“Approximating L^2-invariants by their finite-dimensional analogues”
by
Wolfgang Lück

Abstract: Let X be a finite connected CW-complex. Suppose that its fundamental group π is residually finite, i.e., there is a nested sequence $\ldots \subset \Gamma_{m+1} \subset \Gamma_m \subset \ldots \subset \pi$ of in π normal subgroups of finite index whose intersection is trivial. Then we show that the p-th L^2-Betti number of X is the limit of the sequence $b_p(X_m)/[\pi : \Gamma_m]$ where $b_p(X_m)$ is the (ordinary) p-th Betti number of the finite covering of X associated with Γ_m.

Introduction and statement of results

Let X be a finite connected CW-complex with fundamental group π and $A \subset X$ be a CW-subcomplex. Let $b_p^{(2)}(X,A)$ be the p-th L^2-Betti number. Suppose π is residually finite, i.e., there is a nested sequence of in π normal subgroups $\ldots \subset \Gamma_{m+1} \subset \Gamma_m \subset \ldots \subset \pi$ such that the index $[\pi : \Gamma_m]$ is finite for all $m \geq 0$ and the intersection $\cap_{m\geq 0}\Gamma_m$ is the trivial group. Examples of residually finite groups are fundamental groups of compact Haken 3-manifolds and finitely generated groups possessing a faithful representation into $GL(n,F)$ for some field F. Consider any such sequence $(\Gamma_m)_{m\geq 0}$. Let $p_m : X_m \longrightarrow X$ be the covering of X associated with $\Gamma_m \subset \pi$ and put $A_m = p_m^{-1}(A)$. Denote by $b_p(X_m, A_m)$ the (ordinary) p-th Betti number of (X_m, A_m). The following theorem answers a query of Gromov [14, pages 13, 153].

Theorem 0.1 (Kazhdan's equality) *Under the conditions above*

$$\lim_{m\to\infty} \frac{b_p(X_m, A_m)}{[\pi : \Gamma_m]} \;=\; b_p^{(2)}(X,A). \qquad \blacksquare$$

The inequality $\limsup_{m\to\infty} \frac{b_p(X_m,A_m)}{[\pi:\Gamma_m]} \;\leq\; b_p^{(2)}(X,A)$ for X a closed manifold and empty A is discussed by Gromov [14, pages 13, 153] and is essentially due to Kazhdan [20].

There are estimates on the sum of the ordinary Betti numbers by the volume and by the simplicial volume for a closed Riemannian manifold which satisfies certain pinching conditions on its curvature [12, page 12]. Provided that the fundamental group is residually finite, these hold also for the sum of the L^2-Betti numbers by Theorem 0.1.

Suppose that M is a closed Riemannian manifold. Let $L^2\Omega^k(\widetilde{M})$ be the Hilbert space of square integrable p-forms on the universal covering $\widetilde{M}$. The Laplace operator $\widetilde{\Delta}^p$ acting on $L^2\Omega^k(\widetilde{M})$ is essentially self-adjoint. Hence the spectral theorem applies and there is a spectral family $\{P^p_\lambda | \lambda \in [0,\infty)\}$ associated to $\widetilde{\Delta}^p$. Let $P^p_\lambda(x,y)$ be a Schwartz kernel of P^p_λ. Denote by $\mathrm{tr}_{\mathbb{R}}\left(P^p_\lambda(x,x)\right)$ the trace of the linear endomorphism of the finite-dimensional real vector space $\wedge^p T^*_x\widetilde{M}$. Let $\mathcal{F} \subset \widetilde{M}$ be a fundamental domain for the π-action. Define the *analytic p-th spectral density function* of M by

$$F^p \;:\; [0,\infty) \longrightarrow [0,\infty) \qquad \lambda \;\mapsto\; \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}\left(P^p_\lambda(x,x)\right) \cdot dvol.$$

Notice that $F^p(0) = b_p^{(2)}(M)$ and that the spectrum of $\widetilde{\Delta^p}$ has a gap at zero if and only if $F^p(\lambda) - F^p(0) = 0$ for some $\lambda > 0$.

Suppose that $\pi = \pi_1(M)$ is residually finite with a nested sequence of normal subgroups of finite index $(\Gamma_m)_{m \ge 0}$ with trivial intersection as above. Fix a CW-structure on M. Equip all finite coverings M_m with the induced CW-structure. The cellular $\mathbb{C}[\Gamma_m\backslash\pi]$-cochain complex $C^p(M_m;\mathbb{C})$ inherits a $\mathbb{C}$-basis and in particular a Hilbert space structure. Hence the adjoint $(c^p)^*$ of its differential c^p is defined. The combinatorial Laplace operator $\Delta^p_m : C^p(M_m;\mathbb{C}) \longrightarrow C^p(M_m;\mathbb{C})$ is defined by $c^{p-1}(c^{p-1})^* + (c^p)^*c^p$. Denote by $N^p_m(\lambda)$ the number of eigenvalues μ of Δ^p_m satisfying $\mu \le \lambda$ counted with multiplicity. Notice that the spectrum of $\widetilde{\Delta^p}$ has a gap at zero if and only if $F^p(\lambda) = F^p(0)$ for some $\lambda > 0$. We get under the conditions above

Theorem 0.2 (Kazhdan's criterion) *The spectrum of $\widetilde{\Delta}^p$ has a gap at zero if and only if there is a $\lambda > 0$ such that*

$$\lim_{m\to\infty} \frac{N^p_m(\lambda) - N^p_m(0)}{[\pi : \Gamma_m]} = 0. \qquad \blacksquare$$

Theorem 0.3 (Logarithmic spectral density estimate) *There are constants $C > 0$ and $\epsilon > 0$ (depending on M but not on λ) such that for all $\lambda \in (0,\epsilon)$*

$$F^p(\lambda) - F^p(0) \le \frac{C}{-\ln(\lambda)}. \qquad \blacksquare$$

Notice that zero is not in the spectrum of $\widetilde{\Delta^p}$ if and only if $F^p(0) = b_p^{(2)}(M) = 0$ and the spectrum of $\widetilde{\Delta^p}$ has a gap at zero, or equivalently, if and only if $F^p(\lambda) = 0$ for some

$\lambda > 0$. Hence Theorem 0.1 and Theorem 0.2 together imply the necessary and the sufficient Kazhdan' criterion in [14, page 14] for the question whether zero is not in the spectrum of $\widetilde{\Delta^p}$. Theorem 0.3 gives some evidence for [22, Conjecture 9.1] that the Novikov-Shubin invariants of M are always positive. This means that one can improve the inequality in Theorem 0.3 for some $\alpha > 0$ to

$$F^p(\lambda) - F^p(0) \leq C\lambda^\alpha.$$

All we have said about a closed Riemannian manifold M can be extended to compact Riemannian manifolds with boundary by imposing absolute boundary conditions.

We give two applications to "middle" algebraic K-groups of $\mathbb{Z}\pi$, provided that π is countable and residually finite. Let $\mathcal{N}(\pi) = B(l^2(\pi), l^2(\pi)^\pi$ be the von Neumann algebra of π and denote by $\operatorname{tr}^c_{\mathcal{N}(\pi)}$ its unique center valued trace.

Theorem 0.4 (Swan' theorem for countable residually finite groups)

1. *Let $p \in M(n,n,\mathbb{Z}\pi)$ be an idempotent. If I_1 is the unit element in $\mathcal{N}(\pi)$ and r is the rank of the abelian group $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \operatorname{im}(p)$, then*

$$\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) = r \cdot I_1.$$

2. *The change of rings homomorphism*

$$\widetilde{K}_0(\mathbb{Z}\pi) \longrightarrow \widetilde{K}_0(\mathcal{N}(\pi))$$

is trivial. ■

Roughly speaking, Theorem 0.4 means for a finitely generated projective $\mathbb{Z}\pi$-module P that $l^2(\pi) \otimes_{\mathbb{Z}\pi} P$ is π-isometrically isomorphic to the Hilbert $\mathcal{N}(\pi)$-module mbox$\oplus_{i=1}^r l^2(\pi)$ for r the rank of $\mathbb{Z} \otimes_{\mathbb{Z}\pi} P$. Theorem 0.4 was already proven by Swan [31, Theorem 8.1] for finite π. Notice that for finite π the von Neumann algebra $\mathcal{N}(\pi)$ reduces to $\mathbb{C}\pi$ and $|\pi| \cdot \operatorname{tr}^c_{\mathcal{N}(\pi)}(p)$ for an idempotent $p \in M(n,n,\mathbb{Z}\pi)$ is the character of the complex π-representation $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{im}(p)$. Bass [2, section 9] shows for a torsionfree linear finitely generated group Γ the even stronger statement that the Hattori-Stallings rank of a finitely generated projective $\mathbb{Z}\Gamma$-module is the the Hattori-Stallings rank of the free $\mathbb{Z}\Gamma$-module of rank $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}\Gamma} P)$.

