is also $(p+q-1)$-connected. There is a commutative diagram for $n \geq 1$

$$\begin{array}{ccc}
\pi_{n-1}(F(\{\bullet\},Y_1,Y_0),*) & \xrightarrow{i} & \pi_{n-1}(F(\{\bullet\},Y,Y_2),*) \\
\downarrow \cong & & \cong \downarrow \\
\pi_n(Y_1,Y_0,y_0) & \longrightarrow & \pi_n(Y,Y_2,y_0)
\end{array}$$

whose lower horizontal arrow is induced by the inclusion and vertical arrows are bijections by a version of (9.34) for pairs. Hence the lower vertical arrow is surjective for $1 \leq n = p+q-2$ and bijective for $1 \leq n \leq p+q-3$. This finishes the proof of Theorem 11.2. □

11.3. **The Excision Theorem for** $n = 0$**.** Note that in Theorem 11.2 only $n \geq 1$ is considered. We also want to treat the case $n = 0$.

**Proposition 11.7.** *Let $Y$ be a topological space $Y$ with open subspaces $Y_0$, $Y_1$, and $Y_0$ satisfying $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$ and $Y_0 \neq \emptyset$.*

*Then the canonical map $\iota\colon \pi_0(Y_2, Y_0) \to \pi_0(Y, Y_1)$ is bijective.*

*Proof.* Since very element in $Y$ belongs to $Y_1$ or $Y_2$, the map $\iota$ is obviously surjective. Injectivity is proved as follows.

Consider elements $C_1, C_2 \in \pi_0(Y_2)$. Let $\overline{C_1}$ and $\overline{C_2}$ be the classes represented by them in $\pi_0(Y_2, Y_0)$. Suppose that they have the same image under $\iota$. Then we have to show $\overline{C_1} = \overline{C_2}$.

We first treat the case, where $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is different from the base point in $\pi_0(Y, Y_1)$. Then the images of $C_1$ and $C_2$ under the map $\pi_0(Y_2) \to \pi_0(Y)$ agree. Hence we can find a path $w\colon I \to Y$ with $w(0) \in C_1$ and $w(1) \in C_2$. Since $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is different from the base point in $\pi_0(Y, Y_1)$, this path cannot meet $w^{-1}(Y_1)$. Hence it is a path $w\colon I \to Y_2$. This implies $C_1 = C_2 \in \pi_0(Y_2)$ and hence $\overline{C_1} = \overline{C_2}$.

Next we treat the case, where $\iota(\overline{C_1}) = \iota(\overline{C_2})$ is the base point in $\pi_0(Y, Y_1)$. It suffices to show that then $\overline{C_1}$ is the base point $*$ in $\pi_0(Y_2, Y_0)$. As $\iota(\overline{C_1})$ is the base point in $\pi_0(Y, Y_1)$, there is a path component $D$ in $\pi_0(Y_1)$ such that the image of $C_1$ under $\pi_0(Y_2) \to \pi_0(Y)$ and the image of $D$ under $\pi_0(Y_1) \to \pi_0(Y)$ agree. Hence we can find a path $w\colon I \to Y$ with $w(0) \in C_1$ and $w(1) \in Y_1$. If $w(0) \in Y_0$ holds, $\overline{C_1}$ is obviously the base point in $\pi_0(Y_2, Y_0)$. Hence we can assume without loss of generality that $y_1 \notin Y_0$ holds. Since $y_1 \notin Y_0$ and $y_1 \in Y_1$ hold, we have $y_1 \notin Y_2$. If $w^{-1}(Y_2)$ is empty, $w$ is a path in $Y_0$ and hence $\overline{C_1}$ is the base point in $\pi_0(Y_2, Y_0)$. Hence we can assume without loss of generality that $w^{-1}(Y_2)$ is not empty and $y_1 \notin Y_2$ holds.

Let $t_0$ be the infimum of $w^{-1}(Y_2) \subseteq I$. As $w^{-1}(Y_2)$ is open, we have $0 \leq t_0 < 1$. Since $0 \notin w^{-1}(Y_2)$ holds, we get $t_0 \notin w^{-1}(Y_2)$. Hence $[0, t_0] \subseteq W_1$ holds. There exists $t_1 \in I$ with $t_0 < t_1$ such that $[0, t_1] \in w^{-1}(Y_1)$ holds. Now choose $t_2 \in [0, 1]$ satisfying $t_0 < t_2 < t_1$ and $t_2 \in w^{-1}(Y_2)$. Note that then $t_2 \in w^{-1}(Y_1) \cap w^{-1}(Y_2) = w^{-1}(Y_0)$ holds. Consider the path $v\colon I \to Y$ sending $s$ to $w(st_2)$. Then $v$ is a path in $Y_2$ from $v(0) = x_2$ to $v(1) \in Y_0$. This implies that $\overline{C_1}$ is the base point in $\pi_0(Y_2, Y_0)$. This finishes the proof of Proposition 11.7. □

Another shorter proof of Proposition 11.7 comes from the following observation. The map on singular homology $\mu\colon H_0(Y_2, Y_0; \mathbb{Z}) \xrightarrow{\cong} H_0(Y, Y_1; \mathbb{Z})$ induced by the inclusion is an isomorphism by excision. The abelian group $H_0(Y_2, Y_0; \mathbb{Z})$ is a free $\mathbb{Z}$-modules with basis $B_1$ which is the complement of the image of $\pi_0(Y_0) \to \pi_0(Y_2)$ in $\pi_0(Y_2)$. The abelian group $H_0(Y, Y_1; \mathbb{Z})$ is a free $\mathbb{Z}$-modules with basis $B_2$ which is the complement of the image of $\pi_0(Y_1) \to \pi_0(Y)$ in $\pi_0(Y)$. The map $\mu$ sends an element in $\pi_0(Y_2) \setminus \pi_0(Y_0)$ to an element in $\pi_0(Y) \setminus \pi_0(Y_1)$ or to 0.

### 11.4. **Some applications of the Excision Theorem of Blakers-Massey.**

**Proposition 11.8.** *Let $Y$ be a topological space $Y$ with subspaces $Y_0$, $Y_1$, and $Y_2$ satisfying $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$, and $Y_0 \neq \emptyset$. Consider $m, n \in \mathbb{Z}^{\geq 0}$.*

- *(i) Suppose that $(Y_2, Y_0)$ is $n$-connected. Then $(Y, Y_1)$ is $n$-connected;*
- *(ii) Suppose that $(Y_1, Y_0)$ is $m$-connected and $(Y_2, Y_0)$ is $n$-connected. Then:*
  - *(a) The map $\pi_0(Y_2, Y_0) \to \pi_0(Y, Y_1)$ is bijective;*
  - *(b) For every base point $y_0$ the following holds: The map $\pi_i(Y_2, Y_0, y_0) \to \pi_i(Y, Y_1, y_0)$ induced by the inclusion is bijective for $1 \leq i \leq m+n-1$ and surjective for $i = m + n$.*

*Proof.* We only give the proof for assertion (i), the one for assertion (ii) is analogous, if one takes Theorem 11.2 into account.

Recall that $(Y, Y_1)$ is $n$-connected if and only if $\pi_0(Y, Y_1)$ is trivial, i.e., consist of one element, and for every element $i \in \{1, 2, \dots, n\}$ and every base point $y_1 \in Y_1$ the set $\pi_i(Y, Y_1, y_1)$ is trivial. The analogous statement holds for $(Y_2, Y_0)$.

Proposition 11.7 implies that $\pi_0(Y, Y_1)$ is trivial.

Consider $i$ with $1 \le i$. We conclude from Theorem 11.2 that $\pi_i(Y, Y_1, y_0)$ is trivial for every base point $y_0 \in Y_0$. We need to check that $\pi_i(Y, Y_1, y_1)$ is trivial for every element $i \in \{1, 2, \dots, n\}$ and every base point $y_1 \in Y_1$. Since the map $\pi_0(Y_0) \to \pi_0(Y_1)$ is surjective by assumption, we can connected $y_1$ by a path in $Y_1$ to a point $y_0$ in $Y_0$. The obvious version of (2.6) for pointed pairs implies $\pi_i(Y, Y_1, y_1) \cong \pi_i(Y, Y_1, y_0)$ and hence $\pi_i(Y, Y_1, y_1)$ is trivial. □

**Proposition 11.9.** *Consider $m, n \in \mathbb{Z}^{\ge 0} \amalg \{\infty\}$ and a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \overline{i}} \\ X & \xrightarrow[\overline{f}]{} & Y \end{array}$$

*such that $i\colon A \to X$ is a cofibration.*

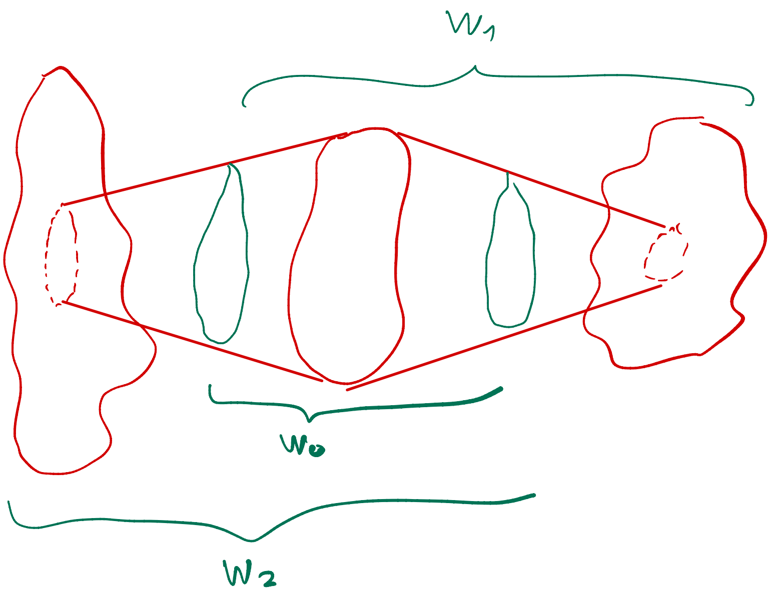
(i) *If $f$ is $n$-connected, then $\overline{f}$ is $n$-connected;*
(ii) *If $f$ is $n$-connected and $i$ is $m$-connected, then*
- *The map $\pi_0(\overline{f}, f)\colon \pi_0(X, A) \to \pi_0(Y, B)$ is bijective;*
- *For every $a \in A$ the map $\pi_k(\overline{f}, f, a)\colon \pi_k(X, A, a) \to \pi_k(Y, B, f(a))$ is bijective for $1 \le k \le m + n - 1$ and surjective for $1 \le k = m + n$,*

*where we use the convention that $m+n-1$ and $m+n$ mean $\infty$ if $m = \infty$ or $n = \infty$ holds.*

*Proof.* Because of Subsection 8.8 and Theorem 8.29 we can replace $f$ and $i$ by the inclusions into their mapping cylinders. Hence it suffices to consider in Proposition 11.9 the diagram of closed subspaces

$$\begin{array}{ccc} A & \longrightarrow & \operatorname{cyl}(f) \\ \downarrow & & \downarrow \\ \operatorname{cyl}(i) & \longrightarrow & \operatorname{cyl}(i) \cup_A \operatorname{cyl}(f). \end{array}$$

Now one easily constructs open subsets $W_0$, $W_1$, and $W_2$ of $\operatorname{cyl}(i) \cup_A \operatorname{cyl}(f)$ such that $A \subseteq W_0$, $\operatorname{cyl}(i) \subseteq W_1$, and $\operatorname{cyl}(f) \subseteq W_2$ hold, the corresponding inclusions are homotopy equivalences, and we have $\operatorname{cyl}(i) \cup_A \operatorname{cyl}(f) = W_1 \cup W_2$ and $W_0 = W_1 \cap W_2$.




Hence it suffices to show if we put $W = \operatorname{cyl}(i) \cup_A \operatorname{cyl}(f)$ and $(W_1, W_0)$ is $m$-connected and $(W_2, W_0)$ is $n$-connected for $m, n \in \mathbb{Z}^{\geq 0}$:

- The pair $(W, W_2)$ is $n$-connected;
- The map induced by the inclusion $\pi_0(W_1, W_0) \to \pi_0(W, W_2)$ is bijective;
- The map induced by the inclusion $\pi_k(W_1, W_0, a) \to \pi_k(W, W_2, a)$ is bijective for $1 \leq k \leq m + n - 1$ and surjective for $1 \leq ik = m + n$.

This has already been done in Proposition 11.8. □

We leave it to the reader to proof the following generalization of Proposition 11.9 (i).

**Proposition 11.10.** *Let the following two diagrams be pushouts*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y_0 & \xrightarrow{k_1} & Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ Y_2 & \xrightarrow[l_2]{} & Y \end{array}$$

*where the left vertical arrows $i_2$ and $k_2$ are cofibrations. Let $f_i \colon X_i \to Y_i$ be maps for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Denote by $f \colon X \to Y$ the map induced by $f_0$, $f_1$, and $f_2$ and the pushout property.*

*Consider $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Suppose that $f_1$ and $f_2$ are $n$-connected and $f_0$ is $(n-1)$-connected with the convention $\infty - 1 = \infty$.*

*Then $f$ is $n$-connected.*

**Proposition 11.11.** *Consider $m, n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$ and a cofibration $i \colon A \to X$. Suppose that $i$ is $m$-connected and $A$ is $n$-connected. Let $\operatorname{pr} \colon X \to X/A$ be the canonical projection. Consider any $a \in A$*

*Then the map*

$$\pi_k(\operatorname{pr}, a) \colon \pi_k(X, A, a) \to \pi_k(X/A, \{*\}, *) = \pi_k(X/A, *)$$

*is bijective for $0 \leq k \leq m + n$ and surjective for $k = m + n + 1$, where we use the convention that $m + n$ and $m + n + 1$ mean $\infty$ if $m = \infty$ or $n = \infty$ holds.*

*Proof.* Consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{j} & \operatorname{cone}(A) \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & X \cup_A \operatorname{cone}(A) \end{array}$$

for $j$ the inclusion. The map $j$ is $(n+1)$-connected as $A$ is $n$-connected. Proposition 11.9 implies that $\pi_k(X,A,a) \to \pi_k(X \cup_A \operatorname{cone}(A), \operatorname{cone}(A), *)$ induced by the inclusion is bijective, if $0 \le k \le m+m$ hold, and is surjective for $k = m+n+1$. The projection $\operatorname{pr}\colon X \cup_A \operatorname{cone}(A) \to X/A$ is a homotopy equivalence by Theorem 8.28 and hence induces an isomorphism $\pi_k(\operatorname{pr})\colon \pi_l(X \cup_A \operatorname{cone}(A), *) \xrightarrow{\cong} \pi_k(X/A, *)$ for every $k \ge 0$ by Lemma 8.25. □

The next two results are consequence of Proposition 11.9. Their rather elementary proof is left to the reader and can be found in [24, Theorem 6.10.5 on page 154 and Proposition 6.10.9 on page 156].

**Proposition 11.12.** *Let $X$ and $Y$ be well pointed spaces. Consider $m, n \in \mathbb{Z}^{\ge 1}$. Suppose that $X$ is $m$-connected and $Y$ is $n$-connected.*

- *(i) The inclusion $X \vee Y \to X \times Y$ induces an isomorphism $\pi_k(X \vee Y) \xrightarrow{\cong} \pi_k(X \times Y)$ for $0 \le k \le m+n$;*
- *(ii) $\pi_k(X \times Y, X \vee Y)$ and $\pi_k(X \wedge Y)$ are trivial for $0 \le k \le m+n+1$;*
- *(iii) The canonical map $\pi_k(X \vee Y) \to \pi_k(X) \times \pi_k(Y) \to$ is bijective for $0 \le k \le m+n$.*

Note that in Proposition 11.12 we assume that $m, n \ge 1$ holds which implies that $X$ and $Y$ are simply connected. This assumption is need as the Example 3.6 shows.

The join $X * Y$ of $X$ and $Y$ is defined by the pushout

(11.13)
$$\begin{array}{ccc} X \times Y & \longrightarrow & X \times \operatorname{cone}(Y) \\ \downarrow & & \downarrow \\ \operatorname{cone}(X) \times Y & \longrightarrow & X * Y. \end{array}$$

One can describe $X * Y$ also as the quotient space of $X \times I \times Y$ under the equivalence relation generated by $(x,0,y_0) \sim (x,0,y_1)$ and $(x_0,1,y) \sim (x_1,1,y)$ for $x, x_0, x_1 \in X$ and $y, y_0, y_1 \in Y$. Intuitively it says that each point in $X$ is connected to each point in $Y$ by a unit interval. One easily checks that $S^m * S^n$ is homeomorphic to $S^{m+n+1}$ and $S^0 * X$ is homeomorphic to $\Sigma X$.

The proof of the next proposition is left to the reader and can be found in [24, Proposition 6.10.9 on page 156].

**Proposition 11.14.** *Consider $m, n \in \mathbb{Z}^{\ge -1}$. Let $X$ and $Y$ be spaces such that $X$ is $m$-connected and $Y$ is $n$-connected, where $(-1)$-connected means that there is no condition.*

*Then their join is $(m+n+2)$-connected.*

11.5. **The Freudenthal Suspension Theorem.** Let $(X,x)$ be a pointed space. The pointed suspension is a functor and hence yields a map

$$[(S^n,s),(X,x)]^0 \to [(\Sigma(S^n,s),*),(\Sigma(X,x),*)]^0.$$

Using the standard identification $(\Sigma(S^n,s),*) = (S^{n+1},s)$ we obtain a group homomorphism called *suspension homomorphism* for $k \in \mathbb{Z}^{\ge 1}$

(11.15)
$$\sigma_k(X,x)\colon \pi_k(X,x) \to \pi_{k+1}(\Sigma(X,x),*)$$

Note that $\sigma_k$ is also defined for $k = 0$ but not a group homomorphism in this case.

**Theorem 11.16** (Freudenthal Suspension Theorem)**.** *Let $(X, x)$ be a well pointed space. Consider $n \in \mathbb{Z}^{\geq 0}$. Suppose that $X$ is $n$-connected. Then the suspension homomorphism*

$$\sigma_k(X, x)\colon \pi_k(X, x) \to \pi_{k+1}(\Sigma(X, x), *)$$

*is bijective for $0 \leq k \leq 2n$ and surjective for $k = 2n + 1$.*

*Proof.* If $X$ is path connected, then $\Sigma(X, x)$ is simply connected and hence the map $\sigma_0(X, x)\colon \pi_0(X, x) \to \pi_1(\Sigma(X, x), *)$ is obviously bijective. Hence we can assume $k \geq 1$ in the sequel.

There is a homeomorphism $u\colon \operatorname{cone}(X, x)/X \xrightarrow{\cong} \Sigma(X, x)$. The following diagram commutes

$$\begin{array}{ccc} \pi_k(X, x) & \xrightarrow{\sigma_k(X,x)} & \pi_{k+1}(\Sigma(X, x), *) \\ \uparrow{\scriptstyle \partial_{k+1}} & & \uparrow{\scriptstyle \pi_{k+1}(u,*)} \\ \pi_{k+1}(\operatorname{cone}(X, x), X, x) & \xrightarrow[\pi_{k+1}(\mathrm{pr},x)]{} \pi_{k+1}(\operatorname{cone}(X, x)/X, \{*\}, *) = & \pi_{k+1}(\operatorname{cone}(X, x)/X, *) \end{array}$$

where $\mathrm{pr}\colon \operatorname{cone}(X, x) \to \operatorname{cone}(X)/X$ is the projection and where the left vertical arrow $\partial_{n+1}$ is the boundary operator of the long exact homotopy sequence of the pair $(\operatorname{cone}(X, x), X)$, see Remark 2.13, and is bijective, since $\operatorname{cone}(X, x)$ is contractible. The right vertical arrow is bijective, as $u$ is a pointed homeomorphism. Hence it remains to show that the map $\pi_l(\mathrm{pr}, x)\colon \pi_l(\operatorname{cone}(X, x), X, x) \to \pi_l(\operatorname{cone}(X), x/X, \{*\}, *)$ is bijective for $2 \leq l \leq 2n+1$ and surjective for $l = 2n+2$. This follows from Proposition 11.11. This finishes the proof of the Freudenthal Suspension Theorem 11.16. $\square$

**Remark 11.17.** We have the degree homomorphism $\deg_n\colon \pi_n(S^n, s) \to \mathbb{Z}$, which is known to be bijective for $n = 1$ by elementary covering theory and compatible with the suspension homomorphisms. We conclude that $S^n$ is simply connected for $n \geq 2$ from the Seifert-van Kampen Theorem. Note that the Freudenthal Suspension Theorem 11.16 implies for $n \geq 1$ that the suspension homomorphism $\sigma_k(S^n, s)\colon \pi_k(S^n, s) \to \pi_{k+1}(S^{n+1}, s)$ is bijective for $0 \leq k \leq 2n - 2$ and surjective for $k = 2n - 1$. This gives another proof of Theorem 3.4, which does not use differential topology.

**Remark 11.18.** Let $\mathcal{H}_*$ be a (generalized) homology theory. Then we have for every $n \in \mathbb{Z}$ the suspension isomorphism

$$\sigma_n(X, x)\colon \mathcal{H}_n(X, \{x\}) \xrightarrow{\cong} \mathcal{H}_{n+1}(\Sigma(X, x), \{*\})$$

whereas in Freudenthal Suspension Theorem 11.16 the suspension homomorphism is only bijective in a range depending on the connectivity of $X$. The connectivity assumptions appearing in Theorem 11.16 are necessary and actually sharp. For instance, we know $\pi_3(S^2) \cong \mathbb{Z}$ from Theorem 10.5, and one can show that $\pi_4(S^3)$ is cyclic of order two and that the suspension homomorphism $\sigma_3(S^2, s)\colon \pi_3(S^2) \to \pi_4(S^3)$ is surjective and obviously not injective. Moreover, the suspension homomorphism $\sigma_2(S^1)\colon \pi_2(S^1) \to \pi_3(S^2)$ is not surjective as its domain is trivial and its codomain is not trivial.

11.6. **Stable homotopy groups.** Let $(X,x)$ be a pointed space and $n \in \mathbb{Z}^{\geq 0}$. Consider the sequence given by the suspension homomorphisms of (11.15)

$$\pi_n(X) \xrightarrow{\sigma_n(X)} \pi_{n+1}(\Sigma X) \xrightarrow{\sigma_{n+1}(\Sigma X)} \pi_{n+2}(\Sigma^2 X) \xrightarrow{\sigma_{n+2}(\Sigma^2 X)} \pi_{n+3}(\Sigma^3 X) \xrightarrow{\sigma_{n+3}(\Sigma^3 X)} \cdots \tag{11.19}$$

where we omit the base points. Recall that $\Sigma^m X$ is $(m-1)$-connected for $m \in \mathbb{Z}^{\geq 1}$ by Proposition 11.11 and hence by the Freudenthal Suspension Theorem 11.16 the map $\sigma_{n+m}(\Sigma^m X)\colon \pi_{n+m}(\Sigma^m X) \to \pi_{m+n+1}(\Sigma^{m+1} X)$ is surjective for $m = (n+1)$ and bijective for $m \geq (n+2)$. So after finitely many steps all these suspension homomorphism are isomorphism of abelian groups.

**Definition 11.20** (Stable homotopy groups)**.** Define the abelian group $\pi_n^s(X,x)$, called *nth stable homotopy group* of $(X,x)$ to be the direct limit of the sequence (11.19).

Given a (unpointed) space $Y$, define

$$\pi_n^s(Y) := \pi_n^s(Y_+)$$

where $Y_+$ is the pointed space $(Y \amalg \{*\}, *)$ obtained from $Y$ by adjoining an extra base point.

Obviously $\pi_n^s(X,x)$ is a functor from the category of pointed spaces to the category of abelian groups. Moreover, if the two pointed maps $f_0.f_1\colon (X,x) \to (Y,y)$ are pointed homotopic, then the induced homomorphisms $\pi_n^s(f_0)$ and $\pi_n^s(f_1)$ from $\pi_n^s(X,x)$ to $\pi_n^s(Y,y)$ agree. The stable homotopy groups come with a natural map

$$\iota_n(X,x)\colon \pi_n(X,x) \to \pi_n^s(X,x) \tag{11.21}$$

and with a natural suspension homomorphism

$$\sigma_n^s(X,x)\colon \pi_n^s(X,x) \to \pi_{n+1}^s(\Sigma(X,x),*). \tag{11.22}$$

The map $\iota_n(X,x)$ is in general neither injective nor surjective. If $X$ is $m$-connected for $m \in \mathbb{Z}^{\geq 0}$, then $\iota_n(X,x)$ is surjective if $n = 2m+1$ and is bijective if $n \leq 2m$ by the Freudenthal Suspension Theorem 11.16. The construction of the stable homotopy groups is designed so that $\sigma_n^s(X,x)$ of (11.22) is bijective for every pointed space $(X,x)$ and $n \geq 0$.

Given a (unpointed) topological pair $(X,A)$, we define

$$\pi_n^s(X,A) = \pi_n^s(X_+ \cup_{A_+} \operatorname{cone}(A_+,*),*). \tag{11.23}$$

Thus we obtain a functor from the category of pairs to the category of abelian groups which is homotopy invariant, i.e., for two maps of pairs $f_0.f_1\colon (X,A) \to (Y,B)$ the induced homomorphisms $\pi_n^s(f_0)$ and $\pi_n^s(f_1)$ from $\pi_n^s(X,A)$ to $\pi_n^s(Y,B)$ agree if $f_0$ and $f_1$ are homotopic as maps of pairs.

We record the following theorem whose proof we will give later when we are dealing more generally with spectra.

**Theorem 11.24** (Stable homotopy groups form a (generalized) homology theory)**.** *There exist natural transformation $\partial_{n+1}(X,A)\colon \pi_{n+1}^s(X,A) \to \pi_n^s(A)$ for $n \in \mathbb{Z}^{\geq 0}$ such that stable homotopy $\pi_*^s$ defines a homology theory on the category of pairs satisfying the disjoint union axiom.*

Obviously $\pi_*^s$ also satisfies the weak homotopy equivalence axiom saying that a weak homotopy equivalence induces isomorphisms on the stable homotopy groups. It does not satisfy the dimension axiom.

**Definition 11.25** (Stable stems)**.** Define the $n$-th stable stem $\pi_n^s$ to be $\pi_n^s(\{\bullet\}) = \pi_n^s(S^0,*)$ for $n \geq 0$.

Note that $\pi_n^s$ is the direct limit of the directed system

$$\pi_n(S^0,*) \xrightarrow{\sigma_n(S^0,*)} \pi_{n+1}(S^1,*) \xrightarrow{\sigma_{n+1}(S^1,*)} \pi_{n+2}(S^2,*) \xrightarrow{\sigma_{n+2}(S^2,*)} \pi_{n+3}(S^3,*) \xrightarrow{\sigma_{n+3}(S^3,*)} \cdots . \tag{11.26}$$

where have used the standard identification $(S^{n+1},*) = (\Sigma(S^n,*),*)$. Recall that the map $\sigma_{n+m}(S^m,*)\colon \pi_{n+m}(S^m,*) \to \pi_{n+m+1}(S^{m+1},*)$ is surjective for $m = (n+1)$ and bijective for $m \geq (n+2)$.

