“L^2-torsion and 3-manifolds”
by
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Abstract: We introduce for a finite CW-complex whose L^2-Betti numbers are all trivial and whose Novikov-Shubin invariants are all positive a positive real number called *combinatorial L^2-torsion.* It behaves like a “multiplicative Euler characteristic”. Tools for the computations of L^2-Betti numbers, Novikov-Shubin invariants, Fuglede-Kadison determinant and combinatorial L^2-torsion are given. For example combinatorial L^2-torsion can be computed for an irreducible Haken 3-manifold from a presentation of the fundamental group without using further topological information. There are the conjectures that combinatorial L^2-torsion agrees with analytic L^2-torsion for closed manifolds and with Gromov's simplicial volume up to a constant for prime 3-manifolds whose boundary is empty or a disjoint union of incompressible tori.

0. Introduction

In this article we assign to a pair of finite CW-complexes (X, A) a positive real number called *combinatorial L^2-torsion* provided that (X, A) is *admissible,* i.e., all its L^2-Betti numbers vanish and all its Novikov-Shubin invariants are positive. Examples are compact irreducible Haken 3-manifolds whose boundary is empty or a disjoint union of incompressible tori, compact manifolds with fixed point free S^1-action such that the inclusion of an orbit induces an injection on the fundamental group and closed odd-dimensional hyperbolic manifolds. The combinatorial L^2-torsion behaves like a “multiplicative Euler characteristic”, it is a simple homotopy invariant, satisfies sum, fibration and product formulas and is multiplicative under finite coverings.

One motivation for the study of this invariant are the following conjectures. Combinatorial L^2-torsion is designed to be the topological counterpart of analytic L^2-torsion defined by Lott [21] and Matthai citeMathai (1991) and there is the conjecture that combinatorial and analytic L^2-torsion agree for a closed manifold. In particular this would imply that the combinatorial L^2-torsion of a closed odd-dimensional hyperbolic manifold is its volume up to a dimension constant. For a closed orientable aspherical manifold we conjecture that the combinatorial L^2-torsion is one if Gromov's simplicial volume vanishes. Moreover, we conjecture that the logarithm of the combinatorial L^2-torsion agrees with Gromov's simplicial volume up to a constant for prime 3-manifolds whose boundary is empty or a disjoint union of incompressible tori.

One of the main items of this article is to give tools for the computation of combinatorial L^2-torsion. Computations of analytic L^2-torsion or simplicial volume are very hard in general whereas combinatorial L^2-torsion much easier to handle.

Combinatorial L^2-torsion is a generalization of the classical notion of Reidemeister torsion and is a special case of Reidemeister-von Neumann torsion introduced in Lück-Rothenberg [26] In Section 1 we give a quick definition of combinatorial L^2-torsion. We list its main properties. Roughly speaking, it behaves like a "multiplicative Euler characteristic".

In Section 2 we consider a compact connected orientable irreducible Haken 3-manifold M whose boundary is empty or a disjoint union of incompressible tori. We show that its combinatorial L^2-torsion is the product of the invariants of the hyperbolic pieces of finite volume in the JSJT-decomposition, i.e., in the decomposition into Seifert pieces and hyperbolic pieces along incompressible tori. We explain the conjecture that the logarithm of the combinatorial L^2-torsion of the hyperbolic pieces of finite volume is just the volume up to a constant. This would imply that the combinatorial L^2-torsion measures the size of the hyperbolic pieces of finite volume in the JSJT-decomposition and is Gromov's simplicial volume up to a constant. We show how to compute the combinatorial torsion from a presentation of the fundamental group of M without using further information on M.

In Section 3 we give a survey about the combinatorial L^2-torsion and its relations to Reidemeister von Neumann torsion, analytic L^2-torsion and simplicial volume. We explain L^2-Betti number and Novikov-Shubin invariant and the role of the condition admissible.

In Section 4 we study L^2-invariants like L^2-Betti number, Novikov-Shubin invariant and Fuglede-Kadison determinant of bounded π-equivariant operators $\oplus_{i=1}^n l^2(\pi) \longrightarrow \oplus_{j=1}^m l^2(\pi)$. We define them from the operator point of view. These definitions are suitable for theoretical considerations, but hard to deal with for concrete calculations. The point is that they require the knowledge of the spectral density function of the operator i n question. It measures the distribution of the spectrum over the non-negative numbers and is very hard to determine. However, the situation simplifies if the operator comes from a matrix A over the complex group ring $\mathbb{C}\pi$ of a group π. This is the case in the definition of combinatorial L^2-torsion where the operator comes from the differentials in the cellular chain complex of the universal covering. We assign to A and any positive real number K which is greater or equal to the operator norm a monotone decreasing sequence of non-negative real numbers $c(A,K)_p$, called *characteristic sequence.* It is given by the formula

$$c(A,K)_p = \mathrm{tr}_{\mathbb{C}\pi}\left(\left(1 - K^{-2} \cdot AA^*\right)^p\right)$$

where the $\mathbb{C}\pi$-trace $\mathrm{tr}_{\mathbb{C}\pi}$ of an element $\sum_{g\in\pi} \lambda_g \cdot g$ is defined to be λ_e for $e \in \pi$ the unit element. Notice that these traces can be computed by a computer program if one has an algorithm to solve the word problem in π. If π is the fundamental group of an appropriate manifold like a 3-manifold or a hyperbolic manifold, this can be done. Let $\alpha(A)$ be the Novikov-Shubin invariant of A. In the case where A comes from the cellular chain complex of the universal covering of an appropriate manifold M it can be computed from the topology of M and is a homotopy invariant of M. Let $b(A)$ and $\det(A)$ be the L^2-Betti number and the Fuglede-Kadison determinant of the operator given by A. Then there is for any real number $0 < \alpha < \alpha(A)$ a constant C such that we have for all positive integers L:

$$0 \leq c(A,K)_L - b(A) \leq \frac{C}{L^\alpha}$$

and

$$0 \leq -2 \cdot \ln(\det(A)) + 2 \cdot (n - b(A)) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{p} \cdot (c(A,K)_p - b(A)) \leq \frac{C}{L^\alpha}$$

In particular $b(A)$ and $\det(A)$ can be computed as limits of monotone decreasing sequences of real numbers and the speed of convergence can e read off from the Novikov-Shubin invariants. At any rate one gets estimates from above for the L^2-Betti numbers and the Fuglede-Kadison determinant if one can compute the first L elements of the characteristic sequence.

The paper is organized as follows:

1. L^2-torsion
2. 3-manifolds
3. Relation to other L^2-torsion invariants and Gromov's simplicial volume
4. L^2-invariants for operators

References

The first two sections are of topological nature and the third one is a kind of survey on L^2-invariants. Section 4 contains the operator theoretic part and is independent of the other sections.

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1. L^2-torsion

We want to assign to a pair of finite CW-complexes (X,A) a positive real number called *(combinatorial) L^2-torsion*:

$$\rho(X,A) \in \mathbb{R}^{>0}$$

provided that (X,A) is admissible, i.e., all its L^2-Betti numbers vanish and all its Novikov-Shubin invariants are positive. We give examples of admissible pairs and list the basic properties of the L^2-torsion which behaves like a "multiplicative Euler characteristic".

In order to define the invariant we need the notion of L^2-Betti number and Fuglede-Kadison determinant. The following definitions are not the original ones and will be identified with them later in Theorem 4.4. Define the $\mathbb{C}\pi$-*trace* of an element $\sum_{g\in\pi}\lambda_g\cdot g\in\mathbb{C}\pi$ to be the complex number:

$$\operatorname{tr}_{\mathbb{C}\pi}(\sum_{g\in\pi}\lambda_g\cdot g)=\lambda_e$$

where e is the unit element in π. Let $B=(b_{i,j})$ be a (n,m)-matrix with entries in $\mathbb{C}\pi$. If $n=m$, define its $\mathbb{C}\pi$-trace by:

$$\operatorname{tr}_{\mathbb{C}\pi}(B)=\sum_{i=1}^{n}\operatorname{tr}_{\mathbb{C}\pi}(b_{i,i}).$$

Let the adjoint B^* be given by $(\overline{b_{j,i}})$ where $\overline{\sum_{g\in\pi}\lambda_g\cdot g}$ is defined to be $\sum_{g\in\pi}\overline{\lambda_g}\cdot g^{-1}$. The operator norm of the bounded π-equivariant operator $R_B:\oplus_{i=1}^{n}l^2(\pi)\longrightarrow\oplus_{i=1}^{m}l^2(\pi)$ induced by right multiplication with B is denoted by $\parallel R_B\parallel_\infty$. In the sequel K is any positive number satisfying $K\geq\parallel R_B\parallel_\infty$. For $u=\sum_{w\in\pi}\lambda_w\cdot w\in\mathbb{C}\pi$ define $\parallel u\parallel_1$ by $\sum_{w\in\pi}|\lambda_w|$. Then a possible choice for K is given by:

$$K=m\cdot\sum_{j=1}^{n}\max\left\{\parallel b_{i,j}\parallel_1|\ 1\leq i\leq m\right\}.$$

