On the Conjectures of Bost and of Baum-Connes and the generalized Trace Conjecture

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Outline

- Equivariant homology theory.
- Classifying G-space for proper actions.
- Conjecture due to Bost and to Baum-Connes.
- Inheritance properties under directed colimits.
- Equivariant Chern characters.
- (Generalized) Trace Conjectured.
- Convention: group will always mean discrete group.

Equivariant homology theories

Definition (G-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}^?_*$ assigns to every group G a G-homology theory $\mathcal{H}^G_*$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\mathrm{ind}_\alpha\colon \mathcal{H}^H_n(X, A) \to \mathcal{H}^G_n(\mathrm{ind}_\alpha(X, A))$$

satisfying:

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then ind_α is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a $\mathcal{K}_*$ non-equivariant homology theory, put

$$\begin{array}{rcll} \mathcal{H}^G_*(X) & := & \mathcal{K}_*(X/G); & \\ \mathcal{H}^G_*(X) & := & \mathcal{K}_*(EG \times_G X) & \text{Borel homology}. \end{array}$$

- Equivariant bordism $\Omega^?_*(X)$;
- Equivariant topological K-homology $K^?_*(X)$ in the sense of Kasparov.
  Recall for $H \subseteq G$ finite

$$K^G_n(G/H) \cong K^H_n(\{\bullet\}) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even}; \\ \{0\} & n \text{ odd}. \end{cases}$$

Classifying spaces for proper actions

Definition (Classifying G-space for proper G-actions, tom Dieck(1974))

A model for the *classifying G-space for proper G-actions* is a proper G-CW-complex $\underline{E}G$ such that for any proper G-CW-complex Y there is up to G-homotopy precisely one G-map $Y \to \underline{E}G$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

- *There exists a model for $\underline{E}G$;*
- *Two models for $\underline{E}G$ are G-homotopy equivalent;*
- *A proper G-CW-complex X is a model for $\underline{E}G$ if and only if for each $H \in \mathcal{F}$ the H-fixed point set X^H is contractible.*

- We have $EG = \underline{E}G$ if and only if G is torsionfree.
- We have $\underline{E}G = \{\bullet\}$ if and only if G is finite.
- A model for $\underline{E}D_\infty$ is the real line with the obvious $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$-action.
  Every model for ED_∞ is infinite dimensional, e.g., the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small.
- On the other hand any CW-complex is homotopy equivalent to $G\backslash\underline{E}G$ for some group G (see Leary-Nucinkis (2001)).

Conjectures due to Bost and Baum-Connes

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Bost Conjecture)

The Bost Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) \to K_n(l^1(G))$$

is bijective for all $n \in \mathbb{Z}$.

- These conjecture have versions, where one allows coefficients in a G-C^*algebra A

$$\begin{aligned} K_n^G(\underline{E}G; A) &\to K_n(A \rtimes_{C_r^*} G); \\ K_n^G(\underline{E}G; A) &\to K_n(A \rtimes_{l^1} G). \end{aligned}$$

- There is a natural map

$$\iota\colon K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$$

map.
The composite of the assembly map appearing in the Bost Conjecture with ι is the assembly map appearing in the Baum-Connes Conjecture.

- We will see that the Bost Conjecture has a better chance to be true than the Baum-Connes Conjecture.
- On the other hand the Baum-Connes Conjecture has a higher potential for applications since it is related to index theory and thus has interesting consequences for instance to the Conjectures due to Bass, Gromov-Lawson-Rosenberg, Novikov, Kadison, Kaplansky.
- These conjecture have been proved for interesting classes of groups. Prominent papers have been written for instance by Connes, Gromov, Higson, Kasparov, Lafforgue, Mineyev, Skandalis, Yu, Weinberger and others.

Inheritance properties under colimits

- Let $\psi \colon H \to G$ be a (not necessarily injective) group homomorphism.
  Given G-CW-complex Y, let $\psi^* Y$ be the H-CW-complex obtained from Y by restricting the G-action to a H-action via ψ.
  Given H-CW-complex X, let $\psi_* X$ be the G-CW-complex obtained from Y by induction with ψ, i.e., $\psi_* X = G \times_\psi X$.
- Consider a directed system of groups $\{G_i \mid i \in I\}$ with (not necessarily injective) structure maps $\psi_i \colon G_i \to G$ for $i \in I$. Put $G = \operatorname{colim}_{i \in I} G_i$.
- Let X be a G-CW-complex.

