ALGEBRAIC TOPOLOGY I (WS 24/25)

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Abstract. This manuscripts is based on the lecture course *Algebraic Topology I* from the winter term 24/25.

1. Introduction

This manuscripts is based on the lecture course *Algebraic Topology I* from the winter term 24/25.

1.1. Prerequisites.

- Topological spaces;
- CW-complexes;
- Coverings;
- Chain complexes and modules over a ring;
- Singular and cellular (co-)homology;
- Basics about smooth manifolds;
- Basics about bundles and vector bundles;

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2. Basic definitions and properties of homotopy groups

2.1. **Review of the fundamental group.** We briefly recall the notion and the basic properties of the *fundamental group* $\pi_1(X,x)$ of a pointed space (X,x)

Let $X = (X,x)$ be a *pointed space*, i.e., a topological space X with an explicit choice of a so called *base point* $x \in X$. Denote by I the unit interval $[0,1]$. A *loop at x in X* is a map of pairs $w\colon (I,\partial I) \to (X,\{x\})$. Elements in $\pi_1(X,x)$ are homotopy classes of loops at x in X. Note that this means that two loops $w,w'\colon (I,\partial I) \to (X,\{x\})$ are homotopic if there is a homotopy $h\colon I\times I \to X$ such that $h(s,0) = w(s)$, $h(s,1) = w'(s)$, and $h(0,t) = h(1,t) = x$ hold for all $s,t \in I$. Given two loops v,w at x in X, we get a new loop $v * w$ by putting

$$v * w(s) = \begin{cases} v(2s) & \text{if } s \in [0,1/2]; \\ w(2s-1) & \text{if } s \in [1/2,1]. \end{cases}$$

The group structure on $\pi_1(X,x)$ is given by the formula $[v]\cdot[w] = [v*w]$. The unit element is given by the constant loop $c_x\colon (I,\partial I) \to (X,\{x\})$ sending $s \in I$ to x and the inverse of $[w]$ is given by $[w^-]$ for $w^-\colon (I,\partial I) \to (X,\{x\}),\ s \mapsto w(1-s)$.

Here are some basic properties of the fundamental group:

- A pointed map $f\colon (X,x) \to (Y,y)$ induces a group homomorphism
$$\pi_1(f,x)\colon \pi_1(X,x) \to \pi_1(Y,y), \quad [w] \mapsto [f\circ w]$$
which depends only on the pointed homotopy class of f;
- We get a functor from the category of pointed spaces to the category of groups;
- Given pointed spaces (X_i,x_i) for $i = 0,1$, we get from the two projections $\mathrm{pr}_i\colon (X_0\times X_1,(x_0,x_1)) \to (X_i,x_i)$ for $i = 0,1$ an isomorphism
$$\pi_1(\mathrm{pr}_0,(x_0,x_1)) \times \pi_1(\mathrm{pr}_1,(x_0,x_1))\colon \pi_1(X_0\times X_1,(x_0,x_1)) \xrightarrow{\cong} \pi_1(X_0,x_0)\times\pi_1(X_1,x_1);$$
- Let $p\colon X \to Y$ be a covering. Choose $x \in X$ and put $y = p(x)$. Then the induced map $\pi_1(p,x)\colon \pi_1(X,x) \to \pi_1(Y,y)$ is injective.

  If p is actually a G-covering for the group G and X is path connected, then we obtain an exact sequence of groups
$$1 \to \pi_1(X,x) \xrightarrow{\pi_1(p,x)} \pi_1(Y,y) \xrightarrow{\partial} G \to 1;$$
- The mapping degree induces an isomorphism $\pi_1(S^1) \xrightarrow{\cong} \mathbb{Z}$;
- There is a *Seifert-van Kampen Theorem.* It allows to read off a presentation of the fundamental group from the 2-skeleton X_2 and implies that the inclusion $X_2 \to X$ induces an isomorphism $\pi_1(X_2,x) \to \pi_1(X,x)$ for any choice of base point $x \in X$. In particular $\pi_1(X,x)$ vanishes if X is a CW-complex which has no 1-cells. Moreover, $\pi_1(\bigvee_{i=1}^r S^1,x)$ is the free group of rank r. So in general $\pi_1(X)$ is not abelian. Actually any group occurs as $\pi_1(X,x)$ for a 2-dimensional path connected CW-complex X;
- We get a functor T_1 from the fundamental groupoid $\Pi(X)$ to the category of groups by sending an object in $\Pi(X)$ which is a point $x \in X$ to $\pi_1(X,x)$. A morphism $[u]\colon x \to y$ in $\Pi(X)$ is a homotopy class $[u]$ relative endpoints of paths $u\colon I \to X$ from x to y. It is send to the group homomorphism $T_1([u])\colon \pi_1(X,x) \to \pi_1(X,y)$ mapping $[w]$ to $[u^- * w * u]$. Recall that the composite of the morphism $[u]\colon x \to y$ and $[v]\colon y \to z$ in $\Pi(X)$ is given by $[v]\circ[u] = [u*v]$. One easily checks $T_1([v]\circ[u]) = T_1([v])\circ T_1([u])$. Recall that there is a canonical isomorphism of $\pi_1(X,x)$ to the opposite of the group $\mathrm{aut}_{\Pi(X)}(x)$;

- Consider two maps $f_0, f_1 \colon X \to Y$. Let $h\colon X \times I \to Y$ be a homotopy between f_0 and f_1. Choose a base point x and put $y_i = f_i(x)$ for $i = 0, 1$. Let $u\colon I \to Y$ be the path from y_0 to y_1 given by $u(t) = h(x,t)$. We obtain a group isomorphism $T_1([u])\colon \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(Y, y_1)$ and the following diagram of groups commutes

(2.1)
$$\begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \nearrow^{\pi_1(f_0,x_0)} & \\ \pi_1(X,x) & & \cong \downarrow T_1([u]) \\ & \searrow_{\pi_1(f_1,x_1)} & \\ & & \pi_1(Y, y_1). \end{array}$$

Now consider a *pointed pair* (X, A, x), i.e., a topological pair (X, A) together with a choice of a base point $x \in A$. Define the set $\pi_1(X, A, x)$ as the set of homotopy classes relative $\{0\}$ of maps of pairs $w\colon (I, \partial I) \to (X, A)$ satisfying $w(0) = x$, or, equivalently, of homotopy classes of maps of triads $(I; \{0\}, \{1\}) \to (X, \{x\}, A)$. Note that $w(1)$ is not necessarily equal to x and is only required to lie in A. If $A = \{x\}$, then $\pi_1(X, A, x)$ agrees with $\pi_1(X, x)$. In general there is no group structure on $\pi_1(X, A, x)$.

Define $\pi_0(X)$ as the *set of path components of* X. Note that this is the same as the homotopy classes of maps $\{\bullet\} \to X$. If (X, x) is pointed map, we sometimes write $\pi_0(X, x)$ instead of $\pi_0(X)$ to indicate that the set $\pi_0(X)$ has a preferred base point, namely the path component containing x.

Next we construct the (in some sense exact) sequence

(2.2)
$$\pi_1(A,x) \xrightarrow{\pi_1(i,x)} \pi_1(X,x) \xrightarrow{\pi_1(j,x)} \pi_1(X,A,x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X,A) \to \{*\}.$$

The map $\pi_1(i,x)$ is the group homomorphism given by the inclusion $i\colon (A,x) \to (X,x)$. The map of sets $\pi_1(j,x)\colon \pi_1(X,x) \to \pi_1(X,A,x)$ is the obvious map given by forgetting that $w(1) = x$ holds in connection with $\pi_1(X,x)$. The map ∂_1 sends $[w]$ represented by $w\colon (I, \partial I) \to (X, A)$ to the path component of A containing $w(1)$. The map of sets $\pi_0(i)$ sends the path component C of A to the path component D of X containing $i(C)$. The pointed set $\pi_0(X, A)$ is the quotient of the set $\pi_0(X)$ by collapsing the image of $\pi_0(i)\colon \pi_0(A) \to \pi_0(X)$ to one element and $\pi_0(j)$ is the obvious projection.

This sequence is exact in the following sense. The image of $\pi_1(i,x)$ is the preimage under $\pi_1(j,x)$ of the element in $\pi_1(X,A,x)$ given by the constant map $c_x\colon I \to X$. The image of $\pi_1(j,x)$ is the preimage under ∂_1 of the path component of A containing x. The image of ∂_1 is the preimage under $\pi_0(i)$ of the path component of X containing x. The image of $\pi_0(i)$ is the preimage under $\pi_0(j)$ of the preferred base point in $\pi_0(X,A)$. The map $\pi_0(j)$ is surjective.

2.2. **Basic definitions and the group structure on homotopy groups.** Next we want to generalize the notion of the fundamental group to the notion of the homotopy group in degree n for all integers $n \geq 1$. The basic idea is to replace $I = [0,1]$ and $\partial I = \{0,1\}$ by the *n-dimensional cube*

$$I^n = \prod_{i=1}^{n} [0,1] = \{(s_1, s_2, \ldots, s_n) \mid s_i \in [0,1]\}$$

where we define

$$\partial I^n = \{(s_1, s_2, \ldots, s_n) \mid s_i \in I, \exists i \in \{1, 2, \ldots, n\} \text{ with } s_i \in \{0, 1\}\}.$$

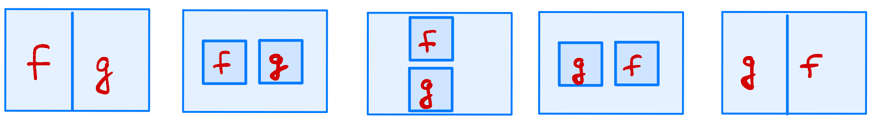
Given a pointed space X, we define the set $\pi_n(X, x)$ to be the set of homotopy classes $[f]$ of maps of pairs $f\colon (I^n, \partial I^n) \to (X, \{x\})$. Given two elements $[f]$ and $[g]$, we define their product $[f] \cdot [g]$ by the homotopy class of the map of pairs $f * g\colon (I^n, \partial I^n) \to (X, \{x\})$ defined by

$$f * g(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{if } s_1 \in [0, 1/2]; \\ g(2s_1 - 1, s_2, \ldots, s_n) & \text{if } s_1 \in [1/2, 1]. \end{cases} \tag{2.3}$$

The unit is given by the homotopy class $[c_x]$ of the constant map $c_x\colon (I^n, \partial I^n) \to (X, \{x\})$. The inverse of $[f]$ is the class $[f^-]$ for the map $f^-\colon (I^n, \partial I^n) \to (X, \{x\})$ sending $(s_1, s_2, \ldots, s_n)$ to $(1 - s_1, s_2, \ldots, s_n)$. The proof that this defines a group $\pi_n(X, x)$ called *n-homotopy group* of the pointed space (X, x) is the essentially the same as the one for $\pi_1(X)$. The construction above for $n = 1$ agrees with the definition of $\pi_1(X, x)$ presented in Subsection 2.1. If we define I^0 to be $\{\bullet\}$ and $\partial I^0 = \emptyset$, the definition of the set $\pi_0(X, x)$ above agrees with the definition of $\pi_0(X)$ as the set of path components of X. Recall that $\pi_0(X)$ has no group structure in general and the $\pi_1(X, x)$ is not necessarily commutative. However, the following lemma is true.

Lemma 2.4. *The group $\pi_n(X, x)$ is abelian for $n \geq 2$.*

Proof. The basic observation is that in the cube I^n for $n \geq 2$ there is enough room to show $[f] \cdot [g] = [g] \cdot [f]$. The desired homotopy is indicated for $n = 2$ by the following sequence of pictures:

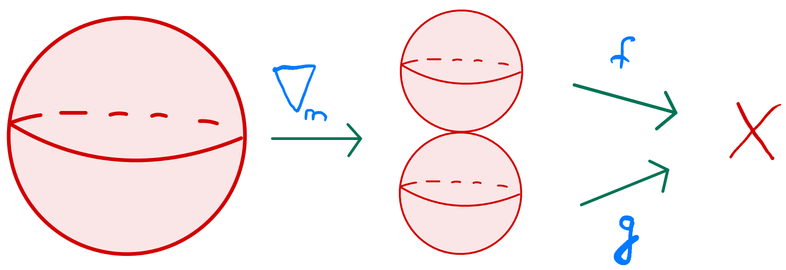


The homotopy begins by shrinking the domains of f and g to smaller subcubes of I^n, where the region outside these subcubes is mapped to the basepoint. After this has been done, there is room to slide the two subcubes around anywhere in I^n as long as they stay disjoint. Hence for $n \geq 2$ they can be slided past each other, interchanging their positions. Then to finish the homotopy, the domains of f and g can be enlarged back to their original size. The whole process can actually be done using just the coordinates s_1 and s_2, keeping the other coordinates fixed. ☐

Any map of pairs $f\colon (I^n, \partial I^n) \to (X, \{x\})$ factorizes in a unique way over the projection $\mathrm{pr}\colon I^n \to I^n/\partial I^n$ to a pointed map $\overline{f}\colon (I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x)$. Obviously this is compatible with the notion of a homotopy of maps of pairs $(I^n, \partial I^n) \to (X, \{x\})$ and of a pointed homotopy of pointed maps $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (X, x)$. There is an obvious homeomorphism of pairs $(I^n/\partial I^n, \partial I^n/\partial I^n) \to (S^n, \{s\})$ for the fixed base point $s = (1, 0, \ldots, 0) \in S^n$. Hence we can interprete an element in $\pi_n(X, x)$ as a pointed homotopy of pointed maps $(S^n, s) \to (X, x)$. The multiplication in this picture is given as follows. Consider pointed maps $f_i\colon (S^n, s) \to (X, x)$ for $i = 0, 1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X, x)$. They define a pointed map $f_0 \vee f_1\colon (S^n \vee S^n, s) \to (X, x)$. Let

$$\nabla_n\colon S^n \to S^n \vee S^n \tag{2.5}$$

be the so-called *pinching map* which is obtained by collapsing the equator $S^{n-1} \subseteq S^n$ given by $\{(x_0, x_1, \ldots, x_n) \in S^n \mid x_n = 0\}$ to a point. Then $[f_0]\cdot[f_1]$ is represented by the composite $f_0 \vee f_0 \circ \nabla_n$, as illustrated in the following picture



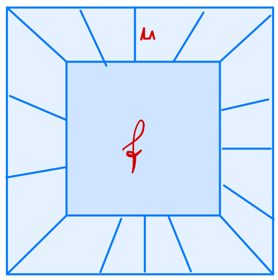
The interpretation in terms of pointed maps $(S^n, s) \to (X, x)$ is useful for some theoretical considerations and in connection with CW-complexes, whereas the picture in terms of maps of pairs $(I^n, \partial I^n) \to (X, \{x\})$ is better suited for some constructions and proofs, e.g., the proof of Lemma 2.4.

