The Novikov Conjecture
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Introduction

Manifolds are the central geometric objects in modern mathematics. An attempt to understand the nature of manifolds leads to many interesting questions. One of the most obvious questions is the following.

Let M and N be manifolds: how can we decide whether M and N are homotopy equivalent or homeomorphic or diffeomorphic (if the manifolds are smooth)?

The prototype of a beautiful answer is given by the Poincaré Conjecture. If N is S^n, the n-dimensional sphere, and M is an arbitrary closed manifold, then it is easy to decide whether M is homotopy equivalent to S^n. This is the case if and only if M is simply connected (assuming $n > 1$, the case $n = 1$ is trivial since every closed connected 1-dimensional manifold is diffeomorphic to S^1) and has the homology of S^n. The *Poincaré Conjecture* states that this is also sufficient for the existence of a homeomorphism from M to S^n. For $n = 2$ this follows from the well-known classification of surfaces. For $n > 4$ this was proved by Smale and Newman in the sixties of the last century, Freedman solved the case in $n = 4$ in 1982 and recently Perelman announced a proof for $n = 3$, but this proof has still to be checked thoroughly by the experts. In the smooth category it is not true that manifolds homotopy equivalent to S^n are diffeomorphic. The first examples were published by Milnor in 1956 and together with Kervaire he analyzed the situation systematically in the sixties.

For spheres one only needs very little information to determine the homeomorphism type: the vanishing of the fundamental group and control of the homology groups. Another natural class of manifolds is given by aspherical manifolds. A CW-complex is called *aspherical* if the homotopy groups vanish in dimension > 1, or, equivalently, if its universal covering is contractible. The *Borel Conjecture*, which is closely related to the Novikov Conjecture, implies that the fundamental group determines the homeomorphism type of an aspherical closed manifold.

For more general manifolds with prescribed fundamental group the classification is in general unknown even if the fundamental group is trivial. In this situation it is natural to construct as many invariants as possible hoping that at least for certain particularly important classes of manifolds one can classify them in terms of theses invariants. The most important invariants after homotopy and (co)homology groups are certainly characteristic classes which were defined and systematically treated in the fifties. There are two types of characteristic classes for smooth manifolds: the *Stiefel-Whitney classes* $w_k(M)$ in $H^k(M;\mathbb{Z}/2)$ and the *Pontrjagin classes* $p_k(M) \in H^{4k}(M;\mathbb{Z})$. The nature of these classes is rather different. The Stiefel-Whitney classes of a closed manifold can be expressed in terms of cohomology operations and so are homotopy invariants, the Pontrjagin classes are diffeomorphism invariants (for smooth manifolds, and only for those they are a priori defined), but not homeomorphism or even homotopy invariants in general. Only very special linear combinations of the Pontrjagin classes are actually homotopy invariants.

For example, the first Pontrjagin class of a closed oriented 4-manifold $p_1(M)$ is a homotopy invariant. The reason is that $\langle p_1(M), [M]\rangle \;=\; 3\cdot \operatorname{sign}(M)$, where $\operatorname{sign}(M)$ is the *signature* of the intersection form on $H^2(M;\mathbb{Q})$. The signature is by construction a homotopy invariant. More generally, Hirzebruch has defined a certain rational polynomial in the Pontrjagin classes (for a definition of Pontrjagin classes see [171]) , the *L-class*

$$\mathcal{L}(M) \;=\; \mathcal{L}(p_1(M), p_2(M), \ldots) \;\in\; \bigoplus_{i\geq 0} H^{4i}(M;\mathbb{Q}).$$

Its i-th component is denoted by

$$\mathcal{L}_i(M) \;=\; \mathcal{L}_i(p_1(M), p_2(M), \ldots, p_i(M)) \in H^{4i}(M;\mathbb{Q}).$$

The famous *Signature Theorem* of Hirzebruch says that the evaluation of $\mathcal{L}_k(M)$ on the fundamental class [M] gives the signature of a $4k$-dimensional manifold M:

$$\operatorname{sign}(M) \;=\; \langle \mathcal{L}_k(p_1(M), \ldots, p_k(M)), [M]\rangle.$$

One can show that a polynomial in the Pontrjagin classes gives a homotopy invariant if and only if it is a multiple of the k-th L-class.

