Isomorphism Conjectures in K- and L-theory
Joint UMI/DMV-meeting

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Outline and goal

- Explain the K-theoretic and L-theoretic Farrell-Jones Conjecture and Baum-Connes Conjecture at least for torsionfree groups
- Discuss applications and the potential of these conjectures.
- State our main theorems.

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Conjecture (Farrell-Jones)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.
- Example $G = \mathbb{Z}$: $K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R)$.

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- $K_0(R)$ is the Grothendieck construction applied to the abelian semigroup of finitely generated projective R-modules.
- Let G be a finite group and F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.
- $K_1(R)$ is the abelianization $GL(R)/[GL(R), GL(R)]$ of $GL(R) = \bigcup_{n \geq 1} GL_n(R)$.
- The assignment $A \mapsto [A] \in K_1(R)$ for $A \in GL_n(R)$ is the universal determinant for R.
- $\widetilde{K}_n(R)$ is the cokernel of $K_n(\mathbb{Z}) \to K_n(R)$.
- The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

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Definition (*h*-cobordism)

An *h*-cobordism over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem (Barden, Mazur, Stallings, Kirby-Siebenmann))

Let M_0 be a closed manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M_0)$. Let $(W; M_0, M_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(G)$ vanishes.

- The *s*-Cobordism Theorem implies the Poincaré Conjecture in dimension ≥ 5.
- It is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.

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- In order to illustrate the depth of the Farrell-Jones Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the K-theoretic Farrell-Jones Conjecture for the coefficient ring R.

Lemma

Let R be a regular ring, for instance $\mathbb{Z}$, a field or a principal ideal domain. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

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We get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every compact h-cobordism $W = (W; M_0, M_1)$ of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial.
- If G belongs to $\mathcal{FJ}_K(\mathbb{Z})$, then it is of type FF if and only if it is of type FP. (Serre's problem)

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Conjecture (Kaplansky)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (Bartels-L.-Reich (2007))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree;*
- *G is torsionfree and sofic, e.g., residually amenable;*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

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- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in R.

Theorem (Bartels-L.-Reich (2007))

- *Every hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}_K(R)$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}_K(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}_K(R)$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}_K(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}_K(R)$;*
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- We emphasize that this result holds for all rings R.
- The groups above are certainly wild in Bridson's universe.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, belong to $\mathcal{FJ}_K(R)$ for all R.
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- If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.
- As in the case of algebraic K-theory there is also an analogous version of the L-theoretic Farrell-Jones Conjecture for arbitrary groups G.
- Bartels and L. have a program to extend our result for the K-theoretic Farrell-Jones Conjecture also to the L-theoretic version. This would imply the Novikov and the Borel Conjecture for such groups.
- Bartels and L. have a program to prove $G \in \mathcal{FJ}_L(R)$ if G acts properly and cocompact on a CAT(0)-space. This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

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The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG.
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- The Bost Conjecture is the analogue for $l^1(G)$, i.e., it concerns the assembly map.

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Theorem (Bartels-L.-Echterhoff (2007))

Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of hyperbolic groups G_i (with not necessarily injective structure maps).
Then G satisfies the Farrell-Jones Conjecture with coefficients and the Bost Conjecture with coefficients.

- The proof uses the deep result of Lafforgue that the Bost Conjecture with coefficients is true for every hyperbolic group.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, do satisfy the Bost Conjecture with coefficients.
  So the failure of the Baum-Connes Conjecture with coefficients says that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$ is not bijective.
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- It is not known whether there are counterexamples to the Farrell-Jones Conjecture or the Baum-Connes Conjecture.
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