**Remark 11.27** (Outlook about $\pi_k^s(S^n)$)**.** Obviously it is easier to compute $\pi_n^s$ instead of $\pi_n(S^m)$ for $m > n$. Nevertheless it is an open (and extremely hard) problem to compute $\pi_n^s$ general. At the time of writing it is fair to say that we do not know $\pi_n^s$ in the range $n \geq 100$. Only some asymptotic results are known in that range. There is not even a formula known which might give the answer. There is no obvious pattern in the computations, one has carried out so far. At least one knows that $\pi_n^s$ is finite for $n \geq 1$, see [17] and one knows its values for $n \leq 61$ and also for some other values for $n \leq 99$. For instance we have

(11.28)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_n^s$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/24$ | 0 | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/240$ | $\mathbb{Z}/2^2$ | $\mathbb{Z}/2^3$ | $\mathbb{Z}/6$ | $\mathbb{Z}/504$ |

| $n$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|
| $\pi_n^s$ | 0 | $\mathbb{Z}/3$ | $\mathbb{Z}/2^2$ | $\mathbb{Z}/480 \times \mathbb{Z}/2$ | $\mathbb{Z}/2^2$ | $\mathbb{Z}/2^4$ | $\mathbb{Z}/8 \times \mathbb{Z}/2$ | $\mathbb{Z}/264 \times \mathbb{Z}/2$ |

where $A^m$ means $\bigoplus_{i=1}^m A$. The table above is taken from Toda [23]. More information about the stable stems can be found for instance in [10, 11].

## 12. The Hurewicz Theorem

12.1. **The Hurewicz homomorphism.** Let $(X,x)$ be a pointed space. Next we define for $n \in \mathbb{Z}^{\geq 1}$ a homomorphism of groups, which is natural in $X$ and called *$n$-th Hurewicz map or $n$-th Hurewicz homomorphism.*

$$\operatorname{hur}_n(X,x)\colon \pi_n(X,x) \to H_n(X), \tag{12.1}$$

where $H_n(X)$ denotes singular homology (with coefficients in $\mathbb{Z}$).

Given an element $[f]$ in $\pi_n(X,x)$ represented by a pointed map $f\colon (S^n,s) \to (X,x)$, define $\operatorname{hur}_n(X,x)([f])$ to be the image of the standard fundamental class $[S^n] \in H_n(S^n)$ under the map $H_n(f)\colon H_n(S^n) \to H_n(X)$ induced by $f$. Obviously this definition is independent of the choice of representative $f$ of $[f]$. Let $\nabla_n\colon S^n \to S^n \vee S^n$ be the pinching map, see (2.5). Let $\operatorname{pr}_k\colon S^1 \vee S^1 \to S^1$ be the projection onto the $k$-th functor for $k = 1, 2$. Then the following diagram commutes

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{H_n(\nabla_n)} & H_n(S^n \vee S^n) \\ & \searrow^{\Delta_n} & \downarrow{\cong}\ H_n(\operatorname{pr}_1) \times H_n(\operatorname{pr}_2) \\ & & H_n(S^n) \times H_n(S^n) \end{array}$$

where $\Delta_n$ is the diagonal map sending $z$ to $(z,z)$ and the right vertical arrow is an isomorphism. Note that in $\pi_n(X,x)$ the inverse of $[f]$ is given by $[f \circ u]$ for any map $u\colon (S^n,s) \to (S^n,s)$ of degree $-1$ and $H_n(u)\colon H_n(S^n) \to H_n(S^n)$ sends $[S^n]$ to $-[S^n]$. Now one easily checks that $\operatorname{hur}_n(X,x)$ is a group homomorphism. Obviously it is natural in $(X,x)$.

The elementary proof of the following lemma is left to the reader.

**Lemma 12.2.** *Let $w\colon I \to X$ be a path from $x$ to $y$. Then the following diagram commutes*

$$\begin{array}{ccc} \pi_n(X,x) & & \\ & \searrow^{\mathrm{hur}_n(X,x)} & \\ {\scriptstyle T_n([w])}\downarrow\cong & & H_n(X) \\ & \nearrow_{\mathrm{hur}_n(X,y)} & \\ \pi_n(X,x) & & \end{array}$$

*where $T_n([w])$ is the isomorphism introduced in Subsection 2.3.*

12.2. **The Hurewicz Theorem.** Before we investigate the Hurewicz homomorphism further, we consider the following two special cases. The first one is the case $n = 1$ and has already been dealt with in a previous lecture course.

**Proposition 12.3.** *If $X$ is a path connected space, then for any base point $x$ the map induced by the Hurewicz homomorphism*

$$\pi_1(X,x)_{\mathrm{ab}} \to H_1(X)$$

*for $\pi_1(X,x)_{\mathrm{ab}} = \pi_1(X)/[\pi_1(X,x),\pi_1(X,x)]$ the abelianization of $\pi_1(X,x)$ is an isomorphism.*

**Lemma 12.4.** *For $n \in \mathbb{Z}^{\geq 1}$ the Hurewicz homomorphism $\mathrm{hur}_n(S^n,s)\colon \pi_n(S^n,s) \to H_n(S^n)$ of (12.1) is bijective.*

*Proof.* The map $f\colon \mathbb{Z} \to \pi_n(S^n)$ sending 1 to the class of $[\mathrm{id}_{S^n}]$ is bijective by Theorem 3.4. The composite of $\mathrm{hur}_n(S^n,s)$ and $f$ is the homomorphism $\mathbb{Z} \to H_n(S^n)$ sending 1 to $[S^n]$ and hence bijective. This implies that $\mathrm{hur}_n(S^n,s)$ is bijective. □

Next we prove one of the main results of the course.

**Theorem 12.5** (Hurewicz Theorem)**.** *Consider $n \in \mathbb{Z}^{\geq 2}$. Let $X$ be an $(n-1)$-connected space. Then the Hurewicz homomorphism*

$$\mathrm{hur}_n(X,x)\colon \pi_n(X,x) \to H_n(X)$$

*of (12.1) is bijective for any base point $x \in X$.*

*Proof.* Since $X$ is $n$-connected, it is weakly homotopy equivalent to a $CW$-complex $Y$ which has precisely one 0-cell and no cells of dimension $d$ for $1 \leq d \leq (n-1)$ by Corollary 6.5. The inclusion $i\colon Y_{n+1} \to Y$ induces for the base point $y_0 \in Y_0$ bijections

$$\begin{aligned} \pi_n(i,y_0)\colon \pi_n(Y_{n+1},y_0) &\xrightarrow{\cong} \pi_n(Y,y_0); \\ H_n(i)\colon H_n(Y_{n+1}) &\xrightarrow{\cong} H_n(Y), \end{aligned}$$

by Corollary 4.5 and the long exact homotopy sequence of the pair $(Y, Y_{n+1})$. Hence we can assume without loss of generality that $X$ has precisely one 0-cell $\{x_0\}$ and the dimension $\dim(e)$ of every cell $e$ satisfies $\dim(e) \in \{0, n, (n+1)\}$.

This implies that $X_n = \bigvee_{i\in I} S^n$ and there exists a pushout

$$\begin{array}{ccc} \coprod_{j\in J} S^n & \xrightarrow{\coprod_{j\in J} q_j} & X_n = \bigvee_{i\in I} S^n \\ \downarrow & & \downarrow \\ \coprod_{j\in J} D^{n+1} & \longrightarrow & X. \end{array}$$

The Cellular Approximation Theorem 4.1 implies that each map $q_j \colon S^n \to X_n$ is homotopic to a map $q_j' \colon S^n \to X_n$ sending $s$ to $y_0$. Choose a homotopy $h_j \colon \coprod_{i \in I} S^n \times [0,1] \to X_n$ with $(h_j)_0 = \coprod_{j \in J} q_j$ and $(h_j)_1 = \coprod_{j \in J} q_j'$. Consider the following commutative diagram

$$\begin{array}{ccccc}
\coprod_{j\in J} D^{n+1} & \hookleftarrow & \coprod_{j\in J} S^n & \xrightarrow{\coprod_{j\in J} q_j} & X_n \\
\simeq \downarrow l_0 & & \simeq \downarrow k_0 & & \simeq \downarrow \mathrm{id}_{X_n} \\
\coprod_{j\in J} D^{n+1} \times I & \hookleftarrow & \coprod_{j\in J} S^n \times I & \xrightarrow{\coprod_{j\in J} h_j} & X_n \\
\simeq \uparrow l_1 & & \simeq \uparrow k_1 & & \simeq \uparrow \mathrm{id}_{X_n} \\
\coprod_{j\in J} D^{n+1} & \hookleftarrow & \coprod_{j\in J} S^n & \xrightarrow{\coprod_{j\in J} q_j'} & X_n.
\end{array}$$

where $k_0$ and $l_0$ are the obvious inclusions coming from $0 \in I$ and $k_1$ and $l_1$ are the obvious inclusions coming from $1 \in I$. All vertical arrows are homotopy equivalences and all left horizontal arrows are cofibrations. Hence the induced maps from the pushout of the upper row to the pushout of the middle row as well as the arrow from the pushout of the lower row to the pushout of middle row are homotopy equivalences by Theorem 8.28. Therefore we can assume without loss of generality that $q_j$ sends the base point $s \in S^n$ to $x_0$ and we can write $X$ as a pushout

$$\begin{array}{ccc}
\bigvee_{j\in J} S^n & \xrightarrow{f} & \bigvee_{i\in I} S^n \\
\downarrow & & \downarrow k \\
\bigvee_{j\in J} D^{n+1} & \longrightarrow & X
\end{array}$$

where $f$ respects the base points and $k$ is the inclusion $X_n = \bigvee_{i\in I} S^n \to X_{n+1} = X$. We obtain a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccc}
\pi_n\left(\bigvee_{j\in J} S^n, *\right) & \xrightarrow{\pi_n(f)} & \pi_n\left(\bigvee_{i\in J} S^n, *\right) & \xrightarrow{\pi_n(k)} & \pi_n(X, x_0) & \longrightarrow & \{0\} \\
\downarrow \mathrm{hur}_n\left(\bigvee_{j\in J} S^n, *\right) & & \downarrow \mathrm{hur}_n\left(\bigvee_{j\in J} S^n, *\right) & & \downarrow \mathrm{hur}_n(X, x_0) & & \\
H_n\left(\bigvee_{j\in J} S^n\right) & \xrightarrow{H_n(f)} & H_n\left(\bigvee_{i\in J} S^n\right) & \xrightarrow{H_n(k)} & H_n(X) & \longrightarrow & \{0\}
\end{array}$$

The rower low is exact by excision, the long exact homology sequence for pairs, and the fact that $H_{n-1}\left(\bigvee_{j\in J} S^n\right) = \{0\}$ holds. The upper row is exact by Theorem 11.9, the long exact homotopy sequence for pairs, and the conclusion from Theorem 3.4 and Proposition 11.12 (iii) that $\pi_{n-1}\left(\bigvee_{j\in J} S^n\right) = \{0\}$ holds. Hence by the Five Lemma it suffices to prove that the left vertical arrow and the middle vertical arrow are bijective. The following diagram commutes

$$\begin{array}{ccc}
\bigoplus_{i\in I} \pi_n(S^n, s) & \xrightarrow[\cong]{\bigoplus_{i\in I} \pi_n(k_i, s)} & \pi_n\left(\bigvee_{i\in I} S^n\right) \\
\bigoplus_{i\in I} \mathrm{hur}_n(S^n, s) \downarrow & & \downarrow \mathrm{hur}_n\left(\bigvee_{i\in I} S^n, *\right) \\
\bigoplus_{i\in I} H_n(S^n, s) & \xrightarrow[\bigoplus_{i\in I} H_n(k_i)]{\cong} & H_n\left(\bigvee_{i\in I} S^n\right).
\end{array}$$

and has bijections as horizontal arrows by Proposition 11.12 (iii), where $k_i$ is the inclusion of the summand belonging to $i \in I$. Since the left vertical arrow is bijective by Lemma 12.4, the right vertical arrow is bijective. Lemma 12.2 implies that the

Hurewicz homomorphism $\operatorname{hur}_n(X,x)\colon \pi_n(X,x) \to H_n(X)$ is bijective for all base points $x \in X$. This finishes the proof of the Hurewicz Theorem 12.5. □

**Remark 12.6.** The condition that $X$ simply connected in Theorem 12.5 is necessary. Consider a non-trivial group $G$ for which $G_{\mathrm{ab}}$ is trivial, e.g., the simple finite group $A_n$ for $n \geq 5$. Choose a path connected CW-complex $X$ with $\pi_1(X) \cong G$. Then $X$ is 0-connected and Lemma 12.3 implies that the Hurewicz homomorphism $\operatorname{hur}_1(X,x)\colon \pi_1(X,x) \to H_1(X)$ is not injective.

The condition that $X$ is simply connected is also necessary in the following Proposition 12.7.

**Proposition 12.7.** *Let $X$ be a simply connected space and $n \in \mathbb{Z}^{\geq 1}$. Then*

*(i) The following assertions are equivalent:*
- *$X$ is $n$-connected;*
- *$H_i(X) = 0$ holds for $1 \leq i \leq n$;*
- *$H_i(X) = 0$ holds for $2 \leq i \leq n$;*

*(ii) The following assertions are equivalent:*
- *$X$ is weakly contractible;*
- *$H_i(X) = 0$ holds for $1 \leq i$;*
- *$H_i(X) = 0$ holds for $2 \leq i$.*

*Proof.* (i) This follows by induction over $n = 1, 2, \ldots$. The induction beginning $n = 1$ follows from the conclusion of Proposition 12.3 that $H_1(X) = 0$ vanishes for a simply connected space $X$. The induction step from $(n-1) \geq 1$ to $n$ follows from Theorem 12.5.

(ii) This follows from assertion (i). □

We record the following stronger version of the Hurewicz Theorem whose proof can be found in [20, Theorem 10.25 on page 185].

**Theorem 12.8** (Improved Hurewicz Theorem)**.** *Consider $n \in \mathbb{Z}^{\geq 2}$. Let $X$ be an $(n-1)$-connected space. Then for any base point $x \in X$ the Hurewicz homomorphism*

$$\operatorname{hur}_m(X,x)\colon \pi_m(X,x) \to H_n(X)$$

*of* (12.1) *is bijective for $m = n$ and surjective for $m = n+1$.*

12.3. **The relative Hurewicz Theorem.** There is also a relative version of the Hurewicz map for a pointed pair $(X, A, a)$ for $n \geq 1$

$$\operatorname{hur}_n(X,A,a)\colon \pi_n(X,A,a) \to H_n(X,A), \tag{12.9}$$

which sends $[f] \in \pi_n(X,A,x)$ represented by a map of triples $f\colon (D^n, S^{n-1}, \{s\}) \to (X, A, \{a\})$ to the image of the standard fundamental class $[D^n, S^{n-1}]$ under the homomorphism $H_n(f)\colon H_n(D^n, S^{n-1}) \to H_n(X,A)$. It is a group homomorphism for $n \geq 2$ and the following diagram commutes for $n \geq 2$

$$\begin{array}{ccccc} \pi_n(X,a) & \longrightarrow & \pi_n(X,A,a) & \longrightarrow & \pi_{n-1}(A,a) \\ \downarrow{\scriptstyle \operatorname{hur}_n(X,a)} & & \downarrow{\scriptstyle \operatorname{hur}_n(X,A,a)} & & \downarrow{\scriptstyle \operatorname{hur}_{n-1}(X,a)} \\ H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A,a), \end{array} \tag{12.10}$$

where the exact upper row and the exact lower row are parts of the long exact sequences associated to the pair $(X, A)$.

**Theorem 12.11** (The relative Hurewicz Theorem)**.** *Consider $n \in \mathbb{Z}^{\geq 2}$. Let $(X, A)$ be a pair. Suppose that $A$ and $X$ are simply connected and $(X, A)$ is $(n-1)$-connected. Then:*

*(i) The Hurewicz homomorphism*

$$\mathrm{hur}_n(X,A,a)\colon \pi_n(X,A,a) \to H_n(X,A)$$

*of* (12.10) *is bijective for any base point* $a \in A$*;*

*(ii) The homology group* $H_i(X,A)$ *vanishes for* $2 \le i \le (n-1)$.

*Proof.* We can arrange that the inclusion $A \to X$ is a cofibration by Subsection 8.8. Let $\mathrm{pr}\colon (X,A,\{a\}) \to (X/A,\{*\})$ be the projection. We obtain a commutative diagram

$$\begin{array}{ccc} \pi_n(X,A,a) & \xrightarrow[\cong]{\pi_n(\mathrm{pr},a)} & \pi_n(X/A,\{*\},*) = \pi_n(X/A,*) \\ \Big\downarrow{\scriptstyle \mathrm{hur}_n(X,A,a)} & & \Big\downarrow{\scriptstyle \mathrm{hur}_n(X/A,*)} \\ H_n(X,A) & \xrightarrow[H_n(\mathrm{pr})]{\cong} & H_n(X/A,\{*\}) = H_n(X/A). \end{array}$$

The upper row is bijective by Proposition 11.11 The lower row is bijective by excision. If $X/A$ is $(n-1)$-connected, then Theorem 12.5 implies that right vertical arrow is also bijective. Hence it suffices to show for $n \in \mathbb{Z}^{\ge 2}$:

- The space $X/A$ is $(n-1)$-connected;
- The homology group $H_i(X/A)$ vanishes for $2 \le i \le (n-1)$.

This is done by induction over $n = 2,3,4,\ldots$.

Since $A$ and $X$ are simply connected, $X/A$ is simply connected by the Seifert-von-Kampen Theorem. Now the induction beginning $n = 2$ follows. The induction step from $(n-1) \ge 2$ to $n$ is done as follows.

By induction hypothesis applied to the $(n-2)$-connected pair $(X,A)$, we know that $\mathrm{hur}_{n-1}(X,A,a)\colon \pi_{n-1}(X,A,a) \to H_{n-1}(X,A)$ is bijective and the homology group $H_i(X,A)$ vanishes for $2 \le i \le (n-2)$. As $\pi_{n-1}(X,A,a)$ vanishes, the homology group $H_i(X,A)$ vanish for $2 \le i \le (n-1)$. As the projection pr induces an isomorphism $H_i(X,A) \xrightarrow{\cong} H_i(X/A)$ for $i \ge 1$, the homology group $H_i(X/A)$ vanishes for $2 \le i \le (n-1)$. This finishes the proof of Theorem 12.11. □

12.4. **Applications of the Hurewicz Theorem.** Next we generalize Proposition 12.7 to maps.

**Proposition 12.12.** *Let* $f\colon X \to Y$ *be a map of simply connected spaces.*

*(i) The following assertions are equivalent for* $n \in \mathbb{Z}^{\ge 1}$*:*
  - $f$ *is* $n$*-connected;*
  - $H_i(f)\colon H_i(X) \to H_i(Y)$ *is bijective for* $2 \le i \le (n-1)$ *and surjective for* $i = n$*;*

*(ii) The following assertions are equivalent:*
  - $f$ *is a weak homotopy equivalence;*
  - $H_i(f)\colon H_i(X) \to H_i(Y)$ *is bijective for* $i \ge 2$.

*Proof.* Since we can replace $f$ by the inclusion into its mapping cylinder, this follows from the relative Hurewicz Theorem 12.11. □

The next theorem is called also sometimes the Whitehead Theorem, see also Theorem 5.1.

**Theorem 12.13** (Whitehead Theorem)**.** *Let* $f\colon X \to Y$ *be a map of simply connected CW-complexes. Then the following assertions are equivalent:*

- $f$ *is a homotopy equivalence;*
- $f$ *is a weak homotopy equivalence;*
- $H_i(f)\colon H_i(X) \to H_i(Y)$ *is bijective for* $i \ge 2$.

*Proof.* This is a direct consequence of Theorem 5.1 and Proposition 12.12. □

The condition that $X$ and $Y$ are simply-connected is necessary in Theorem 12.13. Here is a more general version of Theorem 12.13 which does not need the assumption that $X$ and $Y$ are simply connected.

**Theorem 12.14.** *Let $f\colon X \to Y$ be a map of path connected CW-complexes. Suppose that for one (and hence all) base point $x \in X$ the map $\pi_1(X,x) \to \pi_1(Y,f(x))$ is bijective. We can lift $f$ to a map between the universal coverings $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$, i.e., we have the commutative diagram:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

*Then the following assertions are equivalent:*

- *$f$ is a homotopy equivalence;*
- *$H_i(\widetilde{f})\colon H_i(\widetilde{X}) \to H_i(\widetilde{Y})$ is bijective for $i \geq 2$.*

*Proof.* We conclude from Theorem 2.7 that $f$ is a weak homotopy equivalence if and only if $\widetilde{f}$ is a weak homotopy equivalence. By Theorem 12.13 $\widetilde{f}$ is a weak homotopy equivalence if and only if $H_i(\widetilde{f})\colon H_i(\widetilde{X}) \to H_i(\widetilde{Y})$ is bijective for $i \geq 2$. Now Theorem 12.14 follows from Theorem 5.1. □

**Example 12.15.** Consider the spaces $X = S^n \vee S^n \vee S^{2n}$ and $Y = S^n \times S^n$ for $n \in \mathbb{Z}^{\geq 2}$. Then $X$ and $Y$ are simply connected and $H_i(X) \cong H_i(Y)$ holds for $i \geq 0$. But the cohomology rings of $X$ and $Y$ are not isomorphic and hence there is no homotopy equivalence from $X$ to $Y$.

Note that this does not contradict Theorem 12.13, since there the existence of a map $f\colon X \to Y$ is required which implements the isomorphism $H_i(X) \xrightarrow{\cong} H_i(Y)$.

**Theorem 12.16** (Recognizing the sphere up to homotopy)**.** *Let $X$ be a path connected CW-complex. Then the following assertions are equivalent for $n \geq 2$:*

- *The space $X$ is homotopy equivalent to $S^n$;*
- *The space $X$ is simply connected, $H_i(X)$ vanishes for all $i \geq 2$ with $i \neq n$ and $H_n(X)$ is isomorphic to $\mathbb{Z}$.*

*Proof.* Suppose that the space $X$ is simply connected, $H_i(X)$ vanishes for all $i \geq 2$ with $i \neq n$ and $H_n(X)$ is isomorphic to $\mathbb{Z}$. Proposition 12.7 (i) implies that $X$ is $(n-1)$-connected. We conclude form the Hurewicz Theorem 12.5 that the Hurewicz homomorphism $\mathrm{hur}_n(X,x)\colon \pi_n(X,x) \to H_n(X,x)$ is bijective. Therefore we can find a map $f\colon S^n \to X$ such that $H_n(f)\colon H_n(S^n) \to H_n(X)$ is an isomorphism. Hence $H_i(f)\colon H_i(S^n) \to H_i(X)$ is an isomorphism for $2 \leq i$. Proposition 12.12 (i) implies that $f$ is a homotopy equivalence.

The other implication is obviously true. □

## 13. Moore spaces

**Definition 13.1** (Moore space)**.** Consider $n \geq 1$ and an abelian group $G$. A *Moore space $(X,\varphi)$ of type $(G,n)$* consists of a path connected $CW$-complex $X$ and an isomorphisms $\varphi\colon H_n(X) \xrightarrow{\cong} G$ such that $H_i(X) = \{0\}$ for $i \notin \{0,n\}$ holds and that $X$ is simply connected if $n \geq 2$.

Sometimes $(X,x,\varphi)$ is denoted by $M(G,n)$

**Lemma 13.2.** *Consider $n \geq 1$ and a group $G$ which is assumed to be abelian if $n \geq 2$ holds. Then:*

*(i) There exists an $(n+1)$-dimensional CW-complex $X$ with the following properties:*
- *The space $X$ is the reduced mapping cone of some map $f\colon \bigvee_{i\in I} S^n \to \bigvee_{j\in J} S^n$;*
- *There is an exact sequence of groups*
$$0 \to \pi_n\bigl(\bigvee_{j\in J} S^n\bigr) \xrightarrow{\pi_n(f,x)} \pi_n\bigl(\bigvee_{i\in I} S^n\bigr) \xrightarrow{\pi_n(k)} \pi_n(X,x) \to 0$$
*for $k\colon X_n = \bigvee_{j\in J} S^n \to X$ the inclusion and $n \geq 2$;*
- *We have $\{\bullet\} = X_0 = X_{n-1}$;*
- *The space $X$ is $(n-1)$-connected;*
- *We have $\pi_n(X,x) \cong G$ for any base point $x \in X$;*
- *The homology group $H_i(X)$ vanishes for $i \geq (n+1)$ if $G$ abelian;*
- *The homology group $H_i(X)$ vanishes for $1 \leq i \leq (n-1)$;*
- *If $G$ is finitely presented or if $G$ is finitely generated abelian, then $X$ can be choose to be a finite CW-complex.*

*(ii) Suppose that $n \geq 2$ holds. Let $X$ be the space constructed in the proof of assertion (i). Let $x \in X$ be any base point. Let $(Y,y)$ be any pointed CW-complex. Let $\psi\colon \pi_n(X,x) \xrightarrow{\cong} \pi_n(Y,y)$ be any group homomorphism.*

*Then there is a pointed map $u\colon (X,x) \to (Y,y)$ satisfying $\pi_n(u,x) = \psi$.*

*Proof.* (i) For $n = 1$ one can choose a presentation of the group and consider the associated presentation $CW$-complex, which is path connected and satisfies $\pi_1(X,x) \cong G$ for any base point $x$. In general $H_2(X)$ is not trivial. This can be arranged if $G$ is abelian. Choose an exact sequence of abelian groups $0 \to \bigoplus_{i\in I}\mathbb{Z} \xrightarrow{\alpha} \bigoplus_{j\in J}\mathbb{Z} \to G \to 0$. If $G$ is finitely generated abelian, one can choose $I$ and $J$ to be finite. Then one can find a map $f\colon \bigvee_{i\in I} S^1 \to \bigvee_{i\in I} S^1$ such that the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{i\in I}\mathbb{Z} & \xrightarrow{\alpha} & \bigoplus_{j\in J}\mathbb{Z} \\ \downarrow\cong & & \downarrow\cong \\ H_1\bigl(\bigvee_{i\in I} S^n\bigr) & \xrightarrow[H_1(f)]{} & H_1\bigl(\bigvee_{j\in J} S^n\bigr) \end{array}$$

where the vertical maps are the obvious isomorphisms. We have the short exact sequence

$$H_2\bigl(\bigvee_{j\in j} S^n\bigr) \to H_2(X) \to H_1\bigl(\bigvee_{i\in I} S^n\bigr) \xrightarrow{H_1(f)} H_1\bigl(\bigvee_{j\in j} S^n\bigr) \to H_1(X)$$
$$\to H_0\bigl(\bigvee_{i\in I} S^n\bigr) \xrightarrow{H_0(f)} H_0\bigl(\bigvee_{j\in j} S^n\bigr)$$

This implies $H_2(X) = \{0\}$ and $H_1(X) \cong G$. Since $X$ is 2-dimensional, we get $H_i(X) = \{0\}$ for $i \geq 2$.