2.3. **Functorial properties of homotopy groups.** Obviously a map of pointed spaces $f\colon (X, x) \to (Y, y)$ induces a group homomorphism $\pi_n(f, x)\colon \pi_n(X, x) \to \pi_n(Y, y)$ for $n \geq 1$ by composition. We get a functor from the category of pointed spaces to the category of abelian groups by sending (X, x) to $\pi_n(X, x)$ for $n \geq 2$, whereas for $n = 1$ we get a functor from the category of pointed spaces to the category of groups by sending (X, x) to $\pi_1(X, x)$ for $n = 1$. We get a functor from the category of topological spaces to sets by sending X to $\pi_0(X)$.

Obviously $\pi_n(f, x)$ depends only on the pointed homotopy class of f and $\pi_0(f)$ depends only on the homotopy class of f.

Next we construct for every $n \geq 2$ a functor T_n from $\Pi(X)$ to the category of abelian groups. It sends an object in $\Pi(X)$, which is a point x in X, to the abelian group $\pi_n(X, x)$. Consider a morphism $[u]\colon x \to y$ in $\Pi(X)$ represented by a path u in X from x to y. It is sent to the homomorphism of abelian groups

$T_n([u])\colon \pi_n(X,x) \to \pi_n(X,y)$ defined as follows. Consider $[f] \in \pi_n(X,x)$ represented by the map $f\colon (I^n,\partial I^n) \to (X,\{x\})$. Consider a new map $uf\colon (I^n,\partial I^n) \to (X,\{x\})$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path u on each radial segment in the shell between this smaller cube and ∂I^n, as indicated in the picture below



We leave it to the reader to figure out the elementary proof that this definition is independent of all the choices and indeed yields a functor T_n from $\Pi(X)$ to the category of abelian groups.

Recall that there is a canonical isomorphism of $\pi_1(X,x)$ to the opposite of the group $\operatorname{aut}_{\Pi(X)}(x)$. Hence we obtain from the functor T_n above the structure of a $\mathbb{Z}[\pi_1(X,x)]$-module on $\pi_n(X,x)$ for $n \geq 2$.

Consider two maps $f_0, f_1\colon X \to Y$. Let $h\colon X \times I \to Y$ be a homotopy between f_0 and f_1. Choose a base point x and put $y_i = f_i(x)$ for $i = 0,1$. Let $u\colon I \to Y$ be the path from y_0 to y_1 given by $u(t) = h(x,t)$. For $n \geq 2$ we obtain an isomorphism of abelian groups $T_n([u])\colon \pi_n(Y,y_0) \xrightarrow{\cong} \pi_n(Y,y_1)$ and the following diagram of abelian groups commutes

(2.6)
$$\begin{array}{ccc} & & \pi_n(Y,y_0) \\ & \nearrow^{\pi_n(f_0,x_0)} & \\ \pi_n(X,x) & & \cong \big\downarrow T_n([u]) \\ & \searrow_{\pi_n(f_1,x_1)} & \\ & & \pi_n(Y,y_1). \end{array}$$

A consequence of (2.1) and (2.6) is that a homotopy equivalence $f\colon X \to Y$ induces for every $x \in X$ and $n \geq 1$ a bijection $\pi_n(f,x)\colon \pi_1(X,x) \xrightarrow{\cong} \pi_n(Y,f(x))$. Moreover, for a path connected space X the isomorphism class of $\pi_n(X,x)$ is independent of the choice of $x \in X$. Therefore we sometimes write $\pi_n(X)$ instead of $\pi_n(X,x)$

Given pointed spaces (X_i,x_i) for $i = 0,1$, we get from the two projections $\operatorname{pr}_i\colon (X_0 \times X_1,(x_0,x_1)) \to (X_i,x_i)$ for $i = 0,1$ a group isomorphism

$$\pi_n(\operatorname{pr}_0,(x_0,x_1)) \times \pi_n(\operatorname{pr}_1,(x_0,x_1))\colon \pi_n(X_0 \times X_1,(x_0,x_1)) \xrightarrow{\cong} \pi_n(X_0,x_0) \times \pi_n(X_1,x_1)$$

for every $n \geq 1$.

2.4. **Homotopy groups and coverings.**

Theorem 2.7 (Homotopy groups and covering)**.** *Let $p\colon X \to Y$ be a covering. Choose a base point $x \in X$ and put $y = p(x)$. Then for $n \geq 2$ the map induced by p*

$$\pi_n(p,x)\colon \pi_n(X,x) \to \pi_n(Y,y)$$

is bijective.

Proof. Consider a map $f\colon S^n \to Y$ sending the base point s to y. Since $n \geq 2$ holds by assumption, S^n is simply connected. Hence the image of $\pi_1(f,x)$ is contained in the image $\pi_1(p,x)$. A standard theorem about coverings and liftings implies that we can find a lift $\widetilde{f}\colon (S^n,s) \to (X,x)$ of f, i.e., a pointed map $\widetilde{f}$ satisfying $p \circ \widetilde{f} = f$. This shows that $\pi_n(p,x)$ is surjective for $n \geq 2$.

Injectivity follows from the standard theorem about lifting homotopies along coverings, the argument is the same as for the injectivity of $\pi_1(p,x)$. This standard theorem says that for a map $u\colon Z \to X$ and a homotopy $h\colon Z \times I \to X$ with $h_0 = p \circ u$ we can find precisely one homotopy $\widetilde{h}\colon Z \to X$ with $p \circ \widetilde{h} = h$ and $\widetilde{h}_0 = u$. □

Theorem 2.7 implies for a connected CW-complex X that for the universal covering $p\colon \widetilde{X} \to X$ and any choice of base points $\widetilde{x} \in \widetilde{X}$ and $x \in X$ with $p(\widetilde{x}) = x$ the map $\pi_n(p,\widetilde{x})\colon \pi_n(\widetilde{X},\widetilde{x}) \to \pi_n(X,x)$ is bijective for $n \geq 2$. If we additionally assume that $\widetilde{X}$ is contractible, we get $\pi_n(X,x) = 0$ for $n \geq 2$. In particular we get for any base point $s \in S^1$ and $n \geq 1$

$$\pi_n(S^1,s) \cong \begin{cases} \mathbb{Z} & \text{if } n = 1; \\ \{1\} & \text{if } n \geq 2, \end{cases} \tag{2.8}$$

since the universal covering of S^1 is given by the map $\mathbb{R} \to S^1$ sending $t \in \mathbb{R}$ to $\exp(2\pi i t)$.

2.5. **The long exact sequence of a pair and a triple.** Consider a pointed pair (X,A,x), i.e., a pair of topological spaces (X,A) together with a base point $x \in A$. We can consider I^{n-1} as the subspace of I^n given by those points $(s_1,s_2,\ldots,s_n)$ satisfying $s_n = 0$. Let J_{n-1} be the subspace of ∂I^n which is the closure of $\partial I^n \setminus I^{n-1}$ in ∂I^n. Explicitly we get

$$\begin{aligned} J_{n-1} = (\partial I^n \setminus I^{n-1}) \cup \partial I^{n-1} = \{&(s_1,s_2,\ldots,s_n) \in I^n \\ &\mid (\exists i \in \{1,2,\ldots,(n-1)\} \text{ with } s_i \in \{0,1\}) \text{ or } \ (s_n = 1)\}. \end{aligned}$$

Obviously $I_{n-1} \cup J_{n-1} = \partial I^n$ and $I_{n-1} \cap J_{n-1} = \partial I^{n-1}$. For $n \geq 1$ we define the set $\pi_n(X,A,x)$ as the set homotopy classes $[f]$ of maps of triples $f\colon (I^n, \partial I^n, J_{n-1}) \to (X,A,\{x\})$. For $n \geq 2$, this becomes a group by defining $[f_0]\cdot[f_1]$ by the class $[f_0 * f_1]$ for the maps of triples $f_0 * f_1\colon (I^n,\partial I^n, J_{n-1}) \to (X,A,\{x\})$ defined in (2.3). There is no reasonable group structure on $\pi_1(X,A,x)$. It is not hard to check that this group structure on $\pi_n(X,A,x)$ for $n \geq 2$ is well-defined and that the following result is true.

Lemma 2.9. *The group $\pi_n(X,A,x)$ is abelian for $n \geq 3$.*

Note that there is an obvious identification of $\pi_n(X,\{x\},x)$ defined above and of $\pi_n(X,x)$ defined in Subsection 2.2.

Obviously we obtain a functor from the category of pointed pairs to the category of groups by $\pi_2(X,A,x)$ and a functor from the category of pointed pairs to the category of abelian groups by $\pi_n(X,A,x)$ for $n \geq 3$. If two maps $f_0,f_1\colon (X,A,x) \to (Y,B,y)$ of pointed pairs are homotopic as maps of pointed pairs, then $\pi_n(f_0,x) = \pi_n(f_1,x)$ holds for $n \geq 1$. Given a pair (X,A), one can define a functor T_n from the fundamental groupoid $\Pi(A)$ of A to the category of groups or abelian groups by assigning to a point $x \in A$ the homotopy group $\pi_2(X,A,x)$ or $\pi_n(X,A,x)$ for $n \geq 3$, the construction appearing in Subsection 2.3 for a space X carries directly over. In particular $\pi_n(X,A,x)$ inherits the structure of a $\mathbb{Z}[\pi_1(A,x)]$-module for $n \geq 3$.

A map of triples $f\colon (I^n, \partial I^n, J_{n-1}) \to (X, A, \{x\})$ factorizes uniquely through the projection $\mathrm{pr}\colon (I^n, \partial I^n, J_{n-1}) \to (I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1})$ to a map of pointed pairs $(I^n/J_{n-1}, \partial I^n/J_{n-1}, J_{n-1}/J_{n-1}) \to (X,A,x)$. There is a homeomorphism $(I^n/J_{n-1}, \partial I^n/J_{n-1}, \{J_{n-1}/J_{n-1}\}) \xrightarrow{\cong} (D^n, S^{n-1}, \{s\})$ of triples. Hence one can define $\pi_n(X,A,x)$ also the set of homotopy classes of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \to (X,A,x)$. The multiplication in this picture is given as follows. Consider pointed maps of pointed pairs $f_i\colon (D^n, S^{n-1}, s) \to (X,A,x)$ for $i = 0,1$. Let $[f_0]$ and $[f_1]$ be their classes in $\pi_n(X,A,x)$. They define a pointed map of pointed pairs $f_0 \vee f_1\colon (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s) \to (X,A,x)$. Let

$$\nabla'_n\colon D^n \to D^n \vee D^n \tag{2.10}$$

be the so-called *pinching map* which is obtained by collapsing $D^{n-1} \subseteq D^n$ given by $\{(x_1, \ldots, x_n) \in D^n \mid x_n = 0\}$ to a point. Note that ∇'_n is a map of pointed pairs $(D^n, S^{n-1}, s) \to (D^n \vee D^n, S^{n-1} \vee S^{n-1}, s)$ and its restriction to (S^{n-1}, s) is the pinching map defined in (2.5). Then $[f_0] \cdot [f_1]$ is represented by the composite $f_0 \vee f_0 \circ \nabla'_n$.

Define for $n \geq 2$ a group homomorphism $\partial_n\colon \pi_n(X,A,x) \to \pi_1(A,x)$ by sending the class $[f]$ of the map of pointed pairs $f\colon (D^n, S^{n-1}, s) \to (X,A,s)$ to the pointed homotopy class of maps of pointed spaces obtained by restriction to (S^{n-1}, s). Let $i\colon A \to X$ and $j\colon X \to (X,A)$ be the canonical inclusions.

Theorem 2.11. *We obtain a long exact sequence of groups infinite to the left*

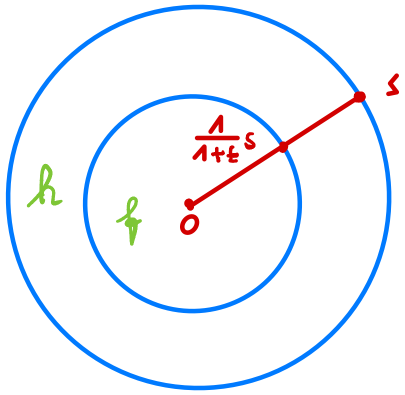
$$\begin{aligned} \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A,x) &\xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X,x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X,A,x) \\ &\xrightarrow{\partial_{n+1}} \pi_n(A,x) \xrightarrow{\pi_n(i,x)} \pi_n(X,x) \xrightarrow{\pi_n(j,x)} \\ &\cdots \xrightarrow{\pi_2(j,x)} \pi_2(X,A,x) \xrightarrow{\partial_2} \pi_1(A,x) \xrightarrow{\pi_1(i,x)} \pi_1(X,x). \end{aligned}$$

Proof. We only show exactness at $\pi_n(X,A,x)$, the proofs at the other places are analogous. Consider a pointed map $f\colon (S^n, s) \to (X,x)$. The image of the class $[f]$ under the composite $\pi_n(X,x) \xrightarrow{\pi_n(j,x)} \pi_n(X,A,x) \xrightarrow{\partial_n} \pi_{n-1}(A,x)$ is by construction represented by the constant map $c_x\colon S^{n-1} \to A$ and hence zero. This shows $\mathrm{im}(\pi_n(j,x)) \subseteq \ker(\partial_n)$. It remains to prove $\ker(\partial_n) \subseteq \mathrm{im}(\pi_n(j,x))$.

Consider a map of pointed pairs $f\colon (D^n, S^{n-1}, s) \to (X,A,x)$ such that $[f]$ lies in the kernel of $\partial_n\colon \pi_n(X,A,x) \to \pi_{n-1}(A,x)$. Then the map of pointed spaces $f|_{S^{n-1}}\colon (S^{n-1}, s) \to (A,x)$ is nullhomotopic as pointed map of pointed spaces. Choose such a nullhomotopy $h\colon S^{n-1} \times I \to A$ with $h_0 = f|_{S^{n-1}}$ and $h_1 = c_x$ for the constant function. Note that $h(s,t) = x$ holds for $t \in I$. Define a homotopy $k\colon D^n \times I \to X$ as follows:

$$k(z,t) = \begin{cases} f((t+1)z) & \text{if } ||z|| \leq \frac{1}{1+t}; \\ h\big(\frac{z}{||z||}, 2||z|| - \frac{2}{1+t}\big) & \text{if } ||z|| \geq \frac{1}{1+t}. \end{cases}$$

Roughly speaking, k_t is given on the disk $\frac{1}{1+t} \cdot D^n$ of radius $\frac{1}{1+t}$ by f with an appropriate scaling of z and on the anulus between $\frac{1}{1+t} \cdot S^{n-1}$ and S^{n-1} by the restriction of the homotopy h to $S^1 \times [2 - 2/(1+t), 1]$



We have $k(z,0) = f(z)$ for $z \in D^n$, $k(s,t) = x$ for $t \in I$, $k(z,t) \in A$ for $z \in S^{n-1}$ and $t \in I$, and $k(z,1) = x$ for $z \in S^{n-1}$. Hence k is a homotopy of pointed maps of pointed pairs $(D^n, S^{n-1}, s) \to (X, A, x)$ between $k_0 = f$ and k_1. Therefore $[f] = [k_1]$ holds in $\pi_n(X, A, x)$. Since $k_1(z) = x$ holds for $z \in S^{n-1}$, the class $[k_1]$ lies in the image of $\pi_n(j, x)\colon \pi_n(X, x) \to \pi_n(X, A, x)$. Hence we get $\operatorname{im}(\pi_n(j, x)) = \ker(\partial_n)$. □

Remark 2.12. Let G be any group. Then we can find a path connected pointed 2-dimensional CW-complex (A, x) with $\pi_1(A, x) \cong G$. Let X be the cone over A. Then we obtain a path connected pointed 3-dimensional CW-complex (X, A, x) such that $\pi_3(X, A, x) \cong \pi_2(A, x) \cong G$ holds by Theorem 2.11.

