Topological Rigidity

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Outline and goal

- Present a list of prominent conjectures such as the one due to Borel, Kaplansky, Moody and Novikov.
- Discuss the Farrell-Jones Conjecture and that it implies the other ones.
- State our main theorem which is joint work with Bartels. It says that these conjectures are true for an interesting class of groups including hyperbolic groups and CAT(0)-groups.
- Discuss topological rigid manifolds.
- Discuss some computational aspects.
- Give a brief indication of the idea of the proof if time allows.

Some prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Projective class groups)

Let R be a regular ring. Suppose that G is torsionfree. Then:

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective;*
- *If R is a principal ideal domain, then* $\widetilde{K}_0(RG) = 0$.

- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.,e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
- Let G be a finitely presented group. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.

Conjecture (Serre)

Every group of type FP is of type FF.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ vanishes.

- Fix $n \geq 6$. The vanishing of $\mathrm{Wh}(G)$ is equivalent to the following geometric statement: Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial.

Conjecture (Moody's Induction Conjecture)

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G*

$$\operatorname{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Then the map*

$$\operatorname{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

- If G is torsionfree, the Induction Conjecture says that everything comes from the trivial subgroup and we rediscover some of the previous conjectures.

Conjecture (L^2-torsion)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of a closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering.
- If M is hyperbolic and has odd dimension, its L^2-torsion is up to a non-zero dimension constant its volume.
- The conjecture above allows to extend the notion of volume to word hyperbolic groups whose L^2-Betti numbers all vanish.
- It also gives interesting invariants for group automorphisms.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f).

Definition (Aspherical)

A connected *CW*-complex X is called aspherical if it satisfies one of the following equivalent conditions:

- $\pi_n(X) = 0$ for $n \geq 2$;
- $\widetilde{X}$ is contractible;
- $X = B\pi_1(X)$;

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic.

- This is the topological version of Mostow rigidity.
- Examples due to Farrell-Jones (1989) show that the Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.

Definition (Poincaré duality group)

A group is called a *Poincaré duality group of dimension n* if the following conditions holds:

- G is finitely presented;
- G is of type FP;
- We get an isomorphism of abelian groups

$$H^i(G;\mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

Conjecture (Poincaré duality groups)

Let G be a Poincaré duality group. Then there is an aspherical closed homology ANR-*manifold with $G \cong \pi_1(M)$.*

- One may also hope that M in the conjecture above can be taken to be a closed topological manifold.
- This is decided by Quinn's resolutions obstruction, an invariant taking values in $1 + 8 \cdot \mathbb{Z}$.
- There are simply connected closed homology ANR-manifolds with non-trivial resolution obstruction. (see Bryant-Ferry-Mio-Weinberger (1995)).

Conjecture (Vanishing of the resolution obstruction in the aspherical case)

The resolution obstruction for an aspherical closed homology ANR-*manifold is always trivial.*
In particular every aspherical closed homology ANR-*manifold is homotopy equivalent to a closed topological manifold.*

The Farrell-Jones Conjecture

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG.
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.

Conjecture (L-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG;\mathbf{L}_R^{\langle -\infty\rangle}\big) \to L_n^{\langle -\infty\rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$

$$\mathcal{S}^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Browder, Novikov, Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\text{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$ if both the K-theoretic and L-theoretic Farrell-Jones Conjectures hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.

Conjecture (Farrell-Jones Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in an additive G-category $\mathcal{A}$ for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_{\mathcal{A}}) \to H_n^G(pt, \mathbf{K}_{\mathcal{A}}) = K_n(\mathcal{A} * G)$$

is bijective for all $n \in \mathbb{Z}$.

- $E_{\mathcal{VC}\text{yc}}(G)$ is the classifying space of the family of virtually cyclic subgroups.
- $H_*^G(-; \mathbf{K}_{\mathcal{A}})$ is the G-homology theory satisfying for every $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}_{\mathcal{A}}) = K_n(\mathcal{A} * H).$$

Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture for any additive G-category $\mathcal{A}$, then the conjectures mentioned above except for the one about the resolution obstruction follow for G, i.e., the following conjecture are true:

- *Kaplansky Conjecture;*
- *Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$;*
- *Serre's Conjecture;*
- *Vanishing of the Whitehead group;*
- *Moody's Induction Conjecture;*
- *Homotopy invariance of L^2-torsion*
- *Novikov Conjecture.*
- *Borel Conjecture in dimension ≥ 5;*
- *Conjecture about Poincaré duality groups in dimensions $n \neq 3, 4$.*

The status of the Farrell-Jones Conjecture

Theorem (Main Theorem Bartels-Lück(2008))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures hold with coefficients in any additive G-category (with involution) has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If we demand on the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all $\mathrm{CAT}(0)$-groups.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- On examples is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.
- Bartels-Echterhoff-Lück(2007) show that the Bost Conjecture with coefficients in C^*-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0\big(\mathcal{A} \rtimes_{l^1} G\big) \to K_0\big(\mathcal{A} \rtimes_{C_r^*} G\big)$$

is not bijective for all G-C^*-algebras $\mathcal{A}$.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems and have some exotic properties which may cause the failure of the Farrell-Jones Conjecture.

Topological rigidity

Definition (Topological rigid)

A closed topological manifold N is called topologically rigid if every homotopy equivalence $f\colon M \to N$ with a closed topological manifold M as source and N as target is homotopic to a homeomorphism.