The sequence $\operatorname{tr}_{\mathbb{C}\pi}\left((1-K^{-2}\cdot BB^*)^p\right)$ is a monotone decreasing sequence of non-negative real numbers and we define the L^2-*Betti number* of B to be the non-negative real number:

$$b(B)=\lim_{p\to\infty}\operatorname{tr}_{\mathbb{C}\pi}\left((1-K^{-2}\cdot BB^*)^p\right).$$

Define the *Fuglede-Kadison determinant* of B to be the positive real number:

$$det(B)=K^{(n-b(B))}\cdot\exp\left(-\frac{1}{2}\cdot\sum_{p=1}^{\infty}\frac{1}{p}\cdot\left(\operatorname{tr}_{\mathbb{C}\pi}\left((1-K^{-2}\cdot BB^*)^p\right)-b(B)\right)\right)$$

if the infinite sum of non-negative real numbers $\sum_{p=1}^{\infty}\frac{1}{p}\cdot\left(\operatorname{tr}_{\mathbb{C}\pi}\left((1-K^{-2}\cdot BB^*)^p\right)-b(B)\right)$ converges to a real number and to be zero otherwise. The L^2-Betti number $b(B)$ and the Fuglede-Kadison determinant $\det(B)$ These invariants are infinite-dimensional generalizations of classical notions. Namely, suppose that π is finite. Then $|\pi|\cdot b(B)$ is the complex dimension of the kernel of the $\mathbb{C}$-linear map $R_B:\oplus_{i=1}^{n}\mathbb{C}\pi\longrightarrow\oplus_{j=1}^{m}\mathbb{C}\pi$ and $\det(B)^{2\cdot|\pi|}$ is the ordinary determinant of BB^* provided $b(B)$ is zero.

In order to be able to define the invariant and to ensure that it is an invariant of the simple homotopy type we need the following condition:

Definition 1.1 *An* admissible pair (X,A) *is a pair of finite CW-complexes* (X,A) *such that for each path component* C *of* X *all* L^2*-Betti numbers* $b_p(C,C\cap A)$ *vanish and all Novikov-Shubin invariants* $\alpha_p(C,C\cap A)$ *are positive.* ■

We will explain L^2-Betti numbers and Novikov-Shubin invariants and discuss the condition being admissible later. At the moment it suffices to know the following examples (see [11], [21, Proposition 4.6], [22, Theorem 4.8, Theorem 7.1]):

Theorem 1.2 *A compact connected orientable manifold M is admissible if it satisfies one of the following conditions:*

1.) The dimension of M is three and M satisfies:

(a) $\pi_1(M)$ is infinite.

(b) M is homotopy equivalent to an irreducible 3-manifold or $S^1 \times S^2$ or $\mathbf{RP}^3 \sharp \mathbf{RP}^3$.

(c) If the boundary of M is non-empty, it consists of tori.

(d) If the boundary of M is empty, M is finitely covered by a 3-manifold which is homotopy equivalent to a hyperbolic, Seifert or Haken 3-manifold.

2.) There is a fixed point free S^1-action on M such that for one (and hence all) orbits S^1/H in M the inclusion induces an injection on the fundamental groups.

3.) M is a closed hyperbolic manifold of odd-dimension. ■

Notice that the conditions a.) to c.) in item 1.) are necessary for a connected compact orientable 3-manifold M to have vanishing L^2-Betti numbers [22, Corollary 7.7] and the condition d.) would always be true if Waldhausen's conjecture or Thurston's geometrization conjecture holds. Moreover, there is the conjecture that the Novikov-Shubin invariants of all connected compact manifolds are positive rational numbers [22, Conjecture 9.1].

Let (X, A) be an admissible pair. Assume that X is connected. Denote the universal covering of X by $p : \widetilde{X} \longrightarrow X$ and define $\widetilde{A}$ to be $p^{-1}(A)$. Let $C(\widetilde{X}, \widetilde{A})$ be the celullar $\mathbb{Z}\pi$-chain complex where π is the fundamental group of X. Let c_p be the p-th differential of $C(\widetilde{X}, \widetilde{A})$. We define for each p the *combinatorial $\mathbb{Z}\pi$-Laplace operator*

$$\Delta_p : C_p(\widetilde{X}, \widetilde{A}) \longrightarrow C_p(\widetilde{X}, \widetilde{A})$$

by $\Delta_p = c_{p+1} \circ c_{p+1}^* + c_p^* \circ c_p$. We mention already here that we will show in Theorem 4.4 that the p-th L^2-Betti number of (X, A) can be computed by:

$$b_p(X, A) = b(\Delta_p)$$

and the Novikov-Shubin invariants $\alpha_p(X, A)$ of (X, A) satisfy:

$$2 \cdot \min\{\alpha_{p-1}(X, A), \alpha_p(X, A)\} = \alpha(\Delta_p) \leq$$

$$\sup \left\{ \beta \in \mathbb{R}^{\geq 0} \,\middle|\, \lim_{p\to\infty} p^\beta \cdot \left(\mathrm{tr}_{\mathbb{Z}\pi}\left(\left(1 - K^{-2}\cdot\Delta_p^*\Delta_p\right)^p\right) - b(\Delta_p)\right) \ = 0 \right\}$$

and we conjecture that the inequality above is an equality. Hence the condition admissible ensures that $b(\Delta_p) = 0$ and the supremum above is positive for all p. We will prove in Theorem 4.4.5 for admissible (X, A) that $det(\Delta_p)$ is a positive real number for all p. Hence we can define:

Definition 1.3 *The* combinatorial L^2-torsion *of an admissible pair* (X, A) *is defined to be the positive real number*

$$\rho(X, A) = \prod_{n=0}^{\infty} \det(\Delta_p)^{(-1)^{(n+1)}\cdot n}. \qquad \blacksquare$$

For non-connected X define $\rho(X, A)$ by the product of the L^2-torsions $\rho(C, C \cap A)$ where C runs over the path components of X.

One may think of combinatorial L^2-torsion as a generalization of classical Reidemeister torsion. In the classical case one has to pick a finite-dimensional unitary representation. Here we use the regular representation $l^2(\pi)$. In the classical case the homology with coefficients in this representation has to be trivial. The condition admissible resembles this assumption.

Next we state the basic properties of this invariant. It behaves like a "multiplicative Euler characteristic". We make comments on the proofs later when we will put this invariant into context with other torsion invariants in Section 3.

Theorem 1.4 (Homotopy invariance) *Let* $f : (X, A) \longrightarrow (Y, B)$ *be a homotopy equivalence of pairs of finite CW-complexes. Suppose one of them is admissible. Then both are admissible and there is a natural homomorphism:*

$$\Phi = \Phi(\pi_1(Y)) : Wh(\pi_1(Y)) \longrightarrow \mathbb{R}^{>0} \qquad [B] \mapsto \det(B)$$

such that

$$\frac{\rho(Y, B)}{\rho(X, A)} = \Phi(\tau(f))$$

where $Wh(\pi_1(Y))$ *is the Whitehead group of* $\pi_1(Y)$ *and* $\tau(f)$ *is the Whitehead torsion of* f.

$\blacksquare$

Theorem 1.4 shows that $\rho(X, A)$ is a simple homotopy invariant for admissible pairs. This enables us to define L^2-torsion $\rho(M, \partial_0 M)$ also for a compact manifold M whose boundary ∂M is the union of two codimension zero submanifolds $\partial_0 M$ and $\partial_1 M$ satisfying $\partial(\partial_0 M) = \partial_0 M \cap \partial_1 M = \partial(\partial M_1)$. Namely, define it by the L^2-torsion of (K, K_0) for any triangulation $(K; K_0, K_1) \longrightarrow (M; \partial_0 M, \partial_1 M)$. This makes sense because two such triangulations differ by a simple homotopy equivalence, i.e., a homotopy equivalence with trivial Whitehead torsion. In connection with Theorem 1.4 the following conjecture is interesting:

Conjecture 1.5 *1.) The map $\Phi : Wh(\pi) \longrightarrow \mathbb{R}^{>0}$ is trivial for finitely presented π.*

2.) The L^2-torsion $\rho(X, A)$ is a homotopy invariant for admissible pairs. ■

The assertions 1.) and 2.) of the conjecture above are equivalent. Namely, Theorem 1.4 shows that 1.) implies 2.). The other implication is proven as follows. Choose a finite CW-complex Y with fundamental group π. Then $Y \times S^1$ is admissible by Theorem 1.2. If $i_* : \mathrm{Wh}(\pi) \longrightarrow \mathrm{Wh}(\pi \times \mathbb{Z})$ is the injection induced by the inclusion, the composition of $\Phi(\pi \times \mathbb{Z})$ with i_* is $\Phi(\pi)$. Hence it suffices to prove the claim for $\Phi(\pi \times \mathbb{Z})$. As any element in $\mathrm{Wh}(\pi \times \mathbb{Z})$ is the Whitehead torsion of a homotopy equivalence $f : X \longrightarrow Y \times S^1$, assertion 1.) follows from Theorem 1.4. For residually finite π conjecture 1.5 is proven in [24].