Suppose $n \geq 2$. Then $G$ is an abelian group and we can choose an exact sequence of abelian groups $0 \to \bigoplus_{i\in I}\mathbb{Z} \xrightarrow{\alpha} \bigoplus_{j\in J}\mathbb{Z} \to G \to 0$. If $G$ is finitely generated, one can choose $I$ and $J$ to be finite. Let $\{e_i \mid i \in I\}$ and $\{e'_j \mid j \in J\}$ be the standard basis of $\bigoplus_{i\in I}\mathbb{Z}$ and $\bigoplus_{j\in J}\mathbb{Z}$. Then there is a set of integers $\{d_{i,j} \mid i \in I, j \in J\}$ such that the set $\{j \in J \mid d_{i,j} \neq 0\}$ is finite for every $i \in I$ and $\alpha(e_i) = \sum_{j\in J} d_{i,j} \cdot e'_j$

holds. The canonical maps

$$\bigoplus_{i \in I} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\bigl(\bigvee_{i \in I} S^n, s\bigr); \tag{13.3}$$

$$\bigoplus_{j \in J} \pi_n(S^n, s) \xrightarrow{\cong} \pi_n\bigl(\bigvee_{j \in J} S^n, s\bigr), \tag{13.4}$$

are bijective by Proposition 11.12 (iii). Because of Theorem 3.4 there is a pointed map $f_i\colon (S^n, s) \to \bigl(\bigvee_{j \in J} S^n, s\bigr)$ such that for every $j \in J$ its composite with the projection $\mathrm{pr}_k\colon \bigvee_{j \in J} S^n \to S^n$ to the factor $k$ belonging to $k \in J$ has degree $d_{i,k}$. Define the pointed map

$$f := \bigvee_{i \in I} f_i\colon \bigvee_{i \in I} S^n \to \bigvee_{j \in J} S^n.$$

Under the obvious identifications coming from the isomorphisms (13.3) and (13.4) the homomorphism $\alpha$ can be identified with

$$\pi_n(f, s)\colon \pi_n\bigl(\bigvee_{i \in I} S^n, s\bigr) \to \pi_n\bigl(\bigvee_{j \in J} S^n, s\bigr).$$

Hence the cokernel of the latter map is isomorphic to $G$ and its kernel is trivial. Let $X$ be the reduced mapping cone of $f$ with the preferred base point $x \in X_0$. The sequence

$$0 \to \pi_n\bigl(\bigvee_{i \in I} S^n, s\bigr) \xrightarrow{\pi_n(f,s)} \pi_n\bigl(\bigvee_{j \in J} S^n, s\bigr) \to \pi_n(X, x) \to 0$$

is exact by Theorem 2.11, Theorem 3.4, and Theorem 11.11. Hence $X$ is a path connected $CW$-complex which is has precisely one zero cell, no cells $e$ of dimension $1 \le \dim(e) \le n-1$, is $(n-1)$-connected by Corollary 4.5, satisfies $\pi_n(X, x) \cong G$ for the base point $x \in X_0$ and hence for all base points in $X$ by the diagram (2.6), and the homology groups $H_i(X)$ vanish for $1 \le i \le (n-1)$. It remains to show that $H_i(X)$ vanishes for $i \ge (n+1)$. As $X$ is $(n+1)$-dimensional, it suffices to do this for $i = (n+1)$. The following diagram commutes

$$\begin{array}{ccc}
 & & H_{n+1}\bigl(\bigvee_{j \in J} S^n\bigr) = \{0\} \\
 & & \downarrow H_{n+1}(k) \\
\{0\} & & H_{n+1}(X) \\
\downarrow & & \downarrow \partial_{n+1} \\
\pi_n\bigl(\bigvee_{i \in I} S^n, s\bigr) & \xrightarrow[\cong]{\mathrm{hur}_n\left(\bigvee_{i \in I} S^n, s\right)} & H_n\bigl(\bigvee_{i \in I} S^n\bigr) \\
\downarrow \pi_n(f,s) & & \downarrow H_n(f) \\
\pi_n\bigl(\bigvee_{j \in J} S^n, s\bigr) & \xrightarrow[\cong]{\mathrm{hur}_n\left(\bigvee_{j \in J} S^n, s\right)} & H_n\bigl(\bigvee_{j \in J} S^n\bigr) \\
\downarrow \pi_n(k,x) & & \downarrow H_n(k) \\
\pi_n(X, x) & \xrightarrow[\cong]{\mathrm{hur}_n(X,x)} & H_n(X) \\
\downarrow & & \downarrow \\
\{0\} & & \{0\}
\end{array}$$

where the column are exact and the horizontal arrows are bijective by Hurewicz Theorem 12.5 since $X$, $\bigvee_{i \in I} S^n$, and $\bigvee_{i \in I} S^n$ are $(n-1)$-connected by Corollary 4.5. Hence $H_{n+1}(X)$ vanishes.

(ii). We start with the case $n \geq 2$. Recall that $X$ is the reduced mapping cone of a specific map

$$f := \bigvee_{i \in I} f_i \colon \bigvee_{i \in I} S^n \to \bigvee_{j \in J} S^n$$

such that we have an exact sequence of abelian groups

$$0 \to \pi_n\big(\bigvee_{i \in I} S^n\big) \xrightarrow{\pi_n(f,x)} \pi_n\big(\bigvee_{j \in J} S^n\big) \xrightarrow{\pi_n(k)} \pi_n(X,x) \to 0$$

for $k \colon X_n = \bigvee_{j \in J} S^n \to X$ the inclusion and that we have isomorphisms (13.3) and (13.4). Because of (2.6) we can assume without loss of generality that the base point of $X$ is the standard base point $s$ in $X_n = \bigvee_{j \in J} S^n \subseteq X$. For every $j \in J$ let $a_j \colon (S^n, s) \to (Y,y)$ be the pointed map whose class in $\pi_n(Y,y)$ is the image of the standard generator of $\pi_n(S^n, s)$ under the composite

$$\pi_n(S^n,s) \xrightarrow{l_j} \bigoplus_{j \in J} \pi_n(S^n,s) \xrightarrow{\cong} \pi_n\big(\bigvee_{j \in J} S^n, s\big) \xrightarrow{\pi_n(k)} \pi_n(X,x) \xrightarrow{\psi} \pi_n(Y,y),$$

where $l_j$ is the inclusion of the $j$-th summand. Define the pointed map

$$u_n := \bigvee_{j \in J} a_j \colon (X_n, x) = \big(\bigvee_{j \in J} S^n, s\big) \to (Y,y).$$

Then the composite of the homomorphism $\pi_n(u_n, s) \colon \pi\big(\bigvee_{j \in J} S^n, s\big) \to \pi_n(Y,y)$ with the homomorphism $\pi_n(f,s) \colon \pi_n(\bigvee_{i \in I} S^n, s) \to \pi_n\big(\bigvee_{j \in J} S^n, s\big)$ is trivial. This implies that the composite $u_n \circ f$ is pointed nullhomotopic. Hence $u_n$ extends to a pointed map $u \colon (X,x) \to (Y,y)$. We get by construction $\pi_n(u,x) = \psi$. □

**Theorem 13.5** (Existence and uniqueness of Moore spaces)**.** *Consider $n \in \mathbb{Z}^{\geq 1}$ and two abelian groups $G$ and $G'$ Then:*

*(i) There exists a Moore space $(X, \phi)$ of type $(G,n)$ such that the $X_{n-1} = X_0 = \{x\}$ holds;*
*(ii) If $n \geq 2$ and $(X,\phi)$ and $(X', \phi')$ are Moore spaces of type $(G,n)$, then there is a homotopy equivalence $f \colon X \to X'$ satisfying $\phi' \circ H_n(f) = \phi$.*

*Proof.* (i) This follows from Lemma 13.2 (i).

(ii) We can suppose without loss of generality that $X$ is a $CW$-complex as it occurs in Lemma 13.2 (i). Then we obtain from Lemma 13.2 (ii) and Hurewicz Theorem 12.5 a map $f \colon X \to X'$ such that the following diagram with isomorphisms as vertical maps commutes

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\pi_n(f)} & \pi_n(X') \\ {\scriptstyle \mathrm{hur}_n}\downarrow\cong & & \cong\downarrow{\scriptstyle \mathrm{hur}_n} \\ H_n(X) & \xrightarrow{H_n(f)} & H_n(X') \\ {\scriptstyle \phi}\downarrow\cong & & \cong\downarrow{\scriptstyle \phi'} \\ G & \xrightarrow{\alpha} & G'. \end{array}$$

This implies that $H_i(f) \colon H_i(X) \to H_i(X')$ is bijective for $i \geq 1$. Since $X$ and $X'$ are simply connected, $f$ is a homotopy equivalence by Whitehead Theorem 12.13. □

## 14. Eilenberg-MacLane spaces

**Definition 14.1** (Eilenberg-MacLane space)**.** Consider $n \geq 1$ and a group $G$ which is assumed to be abelian if $n \geq 2$ holds. An *Eilenberg MacLane space* $(X, x, \psi)$ *of type* $(G, n)$ consists of a path connected pointed $CW$-complex $(X, x)$ and a group isomorphism $\psi \colon \pi_n(X, x) \xrightarrow{\cong} G$ such that $\pi_i(X, x) = \{0\}$ holds for $1 \leq i$ with $i \neq n$.

Sometimes $(X, x, \varphi)$ is denoted by $K(G, n)$.

**Lemma 14.2.** *Let $(A, a)$ be a path connected pointed $CW$-complex and $n \in \mathbb{Z}^{\geq 0}$. Then there is a $CW$-pair $(X, A)$ such that $X$ is path connected and obtained from $A$ by attaching cells of dimension $\geq (n + 2)$, the inclusion $j \colon A \to X$ induces an isomorphism $\pi_i(A, a) \to \pi_i(X, a)$ for $1 \leq i \leq n$, and $\pi_i(X, a) = \{1\}$ holds for $i \geq (n + 1)$.*

*Proof.* Consider the pair $(\operatorname{cone}(A), A)$. Apply Theorem 6.3 to it for the natural number $(n + 1)$. Then we obtain a $CW$-pair $(X, A)$ and a map of pairs $(f, \operatorname{id}_A) \to (\operatorname{cone}(A), A)$ such that the pair $(X, A)$ is $(n+1)$-connected and the homomorphism $\pi_i(f, a) \colon \pi_i(X, a) \to \pi_i(\operatorname{cone}(f), a)$ is injective for $i = (n + 1)$ and bijective for $i \geq (n + 2)$. Since $\operatorname{cone}(A)$ is contractible, this implies $\pi_i(X, a) = \{1\}$ for $i \geq (n + 1)$. As $(X, A)$ is $(n + 1)$-connected, the map $\pi_i(A, a) \to \pi_i(X, a)$ is bijective for $1 \leq i \leq n$. □

**Theorem 14.3** (Existence and uniqueness of Eilenberg-MacLane spaces)**.** *Consider $n \in \mathbb{Z}^{\geq 1}$ and two groups $G$ and $G'$ which are assumed to be abelian if $n \geq 2$ holds. Then:*

*(i) There exists an Eilenberg-MacLane space $(X, x, \varphi)$ of type $(G, n)$ such that the $X_{n-1} = X_0 = \{x\}$ holds;*

*(ii) Let $(X, x)$ be a pointed $n$-connected $CW$-complex and $(X', x', \varphi')$ be an Eilenberg-Maclane space of type $(G', n)$.*

*We obtain a bijection*

$$[(X, x), (X', x')]^0 \xrightarrow{\cong} \hom(\pi_n(X, x), G'), \quad [f] \mapsto \varphi' \circ \pi_n(f, x).$$

*Moreover, the forgetful map $[(X, x), (X', x')]^0 \to [X, X']$ is bijective for $n \geq 2$;*

*(iii) Let $(X, x, \varphi)$ be an Eilenberg-Maclane space of type $(G, n)$ and $(X', x', \varphi')$ be an Eilenberg-Maclane space of type $(G', n)$.*

*We obtain a bijection*

$$[(X, x), (X', x')]^0 \xrightarrow{\cong} \hom(G, G'), \quad [f] \mapsto \varphi' \circ \pi_n(f, x) \circ \varphi^{-1}.$$

*Moreover, the forgetful map $[(X, x), (X', x')]^0 \to [X, X']$ is bijective for $n \geq 2$;*

*(iv) Let $(X, x, \varphi)$ and $(X', x', \varphi')$ be two Eilenberg-MacLane spaces of type $(G, n)$. Then there exists a pointed homotopy equivalence $f \colon (X, x) \to (X', x')$ which is up to pointed homotopy equivalence uniquely determined by the property that $\varphi' \circ \pi_1(f, x) = \varphi$ holds.*

*Proof.* (i) This follows from Lemma 13.2 (i) and Lemma 14.2.

(ii) The forgetful map $[(X, x), (X', x')]^0 \to [X, X']$ is bijective for $n \geq 2$ because of (8.24), since $X'$ is simply connected.

We can find by Corollary 6.5 a $CW$-pair $(Y, y)$ together with a pointed homotopy equivalence $(Y, y) \to (X, x)$ such that $Y_{n-1} = Y_0 = \{y\}$ holds. Hence we can assume without generality that $X_{n-1} = X_0 = \{x\}$ holds. Then $X_n$ looks like $\bigvee_{i \in I} S^n$ and we have canonical isomorphisms

$$\bigoplus_{i\in I}\pi_n(S^n,s) \xrightarrow{\cong} \pi_n(\bigvee_{i\in I} S^n,x);$$
$$\mathbb{Z} \xrightarrow{\cong} \pi_n(S^n,s).$$

Consider a homomorphism $v\colon \pi_n(X,x)\to G'$. For every $i\in I$ choose a pointed map $u_i\colon (S^n,s)\to (X',x')$ such that the composite

$$\pi_n(S^n,s)\xrightarrow{j_i}\bigoplus_{i\in I}\pi_n(S^n,s)\xrightarrow{\cong}\pi_n(X_n,x)\xrightarrow{\pi_n(k)}\pi_n(X,x)\xrightarrow{v} G'\xrightarrow{\phi'^{-1}}\pi_n(X',x')$$

sends $[\mathrm{id}_{S^n}]$ to $[u_i]$, where $j_i$ is the inclusion of the $i$-th summand and $k\colon X_n\to X$ is the inclusion. Define the map

$$f_n=\bigvee_{i\in I} u_i\colon X_n=\bigvee_{i\in I} S^n\to X'.$$

It sends the basepoint $x$ of $X_n$ to the base points $x'$ of $X'$. The map $\pi_n(f_n,x)\colon \pi_n(X_n,x)\to \pi_n(X',x')$ agrees with the composite

$$\pi_n(X_n,x)\xrightarrow{\pi_n(k)}\pi_n(X,x)\xrightarrow{v}G'\xrightarrow{(\varphi')^{-1}}\pi_n(X',x').$$

We can define inductively maps $f_j\colon (X_j,x)\to (X',x')$ for $j=n,(n+1),(n+2),\dots$ satisfying $f_{j+1}|_{X_j}=f_j$ for $j=n,(n+1),(n+2),\dots$, since the attaching map $q\colon S^n\to X_n$ of any $(n+1)$-cell of $X$ lies in the kernel of $\pi_n(k)\colon \pi_n(X_n,x)\to\pi_n(X)$ and $\pi_j(X',x')=0$ holds for $j=(n+1),(n+2),\dots$. Define the map

$$f:=\operatorname{colim}_{j\to\infty} f_j\colon X=\operatorname{colim}_{j\to\infty}X_j\to X'$$

Then $f(x)=x'$ holds and $v$ agrees with the composite $\pi_n(X,x)\xrightarrow{\pi_n(f,x)}\pi_n(X',x')\xrightarrow{\varphi'} G'$. This proves surjectivity.

Injectivity is proved as follows. Consider two pointed maps $f_0,f_1\colon (X,x)\to (X',x')$ such that $\pi_n(f_0,x)=\pi_n(f_1,x)$ holds. We have to construct a pointed homotopy equivalence $h\colon (X,x)\times I\to (X',x')$ between $f_0$ and $f_1$. We construct inductive maps $h_j\colon X_j\times I\cup X\times\{0,1\}\to X'$ for $j=0,1,2,\dots$ such that

$$h_0\colon X_0\times I\cup X\times\{0,1\}=\{x\}\times I\cup X\times\{0,1\}\to X'$$

sends every element in $\{x\}\times I$ to $x$ and is given on $X\times\{k\}$ by $f_k$ for $k=0,1$ and we have for $j=0,1,2,\dots$

$$h_{j+1}|_{X_j\times I\cup X\times\{0,1\}}=h_j.$$

Since $X\times I$ is $\operatorname{colim}_{j\to\infty}X_j\times I\cup X\times\{0,1\}$, we can define the desired pointed homotopy $h$ by $\operatorname{colim}_{j\to\infty}h_j$.

It remains to construct the map $h_j$ for $j=0,1,2,\dots$. We have constructed $h_0$ already. Since $X_{n-1}=X_0$ holds, we have $X_{n-1}\times I\cup X\times\{0,1\}=X_0\times I\cup X\times\{0,1\}$ and can define $h_j=h_0$ for $1\le j\le (n-1)$ Next we construct $h_n$. We have $X_n=\bigvee_{i\in I}S^n$. We have to specify for each $i\in I$ a map $h_{n,i}\colon S_i^{n-1}\times I\to X'$ such that $h_{n,i}$ sends an element in $\{s\}\times I$ to $x'$ and its restriction to $S_i^{n-1}\times\{k\}$ is $f_k|_{S_i^{n-1}}$ for $k=0,1$, where $S_i^n$ is the $i$-th summand in $\bigvee_{i\in I}S^n$, since then the collection of the maps $h_{n,i}$-s yields the desired map $h_n$ by $h_n|_{X_0\times I\cup X\times\{0,1\}}=h_0$ and $h_n|_{S_i^n\times I}=h_{n,i}$. The existence of $h_{n,i}$ follows from $\pi_n(f_0,x)=\pi_n(f_1,x)$ since this implies that the pointed maps $f_0|_{S_i^{n-1}}$ and $f_1|_{S_i^{n-1}}$ from $(S_j^{n-1},s)\to(X',x')$ are pointed homotopic. This finishes the construction of $h_n$.

Since $X_{i+1}\times I\cup X\times\{0,1\}$ is obtained from $X_i\times I\cup X\times\{0,1\}$ by attaching cells of dimension $(i+2)$ and $\pi_{i+1}(X',x')$ vanishes, we can extend $h_i$ to $h_{i+1}$ for $i=n,(n+1),(n+1),\dots$. This finishes the proof of assertion (ii).

(iii) This follows from assertion (ii).

(iv) This follows from assertion (iii). □

**Remark 14.4** (Eilenberg MacLane space of type $(G, 1)$ and unpointed homotopy classes)**.** In Theorem 14.3 we have treated unpointed homotopy classes only for $n \geq 2$. We briefly discuss what happens in the case $n = 1$.

Consider the situation of assertion (ii) of Theorem 14.3. Then one obtains a bijection from $[X, X']$ to the set $[\Pi(X), \Pi(X')]$ of natural equivalence classes of functors from $\Pi(X)$ to $\Pi(X')$ by sending $[f]$ to $[\Pi(f)]$. In terms of fundamental groups one obtains a bijection for $\operatorname{Inn}(G')$ the group of inner automorphisms of $G'$

$$[X, X'] \xrightarrow{\cong} \operatorname{Inn}(G')\backslash \hom(\pi_1(X, x), G')$$

defined as follows. For $[f]$ we can choose a representative $f$ with $f(x) = x'$ and associate to $[f]$ the class of $\varphi' \circ \pi_1(f, x)$. These claims follow from Theorem 14.3 (ii) using the bijection (8.24).

In the situation of assertion (iii) of Theorem 14.3 we obtain a bijection

$$[X, X'] \xrightarrow{\cong} \operatorname{Inn}(G')\backslash \hom(G, G').$$

Note that $\operatorname{Inn}(G')$ is trivial if and only if $G'$ is abelian. So for abelian $G'$ we get also for $n = 1$ that the forgetful map $[(X, x), (X', x')] \to [X, X']$ is bijective in assertions (ii) and (iii) of Theorem 14.3.

Consider an abelian group $G$ and $n \in \mathbb{Z}^{\geq 1}$. Let $(X, x, \varphi)$ be an Eilenberg-Maclane space of type $(G, n)$. Then the Hurewicz homomorphism $\operatorname{hur}_n(X, x, \varphi) \colon \pi_n(X, x) \xrightarrow{\cong} H_n(X)$ is bijective by Theorem 12.5. Moreover $H_i(X)$ is trivial for $1 \leq i < n$ by Proposition 12.7 (i) and $H_0(X) \cong \mathbb{Z}$. By the Universal Coefficient Theorem we obtain an isomorphism $\alpha_n \colon H^n(X; G) \xrightarrow{\cong} \hom_{\mathbb{Z}}(H_n(X), G)$. Hence the following composite is an isomorphism

$$\beta \colon \hom_{\mathbb{Z}}(G, G) \xrightarrow{\hom_{\mathbb{Z}}(\varphi, \mathrm{id}_G)} \hom_{\mathbb{Z}}(\pi_n(X, x), G) \xrightarrow{\hom_{\mathbb{Z}}(\operatorname{hur}_n(X,x)^{-1}, \mathrm{id}_G)} \hom_{\mathbb{Z}}(H_n(X), G) \xrightarrow{\alpha_n^{-1}} H^n(X; G).$$

Let

(14.5) $$\iota_n(X, x, \varphi) \in H^n(X; G)$$

be the element which us uniquely determined by $\beta_n(\mathrm{id}_G) = \iota_n(X, x, \varphi)$. Let $Y$ be a $CW$-complex. Consider the map

(14.6) $$\gamma_n(Y) \colon [Y, X] \to H^n(Y; G), \quad [f] \mapsto H^n(f; G)(\iota_n(X, x, \varphi)).$$

We will later give the proof of the following theorem.

**Theorem 14.7.** *Consider $n \in \mathbb{Z}^{\geq 1}$. Let $G$ be an abelian group. Let $(X, x, \varphi)$ be an Eilenberg-MacLane space of type $(G, n)$. Let $Y$ be a $CW$-complex.*

*Then the map*

$$\gamma_n(Y) \colon [Y, X] \to H^n(Y; G)$$

*defined in* (14.6) *is bijective.*

**Example 14.8** (Homotopy classes of maps to $S^1$)**.** We conclude from Remark 14.4 or from Theorem 14.7 that we obtain for a $CW$-complex $Y$ a bijection of groups

$$[Y, S^1] \xrightarrow{\cong} H^1(Y; \mathbb{Z})$$

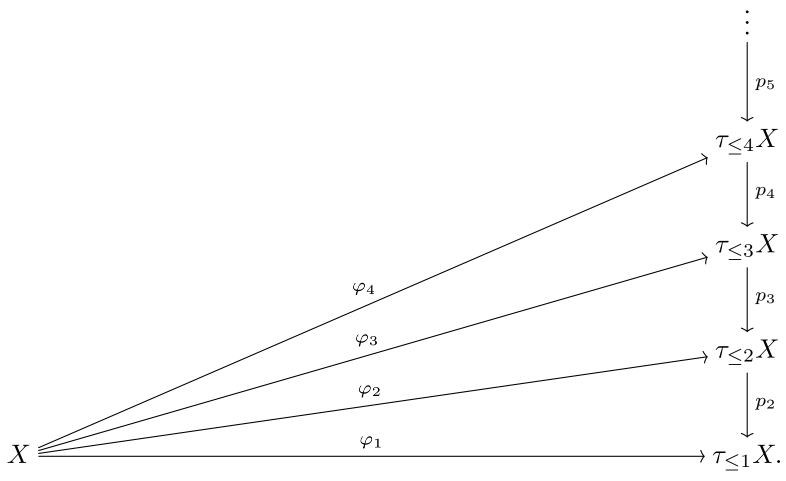
by sending $[f]$ to the image of a fixed generator of the infinite cyclic group $H^1(S^1; \mathbb{Z})$ under the homomorphism $H^1(f; \mathbb{Z}) \colon H^1(S^1; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$.

## 15. Postnikov towers

Let $X$ be a connected $CW$-complex $X$. A *Postnikov tower* for $X$ consists of a sequence of spaces $\tau_{\le k}X$ for $k \in \mathbb{Z}^{\ge 1}$, a sequence of maps $\varphi_k \colon X \to \tau_{\le k}X$ for $k \in \mathbb{Z}^{\ge 1}$, and a sequence of fibrations $p_k \colon \tau_{\le k}X \to \tau_{\le k-1}X$ for $k \in \mathbb{Z}^{\ge 2}$ with the following properties:

- $\pi_i(\tau_{\le k}) = \{0\}$ for $i \ge k+1$;
- The map $\varphi_k$ induces isomorphisms $\pi_i(\varphi_k) \colon \pi_i(X) \xrightarrow{\cong} \pi_i(\tau_{\le k}X)$ for $1 \le i \le k$;
- We have $p_{k+1} \circ \varphi_{k+1} = \varphi_k$ for $k \in \mathbb{Z}^{\ge 1}$,
- Each space $X_n$ has the homotopy type of a $CW$-complex.

The following diagram commutes




Note that $\tau_{\le 1}X$ is necessarily a model for $K(\pi_1(X), 1)$ and each map $\varphi_k$ is $k$-connected and has $K(\pi_k(X), k)$ as fiber.

**Remark 15.1** (The construction of a Postnikov towerr)**.** The details of the construction of a Postnikov tower can be found for instance in [26, Chapter IX]. The basic idea is the following.

For $k \in \mathbb{Z}^{\ge 0}$ we can construct a $CW$-complex $X_k'$ obtained from $X$ by attaching cells of dimension $\ge (k+2)$ such that inclusion $\varphi_k' \colon X \to X_k'$ induces isomorphisms $\pi_i(\varphi_k) \colon \pi_i(X) \to \pi_i(X_k')$ for $i = 0, 1, 2, \ldots, k$ and $\pi_i(X_k') = \{0\}$ holds for $i \ge (k+1)$, see Lemma 14.2. The inclusion $\varphi_i' \colon X \to X_{k-1}$ extends to a map $p_k' \colon X_k' \to X_{k-1}'$ since $X_k'$ is obtained from $X$ by attaching cells of dimension $\ge (k+2)$ and $\pi_i(X_{k-1})$ vanishes for $i \ge k$.

So we get a commutative diagram

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow{\scriptstyle p_5} \\ & & X'_4 \\ & \nearrow^{\varphi'_4} & \downarrow{\scriptstyle p'_4} \\ & & X'_3 \\ & \nearrow^{\varphi'_3} & \downarrow{\scriptstyle p'_3} \\ & & X'_2 \\ & \nearrow^{\varphi'_2} & \downarrow{\scriptstyle p'_2} \\ X & \xrightarrow{\varphi'_1} & X'_1. \end{array}$$

such that

- $\pi_i(X'_k) = \{0\}$ for $i \geq k+1$;
- The map $\varphi'_k$ induces isomorphisms $\pi_i(\varphi'_k)\colon \pi_i(X) \xrightarrow{\cong} \pi_i(X'_k)$ for $1 \leq i \leq k$.

Then by turning a map into a fibration starting with $p'_2$ and working inductively upwards, we obtain the desired Postnikov tower.

There is a canoncial map from $X$ to the inverse limit $\operatorname{invlim}_{k\to\infty}\{\tau_{\leq k}X, p_k\}$ which is a weak homotopy equivalence.