- We have the canonical G-map

$$ad\colon (\psi_i)_*\psi_i^*X = G \times_{G_i} X \to X, \quad (g,x) \mapsto gx.$$

- Define a homomorphism

$$t_n^G(X)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*X) \to \mathcal{H}_n^G(X)$$

by the colimit of the system of maps indexed by $i \in I$

$$\mathcal{H}_n^{G_i}(\psi_i^*X) \xrightarrow{\operatorname{ind}_{\psi_i}} \mathcal{H}_n^G((\psi_i)_*\psi_i^*X) \xrightarrow{\mathcal{H}_n^G(ad)} \mathcal{H}_n^G(X).$$

Definition (Strongly continuous equivariant homology theory)

An equivariant homology theory $\mathcal{H}^?_*$ is called strongly continuous if for every group G and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ the map

$$t_n^G(\{\bullet\})\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

Theorem (Bartels-Echterhoff-Lück (2007))

Consider a directed system of groups $\{G_i \mid i \in I\}$ *with* $G = \operatorname{colim}_{i \in I} G_i$*. Let* X *be a* G*-CW-complex. Suppose that* $\mathcal{H}_*^?$ *is strongly continuous. Then the homomorphism*

$$t_n^G(X) \colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* X) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective for every $n \in \mathbb{Z}$*.*

Idea of proof.

- Show that t_*^G is a transformation of G-homology theories.
- Prove that the strong continuity implies that $t_n^G(G/H)$ is bijective for all $n \in \mathbb{Z}$ and $H \subseteq G$.
- Then a general comparison theorem gives the result.

□

Theorem (Bartels-Echterhoff-Lück (2007))

Let $\{G_i \mid i \in I\}$ *be a directed system of groups with* $G = \operatorname{colim}_{i \in I} G_i$ *and (not necessarily injective) structure maps* $\psi_i \colon G_i \to G$*. Suppose that* $\mathcal{H}^?_*$ *is strongly continuous and for every* $i \in I$ *and subgroup* $H \subseteq G_i$ *the assembly map*

$$H_n^H(\underline{E}H) \to H_n^H(\{\bullet\})$$

is bijective.

Then for every subgroup $K \subseteq G$ *(and in particular for* $K = G$*) also the assembly map*

$$H_n^K(\underline{E}K) \to H_n^K(\{\bullet\})$$

is bijective.

Lemma (Davis-Lück(1998))

There are equivariant homology theories $\mathcal{H}_*^?(-;C_r^*)$ *and* $\mathcal{H}_*^?(-;l^1)$ *defined for all equivariant CW-complexes with the following properties:*

- *If* $H \subseteq G$ *is a (not necessarily finite) subgroup, then*

$$\begin{array}{ccccc} \mathcal{H}_n^G(G/H;C_r^*) & \cong & \mathcal{H}_n^H(\{\bullet\};C_r^*) & \cong & K_n(C_r^*(H)); \\ \mathcal{H}_n^G(G/H;l^1) & \cong & \mathcal{H}_n^H(\{\bullet\};l^1) & \cong & K_n(l^1(H)); \end{array}$$

- $\mathcal{H}_*^?(-,l^1)$ *is strongly continuous;*
- *Both* $\mathcal{H}_*^?(-;C_r^*)$ *and* $\mathcal{H}_*^?(-;l^1)$ *agree for proper equivariant CW-complexes with equivariant topological K-theory* $K_*^?$ *in the sense of Kasparov.*

- One ingredient in the proof of the strong continuity of $\mathcal{H}_*^?(-; l^1)$ is to show

$$\operatorname*{colim}_{i \in I} K_n(l^1(G_i)) \cong K_n(l^1(G)).$$

- This statement does not make sense for the reduced group C^*-algebra since it is not functorial under arbitrary group homomorphisms.
- For instance, $C_r^*(\mathbb{Z} * \mathbb{Z})$ is a simple C^*-algebra and hence no epimorphism $C_r^*(\mathbb{Z} * \mathbb{Z}) \to C_r^*(\{1\})$ exists.
- Hence $\mathcal{H}_*^?(-; C_r^*)$ is not strongly continuous.

Theorem (Inheritance under colimits for the Bost Conjecture, Bartels-Echterhoff-Lück (2007))

Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and (not necessarily injective) structure maps $\psi_i \colon G_i \to G$. Suppose that the Bost Conjecture with C^*-coefficients holds for all groups G_i. Then the Bost Conjecture with C^*-coefficients holds for G.

Theorem (Lafforgue (2002))

The Bost Conjecture holds with C^*-coefficients holds for all hyperbolic groups.

Corollary

Let $\{G_i \mid i \in I\}$ be a directed system of hyperbolic groups with (not necessarily injective structure maps).
Then the Bost Conjecture holds with C^*-coefficients holds for $\operatorname{colim}_{i \in I} G_i$.

- Many recent constructions of groups with exotic properties are given by colimits of directed systems of hyperbolic groups. Examples are:
- groups with expanders;
- Lacunary hyperbolic groups in the sense of Olshanskii-Osin-Sapir;
- Tarski monsters, i.e., groups which are not virtually cyclic and whose proper subgroups are all cyclic;
- Certain infinite torsion groups.

