Arbeitsgruppe Topologie

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April 2008

Proposed Projects

- Rigidity (Bartels, Joachim, Lück, Sauer)
- L^2-invariants, measure theory, and dynamcis (Deninger, Löh, Lück, Sauer, Wegner)
- Equivariant homotopy and (co-)homology (Joachim, Lück, Schürmann)

Rigidity

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.
In particular, M and N are homeomorphic.

- This is the topological version of Mostow rigidity.
- The Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.
- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for principal ideal domains and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the principal ideal domain R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is also an L-theory version.

- If a group G satisfies the Farrell-Jones Conjecture for both algebraic K- and L-theory, then it satisfies also the following prominent conjectures:
  - Kaplansky Conjecture;
  - $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ are trivial;
  - Borel Conjecture;
  - Bass Conjecture;
  - Novikov Conjecture.
- The Farrell-Jones Conjecture is related to its analogue for the topological K-theory of the reduced group C^*-algebra, the Baum-Connes Conjecture.

Theorem (Bartels-Lück(2008))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds. Then $\mathcal{FJ}$ has the following properties:

- *Hyperbolic groups and virtually nilpotent groups belong to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for all $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *The class $\mathcal{FJ}$ contains* CAT(0)*-groups.*

- Limit groups in the sense of Sela are CAT(0)-groups
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- On example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients
- Our results show that these groups do satisfy the Farrell-Jones Conjecture and hence also the other conjectures mentioned above.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- By our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- The Farrell-Jones Conjecture is the basic tool for concrete calculations of $K_n(RG)$ and $L_n(RG)$.
- The proof of the main theorem uses controlled topology and algebra, i.e., topology and algebra parametrized by a metric space.
- Moreover, the construction of a flow space associated to a CAT(0)-group plays an important role.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Cocompact lattices in connected Lie groups;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
- There are analogues of the Farrell-Jones Conjecture for so called pseudo-isotopy spaces and for Waldhausen's A-theory. They have interesting applications to automorphism groups of closed manifolds.

Conjecture (folk)

Let G_0 and G_1 be two virtually nilpotent groups. Let L_0 and L_1 be the simply connected nilpotent Lie groups given by their Mal'cev completion. If G_0 and G_1 are quasi-isometric, then L_0 and L_1 are isomorphic as Lie groups.

Theorem (Sauer (2006))

If two virtually nilpotent groups are quasi-isometric, then their real cohomology rings are isomorphic.

L^2-invariants, measure theory, and dynamcis

- L^2-invariants arise from classical invariants for closed manifolds by passing to the universal covering and taking the action of the fundamental group into account.
- Recall the notion of a Betti number of a closed manifold M

$$b_p(M) = \dim_{\mathbb{C}}(H_p(M;\mathbb{C}).$$

- Define the L^2-Betti number of $\widetilde{M}$

$$b_p^{(2)}(\widetilde{M}) \;=\; \dim_{\mathcal{N}(\pi)}\left(H_p^{(2)}(\widetilde{M};\mathcal{N}(\pi))\right).$$

- The classical Betti number can be expressed in terms of the heat kernel on M

$$b_p(M) = \lim_{t\to\infty} \int \operatorname{tr}(e^{-t\Delta}(x,x))\; dvol_M.$$

- Analogously the L^2-Betti number can be expressed in terms of the heat kernel on $\widetilde{M}$

$$b_p(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}(e^{-t\widetilde{\Delta}}(\widetilde{x},\widetilde{x}))\; dvol_{\widetilde{M}}.$$

- The only relationship between classical Betti numbers and L^2-Betti numbers is the Euler-Poincaré formula

$$\chi(M) = \sum_{p\geq 0}(-1)^p \cdot b_p(M) = \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\widetilde{M}).$$

Conjecture (Atiyah Conjecture for torsionfree groups)

A torsionfree group G satisfies the Atiyah Conjecture if for any matrix $A \in M(m, n, \mathbb{Q}G)$ the von Neumann dimension of the kernel of the G-equivariant bounded operator $r_A^{(2)} \colon l^2(G)^m \to l^2(G)^n, \ \ x \mapsto xA$ is an integer.

- This is for a finitely presented torsionfree group G equivalent to the statement that for any closed manifold M with $\pi_1(M) \cong G$ we have

$$b_P^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- Other interesting conjectures are due to Singer and Hopf.
- L^2-invariants have successfully been used in group theory.

Definition (Measure equivalence)

Two countable groups G_0 and G_1 are called measure equivalent if there exist commuting measure-preserving free actions of G_0 and G_1 on some infinite Lebesgue measure space (Ω, m) such that the actions of both G_0 and G_1 admit finite measure fundamental domains.

- The notion of measure equivalence can be viewed as the measure theoretic analogue of the metric notion of quasi-isometric groups.
- In general measure theoretic methods seem to be very promising in attacking certain geometric problems.