Secondly, we show

Theorem 0.5 (Homotopy invariance of combinatorial L^2-torsion)

1. *If π is a countable residually finite group, then*

$$\Phi : \mathrm{Wh}(\pi) \longrightarrow \mathbb{R}^{>0}$$

given by the Fuglede Kadison determinant is trivial.

2. *The combinatorial L^2-torsion of a finite CW-complex with residually finite fundamental group whose L^2-Betti numbers are all trivial and whose Novikov-Shubin invariants are all positive is a homotopy invariant.*

Combinatorial L^2-torsion was introduced and studied in [4],[23] and [25]. The equivalence of the two assertions in Theorem 0.5 is proven in [23, Theorem 1.4]. In this context we mention the conjecture that the combinatorial L^2-torsion agrees with the analytic L^2-torsion defined by Lott [21] and Mathai [28]. The analytic L^2-torsion is known to be a diffeomorphism invariant of closed Riemannian manifolds whose L^2-Betti numbers all vanish and whose Novikov-Shubin invariants are all positive. The considerations above make it plausible that the analytic L^2-torsion is even a homotopy invariant. Notice that the isomorphism conjectures by Farrell and Jones [10] imply both Theorem 0.4 and Theorem 0.5.

Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group which is Haken, Seifert or hyperbolic. Suppose that its boundary is empty or a disjoint union of incompressible tori. Let $M_1, \ldots M_r$ be the hyperbolic pieces in the decomposition by incompressible tori into Seifert and hyperbolic pieces. There is the [23, Conjecture 2.3] for the combinatorial L^2-torsion $\rho(M)$

$$\ln(\rho(M)) = \frac{-1}{3\pi} \cdot \sum_{i=1}^{r} Vol(M_i).$$

This implies in particular that $\ln(\rho(M))$ is always non-positive. We can prove this last statement in

Theorem 0.6 (Bound on combinatorial L^2-torsion for 3-manifolds) *Let M be a 3-manifold satisfying the conditions above or let M be a connected finite 2-dimensional CW-complex with residually finite fundamental group whose L^2-Betti numbers are all trivial and whose Novikov-Shubin invariants are all positive. Then we get for the combinatorial L^2-torsion*

$$\rho(M) \leq 1.$$

The paper is organized as follows :

1. Review of L^2-Betti numbers and residually finite groups

We give the basic definitions of L^2-Betti numbers and of residually finite groups and state some important facts. For the remainder of this section let X be a finite connected CW-complex with fundamental group π and $A \subset X$ be a CW-subcomplex. Let $p : \widetilde{X} \longrightarrow X$ be the universal covering and put $\widetilde{A} = p^{-1}(A)$. We let operate π from the left on the universal covering and on its cellular chain complex.

We recall the basic definitions about L^2-Betti numbers. Let $l^2(\pi)$ be the Hilbert space of formal sums $\sum_{g\in\pi} \lambda_g \cdot g$ with complex coefficients λ_g satisfying $\sum_{g\in\pi} |\lambda_g|^2 < \infty$. The *von Neumann algebra* of π

$$\mathcal{N}(\pi) = B(l^2(\pi), l^2(\pi))^{\pi}$$

is the algebra of bounded π-equivariant operators from $l^2(\pi)$ to $l^2(\pi)$. The *von Neumann trace* of an element $f \in \mathcal{N}(\pi)$ is defined by

$$\mathrm{tr}_{\mathcal{N}(\pi)}(f) \;=\; \langle f(e), e\rangle$$

for $e \in \pi$ the unit element. For a bounded π-equivariant operator $f : \oplus_{i=1}^n l^2(\pi) \longrightarrow \oplus_{i=1}^n l^2(\pi)$ define

$$\mathrm{tr}_{\mathcal{N}(\pi)}(f) \;=\; \sum_{i=1}^n \mathrm{tr}_{\mathcal{N}(\pi)}\left(f_{i,i}\right).$$

A *finitely generated Hilbert $\mathcal{N}(\pi)$-module* P is a Hilbert space with isometric π-action such there exists an isometric π-equivariant embedding into $\oplus_{i=1}^r l^2(\pi)$ (which is not part of the structure) for some $r \in \mathbb{N}$. Let $\mathrm{pr} : \oplus_{i=1}^r l^2(\pi) \longrightarrow \oplus_{i=1}^r l^2(\pi)$ be a projection whose image is isometrically π-equivariantly isomorphic to P. The *von Neumann dimension* of P is defined by

$$\dim_{\mathcal{N}(\pi)}(P) \;=\; \mathrm{tr}_{\mathcal{N}(\pi)}(\mathrm{pr}).$$

If we tensor the $\mathbb{Z}\pi$-chain complex $C(\widetilde{X}, \widetilde{A})$ with $l^2(\pi)$, we obtain a chain complex $C^{(2)}(X, A)$ of finitely generated Hilbert $\mathcal{N}(\pi)$-modules with bounded π-equivariant operators $c_p^{(2)}$ as differentials. Define the *p-th L^2-homology* $H_p^{(2)}(X, A)$ by the quotient $\ker(c_p^{(2)})/\operatorname{clos}(\operatorname{im}(c_{p+1}^{(2)}))$. Since we divide by the closure of the image, this is again a finitely generated Hilbert $\mathcal{N}(\pi)$-module. Its von Neumann dimension is the *p-th L^2-Betti number* of (X, A). Notice that L^2-homology and L^2-cohomology are the same up to isometric π-isomorphism. We give some basic facts about L^2-Betti numbers. For more information we refer to [14, section 8].

- The L^2-Betti numbers are homotopy invariants.
- If $\chi(X, A)$ is the Euler characteristic, then [1]

$$\chi(X, A) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(X, A).$$

- Let M be a compact Riemannian manifold. The space of harmonic L^2-integrable p-forms on $\widetilde{M}$ satisfying absolute boundary conditions is isometrically π-isomorphic to $H_p^{(2)}(M)$ and its von Neumann dimension agrees with $b_p^{(2)}(M)$ [8] for closed M, [22, section 6].
- The von Neumann dimension is additive under exact sequences of finitely generated Hilbert $\mathcal{N}(\pi)$-modules. The von Neumann dimension of P is zero if and only if P is zero. The von Neumann dimension is continuous, i.e., if $P_1 \supset P_2 \supset \ldots$ is a nested sequence of finitely generated Hilbert $\mathcal{A}$-modules then

$$\dim_{\mathcal{A}} \left(\bigcap_{n=1}^{\infty} P_n \right) = \lim_{n \to \infty} \dim_{\mathcal{A}}(P_n).$$

- If the compact manifold M fibers over S^1, then $b_p(M) = 0$ for all $p \geq 0$ [24].
- If $\pi_1(X)$ is an extension of a finitely presented group which contains $\mathbb{Z}$ as a subgroup by an infinite finitely presented group, then $b_1^{(2)}(X) = 0$ [24].
- If π contains a normal infinite amenable subgroup and $B\pi$ is of finite type, then $b_p^{(2)}(B\pi) = 0$ for all $p \geq 0$ [5].
- Let M be a closed Riemannian manifold of dimension n with sectional curvature K pinched by

$$-1 \leq K \leq -c^2 < 1 - \left(\frac{n-2}{n-1} \right)^2.$$

  Then the L^2-Betti number $b_p(M)$ vanishes if $|p - n/2| \geq 1$ [7].
- If M is a symmetric space of non-compact type, then the L^2-cohomology vanishes possibly except for the middle dimension.

- If M is Kähler hyperbolic in the sense of [13], then the L^2-cohomology vanishes except for the middle dimension where it is non-trivial [13].

- Let M be a compact 3-manifold with infinite fundamental group whose prime decomposition $M = M_1 \sharp M_2 \sharp \ldots \sharp M_r$ consists of non-exceptional manifolds M_i (i.e., which are finitely covered by a manifold which is homotopy equivalent to a Haken, Seifert or hyperbolic manifold). Then [22]

$$\begin{aligned}
b_0^{(2)}(M) &= 0 \\
b_1^{(2)}(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) + \mid \{C \in \pi_0(\partial M) \; s.t. \; C \cong S^2\} \mid \\
b_2^{(2)}(M) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \mid \{C \in \pi_0(\partial M) \; s.t. \; C \cong S^2\} \mid \\
b_3^{(2)}(M) &= 0.
\end{aligned}$$

Two of the outstanding conjectures on L^2-cohomology are Singer's conjecture that the L^2-Betti numbers of a closed aspherical manifold vanish possible except in the middle dimension and Atiyah's conjecture that the L^2-Betti numbers of a compact manifold are rational and, provided that the fundamental group is torsionfree, are integers. Atiyah's conjecture implies Kaplanski's conjecture that the rational group ring of a group has no zero-divisors if and only if the group is torsionfree.