Theorem 1.6 (Sum formula) *Consider the push out of finite CW-complexes such that j_1 is an inclusion of CW-complexes and j_2 is cellular:*

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ j_2 \downarrow & & \downarrow i_1 \\ X_2 & \xrightarrow{i_2} & X \end{array}$$

Assume that X_0, X_1 and X_2 are admissible and that for $i = 0, 1, 2$ the map $\pi_1(X_i) \longrightarrow \pi_1(X)$ induced by the inclusion is injective for all base points in X_i. Then X is admissible and we get:

$$\rho(X) = \rho(X_1) \cdot \rho(X_2) \cdot (\rho(X_0))^{-1}. \quad ■$$

Theorem 1.7 (Pair formula) *Let (X, A) be a pair of finite CW-complexes such that the map induced by the inclusion $\pi_1(A) \longrightarrow \pi_1(X)$ is injective for all base points in A. Suppose that two of the three pairs $A = (A, \emptyset)$, $X = (X, \emptyset)$ and (X, A) are admissible. Then all three pairs are admissible and we get:*

$$\rho(X) = \rho(A) \cdot \rho(X, A). \quad ■$$

Theorem 1.8 (Fibration formula) *Let $F \longrightarrow (E, E_0) \longrightarrow (B, B_0)$ be a fibration of pairs of finite CW-complexes (E, E_0) and (B, B_0) with a connected finite CW-complex F as fiber.*

Suppose that F is admissible and the inclusion induces an injection $\pi_1(F) \longrightarrow E$. Then (E, E_0) is admissible and we get:

$$\rho(E, E_0) = \rho(F)^{\chi(B,B_0)}$$

where $\chi(B, B_0)$ is the Euler characteristic of (B, B_0). ■

Theorem 1.9 (Product formula) *Let (X, A) and (Y, B) be pairs of finite CW-complexes. Suppose that (X, A) is admissible. Then $(X, A) \times (Y, B)$ is admissible and we get:*

$$\rho\left((X, A) \times (Y, B)\right) = \rho(X, A)^{\chi(Y,B)}. \quad ■$$

Theorem 1.10 (Multiplicativity under finite coverings) *Let $p : X \longrightarrow Y$ be a finite d-sheeted covering of finite CW-complexes. Suppose that X or Y is admissible. Then both X and Y are admissible and we get:*

$$\rho(X) = \rho(Y)^d. \quad ■$$

Theorem 1.11 (Poincaré duality) *Let M be a connected compact orientable manifold. Then M is admissible if and only if $(M, \partial M)$ is admissible. Suppose M is admissible. Then we get:*

$$\rho(M) = \rho(M, \partial M)^{(-1)^{m+1}}.$$

In particular we get for an admissible closed manifold of even dimension $\rho(M) = 1$. ■

Theorem 1.12 (S^1-actions) *Let M be a connected compact manifold. Suppose that there is a fixed point free smooth S^1-action on M such that for one (and hence all) orbits S^1/H in M the inclusion induces an injection on the fundamental groups. Then M is admissible and*

$$\rho(M) = 1. \quad ■$$

2. 3-manifolds

Let M be a compact orientable connected irreducible 3-manifold with infinite fundamental group and incompressible boundary. Jaco and Shalen [19] and Johannson [20] have shown for such a manifold that there is a finite family of disjoint, incompressible 2-sided tori in M which splits M into pieces which are either Seifert 3-manifolds or admit no embedded incompressible torus except possibly parallel to the boundary. A minimal such family of tori is unique up to isotopy. If all pieces of the second kind admit a hyperbolic structure, i.e., a complete Riemannian metric on its interior with sectional curvature which is constant -1, then we call such a decomposition the JSJT-decomposition of M.

Let N be a piece of the second kind. If N is Haken, i.e., ∂N is non-empty o r N contains an embedded incompressible orientable surface different from S^2, the torus theorem says that N is atoroidal, i.e any subgroup of $\pi_1(N)$ which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ is conjugate into the fundamental group of a boundary component. Now Thurston has proven that any such manifold has a hyperbolic structure. (For more information about 3-manifolds we refer to the survey article of Scott [33].) Hence M has a JSJT-decomposition if M is Haken. Thurston's geometrization conjecture says that M always has a JSJT-decomposition.

Notice that $\rho(T^2)$ is 1 by Theorem 1.11 or Theorem 1.12. Since any connected Seifert 3-manifold is finitely covered by a compact orientable connected 3-manifold admitting a free S^1-action such that the map on the fundamental groups induced by the inclusion of an orbit into the manifold is injective, we conclude from Theorem 1.10 and Theorem 1.12 that $\rho(M_i)$ is 1 for any Seifert piece M_i in the JSJT-decomposition of M. Notice that the volume of a compact connected orientable irreducible hyperbolic 3-manifold is finite, if and only if ∂M is empty or is a disjoint union of incompressible tori and M is not $T^2 \times I$. (see Morgan [29], Theorem B on page 52). We derive from Theorem 1.2 (or more precisely from the proof of [22, Theorem 7.1]) and Theorem 1.6:

Theorem 2.1 *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group such that ∂M is empty or a disjoint union of incompressible tori. Suppose that M has a JSJT-decomposition. Then M is admissible and we get:*

$$\rho(M) = \prod_{i=1}^{r} \rho(M_i)$$

where $M_1, \ldots M_r$ are the hyperbolic pieces of finite volume in the JSJT-decomposition of M. ■

For the hyperbolic pieces of finite volume we expect the following answer:

Conjecture 2.2 *Let M be a compact connected orientable hyperbolic manifold of dimension m. Assume that either m is odd and ∂M is empty or that m equals 3, the boundary ∂M is a disjoint union of incompressible tori and M is not $T^2 \times I$. There is a constant C_m depending only on m satisfying:*

$$\ln(\rho(M)) = C_m \cdot Vol(M)$$

if $Vol(M)$ is the volume. The constant C_m is zero if and only if m is even. ■

Notice that this conjecture is true for even m because then $C_m = 0$ follows from Theorem 1.11. Hence the only interesting case is the case where m is odd. We will later give evidence for this conjecture when we explain the relationship of L^2-torsion and its analytic counterpart in Section 3. The constant C_3 is expected to be $-\frac{1}{3\pi}$.

Given a compact orientable manifold M, let $\| M \|$ be its *simplicial volume* as defined by Gromov [15].

Conjecture 2.3 *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group possessing a JSJT-decomposition such that the boundary of M is empty or a disjoint union of incompressible tori. Then M is admissable and there are constants C_3 and D_3 different from zero satisfying:*

$$\ln(\rho(M)) = C_3 \cdot \sum_{i=1}^{r} Vol(M_i) = D_3 \cdot \| M \|$$

where $M_1, \ldots M_r$ are the hyperbolic pieces of finite volume in the JSJT-decomposition of M. In particular $\rho(M)$ is 1 if and only if there are no hyperbolic pieces of finite volume in the JSJT-decomposition. ■

Notice that the Conjecture 2.2 together with Theorem 2.1 implies

$$\ln(\rho(M)) = C_3 \cdot \sum_{i=1}^{r} \mathrm{Vol}(M_i).$$

It is already known [34] that there is a constant D_3' different from zero satisfying:

$$\| M \| = D_3' \cdot \sum_{i=1}^{r} \mathrm{Vol}(M_i).$$

Hence Conjecture 2.2 for $m = 3$ and Conjecture 2.3 are equivalent. We will generalize them in Section 3 .

In general analytic L^2-torsion and the simplicial volume are very hard to compute. In comparision with them one can get good information on the combinatorial L^2-torsion as illustrated by the results of Section 1. Next we want to explain how one can read off the combinatorial L^2-torsion in terms of a presentation $\pi = \langle s_1, s_2, \ldots s_g \mid R_1, R_2, \ldots R_r \rangle$ of the fundamental group π of M. We begin with recalling Fox derivatives.

Let $*_g\mathbb{Z}$ be the free group of rank g with standard generators $\overline{s_1}, \overline{s_2}, \ldots \overline{s_g}$. Denote by $\phi : *_g\mathbb{Z} \longrightarrow \pi$ the epimorphism sending $\overline{s_i}$ to s_i. The *i-th partial Fox derivative* is the map

$$\frac{\partial}{\partial s_i} : \mathbb{Z}[*_g\mathbb{Z}] \longrightarrow \mathbb{Z}\pi$$

which is uniquely determined by the following properties for $u, v \in \mathbb{Z}[*_g\mathbb{Z}]$

1.) $\frac{\partial}{\partial s_i}(u+v) = \frac{\partial}{\partial s_i}(u) + \frac{\partial}{\partial s_i}(v)$

2.) $\frac{\partial}{\partial s_i}(u \cdot v) = \frac{\partial}{\partial s_i}(u) \cdot \epsilon(v) + \phi(u) \cdot \frac{\partial}{\partial s_i}(v)$

3.) $\frac{\partial}{\partial s_i}(\overline{s_i}) = \delta_{i,j}$

where $\epsilon : \mathbb{Z}[*_g\mathbb{Z}] \longrightarrow \mathbb{Z}$ is the augmentation sending $\sum_g \lambda_g \cdot g$ to $\sum_g \lambda_g$, the ring homomorphism induced by ϕ from $\mathbb{Z}[*_g\mathbb{Z}]$ to $\mathbb{Z}\pi$ is denoted by ϕ again and the Kronecker symbol $\delta_{i,j}$ is 0 for $i \neq j$ and 1 otherwise. Here are further properties useful for concrete calculations where $m, n \in \mathbb{Z}, n \geq 1$ and $w \in *_g\mathbb{Z}$:

4.) $\frac{\partial}{\partial s_i}(m) = 0$

5.) $\frac{\partial}{\partial s_i}(w^{-1}) = -w^{-1} \cdot \frac{\partial}{\partial s_i}(w)$

6.) $\frac{\partial}{\partial s_i}(w^n) = (1 + w + \ldots + w^{n-1}) \cdot \frac{\partial}{\partial s_i}(w)$

7.) $\frac{\partial}{\partial s_i}(w^{-n}) = -(w^{-1} + \ldots w^{-n}) \cdot \frac{\partial}{\partial s_i}(w)$

The *Fox matrix* of the given presentation is the (r, g)-matrix with entries in $\mathbb{Z}\pi$

$$F = \begin{pmatrix} \frac{\partial R_1}{\partial s_1} & \cdots & \frac{\partial R_1}{\partial s_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial R_r}{\partial s_1} & \cdots & \frac{\partial R_r}{\partial s_g} \end{pmatrix}$$

Theorem 2.4 *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π. Let $\pi = \langle s_1, s_2, \ldots s_g \mid R_1, R_2, \ldots R_r \rangle$ be a presentation of π. Denote by $\alpha_2(M)$ the second Novikov-Shubin invariant of M. Now there are two cases:*

1.) Suppose ∂M is non-empty. We make the assumption that ∂M

2.) Suppose ∂M is empty. We make the assumption that a finite covering of M is homotopy equivalent to a hyperbolic, Seifert or Haken 3-manifold and that the given presentation comes from a Heegaard decomposition. Then M

Let K be any positive real number satisfying $K \geq \| R_A \|_\infty$ where $\| R_A \|_\infty$ is the operator norm of the bounded π-equivariant R_i whose Fox derivatives appear in A.

Then the sum of non-negative rational numbers $\sum_{p=1}^{L} \frac{1}{p} \cdot tr_{\mathbb{Z}\pi}\left((1 - K^{-2} \cdot AA^*)^p\right)$ *converges to the real number* $\ln(\rho(M)) + 2(g-1) \cdot \ln(K)$. *More precisely, there is a constant* C *such that we get for all* $L \geq 1$*:*

$$0 \leq \ln(\rho(M)) + 2(g-1) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{p} \cdot tr_{\mathbb{Z}\pi}\left((1 - K^{-2} \cdot AA^*)^p\right) \leq \frac{C}{L^{\alpha}}$$

Proof : 1.) We deal first with the case where ∂M is non-empty. As M is irreducible and has infinite fundamental group, M is aspherical by the Sphere

Theorem [18, page 40]. Since M is 3-dimensional and has boundary there is a homotopy equivalence $g : Y \longrightarrow M$ for a finite 2-dimensional aspherical CW-complex Y. Let X be the 2-dimensional CW-complex with fundamental group π and 1 cell of dimension zero, g cells of dimension one and r cells of dimension two associated to the given presentation. Let $f : X \longrightarrow Y$ be a map inducing the identity on the fundamental groups. Let $C(\widetilde{f}) : C(\widetilde{X}) \longrightarrow C(\widetilde{Y})$ be the induced $\mathbb{Z}\pi$-chain map. We have $H_1(C(\widetilde{X})) = \{0\}$ and $H_p(C(\widetilde{Y})) = \{0\}$ for $p = 1, 2$ and $C(\widetilde{f})$ induces an isomorphism on H_0. We derive from the long exact homology sequence that the algebraic mapping cone of $C(\widetilde{f})$ is a finitely generated free 3-dimensional $\mathbb{Z}\pi$-chain complex whose homology is trivial except in dimension 3 where it is $H_2(C(\widetilde{X}))$. Hence there is an isomorphism of $\mathbb{Z}\pi$-modules

$$H_2(C(\widetilde{X})) \oplus (\oplus_{i=1}^{a} \mathbb{Z}\pi) \longrightarrow \oplus_{i=1}^{b} \mathbb{Z}\pi$$

where $a = g + \dim_{\mathbb{Z}\pi}(C_2(\widetilde{Y})) + 1$ and $b = r + 1 + \dim_{\mathbb{Z}\pi}(C_1(\widetilde{Y}))$. Since the Euler characteristic of Y satisfies $\chi(Y) = \chi(M) = \chi(\partial M)/2 = 0$ and $g = r + 1$ holds, we conclude $a = b$. This implies by Kaplansky's theorem that $H_2(C(\widetilde{X})) = \{0\}$.

We give a short proof of this fact using dimension theory over the von Neumann algebra of π as explained in Section 4. Namely, let $P : \oplus_{i=1}^{a} \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^{a} \mathbb{Z}\pi$ be any projection whose kernel is $\mathbb{Z}\pi$-isomorphic to $H_2(C(\widetilde{X}))$ and whose image is $\mathbb{Z}\pi$-isomorphic to $\oplus_{i=1}^{a} \mathbb{Z}\pi$. Let $P^{(2)} : \oplus_{i=1}^{a} l^2(\pi) \longrightarrow \oplus_{i=1}^{a} l^2(\pi)$ be the induced bounded π-equivariant operator. Then we get:

$$\dim(ker(P^{(2)})) + \dim(im(P^{(2)})) = a$$

$$\dim(im(P^{(2)})) = a$$

We conclude $\dim(ker(P^{(2)})) = 0$ so that the kernel of $P^{(2)}$ is trivial. Hence the kernel of P which is $H_2(C(\widetilde{X}))$ is trivial.

We conclude that $f : X \longrightarrow Y$ and hence $g \circ f : X \longrightarrow M$ are homotopy equivalences. If $\overline{M} \longrightarrow M$ is a finite d-sheeted covering of M, the composition of the map $\phi(\pi_1(\overline{M})) : Wh(\pi_1(\overline{M})) \longrightarrow \mathbb{R}^{>0}$ of Theorem 1.4 with $p^* : Wh(\pi_1(M)) \longrightarrow Wh(\pi_1(\overline{M}))$ induced by restriction is the same as the composition of $\phi(\pi_1(M)) : Wh(\pi_1(M)) \longrightarrow \mathbb{R}^{>0}$ with the map $\mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$ sending r to r^d. From our assumptions on M and [35, Theorem 19.4 on page 249 and Theorem 19.5 on page 250] and [13] we conclude for an appropriate finite covering $\overline{M}$ of M that the Whitehead group $Wh(\pi_1(\overline{M}))$ vanishes. Hence the map $\phi(\pi_1(M)) : Wh(\pi_1(M)) \longrightarrow \mathbb{R}^{>0}$ is trivial. We derive from Theorem 1.4

$$\rho(M) = \rho(X).$$

The cellular chain complex $C(\widetilde{X})$ looks like

$$\ldots \{0\} \longrightarrow \oplus_{i=1}^{r} \mathbb{Z}\pi \xrightarrow{F} \oplus_{i=1}^{g} \mathbb{Z}\pi \xrightarrow{\oplus_{j=1}^{g} R_{s_j-1}} \mathbb{Z}\pi$$

where $R_{s_j-1} : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ is given by right multiplication with $s_j - 1$ and F is the Fox matrix of the given presentation. Consider the $\mathbb{Z}\pi$-chain complex C concentrated in dimensions 1 and 0

$$\mathbb{Z}\pi \xrightarrow{R_{s_g-1}} \mathbb{Z}\pi$$

and the $\mathbb{Z}\pi$-chain complex D concentrated in dimensions 2 and 1

$$\oplus_{i=1}^{r} \mathbb{Z}\pi \xrightarrow{A} \oplus_{j=1}^{g-1} \mathbb{Z}\pi$$

where A is obtained from F by deleting the g-th column. There is an obvious exact sequence of $\mathbb{Z}\pi$-chain complexes $0 \longrightarrow C \longrightarrow C(\widetilde{X}) \longrightarrow D \longrightarrow 0$. It induces an exact sequence of Hilbert $\mathcal{N}(\pi)$-chain complexes by tensoring with $l^2(\pi)$. Since D is obtained from the cellular $\mathbb{Z}[\mathbb{Z}]$-chain complex of the universal covering of S^1 by induction with respect to the injective homomorphism $\mathbb{Z} \longrightarrow \pi$ sending the generator in $\mathbb{Z}$ to s_g, we conclude from [22, Lemma 4.6, Example 4.11] and Theorem 1.12: (Notice that the Novikov-Shubin invariants defined here are two times the one defined in [22].)

$$b_p(S^1) = 0$$

$$\alpha_1(D) = \alpha_1(S^1) = 2.$$

$$\rho(D) = \rho(S^1) = 1$$

We derive from the long weakly exact L^2-homology sequence [6, Theorem 2.1], the additivity inequalities for the Novikov-Shubin invariants [22, Theorem 2.2] and the chain complex analogue of Theorem 1.6:

$$b_p(C) = b_{p+1}(D)$$

$$\frac{1}{\alpha_2(C)} \leq \frac{1}{\alpha_2(C(\widetilde{X}))} + \frac{1}{\alpha_1(D)}$$

$$\rho(X) = \rho(C) \cdot \rho(D)$$

This implies:

$$b_p(C) = 0$$

$$\alpha(R_A) \geq \frac{2 \cdot \alpha_2(M)}{\alpha_2(M) + 2}$$
$$\rho(M) = \rho(X) = \rho(C) = \det(R_A^* R_A)^{-2} \cdot \det(R_A R_A^*)^{+1}$$

We conclude from Lemma 4.2.1:

$$\ln(\rho(M) = -2 \cdot \ln(\det(R_A)).$$

Now the claim for non-empty ∂M follows from Theorem 4.4.5.