- Certain groups with expanders yield counterexamples to the surjectivity of the assembly map appearing Baum-Connes Conjecture with coefficients by a construction due to Higson-Lafforgue-Skandalis (2002).
- These implies that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is not surjective in general.
- The main critical point concerning the Baum-Connes Conjecture is that the reduced group C^*-algebra of a group lacks certain functorial properties which are present on the left side of the assembly map. This is not true if one deals with $l^1(G)$ or groups rings RG.

- The counterexamples above raised the hope that one may find counterexamples to the conjectures due to Baum-Connes, Borel, Bost, Farrell-Jones, Novikov.
- The results above due to Bartels-Echterhoff-Lück (2007) and unpublished work by Bartels-Lück (2007) prove all these conjectures (with coefficients) except the Baum-Connes Conjecture for colimits of hyperbolic groups.
- There is no counterexample to the Baum-Connes Conjecture (without coefficients) in the literature.

Equivariant Chern character

Theorem (Artin's Theorem)

Let G be finite. Then the map

$$\bigoplus_{C\subset G} \operatorname{ind}_C^G \colon \bigoplus_{C\subset G} R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(G)$$

is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of G.

- Let C be a finite cyclic group.
- The Artin defect is the cokernel of the map

$$\bigoplus_{D\subset C, D\neq C} \operatorname{ind}_D^C \colon \bigoplus_{D\subset C, D\neq C} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C).$$

- For an appropriate idempotent $\theta_C \in R_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$ the Artin defect is after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

Theorem (Lück(2002))

Let X be a proper G-CW-complex. Let $\mathbb{Z} \subseteq \Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained by inverting the orders of all the finite subgroups of G. Then there is a natural isomorphism

$$\mathrm{ch}^G \colon \bigoplus_{(C)} K_n(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n^G(X) \otimes_{\mathbb{Z}} \Lambda^G,$$

where (C) runs through the conjugacy classes of finite cyclic subgroups and $W_G C = N_G C / C \cdot C_G C$.

Example (Improvement of Artin's Theorem)

Consider the special case where G is finite and $X = \{\bullet\}$ Then we get an improvement of Artin's theorem, namely,

$$\bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right] \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right]$$

Example ($X = \underline{E}G$)

In the special case $X = \underline{E}G$ we get an isomorphism

$$\bigoplus_{(C)} K_n(BC_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda^G,$$

Conjecture (Trace Conjecture for G)

The image of the trace map

$$K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{R}$$

is the additive subgroup of $\mathbb{R}$ *generated by* $\{\frac{1}{|H|} \mid H \subset G, |H| < \infty\}$.

Lemma

Let G be torsionfree. Then the Baum-Connes Conjecture for G implies the Trace Conjecture for G.

Proof.

The following diagram commutes because of the L^2-index theorem due to Atiyah(1974).

$$\begin{array}{ccccc}
K_0^G(EG) & \longrightarrow & K_0(C_r^*(G)) & \xrightarrow{\mathrm{tr}} & \mathbb{R} \\
\downarrow \cong & & & & \uparrow \\
K_0(BG) & \longrightarrow & K_0(\{\bullet\}) & \xrightarrow{\cong} & \mathbb{Z}
\end{array}$$

□

Theorem (Roy(1999))

The Trace Conjecture is false in general.

Conjecture (Modified Trace Conjecture)

Let $\Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting the orders of finite subgroups of G. Then the image of the trace map

$$K_0(C_r^*(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.

Theorem (Image of the trace Lueck(2002))

The image of the composite

$$K_0^G(\underline{E}G) \xrightarrow{\text{asmb}} K_0(C_r^*(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.
In particular the Baum-Connes Conjecture implies the Modified Trace Conjecture.

- Problem: What is the image of the trace map in terms of G?
- Take $X = \underline{E}G$. Elements in $K_*(\underline{E}G)$ are given by elliptic G-operators P over cocompact proper G-manifolds with Riemannian metrics.
- Problem: What is the concrete preimage of its class under ch_*^G?
- One term could be the index of P^C on M^C giving an element in $K_0(C_GC \backslash \underline{E}^C)$ which is $K_0(BC_GC)$ after tensoring with Λ^G.
- Another term could come from the normal data of M^C in M which yields an element in $\theta_C \cdot R_{\mathbb{C}}(C)$.
- The failure of the Trace Conjecture shows that this is more complicated than one anticipates. The answer to the question above would lead to a kind of orbifold L^2-index theorem whose possible denominators, however, are not of the expected shape $\frac{n}{|H|}$ for $H \subseteq G$ finite.