A group π is called *residually finite* if for any $g \in \pi$ there is a finite group G and a homomorphism $\phi : \pi \longrightarrow G$ satisfying $\phi(g) \neq 1$. If a group π is countable and residually finite, then there is a nested sequences of normal subgroups $\ldots \subset \Gamma_{m+1} \subset \Gamma_m \subset \ldots \subset \pi$ such that $\Gamma_m \subset \pi$ is of finite index and $\cap_{m=0}^{\infty} \Gamma_m = \{1\}$. We list some basic facts about residually finite groups, for more information we refer to the survey article [27].

- The free product of two residually finite groups is again residually finite [16], [6, page 27].

- A finitely generated residually finite group has a solvable word problem [29].

- The automorphism group of a finitely generated residually finite group is residually finite [3].

- A finitely generated residually finite group is hopfian, i.e, any surjective endomorphism is an automorphism [26], [30, Corollary 41.44].

- Let π be a finitely generated group possessing a faithful representation into $GL(n, F)$ for F a field. Then π is residually finite [26], [32, Theorem 4.2].
- Let π be a finitely generated group. Let π^{rf} be the quotient of π by the normal subgroup which is the intersection of all normal subgroups of π of finite index. The group π^{rf} is residually finite and any finite-dimensional representation of π in a field factorizes over the canonical projection $\pi \longrightarrow \pi^{rf}$.
- The fundamental group of a compact 3-manifold whose prime decomposition consists of non-exceptional manifolds (i.e., which are finitely covered by a manifold which is homotopy equivalent to a Haken, Seifert or hyperbolic manifold) is residually finite [17, page 380] .
- There is an infinite group with four generators and four relations which has no finite quotient except the trivial one [18].

2. Proof of Kazhdan's equality

For the remainder of this section we fix the following

Data 2.1

- *a countable residually finite group* π.
- *a nested sequence* $\ldots \subset \Gamma_{m+1} \subset \Gamma_m \subset \ldots \subset \pi$ *such that* $\Gamma_m \subset \pi$ *is a normal subgroup of finite index* $[\pi : \Gamma_m]$ *and* $\cap_{m=0}^{\infty} \Gamma_m = \{1\}$.
- *a* $\mathbb{Z}\pi$*-linear map* $f : \oplus_{i=1}^{a} \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^{b} \mathbb{Z}\pi$ *(of left* $\mathbb{Z}\pi$*-modules).*

We introduce the following notation. We have explained $\mathcal{N}(\pi)$, $\mathrm{tr}_{\mathcal{N}(\pi)}$ and $\dim_{\mathcal{N}(\pi)}$ in section 1.

Notation 2.2

- *Let* $B \in M(a, b, \mathbb{Z}\pi)$ *be the matrix describing* f*, i.e.* $f(x)$ *is given by* xB*.*

- *Let* $f_m : \oplus_{i=1}^a \mathbb{C}[\Gamma_m\backslash\pi] \longrightarrow \oplus_{i=1}^b \mathbb{C}[\Gamma_m\backslash\pi]$ *be the* $\mathbb{C}$*-linear map induced by* f*. Denote by* $f^{(2)} : \oplus_{i=1}^a l^2(\pi) \longrightarrow \oplus_{j=1}^b l^2(\pi)$ *the bounded* π*-equivariant operator induced by* f*.*
- *Let* $\{P(\lambda) \mid \lambda \in [0,\infty)\}$ *be the right continuous spectral family of the positive operator* $(f^{(2)})^* f^{(2)}$*. The* spectral density function *of* $f^{(2)}$ *is defined by*

$$F : [0,\infty) \longrightarrow [0,a] \qquad \lambda \;\mapsto\; \dim_{\mathcal{N}(\pi)}(\operatorname{im}(P(\lambda))) = \operatorname{tr}_{\mathcal{N}(\pi)}(P(\lambda)).$$

- *The* L^2-Betti number *of* f *is defined by*

$$b^{(2)}(f) = \dim_{\mathcal{N}(\pi)}(\ker(f^{(2)})) = F(0).$$

- *Let* $E_m(\lambda)$ *be the ordered set of eigenvalues* μ *of* $f_m^* f_m$ *satisfying* $\mu \le \lambda$ *listed with multiplicity.*
- *Define*

$$\begin{aligned}
F_m^{rf}(\lambda) &= \frac{|E_m(\lambda)|}{[\pi : \Gamma_m]} \\
\overline{F^{rf}}(\lambda) &= \limsup_{m\to\infty} F_m^{rf}(\lambda) \\
\underline{F^{rf}}(\lambda) &= \liminf_{m\to\infty} F_m^{rf}(\lambda) \\
\overline{F^{rf}}^+(\lambda) &= \lim_{\delta\to 0+} \overline{F^{rf}}(\lambda+\delta) \\
\underline{F^{rf}}^+(\lambda) &= \lim_{\delta\to 0+} \underline{F^{rf}}(\lambda+\delta). \qquad \blacksquare
\end{aligned}$$

Notice that all functions appearing in Notation 2.2 are monotone increasing and, possibly except $\underline{F^{rf}}$ and $\overline{F^{rf}}$, are right continuous. The main technical result of this paper is

Theorem 2.3 *1.* $F(\lambda) = \overline{F^{rf}}^+(\lambda) = \underline{F^{rf}}^+(\lambda)$*.*

2. The functions $\overline{F^{rf}}$ *and* $\underline{F^{rf}}$ *are right continuous at zero. We have*

$$\overline{F^{rf}}(0) = \overline{F^{rf}}^+(0) = F(0) = \underline{F^{rf}}(0) = \underline{F^{rf}}^+(0) = \lim_{m\to\infty} F_m^{rf}(0) = \lim_{m\to\infty} \frac{|E_m(0)|}{[\pi : \Gamma_m]}.$$

3. We have for $0 < \lambda < 1$ *if* K *is the constant introduced in 2.4*

$$F(\lambda) - F(0) \le \frac{a \cdot \ln(K^2)}{-\ln(\lambda)}.$$

Before we give its proof, we explain how it implies Theorem 0.1, Theorem 0.2 and Theorem 0.3.

We begin with Theorem 0.1. Let c_{p+1} and c_p be the differentials of $C(\widetilde{X},\widetilde{A})$ Denote by n_p the number of p-cells in $X-A$. Then we get for any finite-dimensional complex π-representation V if c_p^V is the linar map obtained by tensoring with V

$$\begin{aligned} b_p(X,A;V) &= \dim_{\mathbb{C}}\left(\ker(c_p^V)/\mathrm{im}(c_{p+1}^V)\right) \\ &= \dim_{\mathbb{C}}\left(\ker(c_p^V)\right) - \dim_{\mathbb{C}}\left(\mathrm{im}(c_{p+1}^V)\right) \\ &= \dim_{\mathbb{C}}\left(\ker(c_p^V)\right) + \dim_{\mathbb{C}}\left(\ker(c_{p+1}^V)\right) - n_{p+1}\cdot\dim_{\mathbb{C}}(V) \\ &= \dim_{\mathbb{C}}\left(\ker((c_p\oplus_{p+1})^V)\right) - n_{p+1}\cdot\dim_{\mathbb{C}}(V). \end{aligned}$$

Similiarly we get

$$b_p^{(2)}(X,A) = b^{(2)}\left((c_p\oplus c_{p+1})^{(2)}\right) - n_{p+1}.$$

Now Theorem 0.1 follows from Theorem 2.3.2 applied to $f=c_p\oplus c_{p+1}$ since

$$\frac{b_p(X_m,A_m)}{[\pi:\Gamma_m]} = \frac{b_p(X,A;\mathbb{C}[\Gamma_m\backslash\pi])}{[\pi:\Gamma_m]} = F_m^{rf}(0) - n_{p+1}$$

and

$$b_p^{(2)}(X,A) = F(0) - n_{p+1}.$$

Next we prove Theorem 0.2 and 0.3. Let $f: C_p(\widetilde{M}) \longrightarrow C_p(\widetilde{M})$ be the $\mathbb{C}\pi$-linear map $c_{p+1}c_p^* + c_p^*c_p$ for a given non-negative integer p. Then F and the analytic spectral density function F^p defined in the introduction are dilitationally equivalent [9],[15] and [22] , i.e., there are constants $\epsilon>0$ and $D>1$ such that for all $\lambda\in(0,\epsilon)$

$$F(D^{-1}\cdot\lambda) \le F^p(\lambda) \le F(D\cdot\lambda).$$

Hence it suffices to prove the claims in Theorems 0.2 and 0.3 for F instead of F^p. These follow from Theorem 2.3.