Recall that $p_k\colon \tau_{\leq k}X \to \tau_{\leq k-1}X$ has a fiber $K(\pi_k(X), k)$. Suppose that $X$ is a simple space, i.e., the action of the fundamental group $\pi_1(X)$ is the homotopy groups $\pi_n(X)$ is trivial for $n \in \mathbb{Z}^{\geq 1}$. (Note that this implies that $\pi_1(X)$ is abelian.) Then one can actually extend $p_k$ to a fiber sequence

$$K(\pi_k(X), k) \to \tau_{\leq k}X \xrightarrow{p_k} \tau_{\leq k-1}X \to K(\pi_k(X), k+1).$$

It determines a class

$$[p_k] \in [\tau_{\leq k}X; K(\pi_k(X), k+1)] = H^{k+1}(\tau_{\leq k-1}X; \pi_k(X)), \tag{15.2}$$

called *$k$-invariant of the Postnikov tower* which determines $p_k$ up to strong fiber homotopy equivalence.

**Example 15.3.** The first few terms of the Postnikov tower for the sphere $S^2$ can be understood explicitly. The first homotopy groups of the sphere are given by

$$\pi_n(S^2) \cong \begin{cases} \{0\} & n = 0, 1; \\ \mathbb{Z} & n = 2, 3; \\ \mathbb{Z}/2 & n = 4. \end{cases}$$

Hence $\tau_{\leq 2}S^2$ is $K(\mathbb{Z}, 2)$ for which $\mathbb{CP}^\infty$ is a model. The fibration $p_3\colon \tau_{\leq 3}S^2 \to K(\mathbb{Z}/2)$ is classified by the 3-invariant which is an element in $H^4(\tau_{\leq 2}X; \pi_3(X)) \cong H^4(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$. This invariant is trivial if and only if $\tau_{\leq 3}X \simeq K(\mathbb{Z}/2, \mathbb{Z}) \times K(\mathbb{Z}, 3)$. Actually, it is known that the 3-invariant is non-trivial, see [26, Example 1 in IX.5 on page 437].

## 16. Spectra

16.1. **Basics about spectra.** Note that in the sequel we often omit the base points from the notation. Moreover, pointed space means always well pointed space. Recall that we are working in the category of compactly generated spaces.

**Definition 16.1** (Spectrum)**.** A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1)$$

for $n \in \mathbb{Z}$. A *map of spectra* $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$.

Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category, see [1, III.2.].

Note that we are not requiring that $E(n) = \{\bullet\}$ holds for $n \leq -1$.

**Example 16.2** (Suspension spectrum of a pointed space)**.** Given a pointed space $X$, define its *suspension spectrum* $\Sigma^\infty X$ by $\Sigma^\infty X(n) = \{\bullet\}$ for $n \leq -1$ and $\Sigma^\infty X(n) = X \wedge S^n$ for $n \geq 0$. Note that $\Sigma^\infty X(0) = X \wedge S^0$ can be identified with $X$ itself. Since $S^{n+1}$ can be identified with $S^n \wedge S^1$, we can define the $n$-structure map to be

$$\begin{aligned}\mathrm{id}_{(X\wedge S^n)\wedge S^1}\colon \Sigma^\infty X(n) \wedge S^1 &= (X \wedge S^n) \wedge S^1 \\ &\to (X \wedge S^n) \wedge S^1 = X \wedge (S^n \wedge S^1) = X \wedge S^{n+1} = \Sigma^\infty X(n+1).\end{aligned}$$

**Example 16.3** (Sphere spectrum)**.** If we take $X = S^0$ in Example 16.2, we obtain the *sphere spectrum* $\mathbf{S}$. Note that $S(n) = \{\bullet\}$ for $n \leq -1$ and $S(n) = S^n$ for $n \geq 0$ hold and that the $n$-th structure map comes from the identification $S^n \wedge S^1 = S^{n+1}$.

**Example 16.4** (Eilenberg-MacLane spectrum)**.** Given an abelian group $G$, we define the associated *Eilenberg Mac-Lane spectrum* $\mathbf{K}(G)$ as follows. We put $\mathbf{K}(G)(n) = \{\bullet\}$ for $n \leq 0$ and put $\mathbf{K}(G)(n) = K(G,n)$ for some model $K(G,n)$ of the Eilenberg MacLane space of type $(G,n)$ for $n \geq 1$. In order to define the $n$-th structure map for $n \geq 0$, it suffices to specify a map $\overline{\sigma(n)}\colon K(G,n) \to \Omega K(G,(n+1))$ because of the adjunction (9.32). Recall that we have a preferred isomorphism $\partial_{n+1}(K(G,n+1))\colon \pi_{n+1}(K(G,n+1)) \xrightarrow{\cong} \pi_n(\Omega K(G,n+1))$, see (10.7). We conclude from Theorem 14.3 (iii) that there is a homotopy equivalence $\overline{\sigma(n)}\colon K(G,n) \to \Omega K(G,n+1)$ which is uniquely determined by the property that under the identifications $\pi_n(K(G,n)) = G$ and $\pi_{n+1}(K(G,n+1)) = G$ the map $\pi_n(\overline{\sigma(n)})\colon \pi_n(K(G,n)) \to \pi_n(\Omega K(G,(n+1)))$ and the preferred isomorphism $\partial_{n+1}$ are inverse to one another.

**Definition 16.5** (Homotopy groups of a spectrum)**.** For $n \in \mathbb{Z}$ the *nth homotopy groups of a spectrum* $\mathbf{E}$ is defined by

$$\pi_n(\mathbf{E}) := \operatorname{colim}_{k\to\infty} \pi_{n+k}(E(k))$$

where the $k$th structure map of the system $\pi_{n+k}(E(k))$ is given by the composite

$$\begin{aligned}a_{n,k}(\mathbf{E})\colon \pi_{n+k}(E(k)) &\xrightarrow{\sigma_{n+k}(E(k))} \pi_{n+k+1}(S^1 \wedge E(k)) \\ &\xrightarrow{\pi_{n+k+1}(\mathrm{flip})} \pi_{n+k+1}(E(k) \wedge S^1) \\ &\xrightarrow{\pi_{n+k+1}(\sigma(k))} \pi_{n+k+1}(E(k+1))\end{aligned}$$

of the suspension homomorphism $\sigma_{n+k}(E(k))$ of (11.15), the map induced by the flip map $\mathrm{flip}\colon S^1 \wedge E(k) \xrightarrow{\cong} E(k) \wedge S^1$, and the homomorphism induced by the structure map $\sigma(k)$.

A *weak equivalence of spectra* is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

A spectrum $\mathbf{E}$ is called an $\Omega$-*spectrum* if the adjoint $\overline{\sigma(n)}\colon E_n \to \Omega E_{n+1}$ of $\sigma(n)$ under the adjunction (9.32) induces for every $n \geq 1$ and every $n \in \mathbb{Z}$ a bijection $\pi_n(\overline{\sigma(n)})\colon \pi_n(E(n)) \to \pi_n(\Omega E(n+1))$. The Eilenberg MacLane spectrum $\mathbf{K}(G)$ is an $\Omega$-spectrum.

Note that it may happen that $\pi_n(\mathbf{E})$ is non-trivial for some $n \in \mathbb{Z}$ with $n \leq -1$. Each of the groups $\pi_n(\mathbf{E})$ is abelian.

Note that for $k, n \in \mathbb{Z}$ with $k + n \geq 1$ there is a natural map

$$\psi_{k,n}\colon \pi_{k+n}(E(k)) \to \pi_n(\mathbf{E}). \tag{16.6}$$

In general this map is not bijective. It is bijective if $\mathbf{E}$ is an $\Omega$-spectrum since in view of the adjunction (9.32) one can compute $\pi_n(\mathbf{E})$ of a spectrum as the colimit of the sequence

$$\pi_n(E(0)) \xrightarrow{\pi_n(\overline{\sigma(0)})} \pi_n(\Omega E(1)) \xrightarrow{\pi_n(\Omega\overline{\sigma(1)})} \pi_n(\Omega^2 E(2)) \xrightarrow{\pi_n(\Omega\overline{\sigma(1)})} \cdots. \tag{16.7}$$

Note that our construction of the Eilenberg-Maclane spectrum $\mathbf{K}(G)$ of Example 16.4 depends on choices. In any case we get a preferred isomorphism $\pi_0(\mathbf{K}(G)) \xrightarrow{\cong} G$ and $\pi_n(K(G)) = \{0\}$ for $n \neq 0$. Moreover, for any two such constructions with different choices of $\mathbf{K}(G)$, there is a weak homotopy equivalence between the resulting spectra inducing the identity on $\pi_0$ under the identification of $\pi_0$ with $G$ above.

Note that for a pointed space $X$ the $n$-stable homotopy group $\pi_n^s(X_+)$ of Definition 11.20 agrees with $\pi_n^s(\Sigma^\infty X)$ of Definition 16.5.

Given a spectrum $\mathbf{E}$ and a pointed space $X$, we can define their *smash product* to be the spectrum $X \wedge \mathbf{E}$ whose $n$-th spaces is $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ and whose $n$-th structure map is $\mathrm{id}_X \wedge \sigma(n)\colon X \wedge E(n) \to X \wedge E(n+1)$. Next introduce the associated *mapping spectrum* $\mathrm{map}(X;\mathbf{E})^0$. Its $n$-th space is given by $\mathrm{map}(X;\mathbf{E})^0(n) = \mathrm{map}(X, E(n))^0$ for $n \in \mathbb{Z}$. Its $n$-th structure map for $n \in \mathbb{Z}$ is defined to be the composite

$$\mathrm{map}(X, E(n))^0 \xrightarrow{\mathrm{map}(\mathrm{id}_X, \overline{\sigma(n)})^0} \mathrm{map}(X;\Omega E(n+1))^0 \xrightarrow{i_n} \Omega\,\mathrm{map}(X, E(n+1)^0)$$

having the adjunction 9.32 in mind, Here $i_n$ is the homeomorphism which assigns to the pointed map $f\colon X \to \Omega E(n) = \mathrm{map}(S^1, E(n))^0$ the pointed map $g\colon S^1 \to \mathrm{map}(X, E(n))$ sending $s \in S^1$ to the map $X \to E(n)$, $x \mapsto f(x)(s)$. It can be also witten as the composite

$$\begin{aligned}\mathrm{map}(X, \Omega E(n))^0 &= \mathrm{map}(X, \mathrm{map}(S^1, E(n))^0)^0 \xrightarrow{\cong} \mathrm{map}(X \wedge S^1, E(n))^0 \\ &\xrightarrow{\cong} \mathrm{map}(S^1 \wedge X, E(n))^0 \xrightarrow{\cong} \mathrm{map}(S^1, \mathrm{map}(X, E(n))^0)^0 = \Omega\,\mathrm{map}(X, E(n))^0.\end{aligned}$$

16.2. **Homology and cohomology theories for pointed spaces and pairs.** Fix a commutative ring $R$.

**Definition 16.8** (Homology theory for pointed spaces)**.** A *homology theory for pointed spaces with values in $R$-modules* $\widetilde{\mathcal{H}}_* = (\widetilde{\mathcal{H}}_*, s_*)$ consists of a covariant functor $\widetilde{\mathcal{H}}_*$ from the category $\mathsf{Top}^0$ of pointed spaces to the category of $\mathbb{Z}$-graded $R$-modules together with a natural transformation $s_*\colon \widetilde{\mathcal{H}}_*(-) \to \widetilde{\mathcal{H}}_{*+1}(S^1 \wedge -)$ such that the following conditions are satisfied:

- *Pointed homotopy invariance*
  Let $f$ and $g$ be pointed maps $(X,x) \to (Y,y)$ which are pointed homotopic. Then for every $n \in \mathbb{Z}$ the $R$-homomorphisms $\widetilde{\mathcal{H}}_n(f)$ and $\widetilde{\mathcal{H}}_n(g)$ from $\widetilde{\mathcal{H}}_n(X,x)$ to $\widetilde{\mathcal{H}}_n(Y,y)$ agree:

- *Exactness*
Let $f\colon (X,x) \to (Y,y)$ be a pointed map. Let $i\colon Y \to \operatorname{cone}(f,x)$ be the inclusion into the pointed mapping cone of $f$. Then for every $n \in \mathbb{Z}$ the sequence of $R$-modules
$$\widetilde{\mathcal{H}}_n(X,x) \xrightarrow{\widetilde{\mathcal{H}}_n(f)} \widetilde{\mathcal{H}}_n(Y,y) \xrightarrow{\widetilde{\mathcal{H}}_n(i)} \widetilde{\mathcal{H}}_n(\operatorname{cone}(f),*)$$
is exact;
- *Suspension isomorphism*
For every pointed space $(X,x)$ and every $n \in \mathbb{Z}$ the map
$$s_n(X,x)\colon \widetilde{\mathcal{H}}_n(X,x) \xrightarrow{\cong} \widetilde{\mathcal{H}}_{n+1}(S^1 \wedge X,*)$$
is bijective.

We say that $\widetilde{\mathcal{H}}_*$ satisfies the *one point union axiom* if for any collection of pointed spaces $\{(X_i,x_i) \mid i \in I\}$ and every $n \in \mathbb{Z}$ the map
$$\bigoplus_{i \in I} \widetilde{\mathcal{H}}_n(j_i)\colon \bigoplus_{i \in I} \widetilde{\mathcal{H}}_n(X_i,x_i) \xrightarrow{\cong} \widetilde{\mathcal{H}}_n\big(\bigvee_{i \in I}(X_i,x_i)\big)$$
is bijective for $j_i\colon (X_i,x_i) \to \bigvee_{i \in I}(X_i,x_i)$ the inclusion of the $i$-th summand.

We say that $\widetilde{\mathcal{H}}_*$ satisfies the *dimension axiom* if we have $\widetilde{\mathcal{H}}_n(S^0,s) = 0$ for $n \neq 0$.

**Definition 16.9** (Cohomology theory for pointed spaces)**.** A *cohomology theory for pointed spaces with values in $R$-modules* $\widetilde{\mathcal{H}}^* = (\widetilde{\mathcal{H}}^*, s^*)$ consists of a contravariant functor $\widetilde{\mathcal{H}}_*$ from the category $\mathsf{Top}^0$ of pointed spaces to the category of $\mathbb{Z}$-graded $R$-modules together with a natural transformation $s^*\colon \widetilde{\mathcal{H}}^*(-) \to \widetilde{\mathcal{H}}^{*+1}(S^1 \wedge -)$ such that the following conditions are satisfied:

- *Pointed homotopy invariance*
Let $f$ and $g$ be pointed maps $(X,x) \to (Y,y)$ which are pointed homotopic. Then for every $n \in \mathbb{Z}$ the $R$-homomorphisms $\widetilde{\mathcal{H}}^n(f)$ and $\widetilde{\mathcal{H}}^n(g)$ from $\widetilde{\mathcal{H}}^n(Y,y)$ to $\widetilde{\mathcal{H}}^n(X,x)$ agree;
- *Exactness*
Let $f\colon (X,x) \to (Y,y)$ be a pointed map. Let $i\colon Y \to \operatorname{cone}(f,x)$ be the inclusion into the pointed mapping cone of $f$. Then for every $n \in \mathbb{Z}$ the sequence of $R$-modules
$$\widetilde{\mathcal{H}}^n(\operatorname{cone}(f,x),*) \xrightarrow{\widetilde{\mathcal{H}}^n(i)} \widetilde{\mathcal{H}}^n(Y,y) \xrightarrow{\widetilde{\mathcal{H}}^n(f)} \widetilde{\mathcal{H}}^n(X,x)$$
is exact;
- *Suspension isomorphism*
For every pointed space $(X,x)$ and every $n \in \mathbb{Z}$ the map
$$s^n(X,x)\colon \widetilde{\mathcal{H}}^n(X,x) \xrightarrow{\cong} \widetilde{\mathcal{H}}^{n+1}(S^1 \wedge X)$$
is bijective.

We say that $\widetilde{\mathcal{H}}^*$ satisfies the *one point union axiom* if for any collection of pointed spaces $\{(X_i,x_i) \mid i \in I\}$ and every $n \in \mathbb{Z}$ the map
$$\prod_{i \in I} \widetilde{\mathcal{H}}_n(j_i)\colon \widetilde{\mathcal{H}}_n\big(\bigvee_{i \in I}(X_i,x_i)\big) \to \prod_{i \in I} \widetilde{\mathcal{H}}^n(X_i,x_i)$$
is bijective for $j_i\colon (X_i,x_i) \to \bigvee_{i \in I}(X_i,x_i)$ the inclusion of the $i$-th summand.

We say that $\widetilde{\mathcal{H}}^*$ satisfies the *dimension axiom* if we have $\widetilde{\mathcal{H}}^n(S^0,s) = 0$ for $n \neq 0$.

A cohomology theory for pointed spaces is to be understood to be cohomology theory for pointed spaces with values in $\mathbb{Z}$-modules, and analogously for pairs and homology theories.

**Remark 16.10** (Correspondence between (co-)homology theories for pointed space and pairs)**.** There is a one-to-one correspondence between homology theories for pointed spaces with values in $R$-modules and homology theories for pairs with values in $R$-modules. Let $\widetilde{\mathcal{H}}_*$ be a homology theory for pointed spaces with values in $R$-modules. Then we can define a homology theory $\mathcal{H}_*$ for pairs with values in $R$-modules as follows. For a pair $(X,A)$ define

$$\mathcal{H}_n(X,A) := \widetilde{\mathcal{H}}_n(X_+ \cup_{A_+} \operatorname{cone}(A_+,*)),$$

where $X_+ = X \amalg \{*\}$ is the pointed space obtained from $X$ by adjoining an extra base point. If $A$ is empty, we get $\mathcal{H}_n(X) = \widetilde{\mathcal{H}}_n(X_+,*)$. Recall that we also have to specify for a pair $(X,A)$ a boundary operator $\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$ for $n \in \mathbb{Z}$. It is defined by the composite

$$\begin{aligned}\partial_n(X,A)\colon \mathcal{H}_n(X,A) = \widetilde{\mathcal{H}}_n(X_+ \cup_{X_+} \operatorname{cone}(A_+,*),*) &\xrightarrow{\mathcal{H}_n(\mathrm{pr})} \mathcal{H}_n(S^1 \wedge A,*) \\ &\xrightarrow{s_{n-1}(A_+,*)^{-1}} \widetilde{\mathcal{H}}_{n-1}(A_+,*) = \mathcal{H}_{n-1}(A)\end{aligned}$$

for the projection $\mathrm{pr}\colon X_+ \cup_{X_+} \operatorname{cone}(A_+,*) \to S^1 \wedge A_+$. We leave the elementary proof that $(\mathcal{H}_*, \partial_*)$ is a homology theory to the reader.

Given a homology theory for pairs with values in $R$-modules $(\mathcal{H}_*, \partial_*)$, we can define a homology theory for pointed spaces with values in $R$-modules $\widetilde{\mathcal{H}}_*$ by $\widetilde{\mathcal{H}}_*(X,x) = \mathcal{H}(X,\{x\})$. We leave the construction of the natural transformation $s_*(X,x)\colon \widetilde{\mathcal{H}}_n(X,x) \to \widetilde{\mathcal{H}}_{n+1}(S^1 \wedge X,*)$ and the proof that $(\widetilde{\mathcal{H}}_*, s_*)$ is a homology theory for pointed spaces with values in $R$-modules to the reader.

The analogous statements and construction yields a one-to-one-correspondence between cohomology theories for pointed spaces with values in $R$-modules and cohomology theories for pairs with values in $R$-modules.

More details can be found for instance in [24, Section 7.6 on page 176-177].

**Proposition 16.11.** *Let $X$ be a (compactly generated) topological Hausdorff space with a sequence of closed subspaces $X_0 \subset X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ such that $X$ is the union of the $X_i$-s and carries the colimit topology. Then:*

*(i) Suppose that the homology theory with values in $R$-modules $\mathcal{H}_*$ satisfies the disjoint union axiom for countable index sets.*
*Then there is for every $n \in \mathbb{Z}$ a natural $R$-isomorphism*

$$\operatorname{colim}_{k\to\infty} \mathcal{H}_n(X_k) \xrightarrow{\cong} \mathcal{H}_n(X);$$

*(ii) Suppose that the cohomology theory $\mathcal{H}^*$ with values in $R$-modules satisfies the disjoint union axiom for countable index sets. Then there is for every $n \in \mathbb{Z}$ a natural short exact sequence*

$$0 \to \operatorname{invlim}^1_{k\to\infty} \mathcal{H}^{n-1}(X_k) \to \mathcal{H}^n(X) \to \operatorname{invlim}_{k\to\infty} \mathcal{H}_n(X_k) \to 0.$$

*Proof.* The proof can be found in [20, Proposition 7.53 on page 121 and Proposition 7.66 on page 127] in the special case that $X$ is a $CW$-complex and $X_k$ is its $k$-skeleton. The proof carries directly over to our more general setting. □

**Proposition 16.12.**

*(i) Let $\mathbf{t}_*\colon \mathcal{H}_* \to \mathcal{K}_*$ be a transformation of homology theories with values in $R$-modules satisfying the disjoint union axiom. Suppose that the homomorphism $\mathbf{t}_n(\{\bullet\})\colon \mathcal{H}_n(\{\bullet\}) \to \mathcal{H}_n(\{\bullet\})$ is bijective for all $n \in \mathbb{Z}$.*
*Then $\mathbf{t}_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{K}_n(X,A)$ is bijective for every $CW$-pair $(X,A)$ and $n \in \mathbb{Z}$;*

(ii) *Let* $\mathbf{t}^*\colon \mathcal{H}^* \to \mathcal{K}^*$ *be a transformation of cohomology theories with values in* $R$*-modules satisfying the disjoint union axiom. Suppose that the homomorphism* $\mathbf{t}^n(\{\bullet\})\colon \mathcal{H}^n(\{\bullet\}) \to \mathcal{K}^n(\{\bullet\})$ *is bijective for all* $n \in \mathbb{Z}$.

*Then* $\mathbf{t}^n(X,A)\colon \mathcal{H}^n(X,A) \to \mathcal{K}^n(X,A)$ *is bijective for every* $CW$*-pair* $(X,A)$ *and* $n \in \mathbb{Z}$.

*Proof.* By the long exact sequence of a pair and the Five-Lemma one can reduce the claim to the case $A = \emptyset$. The claim follows for zero-dimensional $CW$-complexes $X$ from the disjoint union axiom. Inductively over the dimension one proves the claim for finite-dimensional $CW$-complexes, where in the induction step the Mayer-Vietoris sequence and homotopy invariance comes in. Using Proposition 16.11 one obtains the general case from the finite-dimensional case. □

16.3. **The homology and cohomology theory assigned to a spectrum.**

**Lemma 16.13.** *For a spectrum* $\mathbf{E}$ *and* $n \in \mathbb{Z}$ *there are equivalences*

$$\pi_n(\mathbf{E}) \simeq \pi_{n+1}(S^1 \wedge \mathbf{E}) \quad \text{and} \quad \pi_n(\mathbf{E}) \simeq \pi_{n-1}(\operatorname{map}(S^1,\mathbf{E})^0) \tag{16.14}$$

*which are natural in* $\mathbf{E}$.

*Proof.* We claim that the maps

$$(-1)^k\sigma_{n+k}(E(k))\colon \pi_{n+k}(E(k)) \to \pi_{n+k+1}(S^1 \wedge E(k)) \tag{16.15}$$

assemble to an equivalence

$$\begin{aligned} \pi_n(\mathbf{E}) &= \operatorname{colim}_{k\to\infty} \pi_{n+k}(E(k)) \\ &\to \operatorname{colim}_{k\to\infty} \pi_{n+k+1}(S^1 \wedge E(k)) = \pi_{n+1}(S^1 \wedge \mathbf{E}). \end{aligned} \tag{16.16}$$

The following commutative diagram shows that the maps $(-1)^k\sigma_{n+k}(E(k))$ commute with the structure maps of the respective colimits and induce the map (16.16):

$$\begin{array}{ccc}
\pi_{n+k}(E(k)) & \xrightarrow{-\sigma_{n+k}(E(k))} & \pi_{n+k+1}(S^1 \wedge E(k)) \\
\downarrow{\scriptstyle \sigma_{n+k}(E(k))} & \overset{-\operatorname{id}}{\dashrightarrow} & \downarrow{\scriptstyle \sigma_{n+k+1}(S^1 \wedge E(k))} \\
\pi_{n+k+1}(S^1 \wedge E(k)) & & \pi_{n+k+2}(S^1 \wedge (S^1 \wedge E(k))) \\
\downarrow{\scriptstyle \pi_{n+k+1}(\mathrm{flip})} & & \downarrow{\scriptstyle \pi_{n+k+2}(\mathrm{flip})} \\
\pi_{n+k+1}(E(k) \wedge S^1) & & \pi_{n+k+2}((S^1 \wedge E(k)) \wedge S^1) \\
\downarrow{\scriptstyle \pi_{n+k+1}(\sigma(k))} & & \downarrow{\scriptstyle \pi_{n+k+2}(\sigma(k))} \\
\pi_{n+k+1}(E(k+1)) & \xrightarrow{\sigma_{n+k+1}(E(k+1))} & \pi_{n+k+2}((S^1 \wedge E(k)) \wedge S^1).
\end{array} \tag{16.17}$$

To prove commutativity of diagram 16.17, one starts with an element $[f] \in \pi_{n+k}(E(k))$ represented by a pointed map $f\colon S^{n+k} \to E(k)$. Its image under the composite of the left vertical arrows is $[g] \in \pi_{n+k+2}(S^1 \wedge E(k+1))$ for the pointed map

$$\begin{aligned} g\colon S^1 \wedge S^1 \wedge S^{n+k} &\xrightarrow{\operatorname{id}_{S^1} \wedge \operatorname{id}_{S^1} \wedge f} S^1 \wedge S^1 \wedge E(k) \\ &\xrightarrow{\operatorname{id}_{S^1} \wedge \mathrm{flip}} S^1 \wedge (E(k) \wedge S^1) \xrightarrow{\operatorname{id}_{S^1} \wedge \sigma(k)} S^1 \wedge E(k+1). \end{aligned}$$

However, the image of $[f] \in \pi_{n+k}(E(k))$ under the composite of the right vertical arrows is $[g] \in \pi_{n+k+2}(S^1 \wedge E(k+1))$ is $g \circ (\mathrm{flip} \wedge \operatorname{id}_{S^{n+k}})$ (and not $[g]$). Since the homomorphism $\mathrm{flip} \wedge \operatorname{id}_{S^{n+k}}\colon S^1 \wedge S^1 \wedge S^{n+k} \to S^1 \wedge S^1 \wedge S^{n+k}$ has degree $-1$, Theorem 3.4 implies that $[g \circ (\mathrm{flip} \wedge \operatorname{id}_{S^{n+k}})] = -[g]$ holds.