Remark 2.13. One can combine the exact sequences appearing in Theorem 2.2 and Theorem 2.11 to an exact sequence

$$(2.14) \quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(A, x) \xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X, A, x)$$
$$\xrightarrow{\partial_{n+1}} \pi_n(A, x) \xrightarrow{\pi_n(i,x)} \pi_n(X, x) \xrightarrow{\pi_n(j,x)} \cdots \xrightarrow{\pi_2(j,x)} \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x)$$
$$\xrightarrow{\pi_1(i,x)} \pi_1(X, x) \xrightarrow{\partial_1} \pi_0(A) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(j)} \pi_0(X, A) \to \{*\}$$

which is compatible with the group structures as long as these exist.

It is not hard to check that one obtains for a triple (X, B, A) and a base point $x \in A$ an exact sequence of the shape

$$(2.15) \quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(B, A, x) \xrightarrow{\pi_{n+1}(i,x)} \pi_{n+1}(X, A, x) \xrightarrow{\pi_{n+1}(j,x)} \pi_{n+1}(X, B, x)$$
$$\xrightarrow{\partial_{n+1}} \pi_n(B, A, x) \xrightarrow{\pi_n(i,x)} \pi_n(X, A, x) \xrightarrow{\pi_n(j,x)}$$
$$\cdots \xrightarrow{\pi_2(j,x)} \pi_2(X, B, x) \xrightarrow{\partial_2} \pi_1(B, A, x) \xrightarrow{\pi_1(i,x)} \pi_1(X, A, x) \xrightarrow{\pi_1(j,x)} \pi_1(X, B, x)$$
$$\xrightarrow{\partial_1} \pi_0(B, A) \xrightarrow{\pi_0(i)} \pi_0(X, A) \xrightarrow{\pi_0(j)} \pi_0(X, B) \to \{*\}$$

which is compatible with the group structures as long as these exist.

Remark 2.16 (Long exact homotopy sequence of a pointed map)**.** Let $f\colon (X, x) \to (Y, y)$ be a map of pointed spaces. Denote by $\operatorname{cyl}(f)$ its mapping cylinder. Note that we obtain a pointed pair $(\mathrm{cyl}(f), X, x)$. The canoncial projection $\mathrm{cyl}(f) \to Y$ is a homotopy equivalence and satisfies $\mathrm{pr}(x) = y$. Hence it induces an isomorphism of groups $\pi_n(\mathrm{pr}, x)\colon \pi_n(\mathrm{cyl}(f), x) \xrightarrow{\cong} \pi_n(Y, y)$ for $n \geq 1$ and a bijection $\pi_0(\mathrm{cyl}(f)) \xrightarrow{\cong} \pi_0(Y)$. Define $\pi_n(f, x) = \pi_n(\mathrm{cyl}(f), X, x)$ for $n \geq 1$. Let $\pi_0(f)$ be the quotient of $\pi_0(Y)$ obtained by collapsing the image of $\pi_0(f)\colon \pi_0(X) \to \pi_0(f)$. Then the long exact sequence of the pointed pair $(\mathrm{cyl}(f), X, x)$ of (2.14) yields the long exact homotopy sequence of the map f

$$(2.17) \quad \cdots \xrightarrow{\partial_{n+2}} \pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(f,x)} \pi_{n+1}(Y, y) \to \pi_{n+1}(f, x) \xrightarrow{\partial_{n+1}} \pi_n(X, x) \xrightarrow{\pi_n(f,x)} \pi_n(Y, y) \to \cdots \to \pi_2(f, x) \xrightarrow{\partial_2} \pi_1(X, x) \xrightarrow{\pi_1(f,x)} \pi_1(Y, y) \xrightarrow{\partial_1} \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y) \to \pi_0(f) \to \{1\}.$$

Note that $\pi_n(f, x)$ can have two different meanings in the notation above.

2.6. Connectivity.

Definition 2.18 (Connectivity)**.** A space X is called 0-*connected* if $\pi_0(X)$ consists of one point, or, equivalently, X is path connected. It is called n-*connected* for $n \geq 1$ if X is path connected and $\pi_k(X, x)$ is trivial for every base point x and $1 \leq k \leq n$. It is called ∞-*connected* or *weakly contractible* if it is path connected and $\pi_k(X, x)$ is trivial for every base point x and $k \geq 1$.

A map $f\colon X \to X$ is called 0-*connected* if the induced map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective. It is called n-*connected* for $n \geq 1$, if the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is bijective and for every base point x the map $\pi_k(f, x)\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is bijective for $1 \leq k < n$ and surjective for $k = n$. It is called ∞-*connected* or a *weak homotopy equivalence* if the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is bijective and for every base point x and $k \geq 1$ the map $\pi_k(f, x)\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is bijective. Note that f is n-*connected* if and only if $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective and the group $\pi_k(f, x)$ defined in Remark 2.16 is trivial for $1 \leq k \leq n$.

A pair (X, A) is called n-*connected* for $n \geq 0$ or $n = \infty$, if the inclusion $i\colon A \to X$ is n-connected. This is equivalent to the condition that $\pi_0(X, A)$ and $\pi_k(X, A, x)$ for $1 \leq k \leq n$ are trivial.

Remark 2.19. One easily checks that the following assertions are equivalent for a pointed space (X, x) and $n \geq 1$:

- $\pi_n(X, x)$ is trivial for any base point $x \in X$;
- Every map $S^n \to X$ is nullhomotopic;
- Every map $S^n \to X$ extends to a map $D^{n+1} \to X$.

This implies that the following assertions are equivalent for a space X and $n \geq 0$ or $n = \infty$:

- X is n-connected;
- Given any k with $0 \leq k \leq n$, every map $S^k \to X$ is nullhomotopic;
- Given any k with $0 \leq k \leq n$, every map $S^k \to X$ extends to a map $D^{k+1} \to X$.

Moreover, the following assertions are equivalent for a pair (X, A) and $n \geq 0$ or $n = \infty$:

- (X, A) is n-connected;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic relative S^{k-1} to a map $D^k \to A$;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic through such maps to a map $D^k \to A$;
- Given any k with $0 \leq k \leq n$, every map $(D^k, S^{k-1}) \to (X, A)$ is homotopic through such maps to a constant map $D^k \to A$.

2.7. **Homotopy groups and colimits.**

Theorem 2.20 (Homotopy groups and colimits)**.** *Let X be a topological Hausdorff space with a sequence of closed subspaces $X_0 \subset X_1 \subseteq \cdots \subseteq X$ such that X is the union of the X_i-s and carries the colimit topology.*

Then for every $x_0 \in X$ and $n \geq 1$ the canonical group homomorphism induced by the inclusions $j_k \colon X_k \to X$

$$\operatorname{colim}_{k\to\infty} \pi_n(j_k, x_0) \colon \operatorname{colim}_{k\to\infty} \pi_n(X_k, x_0) \to \pi_n(X, x_0)$$

is bijective. Also the map of sets

$$\operatorname{colim}_{k\to\infty} \pi_0(j_k) \colon \operatorname{colim}_{k\to\infty} \pi_0(X_k) \to \pi_0(X, x_0)$$

is bijective.

Proof. We first prove that for any compact subset $C \subseteq X$ there exists a natural number k with $C \subseteq X_k$. Suppose that for every $k \geq 0$ we have $C \nsubseteq X_k$. Then we can choose a sequence of $x_0, x_1, x_2, \ldots$ in C and a strictly monotone increasing function $j \colon \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ with $x_i \in X_{j(i)} \setminus X_{j(i-1)}$ for $i = 1, 2, \ldots$. Put $S = \{x_0, x_1, x_2, \ldots\}$. Obviously S is infinite. Let $T \subseteq S$ be any subset. Note that the intersection $T \cap X_k$ is finite and hence a closed subset of X_k for $k = 0, 1, 2, \ldots$. Since X carries the colimit topology, T is closed in X. Hence S is a discrete subset of X. As C is compact and S is a closed subset of C, the set S is compact. As S is a discrete and compact set, it must be finite, a contradiction.

We only treat the case $n \geq 1$, the case $n = 0$ is analogous. Consider an element $[f] \in \pi_n(X, x_0)$ represented by a pointed map $f \colon (S^n, s) \to (X, x_0)$. Then image of f lies already in X_i for some $i \geq 0$. Hence $[f]$ lies in the image of the map $\pi_n(X_i, x_0) \to \pi_n(X, x_0)$ induced by the inclusion $X_i \to X$. This implies that $[f]$ lies in the image of $\operatorname{colim}_{k\to\infty} \pi_n(j_k, x_0) \colon \operatorname{colim}_{k\to\infty} \pi_n(X_k, x_0) \to \pi_n(X, x_0)$. Hence this map is surjective. To prove injectivity, we consider an element $[g]$ in its kernel. There exists $i \geq 0$ and an element $[g'] \in \pi_n(X_i, x_0)$ such that the structure map $\pi_n(X_i, x_0) \to \operatorname{colim}_{j\to\infty} \pi_n(X_j, x_0)$ sends $[g']$ to $[g]$. The element $[g']$ lies in the kernel of the map $\pi_n(X_i, x_0) \to \pi_n(X, x_0)$ induced by the inclusion $X_i \to X$. If $g' \colon (S^n, s) \to (X_i, x_0)$ is a representative of $[g']$, there is a nullhomotopy $h \colon S^n \times I \to X$ for it. The image of h lies already in X_j for some j with $i \leq j$. Hence the image of $[g']$ under the map $\pi_n(X_i, x_0) \to \pi_n(X_j, x_0)$ induced by the inclusion $X_i \to X_j$ is trivial. This implies that $[g]$ is trivial. □

3. Hopf's Degree Theorem

In this section we give the proof of the following theorem.

Theorem 3.1 (Hopf's Degree Theorem)**.** *Let M be a connected oriented closed smooth manifold of dimension $n \geq 1$. Then the degree defines a bijection*

$$\deg \colon [M, S^n] \to \mathbb{Z}.$$

3.1. **Some basics about differential topology and the mapping degree.** Its proof needs some preparation. We recall some basic facts about differential topology and the mapping degree.

- Let M and N be smooth manifolds. Then a (continuous) map $f \colon M \to N$ is homotopic to a smooth map. If two smooth maps $M \to N$ are homotopic, then one can find a smooth homotopy between them.
- Let M and N be smooth manifolds and $L \subseteq N \setminus \partial N$ be a smooth submanifold without boundary. Then any smooth map $f \colon M \to N$ with $f(\partial M) \cap L = \emptyset$ is smoothly homotopic relativ ∂M to a map $g \colon M \to N$ which is *transversal to* L at every $x \in M$, i.e., we have either $f(x) \notin L$ or we

have $f(x) \in L$ and $T_xf(T_xM) + T_{f(x)}L = T_{f(x)}N$. If $\dim(M) + \dim(L) < \dim(N)$ holds, then f is transversal to L if and only if $f(M) \cap L = \emptyset$.

- If $L = \{y\}$ for $y \in N \setminus \partial N$, then we say that y is a *regular value* of f if f is transversal to $\{y\}$.
- Every smooth map $f\colon M \to N$ has a regular value $y \in N \setminus \partial N$. Actually the points in $N \setminus \partial N$ for which y is not a regular value has measure zero in N by the Theorem of Sard.

  If $y \in N \setminus \partial N$ is a regular value of f, M is compact, and $\dim(M) = \dim(N)$, then $f^{-1}(y)$ is finite and for every $x \in f^{-1}(y)$ the differential induces an isomorphism $T_xf\colon T_xM \to T_yN$.
- Let $f\colon M \to N$ be a map of connected oriented compact smooth oriented manifolds of dimension n such that $f(\partial M) \subseteq \partial N$ holds. Let $y \in N \setminus \partial N$ be any regular value. For $x \in f^{-1}(y) \subseteq M \setminus \partial M$ the orientations on M and N yield orientations on the finite dimensional vector spaces T_xM and T_yN. Define $\epsilon(x) \in \{\pm 1\}$ to be 1 if $T_xf\colon T_xM \xrightarrow{\cong} T_yN$ respects these orientations and to be -1 otherwise.

  Recall degree of f is the natural number for which $H_n(f)\colon H_n(M,\partial M) \to H_n(N,\partial N)$ sends $[M,\partial M]$ to $\deg(f)\cdot[N,\partial N]$. We get

$$\deg(f) = \sum_{x \in f^{-1}(y)} \epsilon(x). \tag{3.2}$$

  This formula is well-known for $\partial M = \partial N = \emptyset$. The proof in this case extends directly to the more general case above. Or one considers the map of closed oriented manifolds $f \cup_{\partial f} f\colon M \cup_{\partial M} M \to N \cup_{\partial N} N$ for $\partial f\colon \partial M \to \partial N$ given by $f|_{\partial M}$.
- Let M be a smooth Riemannian manifold and $x \in M \setminus \partial M$. Then there is an $\epsilon > 0$, an open subset U of M containing x, and a diffeomorphism called *exponential map*

$$\exp_x\colon D^\circ_\epsilon T_xM := \{v \in T_xM \mid ||x|| < \epsilon\} \to U \tag{3.3}$$

  such that the differential $T_0\exp_x\colon T_0(T_xM) \to T_xM$ of $\exp_x$ at $0 \in T_xM$ becomes the identity under the canoncial identification $T_0(T_xM) = T_xM$.

3.2. **The proof of Hopf's Degree Theorem.** We prove Hopf's Degree Theorem 3.1 by induction over the dimension $n = \dim(M)$. If $n = 1$, then M is diffeomorphic to S^1 and elementary covering theory shows that the degree induces a bijection $\deg\colon [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. The induction step from $(n-1)$ to $n \geq 2$ is done as follows.