The proof of Theorem 2.3 needs some preparations.

For $u=\sum_{g\in\pi}\lambda_g\cdot g\in\mathbb{C}\pi$ define $|u|_1=\sum_{g\in\pi}|\lambda_g|$. Recall that

$B\in M(a,b,\mathbb{Z}\pi)$ describes f. For the sequel fix a real number K satisfying

2.4 $K \ge a\cdot\sum_{j=1}^b \max\{|B_{i,j}|_1 \mid i=1,2\ldots a\}$.

Lemma 2.5 *The number K is greater or equal to the operator norm of $f^{(2)}$ and f_m.*

Proof: We give the proof only for $f^{(2)}$, the one for f_m is completely analogous. Consider $x \in l^2(\pi)$ and $u = \sum_{g \in \pi} \lambda_g \cdot g \in \mathbb{C}\pi$. Then

$$|ux| = \left| \sum_{g \in \pi} \lambda_g \cdot g \cdot x \right| \leq \sum_{g \in \pi} |\lambda_g| \cdot |g \cdot x| \leq \sum_{g \in \pi} |\lambda_g| \cdot |x| \leq |u|_1 \cdot |x| \, .$$

We conclude for $x = (x_1, x_2, \ldots x_a) \in \oplus_{i=1}^a l^2(\pi)$

$$\begin{aligned}
\left| f^{(2)}(x) \right|^2 &= \sum_{j=1}^b \left| \sum_{i=1}^a x_i \cdot B_{i,j} \right|^2 \\
&\leq \sum_{j=1}^b \left(\sum_{i=1}^a |x_i \cdot B_{i,j}| \right)^2 \\
&\leq \sum_{j=1}^b \left(\sum_{i=1}^a |x_i| \cdot |B_{i,j}|_1 \right)^2 \\
&\leq \sum_{j=1}^b \left(\sum_{i=1}^a |x_i| \cdot \max\{ |B_{i,j}|_1 \mid 1 \leq i \leq a \} \right)^2 \\
&\leq \sum_{j=1}^b \left(\max\{ |B_{i,j}|_1 \mid 1 \leq i \leq a \} \right)^2 \cdot \left(\sum_{i=1}^a |x_i| \right)^2 \\
&\leq \sum_{j=1}^b \left(\max\{ |B_{i,j}|_1 \mid 1 \leq i \leq a \} \right)^2 \cdot a^2 \cdot \sum_{i=1}^a |x_i|^2 \\
&\leq \left(\sum_{j=1}^b \max\{ |B_{i,j}|_1 \mid 1 \leq i \leq a \} \cdot a \right)^2 \cdot |x|^2
\end{aligned}$$

This finishes the proof of Lemma 2.5. ■

For $\sum_{g \in \pi} \lambda_g \cdot g \in \mathbb{Z}\pi$ define

$$\operatorname{tr}_{\mathbb{Z}\pi} \left(\sum_{g \in \pi} \lambda_g \cdot g \right) = \lambda_e$$

where $e \in \pi$ is the unit element. Recall that $B \in M(a, b, \mathbb{Z}\pi)$ is the matrix describing f. Put

$$\operatorname{tr}_{\mathbb{Z}\pi}(f) = \sum_{i=1}^a \operatorname{tr}_{\mathbb{Z}\pi}(B_{i,i})$$

provided $a = b$. Denote by B^* the matrix $(\overline{B_{j,i}})$ in $M(b,a,\mathbb{Z}\pi)$ if $^- : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ is the involution sending $\sum_{g\in\pi} \lambda_g \cdot g$ to $\sum_{g\in\pi} \lambda_g \cdot g^{-1}$. Let $f^* : \oplus_{j=1}^b \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^a \mathbb{Z}\pi$ be the $\mathbb{Z}\pi$-linear map described by B^*. Notice that $(f^{(2)})^* = (f^*)^{(2)}$. One easily checks that f^*f is described by BB^* and

$$\mathrm{tr}_{\mathcal{N}(\pi)}\left((f^{(2)})^* f^{(2)}\right) = \mathrm{tr}_{\mathbb{Z}\pi}\left(f^* f\right).$$

The proof of the following lemma is the only place where we need the assumption that $\Gamma_m \subset \pi$ is normal.

Lemma 2.6 *Let $p(\mu)$ be a polynomial. There is a number $m_0\left((\Gamma_m)_{m\geq 0}, p\right)$ such that for all $m \geq m_0\left((\Gamma_m)_{m\geq 0}, p\right)$*

$$\mathrm{tr}_{\mathbb{Z}\pi}\left(p(f^* f)\right) = \frac{1}{[\pi : \Gamma_m]} \cdot \mathrm{tr}_{\mathbb{C}}\left(p(f_m^* f_m)\right).$$

Proof: Recall that the matrix $B \in M(a,b,\mathbb{Z}\pi)$ describes f. Fix elements $g_0, g_1, \ldots, g_r \in \pi$ and $\lambda_0, \lambda_1, \ldots, \lambda_r \in \mathbb{C}$ such that $g_0 = e$, $g_i \neq e$ and $\lambda_i \neq 0$ for $1 \leq i \leq r$ and

$$\sum_{j=1}^{a} \left(p(BB^*)\right)_{j,j} = \sum_{i=0}^{r} \lambda_i g_i.$$

We get:

$$\mathrm{tr}_{\mathbb{Z}\pi}\left(p(f^* f)\right) = \lambda_0$$

and

$$\mathrm{tr}_{\mathbb{C}}\left(p(f_m^* f_m)\right) = \mathrm{tr}_{\mathbb{C}}\left(\sum_{i=1}^{r} \lambda_i \cdot r(g_i) : \mathbb{C}[\Gamma_m\backslash\pi] \longrightarrow \mathbb{C}[\Gamma_m\backslash\pi]\right)$$

where $r(g_i)$ is right multiplication with g_i. Since the intersection of the Γ_m-s is trivial, there is a number m_0 such that for $m \geq m_0$ none of the elements g_i for $1 \leq i \leq r$ lies in Γ_m. Since $\Gamma_m \subset \pi$ is normal, we conclude for $m \geq m_0$ and $1 \leq i \leq r$

$$\mathrm{tr}_{\mathbb{C}}(r(g_i) : \mathbb{C}[\Gamma_m\backslash\pi] \longrightarrow \mathbb{C}[\Gamma_m\backslash\pi]) = 0.$$

This implies for $m \geq m_0$

$$\mathrm{tr}_{\mathbb{Z}\pi}\left(p(f^* f)\right) = \frac{1}{[\pi : \Gamma_m]} \cdot \mathrm{tr}_{\mathbb{C}}\left(p(f_m^* f_m)\right). \quad \blacksquare$$

Lemma 2.7 *Let $p_n(\mu)$ be a sequence of polynomials such that for the characteristic function $\chi_{[0,\lambda]}(\mu)$ of the interval $[0,\lambda]$ and an appropriate real number L*

$$\lim_{n\to\infty} p_n(\mu) = \chi_{[0,\lambda]}(\mu) \quad \textit{and} \quad |p_n(\mu)| \leq L$$

holds for each $\mu \in [0, |f^{(2)}|^2]$. Then

$$\lim_{n\to\infty} \mathrm{tr}_{\mathbb{Z}\pi}\left(p_n(f^*f)\right) = F(\lambda).$$

Proof: Let $\{P(\lambda) \mid \lambda \in [0, |f^{(2)}|^2\}$ be the right continuous spectral family of $(f^{(2)})^*f^{(2)}$. We get using Lebesgues Theorem of Majorized Convergence and the fact that the von Neumann trace is linear, monotone and ultraweakly continuous

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{tr}_{\mathbb{Z}\pi}\left(p_n(f^*f)\right) &= \lim_{n\to\infty} \mathrm{tr}_{\mathcal{N}(\pi)}\left(p_n((f^{(2)})^*f^{(2)})\right) \\
&= \lim_{n\to\infty} \mathrm{tr}_{\mathcal{N}(\pi)}\left(\int_0^{||f||^2} p_n(\lambda)\, dP(\lambda)\right) \\
&= \lim_{n\to\infty} \int_0^{||f||^2} p_n(\lambda)\, dF(\lambda) \\
&= \int_0^{||f||^2} \left(\lim_{n\to\infty} p_n(\lambda)\right) dF(\lambda) \\
&= \int_0^{||f||^2} \chi_{[0,\lambda]}\, dF(\lambda) \\
&= F(\lambda)
\end{aligned}$$