It remains to show that the map (16.16) is an equivalence. Recall the general fact that for a directed system of abelian groups $A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$ every element in the colimit can be written as $\psi_m(a_m)$ for some $m \in \mathbb{Z}^{\geq 0}$ and some

$a_m \in A_m$ for the structure map $\psi_m \colon A_m \to \operatorname{colim}_{k\to\infty} A_k$ and the element $\psi_m(a_m)$ is zero in the colimit if and only if there exists $n \in \mathbb{Z}^{\geq 0}$ with $n \geq m$ such that the composite $\phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m \colon A_m \to A_n$ sends $a_m$ to zero. Now note that the upper left triangle in diagram 16.17 with the dashed map commutes, which implies injectivity of (16.16). Surjectivity is more difficult. We only show that the twofold suspension $\pi_n(\mathbf{E}) \to \pi_{n+1}(S^1 \wedge \mathbf{E}) \to \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$ is surjective. As both maps are injective this proves that they are equivalences. Consider $f \colon S^{n+k} \to S^2 \wedge E(k)$. We will show that $[f] \in \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$ is the image of $[g] \in \pi_n(\mathbf{E})$ for the map

$$g \colon S^{n+k+2} \xrightarrow{f} S^2 \wedge E(k) \xrightarrow{\text{flip}} E(k) \wedge S^2 \xrightarrow{\sigma^2} E(k+2). \tag{16.18}$$

Consider the the diagram

$$\begin{array}{ccc}
S^2 \wedge S^{n+k+2} & \xrightarrow{\quad\text{flip}\quad} & S^{n+k+2} \wedge S^2 \\
\big\downarrow{\scriptstyle \mathrm{id} \wedge f} & & \big\downarrow{\scriptstyle f \wedge \mathrm{id}} \\
S^2 \wedge (S^2 \wedge E(k)) & \xrightarrow{\quad\text{flip}\quad} & (S^2 \wedge E(k)) \wedge S^2 \\
\big\downarrow{\scriptstyle \mathrm{id} \wedge \text{flip}} & \swarrow{\scriptstyle \alpha} & \big\downarrow{\scriptstyle \mathrm{id} \wedge \sigma^2} \\
S^2 \wedge E(k) \wedge S^2 & \xrightarrow{\quad\mathrm{id} \wedge \sigma^2\quad} & S^2 \wedge E(k+2).
\end{array} \tag{16.19}$$

The map $\alpha \colon S^2 \wedge E(k) \wedge S^2 \to S^2 \wedge E(k) \wedge S^2$ is the map swapping the first and last factor. As this swap map of $S^2 \wedge S^2$ has degree 1, we see that $\alpha \simeq \mathrm{id}$. Similarly, the flip map $S^2 \wedge S^{n+k+2} \to S^{n+k+2} \wedge S^2$ is homotopic to the identity. This is the reason why we have to suspend twice! Now the upper square and middle triangle in diagram 16.19 obviously commute. The bottom right triangle commutes up to homotopy using $\mathrm{id} \simeq \alpha$. The right vertical composite represents $[f] \in \pi_{n+2}(S^2 \wedge E)$. The bottom left composite is the image of $[g]$ under $\pi_n(\mathbf{E}) \to \pi_{n+2}(S^1 \wedge S^1 \wedge \mathbf{E})$.

The proof of the equivalence $\pi_n(\mathbf{E}) \simeq \pi_{n-1}(\operatorname{map}(S^1, \mathbf{E})^0)$ is easy using the equivalence $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$ for a pointed space $X$. □

**Definition 16.20.** Let $\mathbf{E}$ be a spectrum and $X$ be a pointed space. Define the *reduced* $\mathbf{E}$*-(co)homology of* $X$ by

$$\widetilde{H}_n(X; \mathbf{E}) = \pi_n(X \wedge \mathbf{E}) \quad \text{and} \quad \widetilde{H}^n(X; \mathbf{E}) = \pi_{-n}(\operatorname{map}(X, \mathbf{E})^0).$$

**Theorem 16.21** (The homology and cohomology theory assigned to a spectrum)**.**

*Let* $\mathbf{E}$ *be a spectrum. Then:*

*(i) The reduced* $\mathbf{E}$*-homology* $\widetilde{H}_*(-, \mathbf{E})$ *is a homology theory of pointed spaces with values in* $\mathbb{Z}$*-modules. Its associated homology theory on pairs of spaces satisfies the disjoint union axiom. For every* $n \in \mathbb{Z}$ *there is an isomorphism*

$$\alpha_n(\mathbf{E}) \colon H_n(\{\bullet\}; \mathbf{E}) \xrightarrow{\cong} \pi_n(\mathbf{E});$$

*(ii) The reduced* $\mathbf{E}$*-cohomology* $\widetilde{H}^*(-; \mathbf{E})$ *is a cohomology theory of pointed spaces with values in* $\mathbb{Z}$*-modules. Its associated cohomology theory on pairs of spaces satisfies*

$$\alpha^n(\mathbf{E}) \colon H^n(\{\bullet\}; \mathbf{E}) \xrightarrow{\cong} \pi_{-n}(\mathbf{E}).$$

*If* $\mathbf{E}$ *is an* $\Omega$*-spectrum and we consider as input only pointed CW-complexes, then* $H^*(-; \mathbf{E})$ *satisfies the disjoint union axiom.*

*Proof.* (i). Because of Remark 16.10 it suffices to construct a homology theory for pointed spaces $\widetilde{\mathcal{H}}_*$.

We begin by showing how $\widetilde{H}_n(X;\mathbf{E}) = \pi_n(X \wedge \mathbf{E})$ from definition 16.20 is a homology theory for pointed spaces. Is is obviously a functor from pointed spaces to $\mathbb{Z}$-graded abelian groups. Furthermore, it sends pointed homotopic maps to the same map on $\widetilde{H}_n(-;\mathbf{E})$ As suspension isomorphism we use the isomorphism

$$s_n(x)\colon \widetilde{H}_n(X;\mathbf{E}) = \pi_n(X \wedge \mathbf{E}) \simeq \pi_{n+1}(S^1 \wedge X \wedge \mathbf{E}) = \widetilde{H}_{n+1}(X;\mathbf{E})$$

from lemma 16.13 for the spectrum $X \wedge \mathbf{E}$, which is natural in $X$.

Next we prove exactness. Consider a pointed map $f\colon X \to Y$. We have to prove the exactness of the sequence

$$\pi_n(X \wedge \mathbf{E}) \xrightarrow{\pi_n(f\wedge \mathrm{id}_{\mathbf{E}})} \pi_n(Y \wedge \mathbf{E}) \xrightarrow{\pi_n(i\wedge \mathrm{id}_{\mathbf{E}})} \pi_n(\mathrm{cone}(f) \wedge \mathbf{E})$$

for $i\colon Y \to \mathrm{cone}(f)$ the inclusion into the pointed mapping cone of $f$. Since the composite $i \circ f$ is pointed nullhomotopic, we get $\mathrm{im}(\pi_n(f \wedge \mathrm{id}_{\mathbf{E}})) \subseteq \ker(\pi_n(i \wedge \mathrm{id}_{\mathbf{E}}))$. It remains to show $\ker(\pi_n(i \wedge \mathrm{id}_{\mathbf{E}})) \subseteq \mathrm{im}(\pi_n(f \wedge \mathrm{id}_{\mathbf{E}}))$.

Consider an element $z \in \ker(\pi_n(i \wedge \mathrm{id}_{\mathbf{E}}))$. Then we can find $k \in \mathbb{Z}^{\geq 0}$ and a pointed map $g\colon S^{n+k} \to Y \wedge E(k)$ such that $[g] \in \pi_{n+k}(Y \wedge E(k))$ represents $z \in \ker(\pi_n(i \wedge \mathrm{id}_{\mathbf{E}}))$ and $\pi_{n+k}(i \wedge \mathrm{id}_{E(k)})\colon \pi_{n+k}(Y \wedge E(k)) \to \pi_{n+k}(\mathrm{cone}(f) \wedge E(n))$ sends $[g]$ to zero. Let $h\colon \mathrm{cone}(f) \wedge E(n) \wedge I \to \mathrm{cone}(f) \wedge E(n)$ be a pointed homotopy with $h_0 = i \circ g$ and $h_1$ the constant map. Next we construct the following diagram

(16.22)
$$\begin{array}{ccccc}
S^{n+k} & \xrightarrow{g} & Y \wedge E(n) & \overset{i\wedge \mathrm{id}_{E(n)}}{\searrow} & \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i\wedge \mathrm{id}_{E(n)}} & & \\
\mathrm{cone}(\mathrm{id}_{S^{n+k}}) & \xrightarrow{H} & \mathrm{cone}(f \wedge \mathrm{id}_{E(n)}) & \xrightarrow[\cong]{\phi} & \mathrm{cone}(f) \wedge E(n) \\
\downarrow{\scriptstyle p(\mathrm{id}_{S^n})} & & \downarrow{\scriptstyle p(f\wedge \mathrm{id}_{E(n)})} & & \\
S^{n+k} \wedge S^1 & \xrightarrow{\beta} & X \wedge E(n) \wedge S^1 & \xrightarrow{\mathrm{id}_X \wedge \sigma(n)} & X \wedge E(n+1) \\
\downarrow{\scriptstyle \mathrm{id}_{S^{n+k}\wedge S^1}} & & \downarrow{\scriptstyle f\wedge \mathrm{id}_{E(n)} \wedge \mathrm{id}_{S^1}} & & \downarrow{\scriptstyle f\wedge \mathrm{id}_{E(n+1)}} \\
S^{n+k} \wedge S^1 & \xrightarrow{g\wedge \mathrm{id}_{S^1}} & Y \wedge E(n) \wedge S^1 & \xrightarrow{\mathrm{id}_Y \wedge \sigma(n)} & Y \wedge E(n+1)
\end{array}$$

The left column is part of the cofibration sequence of the pointed map $\mathrm{id}_{S^n}$, whereas the middle column is part of the cofibration sequence of the pointed map $f \wedge \mathrm{id}_{E(n)}$, see Theorem 8.40. The map $H$ is given by the map $g$ and the homotopy $h$ and makes the uppermost left square commutative. The map $\beta$ is the map uniquely determined by the property that the left middle square commutes. The map $\phi$ is the canoncial homeomorphism and makes the corresponding triangle commutative. The lowermost right square commutes. The left lowermost square does not commute but it does commute up to pointed homotopy. The elementary verification of this fact is left to the reader or can be extracted from [20, Lemma 8.31 on page 143].

Now the composite $(\mathrm{id}_Y \wedge \sigma(n)) \circ (g \wedge \mathrm{id}_{S^1})\colon S^{n+k} \wedge S^1 \to Y \wedge E(n+1)$ is another representative of $z \in \ker(\pi_n(i \wedge \mathrm{id}_{\mathbf{E}}))$. We conclude from the diagram 16.22 that the composite $(\mathrm{id}_X \wedge \sigma(n)) \circ \beta\colon S^{n+k} \wedge S^1 \to X \wedge E(n+1)$ represents an element in $\pi_n(X \wedge \mathbf{E})$ which sent by $\pi_n(f \wedge \mathrm{id}_{\mathbf{E}})$ to $z$. This finishes the proof of exactness and hence of the assertion that $(\widetilde{\mathcal{H}}_*(-;\mathbf{E}), s_*)$ defines homology theory for pointed spaces in the sense of Defintion 16.9.

It remains to check that $\widetilde{\mathcal{H}}_*(-;\mathbf{E})$ satisfies the one point union axiom, i.e., that the map

(16.23)
$$\bigoplus_{i\in I} \pi_n(X_i \wedge E) \to \pi_n(\bigvee_{i\in I} X_i \wedge E)$$

is an equivalence. Recall from exercise 47 on sheet 12 that for two spectra $\mathbf{E}_1$ and $\mathbf{E}_2$ the canonical map $\mathbf{E}_1 \vee \mathbf{E}_2 \to \mathbf{E}_1 \times \mathbf{E}_2$ is a weak homotopy equivalence. This shows that the map (16.23) is an equivalence if $I$ is finite. For general $I$, use the equivalences

$$\pi_n(\bigvee_{i\in I} X_i \wedge E) \simeq \operatorname{colim}_{F\subseteq I \text{ finite}} \pi_n(\bigvee_{i\in F} X_i \wedge E) \quad \text{and} \tag{16.24}$$

$$\bigoplus_{i\in I} \pi_n(X_i \wedge E) \simeq \operatorname{colim}_{F\subseteq I \text{ finite}} \bigoplus_{i\in F} \pi_n(X_i \wedge E). \tag{16.25}$$

The first equivalence follows from the following argument: Consider a compact subset $C \subseteq \bigvee_{i\in I}(X_i, x_i)$. We want to show that there is a finite subset $J \subseteq I$ with $C \subseteq I$. Suppose that this is not the case. Then we can find a sequence of elements $j(1), j(2), j(3), \ldots$ of pairwise distincts elements in $I$ and a sequence of pairwise distinct points $s_1, s_2, s_3, \ldots$ in $C$ satisfying $s_i \in C \cap X_{j(i)} \setminus \{x_{j(i)}\}$. Consider the set $S = \{s_1, s_2, s_3, \ldots, \}$. Let $T \subseteq S$ be any subset of $S$. Then $T \cap X_i$ is either empty of consists of one point for $i \in I$. Since each $X_i$ is Hausdorff, $T \cap X_i$ is closed in $X_i$ for every $i \in I$. This implies that $T$ is a closed subset of $\bigvee_{i\in I} X_i$. Hence $S$ is a discrete subset of $\bigvee_{i\in I} X_i$ and contained in a compact subset $C$ of $X$. This implies that $S$ is finite, a contradiction.

The proof that the reduced $\mathbf{E}$-cohomology

$$\widetilde{\mathcal{H}}^n(X;\mathbf{E}) = \pi_{-n}(\operatorname{map}(X,\mathbf{E})^0)$$

is a cohomology theory for pointed spaces is analogous to the one for homology except that some care is necessary for the disjoint union axiom. The additional difficulty is that we have a homeomorphism

$$\operatorname{map}\bigl(\bigvee_{i\in I} X_i; \mathbf{E}(n)\bigr)^0 = \prod_{i\in I} \operatorname{map}(X_i, \mathbf{E}(n))^0$$

and hence we get for $k, n$ an isomorphism

$$\pi_{n+k}\bigl(\operatorname{map}\bigl(\bigvee_{i\in I} X_i; \mathbf{E}(n)\bigr)^0\bigr) \xrightarrow{\cong} \prod_{i\in I} \pi_{n+k}\bigl(\operatorname{map}(X_i; \mathbf{E}(n))^0\bigr)$$

but colimits and products do not commute. Therefore we need the assumption that $\mathbf{E}$ is an $\Omega$-spectrum, Namely, with this assumption, structure map of (16.6)

$$\psi_{n,k}\colon \pi_{n+k}(\operatorname{map}(Y,\mathbf{E}(k))^0) \to \pi_n(\operatorname{map}(Y,\mathbf{E})^0)$$

is an isomorphism for every pointed space $(Y, y)$ and we do not have to take the colimit, since $\operatorname{map}(X;\mathbf{E})^0$ is an $\Omega$-spectrum by Theorem 16.6 and Theorem 10.1 applied to the fibration $\operatorname{map}(X, E(n))^0 \to \operatorname{map}(X, E(n)) \to E(n)$, as $\mathbf{E}$ is an $\Omega$-spectrum and $X$ is a $CW$-complex. □

**Example 16.26** (Sphere spectrum and stable homotopy)**.** Let $\mathbf{S}$ be the sphere spectrum of Example 16.3. Then the associated homology theory $\mathcal{H}_*(-;\mathbf{S})$ agrees with the stable homotopy theory $\pi_*^s(-)$ introduced in Defintion 11.20 and Theorem 11.24 follows from Theorem 16.21 (i).

**Example 16.27** (The Eilenberg-MacLane spectrum and singular homology)**.** We have introduced for an abelian group $G$ the Eilenberg-Maclane spectrum $\mathbf{K}(G)$ in Definition 16.4. Theorem 16.21 (ii) we obtain a cohomology theory which satisfies the disjoint union axiom and the dimension axiom and $\mathcal{H}^0(\{\bullet\}) \cong G$. Singular cohomology $H^*(-;G)$ with coefficients in $G$ is also a cohomology theory which satisfies the disjoint union axiom and the dimension axiom and $H^0(\{\bullet\};G) \cong G$. We obtain from the maps $\gamma^n(Y)\colon [Y, K(G,n)] \to H^n(Y;G)$ of (14.6) a natural transformation of cohomology theories $\gamma^*\colon \mathcal{H}^*(-;\mathbf{K}(G)) \to H^*(;G)$ which induces an isomorphism $\gamma^0(\{\bullet\})\colon \mathcal{H}^*(\{\bullet\};\mathbf{K}(G)) \to H^*(\{\bullet\};G)$. Proposition 16.12 (ii) implies that we get a natural equivalence of cohomology theories

$$\gamma^*\colon \mathcal{H}^*(-;\mathbf{K}(G)) \xrightarrow{\cong} H^*(-;G).$$

In particular we see that for every $n \in \mathbb{Z}$ the map $\gamma^n(X)\colon [X, K(G,n)] \to H^n(Y;G)$ of (14.6) bijective for every $CW$-complex $X$, as predicted in Theorem 14.7.

We mention without proof that $\mathcal{H}_*(-;\mathbf{K}(A))$ can be identified with singular homology $H_*(-;A)$ with coefficients in $A$.

**Example 16.28** (Hopf's Theorem revisited)**.** Let $M$ be a closed smooth manifold of dimension $d$. Let $K(\mathbb{Z},d)$ be a model for the Eilenberg-MacLane space of type $(\mathbb{Z},d)$. Choose a map $f\colon S^d \to K(\mathbb{Z},d)$ inducing an isomorphism $\pi_d(f)\colon \pi_d(S^d) \xrightarrow{\cong} \pi_d(K(\mathbb{Z},d))$. Since $f$ is $(d+1)$-connected and any smooth $d$-dimensional manifold carries a $d$-dimensional $CW$-structure, we obtain a bijection $[M,S^d] \xrightarrow{\cong} [M,K(\mathbb{Z},d)]$ by sending $[g]$ to $[f \circ g]$ from the Whitehead Theorem 5.1 (i). Composing it with the bijection $\gamma^n(Y)\colon [Y,K(G,n)] \xrightarrow{\cong} H^d(Y;G)$ of (14.6) yields a bijection

$$\nu\colon [M,S^d] \xrightarrow{\cong} H^d(M), \quad [f] \mapsto H^d(f)([S^d])$$

for the fundamental class $[S^d] \in H^d(S^d)$.

Suppose that $M$ is oriented. If we compose $\nu$ with the bijective homomorphism $H^d(M) \xrightarrow{\cong} \mathbb{Z}$ sending $u$ to $\langle u,[M]\rangle$ for the fundamental class $[M] \in H_d(M)$, then we obtain a bijection

$$[M,S^d] \xrightarrow{\cong} \mathbb{Z}, \quad [f] \mapsto \deg(f).$$

Thus we rediscover Hopf's Degree Theorem 3.1.

Suppose that $M$ is not orientable. We mention without giving the proof that $H^d(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ and $\nu\colon [M,S^d] \xrightarrow{\cong} \mathbb{Z}/2$ sends $f\colon M \to S^d$ to zero, if $H_d(f,\mathbb{Z}/2)\colon H_d(M,\mathbb{Z}/2) \to H_d(S^d;\mathbb{Z}/2)$ is trivial, and to the generator if $H_d(f,\mathbb{Z}/2)\colon H_d(M,\mathbb{Z}/2) \to H_d(S^d;\mathbb{Z}/2)$ is bijective.

16.4. **Brown's Representation Theorem.** Let $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ be a map of spectra. It induces in the obvious way a natural transformation of homology theories with values in $\mathbb{Z}$-modules

$$t_*^{\mathbf{f}}\colon \mathcal{H}_*(-;\mathbf{E}) \to \mathcal{H}_*(-;\mathbf{F}) \tag{16.29}$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{H}_n(\{\bullet\};\mathbf{E}) & \xrightarrow{t_n^{\mathbf{f}}(\{\bullet\})} & \mathcal{H}_n(\{\bullet\};\mathbf{F}) \\
\cong\downarrow & & \downarrow\cong \\
\pi_n(\mathbf{E}) & \xrightarrow[\pi_n(\mathbf{f})]{} & \pi_n(\mathbf{E}).
\end{array}$$

It also induces in the obvious way a natural transformation of cohomology theories with values in $\mathbb{Z}$-modules

$$t^*_{\mathbf{f}}\colon \mathcal{H}^*(-;\mathbf{E}) \to \mathcal{H}^*(-;\mathbf{F}) \tag{16.30}$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{H}^n(\{\bullet\};\mathbf{E}) & \xrightarrow{t^n_{\mathbf{f}}(\{\bullet\})} & \mathcal{H}^n(\{\bullet\};\mathbf{F}) \\
\cong\downarrow & & \downarrow\cong \\
\pi_n(\mathbf{E}) & \xrightarrow[\pi_n(\mathbf{f})]{} & \pi_n(\mathbf{E}).
\end{array}$$

A proof of the next theorem can be found in [20, Theorem 9.27 on page 164 and Theorem 9.28 on page 165]

**Theorem 16.31** (Brown's Representation Theorem).

(i) *Let $\mathcal{K}^*$ be a cohomology theory with values in $\mathbb{Z}$-modules defined on the category of CW-pairs satisfying the disjoint union axiom. Then there is an $\Omega$-spectrum $\mathbf{E}$ and a natural equivalence of cohomology theories*

$$t^*\colon \mathcal{H}^*(-;\mathbf{E}) \xrightarrow{\cong} \mathcal{K}^*;$$

(ii) *Consider two $\Omega$-spectra $\mathbf{E}$ and $\mathbf{F}$. Let $t^*\colon \mathcal{H}^*(-;\mathbf{E}) \to \mathcal{H}^*(-;\mathbf{F})$ be a natural*

*transformation of cohomology theories.*

*Then there is a map of spectra $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ such that for every CW-pair $(X,A)$ and $n \in \mathbb{Z}$ the maps $t^n(X,A)$ and $\mathcal{H}^n(X,A;\mathbf{f})$ from $\mathcal{H}^n(-;\mathbf{E})$ to $\mathcal{H}^n(-;\mathbf{F})$ agree.*

*If $t^*\colon \mathcal{H}^*(-;\mathbf{E}) \to \mathcal{H}^*(-;\mathbf{F})$ is a natural equivalence of cohomology theories with values in $\mathbb{Z}$-modules, then $\mathbf{f}$ is a weak homotopy equivalence.*

One knows for a homology theory $\mathcal{H}_*$ with values in $\mathbb{Z}$-modules satisfying the disjoint union that it can be identified on *CW*-pairs with $\mathcal{H}_*(-;\mathbf{E})$ for some spectrum $\mathbf{E}$ but in contrast to cohomology with values in $\mathbb{Z}$-modules $\mathbf{E}$ is not uniquely determined by this property up to weak homotopy equivalence.

16.5. **Basics about vector bundles.** Vector bundles are always to be understood to be finite dimensional real or complex vector bundles. For a vector bundle $\xi$ we denote by $p_\xi\colon E \to B$ its bundle projection. For a finite dimensional real or complex vector space $V$ and a *CW*-complex $B$ we denote by $\underline{V}_B$ the trivial vector bundle over $B$ whose bundle projection $B \times V \to B$ is the canonical projection onto $B$. If $B$ is clear from the context, we simply write $\underline{V}$. If $V$ is oriented, then $\underline{V}$ inherits an orientation. We will equip $\mathbb{R}^k$ always with the standard orientation and $\mathbb{C}^k$ considered as a real vector spaces with the preferred orientation coming from $\{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n\}$ for any complex basis $\{v_1, v_2, \ldots, v_n\}$. Given two bundles $\xi$ and $\eta$ with projections $p_\xi\colon E_\xi \to B_\xi$ and $p_\eta\colon E_\eta \to B_\eta$, a bundle morphism $(f, \overline{f})$ consists of two maps for which the following diagram is commutative

$$\begin{array}{ccc} E_\xi & \xrightarrow{\overline{f}} & E_\eta \\ \big\downarrow{\scriptstyle p_\xi} & & \big\downarrow{\scriptstyle p_\eta} \\ B_\xi & \xrightarrow{f} & B_\eta \end{array}$$

and for each $b \in B_\xi$ the maps induced by $\overline{f}$ from the fiber $p_\xi^{-1}(b)$ of $\xi$ over $b$ to the fiber $p_\eta^{-1}(f(b))$ of $\eta$ over $f(b)$ is a linear isomorphism. We call two bundles $\xi$ and $\eta$ over the same basis space $B$ *isomorphic over* $B$, if there is a bundle map $(f, \overline{f})$ with $f = \mathrm{id}_B$.

The proof of the next result can be found for instance in [9, Theorem 4.7 on page 30].

**Proposition 16.32.** *Let $X$ and $Y$ be CW-complexes. Let $\eta$ be a vector bundle over the CW-complex $Y$. Let $f, g\colon X \to Y$ be maps which are homotopic.*

*Then the vector bundles $f^*\eta$ and $g^*\eta$ over $X$ obtained from the pull back construction applied to $f$ and $g$ are isomorphic over $X$. Moreover, if $\eta$ is oriented, then $f^*\eta$ and $g^*\eta$ inherits orientations and are oriented isomorphic over $X$.*

Let $X$ be a *CW*-complex and let $\mathrm{VB}_k(X)$ be the set of isomorphism classes $[\xi]$ of $k$-dimensional real vector bundles $\xi$ over $X$. There is a universal $k$-dimensional bundle $\gamma_k$ over a $CW$-complex $\mathrm{BO}(k)$ such that for any $k$-dimensional real vector bundle $\xi$ there is a map $c_\xi\colon X \to \mathrm{BO}(k)$ uniquely determined up to homotopy by the property that $\xi$ is isomorphic over $X$ to $c_\xi^*\gamma_k$. Moreover, the homotopy class $[c_\xi]$ depends only on the isomorphism class $[\xi]$. Sometimes $c_\xi$ is called the *classifying map of* $\xi$. The space $\mathrm{BO}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\mathrm{O}(k)$-bundle $\mathrm{EO}(k) \to \mathrm{BO}(k)$ with contractible total space $\mathrm{EO}(k)$. The $k$-dimensional vector bundle $\gamma_k$ is given by the canoncial projection $\mathrm{EO}(k) \times_{\mathrm{O}(k)} \mathbb{R}^k \to \mathrm{BO}(k)$.

All this has an analog for oriented $k$-dimensional real vector bundles. Namely, let $X$ be a $CW$-complex and let $\overline{\mathrm{VB}}_k(X)$ be the set of oriented isomorphism classes $[\xi]$ of oriented $k$-dimensional real vector bundles $\xi$ over $X$. There is a universal oriented $k$-dimensional real bundle $\overline{\gamma}_k$ over a $CW$-complex $\mathrm{BSO}(k)$ such that for any oriented $k$-dimensional real vector bundle $\xi$ there is a map $\overline{c}_\xi\colon X \to \mathrm{BSO}(k)$ uniquely determined up to homotopy by the property that $\xi$ is oriented isomorphic over $X$ to $\overline{c}_\xi^*\overline{\gamma}_k$. Moreover, the homotopy class $[\overline{c}_\xi]$ depends only on the oriented isomorphism class $[\xi]$. The space $\mathrm{BSO}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\mathrm{SO}(k)$-bundle $\mathrm{ESO}(k) \to \mathrm{BSO}(k)$ with contractible total space $\mathrm{ESO}(k)$. The oriented $k$-dimensional real vector bundle $\overline{\gamma}_k$ is given by the canoncial projection $\mathrm{ESO}(k) \times_{\mathrm{SO}(k)} \mathbb{R}^k \to \mathrm{BSO}(k)$ and inherits an orientation from the standard orientation of $\mathbb{R}^k$.