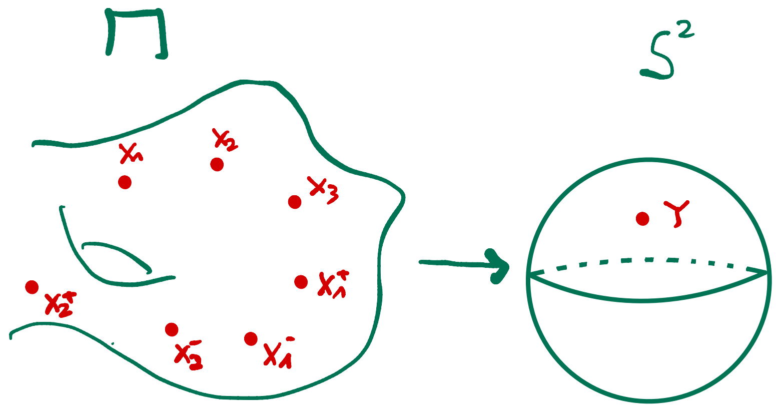
Fix $x \in M$ and an embedding $i\colon D^n \subseteq M$ such that $i(0) = x$ holds and $T_0i\colon T_0D^n \xrightarrow{\cong} T_xM$ is compatible with the orientations coming from the standard orientation on D^n and the given orientation on M. Define the collaps map $c\colon M \to D^n/S^{n-1} \cong S^n$ by sending $i(x)$ for $x \in D^n$ to the element given by x in D^n/S^{n-1} and every point $y \in M \setminus i(D^n)$ to the point S^{n-1}/S^{n-1} in D^n/S^{n-1}. We conclude from (3.2) applied to the regular value $z \in D^n/S^{n-1} = S^n$ given by $0 \in D^n$ of c that $\deg(c) = 1$. Given any $d \in \mathbb{Z}$, there exists a selfmap $u_d\colon S^n \to S^n$ with $\deg(u_d) = d$. It can be constructed as the $(n-1)$-fold suspension of the map $S^1 \to S^1$ sending z to z^d. Then $\deg(u_d \circ c) = d$. This shows that $\deg\colon [M, S^n] \to \mathbb{Z}$ is surjective.

In order to show that $\deg\colon [M, S^n] \to \mathbb{Z}$ is injective, we have to show that two smooth maps $f, g\colon M \to S^n$ with $\deg(f) = \deg(g)$ are homotopic. Since there is diffeomorphism $u\colon S^n \to S^n$ with degree -1 and $\deg(u \circ f) = -\deg(f)$, we can assume in the sequel that $d = \deg(f) = \deg(g)$ satisfies $d \geq 0$.

We can change f and g up to homotopy and find $y \in S^n$ such that both f and g are smooth and have y as regular value. Then we can write

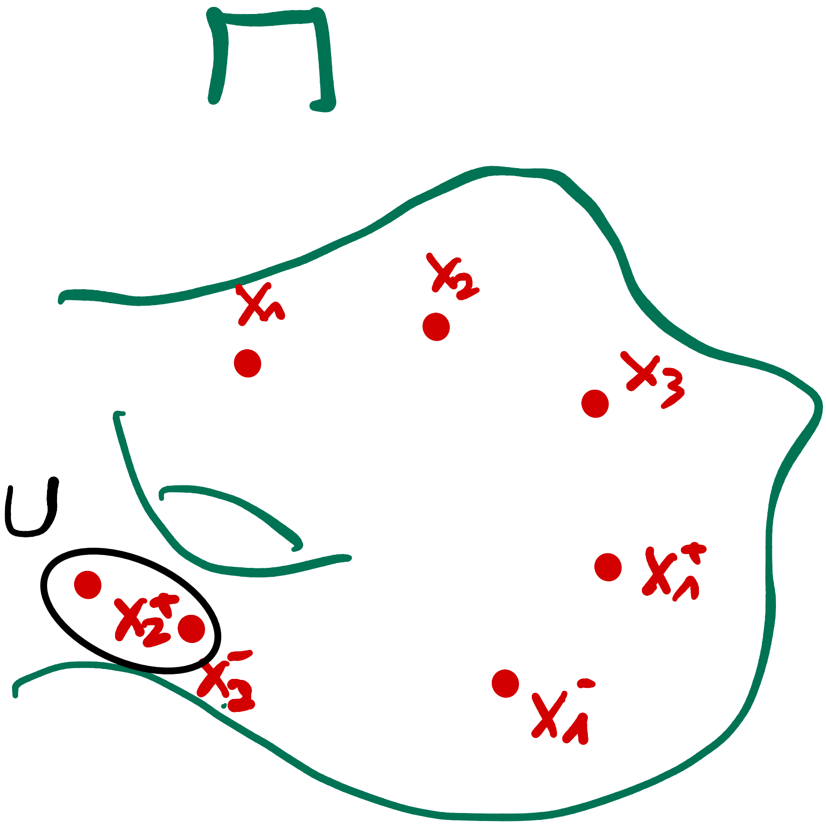
$$f^{-1}(y) = \{x_1, x_2, \dots, x_d\} \amalg \{x_1^+, x_1^-, \dots, x_m^+, x_m^-\}$$

for some $m \geq 0$ such that $\epsilon(x_i) = 1$ for $i = 1, 2, \dots, d$ and $\epsilon(x_j^\pm) = \pm 1$ holds for $j = 1, 2, \dots, m$.



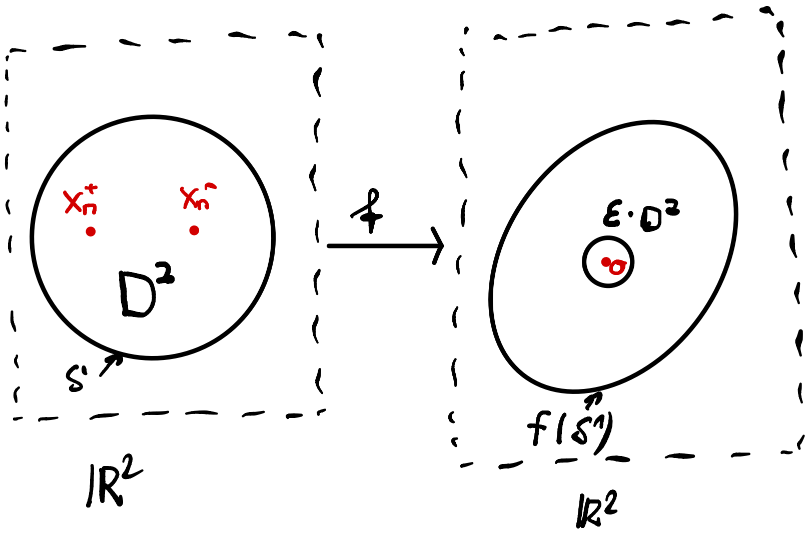
We next describe a procedure how to change f up to homotopy so that $m = 0$, or, equivalently $f^{-1}(y) = \{x_1, x_2, \dots, x_d\}$ holds. This will be done by an inductive procedure where we change f up to homotopy such that $m \geq 1$ becomes $(m-1)$, in other words, we get rid of the points x_m^+ and x_m^-.

Choose an embedded arc in M joining x_m^+ and x_m^- that does not meet any of the other points in $f^{-1}(y)$. Let U be an open neighbourhood of x_m^- that is diffeomorphic to $\mathbb{R}^n$. Now perform a local homotopy of f along this arc to move x_{2m-1} so close to x_{2m} such that x_m^+ lies in U.

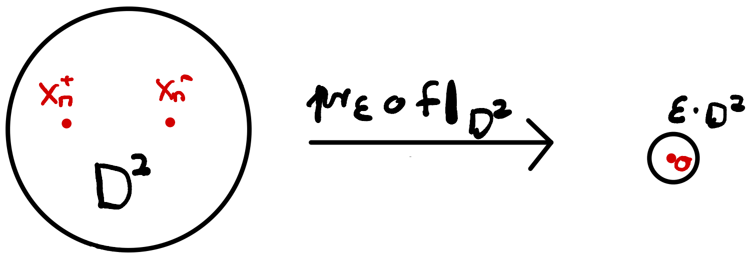


Hence it suffices to prove the following: Given a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that f is transversal to $0 \in \mathbb{R}^n$, the preimage $f^{-1}(0)$ consists of precisely two points x_0 and x_1 belonging to the interior of the disk $D^n \subseteq \mathbb{R}^n$, the differential $T_{x_0}f\colon T_{x_0}\mathbb{R}^n \to T_0\mathbb{R}^n$ is bijective and reverses the standard orientations, and the differential $T_{x_1}f\colon T_{x_1}\mathbb{R}^n \to T_0\mathbb{R}^n$ is bijective and preserves the standard orientations, then we can change f up to homotopy relative $\mathbb{R}^n \setminus D^n$ so that $f^{-1}(0)$ is empty.

Choose $\epsilon > 0$ so small that the image of $S^{n-1} \subseteq \mathbb{R}^n$ under f does not meet the interior of $\epsilon \cdot D^n$.



Let $\mathrm{pr}_\epsilon \colon \mathbb{R}^n \to \epsilon \cdot D_n$ be the retraction that sends $x \in \mathbb{R}^n$ to $\frac{\epsilon}{||x||} \cdot x$ if $||x|| \geq \epsilon$, and to x if $||x|| \leq \epsilon$. Then $\mathrm{pr}_\epsilon \circ f$ induces a map of compact oriented manifolds $(D^n, S^{n-1}) \to (\epsilon \cdot D^n, \epsilon \cdot S^{n-1})$. By inspecting the preimage of $0 \in \epsilon \cdot D^n$ we conclude from (3.2) that its degree is zero.



Since the following diagram commutes and the vertical maps given by boundary homomorphisms of pairs are isomorphism of infinite cyclic groups respecting the fundamental classes

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow{H_n(f)} & H_n(\epsilon \cdot D^n, \epsilon \cdot S^{n-1}) \\ \cong \downarrow & & \downarrow \cong \\ H_n(S^{n-1}) & \xrightarrow[H_n(f|_{S^{n-1}})]{} & H_n(\epsilon \cdot S^{n-1}) \end{array}$$

the induced map $(\mathrm{pr}_\epsilon \circ f)|_{S^{n-1}} \colon S^{n-1} \to \epsilon \cdot S^{n-1}$ has degree zero and hence is nullhomotopic by the induction hypothesis. This implies that the map $f_0 \colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ induced by f is nullhomotopic and hence extends to a map $f_1 \colon D^n \to \mathbb{R}^n \setminus \{0\}$. Let $f' \colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$ be the map whose restriction to D^n is f_1 and whose restriction to $\mathbb{R}^n \setminus D^n$ agrees with the restriction of f to $\mathbb{R}^n \setminus D^n$. We obtain a homotopy $h \colon f \simeq f'$ of maps $\mathbb{R}^n \to \mathbb{R}^n$ by $h(x,t) = t \cdot f'(x) + (1-t) \cdot f$ that is stationary outside the interior of D^n. Since the image of f' does not contain zero, the claim follows.

This argument applies also to g. If $d = 0$, then $\mathrm{im}(f)$ and $\mathrm{im}(g)$ are contained in the contractible subspace $S^n \setminus \{y\}$ of S^n and hence f and f' are homotopic. It remains to consider the case $d \geq 1$. Then we can find finite subsets $\{x_1, x_2, \ldots, x_d\}$ and $\{x'_1, x'_2, \ldots, x'_d\}$ of M such that $f^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$ and $g^{-1}(y) = \{x'_1, x'_2, \ldots, x'_d\}$ holds and the differentials $T_{x_i}f \colon T_{x_i}M \to T_yS^n$ and $T_{x'_i}g \colon T_{x'_i}M \to T_yS^n$ are orientation preserving isomorphisms for $i = 1, 2, \ldots, n$. Now we can construct a diffeomorphism $a \colon M \to M$ which is homotopic to the identity and satisfies $w(x_i) = x'_i$ for $i = 1, 2, \ldots, d$. Then g and $g' = g \circ a$ are homotopic, $f^{-1}(y) = g'^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$ and the differentials $T_{x_i}f \colon T_{x_i}M \to T_yS^n$ and $T_{x_i}g' \colon T_{x_i}M \to T_yS^n$ are orientation preserving isomorphisms for $i = 1, 2, \ldots, d$. It remains to show that f and g' are homotopic.

For this purpose we need the following construction. Let $u_0, u_1 \colon \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ be linear $\mathbb{R}$-isomorphisms which are orientation preserving. Then we can find a homotopy $h \colon \mathbb{R}^n \times I \to \mathbb{R}^n$ such that $h_0 = u_0$ and $h_1 = u_1$ holds and $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ is a orientation preserving automorphism for $t \in I$. This follows from the fact that $\{A \in GL_n(\mathbb{R}) \mid \det(A) > 0\}$ is path connected for $n \geq 1$. Define the homotopy

$$H \colon \mathbb{R}^n \times I, \quad (v,t) \mapsto \begin{cases} h_t(v) & \text{if } ||v|| \leq 1; \\ h_{(2-||v||)\cdot t}(v) & \text{if } 1 \leq ||v|| \leq 2; \\ u_0(v) & \text{if } ||v|| \geq 2. \end{cases}$$

Then we have

$$\begin{aligned} H_t^{-1}(0) &= 0 \quad \text{for } t \in I; \\ H_0 &= u_0; \\ H_t(v) &= u_0(v) \quad \text{for } t \in I \text{ and } ||v|| \geq 2; \\ H_1(v) &= \begin{cases} u_1(v) & \text{if } ||v|| \leq 1; \\ h_{(2-||v||)}(v) & \text{if } 1 \leq ||v|| \leq 2; \\ u_0(v) & \text{if } ||v|| \geq 2. \end{cases} \end{aligned}$$

So H is a homotopy between $H_0 = u_0$ and H_1 which is stationary on $\{v \in V \mid ||v|| \geq 2\}$ and satisfies $H_t^{-1}(0) = 0$ for $t \in I$ and $H_1(v) = u_1(v)$ for $||v|| \leq 1$.

Using this construction and the exponential map (3.3), we can change g' by a homotopy to a map $g'' \colon M \to S^n$, such that for $i = 1, 2, \ldots, d$ there are disjoint embedded disks $D^n_i \subseteq M$ such that $0 \in D^n_i$ corresponds to x_i, $f|_{D^n_i} = g''|_{D^n-i}$ holds and we have $f^{-1}(y) = (g'')^{-1}(y) = \{x_1, x_2, \ldots, x_d\}$. Let X be the complement in M of the disjoint union $\coprod_{i=1}^d D^n_i \setminus \partial S^{n-1}_i$. This is a manifold with boundary $\partial X =$

$\coprod_{i=1}^d S_i^{n-1}$ such that $f(X)$ and $g''(X)$ are contained in $S^n \setminus \{y\}$ and $f|_{\partial X} = g''|_{\partial X}$ holds. As $S^n \setminus \{y\}$ is contractible, the maps $f|_X$ and $g''|_X$ from X to S^n are homotopic relative ∂X. Recall that f and g'' agree on $\coprod_{i=1}^d D_i^n$. Hence f and g'' are homotopic as maps $M \to S^n$. This implies that the maps f and g from M to S^n are homotopic. This finishes the proof of Hopf's Degree Theorem 3.1.