2.) Suppose M is closed and the presentation comes from a Heegard decomposition. Then the cellular $\mathbb{Z}\pi$-chain complex of $\widetilde{M}$ looks like:

$$\ldots \mathbb{Z}\pi \xrightarrow{(R_{s_i^{-1}-1})_{i=1,\ldots,r}} \oplus_{i=1}^r \mathbb{Z}\pi \xrightarrow{F} \oplus_{i=1}^g \mathbb{Z}\pi \xrightarrow{\oplus_{i=1}^g R_{s_j-1}} \mathbb{Z}\pi$$

Now one shows as above by constructing appropriate exact sequences of $\mathbb{Z}\pi$-chain complexes that $\alpha(R_A) \leq \frac{\alpha_2(M)}{\alpha_2(M)+1}$ and $\ln(\rho(M)) = -2 \cdot \ln(\det(R_A))$ holds. This finishes the proof of Theorem 2.4. ■

Remark 2.5 The main problem in orientable surface as fiber. Then there is an exact sequence of groups

$$\{1\} \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow \{1\}$$

In order to check whether a word represents the unit element in $\pi_1(M)$, one maps it to $\mathbb{Z}$ first and checks whether it represents the unit element there. If it does, it can be rewritten as a word in standard generators of $\pi_1(F)$ and one has to solve the word problem in $\pi_1(F)$. This is always possible and in particular easy if ∂F is non-empty since in this case $\pi_1(F)$ is a free group. In this context Conjecture 2.2 is interesting. Namely, it would give together with Theorem 2.4 a way of computing the volume of a hyperbolic 3-manifold which is the mapping torus of a . ■

Remark 2.6 Notice that the complement $M(K)$ of a knot in S^3 is admissible by Theorem 1.2. Hence one obtains a new nvariant for knots in S^3 by

$$\rho(K) := \rho(M(K))$$

If the knot is trivial, then $\rho(K) = 1$. Otherwise M(K) does satisfies the hypothesis of Theorem 2.4 and one can compute the invariant from a presentation of the knot group. The invariant is 1 for torus knots by Theorem 2.1. It should give up to a constant the volume for hyperbolic knots because of Conjecture 2.2. One can heck that this invariant is multiplicative under the connected sum of knots in S^3. The invariant for a knot and its mirror image are the ame. In a certain sense this invariant is the Alexander polynomial, but not for the maximal abelian covering but for the universal covering of the knot complement [26, Example 4.7]. ■

Example 2.7 Let K_8 be the figure eight knot. Its complement is a hyperbolic 3-manifold. It fibers over S^1 and the fiber is a surface whose fundamental group is the free group F_2 in two generators s_1 and s_2. The monodromy is up to homotopy determined by the induced automorphism of F_2 up to inner automorphisms of F_2. Two automorphisms of F_2 differ by an inner automorphism if and only if they induces the same map on the abelianization $\mathbb{Z}\oplus\mathbb{Z}$ [27, Proposition I.4.5]. The automorphism of F_2 belonging to the figure eight knot induces on $\mathbb{Z}\oplus\mathbb{Z}$ the automorphism given by the matrix [2, page 73]

$$\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$$

Hence the automorphism of F_2 given by mbox$s_1 \mapsto s_2$ and $s_2 \mapsto s_2^3 s_1^{-1}$ is appropriate for the computation of the knot group π of the figure eight knot. We get the presentation:

$$\pi \;=\; \langle s_1, s_2, t \mid t s_1 t^{-1} s_2^{-1} = t s_2 t^{-1} s_1 s_2^{-3} = 1 \rangle$$

If we delete from the Fox matrix the column belonging to s_2, we obtain the matrix

$$A \;=\; \begin{pmatrix} t & 1 - s_2 \\ s_2^3 s_1^{-1} & 1 - s_2^3 s_1^{-1} \end{pmatrix}$$

The number $K = 4$ is greater or equal to the operator norm of the bounded π-equivariant operator induced by A. Define:

$$B \;=\; \begin{pmatrix} 13 + s_2 + s_2^{-1} & -1 + s_2 + s_1 s_2^3 - s_2 s_1 s_2^{-3} - t s_1 s_2^{-3} \\ -1 + s_2^{-1} + s_2^3 s_1^{-1} - s_2^3 s_1^{-1} s_2^{-1} - s_2^3 s_1^{-1} t^{-1} & 13 + s_2^3 s_1^{-1} + s_1 s_2^{-3} \end{pmatrix}$$

Since $B = 16 - AA^*$, we get:

$$\operatorname{tr}\left(\left(1 - K^{-2} \cdot AA^*\right)^p\right) = 16^{-p} \cdot \operatorname{tr}_{\mathbb{Z}\pi}(B^p)$$

and

$$\ln(\rho(K_8)) = -8\ln(2) + \sum_{p=1}^{\infty} \frac{1}{p \cdot 16^p} \cdot \operatorname{tr}_{\mathbb{Z}\pi}(B^p)$$

As mentioned before we have not yet implemented a computer programm to compute the numbers $\operatorname{tr}_{\mathbb{Z}\pi}(B^p)$. We get 26 for $p = 1$ and 352 for $p = 2$ by a direct calculation. Since the volume of the complement of the figure eight knot is known, this is a good test for Conjecture 2.3. The second Novikov-Shubin invariant of the knot complement of an hyperbolic knot such as K_8 is bounded from below by $\frac{4}{3}$ [22, Theorem 6.17] and probably is equal to 2. (Notice that the Novikov-Shubin invariants defined here are two times the one defined in [22].) ■

3. Relation to other L^2-torsion invariants and Gromov's simplicial volume

In this section we relate the combinatorial L^2-torsion to Reidemeister-von Neumann torsion and explain the proofs of the results of Section 1 and the condition "admissible". Then we discuss the connection to analytic L^2-torsion and simplicial volume and state some interesting conjectures.

The combinatorial L^2-torsion may be viewed as a special case of the Reidemeister-von Neumann torsion of [26, Definition 3.9]. It assigns to a pair of finite CW-complexes (X, A) and a unitary representation $\mu : \pi_1(X) \longrightarrow Iso_{\mathcal{A}}(V)$ into a finitely generated Hilbert module V over a finite von Neumann algebra $\mathcal{A}$ an element in the weak K_1-group of $\mathcal{A}$:

$$\rho^{RN}(X, A; V) \in K_1^w(\mathcal{A})$$

To keep the discussion simple we assume from now on that $\mathcal{A}$ is the von Neumann algebra $\mathcal{N}(\pi)$ of the fundamental group $\pi = \pi_1(X)$ and μ is the regular representation $l^2(\pi)$ and we abbreviate $\rho^{RN}(X, A) := \rho^{RN}(X, A; l^2(\pi))$. The K_1-group $K_1^w(\mathcal{N}(\pi))$ is the K_1-group of weak endomorphisms of finitely generated Hilbert $\mathcal{N}(\pi)$-modules where an operator is called weak isomorphism if its kernel is trivial and its image is dense [26, page 221]. We remark that an endomorphism T is a weak isomorphism if and only if its L^2-Betti number $b(T)$ (which we will define for an operator in Section 4) is zero. Define $K_1^{w,\alpha}(N(\pi))$ to be the K_1-group of weak endomorphisms of finitely generated Hilbert $\mathcal{N}(\pi)$-modules which have positive Novikov-Shubin invariant (which we will define for an operator in Section 4). Now Lemma 4.2 shows that the Fuglede-Kadison determinant induces a split epimorphism

$$\det : K_1^{w,\alpha}(\mathcal{N}(\pi)) \longrightarrow \mathbb{R}^{>0}$$

The Reidemeister-von Neumann torsion $\rho^{RN}(X, A)$ takes values in $K_1^{w,\alpha}(\mathcal{N}(\pi))$ if and only if all Novikov-Shubin invariants of (X, A) are positive. Hence the construction in [26] yields a well-defined element

$$\rho^{RN}(X, A) \in K_1^{w,\alpha}(\mathcal{N}(\pi))$$

provided all Novikov-Shubin invariants of (X, A) are positive. Moreover, all the results of [26] remain true when interpreted in $K_1^{w,\alpha}(\mathcal{N}(\pi))$ provided that all Novikov-Shubin invariants of (X, A) are positive. The combinatorial L^2-torsion $\rho(X, A) \in \mathbb{R}^{>0}$ Hence the results of Section 1, namely homotopy invariance 1.4, sum formula 1.6, pair formula 1.7, product formula refproduct formula, multiplicativity under finite coverings 1.10, Poincaré duality 1.11 and the result on S^1-actions 1.12 follow from the analogous statements for Reidemeister-von Neumann torsion [26, Theorem 3.11, Theorem 3.14, Lemma 3.17, Theorem 3.16, Theorem 5.13, Corollary 3.21] except for the statements about the positivity of the Novikov-Shubin invariants which are not at all treated in [26]. The homotopy invariance of L^2-Betti numbers and Novikov-Shubin invariants follows from [10], [17], [22]. The statements on them in the sum formula 1.6, the pair formula 1.7 and the product formula 1.9 come from the long weakly exact L^2-homology sequence [6, Theorem 2.1] and the additivity inequalities for the Novikov-Shubin invariants [22, Theorem 2.2]. Poincaré duality 1.11 is a consequence of [22, Proposition 4.2]. The fibration formula 1.8 follows from the sum formula 1.6 by induction over the cells of B. These remarks give the proofs of the results of Section 1.