All this has an analog for $k$-dimensional complex vector bundles. Namely, let $X$ be a $CW$-complex and let $\mathrm{VB}_k^{\mathbb{C}}(X)$ be the set of isomorphism classes $[\xi]$ of $k$-dimensional complex vector bundles $\xi$ over $X$. There is a universal $k$-dimensional complex bundle $\gamma_k^{\mathbb{C}}$ over a $CW$-complex $\mathrm{BU}(k)$ such that for any $k$-dimensional complex $k$-vector bundle $\xi$ there is a map $c_\xi^{\mathbb{C}}\colon X \to \mathrm{BU}(k)$ uniquely determined up to homotopy by the property that $\xi$ is isomorphic over $X$ to $(c_\xi^{\mathbb{C}})^*\gamma_k^{\mathbb{C}}$. Moreover, the homotopy class $[c_\xi^{\mathbb{C}}]$ depends only on the isomorphism class $[\xi]$. The space $\mathrm{BU}(k)$ is uniquely up to homotopy determined by the property that there exists a principal $\mathrm{U}(k)$-bundle $\mathrm{EU}(k) \to \mathrm{BU}(k)$ with contractible total space $\mathrm{EU}(k)$. The $k$-dimensional vector bundle $\gamma_k^{\mathbb{C}}$ is given by the canoncial projection $\mathrm{EU}(k)\times_{\mathrm{U}(k)}\mathbb{C}^k \to \mathrm{BU}(k)$.

For the proof of the next theorem we refer, for instance, to [14, Chapter 5]. It is a prototype of a connection between a geometric classification problem to homotopy theory.

**Theorem 16.33** (Classification of vector bundles)**.** *If $X$ is a $CW$-complex, then the maps*

$$\begin{array}{rcll} \mathrm{VB}_k(X) & \to & [X, \mathrm{BO}(k)], & [\xi] \mapsto [c_\xi]; \\ \overline{\mathrm{VB}}_k(X) & \to & [X, \mathrm{BSO}(k)], & [\xi] \mapsto [\overline{c}_\xi]; \\ \mathrm{VB}_k^{\mathbb{C}}(X) & \to & [X, \mathrm{BU}(k)], & [\xi] \mapsto [c_\xi^{\mathbb{C}}], \end{array}$$

*are bijective. Their inverses send $[f]$ to $[f^*\gamma_k]$, $[f^*\overline{\gamma}_k]$ and $[f^*\gamma_k^{\mathbb{C}}]$.*

The spaces $\mathrm{BO}(k)$, $\mathrm{BSO}(k)$, and $\mathrm{BU}(k)$ are path connected and unique up to homotopy.

For a real vector bundle $\xi\colon E \to X$ with Riemannian metric define its *disk bundle* $p_{DE}\colon DE \to X$ by $DE = \{v \in E \mid ||v|| \leq 1\}$ and its *sphere bundle* $p_{SE}\colon SE \to X$ by $SE = \{v \in E \mid ||v|| = 1\}$, where $p_{DE}$ and $p_{SE}$ are the restrictions of $p$. Its *Thom space* $\mathrm{Th}(\xi)$ is defined by $DE/SE$. It has a preferred base point $\infty := SE/SE$. The Thom space can be defined without a choice of a Riemannian metric as follows. Put $\mathrm{Th}(\xi) = E \cup \{\infty\}$ for some extra point $\infty$. Equip $\mathrm{Th}(\xi)$ with the smallest topology for which any open subset $U$ of $E$ is an open subset of $\mathrm{Th}(\xi)$ and a basis of open neighbourhoods for $\infty$ is given by the complements of closed subsets $A \subset E$ for which $A \cap E_x$ is compact for each fiber $E_x$. If $X$ is compact, $E$ is locally compact and $\mathrm{Th}(\xi)$ is the one-point-compactification of $E$. The advantage of this definition is that any bundle map $(f, \overline{f})\colon \xi_0 \to \xi_1$ of vector bundles $\xi_0$ and $\xi_1$ canonically induces a pointed map $\mathrm{Th}(\overline{f}, f)\colon \mathrm{Th}(\xi_0) \to \mathrm{Th}(\xi_1)$. Denote by $\underline{\mathbb{R}^k}$ the trivial vector bundle with fiber $\mathbb{R}^k$. We mention that there are pointed homeomorphisms, see for instance [21, Proposition 12.28].

$$\mathrm{Th}(\xi \times \eta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta); \tag{16.34}$$

$$\mathrm{Th}(\xi \oplus \underline{\mathbb{R}^k}) \cong \mathrm{Th}(\xi) \wedge S^k. \tag{16.35}$$

16.6. **Thom spaces and Thom spectra.**

**Definition 16.36** (Stable system of vector bundles bundles)**.** Given $l \in \mathbb{Z}^{\geq 0}$, an *$l$-dimensional stable system of vector bundles* $\mu = \{(\xi_k, (f_k, \overline{f_k})) \mid k \in \mathbb{Z}^{\geq 0}\}$ is a sequence of vector bundles $\{\xi_k \mid k \in \mathbb{Z}^{\geq 0}\}$ such that $\xi_k$ is a $(k+l)$-dimensional vector bundle with projection $p_{\xi_k}\colon E_k \to B_k$ for a $CW$-complex $B_k$ as basis together with a bundle maps

$$\begin{array}{ccc} E_k \oplus \underline{\mathbb{R}} & \xrightarrow{\overline{f_k}} & E_{k+1} \\ \big\downarrow{\scriptstyle p_k \oplus p_{\underline{\mathbb{R}}}} & & \big\downarrow{\scriptstyle p_{\xi_{k+1}}} \\ B_k & \xrightarrow{f_k} & B_{k+1} \end{array}$$

for $k \in \mathbb{Z}^{\geq 0}$.

We call the system *oriented* if each each vector bundle $\xi_k$ is oriented and each bundle map $(\overline{f_k}, f_k)$ respects the orientations.

Given an $l$-dimensional vector bundle $\xi$ over $B$, we can associated to it an $l$-dimensional stable vector bundle system $\underline{\xi}$ by putting $\underline{\xi}_k = \xi \oplus \underline{\mathbb{R}^k}$ for $k \in \mathbb{Z}^{\geq 0}$ by defining the structure maps $(\mathrm{id}_B, \overline{f_k})$ to be the obvious bundle isomorphism over $B$ from $\xi \oplus \underline{\mathbb{R}^k} \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^{k+1}}$.

**Definition 16.37** (Thom spectrum of a stable system of vector bundles bundles)**.** Consider an $l$-dimensional stable system of vector bundles $\mu = \{(\xi_k, (f_k, \overline{f_k})) \mid k \in \mathbb{Z}^{\geq 0}\}$. Define the associated Thom spectrum $\mathbf{Th}(\mu)$ as follows. Its $k$-th space $\mathrm{Th}(\mu)_k$ is $\{\bullet\}$ for $k < 0$ and $\mathrm{Th}(\mu)_k = \mathrm{Th}(\xi_k)$ for $k \geq 0$. The $k$th-structure map is given by the composite

$$\mathrm{Th}(\mu)_k \wedge S^1 = \mathrm{Th}(\xi_k) \wedge S^1 \stackrel{(16.35)}{=} \mathrm{Th}(\xi_k \oplus \underline{\mathbb{R}}) \xrightarrow{\mathrm{Th}(f_k, \overline{f}_k)} \mathrm{Th}(\xi_{k+1}) = \mathrm{Th}(\mu)_{k+1}.$$

**Example 16.38** (Suspension spectrum)**.** Let $X$ be a $CW$-complex. Consider the zero-dimensional vector bundle $\underline{\mathbb{R}^0}_X$ over $X$. Let $\underline{\underline{\mathbb{R}^0}_X}$ be the associated stable system of bundles maps. We have introduced its Thom spectrum $\mathbf{Th}(\underline{\underline{\mathbb{R}^0}_X})$ in Definition 16.37. Note that its 0-th space is $X_+$. Then $\mathbf{Th}(\underline{\underline{\mathbb{R}^0}_X})$ agrees with the suspension spectrum $\Sigma^\infty X_+$ of Example 16.2. If we take $X = \{\bullet\}$, then $\Sigma^\infty\{\bullet\}_+ = \Sigma^\infty(S^0, s)$ is the sphere spectrum $\mathbf{S}$ of Example 16.3.

**Example 16.39** (The spectra **MO** and **MSO**)**.** Choose for any $k$ a model for the universal $k$-dimensional bundle $\gamma_k$ over $\mathrm{BO}(k)$. By the universal property of $\gamma_{k+1}$ we can choose a bundle map $(f_k, \overline{f_k})\colon \gamma_k \oplus \underline{\mathbb{R}}_{\mathrm{BO}(k)} \to \gamma_{k+1}$. We can arrange by the construction of Subsection 8.8 that each map $f_k\colon \mathrm{BO}(k) \to \mathrm{BO}(k+1)$ is an inclusion of $CW$-complexes and in particular a cofibration. We obtain a stable system $\gamma$ of bundle maps by the collection of the bundles $\gamma_k$ and bundle maps $(f_k, \overline{f_k})$. The associated Thom spectrum of Definition 16.37 is denoted by $\mathbf{MO} = \mathrm{Th}(\mu)$.

Note that $\mathbf{MO}$ depends on some choices. But one can show for the result $\mathbf{MO}'$ for any other choices that there are homotopy equivalences of spectra $\mathbf{MO} \to \mathbf{MO}'$ and

$\mathbf{MO}' \to \mathbf{MO}$ which are homotopy inverse to one another. (Here it is crucial that the maps $f_k$ are cofibrations.) In particular $\mathbf{MO} \to \mathbf{MO}'$ and $\mathbf{MO}' \to \mathbf{MO}$ are weak homotopy equivalences and we get from Proposition 16.12 natural equivalences of (co)homology theories $\mathcal{H}_*(-\mathbf{MO}) \xrightarrow{\cong} \mathcal{H}_*(-\mathbf{MO}')$ and $\mathcal{H}^*(-\mathbf{MO}) \xrightarrow{\cong} \mathcal{H}^*(-\mathbf{MO}')$ which actually are inverse to one another. Moreover, there is actually a canonical construction $\mathbf{MO}$ for which no additional choices are made. Therefore we ignore this ambiguity about $\mathbf{MO}$ in the sequel.

There is an analog $\overline{\mu}$ and $\mathbf{MSO} = \mathbf{Th}(\overline{\mu})$, where one replaces $\gamma_k$ by $\overline{\gamma}_k$ and $\mathrm{BO}(k)$ by $\mathrm{BSO}(k)$.

Note that we have constructed the sphere spectrum $\mathbf{S}$, the Eilenberg-MacLane spectrum $\mathbf{K}(A)$ for an abelian group $A$, and the spectra $\mathbf{MO}$ and $\mathbf{MSO}$ so far. Recall that associated to them are (co-)homology theories in Theorem 16.21. For $\mathbf{S}$ we have identified $\mathcal{H}_*(-;\mathbf{S})$ with the stable homotopy groups $\pi_*^s(-)$, see Example 16.26. These will be identified with more geometric terms, namely with framed bordism, in Theorem 17.15. For $\mathbf{K}(A)$ we have identified $\mathcal{H}^*(-;\mathbf{K}(A))$ with the singular cohomology $H^*(-;A)$ with coefficients in $A$, see Example 16.27. We will identify $\mathcal{H}_*(-;\mathbf{MO})$ and $\mathcal{H}_*(-;\mathbf{MSO})$ with more geometric terms, namely with unoriented and oriented bordism theory, see Theorem 17.11 and Theorem 17.14.

16.7. **Topological $K$-theory.** One can define topological groups

$$\begin{aligned} \mathrm{O} &= \operatorname{colim}_{k\to\infty} \mathrm{O}(k);\\ \mathrm{SO} &= \operatorname{colim}_{k\to\infty} \mathrm{SO}(k);\\ \mathrm{U} &= \operatorname{colim}_{k\to\infty} \mathrm{U}(k), \end{aligned}$$

for the inclusions $\mathrm{O}(k) \to \mathrm{O}(k+1)$, $\mathrm{SO}(k) \to \mathrm{SO}(k+1)$, and $\mathrm{U}(k) \to \mathrm{U}(k+1)$ given by taking the block sum with the $(1,1)$ matrix $(1)$.

There is a principal O-bundle $\mathrm{EO} \to \mathrm{BO}$ over a $CW$-complex BO for which EO is contractible. Up to homotopy one can obtain BO also as $\operatorname{colim}_{k\to\infty} \mathrm{BO}(k)$ if one chooses adequate models for $\mathrm{BO}(k)$ and arranges that each map $\mathrm{BO}(k) \to \mathrm{BO}(k+1)$ is an inclusion of $CW$-complexes and in particular a cofibration. Analogously one can construct spaces BSO and BU. The spaces BO, BSO, and BU are path connected. We have $\pi_1(\mathrm{BO}) \cong /IZ/2$ and the spaces BSO and BU are actually simply connected.

A deep theorem of Bott says that there are weak homotopy equivalences

$$\begin{aligned} \beta^{\mathbb{R}}\colon \mathbb{Z} \times \mathrm{BO} &\xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times \mathrm{BO});\\ \beta^{\mathbb{C}}\colon \mathbb{Z} \times \mathrm{BU} &\xrightarrow{\simeq} \Omega^2(\mathbb{Z} \times \mathrm{BU}), \end{aligned}$$

where $\mathbb{Z}$ is equipped with the discrete topology and the base point $0 \in \mathbb{Z}$ and we choose some base point in the path connected spaces BO and BU.

For $n \in \mathbb{Z}$ define $k(n) \in \{0,1,2,3,4,5,6,7\}$ to be the unique element satisfying $k(n) \equiv n \mod 8$. Define an $\Omega$-spectrum $\mathbf{KO}$ by defining the $n$-th space $\mathbf{KO}_n^{\mathbb{R}}$ to be $\Omega^{8-k(n)}(\mathbb{Z} \times \mathrm{BO})$ if $k(n) \neq 0$ and to be $\mathbb{Z} \times \mathrm{BO}$ if $k(n) = 0$. The $n$-th structure map is

$$\mathrm{id}\colon \Omega^{8-k(n)}(\mathbb{Z} \times \mathrm{BO}) \to \Omega\Omega^{8-(k(n)+1)}(\mathbb{Z} \times \mathrm{BO}) = \Omega^{8-k(n)}(\mathbb{Z} \times \mathrm{BO})$$

if $k(n) \neq 0$ and $\beta^{\mathbb{R}}\colon (\mathbb{Z} \times \mathrm{BO}) \to \Omega\Omega^7(\mathbb{Z} \times \mathrm{BO}) = \Omega^8(\mathbb{Z} \times \mathrm{BO})$. So the spectrum $\mathbf{KO}$ is 8-periodic and looks in tte range from 0 to 8 like

$$\mathbb{Z} \times \mathrm{BO}, \Omega^7(\mathbb{Z} \times \mathrm{BO}), \Omega^6(\mathbb{Z} \times \mathrm{BO}), \ldots, \Omega^1(\mathbb{Z} \times \mathrm{BO}), \mathbb{Z} \times \mathrm{BO}\,.$$

Similarly we define the $\Omega$-spectrum $\mathbf{K}$. Define $\mathbf{K}_n$ to be $\mathbb{Z} \times \mathrm{BU}$ if $n$ is even, and to be $\Omega(\mathbb{Z} \times \mathrm{BU})$ if $n$ is odd. The $n$-th structure map is the identity $\mathrm{id}_{\Omega(\mathbb{Z}\times\mathrm{BU})}$ if

$n$ is odd, and is $\beta^{\mathbb{C}}$ if $n$ is even. So the spectrum $\mathbf{K}$ is 2-periodic and looks in the range 0 to 2 like

$$\mathbb{Z} \times \mathrm{BU}, \Omega(\mathbb{Z} \times \mathrm{BU}), \mathbb{Z} \times \mathrm{BU}\,.$$

Associated to these $\Omega$-spectra are cohomology theories satisfying the disjoint union axiom

$$\begin{array}{rcl} KO^*(X,A) & := & H^*(X,A;\mathbf{KO}); \\ K^*(X,A) & = & H^*(X,A;\mathbf{K}), \end{array}$$

called *real and complex topological $K$-theory.* Note that $KO^*$ is 8-periodic, i.e., there are natural isomorphisms $KO^*(X,A) \xrightarrow{\cong} KO^{*+8}(X,A)$, whereas $K^*$ is 2-periodic, i.e., there are natural isomorphisms $K^*(X,A) \xrightarrow{\cong} K^{*+2}(X,A)$.

Associated to these $\Omega$-spectra are homology theories satisfying the disjoint union axiom

$$\begin{array}{rcl} KO_*(X,A) & := & H_*(X,A;\mathbf{KO}); \\ K_*(X,A) & = & H_*(X,A;\mathbf{KO}), \end{array}$$

called *real and complex topological $K$-homology* Note that $KO_*$ is 8-periodic, i.e., there are natural isomorphisms $KO_*(X,A) \xrightarrow{\cong} KO_{*+8}(X,A)$, whereas $K_*$ is 2-periodic, i.e., there are natural isomorphisms $K_*(X,A) \xrightarrow{\cong} KO_{*+2}(X,A)$.

The coefficients are given for the real case by

$$KO_n(\{\bullet\}) = KO^{8-n}(\{\bullet\}) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \{0\}, \mathbb{Z}, , \{0\}, \{0\}, \{0\}, \mathbb{Z}, \quad \text{for } n = 0,1,2\ldots,7 \tag{16.40}$$

and in the complex case by

$$K_n(\{\bullet\}) = K^n(\{\bullet\}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases} \tag{16.41}$$

If $X$ is a finite $CW$-complex, the abelian group $KO^0(X)$ can be identified with the Grothendieck construction applied to the abelian semi-group of stable isomorphism classes of finite-dimensional real vector bundles over $X$, where two finite-dimensional real vector bundles $\xi$ and $\eta$ over $X$ are called stably isomorphic if $\xi \oplus \underline{\mathbb{R}^k}$ and $\eta\underline{\mathbb{R}^l}$ are isomorphic for some natural numbers $k$ and $l$ and the addition comes from the Whitney sum. The analogous statement holds for the complex case.

**Remark 16.42.** Topological $K$-theory is a very valuable cohomology theory which had many applications to problems in topology. It was later extended to $C^*$-algebras and plays a prominent role in the classification and the theory of $C^*$-algebras and in index theory.

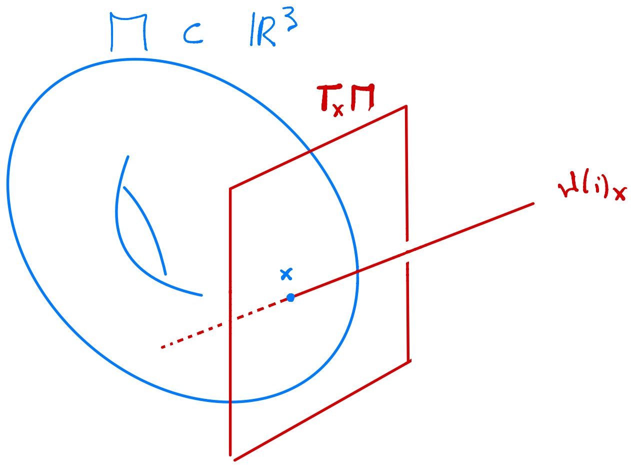
16.8. **Outlook.** The approach to spectra presented above can be called "classical" or "naive". Moreover, we have not defined the notion of a smash product of two spectra and of a ring spectrum.

One can define the smash product $\mathbf{E} \wedge \mathbf{F}$ of two spectra $\mathbf{E}$ and $\mathbf{E}$ in the setting discussed in these notes but it depends on certain choices. Moreover associativity or commutativity of this smash product make only sense up to homotopy. This has led to the notions of highly structured spectra such as symmetric or orthogonal spectra, where the smash product is strictly defined and also associativity and commutativity of the smash product hold strictly.

Moreover, one works with spectra in the setting of higher category theory nowadays. An introduction to higher categories can be found for instance in [12].

## 17. The Pontrjagin-Thom Construction

17.1. **$\xi$-bordism.** Let $(M, i)$ be an embedding $i\colon M^n \to \mathbb{R}^{n+k}$ of a closed $n$-dimensional manifold $M$ into $\mathbb{R}^{n+k}$. Note that $T\mathbb{R}^{n+k}$ comes with an explicit trivialisation $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \xrightarrow{\cong} T\mathbb{R}^{n+k}$ and the standard Euclidean inner product induces a Riemannian metric on $T\mathbb{R}^{n+k}$. Denote by $\nu(i)$ the *normal bundle*, which is the orthogonal complement of $TM$ in $i^*T\mathbb{R}^{n+k}$ or can be thought of as the quotient bundle $i^*T\mathbb{R}^{n+k}/TM$.




Next we apply this construction to bordism. Fix a space $X$ together with a $k$-dimensional vector bundle $\xi$ over $X$. We define the bordism set

$$\Omega_n(\xi) \tag{17.1}$$

of normal $\xi$-bordism classes of normal $\xi$-maps as follows.

**Definition 17.2** (Normal $\xi$-map)**.**
A *normal $\xi$-map* $(M, i, f, \overline{f})$ is a quadruple consisting of:

- A closed manifold $M$ of dimension $n$;
- An embedding $i\colon M \to \mathbb{R}^{n+k}$;
- A map $f\colon M \to X$;
- A bundle map $(f, \overline{f})\colon \nu(i) \to \xi$ covering $f$, where $\nu(i)$ is the normal bundle of the embedding $i$.

**Definition 17.3** (Bordism of normal $\xi$-maps)**.**
A *normal $\xi$-bordism* from the normal $\xi$-map $(M_0, i_0, f_0, \overline{f_0})$ to the normal $\xi$-map $(M_1, i_1, f_1, \overline{f_1})$ is a quadruple $(W, I, F, \overline{F})$ consisting of:

- A compact manifold $W$ of dimension $(n+1)$ whose boundary $\partial W$ is the disjoint union $\partial_0 W \amalg \partial_1 W$;
- An embedding of manifolds with boundary $I\colon W \to \mathbb{R}^{n+k} \times [0,1]$ sending $\partial_m W$ to $\mathbb{R}^{n+k} \times \{m\}$ for $m = 0, 1$;
- Diffeomorphisms $u_m\colon M_m \to \partial_m W$ and $U_m\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k} \times \{m\}$ for $m = 0, 1$ satisfying $I \circ u_m = U_m \circ i_m$;
- A map $F\colon W \to X \times [0,1]$ satisfying $j_m \circ f_m = F \circ u_m$ for $m = 0, 1$ where $j_m\colon X \to X \times [0,1]$ sends $x$ to $(x, m)$;

- A bundle map $(F, \overline{F})\colon \nu(I) \to \xi$ covering $F$ such that $\overline{F} \circ \nu(u_m, U_m) = \overline{f}_m$ holds for $m = 0, 1$ where $(u_m, \nu(u_m, U_m))\colon \nu(i_m) \to \nu(I)$ is the obvious bundle map induced by $Tu_m$ and $TU_m$.

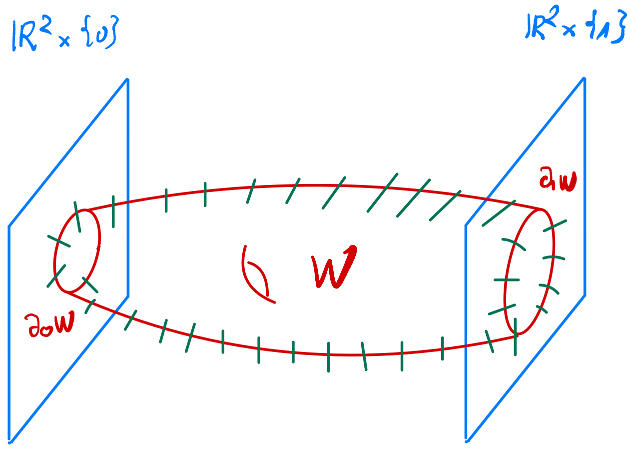
**Remark 17.4.** Note that in the definition above the following implicit identification

$$\nu(\partial W \subseteq \mathbb{R}^{n+k} \times \{0,1\}) = \nu(W \subseteq \mathbb{R}^{n+1} \times [0,1])|_{\partial W}$$

is used, which is based on the convention that at $\{0\}$ we take the inward normal field and at $\{1\}$ the outward normal vector field to get identifications

$$\begin{array}{rcl} T\mathbb{R}^n \times [0,1]|_{\mathbb{R}^n \times \{0,1\}} & = & T\mathbb{R}^n \times \{0,1\} \oplus \mathbb{R}; \\ TW|_{\partial W} & = & T\partial W \oplus \underline{\mathbb{R}}. \end{array}$$

This convention guarantees that we can stack two cobordisms together to prove transitivity of the bordism relation.



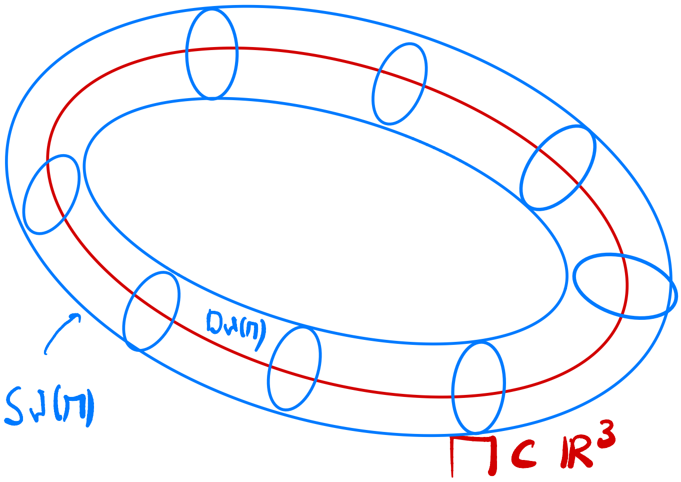

17.2. **The Pontrjagin-Thom construction of $\xi$-bordism.** Consider a normal $\xi$-map $(M, i, f, \overline{f})$, see Definition 17.2. Note that for any vector bundle $\eta$ over a manifold $B$ with total space $E$ there exists a canonical bundle isomorphism $TB \oplus \eta \xrightarrow{\cong} s^*TE$ over $B$, where $s\colon B \to E$ is the zero-section. So we get an identification $TB \oplus \eta = TE|_B$. Let $(N(M), \partial N(M))$ be a tubular neighbourhood of $M$. Recall that there is a diffeomorphism

$$u\colon (D\nu(M), S\nu(M)) \to (N(M), \partial N(M))$$

with the property that its restriction to $M$ is $i$ and under the canonical identification $T(D\nu(M))|_M = TM \oplus \nu(i)$ the composite

$$\nu(i) = \{0\} \oplus \nu(i) \to TM \oplus \nu(i) = T(D\nu(M))|_M \xrightarrow{Tu|_M} i^*T\mathbb{R}^{n+k} \to i^*T\mathbb{R}^{n+k}/TM = \nu(i)$$

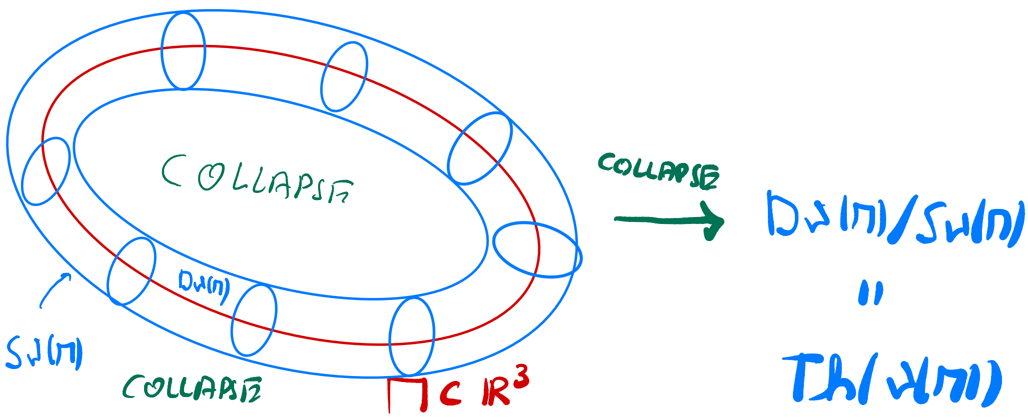
is the identity. Such a tubular neighborhood is unique up to diffeotopy on $\mathbb{R}^{n+k}$ relative $M$. See for instance [2, Theorem 21.11 on page 130 and Theorem 12.13 on page 131].