3.3. The homotopy groups of the n-sphere in the degree $\le n$.

Theorem 3.4. *We get for every $n \ge 1$*

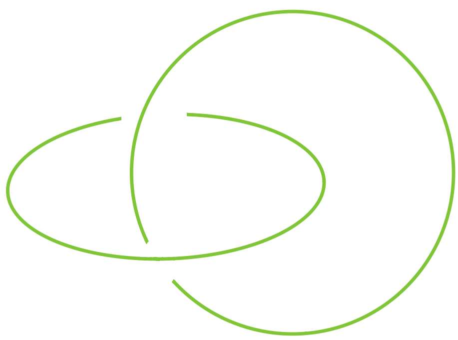
$$\pi_k(S^n) \cong \begin{cases} \{0\} & k < n; \\ \mathbb{Z} & k = n. \end{cases}$$

There is an explicit isomorphism $\mathbb{Z} \xrightarrow{\cong} \pi_n(S^n)$ which sends $1 \in \mathbb{Z}$ to $[\mathrm{id}_{S^n}]$. Its inverse $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ sends $[f]$ to the degree of f.

Proof. Suppose $k < n$. Let $f\colon S^k \to S^n$ be any map. Since we can change any map $f\colon S^k \to S^n$ up to homotopy into a smooth map transversal to $y \in S^n$, we can change f by a homotopy to map $S^n \to S^n \setminus \{y\}$. As $S^n \setminus \{y\}$ is contractible, f is nullhomotopic. This implies $\pi_k(S^n, s)$ for every $s \in S$.

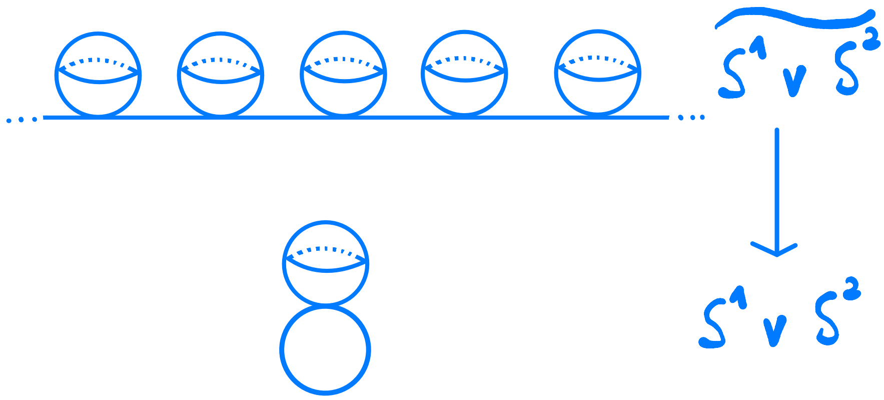
The degree defines a bijection $\deg\colon [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$ because of Hopf's Degree Theorem 3.1 for $n \ge 1$. By inspecting the proof of surjectivity of this map we see that the forgetful map $\pi_n(S^n, s) \to [S^n, S^n]$ is surjective. We conclude from (2.1) and (2.6) that the forgetful map $\pi_n(S^n, s) \to [S^n, S^n]$ is injective. □

Example 3.5 (The Hopf map and $\pi_3(S^2)$)**.** One may think that $\pi_k(S^n, s)$ vanishes for $k > n$ as $H_k(S^n)$ vanishes for $k > n$. But this is not true as the following example due to Hopf shows. We can think of S^3 as the subset of $\mathbb{C}^2$ given by $\{(z_1, z_2) \mid z_1\overline{z_1} + z_2\overline{z_2} = 1\}$. We get an S^1-action on S^3 by $z \cdot (z_1, z_2) = (zz_1zz_2)$. This action is free and the quotient space S^3/S^1 is homeomorphic to S^2. Thus we get a projection $p\colon S^3 \to S^2$. We will later show that $\pi_3(S^2) \cong \mathbb{Z}$ holds with the class $[p]$ of p as generator. One indication that $[p]$ is not zero in $\pi_3(S^2)$ is the observation that the preimages of the north and the south pole of S^2 are two embedded S^1-s in S^3 which are linked.



Example 3.6 ($\pi_n(S^1 \vee S^n)$ is not finitely generated.)**.**

Consider $X = S^1 \vee S^n$ for $n \ge 2$. Its universal covering $\widetilde{X}$ is obtained from $\mathbb{R}$ by glueing to each element in $\mathbb{Z}$ a copy of S^n along the base point.



The map $\widetilde{X} \to \bigvee_{i\in\mathbb{Z}} S^n$ given by collapsing $\mathbb{R}$ to point turns out to be a point homotopy equivalence. This can be seen by a direct inspection. Hence we conclude $\pi_2(X) \cong \pi_2(\widetilde{X}) \cong \pi_2(\bigvee_{i\in\mathbb{Z}} S^n)$ from Theorem 2.7. For each $k \in \mathbb{Z}$ we have the pointed inclusion $j_k \colon S^n \to \bigvee_{i\in\mathbb{Z}} S^n$ of the k-th summand and the pointed projection $\mathrm{pr}_k \colon \bigvee_{i\in\mathbb{Z}} S^n \to S^n$ onto the k-th summand. Obviously $\mathrm{pr}_k \circ j_k$ is the identity and $\mathrm{pr}_k \circ j_l$ is the constant map for $k \neq l$. Hence the map $\bigoplus_{i\in\mathbb{Z}} \pi_n(j_i) \colon \bigoplus_{i\in\mathbb{Z}} \pi_n(S^n) \to \pi_n(\bigvee_{i\in\mathbb{Z}} S^n)$ is injective. As $\pi_n(S^n) \cong \mathbb{Z}$, the abelian group $\pi_n(S^1 \vee S^n)$ is not finitely generated.

Actually, we know that $\pi_n(S^1 \cup S^n)$ is a $\mathbb{Z}[\pi_1(S^1)]$-module and it will turn out that it is $\mathbb{Z}[\pi_1(S^1)]$-isomorphic to $\mathbb{Z}[\pi_1(S^1)]$.

Remark 3.7 (Outlook about $\pi_k(S^n)$ for $k > n$)**.** It is an open (and extremely hard) problem to compute $\pi_k(S^n, s)$ for $2 \leq n < k$ in general. There is not even a formula known which might give the answer. There is no obvious pattern in the computations, one has carried out so far. At least one knows that $\pi_k(S^n)$ is finite for $k > n$ except for $\pi_{4i-1}(S^{2i})$ for $i \geq 1$ which is a direct sum of a copy of $\mathbb{Z}$ and some finite abelian group.

4. The Cellular Approximation Theorem

In this section we want to sketch the proof of the following theorem.

Theorem 4.1 (Cellular Approximation Theorem)**.** *Let (X, A) be a CW-pair and Y be a CW-complex. Let $f \colon X \to Y$ be a map whose restriction $f|_A \colon A \to Y$ to A is cellular. Then f is homotopic relative A to a cellular map $X \to Y$.*

By a colimit argument one can reduce the proof of the Cellular Approximation Theorem 4.1 to the proof of following lemma.

Lemma 4.2. *Consider any $k \in \{0, 1, 2, \ldots\}$. Let $f \colon X \to Y$ be a map of CW-complexes. Suppose that $f(X_{k-1}) \subseteq Y_{k-1}$ holds.*

Then we can change f up to homotopy relative X_{k-1} such that $f(X_k) \subseteq Y_k$ holds.

In order to arrange that $f(X_k) \subseteq Y_k$ holds, we must achieve for every closed k-dimensional cell e of X by a homotopy of $f|_e$ relative ∂e that e does not meet any cell of Y of dimension $> k$. Note that each compact subset of Y meets only finitely many cells. Hence for a closed cell of e of X of dimension k there are only finitely many closed cells $e_1, e_2, \ldots, e_m$ of Y satisfying $f(e) \cap e_i \neq \emptyset$. Choose $\{i \in 1, 2, \ldots, m\}$ such the the dimension of e_i is greater than $\dim(e)$. If such an i does not exists, we are already done for e. If such i exists, we can arrange that $\dim(e_i) \geq \dim(e_j)$ holds for all $j \in \{1, 2, \ldots, m\}$ and we have to change $f|_e$ up to homotopy relative ∂e such that $f(e)$ meets only the cells $e_1, e_2, \ldots, e_{i-1}, e_{i+1}, e_m$ of Y. Therefore it suffices to show the following lemma.

Lemma 4.3. *Consider $0 \leq k < l$. Let (W, V) be pair for which there exists a pushout*

$$\begin{array}{ccc} S^{l-1} & \xrightarrow{q} & V \\ \downarrow & & \downarrow \\ D^l & \xrightarrow{Q} & W. \end{array}$$

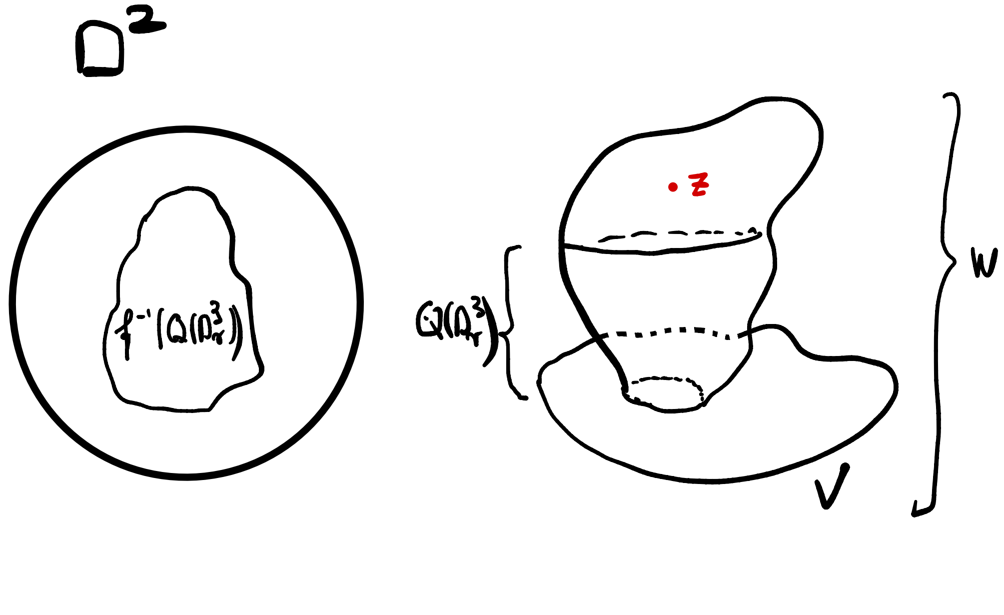
Consider any map $f\colon (D^k, S^{k-1}) \to (W, V)$.

Then f is homotopic relative S^{k-1} to a map $D^k \to V$.

Consider any point $z \in W \setminus V$. Then $(W \setminus \{z\}, V)$ is a strong deformation retraction, i..e, there exists a homotopy $h\colon W \setminus \{z\} \times I \to W \setminus \{z\}$ such that $h(y, 0) = y$ and $h(y, 1) \in V$ hold for $y \in W \setminus \{z\}$ and $h(y, t) = y$ holds for $y \in V$ and $t \in I$. Hence Lemma 4.3 follows from the next lemma.

Lemma 4.4. *Consider the situation of Lemma 4.3. Then there exists $z \in W \setminus V$ such that f is homotopic relative S^{k-1} to a map $D^k \to W \setminus \{z\}$.*

Sketch of proof. Fix $r \in (0, 1)$. Let $D^l_r \subseteq D^l$ be the open ball of radius r, i.e, $\{x \in D^l \mid ||x|| < r\}$. Then one can arrange by an improved version of the Whitney Approximation Theorem that f is homotopy relative to $D^k \setminus f^{-1}(Q(D^l_r))$ to a map $g\colon (D^l, S^{l-1}) \to (W, V)$ such that the map induced by g from the open subset $f^{-1}(Q(D^l_r))$ of D^k to the open subset $Q(D^L_r)$ of W, which we can be equipped with the structure of a smooth manifold diffeomorphic to D^L_r, is smooth.



Since by Sard's Theorem this smooth map g has a regular value z and $k < l$, we get $g(D^k) \subseteq W \setminus \{z\}$. □

This finishes the sketch of the proof of the Cellular Approximation Theorem 4.1.

Corollary 4.5. *Consider $n \geq 0$. Let (X, A) be a CW-pair such that all cells in $X \setminus A$ have dimension $> n$. Then (X, A) is n-connected. In particular (X, X_n) is n-connected for a CW-complex X.*

Proof. We only deal with the case, where A is non-empty. The proof for $A = \emptyset$ follows from the one, where $A = \{x\}$ for any zero-cell $\{x\} \in X$, since X is the disjoint union of its path components and every path component contains a zero-cell.

First we show that $\pi_0(f)\colon \pi_0(A) \to \pi_0(X)$ is surjective for $n = 0$ and bijective for $n \geq 1$. Surjectivity follows from Cellular Approximation Theorem 4.1 applied to any map $\{\bullet\} \to X$ using the fact that $X_0 = A$ holds. Note for the sequel that any path component of a CW-complex must contain a zero-cell. By the Cellular Approximation Theorem 4.1 any path in X connecting two zero-cells in A is homotopic relative endpoints to a path in A as $X_1 = A$ holds if $n \geq 1$. This shows the bijectivity of $\pi_0(f)$ if $n \geq 1$.

It remains to show that $\pi_i(X, A, a) = \{1\}$ holds for any base point $a \in A$ and $i \in \{1, 2, \ldots, n\}$ Since any path component of A contains a zero-cell, diagrams (2.1) and (2.6) imply that we can assume without loss of generality that a is a zero-cell of A. Consider an element $[f] \in \pi_i(X, A, a)$ given by a map of triples $f\colon (D^i, S^{i-1}, \{s\}) \to (X, A, \{a\})$. Equip S^{i-1} with the CW-structure consisting of precisely two cells, namely one 0-cell $\{s\}$ given by the base point s and one $(i-1)$-cell. By the Cellular Approximation Theorem 4.1 the map $f|_{S^{i-1}}\colon S^{i-1} \to A$ is relative $\{s\}$ homotopic to cellular map. One easily checks that this implies that $f\colon (D^i, S^{i-1}, \{s\}) \to (X, A, \{a\})$ is homotopic as a map of triples to a map f' such that $f'|_{S^{i-1}}\colon S^{i-1} \to A$ is cellular. (This is a standard cofibration argument as we will see later, or done by direct inspection.) By the Cellular Approximation Theorem 4.1 the map f' is homotopic relative S^{i-1} to map $f''\colon (D^i, S^{i-1}) \to (X_i, A)$. As $X_i = A$ holds and hence $\pi_1(X_i, X_i, a)$ is trivial by the long exact sequence of the pointed pair (X_i, X_i, a), see Theorem 2.11, we conclude $[f] = [f'] = [f''] = 1$ in $\pi_i(X, A, a)$. □

5. The Whitehead Theorem

In this section we want to prove the following theorem.