- The Poincaré Conjecture is equivalent to the statement that the sphere S^n is topologically rigid.
- The Borel Conjecture is equivalent to the statement that every closed aspherical manifold is topologically rigid.

Theorem (Dimension 3)

Suppose that Thurston's Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental group and the 3-dimensional Poincaré Conjecture are true.
Then every 3-manifold with torsionfree fundamental group is topologically rigid.

- The main input in the proof are Waldhausen's rigidity results for Haken manifolds.
- Conclusion: If $\pi_1(M)$ is torsionfree, then $\pi_1(M)$ determines the homeomorphism type.

Theorem (Connected sums Kreck-Lueck(2006))

Let M and N be manifolds of the same dimension $n \geq 5$ such that neither $\pi_1(M)$ nor $\pi_1(N)$ contains elements of order 2 or that $n = 0, 3 \mod 4$. If both M and N are topologically rigid, then the same is true for their connected sum $M \# N$.

- The proof is based on Cappell's work on splitting obstructions and of UNIL-groups and recent improvements by Banagl, Connolly, Davis, Ranicki.

Theorem (Products of two spheres Kreck-Lueck (2006))

Suppose $k + d \neq 3$. Then the product $S^k \times S^d$ is topologically rigid if and only if both k and d are odd.

Theorem (A surgery construction Kreck-Lueck (2006))

Start with a closed topological manifold M of dimension $n \geq 5$ which is topologically rigid. Choose an embedding $S^1 \times D^{n-1} \to M$ which induces an injection on π_1. Choose a high dimensional knot $K \subseteq S^n$ with complement X such that the inclusion $\partial X \cong S^1 \times S^{n-2} \to X$ induces an isomorphism on π_1. Put

$$N = M - (S^1 \times D^{n-1}) \cup_{S^1 \times S^{n-2}} X.$$

Then N is topologically rigid.

- If M is aspherical, then N is in general not aspherical.

Further applications

Theorem (Product decomposition Lück (2008))

Let M be a closed aspherical manifold of dimension n with fundamental group $G = \pi_1(M)$. Suppose we have a product decomposition

$$G \cong G_1 \times G_2.$$

Suppose that G, G_1 and G_2 satisfy the Farrell-Jones Conjecture.

- *Then G, G_1 and G_2 are Poincaré duality groups whose cohomological dimensions satisfy*

$$n = \mathrm{cd}(G) = \mathrm{cd}(G_1) + \mathrm{cd}(G_2).$$

- If the cohomological dimension $\mathrm{cd}(G_i)$ is different from 3 and 4, then there are topological closed aspherical manifolds M_1 and M_2 with $G_i = \pi_1(M_i)$ and a product decomposition compatible with the one for the fundamental groups

$$M \cong M_1 \times M_2.$$

- This decomposition is unique.

Theorem (High-dimensional spheres as boundary of hyperbolic groups Bartels-Lück-Weinberger(2008))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 5. Then:

- *The following statements are equivalent:*
  - *The boundary ∂G is homeomorphic to S^{n-1};*
  - *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*
- *The following statements are equivalent:*
  - *The boundary ∂G has the integral Čech cohomology of S^{n-1};*
  - *There is a closed aspherical* ANR-*homology manifold M with $G \cong \pi_1(M)$.*

Computational aspects

Theorem (The algebraic K-theory of torsionfree hyperbolic groups)

Let G be a torsionfree hyperbolic group and let R be a ring (with involution). Then we get an isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG);$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG);$$

Theorem (L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

K-theory versus L-theory

- So far the K-theory case has been easier to handle.
- The reason is that at some point a transfer argument comes in. After applying the transfers the element gets controlled on the total space level and then is pushed down to the base space.
- The transfer $p^!$ for a fiber bundle $F\colon E \to B$ has in K-theory the property that $p^! \circ p_*$ is multiplication with the Euler characteristic. In most situations F is contractible and hence obviously $p^! \circ p_*$ is the identity what is needed for the proof.
- In the L-theory case $p^! \circ p_*$ is multiplication with the signature. If the fiber is a sphere, then $p^! \circ p_*$ is zero.
- One needs a construction which makes out of a finite CW-complex with Euler characteristic 1 a finite Poincare complex with signature 1 or a chain complex or module analogue.

- Such a construction is given by the multiplicative hyperbolic form.
- Given a finitely projective R-module P over the commutative ring R, define a symmetric bilinear R-form $H_{\otimes}(P)$ by

$$\left(P \otimes P^*\right) \times \left(P \otimes P^*\right) \to R, \quad (p \otimes \alpha, q \otimes \beta) \mapsto \alpha(q) \cdot \beta(p).$$

  If one replaces $\otimes$ by $\oplus$ and $\cdot$ by $+$, this becomes the standard hyperbolic form.
- The multiplicative hyperbolic form induces a ring homomorphism

$$K_0(R) \to L^0(R), \quad [P] \mapsto [H_{\otimes}(P)].$$

- It is an isomorphism for $R = \mathbb{Z}$.

Comments on the proof

Here are the basic steps of the proof of the main Theorem.

Step 1: Interpret the assembly map as a forget control map. Then the task is to give a way of gaining control.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}$ holds if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- Z-set-condition
  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;
- Long thin coverings
  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists a $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^\beta \times \{x\} \subset U$. Here g^β denotes the β-ball around g in G with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above. This is often done by constructing a certain flow space and use the flow to let a given not yet perfect covering flow into a good one. The construction of the flow space for CAT(0)-space is one of the main ingredients.