The Farrell-Jones Conjecture and its applications

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Outline and goal

- Explain the K-theoretic and L-theoretic Farrell-Jones Conjecture and its potential.
- Discuss applications of these conjectures.
- State our main theorem which is joint work with Bartels and Reich.
- Link the Farrell-Jones Conjecture to the Baum-Connes Conjecture.
- Make a few comments about the proof.

Algebraic K-theory

Conjecture

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.

- In order to illustrate the depth of the Farrell-Jones Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the K-theoretic Farrell-Jones Conjecture for the coefficient ring R.

Lemma

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ *for* $n \leq -1$*;*
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*

Lemma

Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\operatorname{Wh}(G)$ *is trivial.*

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$\begin{aligned} 0 \to H_0(BG; \mathbf{K}_\mathbb{Z}) = \{\pm 1\} \to H_1(BG; \mathbf{K}_\mathbb{Z}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G, G] \to 1. \end{aligned}$$

  This implies

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0.$$

In particular we get for a torsionfree group $G \in \mathcal{FJ}(\mathbb{Z})$

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism $W = (W; M_0, M_1)$ of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial.
- If G belongs to $\mathcal{FJ}(\mathbb{Z})$, then it is of type FF if and only if it is of type FP.

Conjecture

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree.*
- *G is torsionfree and sofic, e.g., residually amenable.*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Conjecture

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $E_{\mathcal{VC}\text{yc}}(G)$ is the classifying space of the family of virtually cyclic subgroups;
- $H_*^G(-; \mathbf{K}_R)$ is the G-homology theory satisfying for every $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}_R) = K_n(RH).$$

- We think of it as an advanced induction theorem (such as Artin's or Brower's induction theorem for representations of finite groups).

Theorem

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}(R)$. Then the map given by induction from finite subgroups of G*

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Suppose that $G \in \mathcal{FJ}(F)$. Then the map*

$$\operatorname{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

Conjecture

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- If G is finite, this is just the Theorem of Swan.
- Another version of it would predict for the quotient field F of R that

$$K_0(RG) \to K_0(FG)$$

factorizes as

$$K_0(RG) \to K_0(R) \to K_0(F) \to K_0(FG).$$

Theorem (Linnell-Farrell)

Let G be a group. Suppose that

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic. (This is true if $G \in \mathcal{FJ}(F)$ for every field F of prime characteristic).
Then the Bass Conjecture is satisfied for every integral domain R.

Conjecture

Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups G and all $n \in \mathbb{Z}$ that

$$NK_n(RG) = 0$$

and that the canonical map from algebraic to homotopy K-theory

$$K_n(RG) \to KH_n(RG)$$

is bijective.

Theorem

Let R be a regular ring with $\mathbb{Q} \subseteq R$. If $G \in \mathcal{FJ}(R)$, then the conjecture above is true.

Conjecture

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

- The L^2-torsion of closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of volume to hyperbolic groups whose L^2-Betti numbers all vanish.

Theorem

Suppose that $G \in \mathcal{FJ}(\mathbb{Z})$. Then G satisfies the Conjecture above.

- Deninger can define a p-adic Fuglede-Kadison determinant for a group G and relate it to p-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
- The surjectivity of the map

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

  plays a role (33 %) in a program to prove the Conjecture emanating from a question of Atiyah that for a closed Riemannian manifold with torsionfree fundamental group the L^2-Betti numbers of its universal covering are all integers.

- Let $\mathcal{FJ}_K(R)$ be the class of groups which satisfy the (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in R.

Theorem (Bartels-L.-Reich (2007))

- *Every hyperbolic group and every virtually nilpotent group belongs to* $\mathcal{FJ}(R)$*;*
- *If* G_1 *and* G_2 *belong to* $\mathcal{FJ}(R)$*, then* $G_1 \times G_2$ *belongs to* $\mathcal{FJ}(R)$*;*
- *Let* $\{G_i \mid i \in I\}$ *be a directed system of groups (with not necessarily injective structure maps) such that* $G_i \in \mathcal{FJ}(R)$ *for* $i \in I$*. Then* $\operatorname{colim}_{i \in I} G_i$ *belongs to* $\mathcal{FJ}(R)$*;*
- *If* H *is a subgroup of* G *and* $G \in \mathcal{FJ}(R)$*, then* $H \in \mathcal{FJ}(R)$*.*