Theorem 5.1 (Whitehead Theorem)**.** *Let $f\colon Y \to Z$ be a map of pairs.*

(i) Consider any $n \in \{0, 1, 2, \ldots\}$. Then the following assertions are equivalent:
   (a) The map induced by composition with f
   $$f_*\colon [X, Y] \to [X, Z], \quad [g] \mapsto [f \circ g]$$
   is bijective for every CW-complex X of dimension $\dim(X) < n$ and is surjective for every CW-complex X of dimension $\dim(X) = n$;
   (b) The map $f\colon Y \to Z$ is n-connected;

(ii) The following assertions are equivalent:
   (a) The map induced by composition with f
   $$f_*\colon [X, Y] \to [X, Z], \quad [g] \mapsto [f \circ g];$$
   is bijective for every CW-complex X;
   (b) The map $f\colon Y \to Z$ is a weak homotopy equivalence.

Its proof needs some preparations.

Lemma 5.2. *Let Y be a space which is n-connected for some $n \in \{0,1,2,\ldots\} \amalg \{\infty\}$. Let (X,A) be a relative CW-complex whose relative dimension* $\dim(X,A)$ *is less or equal to n.*

Then any map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.

Proof. We construct for $k = -1,0,1,2,\ldots$ with $k \le n$ maps $f_k\colon X_k \to Y$ such that $f_{-1}\colon X_{-1} = A \to Y$ is the given map f and we have $f_k|_{X_{k-1}} = f_{k-1}$ for $k \ge 0$. Then Lemma 5.2 is a consequence of the following argument. If $n < \infty$, then we can take $F = f_n$. If $n = \infty$, we define $F = \operatorname{colim}_{k\to\infty} f_k$ having in mind that by the definition of a CW-pair we have $X = \operatorname{colim}_{k\to\infty} X_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to k is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i\in I} S^{k-1} & \xrightarrow{\coprod_{i\in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^{k} & \xrightarrow{\coprod_{i\in I} Q_i} & X_k. \end{array}$$

We conclude from the pushout property that we can construct f_k from f_{k-1} if for any $i \in I$ we can extend the composite $f_{k-1}\circ q_i\colon S^{k-1} \to X_{k-1}$ to a map $D^k \to X_k$. This can be done as Y is by assumption k-connected. □

Lemma 5.3. *Let (Y,B) be a pair which is n-connected for some $n \in \{0,1,2,\ldots\} \amalg \{\infty\}$. Let (X,A) be a relative CW-complex whose relative dimension* $\dim(X,A)$ *is less or equal to n.*

Then any map $f\colon (X,A) \to (Y,B)$ is homotopic relative A to a map $f\colon (X,A) \to (Y,B)$ with $g(X) \subseteq B$.

Proof. We construct for $k = -1,0,1,2,\ldots$ with $k \le n$ a map

$$h_k\colon X_k \times I \cup_{X_k\times\{0\}} X \times \{0\} \to Y$$

such that the following conditions are satisfied:

- $h_{-1}\colon A \times I \cup_{A\times\{0\}} \cup X \times \{0\} \to X$ sends (a,t) to $f(a)$ for $(a,t) \in A \times I$ and $(x,0)$ to $f(x)$ for $x \in X$.
- We have $h_k(x,0) = f(x)$ for $x \in X$;
- We have $h_k(x,1) \in B$ for every $x \in X_k$
- For $0 \le k \le n$ we have $h_k|_{X_{k-1}\times I} = h_{k-1}|_{X_{k-1}\times I}$.

Then Lemma 5.3 is a consequence of the following argument. If $n < \infty$, then $h = h_n$ is the desired homotopy relative A from f to a map with image in B. Suppose $n = \infty$. Since $X = \operatorname{colim}_{k\to\infty} X_k$, we get $X \times I = \operatorname{colim}_{k\to\infty}(X_k \times I)$ and we obtain the desired homotopy h by $\operatorname{colim}_{k\to\infty} h_k$.

The induction beginning $k = -1$ is trivial. The induction step from $(k-1)$ to k is done as follows. Choose a cellular pushout

$$\begin{array}{ccc} \coprod_{i\in I} S^{k-1} & \xrightarrow{\coprod_{i\in I} q_i} & X_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^{k} & \xrightarrow{\coprod_{i\in I} Q_i} & X_k. \end{array}$$

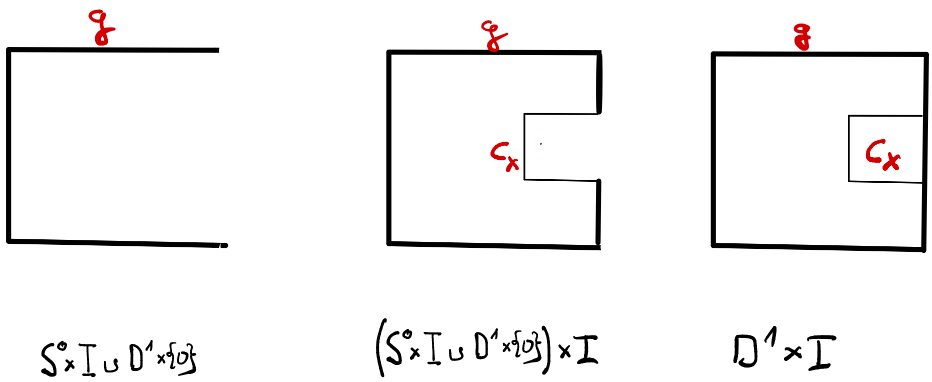
Then we obtain a pushout

$$\begin{array}{ccc}
\coprod_{i\in I} S^{k-1}\times I\cup_{S^{k-1}\times\{0\}} D^k\times\{0\} & \xrightarrow{\coprod_{i\in I} q_i'} & X_{k-1}\times I\cup_{X_{k-1}\times\{0\}} X\times\{0\} \\
\downarrow & & \downarrow \\
\coprod_{i\in I} D^k\times I & \xrightarrow{\coprod_{i\in I} Q_i'} & X_k\times I\cup_{X_k\times\{0\}} X\times\{0\}
\end{array}$$

where q_i' is given by $q_i\times\mathrm{id}_I\cup_{q_i\times\mathrm{id}_{\{0\}}}Q_i\times\mathrm{id}_{\{0\}}$. We conclude from the pushout property that it suffices to construct for every $i\in I$ an extension of the map

$$u=h_{k-1}\circ q_i'\colon S^{k-1}\times I\cup_{S^{k-1}\times\{0\}} D^k\times\{0\}\to Y$$

to a map $U\colon D^k\times I\to Y$ such that $g(D^k\times\{1\})\subseteq B$ holds. Up to homeomorphism the pair $(S^{k-1}\times I\cup_{S^{k-1}\times\{0\}}D^k\times\{0\},S^{k-1}\times\{1\})$ can be identified with (D^k,S^{k-1}). So we can think of u as a map of triples $(D^k,S^{k-1},\{s\})\to(Y,B,\{x\})$ for $x=u(s)$. Hence it defines a element in $\pi_k(Y,B,x)$. As $\pi_k(Y,B,x)$ is by assumption trivial, there is a homotopy of maps of triples $(S^{k-1}\times I\cup_{S^{k-1}\times\{0\}}D^k\times\{0\},S^{k-1}\times\{1\},\{(s,1)\})\to(Y,B,\{x\})$ from u to the constant map c_x. Obviously the latter map extends to the constant map $c_x\colon D^k\times I,D^k\times\{1\},\{(s,1\})\to(Y,B,\{x\})$. Hence we can extend u to a map $U\colon(D^k\times I,D^k\times\{1\},\{s\})\to(Y,B,\{x\})$.



This finishes the proof of Lemma 5.3. □

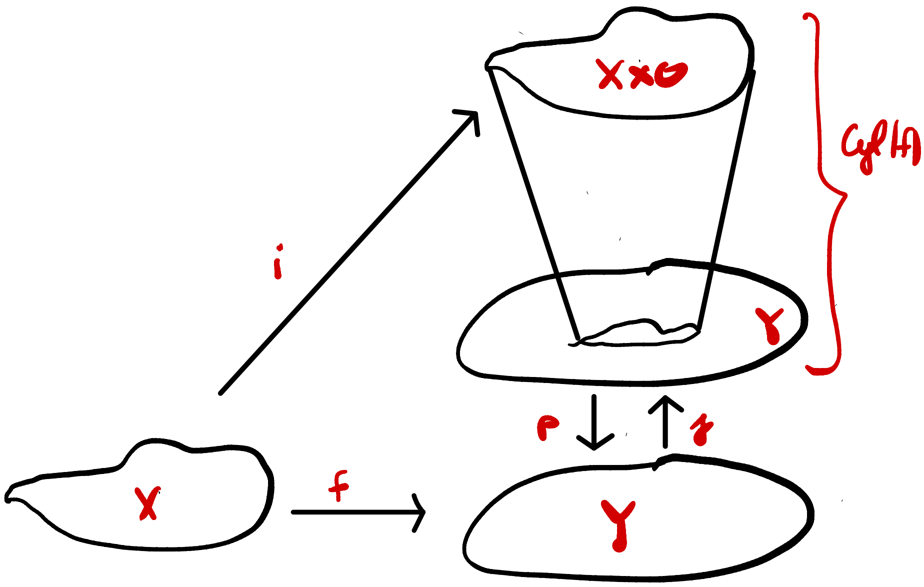
Proof of the Whitehead Theorem 5.1. (i)a $\implies$ (i)b The map $\pi_0(f)\colon\pi_0(Y)\to\pi_0(Z)$ can be identified with the map $f_*\colon[\{\bullet\},Y]\to[\{\bullet\},Z]$. Hence the claim is true for $n=0$. So it suffices to treat the case $n\geq 1$. Then $\pi_0(f)$ is bijective. It remains to show for any $y\in Y$ that $\pi_k(f,y)\colon\pi_k(Y,y)\to\pi_k(Z,f(y))$ is injective for $1\leq k<n$ and surjective for $1\leq k\leq n$.

Consider an element $[v]\in\pi_1(Z,f(y))$ represented by a pointed map $v\colon(S^k,s)\to(Z,f(y))$ for some k with $1\leq k\leq n$. By applying assertion (i)a in the case $X=S^k$, we can choose a map $u\colon S^k\to Y$ and a homotopy $h\colon Y\times I\to Z$ such that $h_0=v$ and $h_1=f\circ u$. Then we conclude from the diagrams (2.1) and (2.6) that $[g]$ is in the image of $\pi_k(f,y)\colon \pi_k(Y,y) \to \pi_k(Z,f(y))$. This shows that $\pi_k(f,y)\colon \pi_k(Y,y) \to \pi_k(Z,f(y))$ is surjective for $1 \le k \le n$.

Next we show that $\pi_k(f,y)\colon \pi_k(Y,y) \to \pi_k(Z,f(y))$ is injective for $1 \le k < n$. Consider $[u] \in \pi_1(Y,y)$ represented by a pointed map $u\colon (S^k,s) \to (Y,y)$ for some k with $1 \le k < n$ which lies in the kernel of $\pi_k(f,y)$. By applying assertion (i)a in the case $X = S^k$, we get a homotopy $h_0\colon S^k \times I \to Y$ with $h_0 = u$ and $h_1 = c_x$. We conclude from the diagrams (2.1) and (2.6) that $[u]$ is the unit in $\pi_k(Y,y)$.

(ii)a $\implies$ (ii)b This follows directly from the already established implication (i)a $\implies$ (i)b.

(i)b $\implies$ (i)a and (ii)b $\implies$ (ii)b Let $\mathrm{cyl}(f)$ be the mapping cylinder of f. Let $i\colon X \to \mathrm{cyl}(f)$ and $j\colon Y \to \mathrm{cyl}(f)$ be the canonical inclusions and $p\colon \mathrm{cyl}(f) \to Y$ be the canonical projection. Then $p \circ i = f$, $p \circ j = \mathrm{id}_Y$, and $j \circ p \simeq \mathrm{id}_{\mathrm{cyl}(f)}$. Hence we can assume without loss of generality that $f\colon Y \to Z$ is an inclusion of pairs, otherwise replace the given $f\colon Y \to Z$ by $i\colon X \to \mathrm{cyl}(f)$.



The surjectivity of $f_*\colon [X,Y] \to [X,Z]$ follows for $\dim(X) \le n$ directly from Lemma 5.3 applied to a map $g\colon (X,\emptyset) \to (Z,Y)$. Finally we prove the injectivity of f_* under the assumption that either $n = \infty$ or $\dim(X) < n < \infty$ holds. Consider $g_0, g_1\colon X \to Y$ and a homotopy $h\colon f \circ g_0 \simeq f \circ g_1$ of maps from X to Z. We obtain a map of pairs $(h, g_0 \amalg g_1)\colon (X \times I, X \times \{0,1\}) \to (Z,Y)$. This map is homotopic relative $X \times \{0,1\}$ to a map $k\colon X \times I \to Y$ by Lemma 5.3 since $\dim(X \times I) \le n$ holds. Obviously k is a homotopy of maps $X \to Y$ between g_0 and g_1. This finishes the proof of the Whitehead Theorem 5.1. □

Corollary 5.4. *Let $f\colon X \to Y$ be a map of CW-complexes. Then f is a homotopy equivalence if and only f is a weak homotopy equivalence.*

Proof. We conclude from the diagrams (2.1) and (2.6) that f is a weak homotopy equivalence if it is a homotopy equivalence. Suppose that f is a weak homotopy equivalence. Theorem 5.1 (ii) implies that $f_*\colon [Y,X] \to [Y,Y]$ is bijective. Let $a\colon Y \to X$ be map with $f_*([a]) = [f \circ a] = \mathrm{id}_Y$. Then a is a weak homotopy equivalence. Theorem 5.1 (ii) again implies that $a_*\colon [X,Y] \to [X,X]$ is bijective. So we can choose a map $b\colon X \to Y$ with $[a \circ b] = [\mathrm{id}_X]$. This implies $b \simeq f \circ a \circ b \simeq f$. Hence a is a homotopy inverse of f and in particular f is a homotopy equivalence. $\square$

Example 5.5 (S^∞)**.** Define the real vector space $\mathbb{R}^\infty := \bigoplus_{i=1}^\infty \mathbb{R}$. It inherits a norm by

$$||(x_1,x_2,x_3,\ldots)|| = \sqrt{\sum_{i=1}^\infty x_i^2}.$$

In particular $\mathbb{R}^\infty$ inherits a metric and the structure of a topological space. We can identify the topological space $\mathbb{R}^n$ with the subspace consisting of points $(x_1,x_2,\ldots)$ for which $x_i = 0$ for $i > n$ holds. Let $S^\infty \subseteq \mathbb{R}^\infty$ be the subspace consisting of points z satisfying $||z|| = 1$. Then S^n can be identified with $S^\infty \cap \mathbb{R}^{n+1}$ for $n \geq 0$. Moreover, we get:

(i) We have the nested sequence $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^\infty$ such that S^∞ is the unions of the S^n-s. The colimit topology with respect to this filtration is not the subspace topology $S^\infty \subseteq \mathbb{R}^\infty$;

(ii) S^∞ equipped with the colimit topology carries a CW-structure with S^n as n-skeleton;

(iii) S^∞ equipped with the subspace topology does not carry the structure of a CW-complex;

(iv) S^∞ equipped with the subspace topology is contractible;

(v) S^∞ equipped with the colimit topology is contractible;

(vi) Consider the identity $S^\infty \to S^\infty$, where we equip the domain with the colimit topology and the codomain with the subspace topology. Then this map is bijective and continuous and is a homotopy equivalence but is not a homeomorphism.