This finishes the proof of Lemma 2.7. ■

Lemma 2.8 *Let $g : V \longrightarrow W$ be a linear map of finite-dimensional Hilbert spaces V and W. Let $p(t) = \det(t\mathrm{id} - g^*g)$ be the characteristic polynomial of g^*g. Write $p(t) = t^k \cdot q(t)$ for a polynomial q satisfying $q(0) \neq 0$. Let K be a positive real number such that $K \geq 1$ and $K \geq ||g||$ and let C be a positive real number such that $C \leq |q(0)|$. Let $E(\lambda)$ be the ordered set of eigenvalues μ of g^*g satisfying $\mu \leq \lambda$ listed with multiplicity. Then we get for $0 < \lambda < 1$:*

$$\frac{|E(\lambda)| - |E(0)|}{\dim_{\mathbb{C}}(V)} \leq \frac{-\ln(C)}{\dim_{\mathbb{C}}(V) \cdot (-\ln(\lambda))} + \frac{\ln(K^2)}{-\ln(\lambda)}.$$

Proof: Let $0 \leq \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{\dim_{\mathbb{C}}(V)}$ be the eigenvalues of g^*g listed with multiplicity. Let r be the integer for which $\mu_i = 0$ for $i \leq r$ and $\mu_i > 0$ for $i > r$ holds. Let s be the integer for which $\mu_i \leq \lambda$ for $i \leq s$ and $\mu_i > \lambda$ for $i > s$ is valid. Then

$$q(t) = \prod_{i=r+1}^{\dim_{\mathbb{C}}(V)} (t - \mu_i)$$

and

$$|E(\lambda)| - |E(0)| = s - r.$$

We conclude for $\lambda < 1$ (and hence $\ln(\lambda) < 0$) since $\mu_i \leq K^2$ for all i

$$\begin{array}{rcl}
\prod_{i=r+1}^{\dim_{\mathbb{C}}(V)} (-\mu_i) & = & q(0) \\
\prod_{i=r+1}^{s} \mu_i & = & |q(0)| \cdot \prod_{i=s+1}^{\dim_{\mathbb{C}}(V)} \mu_i^{-1} \\
\prod_{i=r+1}^{s} \lambda & \geq & C \cdot \prod_{i=s+1}^{\dim_{\mathbb{C}}(V)} K^{-2} \\
\lambda^{|E(\lambda)|-|E(0)|} & \geq & C \cdot K^{-2\dim_{\mathbb{C}}(V)} \\
(|E(\lambda)| - |E(0)|) \cdot \ln(\lambda) & ge & \ln(C) - \dim_{\mathbb{C}}(V) \cdot \ln(K^2) \\
\frac{|E(\lambda)| - |E(0)|}{\dim_{\mathbb{C}}(V)} & \leq & \frac{-\ln(C)}{\dim_{\mathbb{C}}(V) \cdot (-\ln(\lambda))} + \frac{\ln(K^2)}{-\ln(\lambda)}.
\end{array}$$

This finishes the proof of Lemma 2.8. ■

Next we proof Theorem 2.3.1. Fix $\lambda \geq 0$. Define for $n \geq 1$ a continuous function

$$f_n : \mathbb{R} \longrightarrow \mathbb{R} \qquad \mu \mapsto \begin{cases} 1 + 1/n & \mu \leq \lambda \\ 1 + 1/n - n \cdot (\mu - \lambda) & \lambda \leq \mu \leq \lambda + \frac{1}{n} \\ 1/n & \lambda + \frac{1}{n} \leq \mu \end{cases}$$

Obviously $\chi_{[0,\lambda]}(\mu) < f_{n+1}(\mu) < f_n(\mu)$ and $f_n(\mu)$ converges for $n \to \infty$ to $\chi_{[0,\lambda]}(\mu)$ for all $\mu \in [0, \infty)$. For each n choose a polynomial p_n such that $\chi_{0,\lambda]}(\mu) < p_n(\mu) < f_n(\mu)$ holds for all $\mu \in [0, K^2]$. Such polynomial can be found by approximating f_{n+1} sufficiently close. Hence

$$\chi_{[0,\lambda]}(\mu) < p_n(\mu) < 2 \quad \text{and} \quad \lim_{n \to \infty} p_n(\mu) = \chi_{[0,\lambda]}(\mu) \quad \text{for} \quad \mu \in [0, K^2].$$

Recall that $E_m(\lambda)$ is the ordered set of eigenvalues μ of $f_m^* f_m$ satisfying $\mu \leq \lambda$ listed with multiplicity. We conclude since $K^2 \geq ||f_m^* f_m||$ by Lemma 2.5

$$\begin{array}{rl}
& \frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}} \left(p_n(f_m^* f_m) \right) \\
= & \frac{1}{[\pi : \Gamma_m]} \cdot \sum_{\mu \in E_m(K^2)} p_n(\mu)
\end{array}$$

$$
\begin{aligned}
= \quad & \frac{|E_m(\lambda)|}{[\pi : \Gamma_m]} + \frac{1}{[\pi : \Gamma_m]} \cdot \\
& \left(\sum_{\mu \in E_m(\lambda)} (p_n(\mu) - 1) \; + \sum_{\mu \in E_m(\lambda+1/n) - E_m(\lambda)} p_n(\mu) \; + \sum_{\mu \in E_m(K^2) - E_m(\lambda+1/n)} p_n(\mu) \right)
\end{aligned}
$$

This implies

$$
F_m^{rf}(\lambda) = \frac{|E_m(\lambda)|}{[\pi : \Gamma_m]} \le \frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}} \left(p_n(f_m^* f_m)\right)
$$

We conclude further

$$
\begin{aligned}
& \frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}} \left(p_n(f_m^* f_m)\right) \\
\le \quad & \frac{|E_m(\lambda)|}{[\pi : \Gamma_m]} + \frac{\sup\{p_n(\mu) - 1 \mid \mu \in [0, \lambda]\} \cdot |E_m(\lambda)|}{[\pi : \Gamma_m]} \\
& \quad + \frac{\sup\{p_n(\mu) \mid \mu \in [\lambda, \lambda + 1/n]\} \cdot |E_m(\lambda + 1/n) - E_m(\lambda)|}{[\pi : \Gamma_m]} \\
& \quad + \frac{\sup\{p_n(\mu) \mid \mu \in [\lambda + 1/n, K^2]\} \cdot |E_m(K^2) - E_m(\lambda + 1/n)|}{[\pi : \Gamma_m]} \\
\le \quad & \frac{|E_m(\lambda)|}{[\pi : \Gamma_m]} + \frac{1/n \cdot |E_m(\lambda)|}{[\pi : \Gamma_m]} + \frac{(1 + 1/n) \cdot |E_m(\lambda + 1/n) - E_m(\lambda)|}{[\pi : \Gamma_m]} + \\
& \quad \frac{1/n \cdot |E_m(K^2) - E_m(\lambda + 1/n)|}{[\pi : \Gamma_m]} \\
\le \quad & \frac{|E_m(\lambda + 1/n)|}{[\pi : \Gamma_m]} + \frac{1}{n} \cdot \frac{|E_m(K^2)|}{[\pi : \Gamma_m]} \\
\le \quad & F_m^{rf}(\lambda + 1/n) + \frac{1}{n} \cdot a
\end{aligned}
$$

Because of Lemma 2.6 there is for each $n \in \mathbb{N}$ a number m(n) such that for all $m \ge m(n)$

$$
\frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}} \left(p_n(f_m^* f_m)\right) = \operatorname{tr}_{\mathbb{Z}\pi} \left(p_n(f^* f)\right).
$$

Hence we get for $m \ge m(n)$:

$$
F_m^{rf}(\lambda) \le \operatorname{tr}_{\mathbb{Z}\pi} \left(p_n(f^* f)\right) \le F_m^{rf}(\lambda + 1/n) + \frac{1}{n} \cdot a
$$

Taking for fixed n the limit superior respectively limit inferior for $m \to \infty$ gives

$$
\overline{F^{rf}}(\lambda) \le \operatorname{tr}_{\mathbb{Z}\pi} \left(p_n(f^* f)\right) \le \underline{F^{rf}}(\lambda + 1/n) + \frac{a}{n}
$$

Now we take the limit for $n \to \infty$ and obtain because of Lemma 2.7

$$\overline{F^{rf}}(\lambda) \leq F(\lambda) \leq \underline{F^{rf}}^{+}(\lambda)$$