Theorem (L^2-Betti numbers and measure equivalence, Gaboriau (2002))

Let G_0 and G_1 be two countable groups which are measure equivalent. Then there is a constant $C > 0$ such that for all $p \geq 0$

$$b_p^{(2)}(G_0) \;=\; C \cdot b_p^{(2)}(G_1).$$

Conjecture (Measure equivalence and L^2-torsion)

Let G_0 and G_1 be two countable groups which are measure equivalent. Suppose that there are finite models for BG_0 and BG_1 and $b_p^{(2)}(G_0) = b_p^{(2)}(G_1) = 0$ for $p \geq 0$. Then

$$\rho^{(2)}(\widetilde{BG_0}) = 0 \Leftrightarrow \rho^{(2)}(\widetilde{BG_1}) = 0.$$

Conjecture (Simplicial volume and L^2-invariants)

Let M be an aspherical closed oriented manifold. Suppose that its simplicial volume $||M||$ vanishes. Then

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad \text{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

Theorem (Inequality for L^2-Betti numbers, Sauer (2007))

For each $n \geq 0$ there is a constant $C_n > 0$ with the following property: If M is an n-dimensional closed aspherical manifold M, then for all $p \geq 0$

$$b_p^{(2)}(\widetilde{M}) \leq C_n \cdot \operatorname{minvol}(M).$$

Conjecture (Rank 1-rigidity)

Let Γ be a lattice in a Lie group G of $\mathbb{R}$-rank 1 and dimension ≥ 3. If a countable group Λ is measure equivalent to Γ, then Λ is virtually a lattice in G, and the given measure coupling is induced by the standard coupling of lattices.

- The analogous conjecture in the higher rank case was proved by Furman.
- The biggest challenge in proving the above conjecture is the lack of a general Margulis-Zimmer-type cocycle superrigidity theorem for the rank 1 case.

- Deninger will explain his research proposal belonging to this project in his talk.

Equivariant homotopy and (co-)homology

- The general idea is to extend notions from equivariant homotopy and (co-)-homology for actions of finite groups acting to proper actions of infinite groups.
- Equivariant (co-)homology theories
  The axioms and an approach using spectra have been developed. Examples arise from the sources of the Baum-Connes Conjecture and the Farrell-Jones Conjecture.
- Equivariant stable homotopy category

- Equivariant Chern characters
- Specific integral computations
- Equivariant (co-)homotopy
- Segal Conjecture

  It says for a finite group

$$\pi_s^0(BG) \cong A(G)\widehat{_I},$$

  and has been proved by Carlsson (1986).

  Problem and test case: Extend it to infinite groups.

- Computations for (co-)homology of BG using equivariant (co-)-homology

Theorem (Rational computation of $K^*(BG)$, Lueck(2007))

Suppose that there is a cocompact G-CW-model for $\underline{E}G$. Then

$$K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \operatorname{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle; \widehat{\mathbb{Q}_p})\right),$$

- Equivariant Chern characters have been used to analyze the link between the Baum-Connes Conjecture and the Trace Conjecture.

- Characteristic classes of singular spaces, (Schürmann).
- Question: What are the relations of the characteristic classes under a map $f\colon Y \to Z$ of singular spaces such as complex algebraic varieties?
- A typical formula for a suitable partition $\{S\}$ of Z, with F_S a fiber over S, is

$$f_*((T_y(Y)) = \sum_S \chi_y(F_S) \cdot \big(T_y(\overline{S}) - T_y(\overline{S} \backslash S)\big).$$

- In the case , where Y is smooth and Z is a point, it specializes to the theorems of Poincaré-Hopf ($y = -1$), Hirzebruch-Riemann-Roch ($y = 0$) and the signature theorem of Hirzebruch ($y = 1$).

Future projects are:

- Equivariant versions with values in equivariant homology. These are in close relation to equivariant Chern characters and index theorems in non-commutative geometry.
- Functorial characteristic classes with values in symmetric L-theory.
- A universal motivic version of these theories in the algebraic geometric context.

Computations of *ko*-homology groups for classifying spaces, (Joachim)

- Problem: Compute the connective *ko*-theory of BG for a finite group$G$.
- This is an important step towards proving the (unstable) Gromov-Lawson-Rosenberg Conjecture for finite groups.
- Problem: What happens in the twisted context?

- Extend all these results to proper actions of topological groups.
- Minimal requirement: All compact subgroups are Lie groups.
- Some computations concerning Kac-Moody groups raise the question whether there is a version of the Baum-Connes Conjecture for these groups.

Conformal nets as higher von Neumann algebras, (Bartels)

- Quantum field theories can be defined as functors from bordims categories to the category with Hilbert spaces as objects and operators as morphisms.
- These descriptions are not as local as one wishes them to be.
- To repair this, one needs to cut also boundaries into pieces. So one should pass to a 2-category on the bordism side and to the 2-category of von Neumann algebras, bimodules and operators.
- The main project is to pass even to level 3 which gives a delooping of the previous case. Objects are now conformal nets.

- Applications are expected in connection with
  - Reshetikhin-Turaev-quantum field theories and the Jones-polynomial.
  - Construction of conformal field theories.
  - Elliptic cohomology
- This project may well fit into one of the projects of non-commutative geometry or mathematical physics.