- We emphasize that this result holds for all rings R.
- The groups above are certainly wild in Bridson's universe.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, belong to $\mathcal{FJ}_K(R)$ for all R.
- If G is a torsionfree hyperbolic group and R any ring, then we get an isomorphism

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG).$$

- Bartels and L. have a program to prove $G \in \mathcal{FJ}_K(R)$ if G acts properly and cocompact on a CAT(0)-space.
- This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

Algebraic L-theory

Conjecture

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$.
- Let $\mathcal{FJ}_L(R)$ be the class of groups which satisfy the L-theoretic Farrell-Jones Conjecture for the coefficient ring R.

Conjecture

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

Conjecture

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The L-theoretic Farrell-Jones Conjecture for a group G in the case $R = \mathbb{Z}$ implies the Novikov Conjecture in dimension ≥ 5.
- If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.
- As in the case of algebraic K-theory there is also an analogous version of the L-theoretic Farrell-Jones Conjecture for arbitrary groups G.
- Bartels and L. have a program to extend our result for the K-theoretic Farrell-Jones Conjecture also to the L-theoretic version.
- Bartels and L. have a program to prove $G \in \mathcal{FJ}_L(R)$ if G acts properly and cocompact on a CAT(0)-space. This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element;

Theorem (Ranicki)

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\text{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$.

Conjecture

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G;
- There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G;\mathbb{R})).$$

- There is also a version for arbitrary groups

$$K_n^G(E_{\mathcal{F}\text{in}}(G)) \to K_n(C_r^*(G)).$$

- The Bost Conjecture is the analogue for $l^1(G)$, i.e., it concerns the assembly map.

$$K_n^G(E_{\mathcal{F}\text{in}}(G)) \to K_n(l^1(G)).$$

  Its composition with the canonical map $K_n(l^1(G)) \to K_n(C_r^*(G))$ is the Baum-Connes assembly map.
  Both Conjectures have versions, where coefficients in a G-C^*-algebra are allowed.

- Next we discuss some relations relations between these conjectures.

Theorem (Bartels-L.-Echterhoff (2007))

Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of hyperbolic groups G_i (with not necessarily injective structure maps).
Then G satisfies the Bost Conjecture with coefficients.

- The proof uses the deep result of Lafforgue that the Bost Conjecture with coefficients is true for every hyperbolic group.
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis, do satisfy the Bost Conjecture with coefficients.
  So the failure of the Baum-Connes Conjecture with coefficients says that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$ is not bijective.
  The underlying problem with the Baum-Connes Conjecture is the lack of functoriality of the reduced group C^*-algebra.

Theorem (L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

$$
\begin{array}{ccc}
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{\mathbb{Z}})[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{Z}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{\mathbb{R}})[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{R}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{L}^p_{C_r^*(?;\mathbb{R})})[1/2] & \xrightarrow{\cong} & L_n^p(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{R}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{C}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G))[1/2]
\end{array}
$$

- Let M be a closed manifold with $G = \pi_1(M)$

$$\begin{array}{ccccc} H_n(M;\mathbf{L}_{\mathbb{Z}})[1/2] & \xrightarrow{(f_M)_*} & H_n(BG;\mathbf{L}_{\mathbb{Z}})[1/2] & \longrightarrow & L_n(\mathbb{Z}G)[1/2] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ K_n(M)[1/2] & \xrightarrow{(f_M)_*} & K_n(BG)[1/2] & \longrightarrow & K_n(C_r^*(G;\mathbb{R}))[1/2] \end{array}$$

- Hence the surgery sequence

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

has after inverting 2 an interpretation in terms of C^*-algebras provided the L-theoretic Farrell-Jones Conjecture and the Baum-Connes Conjecture hold. (see Higson-Roe)

Comments on the proof

Here are the basic steps of the proof of the main Theorem.

Step 1: Interprete the assembly map as a forget control map.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}(R)$ holds for all rings R if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- *Z*-set-condition
  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;
- Long thin covers
  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^{\beta} \times \{x\} \subset U$. Here g^{β} denotes the β-ball around g in G with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above.