We have for all $\epsilon > 0$

$$F(\lambda) \leq \underline{F^{rf}}^{+}(\lambda) \leq \underline{F^{rf}}(\lambda+\epsilon) \leq \overline{F^{rf}}(\lambda+\epsilon) \leq F(\lambda+\epsilon)$$

Since $\lim_{\epsilon \to 0+} F(\lambda+\epsilon) = F(\lambda)$ we get

$$F(\lambda) = \overline{F^{rf}}^{+}(\lambda) = \underline{F^{rf}}^{+}(\lambda).$$

This finishes the proof of Theorem 2.3.1. ■

Next we show Theorem 2.3.2 and 2.3.3. We want to apply Lemma 2.8 to the functions f_m simultaneously. Let p_m be the characteristic polynomial of f_m and $p_m(t) = t^{r_m} \cdot q_m(t)$ for appropriate r_m and $q_m(t)$ with $q_m(0) \neq 0$. Since f is a $\mathbb{Z}\pi$-linear map, the matrix describing f_m has integers as entries. Hence p_m is an integer polynomial and $|q_m(0)| \geq 1$. We get for $0 < \lambda < 1$ from Lemma 2.8 and Lemma 2.5 with the constants K and $C = 1$ which are independent of m

$$\frac{F_m^{rf}(\lambda) - F_m^{rf}(0)}{a} \leq \frac{-\ln(1)}{a \cdot [\pi : \Gamma_m] \cdot (-\ln(\lambda))} + \frac{\ln(K^2)}{-\ln(\lambda)}$$

and hence

$$F_m^{rf}(\lambda) \leq F_m^{rf}(0) + \frac{a \cdot \ln(K^2)}{-\ln(\lambda)}.$$

Taking the limit inferior and limit superior for $m \to \infty$ gives:

$$\underline{F^{rf}}(\lambda) \leq \underline{F^{rf}}(0) + \frac{a \cdot \ln(K^2)}{-\ln(\lambda)}$$

and

$$\overline{F^{rf}}(\lambda) \leq \overline{F^{rf}}(0) + \frac{a \cdot \ln(K^2)}{-\ln(\lambda)}.$$

Taking the limit for $\lambda \to 0$ gives

$$\underline{F^{rf}}(0) = \underline{F^{rf}}^{+}(0)$$

and

$$\overline{F^{rf}}(0) = \overline{F^{rf}}^{+}(0).$$

We already know $\overline{F^{rf}}^{+}(0) = F(0) = \underline{F^{rf}}(0)$ from Theorem 2.3.1. This proves Theorem 2.3.2. Since $\frac{a \cdot \ln(K^2)}{-\ln(\lambda)}$ is right continuous, we conclude:

$$\overline{F^{rf}}^{+}(\lambda) \leq F(0) + \frac{a \cdot \ln(K^2)}{-\ln(\lambda)}.$$

Now Theorem 2.3.3 follows from Theorem 2.3.1. This finishes the proof of Theorem 2.3. ■

3. Density functions and their invariants

We consider density functions and their invariants like Betti numbers, Novikov-Shubin invariants and determinants. We show that these invariants are not changed if the density function in question is made right continuous. This is useful in connection with Theorem 2.3.1. We apply this to the spectral density function of f (given in Data 2.1) and will express these invariants for $f^{(2)}$ in terms of the f_m-s. We will prove Theorem 0.5 and Theorem 0.6.

Definition 3.1 *A* density function *is a monotone increasing function* $F : [0, \infty) \longrightarrow [0, \infty)$. *Put*

$$b^{(2)}(F) = \lim_{\lambda \to 0+} F(\lambda).$$

Define

$$\alpha(F) = \liminf_{\lambda \to 0^+} \frac{\ln(F(\lambda) - b^{(2)}(F))}{\ln(\lambda)} \quad \in [0, \infty],$$

provided that $F(\lambda) > b^{(2)}(F)$ *holds for all* $\lambda > 0$. *Otherwise, we put* $\alpha(F) = \infty^+$. *Suppose there is a* K *satisfying* $F(\lambda) = F(K^2)$ *for all* $\lambda \geq K^2$. *If the integral*

$$\int_{0+}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda$$

exists as real number, put

$$\det(F) = \exp\left(\frac{1}{2} \cdot \left(\int_{0+}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda\right)\right)$$

If the integral does not exist, define $\det(F) = 0$. ■

It is not hard to check that the definition of $\det(F)$ is independent of the choice of K and that the integral exists as real number if $\alpha(F) > 0$. If $g : \oplus_{i=1}^r l^2(\pi) \longrightarrow \oplus_{i=1}^r l^2(\pi)$ is an invertible bounded π-equivariant operator its *Fuglede Kadison determinant* is defined by [11]

$$\det(g) = \exp\left(\frac{1}{2} \cdot \mathrm{tr}_{\mathcal{N}(\pi)}\left(\ln\left(g^* g\right)\right)\right) \quad \in (0, \infty).$$

It coincides with the determinant of Definition 3.1 applied to the spectral density function of g by [23, Lemma 4.2].

Lemma 3.2 *Let $F : [0, \infty) \longrightarrow [0, \infty)$ be a density function. Define the right continuous density function*

$$F^+ : [0, \infty) \longrightarrow [0, \infty) \qquad \lambda \mapsto \lim_{\delta \to 0+} F(\lambda + \delta).$$

Then

$$b^{(2)}(F) = b^{(2)}(F^+) = F^+(0), \ \alpha(F) = \alpha(F^+) \quad \textit{and} \quad \det(F) = \det(F^+).$$

Proof: Obviously we have $b^{(2)}(F) = b^{(2)}(F^+) = F^+(0)$. We have for all $\lambda > 0$

$$F(\lambda) \leq F^+(\lambda) \leq F(2 \cdot \lambda)$$

and hence for $0 < \lambda < 1$

$$\frac{F(\lambda) - b^{(2)}(F)}{\ln(\lambda)} \geq \frac{F^+(\lambda) - b^{(2)}(F^+)}{\ln(\lambda)} \geq \frac{F(2 \cdot \lambda) - b^{(2)}(F)}{\ln(\lambda)} = \frac{F(2 \cdot \lambda) - b^{(2)}(F)}{\ln(2 \cdot \lambda)} \cdot \left(1 + \frac{\ln(2)}{\ln(\lambda)}\right)$$

Now the we get by taking the limit inferior for $\lambda \to 0+$

$$\alpha(F) \geq \alpha(F^+) \geq \alpha(F)$$

and thus the claim

$$\alpha(F) = \alpha(F^+).$$

For $\epsilon > 0$ and $\delta > 0$ we estimate

$$\begin{aligned}
0 \ &\leq \ \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{\lambda} d\lambda \ - \ \int_\epsilon^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \\
&= \ \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{\lambda + \delta} \cdot \frac{\lambda + \delta}{\lambda} d\lambda \ - \ \int_\epsilon^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \\
&= \ \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{\lambda + \delta} d\lambda \ + \ \delta \cdot \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{(\lambda + \delta) \cdot \lambda} d\lambda \ - \ \int_\epsilon^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \\
&= \ \int_{\epsilon+\delta}^{K^2+\delta} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \ - \ \int_\epsilon^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \ + \ \delta \cdot \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{(\lambda + \delta) \cdot \lambda} d\lambda \\
&= \ \int_{K^2}^{K^2+\delta} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \ - \ \int_\epsilon^{\epsilon+\delta} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \ + \ \delta \cdot \int_\epsilon^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{(\lambda + \delta) \cdot \lambda} d\lambda \\
&\leq \ \delta \cdot \frac{F(K^2)}{K^2} + \delta \cdot K^2 \cdot \frac{F(K^2)}{\epsilon^2} \\
&= \ \delta \cdot \left(\frac{F(K^2)}{K^2} + \frac{K^2 \cdot F(K^2)}{\epsilon^2} \right)
\end{aligned}$$

We conclude from Lebesgue's Theorem of Majorized Convergence

$$\begin{aligned}
&\int_{\epsilon}^{K^2} \frac{F^+(\lambda) - b^{(2)}(F^+)}{\lambda} d\lambda \\
&\qquad = \int_{\epsilon}^{K^2} \lim_{\delta \to 0} \frac{F(\lambda + \delta) - b^{(2)}(F)}{\lambda} d\lambda \\
&\qquad = \lim_{\delta \to 0} \int_{\epsilon}^{K^2} \frac{F(\lambda + \delta) - b^{(2)}(F)}{\lambda} d\lambda \\
&\qquad = \int_{\epsilon}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda
\end{aligned}$$