This sheds light on the homotopy properties of the polynomial $\mathcal{L}_k(M)$ of a $4k$-dimensional manifold M. But what can one say about the other polynomials $\mathcal{L}_1(M), \mathcal{L}_2(M), \mathcal{L}_3(M), \ldots$? Understanding $\mathcal{L}_i(M)$ is — by Poincaré duality — equivalent to understanding the numerical invariants

$$\langle x \cup \mathcal{L}_i(M), [M]\rangle \;\in\; \mathbb{Q} \tag{0.1}$$

for all $x \in H^{n-4i}(M)$, where $n = \dim(M)$. One may ask whether these numerical invariants are homotopy invariant in the following sense: If $g\colon N \to M$ is an orientation preserving homotopy equivalence, then

$$\langle x \cup \mathcal{L}_i(M), [M]\rangle \;\;=\;\; \langle g^*(x) \cup \mathcal{L}_i(N), [N]\rangle. \tag{0.2}$$

In general, these numerical invariants are not homotopy invariants. The Signature Theorem implies that the expression 0.1 is homotopy invariant for all $x \in H^0(M;\mathbb{Q})$. Novikov proved the remarkable result in the sixties that for $\dim(M) = 4k+1$ and $x \in H^1(M)$ the expression 0.1 is homotopy invariant. This motivated Novikov to state the following conjecture.

Let G be a group. Denote by BG its *classifying space* which is up to homotopy uniquely determined by the property that it is an aspherical CW-complex with G as fundamental group. Novikov conjectured that the numerical expression

$$\langle f^*(x) \cup \mathcal{L}_i(M), [M] \rangle \in \mathbb{Q} \tag{0.3}$$

is *homotopy invariant* for every map $f\colon M \to BG$ from a closed oriented n-dimensional manifold M to BG and every class $x \in H^{n-4i}(M;\mathbb{Q})$. More precisely, the famous *Novikov Conjecture* says that if $f'\colon M' \to K$ is another map and $g\colon M \to M'$ is an orientation preserving homotopy equivalence such that $f' \circ g$ is homotopic to f, then

$$\langle f^*(x) \cup \mathcal{L}_i(M), [M] \rangle = \langle (f')^*(x) \cup \mathcal{L}_i(M'), [M']) \rangle.$$

Notice that Novikov's result that 0.2 holds in the case $\dim(M) = 4k+1$ and $x \in H^1(M)$ is a special case of the Novikov Conjecture above since S^1 is a model for $B\mathbb{Z}$ and a cohomology class $x \in H^1(M)$ is the same as a homotopy class of maps $f\colon M \to S^1$, the correspondence is given by associating to the homotopy class of $f\colon M \to S^1$ the pullback $f^*(x)$, where x is a generator of $H^1(S^1)$.

Looking at this conjecture in a naive way one does not see a philosophical reason why it should be true. Even in the case of the polynomial $\mathcal{L}_k$, where $4k$ is the dimension of a manifold, the proof cannot be understood without the signature theorem translating the L-class to a cohomological invariant, the signature. In this situation it is natural to ask for other homotopy invariants (instead of the signature) hoping that one can interpret the expressions 0.3 occurring in the Novikov Conjecture in terms of these invariants. These expressions 0.3 are called *higher signatures*. One can actually express them as signature of certain submanifolds. But this point of view does not give homotopy invariants.

It is natural to collect all higher signatures and form from them a single invariant. This can be done, namely, one considers

$$\operatorname{sign}^G(M,f) := f_*(\mathcal{L}(M) \cap [M]) \in \bigoplus_{i \in \mathbb{Z}, i \geq 0} H_{m-4i}(BG;\mathbb{Q}),$$

the image of the Poincaré dual of the L-class under the map induced from f. An approach to proving the Novikov Conjecture could be to construct a homomorphism

$$A^G\colon \bigoplus_{i \in \mathbb{Z}, i \geq 0} H_{m-4i}(BG;\mathbb{Q}) \to L(G)$$

where $L(G)$ is some Abelian group, such that $A^G(\operatorname{sign}_G(M))$ is a homotopy invariant. Then the Novikov Conjecture would follow if the map A^G is injective. Such maps will be given by so called *assembly maps*.