The *Thom collapse map*

$$c\colon S^{n+k} = \mathbb{R}^{n+k} \amalg \{\infty\} \quad \to \quad \mathrm{Th}(\nu(M)) \tag{17.5}$$

is the pointed map that is given by the diffeomorphism $u^{-1}$ on the interior of $N(M)$ and sends the complement of the interior of $N(M)$ to the preferred base point $\infty$.



The homology group $H_{n+k}(\mathrm{Th}(\nu(M))) \cong H_{n+k}(N(M), \partial N(M))$ is infinite cyclic if $M$ is connected, since $N(M)$ is a connected compact orientable $(n+k)$-dimensional manifold with boundary $\partial N(M)$. The Hurewicz homomorphism

$$\mathrm{hur}_{n+k} \colon \pi_{n+k}(\mathrm{Th}(\nu(i))) \to H_{n+k}(\mathrm{Th}(\nu(i)))$$

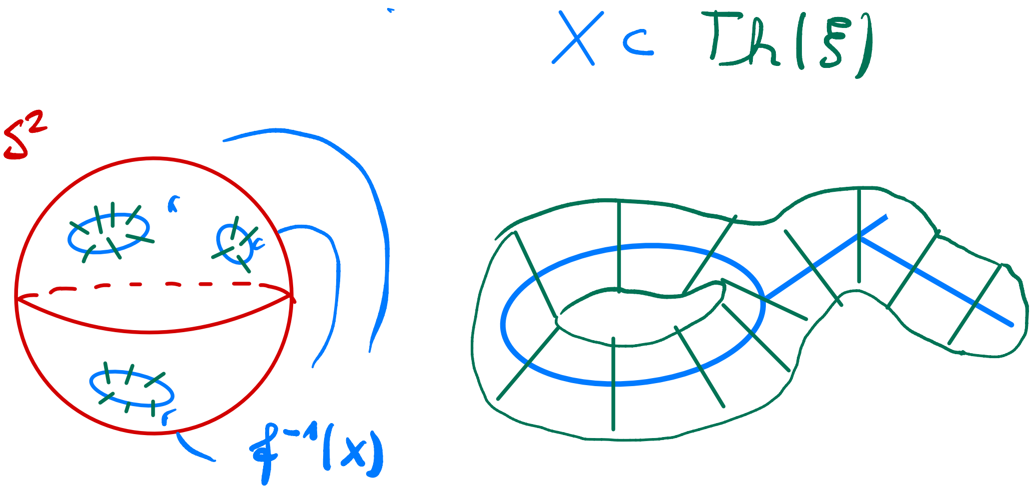
sends the class $[c]$ of $c$ to a generator. This follows from the fact that any point in the interior of $N(M)$ is a regular value of $c$ and has precisely one point in its preimage.

**Theorem 17.6** (Pontrjagin-Thom Construction)**.** *Let $\xi \colon E \to X$ be a $k$-dimensional vector bundle over a $CW$-complex $X$. Then the map*

$$P_n(\xi) \colon \Omega_n(\xi) \to \pi_{n+k}(\mathrm{Th}(\xi)),$$

*which sends the bordism class of $(M, i, f, \overline{f})$ to the homotopy class of the composite $S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M)) \xrightarrow{\mathrm{Th}(f,\overline{f})} \mathrm{Th}(\xi)$, is a well-defined bijection, natural in $\xi$.*

*Proof.* The details can be found in [3, Satz 3.1 on page 28, Satz 4.9 on page 35] or [8, Section 7.2 on page 172]. The basic idea becomes clear after we have explained the construction of the inverse for a finite $CW$-complex $X$. Consider a pointed map $(S^{n+k}, \infty) \to (\mathrm{Th}(\xi), \infty)$. We can change $f$ up to homotopy relative $\{\infty\}$ so that $f$ becomes transverse to $X$. Note that transversality makes sense although $X$ is not a manifold, one needs only the fact that $X$ can identified with the image of a zero-section of a vector bundle. Put $M = f^{-1}(X)$. The transversality construction yields a bundle map $(f|_M, \overline{f|_M}) \colon \nu(M) \to \xi$ covering $f|_M$. Let $i \colon M \to \mathbb{R}^{n+k} = S^{n+k} - \{\infty\}$ be the inclusion. Then the inverse of $P_n(\xi)$ sends the class of $f$ to the class of $(M, i, f|_M, \overline{f|_M})$.




□

17.3. **The Pontrjagin-Thom construction and bordism for stable systems of bundles.** Consider an $n$-dimensional system $\mu$ of vector bundles $\mu$ given by vector bundles $\xi_k$ and bundle morphisms $(f_k, \overline{f}_k)\colon \xi_k \oplus \underline{\mathbb{R}} \to \xi_{k+1}$. For $n, k \in \mathbb{Z}^{\geq 0}$ we next define a map

$$\Omega_n(f_k, \overline{f}_k)\colon \Omega_n(\xi_k) \to \Omega_n(\xi_{k+1}).$$

Consider an element $z$ in $\Omega_n(\xi_k)$ represented by normal $\xi_k$-map $(M, i, u, \overline{u})$. Let $j\colon \mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\} \to \mathbb{R}^{n+k+1}$ be the standard inclusion. Then there is a natural identification of $\nu(j \circ i)$ with $\nu(i) \oplus \underline{\mathbb{R}}$. Consider the bundle map

$$(v, \overline{v})\colon \nu(j \circ i) = \nu(i) \oplus \underline{\mathbb{R}}_M \xrightarrow{(f,\overline{f}) \oplus (u,\underline{u})} \xi_k \oplus \underline{\mathbb{R}}_{B_k} \xrightarrow{(\overline{f_k}, f_k)} \xi_{k+1}$$

where $(u, \underline{u})$ is the obvious bundle map induced by $u$ and $\mathrm{id}_{\mathbb{R}}$. Then we obtain a normal $\xi_{k+1}$-map $(M, j \circ i, v, \overline{v})$. Its bordism class in $\Omega_n(\xi_{k+1})$ is the image of $z$ under $\Omega_n(f_k, \overline{f}_k)$. We omit the proof that $\Omega_n(f_k, \overline{f}_k)$ is well-defined. We define the set

$$\Omega_n(\mu) = \operatorname{colim}_{k \to \infty} \Omega_n(\xi_k) \tag{17.7}$$

with respect to the structure maps $\Omega_n(f_k, \overline{f}_k)$.

The set $\Omega_n(\mu)$ carries in contrast to each of the sets $\Omega_n(\xi_k)$ the structure of an abelian group. The unit is given by the class of the normal $\xi_k$-map $(M, i, v, \overline{v})$ with $M = \emptyset$ for any $k \in \mathbb{Z}^{\geq}$. Consider two elements $z$ and $z'$ in $\Omega_n(\mu)$. We can find representatives $(M, i, u, \overline{u})$ and $(M', i', u', \overline{u'})$ with $k = k'$ and $\mathrm{im}(i) \cap \mathrm{im}(i') = \emptyset$ and define $z_1 + z_2$ by the class of the disjoint union $(M \amalg M', i \amalg i', u \amalg u', \overline{u} \amalg \overline{u'})$. The inverse of a class represented by $(M, i, u, \overline{u})$ is the class represented by $(M, j \circ$

$i, f_k \circ u, \overline{f_k \circ u})$, where $j\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$ is the standard inclusion, we identify $\nu(j \circ i)$ with $\nu(i) \oplus \underline{\mathbb{R}}$ and $(f_k\circ, \overline{f_k \circ u})$ is the composite

$$(j \circ u, \overline{f_k \circ u},)\colon \nu(j \circ i) = \nu(i) \oplus \underline{\mathbb{R}}_M \xrightarrow{(u,\overline{u}) \oplus \underline{-\operatorname{id}_{\mathbb{R}}}_u} \xi_k \oplus \underline{\mathbb{R}}_{B_k} \xrightarrow{(f_k, \overline{f}_k)} \xi_{k+1}$$

for $\underline{-\operatorname{id}_{\mathbb{R}}}_u$ the obvious bundle map induced by $u$ and $-\operatorname{id}_{\mathbb{R}}$.

Let $s_{n,k}\colon \pi_{n+k}(\operatorname{Th}(\xi_k)) \to \pi_{n+k+1}(\operatorname{Th}(\xi_{k+1}))$ be the composite of the suspension homomorphism $\pi_{n+k}(\operatorname{Th}(\xi_k)) \to \pi_{n+k+1}(\operatorname{Th}(\xi_k) \wedge S^1)$ and the homomorphism $\pi_{n+k+1}(\operatorname{Th}(\xi_k) \wedge S^1) \to \pi_{n+k+1}(\operatorname{Th}(\xi_{k+1}))$ induced by the $k$th structure map $\operatorname{Th}(\xi_k) \wedge S^1 \to \operatorname{Th}(\xi_{k+1})$ of the Thom spectrum $\mathbf{Th}(\mu)$ of Definition 16.37. Then we get from the definitions

$$\pi_n(\mathbf{Th}(\mu)) = \operatorname{colim}_{k\to\infty} \pi_{n+k}(\operatorname{Th}(\xi_k))$$

with respect to the structure maps $s_{n,k}$.

One easily checks that we obtain a commutative diagram

$$\begin{array}{ccc} \Omega_n(\xi_k) & \xrightarrow{\Omega_n(f_k, \widetilde{f}_k)} & \Omega_n(\xi_{k+1}) \\ {\scriptstyle P_n(\xi_k)}\downarrow\cong & & \cong\downarrow{\scriptstyle P_n(\xi_{k+1})} \\ \pi_{n+k}(\operatorname{Th}(\xi_k)) & \xrightarrow[s_{n,k}]{} & \pi_{n+k+1}(\operatorname{Th}(\xi_{k+1})). \end{array}$$

Therefore we obtain the following result.

**Theorem 17.8** (Pontrjagin-Thom Construction for stable bundle systems)**.**
*Let $\mu$ be a stable bundle system. Then we obtain an isomorphism of abelian groups*

$$P_n(\mu)\colon \Omega_n(\mu) \to \pi_n(\mathbf{Th}(\mu))$$

*by putting $P_n(\mu) = \operatorname{colim}_{k\to\infty} P_n(\xi_k)$.*

17.4. **Unoriented bordism.** Consider a pair $(X, A)$ and $n \in \mathbb{Z}^{\geq 0}$. A *singular $n$-manifold over* $(X, A)$ is a map $(u, \partial u)\colon (M, \partial M) \to (X, A)$ with target $(X, A)$ and a compact smooth manifold $M$ with boundary $\partial M$ of dimension $n$ as source. Consider two singular $n$-maps $(u_k, \partial u_k)\colon (M_k, \partial M_k) \to (X, A)$ for $k = 0, 1$. A singular $(n + 1)$-dimensional bordism between them consists of:

- A compact smooth manifold $W$ of dimension $(n + 1)$ with boundary $\partial W$;
- A decomposition $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$ for smooth submanifolds $\partial_0 W$, $\partial_1 W$, and $\partial_2 W$ of $\partial W$ satisfying $\partial(\partial_0 W) \cap \partial(\partial_1 W) = \emptyset$ and $\partial(\partial_2 W) = \partial(\partial_0 W) \amalg \partial(\partial_1 W)$;
- A map $(U, \partial U)\colon (W, \partial W) \to (X, A)$;
- Diffeomorphisms $(v_k, \partial v_k)\colon (M_k, \partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ for $k = 0, 1$ such that $\partial U \circ v_k = \partial u_k$ holds for $k = 0, 1$;
- We have $U(\partial_2 W) \subseteq A$.

If $(u_1, \partial u_1)\colon (M_1, \partial M_1) \to (X, A)$ is given by $M_1 = \emptyset$, we call such a $(n + 1)$-dimensional bordism a *nullbordism* for $(u_0, \partial u_0)\colon (M_0, \partial M_0) \to (X, A)$.

If there exists a bordism between two singular $n$-manifolds over $(X, A)$, we call them *bordant*. This turns out to be an equivalence relation, for transitivity one has to glue two bordisms together. So we can define the set $\mathcal{N}_n(X, A)$ to be the set of of bordism classes of singular $n$-manifolds over $(X, A)$.

If $A$ is empty, then for a singular bordism $(u, \partial u)\colon (M, \partial M) \to (X, A)$ over $X = (X, \emptyset)$ we have $\partial M = \emptyset$ and hence $M$ is just a closed manifold with a map $u\colon M \to X$. Also the notion of a bordism simplifies, since $\partial_2 W$ must be empty and hence $\partial W$ is just the disjoint union $\partial_0 W \amalg \partial_1 W$.

The set $\mathcal{N}_n(X, A)$ inherits the structure of an abelian group as follows. The unit is given by the bordism class of the singular $n$-manifold $(u, \partial u)\colon (M, \partial M) \to (X, A)$

for which $M = \emptyset$. Given two singular $n$-manifolds $(u_k, \partial u_k)\colon (M_k, \partial M_k) \to (X, A)$ for $k = 0, 1$, define the sum of their bordism classes to be the bordism class of the disjoint union $(u_0, \partial u_0) \amalg (u_1, \partial u_1)\colon (M_0, \partial M_0) \amalg (M_1, \partial M_1) \to (X, A)$. The inverse of the bordism class of $(u, \partial u)\colon (M, \partial M) \to (X, A)$ is given by the bordism class of $(u, \partial u)\colon (M, \partial M) \to (X, A)$ again, since a nullbordism for their disjoint union $(u, \partial u) \amalg (u, \partial u)\colon (M, \partial M) \amalg (M, \partial M) \to (X, A)$ can be constructed from the cylinder $M \times I$ in the obvious way. Note that this implies that $\mathcal{N}_n(X, A)$ is actually an $\mathbb{F}_2$-vector space.

A map of pairs $(F, f)\colon (X, A) \to (Y, B)$ induces a homomorphism of $\mathbb{F}_2$-vector spaces by sending the bordism class of $(u, \partial u)\colon (M, \partial M) \to (X, A)$ to the bordism class of $(F \circ u, f \circ \partial u)\colon (M, \partial M) \to (X, A)$. We omit the proof that we obtain a covariant functor $\mathcal{N}_n(-)$ from the category of topological pairs to the category of $\mathbb{F}_2$-vector spaces for $n \in \mathbb{Z}^{\geq 0}$. We define $\mathcal{N}_n(X, A)$ for $n \in \mathbb{Z}^{\leq -1}$ to be $\{0\}$. For a pair $(X, A)$ define the homomorphism

$$\partial_n(X, A)\colon \mathcal{N}_n(X, A) \to \mathcal{N}_{n-1}(A) \tag{17.9}$$

by sending the bordism class of $(u, \partial u)\colon (M, \partial M) \to (X, A)$ to the bordism class of $\partial u\colon \partial M \to A$.

**Theorem 17.10** (Singular bordism is a homology theory satisfying the disjoint union axiom)**.**

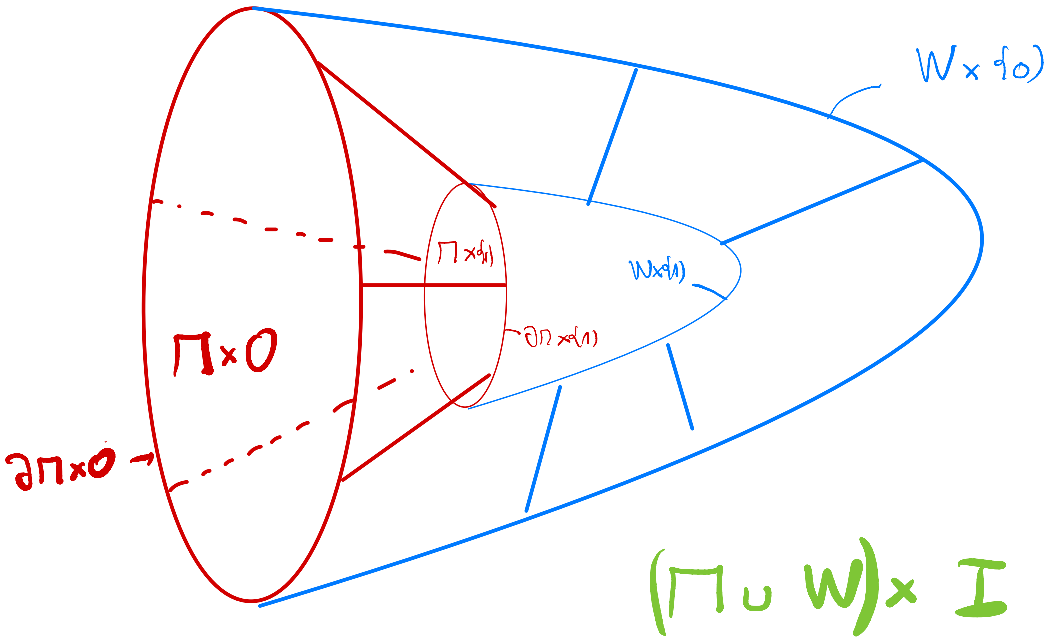
*We obtain a homology theory with values in $\mathbb{F}_2$-vector spaces satisfying the disjoint union axiom by $\mathcal{N}_*(-)$ and $\partial_*(-)$.*

*Sketch of the proof.* We start with homotopy invariance. Consider for $k = 0, 1$ the maps $(F_k, f_k)\colon (X, A) \to (Y, B)$ and a homotopy $h\colon (X, A) \times I \to (Y, B)$ between them. We have to show $\mathcal{N}(F_0, f_0) = \mathcal{N}(F_1, f_1)$. Consider a singular $n$-manifold $(u, \partial u)\colon (M, \partial M) \to (X, A)$. We have to show that $(F_0, f_0) \circ (u, \partial u)\colon (M, \partial M) \to (X, A)$ and $(F_1, f_1) \circ (u, \partial u)\colon (M, \partial M) \to (X, A)$ are bordant. The desired bordism can easily be constructed from the composite $M \times I \xrightarrow{u \times \mathrm{id}_I} X \times I \xrightarrow{h} Y$.

Consider a pair $(X, A)$. We have to show that we obtain a long exact sequence of $\mathbb{F}_2$-vector spaces

$$\cdots \xrightarrow{\partial_{n+2}} \mathcal{N}_{n+1}(A) \xrightarrow{\mathcal{N}_{n+1}(i)} \mathcal{N}_{n+1}(X) \xrightarrow{\mathcal{N}_{n+1}(j)} \mathcal{N}_{n+1}(X, A)$$
$$\cdots \xrightarrow{\partial_{n+1}} \mathcal{N}_n(A) \xrightarrow{\mathcal{N}_n(i)} \mathcal{N}_n(X) \xrightarrow{\mathcal{N}_n(j)} \mathcal{N}_n(X, A) \xrightarrow{\partial_n} \cdots$$

where $i\colon A \to X$ and $j\colon X = (X, \emptyset) \to (X, A)$ are the inclusions. We only explain exactness at $\mathcal{N}_{n+1}(X, A)$. Consider an element in $\mathcal{N}_{n+1}(X)$ given by the bordism class of $u\colon M \to X$. Its image under the composite $\partial_{n+1} \circ \mathcal{N}_{n+1}(j)$ is represented by the singular map with the empty set as domain and hence is zero. This shows $\mathrm{im}(\mathcal{N}_{n+1}(j)) \subseteq \ker(\partial_{n+1})$. It remains to prove $\ker(\partial_{n+1}) \subseteq \mathrm{im}(\mathcal{N}_{n+1}(j))$. Consider a singular $(n+1)$-manifold $(u, \partial u)\colon (M, \partial M) \to (X, A)$ over $(X, A)$ such that its bordism class lies in $\ker(\partial_{n+1})$. Hence we can find a nullbordism for $\partial u_k\colon \partial M \to A$, i.e., a compact manifold $W$ with boundary $\partial W$, a map $U\colon W \to A$ and a diffeomorphism $v\colon \partial M \to \partial W$ with $U \circ v = \partial u$. Then we obtain a singular $n$-manifold over $X$ by $u \cup_v U\colon M \cup_v W \to X$. We claim that its bordism class is sent under $\mathcal{N}_{n+1}(j)\colon \mathcal{N}_{n+1}(X) \to \mathcal{N}_{n+1}(X, A)$ is the bordism class of $(u, \partial u)\colon (M, \partial M) \to (X, A)$ over $(X, A)$. This follows from the fact that one can construct from the composite $M \cup_v W \times I \xrightarrow{(u \cup_v U) \times \mathrm{id}_I} Y \times I \xrightarrow{\mathrm{pr}_Y} Y$ for the $\mathrm{pr}_Y$ the canonical projection a bordism of singular $(n+1)$-manifolds over $(X, A)$ between $u \cup_v U\colon (M \cup_v W, \emptyset) \to (X, A)$ and $(u, \partial u)\colon (M, \partial M) \to (X, A)$. This finishes the proof of exactness at $\mathcal{N}_{n+1}(X, A)$. The proof of exactness at the other places is similar.




The disjoint union axiom follows from the fact that for a compact subset $C$ of the disjoint union $\coprod_{i \in I} X_i$ of the collection of spaces $\{X_i \mid i \in I\}$ there is a finite subset $J \subseteq I$ with $C \subseteq \coprod_{i \in J} X_i$.

We omit the proof that excision holds, i.e., if $X$ is a space with subspaces $A \subseteq B \subseteq X$ satisfying $\overline{A} \subseteq B^\circ$, then the inclusion $i\colon (X \setminus A, B \setminus A) \to (X, A)$ induces for every $n \in \mathbb{Z}$ a bijection $\mathcal{N}_n(i)\colon \mathcal{N}(X \setminus A, B \setminus A) \to \mathcal{N}_n(X, A)$. For a proof of the Mayer-Vietoris sequence for space $X$ with open subspaces $X_0$, $X_1$, and $X_2$ satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$, we refer to [24, Proposition 21.1.7 on page 523]. The existence of such a Mayer-Vietoris sequence is essentially the same as excision. □

**Theorem 17.11** (Unoriented singular bordism and the spectrum **MO**)**.**
*There is a natural equivalence of homology theories*

$$\mathcal{N}_*(-) \xrightarrow{\cong} \mathcal{H}_*(-;\mathbf{MO})$$

*where* $\mathcal{H}_*(-;\mathbf{MO})$ *is the homology theory associated in Theorem 16.21 (i), to the spectrum* **MO** *defined in Example 16.39.*

*Sketch of proof.* We only construct for every space $X$ and $n \in \mathbb{Z}$ an isomorphism of abelian groups $\mathcal{N}_n(X) \xrightarrow{\cong} \mathcal{H}_*(X;\mathbf{MO})$. We leave it to the reader to show that it can be extended to pairs $(X, A)$, is natural in $(X, A)$, and is compatible with the boundary operators of $(X, A)$ and hence defines the desired natural equivalence of homology theories $\mathcal{N}_*(-) \xrightarrow{\cong} \mathcal{H}_*(-;\mathbf{MO})$.

Recall the stable system of vector bundles $\mu$ of Example 16.39 whose $k$-th vector bundle $\gamma_k$ is the universal $k$-dimensional vector bundle over $\mathrm{BO}(k)$. For a space $X$, define the stable system of vector bundles $\mathrm{pr}^* \mu$ as follows. The $k$-th vector bundle is $\mathrm{pr}^*\gamma_k$ over $X \times \mathrm{BO}(k)$ which is obtained from the vector bundle $\gamma_k$ by the pullback construction applied to the projection $\mathrm{pr}\colon X \times \mathrm{BO}(k) \to \mathrm{BO}(k)$. The $k$-th bundle map is given by $\mathrm{pr}^*(f_k, \overline{f_k})$ for the bundle map $(f_k, \overline{f_k})\colon \gamma_k \oplus \underline{\mathbb{R}} \to \gamma_{k+1}$ using the obvious identification $\mathrm{pr}^*\underline{R}_{\mathrm{BO}(k)} = \underline{R}_{X\times\mathrm{BO}(k)}$. Then we can identify the spectrum $\mathbf{Th}(\mathrm{pr}^*\mu)$ with the spectrum $X_+ \wedge \mathbf{Th}(\mu)$ using (16.35). We get from Theorem 17.8 an isomorphism of abelian groups

$$\Omega_n(\mathrm{pr}^*\mu) \xrightarrow{\cong} \pi_n(X_+ \wedge \mathbf{Th}(\mu)) = \mathcal{H}_n(X; \mathbf{MO}).$$

Hence it suffices to construct a bijection

$$\alpha_n\colon \Omega_n(\mathrm{pr}^*\mu) \to \mathcal{N}_n(X). \tag{17.12}$$

One can define $\alpha_n$ as a forgetful map. More precisely, $\alpha_n$ sends the class of a normal $\mu_k$-map $(M, i, f, \overline{f})$ to the bordism class of the singular $n$-manifold $\mathrm{pr}_X \circ f\colon M \to X$ for the projection $\mathrm{pr}\colon X \times \mathrm{BO}(k) \to X$. Obviously $\alpha_n$ is a well-defined homomorphism of abelian groups, It remains to show that $\alpha_n$ is bijective.

We begin with surjectivity. Consider a singular $n$-manifold $f\colon M \to X$ representing an element $z \in \mathcal{N}_n(X)$. We can choose $k \in \mathbb{Z}^{\geq 0}$, actually $k = n+1$ suffices, and an embedding $i\colon M \to \mathbb{R}^{n+k}$. Let $(u, \overline{u})$ be a bundle map from $\nu(i)$ to $\mu_k$. We obtain a bundle map $(f \times u, f \times \overline{u})$ from $\nu(i)$ to $\mathrm{pr}^*\gamma_k$. Then $(M, i, f \times u, f \times \overline{u})$ is a normal $\mathrm{pr}^*\mu_k$-map and hence defines an element $y \in \Omega_n(\mathrm{pr}^*\mu)$. The image of $y$ under $\alpha_n$ is $z$. Hence $\alpha_n$ is surjective.

Next we show injectivity of $\alpha_n$. Consider an element $z \in \Omega_n(\mathrm{pr}^*\mu)$ which is sent to zero under $\alpha_n$. Choose $k \in \mathbb{Z}$ with $k \geq 2n+3$ and a normal $\mathrm{pr}^*\mu_k$-map $(M, i, f \times u, f \times \overline{u})$ for $f\colon M \to X$ and $(u, \overline{u})\colon \nu(i) \to \gamma_k$, whose normal bordism class is $z$. Then $\alpha_n(z)$ is represented by the singular $n$-map $f\colon M \to X$. Hence we can find a compact $(n+1)$-dimensional manifold $W$ with boundary $\partial W$, a diffeomorphism $t\colon M \to \partial W$, and a map $F\colon W \to X$ satisfying $F \circ t = f$. Now we have to find the right data to construct out of $(W, F, w)$ a normal $\mathrm{pr}^*\mu$-nullbordism for $(M, i, f \times u, f \times \overline{u})$.