At this point we mention the article [3] of Carey and Mathai where a construction using the Fuglede-Kadison determinant is described in order to generalize classical Reidemeister torsion to the L^2-setting. There is an error in the definition which was corrected in the preprint [4] and the corrected version is the square root of the combinatorial L^2-torsion defined here. Some of the results of Section 1 are also proven in [4].

Lott introduces in [21, Definition 2] analytic torsion for a closed Riemannian manifold which is the L^2-analogue of the analytic torsion defined by Ray and Singer [32] (see also [28]). The definition makes only sense if all Novikov-Shubin invariants of the closed Riemannian manifold are positive. This condition is necessary to control the large time behaviour of the heat kernel of the universal covering. Combinatorial L^2-torsion is the L^2-analogue of the R-torsion defined in [32] using Milnor's version of Reidemeister torsion. Cheeger [5] and Müller [30] have shown independently that the difference of analytic torsion and the logarithm of R-torsion of a compact Riemannian manifold M is zero provided M has no boundary. This difference has been identified with $\frac{\ln(2)}{2} \cdot \chi(\partial M)$ for a compact Riemannian manifold M in [23]. This leads to the obvious conjecture:

Conjecture 3.1 Let M be a compact Riemannian manifold whose boundary is the disjoint union $\partial M = \partial_0 M \coprod \partial_1 M$ where $\partial_0 M$ and $\partial_1 M$ are disjoint unions of components of ∂M. Then we get for combinatorial and analytic L^2-torsion:

$$\ln(\rho(M, \partial_0 M)) = \rho_{an}(M, \partial_0 M) + \frac{\ln(2)}{2} \cdot \chi(\partial M). \qquad \blacksquare$$

The analytic L^2-torsion of a closed hyperbolic 3-manifold M is computed in [21, Proposition 16] by

$$\rho_{an}(M) = -\frac{1}{3\pi} \cdot Vol(M).$$

Hence the equivalent Conjectures 2.2 and 2.3 would follow from Conjecture 3.1. Notice that Reidemeister-von Neumann torsion takes values in $K_1^{w,\alpha}(\mathcal{N}(\pi))$. We have not carried out the details of defining analytic L^2-torsion taking values in these K-groups but it can be done using the center valued trace of the von Neumann algebra $\mathcal{N}(\pi)$ and by extending the calculations of [25] to $K_1^{w,\alpha}(\mathcal{N}(\pi))$. Notice that the invariants taking values in K_1-groups i nstead of the positive real numbers are much more refined. For example Reidemeister-von Neumann torsion gives the Alexander polynomial $\Delta_K(t)$ of a knot $K \subset S^3$ when applied to the infinite cyclic covering of the knot complement [26, Example 4.7] whereas the combinatorial L^2-torsion gives just the real number $\int_{S^1} \ln\left(\Delta_K(z) \cdot \Delta_K(\overline{z})\right) dvol$.

Before we discuss the condition admissible, we give an overview over the notions of L^2-Betti number, Novikov-Shubin invariant and analytic L^2-torsion. Let M be a compact Riemannian manifold and assume for simplicity that it has no boundary. Let $\widetilde{\Delta}_p$ be the Laplace operator on the universal covering acting on smooth p-forms. Denote by $e^{-t\widetilde{\Delta}_p}(x,y)$ the kernel of the operator $e^{-t\widetilde{\Delta}_p}$. Then $e^{-t\widetilde{\Delta}_p}(x,x)$ is an endomorphism of the finite-dimensional vector space $\Lambda^p T_x^* M$ and hence its trace $\mathrm{tr}\left(e^{-t\widetilde{\Delta}_p}(x,x)\right)$ is a well-defined real number. If $\mathcal{F}$ is a fundamental domain for the $\pi = \pi_1(M)$-action on the universal covering, define the π-trace by:

$$Tr_\pi\left(e^{-t\widetilde{\Delta}_p}\right) =_{\mathcal{F}} \mathrm{tr}\left(e^{-t\widetilde{\Delta}_p}(x,x)\right)\ dvol.$$

The L^2-Betti number is defined by Atiyah [1] as

$$b_p(M) = \lim_{t\to\infty} Tr_\pi\left(e^{-t\overline{\Delta}_p}\right).$$

The p-th Novikov-Shubin invariant is (up to a factor 4) defined by Novikov and Shubin [31] as

$$\widetilde{\alpha}_p(M) = \sup\left\{\beta_p \mid Tr_\pi\left(e^{-t\widetilde{\Delta_p}}\right) - b_p(M) \text{ is } O(t^{-\beta_p/4}) \text{ as } t \to \infty\right\} \in [0,\infty].$$

Given a pair (X,A) of finite CW-complexes, the combinatorial analogues are defined as follows. Let $C(\widetilde{X},\widetilde{A})$ be the cellular $\mathbb{Z}\pi$-chain complex of the universal covering for $\pi = \pi_1(X)$ as explained in Section 1. Tensoring with $l^2(\pi)$ yields the cellular L^2-chain complex $C(X,A;l^2(\pi))$ whose Hilbert $\mathcal{N}(\pi)$-structure is defined using a cellular basis and is independent of this choice. The p-th differential

$$c_p : C_p(X,A;l^2(\pi)) \longrightarrow C_{p-1}(X,A;l^2(\pi))$$

and the combinatorial L^2-Laplace operator

$$\Delta_p = c_p^* \circ c_p + c_{p+1} \circ c_{p+1}^* \ :\ C_p(X,A;l^2(\pi)) \longrightarrow C_p(X,A;l^2(\pi))$$

are bounded π-equivariant operators and we define the combinatorial analogues by applying the definitions of Section 4 to Δ_p and c_p:

$$\begin{aligned} b_p(X,A) &= b(\Delta_p) \\ \alpha_p(X,A) &= \alpha(c_p) \\ \widetilde{\alpha}_p(M) &= \min\{\alpha_p(M), \alpha_{p-1}(M)\}. \end{aligned}$$

Of course the combinatorial and analytic versions of $b_p(M)$ and $\widetilde{\alpha}_p(M)$ agree (see [10], [12] and [17]) and we have [22, Lemma 2.3.1]:

$$2 \cdot \widetilde{\alpha}_p(M) = \alpha(\Delta_p)$$

Analytic L^2-torsion is defined by Lott [21, Definition 2] using a regularization process for the first integral by setting:

$$\begin{aligned}\rho_{an}(M) &= \sum_{p\geq 0}(-1)^p \cdot p \cdot \left(\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \cdot \int_0^{\epsilon} t^{s-1} \cdot \left(Tr_{\pi}\left(e^{-t\widetilde{\Delta}_p}\right) - b_p(M)\right) \, dt \right. \\ &+ \left. \int_{\epsilon}^{\infty} t^{-1} \cdot \left(Tr_{\pi}\left(e^{-t\widetilde{\Delta}_p}\right) - b_p(M)\right) \, dt \right)\end{aligned}$$

where $\Gamma(s)$ is the Γ-function and ϵ is a positive real number. The condition that the Novikov-Shubin invariants are positive ensures that the second integral converges to a real number.

Recall that the construction of analytic L^2-torsion of a compact Riemannian manifold M is only well-defined if all the Novikov-Shubin invariants of M are positive. Reidemeister-von Neumann as defined in [26] does not need this assumption. However, the computations of $K_1^w(\mathcal{N}(\pi))$ in [25] show that these groups vanish if and only if $\mathcal{N}(\pi)$ is a von Neumann algebra of type II. Hence the condition that all Novikov-Shubin invariants are all positive appears also on the topological side, it is needed to ensure that the K_1-groups where the invariant takes values in is always non-trivial. We mention without proof that the von Neumann algebra of a finitely generated group π is of type I if π is a crystallographic group, i.e., contains a finitely generated free abelian group of finite index, and is of type II otherwise. If the fundamental group π is crystallographic, no conditions are needed on the topological side. However, in this case the conjecture that all the Novikov-Shubin invariants of a compact manifold are positive [22, Conjecture 9.1] follows from [21, Proposition 39] and the fact that the Novikov-Shubin nvariants do not change under finite coverings. Moreover, in this case Conjecture 3.1 follows from the corresponding result for analytic and R-torsion of [5], [23] and [30] since for finitely generated free abelian π one can express the L^2-invariants in terms of the original invariants for all 1-dimensional complex representations $\pi \longrightarrow S^1$ and the L^2-invariants are We have explained above the condition that all the Novikov-Shubin invariants are positive. Next we discuss the condition that the L^2-Betti numbers are ll zero. This assumption is not needed for the definition of Reidemeister-von Neumann torsion, combinatorial L^2-torsion or analytic L^2-torsion if one considers a compact Riemannian manifold. However, in general these invariants depend on the Riemannian metric and it turns out that they are independent of the Riemannian metric if and only if all L^2-Betti numbers are trivial. Since we want to deal with the topological aspect of L^2-torsion, we always assume that the L^2-Betti numbers are trivial in order to get a simple homotopy invariant (see Theorem 1.4).