The construction of such a map is rather complicated. A large part of these lecture notes treats the background needed to construct such a map. In particular, one needs the full machinery of surgery theory. We will give an introduction to this important theory. Roughly speaking, surgery deals with the following problem. Let W be a compact m-dimensional manifold whose boundary is either empty or consists of two components M_0 and M_1 and $f\colon W \to X$ a map to a finite CW-complex. If the boundary of W is not empty, we assume that f restricted to M_0 and M_1 is a homotopy equivalence. Then X is a so called *Poincaré complex*, something we also require if the boundary of W is empty. The question is whether we can replace W and f by W' and f' (bordant to (W, f)) such that f' is a homotopy equivalence. If the boundary of W is not empty, then W' is an *h-cobordism* between M_0 and M_1. In general it is not possible to replace (W, f) by (W', f') with f' a homotopy equivalence. Wall has defined abelian groups $L^h_m(\pi_1(X))$ and an obstruction $\theta(W, f) \in L^h_m(\pi_1(X))$ whose vanishing is a necessary and sufficient condition for replacing (W, f) by (W', f') with f' a homotopy equivalence, if $m > 4$. One actually needs some more control, namely a so-called normal structure on W. All this is explained in Chapters 2, 10 - 14 and Chapter 17.

Why is it so interesting to obtain an h-cobordism? If X is simply-connected, and the dimension of W is greater than five, the celebrated *h-cobordism theorem* of Smale says that an h-cobordism W is diffeomorphic to the cylinder over M_0. In particular, M_0 and M_1 are diffeomorphic. There is a corresponding result for topological manifolds. In the situation which is relevant for the Novikov Conjecture, X is not simply-connected and then the h-cobordism theorem does not hold. There is an obstruction, the *Whitehead torsion*, sitting in the *Whitehead group* which is closely related to the algebraic K_1-group. If the dimension of the h-cobordism W is larger than five, then the vanishing of this obstruction is necessary and sufficient for W to be diffeomorphic to the cylinder. This is called the *s-cobordism theorem*. The Whitehead group, the obstruction and the idea of the proof of the s-cobordism theorem are treated in Chapters 5 - 8.

In Chapters 15 - 16 we define the assembly map and apply it to prove the Novikov Conjecture for finitely-generated free Abelian Groups.

What we have presented so far summarizes and explains information which was known around 1970. To get a feeling for how useful the Novikov Conjecture is, we apply it to some classification problems in low dimensions (see Chapter 0).

In the rest of the lecture notes we present some of the most important concepts and results concerning the Novikov Conjecture and other closely related conjectures dating from after 1970. This starts with an introduction to spectra (see Chapter 18) and continues with classifying spaces of families, a generalization of aspherical spaces (see Chapter 19). With this we have prepared a frame in which not only the Novikov Conjecture but other similar and very important conjectures can be formulated: the *Farrell-Jones* and the *Baum-Connes Conjectures*. After introducing equivariant homology theories in Chapter 20, these conjectures and their relation to the Novikov Conjecture are discussed in Chapters 21 - 23. Finally, these lecture notes are finished by Chapter 24 called "Miscellaneous" in which the status of the conjectures is summarized and methods and proofs are presented.

It is interesting to speculate whether the Novikov Conjecture holds for all groups. No counterexamples are known to the authors. An interesting article expressing doubts was published by Gromov [102].

We have added a collection of exercises and hints for their solutions.

From the amount of material presented in these Lecture Notes it is obvious, that we cannot present all of the details. We have tried to explain those things which are realistic for the very young participants of the seminar to master and we have only said a few words (if anything at all) at other places. People who want to understand the details of this fascinating theory will have to consult other books and often the original literature. We hope that they will find our Lecture Notes useful, since we explain some of the central ideas and give a guide for learning the beautiful mathematics related to the Novikov Conjecture and other closely related conjectures and results.

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