For $n \geq 1$ consider the element a_n in S^∞ whose i-th entry is $\sqrt{1-n^{-1}}$ for $i = 1$, n^{-1} for $i = 2,\ldots,n+1$, and is 0 for $i \geq (n+2)$. Let $A = \{a_n \mid n \geq 1\}$. Since the intersection of A with S^n is finite for $n \geq 1$, it is a closed subspace of S^∞ with respect to the colimit topology. Since $(1,0,0,\ldots)$ does not belong to A and $\lim_{n\to\infty} a_n = (1,0,0\ldots)$ holds with respect to the metric above, A is not closed with respect to the subspace topology. This finishes the proof of assertion (i).

We leave the obvious proof of the assertion (ii) is left to the reader.

Assertion (iii) is proved as follows. Suppose that S^∞ with the subspace topology has a CW-structure. Since then S^∞ is a metrizable CW-complex, it must be locally compact by [1, Theorem B on page 81]. This implies there is an $\epsilon > 0$ such that that the intersection of S^∞ with the closed ball of radius ϵ around $(1,0,0,\ldots)$ is compact. Hence we can find $\delta > 0$ such that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sqrt{1-\delta}\cdot e_1 + \sqrt{\delta}\cdot e_n$ with e_i the i-th element of the standard base belongs to the intersection of S^∞ with the closed ball of radius ϵ around $(1,0,0,\ldots)$. Hence it has a subsequence which is a Cauchy sequence. Since this is not the case, we get a contradiction.

Next we prove assertion (iv). Let $s\colon S^\infty \to S^\infty$ be the shift map sending $(x_1,x_2,x_3,\ldots)$ to $(0,x_1,x_2,x_3,\ldots)$ Define

$$h\colon S^\infty \times I \to S^\infty, \quad x \mapsto \frac{t\cdot s(x) + (1-t)\cdot x}{||t\cdot s(x)x + (1-t)\cdot x||}.$$

This is a homotopy between id_{S^∞} and s. Now consider the homotopy

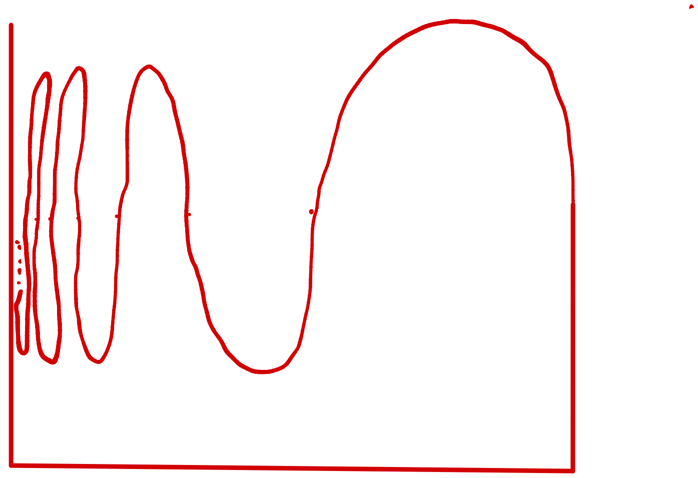
$$k\colon S^\infty \times I \to S^\infty, \quad x \mapsto \frac{(1-t)\cdot s(x) + t\cdot e_1}{||(1-t)\cdot s(x) + t\cdot e_1||}$$

for $e_1 = (1, 0, 0, \ldots)$. Then k is a homotopy between s and the constant map $S^\infty \to S^\infty$ with value e_1. Hence S^∞ with the subspace topology is contractible.

Assertion (v) follows from Theorem 2.20, Theorem 3.4, and Corollary 5.4 using assertion (ii). Alternatively, the proof for assertion (iv) does carry over to assertion (v).

Assertion (vi) is a direct consequence of the other assertions.

Example 5.6 (Warsaw circle)**.** Let W be the *Warsaw circle*, i.e., the compact subsets of $\mathbb{R}^2$ given by the union of $\{(x, \sin(2\pi/x)) \mid x \in (0, 1]\}$, $\{(1, y) \mid y \in [-2, 0]\}$, $\{(x, -2) \mid x \in [0, 1]\}$ and $\{(0, y) \mid y \in [-2, 1]\}$.



Then the projection $p \colon W \to \{\bullet\}$ is a weak homotopy equivalence but not a homotopy equivalence. In particular W is a compact space which is not homotopy equivalent to a CW-complex.

Remark 5.7 (Whitehead Theorem for pairs)**.** There is the following version of the Whitehead Theorem 5.1 (ii) for pairs. Let $(F, f) \colon (Y, B) \to (Z, C)$ be a map of pairs. Then the following assertions are equivalent:

(i) The maps $F \colon Y \to Z$ and $f \colon B \to C$ are weak homotopy equivalences;

(ii) For every CW-pair (X, A) the maps of the homotopy classes of pairs induced by composition with (F, f)
$$(F, f)_* \colon [(X, A), (Y, B)] \to [(X, A), (Z, C)], \quad [(G, g)] \mapsto [(F \circ G, g \circ f)]$$
is bijective.

6. CW-Approximation

Definition 6.1 (n-coconnected maps)**.** A map $f \colon X \to Y$ is called *n-coconnected* for $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$, if for any base point $x \in X$ the map $\pi_i(f, x) \colon \pi_i(X, x) \to \pi_i(Y, f(x))$ is injective if $i = n$, and is bijective if $i > n$.

Consider a natural number n and a map $f \colon X \to Y$. Then f is a weak homotopy equivalence if and only if it is both n-connected and n-coconnected.

Definition 6.2 (n-CW-model for a pair)**.** Consider a topological pair (Y, A) such that A is a CW-complex and $n \in \mathbb{Z}^{\geq 0}$. An *$n$-$CW$-model* for (Y, A) consists of an n-connected pair of CW-complexes (Z, A) together with an n-coconnected map $f\colon Z \to Y$ satisfying $f|_A = \mathrm{id}_A$.

Theorem 6.3 (n-CW-models)**.** *Consider a topological pair (Y, A) such that A is a CW-complex and $n \in \mathbb{Z}^{\geq 0} \amalg \{\infty\}$. Then there exists an n-CW-model*

$$(f, \mathrm{id}_A)\colon (Z, A) \to (Y, A)$$

such that $Z \setminus A$ contains no cells of dimension $\leq n$.

Proof. We construct a sequence of nested spaces $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \cdots$ and maps $f_i\colon Z_i \to Y$ for $i \geq n$ such that the following holds:

- $Z_n = A$ and $f_n = \mathrm{id}_A$;
- $f_i|_{Z_{i-1}} = f_{i-1}$ for $i = (n+1), (n+2), \ldots$;
- There exists for $i \geq n$ a pushout of the shape

$$\begin{array}{ccc} \coprod_{j\in J_i} S^i & \xrightarrow{\coprod_{j_i\in J_i} q_i^j} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{j\in J_i} D^{i+1} & \xrightarrow{\coprod_{j\in J_i} Q_i^j} & Z_{i+1} \end{array}$$

  such that the image of each map q_j does not meet any closed cell in A of dimension $> i$;
- For any base point $z \in Z_{i+1}$ the map $\pi_j(f_{i+1}, z)$ is injective for $i = n$, bijective for $n < j \leq i$, and surjective for $j = (i+1)$.

Before we explain the construction of these data, we explain how we get the desired n-CW-model from it. Namely, we define $Z = \mathrm{colim}_{i\to\infty} Z_i$ and $f = \mathrm{colim}_{i\to\infty} f_i\colon Z \to X$. Then (Z, A) is a CW-pair and the i-skeleton Z_i of Z is the complement of the union of the open cells of dimension $> i$ of A in Z_i. In particular $Z \setminus Z_i$ contains no cells of dimension $\leq i$. Since $Z \setminus A$ contains no k-cells for $0 \leq k \leq n$, the pair (Z, A) is n-connected by Corollary 4.5. We conclude from Corollary 4.5 again that the map $\pi_m(Z_i, z_i) \to \pi_m(Z, z_i)$ induced by the inclusion $Z_i \to Z$ is bijective for $m < i$ and surjective for $m = i$ for any $i \geq n$ and $z_i \in Z_i$. Hence the map f is n-coconnected.

Finally we carry out the construction of the sequence $Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \cdots$ and the sequence of maps $f_i\colon Z_i \to Y$. The induction beginning is obvious, take $Z_n = A$ and $f_n = \mathrm{id}_A$. The induction step how to construct Z_{i+1} and f_{i+1}, when Z_i and f_i have already been established, is done as follows. For each path component C of A choose a zero-cell x_C in A which is contained in C. Then for every element u in the kernel of the map $\pi_i(f_i, x_C)\colon \pi_i(Z_i, x_C) \to \pi_i(Y, x_C)$ choose a pointed map $q_{C,u}\colon (S^i, s) \to (Z_i, x_C)$ with $u = [q_{C,u}]$. The define Z'_{i+1} as the pushout

$$\begin{array}{ccc} \coprod_{\substack{C\in\pi_0(A)\\ u\in\ker(\pi_i(f_i,x_C))}} S^i & \xrightarrow{\coprod_{\substack{C\in\pi_0(A)\\ u\in\ker(\pi_i(f_i,x_C))}} q_{C,u}} & Z_i \\ \downarrow & & \downarrow \\ \coprod_{\substack{C\in\pi_0(A)\\ u\in\ker(\pi_i(f_i,x_C))}} D^{i+1} & \longrightarrow & Z'_{i+1}. \end{array}$$

Since each $[q_{C,u}]$ lies in the kernel of $\pi_i(f_i, x_C)$, each map $q_{C,u}$ can be extended to a map $\overline{q_{C,u}}\colon D^{i+1} \to Y$. By the Cellular Approximation Theorem 4.1 we can additionally arrange that the image of each map $\overline{q_{C,u}}$ has trivial intersection with any open cell of A of dimension $i > i+1$.

The collection of these extensions yield a map $f'_{i+1} \colon Z'_{i+1} \to Y$ by the pushout property. We have for $j \le (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z_i, x_C) & \xrightarrow{\pi_j(f_i, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow_{\pi_j(f'_{i+1}, x_C)} & \\ \pi_j(Z'_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z_i \to Z_{i+1}$. The vertical arrow is bijective for $j \le (i-1)$ and surjective for $j = i$ by Corollary 4.5. Hence $\pi_j(f'_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \le (i-1)$, as $\pi_j(f_i, x_C)$ has these properties by the induction hypothesis. Consider an element v in the kernel of $\pi_i(f'_{i+1}, x_C)$. Choose $u \in \pi_i(Z_i, x_C)$ whose image under the vertical arrow is v. Then u lies in the kernel of $\pi_i(f_i, x_C)$. By construction u lies in the kernel of the vertical arrow. Hence v is trivial. Therefore $\pi_i(f'_{i+1}, x_C)$ is injective. As $\pi_i(f_i, x_C)$ is surjective by the induction hypothesis, $\pi_i(f'_{i+1}, x_C)$ is surjective. This implies that $\pi_j(f'_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \le i$ for all $C \in \pi_0(A)$.

Now consider any $C \in \pi_0(A)$ and any element $[w_C] \in \pi_{i+1}(Y, x_C)$. Choose a map $w_C \colon (S^{i+1}, s) \to (Y, x_C)$ representing $[w_C]$. Define the desired space Z_{i+1} and the desired map $f_{i+1} \colon Z_{i+1} \to Y$ by

$$\begin{aligned} Z_{i+1} &= Z'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} S^{i+1}, \\ f_{i+1} &= f'_{i+1} \vee \bigvee_{\substack{C \in \pi_0(A) \\ [w_C] \in \pi_{i+1}(Y, x_C)}} w_C. \end{aligned}$$

We have for $j \le (i+1)$ and $C \in \pi_0(A)$ the commutative diagram

$$\begin{array}{ccc} \pi_j(Z'_{i+1}, x_C) & \xrightarrow{\pi_j(f'_{i+1}, x_C)} & \pi_j(Y, x_C) \\ \downarrow & \nearrow_{\pi_j(f_{i+1}, x_C)} & \\ \pi_j(Z_{i+1}, x_C) & & \end{array}$$

where the vertical arrow is induced by the inclusion $Z'_{i+1} \to Z_{i+1}$. The left vertical arrow is bijective for $j < i$ and surjective for $j = i$ by Corollary 4.5. It is also injective for $j = i$, since the inclusion $Z'_{i+1} \to Z_{i+1}$ has an obvious retraction $Z_{i+1} \to Z'_{i+1}$. Hence the left vertical arrow is bijective for $j \le i$. This implies that $\pi_j(f_{i+1}, x_C)$ is injective for $i = n$ and bijective for $n < j \le i$ for all $C \in \pi_0(A)$. Moreover, by construction any element $[w_C]$ is in the image of $\pi_j(f_{i+1}, x_C)$. Hence $\pi_j(f_{i+1}, x_C)$ is surjective for all $C \in \pi_0(A)$. Since $\pi_0(A) \to \pi_0(Z_{i+1})$ is surjective, we conclude from the diagrams (2.1) and (2.6) that for any base point $z \in Z_{i+1}$ the map $\pi_j(f_{i+1}, z)$ is injective for $i = n$, bijective for $n < j \le i$, and surjective for $j = (i+1)$.

This finishes the proof of Theorem 6.3. □

Remark 6.4. One can think of the n-CW-model $f \colon (Z, A) \to (Y, A)$ as a sort of homotopy theoretic hybrid of A and X. If $n = 0$ and Y is path connected, then the hybrid looks like Y in the sense that f is a weak homotopy equivalence. As n increases, the hybrid looks more and more like A, and less and less like Y. If we take $n = \infty$, then the inclusion $A \to Z$ is a weak homotopy equivalence and can actually be realized by $Z = A$ and id_A.