We recall the definition of simplicial volume of an m-dimensional oriented closed manifold M [15, Section 0.2]. Let $C_*^{sing}(M,\mathbb{R})$ be the singular chain complex of M with coefficients in the real numbers $\mathbb{R}$. An element c in $C_p^{sing}(M,\mathbb{R})$ is given by a finite $\mathbb{R}$-linear combination $c = \sum_{i=1}^{s} r_i \cdot \sigma_i$ of singular p-simplices σ_i in M. Define the l^1-norm of c by setting

$$\| c \|_1 = \sum_{i=1}^{s} |r_i|$$

The *simplicial volume* of M is defined to be:

$$\| M \| = \inf \left\{ \| c \|_1 \mid c \in C_m^{sing}(M; \mathbb{R}) \text{ is a cocycle representing } [M] \right\}$$

where $[M]$ is the image of the fundamental class of M under the change of ring homomorphism on singular homology $H_m(M;\mathbb{Z}) \longrightarrow H_m(M;\mathbb{R})$. We mention the following extension of a conjecture of Gromov [16, page 154]:

Conjecture 3.2 Let M be a closed aspherical orientable manifold with vanishing simplicial volume. Then M is admissible and its combinatorial L^2-torsion is one. ■

This conjecture is based on a variety of calculations and similiarities of the properties of combinatorial L^2-torsion and simplicial volume. For example, suppose M is a closed aspherical orientable manifold with non-trivial S^1-action. Then its simplicial volume vanishes [15, Section 3.1]. The map induced by evaluation $\pi_1(S^1) \longrightarrow \pi_1(M)$ is injective [8]. Hence M is admissible and $\rho(M) = 1$ by Theorem 1.12. Conjecture 3.2 follows from Theorem 2.1 for a compact connected irreducible 3-manifold with infinite fundamental group whose boundary is empty or a disjoint union of incompressible tori, provided M has a JSJT-decomposition.

4. L^2-invariants for operators

In this section we introduce L^2-invariants for bounded π-equivariant operators between finitely generated Hilbert $\mathcal{N}(\pi)$-modules, namely, L^2-Betti numbers, Novikov-Shubin invariants and Fuglede-Kadison determinants. We investigate their basic properties. We show how their computations simplify if the operator comes from a matrix over the group ring $\mathbb{C}\pi$ with complex coefficients.

Let π be a countable group. Denote by $l^2(\pi)$ the complex Hilbert space of formal sums $\sum_{g\in\pi} \lambda_g \cdot g$ which are square summable, i.e., $\sum_{g\in\pi} \| \lambda_g \|^2$ converges to a real number. This is the Hilbert completion of the pre-Hilbert space $\mathbb{C}\pi$ with respect to the inner product:

$$\langle \sum_{w\in\pi} \lambda_w \cdot w, \sum_{w\in\pi} \mu_w \cdot w \rangle = \sum_{w\in\pi} \lambda_w \cdot \overline{\mu_w}$$

The *von Neumann algebra* $\mathcal{N}(\pi)$ is the algebra $B(l^2(\pi), l^2(\pi))^\pi$ of all bounded π-equivariant operators $l^2(\pi) \longrightarrow l^2(\pi)$. The *standard trace* on $\mathcal{N}(\pi)$ for $T \in \mathcal{N}(\pi)$ is given by

$$\mathrm{tr}(T) = \langle T(e), e \rangle_{l^2(\pi)}$$

where $e \in \pi$ is the unit element. A *Hilbert $\mathcal{N}(\pi)$-module* is a separable Hilbert space M together with a left π-action by unitary operators such that there exists an isometric π-equivariant embedding (which is not part of the structure) into $H \otimes l^2(\pi)$ for a separable Hilbert space H. We call M *finitely generated* if H can be choosen to be $\mathbb{C}^n$ for some positive integer n. Consider a bounded π-equivariant operator $T : \oplus_{i=1}^n l^2(\pi) \longrightarrow \oplus_{i=1}^n l^2(\pi)$. Define its *von Neumann trace* by:

$$\operatorname{tr}(T) = \sum_{i=1}^n \operatorname{tr}(T_{i,i})$$

The *von Neumann dimension*

$$\dim(M) \in \mathbb{R}^{\geq 0}$$

of a finitely generated Hilbert $\mathcal{N}(\pi)$-module M is the non-negative eal number $tr(pr)$ for any projection in $M(n,n,\mathcal{N}(\pi)) = B(\oplus_{i=1}^n l^2(\pi), \oplus_{i=1}^n l^2(\pi))^\pi$ whose image is isometrically π-isomorphic to M. Consider a bounded π-equivariant operator $T : M \longrightarrow N$ of finitely generated Hilbert $\mathcal{N}(\pi)$-modules. The operator $T^*T : M \longrightarrow M$ is a positive operator. Denote by $\{E_\lambda^{T^*T} \mid 0 \leq \lambda < \infty\}$ its right continuous spectral family. The *spectral density function* associated to T is the right continuous monotone increasing function

$$F(T) : \mathbb{R} \longrightarrow \mathbb{R}^{\geq 0} \qquad \lambda \mapsto \operatorname{tr}(E_\lambda^{T^*T})$$

The *Betti number* of T is:

$$b(T) = \dim(\ker(T)) = \dim(\ker(T^*T)) = F(0)$$

which is zero if and only if T is injective. The *Novikov-Shubin invariant* of T

$$\alpha(T) \in \mathbb{R}^{\geq 0} \cup \{\infty\}$$

is defined to be

$$\alpha(T) = \liminf_{\lambda \to 0+} \frac{\ln(F(\lambda) - F(0))}{\ln(\lambda)}$$

provided $F(\lambda) > b(T)$ holds for $\lambda > 0$. Otherwise put $\alpha(T) = \infty$. Roughly speaking, the L^2-Betti number measures the size of the kernel of T and the Novikov-Shubin invariant measures how fast the spectral density function appoaches for $\lambda \to 0+$ ts limit $F(0)$ and thus how concentrated the spectrum of T^*T near 0 is. (Notice that the Novikov-Shubin invariants defined here are two times the one defined in [22].) The *Fuglede-Kadison determinant*

$$\det(T) \in \mathbb{R}^{\geq 0}$$

is defined to be the positive real number

$$\det(T) = \exp\left(\frac{1}{2} \cdot \int_{0+}^{\infty} \ln(\lambda)\; dF\right)$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda)\; dF$ converges and to be 0 otherwise. Here dF is the measure satisfying for $a < b$:

$$dF(]a,b]) = F(b) - F(a)$$

and we use here and in the sequel the convention that $\int_a^b$, $\int_{a+}^b$, $\int_a^\infty$ respectively $\int_{a+}^\infty$ means integration over the interval $[a,b]$, $]a,b]$, $[a,\infty[$ respectively $]a,\infty[$. Notice that there is only a problem of convergence near zero where $\ln(\lambda)$ goes to $-\infty$ but not at ∞ because we have $F(\lambda) = F(\| R_A \|_\infty^2)$ for all $\lambda \geq \| R_A \|_\infty^2$. The next lemma explains why the definition above extends the well-known definition of the Fuglede-Kadison determinant for invertible T [14] and gives a criterion for the convergence of the Lebegues-Stieltjes integral.

Lemma 4.1 *Let $T : M \longrightarrow N$ be a bounded π-equivariant operator of finitely generated Hilbert $\mathcal{N}(\pi)$-modules.*

1.) *We have for $0 < \epsilon \leq a$:*

$$\int_\epsilon^a \ln(\lambda)\, dF = \int_\epsilon^a \frac{1}{\lambda} \cdot (F(\lambda) - F(0)) \cdot d\lambda \, + \ln(a) \cdot (F(a) - F(0)) - \ln(\epsilon) \cdot (F(\epsilon) - F(0))$$

$$\int_{0+}^a \ln(\lambda)\, dF = \lim_{\epsilon \to 0+} \int_\epsilon^a \ln(\lambda)\, dF$$

and

$$\int_{0+}^a \frac{1}{\lambda} \cdot (F(\lambda) - F(0)) \cdot d\lambda = \lim_{\epsilon \to 0+} \int_\epsilon^a \frac{1}{\lambda} \cdot (F(\lambda) - F(0)) \cdot d\lambda$$

2.) *If $\alpha(T) > 0$ and $a \geq \| T \|$, the integrals*

$$\int_{0+}^\infty \ln(\lambda)\, dF$$

and

$$\ln(a) \cdot (F(a) - F(0)) - \int_{0+}^a \frac{1}{\lambda} \cdot (F(\lambda) - F(0)) \cdot d\lambda$$

do converge to the same real number and we have $\det(T) > 0$.