From Levi's Theorem of Monotone Convergence we conclude

$$\begin{aligned}
&\int_{0+}^{K^2} \frac{F^+(\lambda) - b^{(2)}(F^+)}{\lambda} d\lambda \\
&\qquad = \lim_{\epsilon \to \infty} \int_{\epsilon}^{K^2} \frac{F^+(\lambda) - b^{(2)}(F^+)}{\lambda} d\lambda \\
&\qquad = \lim_{\epsilon \to \infty} \int_{\epsilon}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \\
&\qquad = \int_{0+}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda
\end{aligned}$$

This finishes the proof of Lemma 3.2. ■

Now we consider Data 2.1 and use Notation 2.2

Lemma 3.3

1.

$$\int_{0+}^{K^2} \frac{F(\lambda) - F(0)}{\lambda}\, d\lambda \;\leq\; \liminf_{m \to \infty} \int_{0+}^{K^2} \frac{F_m(\lambda) - F_m(0)}{\lambda}\, d\lambda$$

2. If

$$\int_{0+}^{K^2} \sup\left\{ \frac{F_m(\lambda) - F_m(0)}{\lambda} \,\middle|\, m \geq 0 \right\} d\lambda \;<\; \infty$$

then

$$\int_{0+}^{K^2} \frac{F(\lambda) - F(0)}{\lambda}\, d\lambda \;=\; \lim_{m \to \infty} \int_{0+}^{K^2} \frac{F_m(\lambda) - F_m(0)}{\lambda}\, d\lambda$$

3.

$$\int_{0+}^{K^2} \frac{F(\lambda)-F(0)}{\lambda}\, d\lambda \;=\; \lim_{\epsilon\to 0}\,\lim_{m\to\infty}\int_{\epsilon}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda$$

Proof: 1.) We get from Theorem 2.3, Lemma 3.2 and Levi's Theorem of Monotone Convergence

$$\begin{aligned}
& \int_{0+}^{K^2} \frac{F(\lambda)-F(0)}{\lambda}\, d\lambda \\
= \;& \int_{0+}^{K^2} \frac{\underline{\underline{F}}^{rf+}(\lambda)-\underline{\underline{F}}^{rf+}(0)}{\lambda}\, d\lambda \\
= \;& \int_{0+}^{K^2} \frac{\underline{\underline{F}}^{rf}(\lambda)-\underline{\underline{F}}^{rf}(0)}{\lambda}\, d\lambda \\
= \;& \int_{0+}^{K^2} \frac{\liminf_{m\to\infty} F_m(\lambda)-\lim_{m\to\infty} F_m(0)}{\lambda}\, d\lambda \\
= \;& \int_{0+}^{K^2} \liminf_{m\to\infty} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda \\
= \;& \int_{0+}^{K^2} \lim_{m\to\infty}\left(\inf\left\{\frac{F_n(\lambda)-F_n(0)}{\lambda}\,\middle|\, n\geq m\right\}\right) d\lambda \\
= \;& \lim_{m\to\infty}\int_{0+}^{K^2} \inf\left\{\frac{F_n(\lambda)-F_n(0)}{\lambda}\,\middle|\, n\geq m\right\} d\lambda \\
\leq \;& \lim_{m\to\infty}\left(\inf\left\{\int_{0+}^{K^2}\frac{F_n(\lambda)-F_n(0)}{\lambda}\, d\lambda\,\middle|\, n\geq m\right\}\right) \\
= \;& \liminf_{m\to\infty}\int_{0+}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda
\end{aligned}$$

2.) We get from Theorem 2.3, Lemma 3.2 and Lebegues Theorem of Majorized Convergence

$$\begin{aligned}
& \limsup_{m\to\infty}\int_{0+}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda}\, d\lambda \\
= \;& \lim_{m\to\infty}\left(\sup\left\{\int_{0+}^{K^2}\frac{F_n(\lambda)-F_n(0)}{\lambda}\, d\lambda\,\middle|\, n\geq m\right\}\right) \\
\leq \;& \lim_{m\to\infty}\int_{0+}^{K^2} \sup\left\{\frac{F_n(\lambda)-F_n(0)}{\lambda}\,\middle|\, n\geq m\right\} d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \int_{0+}^{K^2} \lim_{m\to\infty} \left(\sup\left\{ \frac{F_n(\lambda)-F_n(0)}{\lambda} \middle| n \geq m \right\} \right) \, d\lambda \\
&= \int_{0+}^{K^2} \limsup_{m\to\infty} \frac{F_m(\lambda)-F_m(0)}{\lambda} \, d\lambda \\
&= \int_{0+}^{K^2} \frac{\limsup_{m\to\infty} F_m(\lambda) - \lim_{m\to\infty} F_m(0)}{\lambda} \, d\lambda \\
&= \int_{0+}^{K^2} \frac{\overline{F^{rf}}(\lambda) - \overline{F^{rf}}(0)}{\lambda} \, d\lambda \\
&= \int_{0+}^{K^2} \frac{\overline{F^{rf}}^{+}(\lambda) - \overline{F^{rf}}^{+}(0)}{\lambda} \, d\lambda \\
&\int_{0+}^{K^2} \frac{F(\lambda)-F(0)}{\lambda} \, d\lambda
\end{aligned}
$$

Now assertion 2.) follows using assertion 1.)

3.) If we substitute the lower bound 0+ in the calculations of assertion 1.) and assertion 2.) by $\epsilon > 0$, they remain true without the additional assumption in assertion 2.) that a certain integral is smaller than ∞. Hence we get

$$
\int_{\epsilon}^{K^2} \frac{F(\lambda)-F(0)}{\lambda} \, d\lambda \;=\; \lim_{m\to\infty} \int_{\epsilon}^{K^2} \frac{F_m(\lambda)-F_m(0)}{\lambda} \, d\lambda
$$

Now we derive from Levi's Theorem of Monotone Convergence

$$
\int_{0+}^{K^2} \frac{F(\lambda)-F(0)}{\lambda} \, d\lambda \;=\; \lim_{\epsilon\to 0} \int_{\epsilon}^{K^2} \frac{F(\lambda)-F(0)}{\lambda} \, d\lambda.
$$

This finishes the proof of Lemma 3.3. ■

The main technical result of this section is

Theorem 3.4 *1. Let $p(t) = \det_{\mathbb{C}}(t - f_m^* f_m)$ be the characteristic polynomial of the $\mathbb{C}$-linear endomorphism $f_m^* f_m$. Write $p(t) = t^r \cdot q(t)$ for appropriate r and q satisfying $q(0) \neq 0$. Then*

$$
\det(F_m) \;=\; \left(\prod_{\mu \in E_m(K^2), \mu > 0} \mu \right)^{\frac{1}{2\cdot[\pi:\Gamma_m])}} \;=\; |q(0)|^{\frac{1}{2\cdot[\pi:\Gamma_m])}} \geq 1
$$

where $E_m(K^2)$ is the ordered set of eigenvalues of $f_m^* f_m$ counted with multiplicity.

2.

$$\det(F) \geq \limsup_{m\to\infty} \det(F_m) \geq 1.$$

3. Suppose there are constants $\alpha > 0$, $\epsilon > 0$ and $C > 0$ such that for all $\lambda \in (0, \epsilon)$ and $m \geq 0$

$$F_m(\lambda) - F_m(0) \leq C \cdot \lambda^\alpha.$$

Then

$$\begin{array}{rcl} \alpha(F) & \geq & \alpha \\ \det(F) & = & \lim_{m\to\infty} \det(F_m). \end{array}$$

Proof: 1.) follows from the definition of $\det(F_m)$ by an elementary calculation since $F_m(\lambda)$ is a step function and

$$p(t) = \prod_{\mu \in E_m(K^2)} (t - \mu).$$

Since f_m is described by a matrix with integral coefficients, $p(t)$ is an integer polynomial. Hence $q(0)$ an integer and $|q(0)| \geq 1$.

2.) From Theorem 2.3.2 and Lemma 3.3.1

$$\begin{array}{rcl} 2 \cdot \ln(\det(F)) & = & \\ & = & \ln(K^2) \cdot (F(K^2) - b^{(2)}(F)) - \displaystyle\int_{0+}^{K^2} \frac{F(\lambda) - b^{(2)}(F)}{\lambda} d\lambda \\ & \geq & \ln(K^2) \cdot (F_m(K^2) - \lim_{m\to\infty} F_m(0)) - \liminf_{m\to\infty} \displaystyle\int_{0+}^{K^2} \frac{F_m(\lambda) - F_m(0)}{\lambda}\, d\lambda \\ & = & \limsup_{m\to\infty} \left(\ln(K^2) \cdot (F_m(K^2) - F_m(0)) - \displaystyle\int_{0+}^{K^2} \frac{F_m(\lambda) - F_m(0)}{\lambda}\, d\lambda \right) \\ & = & \limsup_{m\to\infty}\ 2 \cdot \ln(\det(F_m)) \end{array}$$

and the claim follows.