More precisely, if $k\colon A \to Z$ and $l\colon A \to Y$ are the inclusions and $a \in A$ is a base point, we get a factorization

$$\pi_i(l,a)\colon \pi_1(A,a) \xrightarrow{\pi_i(k,a)} \pi_i(Z,a) \xrightarrow{\pi_i(f,a)} \pi_i(Y,a)$$

such that the following holds:

- If $i < n$, then the first map $\pi_i(k,a)$ is bijective;
- If $i = n$, then the first map $\pi_i(k,a)$ is surjective and the second map $\pi_i(f,a)$ is injective;
- If $i > n$, then the second map $\pi_i(f,a)$ is bijective.

Corollary 6.5. *Consider a CW-pair (X,A) and $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) There is a CW-pair (Z,A) such that (X,A) and (Z,A) are homotopy equivalent relative A and $Z \setminus A$ contains no cells of dimension $\le n$;
(ii) The pair (X,A) is n-connected.

Proof. (i) $\implies$ (ii) This follows from Corollary 4.5.

(ii) $\implies$ (i) We obtain from Theorem 6.3 an n-model $(f, \mathrm{id}_A)\colon (Z,A) \to (X,A)$ such that $Z \setminus A$ contains no cells of dimension $\le n$. Since (Z,A) and (X,A) are n-connected and f is n-coconnected, $f\colon Z \to X$ is a weak homotopy equivalence inducing the identity on A. A version of the Whitehead Theorem 5.1 (ii) relative A implies that (X,A) and (Z,A) are homotopy equivalent relative A. □

In particular any path connected CW-complex is homotopy equivalent to a CW-complex Z having precisely one 0-cell.

Example 6.6. Let X be path connected CW-complex and $x \in X$ a base point. We conclude from Theorem 2.7 that a 1-connected CW-model for $(X, \{x\})$ is given by the universal covering $\widetilde{X} \to X$.

For this section the case $n = 0$ is important which we treat next.

Definition 6.7. Consider a space Y. A *CW-approximation* (X,f) of Y is a CW-complex X together with a weak homotopy equivalence $f\colon X \to Y$.

Theorem 6.8 (Existence and uniqueness of CW-approximations)**.** *Let Y be a topological space. Then:*

(i) There exists a CW-approximation (X,f) of Y;
(ii) Let (X,f) and (X',f') be two CW-approximations of Y. Then there exists a homotopy equivalence $g\colon X \to X'$ for which the following diagram commutes up to homotopy

$$\begin{array}{ccc} X_0 & \xrightarrow{\quad g \quad} & X_1 \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y. & \end{array}$$

The homotopy equivalence g is up to homotopy uniquely determined by the property $f' \circ g \simeq f$.

Proof. (i) Consider a path component C of Y. From Theorem 6.3 applied to the pair $(C, \emptyset)$ and $n = 0$ we obtain a CW-complex X_C and weak homotopy equivalence $f_C\colon X_C \to C$. Then we get from $X = \coprod_{C \in \pi_0(C)} X_C$ and $f = \coprod_{C \in \pi_0(C)} f_C$ a CW-approximation of Y.

(ii) We conclude from the Whitehead Theorem 5.1 (ii) that there exists a map $g\colon X \to X'$ which is uniquely determined up to homotopy by the property $f' \circ g \simeq f$. The map g is a weak homotopy equivalence and hence a homotopy equivalence by Corollary 5.4. □

Remark 6.9. One may think of Theorem 6.8 as the topological analogon of the fact that any positive R-chain complex C_* possesses a *projective R-resolution* $f_*\colon P_* \to C_*$, i.e., a projective positive R-chain complex P_* together with an R-chain map $f_*\colon P_* \to C_*$ inducing an isomorphism on all homology modules, and that for two projective resolutions (P_*, f_*) and (P'_*, f'_*) of C_* there is a R-chain homotopy equivalence $g_*\colon P_* \to P'_*$ which is a up to R-chain homotopy uniquely determined by the property $f'_* \circ g_* \simeq f_*$.

Theorem 6.10. *Let $f\colon X \to Y$ be a weak homotopy equivalence of spaces. Then the induced map on singular homology $H_n(f)\colon H_n(X) \to H_n(Y)$ is bijective for all $n \geq 0$.*

Proof. See [6, Theorem 9.5.3 on page 237]. □

Remark 6.11 (CW-approximations for pairs)**.** Consider a pair (Y, B). Choose a CW-approximation $u\colon A \to B$ for B. Let $\mathrm{cyl}(u)$ be the mapping cylinder of u. It contains the CW-complex A as subspace. Let $g\colon (X, A) \to (\mathrm{cyl}(f), A)$ be a 0-CW-model which exists by Theorem 6.3. Thus we obtain a pair of CW-complexes (X, A) together with a weak homotopy equivalence $g\colon X \to Y$ satisfying $g|_A = \mathrm{id}_A$. Let $p\colon \mathrm{cyl}(f) \to Y$ be the projection which is a homotopy equivalence and satisfies $p|_A = u$. Let $f\colon X \to Y$ be the composite $p \circ v$. Then $f\colon X \to Y$ and $f|_A = u\colon A \to B$ are weak homotopy equivalences. So we obtain a CW-approximation $(f, u)\colon (X, A) \to (Y, B)$ for pairs.

A relative version of the Whitehead Theorem 5.1 (ii), see Remark 5.7, shows that for two such CW-approximations $f\colon (X, A) \to (Y, B)$ and $f'\colon (X', A') \to (Y, B)$ there is a homotopy equivalence of pairs $g\colon (X, A) \to (X', A')$ which is up to homotopy uniquely determined by the property that f and $f' \circ g$ are homotopic as maps of pairs $(X, A) \to (Y, B)$.

7. The category of compactly generated spaces

We briefly recall some basics about compactly generated spaces. More information and proofs can be found in [4]. A topological space X is *compactly generated* if it is a Hausdorff space and a set $A \subseteq X$ is closed if and only if for any compact subset $C \subset X$ the intersection $C \cap A$ is a closed subspace of C.

Every locally compact space, and every space satisfying the first axiom of countability, e.g., a metrizable space, is compactly generated. If $p\colon X \to Y$ is an identification of topological spaces and X is compactly generated and Y is Hausdorff, then Y is compactly generated. A closed subset of a compactly generated space is again compactly generated. For open subsets one has to be careful as it is explained in Subsection 7.1.

7.1. **Open subsets.**

Recall that a topological space B is called *regular* if for any point $x \in X$ and closed set $A \subseteq X$ there exists open subsets U and V with $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$. A Hausdorff space is called *locally compact* if every $x \in X$ possesses a compact neighborhood. Equivalently, for every $x \in X$ and open neighborhood U there exists an open neighborhood V of x such that the closure of V in X is compact and contained in U, see [3, Lemma 8.2 in Section 3-8 on page 185].

Definition 7.1 (Quasi-regular open set and regular space)**.** An open subset $U \subseteq B$ is called *quasi-regular* if for any $x \in X$ there exists an open neighborhood V_x whose closure in B is contained in U.

Lemma 7.2. *(i) Let B be a compactly generated Hausdorff space. A quasi-regular open subset $U \subseteq B$ equipped with the subspace topology is compactly generated;*

(ii) Let $f\colon X \to Y$ be a (continuous) map between (not necessarily compactly generated) spaces. If $V \subseteq Y$ is a quasi-regular open subset, then $f^{-1}(V) \subseteq X$ is a quasi-regular open subset;

(iii) The intersection of finitely many quasi-regular open subsets is again a quasi-regular open subset;

(iv) A space is regular if and only if every open subset is quasi-regular;

(v) Any locally compact Hausdorff space, any metrizable space, and every CW-complex are regular;

(vi) Every open subset of a CW-complex is quasi-regular and, equipped with the subspace topology, compactly generated.

Proof. (i) See [4, page 135].

(ii) Consider a point $x \in f^{-1}(V)$. Choose an open set W of Y such that $f(x) \in W$ and the closure of W in B is contained in V. Then $f^{-1}(W)$ is an open subset of X which contains x and whose closure in X is contained in $f^{-1}(V)$.

(iii) Let U_1, U_2, ..., U_r be quasi-regular open subsets. Consider $x \in U := \bigcap_{i=1}^r U_i$. Choose for every $i = 1, 2 \ldots, r$ an open subset V_i with $x \in V_i$ such that the closure $\overline{V_i}$ of V_i in B is contained in U_i. Put $V := \bigcap_{i=1}^r V_i$. Then $x \in V$ and $\overline{V} \subseteq \cap_{i=1}^r \overline{V_i} \subseteq U$. Hence U is a quasi-regular open subset.

(iv) See [3, Lemma 2.1 in Section 4-2 on page 196].

(v) This is obvious for locally compact spaces. Metrizable spaces are treated in [3, Theorem 2.3 in Section 4-2 on page 198]. Every CW-complex is paracompact, see [2], and hence in particular regular, see [3, Theorem 4.1 in Section 6-4 on page 255].

(vi) This follows from assertions (i), (iv), and (v). □

7.2. **The retraction functor k.** There is a construction which assigns to a topological Hausdorff space X a new topological space $k(X)$ such that X and $k(X)$ have the same underlying sets, $k(X)$ is compactly generated, X and $k(X)$ have the same compact subsets, the identity $k(X) \to X$ is continuous and is a homeomorphism if and only if X is compactly generated. Namely, define the new topology on $k(X)$ by declaring a subset $A \subseteq X$ to be closed if and only if for every compact subset of X the intersection $A \cap C$ is a closed subset of C.

7.3. **Mapping spaces, product spaces, and subspaces.** Given two compactly generated spaces X and Y, denote by $\operatorname{map}(X,Y)_{k.o.}$ the set of maps $X \to Y$ with the compact-open-topology, i.e., a subbasis for the compact-open-topology is given by the sets $W(C,U) = \{f\colon X \to Y \mid f(C) \subseteq U\}$, where C runs through the compact subsets of X and U runs though the open subsets of Y. Note that $\operatorname{map}(X,Y)_{k.o.}$ is not compactly generated in general. We denote by $\operatorname{map}(X,Y)$ the topological space given by $k(\operatorname{map}(X,Y)_{k.o.})$. Sometimes we abbreviate $\operatorname{map}(X,Y)$ by Y^X and denote for a map $f\colon Y \to Z$ the induced map $\operatorname{map}(\operatorname{id}_X, f)\colon \operatorname{map}(X,Y) \to \operatorname{map}(X,Z)$, $g \mapsto f \circ g$ by $f^X\colon Y^X \to Z^X$. If X and Y are compactly generated spaces, then $X \times Y$ stands for $k(X \times_p X)$, where $X \times_p Y$ is the topological space with respect to the "classical" product topology.

If $A \subseteq X$ is a subset of a compactly generated space, the subspace topology means that we take $k(A_{st})$ for A_{st} the topology space given by the "classical" subspace topology on A.

Roughly speaking, all the usual constructions of topologies are made compactly generated by passing from Y to $k(Y)$ in order to stay within the category of compactly generated spaces.

7.4. **Basic features of the category of compactly generated spaces.** The category of compactly generated spaces has the following convenient features:

- A map $f\colon X \to Y$ of compactly generated spaces is continuous if and only if its restriction $f|_C\colon C \to Y$ to any compact subset $C \subseteq X$ is continuous;
- If X, Y, and Z are compactly generated spaces, then the obvious maps
$$\begin{aligned} \operatorname{map}(X, \operatorname{map}(Y,Z)) &\xrightarrow{\cong} \operatorname{map}(X \times Y, Z); \\ \operatorname{map}(X, Y \times Z) &\xrightarrow{\cong} \operatorname{map}(X,Y) \times \operatorname{map}(X,Z), \end{aligned}$$
are homeomorphisms and the map given by composition
$$\operatorname{map}(X,Y) \times \operatorname{map}(Y;Z) \to \operatorname{map}(X,Z)$$
is continuous;
- The product of two identifications is again an identification;
- If X is locally compact and Y compactly generated, then $X \times Y$ and $X \times_p Y$ are the same topological spaces;
- Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of inclusions of compactly generated spaces such that X_i is a closed subspace of X_{i+1} for $i = 0, 1, 2, \ldots$.

  Then the colimit $\operatorname{colim}_{i\to\infty} X_i$ exists in the category of compactly generated Hausdorf spaces. Moreover, if Y is a compactly generated space, then $\operatorname{colim}_{i\to\infty}(X_i \times Y)$ exists in the category of compactly generated spaces and the canonical map
$$\operatorname{colim}_{i\to\infty}(X_i \times Y) \xrightarrow{\cong} \bigl(\operatorname{colim}_{i\to\infty} X_i\bigr) \times Y$$
is a homeomorphism;
- In the category of compactly generated spaces the pushout of a diagram $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ exists if f_1 or f_2 is the inclusion of a closed subspace;
- Given a Hausdorf space Y, the canonical map $Y \to k(Y)$ is a weak homotopy equivalence and induces an isomorphism on singular homology.
- Given a pushout in the category of compactly generated spaces, its product with a compactly generated space is again a pushout in the category of compactly generated spaces.
- The product of two CW-complexes is again a CW-complex;

Remark 7.3 (Compactly generated weak Hausdorff spaces)**.** There is also the category of compactly generated weak Hausdorff spaces, see [5]. The main advantage in contrast to the category of compactly generated Hausdorff spaces, see [4], is that in the category of compactly generated weak Hausdorff spaces colimits for small diagrams, for instance pushouts or filtered colimits, always exist, see [5, Corollary 2.23]. In the category of compactly generated spaces one can define the pushout of a diagram $X_1 \xrightarrow{f_1} X_0 \xleftarrow{f_2} X_2$ only if for the pushout in the classical setting

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \overline{f_2}} \\ X_2 & \xrightarrow{\overline{f_1}} & X \end{array}$$

the space X is Hausdorff, since the retraction functor k digests only Hausdorff spaces. Note that X is Hausdorff if f_1 or f_2 is an inclusion of a closed subspace. Therefore in the case treated in the manuscript this condition is always satisfies and the pushout exists in the category of compactly generated Hausdorff spaces.

The same discussion applies to the colimit $\operatorname{colim}_{i\to\infty} X_i$ of a sequence of inclusions of compactly generated spaces of $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$.

For simplicity we will discuss these issues not anymore and will work in the category of compactly generated Hausdorff spaces throughout this manuscript.

References

[1] R. Fritsch and R. Piccinini. CW-complexes and Euclidean spaces. In *Fourth Conference on Topology (Italian) (Sorrento, 1988)*, number 24 in Rendiconti del Circolo Matematico di Palermo Serie II. Supplemento, pages 79–95. Springer, 1990.
[2] H. Miyazaki. The paracompactness of CW-complexes. *Tôhoku Math. J*, 4:309–313, 1952.
[3] J. R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.
[4] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.
[5] N. P. Strickland. The category of cgwh spaces. Preprint, http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf, 2009.
[6] T. tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

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