Since $k \geq 2n+3$ we can construct an embedding of manifolds with boundary

$$(J, j)\colon (W, \partial W) \to (\mathbb{R}^{n+k} \times [0,1), \mathbb{R}^{n+k} \times \{0\}).$$

Then there is a natural identification $\nu(J)|_{\partial W} = \nu(j)$. The embedding $j \circ t\colon M \to \mathbb{R}^{n+k}$ and the given embedding $i\colon M \to \mathbb{R}^{n+k}$ are related by a diffeotopy $\Phi\colon \mathbb{R}^{n+k} \times \mathbb{R} \to \mathbb{R}^{n+k}$ because of $k \geq 2n+3$. Hence we can find a diffeomorphism $T\colon \mathbb{R}^{n+k} \xrightarrow{\cong} \mathbb{R}^{n+k}$ such that $T \circ i = j \circ t$ holds. We get a bundle isomorphism $(t, \overline{t})\colon \nu(i) \xrightarrow{\cong} \nu(J)|_{\partial W} = \nu(j)$ coming from the differentials of $t$ and $T$. Choose a bundle map $(v, \overline{v})\colon \nu(J) \to \mu_k$. Then the bundle maps $(v, \overline{v}) \circ (t, \overline{t})$ and $(u, \overline{u})$ from $\nu(i)$ to $\gamma_k$ are homotopic. By a cofibration argument we can change $(v, \overline{v})$ up to homotopy of bundle maps $\nu(J) \to \gamma_k$ such that $(v, \overline{v}) \circ (t, \overline{t}) = (u, \overline{u})$ holds. These data yield a normal $\mathrm{pr}^*\mu$-nullbordism $(W, J, F \times v, F \times \overline{v})$ for $(M, i, f, \overline{f})$. Hence $z = 0$. This finishes the proof that the map $\alpha_n$ of (17.12) is bijective and therefore of Theorem 17.11. □

17.5. **The unoriented bordism ring.** There is an external multiplicative structure on $\mathcal{N}_*(-)$ coming from taking the cartesian product. In particular we get for $m, n \in \mathbb{Z}^{\geq 0}$ and every two pairs $(X, A)$ and $(Y, B)$ a natural bilinear pairing

$$\mathcal{N}_m(X, A) \times \mathcal{N}_n(Y, B) \to \mathcal{N}_{m+n}((X, A) \times (Y, B)).$$

This induces on $\mathcal{N}_* = \mathcal{N}_*(\{\bullet\})$ the structure of a commutative $\mathbb{Z}$-graded $\mathbb{F}_2$-algebra whose unit is given by $\mathrm{id}\colon \{\bullet\} \to \{\bullet\}$. Thom [22] has shown that $\mathcal{N}_*$, which is called the *unoriented bordism ring*, is a polynomial ring over $\mathbb{F}_2$ in variables $x_i$ for $i \neq 2^k - 1$ and that for $i$ even one can take the bordism class of $\mathbb{RP}^i$ for $x_i$. Dold [5] has constructed explicit closed manifolds representing $x_i$ for $i$ odd. In particular we get

$$\mathcal{N}_n = \mathbb{F}_2, \{0\}, \mathbb{F}_2, \{0\}, \mathbb{F}_2 \oplus \mathbb{F}_2, \mathbb{F}_2, \quad \text{for } n = 0, 1, 2, 4, 5. \tag{17.13}$$

Moreover, two closed manifolds are cobordant, or, equivalently, determine the same element in $\mathcal{N}_*$, if and only if they have the same Stiefel-Whitney numbers. For the definition of Stiefel-Whitney numbers were refer for instance to [14, Chapter 4].

17.6. **Conventions about orientations.** Let us discuss our orientation conventions for manifolds. For simplicity we will only consider a connected compact orientable $n$-dimensional manifold $M$ with (possibly empty or non-connected) boundary $\partial M$, where orientable means that $H_n(M; \partial M)$ is infinite cyclic. Here is a list of desired properties or standard conventions.

(i) On the vector space $\mathbb{R}^n$ for $n \geq 1$ we use the standard orientation given by the ordered standard basis $\{e_1, e_2, \ldots, e_n\}$, where $e_i$ is the vector

$$(0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

whose only non-zero entry is at position $i$. If $n = 0$, an orientation on $\mathbb{R}^0$ is a choice of an element in $\{+, -\}$;

(ii) For $n \geq 1$ an orientation on a $TM$ is a choice of orientation on every $T_xM$ for $x \in M$ such that for every $x \in M$ there is an open neighbourhood $U$ together with an isomorphism $TM|_U \xrightarrow{\cong} \underline{\mathbb{R}^n}$ of vector bundles over $U$ with the property that for every $x \in U$ the isomorphism $T_xM \xrightarrow{\cong} \mathbb{R}^n$ respects the given orientation on $T_xM$ and the standard orientation of $\mathbb{R}^n$.

For $n = 0$ a choice of an orientation on $TM$ is a choice of an element in $\{+, -\}$.

This makes actually sense for any vector bundle over $M$;

(iii) Since $TD^n$ is $T\mathbb{R}^n|_{D^n}$ and we have the standard trivialisation $\underline{\mathbb{R}^n} \xrightarrow{\cong} T\mathbb{R}^n$, the standard orientation on the vector space $\mathbb{R}^n$ induces a standard orientation on $TD^n$. In particular on $D^1 = [-1, 1]$ we use the orientation on $TD^1$ coming from moving from $-1$ to 1;

(iv) An orientation on $M$ is a choice of a generator $[M, \partial M]$ of the infinite cyclic group $H_n(M, \partial M)$;

(v) There is a preferred one-to-one-correspondence between the orientations on $TM$ and the orientations on $M$ which comes from the identification $H_n(T_xM, T_xM \setminus \{0\}) \xrightarrow{\cong} H_n(M, M \setminus \{x\})$ induced by the exponential map for $x \in M \setminus \partial M$;

(vi) The boundary homomorphism $H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends $[M, \partial M]$ to a class $[\partial M]$ which induces for every path component $C \in \partial M$ a generator $[C] \in H_{n-1}(C)$. Thus an orientation on $M$ induces an orientation on $C$.

(vii) We use the outward normal vector field and the canonical isomorphism $n_v \oplus Ti\colon \underline{\mathbb{R}} \oplus T\partial M \xrightarrow{\cong} TM|_{\partial M}$ in order to assign to an orientation on $T_xM$ an orientation on $T_x\partial M$ for $x \in X$. Thus an orientation on $TM$ induces an orientation on $TC$ for every path component $C$ of $\partial M$;

(viii) On a product $M \times N$ of oriented connected closed smooth manifolds we use the orientation coming from the isomorphism induced by the cross product $H_{\dim(M)}(M; \mathbb{Z}) \otimes_{\mathbb{Z}} H_{\dim(N)}(N; \mathbb{Z}) \xrightarrow{\cong} H_{\dim(M \times N)}(M \times N; \mathbb{Z})$;

(ix) On a direct sum $V \oplus W$ of oriented vector spaces we use the orientation coming from assigning to two ordered basis of $V$ and $W$ the obvious ordered basis of $V \oplus W$ by stacking the basis together.

This yields also a preferred procedure to define a preferred orientation on the Whitney sum $\xi \oplus \eta$ of two orientable vector bundles $\xi$ and $\eta$;

(x) All the items above are compatible with one another;

(xi) These conventions together with the standard orientation on the vector space $\mathbb{R}^n$ yield on $S^1$ respectively $TS^1$ the anticlockwise orientation and on $[-1,1]$ and $T[-1,1]$ respectively the orientation corresponding from moving from $-1$ to 1;

(xii) With these conventions the standard orientation on $T[-1,1]$ induces on $T\partial D^1 = T\partial[-1,1] = T\{-1,1\}$ the orientation which corresponds to $-$ on $-1$ and $+$ on 1.

We leave it to the reader to check that this can be arranged if and only if we use the outward normal field and the convention that in the identification $n_v \oplus Ti\colon \underline{\mathbb{R}} \oplus T\partial M \xrightarrow{\cong} TM|_{\partial M}$ we choose the order $\underline{\mathbb{R}} \oplus T\partial M$ and not the order $T\partial M \oplus \underline{\mathbb{R}}$. Namely ((xii)) forces us to use the outward normal field and the order is determined by ((iii)) and ((xi)).

17.7. **Oriented bordism.** Now we can modify the definition of the unoriented bordism group $\mathcal{N}_n(X,A)$ to the oriented bordism group $\Omega_n(X,A)$. W call a compact manifold $M$ with (possibly empty) boundary $\partial M$ *oriented* if for each path component $C$ of $M$ the homology group $H_n(C;\partial C)$ is infinite cyclic and we have chosen a generator $[C,\partial C] \in H_n(C;\partial C)$. Given an oriented compact manifold $M$, we denote by $M^-$ the oriented compact manifold whose underlying manifold is $M$ but where we use the orientation, where we replace $[C,\partial C]$ by $-[C,\partial C]$.

The difference in the new definition of $\Omega_n(X,A)$ and in the definition of $\mathcal{N}_n(X,A)$ appearing in Subsection 17.4 is that we additionally require for a singular $n$-manifold $(u,\partial u)\colon (M,\partial M) \to (X,A)$ over $(X,A)$ that $M$ is an oriented manifold and in the definition of the bordism relation we additionally require that $W$ is an oriented manifold and the diffeomorphism $(v_k,\partial v_k)\colon (M_k,\partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ preserve the orientions for $k=0,1$. The addition and the unit is defined as before. However, the inverse of the bordism class of $(u,\partial u)\colon (M,\partial M) \to (X,A)$ is given by the bordism class of $(u,\partial u)\colon (M^-,\partial M^-) \to (X,A)$ and *not* by $(u,\partial u)\colon (M,\partial M) \to (X,A)$; we have to reverse the orientations. This has the effect that $\Omega_n(X,A)$ is an abelian group but in general not a $\mathbb{F}_2$-vector space which was the case for $\mathcal{N}_n(X,A)$. The proof that we get a homology theory $\Omega_*(-)$ satisfying the disjoint union axiom is analogous to the proof of Theorem 17.10.

The proof of the next theorem is analogous to the proof of Theorem 17.11.

**Theorem 17.14** (Oriented singular bordism and the spectrum **MSO**). *There is a natural equivalence of homology theories*

$$\Omega_*(-) \xrightarrow{\cong} \mathcal{H}_*(-;\mathbf{MSO})$$

*where $\mathcal{H}_*(-;\mathbf{MSO})$ is the homology theory associated in Theorem 16.21 (i), to the spectrum* **MSO** *defined in Example 16.39.*

17.8. **The oriented bordism ring.** There is an external multiplicative structure on $\Omega_*(-)$ coming from taking the cartesian product. In particular we get for $m,n \in \mathbb{Z}^{\ge 0}$ and every two pairs $(X,A)$ and $(Y,B)$ a natural bilinear pairing

$$\Omega_m(X,A) \times \Omega_n(Y,B) \to \Omega_{m+n}((X,A) \times (Y,B)).$$

This induces on $\Omega_* = \Omega_*(\{\bullet\})$ the structure of a commutative $\mathbb{Z}$-graded ring whose unit is given by $\mathrm{id}\colon \{\bullet\} \to \{\bullet\}$ with the standard orientation $+$ on the domain. Its structure was completely determined by Wall [25]. In particular $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a polynomial $\mathbb{Q}$-algebra whose generators as a polynomial $\mathbb{Q}$-algebra can be taken to be the oriented bordism classes of $\mathbb{CP}^{2n}$ for $n \geq 1$. Each $\Omega_n$ is a finitely generated abelian group in which the order of any nontrivial torsion element is 2.

Moreover, two oriented closed manifolds are oriented cobordant, or, equivalently, determine the same element in $\Omega_*$, if and only if they have the same Pontrjagin and Stiefel-Whitney numbers. For the definition of Pontrjagin and Stiefel-Whitney numbers were refer for instance to [14, Chapter 4 and 16].

Here is some information about $\Omega_n$ in low degrees $n$:

- There is an isomorphism of abelian groups
$$\Omega_0 \xrightarrow{\cong} \mathbb{Z}$$
which sends the bordism class of a 0-dimensional oriented manifold which is just a finite collection of points equipped with a sign $+$ or $-$ to the sum of these signs;
- The signature defines an isomorphism of abelian groups
$$\operatorname{sign}\colon \Omega_4 \xrightarrow{\cong} \mathbb{Z}$$
and the preimage of $1 \in \mathbb{Z}$ is the bordism class of $\mathbb{CP}^2$.
- We have $\Omega_n = \{0\}$ if and only if $n \in \{1,2,3,6,7\}$;
- We have
$$\Omega_n \cong \begin{cases} \mathbb{Z}/2 & n = 5; \\ \mathbb{Z} \oplus \mathbb{Z} & n = 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n = 9; \\ \mathbb{Z}/2 & n = 10. \end{cases}$$

17.9. **Framed bordism.** Let $\xi$ be an $n$-dimensional vector bundle over the space $B$. For $l \in \mathbb{Z}^{\geq 0}$ an *$l$-framing* of $\xi$ is a bundle isomorphism $(\mathrm{id}_B, \overline{u})\colon \underline{\mathbb{R}^{n+l}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^l}$ over $B$. We call an $l_0$-framing $(\mathrm{id}_B, \overline{u}_0)\colon \underline{\mathbb{R}^{n+l_0}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^{l_0}}$ and an $l_1$-framing $(\mathrm{id}_B, \overline{u}_1)\colon \underline{\mathbb{R}^{n+l_1}} \xrightarrow{\cong} \xi \oplus \underline{\mathbb{R}^{l_1}}$ equivalent if there exists $l \in \mathbb{Z}^{\geq 0}$ with $l \geq l_0, l_1$ such that for $i = 0,1$ the two bundle isomorphisms over $B$
$$\underline{\mathbb{R}^{n+l}} = \underline{\mathbb{R}^{n+l_i}} \oplus \underline{\mathbb{R}^{l-l_i}} \xrightarrow{(\mathrm{id}_B, \overline{u}_i) \oplus \mathrm{id}_{\underline{\mathbb{R}^{l-l_i}}}} \xi \oplus \underline{\mathbb{R}^{l_i}} \oplus \underline{\mathbb{R}^{l-l_i}} = \xi \oplus \underline{\mathbb{R}^l}$$
are homotopic through bundle isomorphisms over $B$.

For a compact manifold $M$ a *stable framing* is a stable framing of its tangent bundle $TM$. Of course not every compact manifold admits a framing. Every compact manifold with a stable framing is orientable and inherits from the stable framing an orientation. Let $i\colon (M, \partial M) \to (\mathbb{R}^m \times \mathbb{R}^{\geq 0}, \mathbb{R}^m \times \{0\})$ be an embedding of the compact manifold $M$ (with possibly empty) boundary $\partial M$. Then we will tacitly use in the sequel the fact that there is a one-to-one correspondence between the stable framings of $M$ and the stable framings of the normal bundle $\nu(i)$.

Now one can define for a pair $(X, A)$ its framed bordism group $\Omega_n^{\mathrm{fr}}(X, A)$ analogously to how we modified the definition of unoriented bordism $\mathcal{N}_*(-)$ to oriented bordism $\Omega_*(-)$. The difference in the new definition of $\Omega_n^{\mathrm{fr}}(X, A)$ and in the definition of $\mathcal{N}_n(X, A)$ appearing in Subsection 17.4 is that we additionally require for a singular $n$-manifold $(u, \partial u)\colon (M, \partial M) \to (X, A)$ over $(X, A)$ that $M$ comes with a stable framing and in the definition of the bordism relation we additionally require that $W$ comes with a stable framing and the diffeomorphism $(v_k, \partial v_k)\colon (M_k, \partial M_k) \xrightarrow{\cong} (\partial_k W, \partial(\partial_k W))$ is compatible with the stable framings. The addition and the unit is defined as before. However, the inverse of the class represented is now defined by replacing a given stable framing by the new stable framing obtained by precomposition with the bundle automorphism for $l \in \mathbb{Z}^{\geq 1}$

$$\underline{\mathbb{R}^{n+l}} = \underline{\mathbb{R}^{n+l-1}} \oplus \underline{\mathbb{R}} \xrightarrow{\operatorname{id}_{\underline{\mathbb{R}^{n+l-1}}} \oplus - \operatorname{id}_{\underline{\mathbb{R}}}} \underline{\mathbb{R}^{n+l-1}} \oplus \underline{\mathbb{R}} = \underline{\mathbb{R}^{n+l}}.$$

The proof of the next theorem is a variation of the proof of Theorem 17.11 in view of Example 16.26 and Remark 16.37.

**Theorem 17.15** (Framed bordism and the stable homotopy)**.** *There is a natural equivalence of homology theories*

$$\Omega_*^{\mathrm{fr}}(-) \xrightarrow{\cong} \pi_*^s(-).$$

**Remark 17.16.** One can give rather elementary geometric proofs of the formula

$$\Omega_n^{\mathrm{fr}} \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}/2 & n = 1. \end{cases}$$

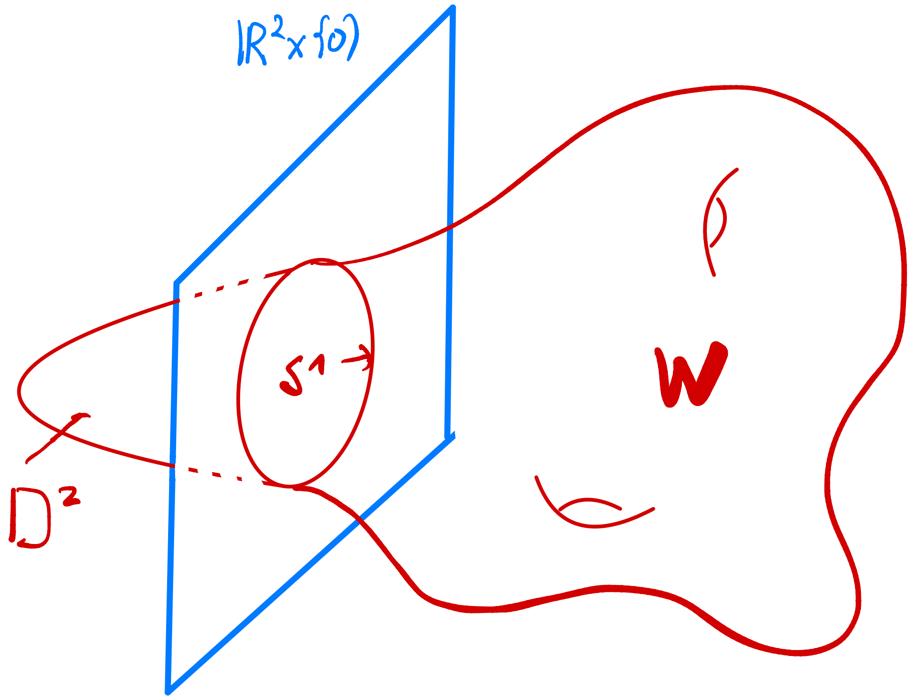
which agrees with the values of the $n$-stem $\pi_n^s$ for $n = 0, 1$ by Theorem 17.15.

It is not hard to check that the forgetful map $\Omega_0^{\mathrm{fr}} \to \Omega_0$ is bijective and we have computed $\Omega_0$ already in Subsection 17.8.

Any connected closed 1-dimensional manifold $M$ is diffeomorphic to $S^1$. Because of Example 10.14 we get $[S^1, \mathrm{SO}] \cong \mathbb{Z}/2$ and hence there are precisely two stable framings on $S^1$. One of them extends to $D^2$ and the other does not. Then one can show that $\mathcal{N}_1 \cong \mathbb{Z}/2$ with the bordism class of $S^1$ equipped with the framing not extending to $D^2$ as generator. From $\pi_1^s \cong \mathbb{Z}/2$, Theorem 10.5, and the Freudenthal Suspension Theorem 11.20 we conclude

$$\pi_{n+1}(S^n) \cong \begin{cases} \{0\} & n = 1; \\ \mathbb{Z} & n = 2; \\ \mathbb{Z}/2 & n \geq 3. \end{cases}$$

Let us sketch the proof that $\mathcal{N}_1 \cong \mathbb{Z}/2$ with the bordism class of $S^1$ with the framing which does not extend to $D^2$ as generator. We first show that $S^1$ with the framing which does not extend to $D^2$ is not framed nullbordant. Suppose the contrary, i.e., that there is a framed nullbordism $W$ for it. We can assume without loss of generality that $W$ is path connected. In the sequel we identify $S^1 = \partial W$. Then $W \cup_{S^1} D^2$ is a closed 2-dimensional manifold. Its first Stiefel-Whitney class vanishes, since $H^1(W \cup_{S^1} D^2; \mathbb{F}_2) \to H^1(W; \mathbb{F}_2)$ is injective and sends it to the Stiefel-Whitney class of $W$ which is trivial. Hence $W \cup_{S^1} D^2$ is orientable. We can choose an embedding $i \colon W \to \mathbb{R}^3$ such that $i$ restricted to $S^1$ is given by the inclusion $S^1 \subset \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$ and $i$ maps $D^2$ to $(\mathbb{R}^3)^{\leq 0}$ and $W$ to $(\mathbb{R}^3)^{\geq 0}$.




Since $W$ and $\mathbb{R}^3$ are orientable, the normal bundle $\nu(i)$ is orientable and hence trivial. We choose an identification $\nu(i) = \underline{\mathbb{R}}$. Note that this yields a specific stable framing on $W$. We get identifications $\nu(i|_{S^1}) = \underline{\mathbb{R}^2}$, $\nu(i|_{D^2}) = \underline{\mathbb{R}}$, and $\nu(i|_W) = \underline{\mathbb{R}}$. So we get on $S^1$ the stable framing which extends to $D^2$ and comes from the specific stable framing on $W$. The problem is that there are different stable framing on $W$. They differ from the specific stable framing by an automorphism of $\underline{\mathbb{R}^k}_W$ for large $k \in \mathbb{Z}^{\geq 0}$. We conclude from Proposition 10.10 (ii) that the set of stable framing of $W$ can be identified with $[W, \mathrm{SO}]$. Hence it suffices to show for the inclusion $k\colon S^1 = \partial W \to W$ that the map $k^*\colon [W, \mathrm{SO}] \to [S^1, \mathrm{SO}]$ sending $[g]$ to $[g \circ k]$ is trivial, since then the induced stable framing on $S^1$ is the same for all possible stable framings on $W$ and hence extends to $D^2$. We have shown $\pi_2(\mathrm{SO}) = \{0\}$ and $\pi_1(\mathrm{SO}) \cong \mathbb{Z}/2$ in Example 10.14. We conclude from Theorem 14.3 (ii) that there is a map $f\colon \mathrm{SO} \to K(\mathbb{Z}/2, 1)$ which is 3-connected. Since $W$ and $S^1$ are $CW$-complexes of dimension $\leq 2$, we conclude from the Whitehead Theorem that it suffices to show that $k^*\colon [W, K(\mathbb{Z}/2, 1)] \to [S^1, K(\mathbb{Z}/2, 1)]$ is trivial. Because of Theorem 14.7 it suffices to show that $H^1(k; \mathbb{Z}/2)\colon H^1(W, \mathbb{Z}/2) \to H^1(\partial W; \mathbb{Z}/2)$ is the trivial map. This follows from the part of the long exact cohomology sequence of $(W, \partial W)$

$$H^1(W, \mathbb{Z}/2) \xrightarrow{H^1(k;\mathbb{Z}/2)} H^1(\partial W; \mathbb{Z}/2) \to H^2(W, \partial W; \mathbb{Z}/2) \to H^2(W; \mathbb{Z}/2)$$

and from the computations $H^1(\partial W; \mathbb{Z}/2) \cong \mathbb{Z}/2$, $H^2(W, \partial W; \mathbb{Z}/2) \cong H_0(W; \mathbb{Z}/2) \cong \mathbb{Z}/2$, and $H^2(W; \mathbb{Z}/2) \cong H_0(W, \partial W; \mathbb{Z}/2) \cong \{0\}$. Thus we have shown that $S^1$ with the framing which does not extend to $D^2$ is not framed nullbordant and hence defines a non-trivial element in $\mathcal{N}_1$.

The framed bordism class of any framed 1-dimensional closed manifold is a $\mathbb{Z}$-linear combination of the elements in $\mathcal{N}_1$ represented by $S^1$ with the framing which extends to $D^2$ and by $S^1$ with the framing that does not extend to $D^2$. Obviously the class of $S^1$ with the stable framing which extends to $D^2$ represents zero in $\mathcal{N}_1$. One easily checks the cylinder over $S^1$ gives a framed nullbordism for the disjoint union of two copies of $S^1$ equipped with the framing which does not extend to $D^2$. Hence the element represented in $\mathcal{N}_1$ by $S^1$ equipped with the framing which does not extend to $D^2$ has order precisely two and generates the abelian group $\mathcal{N}_1$.

## References


[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[2] T. Bröcker and K. Jänich. *Einführung in die Differentialtopologie.* Springer-Verlag, Berlin, 1973. Heidelberger Taschenbücher, Band 143.

[3] T. Bröcker and T. tom Dieck. *Kobordismentheorie.* Springer-Verlag, Berlin, 1970.

[4] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups.* Springer-Verlag, New York, 1985.

[5] A. Dold. Erzeugende der Thomschen Algebra $\mathfrak{N}$. *Math. Z.*, 65:25–35, 1956.

[6] R. Fritsch and R. Piccinini. CW-complexes and Euclidean spaces. In *Fourth Conference on Topology (Italian) (Sorrento, 1988)*, number 24 in Rendiconti del Circolo Matematico di Palermo Serie II. Supplemento, pages 79–95. Springer, 1990.

[7] R. Fritsch and R. A. Piccinini. *Cellular structures in topology.* Cambridge University Press, Cambridge, 1990.

[8] M. W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.

[9] D. Husemoller. *Fibre bundles.* Springer-Verlag, New York, second edition, 1975. Graduate Texts in Mathematics, No. 20.

[10] D. C. Isaksen. Stable stems. *Mem. Amer. Math. Soc.*, 262(1269):viii+159, 2019.

[11] D. C. Isaksen, G. Wang, and Z. Xu. Stable homotopy groups of spheres: from dimension 0 to 90. *Publ. Math. Inst. Hautes Études Sci.*, 137:107–243, 2023.

[12] M. Land. *Introduction to infinity-categories.* Compact Textbooks in Mathematics. Birkhäuser/Springer, Cham, [2021] ©2021.

[13] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[14] J. Milnor and J. D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[15] H. Miyazaki. The paracompactness of $CW$-complexes. *Tôhoku Math. J*, 4:309–313, 1952.

[16] J. R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[17] J.-P. Serre. Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math. (2)*, 58:258–294, 1953.

[18] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[19] N. P. Strickland. The category of cgwh spaces. Preprint, http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf, 2009.

[20] R. M. Switzer. *Algebraic topology—homotopy and homology.* Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[21] R. M. Switzer. *Algebraic topology—homotopy and homology.* Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].

[22] R. Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[23] H. Toda. *Composition methods in homotopy groups of spheres.* Annals of Mathematics Studies, No. 49. Princeton University Press, Princeton, N.J., 1962.

[24] T. tom Dieck. *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[25] C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960.

[26] G. W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1978.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