3.) We get from the assumption and Theorem 2.3 for small $0 < \lambda < 0$.

$$\overline{F^{rf}}(\lambda) - b^{(2)}(\overline{F^{rf}}) = \limsup_{m\to\infty} F_m(\lambda) - \lim_{m\to\infty} F_m(0) \leq \sup\{F_m(\lambda) - F_m(0) |\, m \geq 0\} \leq C \cdot \lambda^\alpha$$

Hence $\alpha(\overline{F^{rf}}) \geq \alpha$ Since $\alpha(F) = \alpha(\overline{F^{rf}}^{+}) = \alpha(\overline{F^{rf}})$ by Theorem 2.3.1 and Lemma 3.2 we get $\alpha(F) \geq \alpha$. Because

$$\int_{0+}^{K^2} \sup \left\{ \frac{F_m(\lambda) - F_m(0)}{\lambda} \middle| m \geq 0 \right\} d\lambda \leq C \cdot \int_{0+}^{K^2} \frac{\lambda^\alpha}{\lambda} d\lambda < \infty$$

holds, $\det(F) = \lim_{m\to\infty} \det(F_m)$ follows from Lemma 3.3.2. This finishes the proof of Theorem 3.4

Now we can prove Theorem 0.5. We have already mentioned in the introduction that assertions 1.) and 2.) are equivalent so that we only have to prove assertion 2.). Suppose that $f : \oplus_{i=1}^a \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^a \mathbb{Z}\pi$ is a $\mathbb{Z}\pi$-automorphism. Choose K such big that the inequality 2.4 holds for both f and f^{-1}. Then the operator norms of $f^{(2)}$, $\left(f^{(2)}\right)^{-1}$, f_m and f_m^{-1} for all $m \geq 0$ are less or equal to K. In particular $F_m(\lambda) = F_m(0) = 0$ holds for $\lambda < K^{-1}$. From Theorem 3.4.3 we get for the Fuglede Kadison determinant of $f^{(2)}$ which is $\det(F)$ if $\det_{\mathbb{C}}$ denotes the determinant of a linear automorphism of a finite-dimensional $\mathbb{C}$-vector space

$$\det(F) = \lim_{m\to\infty} \det{}_{\mathbb{C}}(f_m^* f_m)^{\frac{1}{2\cdot[\pi:\Gamma_m]}}.$$

Since f is described by a matrix with coefficients in $\mathbb{Z}\pi$, the matrix describing f_m has integer entries. Hence $\det_{\mathbb{C}}(f_m) = \pm 1$ and therefore $\det_{\mathbb{C}}(f_m^* f_m) = 1$ for all m. This implies $\det(F) = 1$ and finishes the proof of Theorem 0.5.

Finally we prove Theorem 0.6. The proof of [23, Theorem 2.4] shows that for an appropriate $\mathbb{Z}\pi$-endomorphism $f : \oplus_{i=1}^a \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^a \mathbb{Z}\pi$

$$\rho(M) = \det(F)^{-1}$$

holds. Now apply Theorem 3.4.2 ■

4. Extending Swan's theorem from finite to residually finite groups

This section is devoted to the proof of Theorem 0.4. We mention that it is not hard to prove Theorem 0.4 using Bass [2, remark 6.11]. We begin with explaining the meaning of assertion 1.) and why it does imply assertion 2.)

Denote by $\mathrm{tr}^c_{\mathcal{N}(\pi)}$ the center valued trace of the finite von Neumann algebra $\mathcal{N}(\pi)$ [19, Theorem 8.2.8]. Two idempotents $p \in M(n,n,\mathcal{N}(\pi))$ and $q \in M(m,m,\mathcal{N}(\pi))$ are called *stably equivalent* if $p \oplus 0 = xy$ and $q \oplus 0 = yx$ for some $l \geq m, n$ and $x, y \in M(l,l,\mathcal{N}(\pi))$ holds.

This is equivalent to the statement that $\operatorname{im}(p)$ and $\operatorname{im}(q)$ are isometrically π-isomorphic Hilbert $\mathcal{N}(\pi)$-modules. Moreover, p and q are stably e quivalent if and only if they have the same center valued trace [19, Theorem 8.4.3]. Hence the first assertion in Theorem 0.4 means that for any idempotent $p \in M(n,n,\mathbb{Z}\pi)$ the idempotent $p^{(2)}$ is stably equivalent to the idempotent given by the identity matrix $I_r \in M(r,r,\mathcal{N}(\pi))$ for r the rank of the abelian group $\mathbb{Z} \otimes_{\mathbb{Z}\pi} \operatorname{im}(p)$. Equivalently, $\operatorname{im}(p^{(2)})$ is isometrically π-isomorphic to $\oplus_{i=1}^r l^2(\pi))$. Hence assertion 1.) implies assertion 2.) and it remains to prove assertion 1.)

We have to show for an idempotent $p \in M(n,n,\mathbb{Z}\pi)$ that $\operatorname{tr}^c_{\mathcal{N}(\pi)}(p^{(2)}) = rI_1$ for I_1 the unit element in $\mathcal{N}(\pi)$. Consider an element u in the center $\mathcal{Z}(\mathbb{Z}\pi)$ of $\mathbb{Z}\pi$. Denote by uI_n the diagonal (n,n)-matrix with all diagonal entries equal to u. From Lemma 2.6 applied to $f = uI_n p$ and $f = uI_1$ we obtain the existence of a positive integer m satisfying (in the notation 2.2)

$$\operatorname{tr}_{\mathcal{N}(\pi)}\left(uI_n \cdot p)\right) = \frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}}\left((uI_n \cdot p)_m\right)$$

and

$$\operatorname{tr}_{\mathcal{N}(\pi)}(uI_1) = \frac{1}{[\pi : \Gamma_m]} \cdot \operatorname{tr}_{\mathbb{C}}\left(u_m I_1\right).$$

By Swan's result [31, Theorem 8.1] applied to the finite group $\Gamma_m \backslash \pi$ the image of p_m is a finitely generated free $\mathbb{C}[\Gamma_m \backslash \pi]$-module of rank r. Hence we get

$$\operatorname{tr}_{\mathbb{C}}\left((uI_n \cdot p)_m\right) = \operatorname{tr}_{\mathbb{C}}\left((uI_r)_m\right) = ru_m I_1.$$

This implies

$$\operatorname{tr}_{\mathcal{N}(\pi)}(uI_n \cdot p) = \operatorname{tr}_{\mathcal{N}(\pi)}(ruI_1).$$

From the universal property of the center valued trace [19, Proposition 8.3.10],

$$\operatorname{tr}_{\mathcal{N}(\pi)}\left(uI_n \cdot p\right) = \operatorname{tr}_{\mathcal{N}(\pi)}\left(uI_1 \cdot \operatorname{tr}^c_{\mathcal{N}(\pi)}(p)\right).$$

Hence we get for all $u \in \mathcal{Z}(\mathbb{Z}\pi)$

$$\operatorname{tr}_{\mathcal{N}(\pi)}\left(uI_1 \cdot (\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1)\right) = 0.$$

For $g \in \pi$ denote by (g) the set of elements of π which are conjugated to g. The center valued trace applied to gI_1 gives 0 if (g) is infinite and $\frac{1}{|(g)|} \cdot \left(\sum_{h \in (g)} h\right) I_1$ otherwise. Hence $\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1 = vI_1$ for an appropriate element v in $\mathcal{Z}(\mathbb{Z}\pi)$. We conclude for $u = v^*$

$$\operatorname{tr}_{\mathcal{N}(\pi)}\left((\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1)^* \cdot (\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1)\right) = 0.$$

Since $\operatorname{tr}_{\mathcal{N}(\pi)}$ is positive,

$$(\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1)^* \cdot (\operatorname{tr}^c_{\mathcal{N}(\pi)}(p) - rI_1) = 0$$

and hence

$$\mathrm{tr}^{c}_{\mathcal{N}(\pi)}(p) - rI_1 = 0.$$

This finishes the proof of Theorem 0.4. $\blacksquare$

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Wolfgang Lück
Fachbereich Mathematik
Johannes Gutenberg-Universität
55099 Mainz
Bundesrepublik Deutschland
email: lueck@topologie.mathematik.uni-mainz.de
FAX: 06131 393867

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