3.) *If T is invertible, we get:*

$$\exp\left(\frac{1}{2} \cdot tr(\ln(T^*T))\right) = \det(T).$$

4.) *Let $T' : \ker(T)^\perp \longrightarrow N$ be the injective bounded π-equivariant operator induced by T. Then we get:*

$$\alpha(T) = \alpha(T')$$

and

$$\det(T) = \det(T').$$

Proof : 1.) The second and third equations follow from Levi's theorem of monotone convergence. The first one is a special case of the equality for f a continuously differentiable function on $\mathbb{R}^{>0}$ and $0 < \epsilon \le a$:

$$\int_\epsilon^a f(\lambda)\, dF = -\int_\epsilon^a f'(\lambda)\cdot F(\lambda)\cdot d\lambda \;+\; f(a)\cdot F(a) - f(\epsilon)\cdot F(\epsilon)$$

We give the elementary proof of this certainly known equation because we lack a good reference for its proof. Let δ be any positive real number. Since f, f', F and hence $f'\cdot F$ are Riemannian integrable, we can find a positive integer n such that for the partition $\epsilon = \lambda_0 < \lambda_1 < \ldots \lambda_n = a$ satisfying $\lambda_i - \lambda_{i-1} = \frac{1}{n}$ the following holds.

$$\begin{aligned}
\left|\int_\epsilon^a f'(\lambda)\cdot F(\lambda)\cdot d\lambda - \sum_{i=1}^n f'(\lambda_{i-1})\cdot F(\lambda_{i-1})\cdot(\lambda_i - \lambda_{i-1})\right| &\le \delta \\
\left|\int_\epsilon^a f(\lambda)\cdot dF - \sum_{i=1}^n f(\lambda_i)\cdot(F(\lambda_i) - F(\lambda_{i-1})\right| &\le \delta \\
|\, f'(\lambda_{i-1}) - f'(\xi_i)\,| &\le \delta
\end{aligned}$$

where here and in the sequel ξ_i is an element in $[\lambda_{i-1}, \lambda_i]$. Now we estimate:

$$\begin{aligned}
&\left|\int_\epsilon^a f(\lambda)\, dF + \int_\epsilon^a f'(\lambda)\cdot F(\lambda)\cdot d\lambda \;-\; f(a)\cdot F(a) + f(\epsilon)\cdot F(\epsilon)\right| \\
\le\;& 2\delta + \left|\sum_{i=1}^n f'(\lambda_{i-1})\cdot F(\lambda_{i-1})\cdot(\lambda_i - \lambda_{i-1}) + \sum_{i=1}^n f(\lambda_i)\cdot(F(\lambda_i) - F(\lambda_{i-1}))\right. \\
&\qquad\left. -\; f(a)\cdot F(a) + f(\epsilon)\cdot F(\epsilon)\right| \\
=\;& 2\delta + \left|\sum_{i=1}^n f'(\lambda_{i-1})\cdot F(\lambda_{i-1})\cdot(\lambda_i - \lambda_{i-1}) - \sum_{i=1}^n (f(\lambda_i) - f(\lambda_{i-1}))\cdot F(\lambda_{i-1})\right| \\
=\;& 2\delta + \left|\sum_{i=1}^n \left(f'(\lambda_{i-1}) - \frac{f(\lambda_i) - f(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}}\right)\cdot F(\lambda_{i-1})\cdot(\lambda_i - \lambda_{i-1})\right| \\
=\;& 2\delta + \left|\sum_{i=1}^n (f'(\lambda_{i-1}) - f'(\xi_i))\cdot F(\lambda_{i-1})\cdot(\lambda_i - \lambda_{i-1})\right| \\
\le\;& 2\delta + \sum_{i=1}^n \delta\cdot F(a)\cdot\frac{1}{n} \\
\le\;& (2 + F(a))\cdot\delta
\end{aligned}$$

Since $\delta > 0$ was arbitrary, assertion 1.) follows.

2.) Because of assertion 1.) it suffices to show:

$$\lim_{\epsilon\to 0+}\int_\epsilon^a \frac{1}{\lambda}\cdot(F(\lambda) - F(0))\cdot d\lambda \;< \infty$$

and

$$\lim_{\epsilon\to 0+} \ln(\epsilon)\cdot(F(\epsilon)-F(0)) = 0$$

Since $\alpha(T)$ is assumed to be positive, there is $0<\delta$ and $0<\alpha<\alpha(T)$ such that

$$F(\lambda)-F(0)\leq \lambda^{\alpha}$$

holds for $0\leq\lambda\leq\delta$. We get from l'Hospital's rule:

$$\lim_{\epsilon\to 0+} \ln(\epsilon)\cdot\epsilon^{\alpha} = 0$$

We have

$$\lim_{\epsilon\to 0+}\int_{\epsilon}^{a} \lambda^{\alpha-1} cdot d\lambda \ = \lim_{\epsilon\to 0+}\frac{1}{\alpha}\cdot(a^{\alpha}-\epsilon^{\alpha}) = \frac{1}{\alpha}\cdot a^{\alpha}$$

and assertion 2.) follows.

3.) Since the trace is linear and ultra-weakly continuous, we get:

$$\operatorname{tr}(\ln(T^*T)) = \operatorname{tr}\left(\int_{0+}^{\infty}\ln(\lambda)\ dE_{\lambda}\right) = \int_{0+}^{\infty}\ln(\lambda)\ d\left(\operatorname{tr}(E_{\lambda})\right) = \int_{0+}^{\infty}\ln(\lambda)\ dF$$

4.) If F' is the spectral density function of T', then we have:

$$F(\lambda) = F'(\lambda) + b(F)$$

and the claim follows. This finishes the proof of Lemma 4.1. ■

Lemma 4.1.3 shows that the L^2-Betti numbers and the Fuglede-Kadison determinant are infinite-dimensional analogues of classical notions. Namely, suppose that π is trivial. Let $T: M \longrightarrow N$ be a $\mathbb{C}$-linear map of finite-dimensional complex vector spaces. Then $b(T)$ is just the complex dimension of the kernel of T. Provided that T is injective, $\det(T)$ is the ordinary determinant of T^*T. The Novikov-Shubin invariant of T is ∞ in this case and is only relevant in the infinite-dimensional setting. The next lemma contains the basic properties of the Fuglede-Kadison determinant.

Lemma 4.2 *Let M be a finitely generated Hilbert $\mathcal{N}(\pi)$-module. Let S, T and R be bounded π-equivariant operators $M \longrightarrow M$. Suppose that S and T have trivial kernel. Then we get:*

1.) $\det(S) = \det(S^*) = \sqrt{\det(S^*S)} = \sqrt{\det(SS^*)}$.

2.) If $0\leq S$ (i.e., S is a positive operator), then:

$$\lim_{\epsilon\to 0+}\det(S+\epsilon) = \det(S).$$

3.) If $0 \le S \le T$*, then:*

$$\det(S) \le \det(T).$$

4.) $\det(ST) = \det(S) \cdot \det(T)$.

5.) $\det \begin{pmatrix} S & R \\ 0 & T \end{pmatrix} = \det(S) \cdot \det(T)$.

Proof : 1.) One easily checks for spectral density functions [22, Lemma 1.12.5]:

$$F^S(\lambda) = F^{S^*}(\lambda) = F^{S^*S}(\sqrt{\lambda}) = F^{SS^*}(\sqrt{\lambda})$$

and the claim follows.

2.) We have:

$$F^{\sqrt{S+\epsilon}}(\lambda) = F^{\sqrt{S}}(\lambda - \epsilon)$$

Since $F^{\sqrt{S}}(0) = 0$, we get:

$$\int_{0+}^{\infty} \ln(\lambda) dF^{\sqrt{S+\epsilon}} = \int_{(-\epsilon)+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}}$$

$$= \int_{-\epsilon+}^{0} \ln(\lambda+\epsilon) dF^{\sqrt{S}} + \int_{0+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}}$$

$$= \ln(\epsilon) \cdot F^{\sqrt{S}}(0) + \int_{0+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}} = \int_{0+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}}$$

We conclude from Levi's theorem of monotone convergence for small $b > 0$:

$$\lim_{\epsilon \to 0+} \int_{0+}^{b} \ln(\lambda+\epsilon) dF^{\sqrt{S}} = \int_{0+}^{b} \ln(\lambda) dF^{\sqrt{S}}$$

and from Lebesgues' theorem of majorized convergence:

$$\lim_{\epsilon \to 0+} \int_{b+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}} = \int_{b+}^{\infty} \ln(\lambda) dF^{\sqrt{S}}$$

We conclude:

$$\lim_{\epsilon \to 0+} \int_{0+}^{\infty} \ln(\lambda+\epsilon) dF^{\sqrt{S}} = \int_{0+}^{\infty} \ln(\lambda) dF^{\sqrt{S}}$$

This shows:

$$\lim_{\epsilon \to 0+} \det(\sqrt{S+\epsilon}) \;=\; \det(\sqrt{S})$$

Since we have $\det(S) = \det(\sqrt{S})^2$ by assertion 1), assertion 2.) follows.

3.) As $S+\epsilon$ is invertible for positive ϵ, we can assume by assertion 2.) without loss of generality that S and T are invertible. Since $\det(ST) = \det(S) \cdot \det(T)$ holds for for invertible T and S. Next we want to prove for positive S and positive T the claim:

$$\det(ST^2S) = \det(S)^2 \cdot \det(T)^2.$$

There is a positive constant C satisfying for small $\epsilon > 0$:

$$ST^2S \leq S(T+\epsilon)^2 S \leq ST^2S + C \cdot \epsilon$$

We conclude from assertion 2.) and 3.)

$$\det(ST^2S) = \lim_{\epsilon \to 0} (\det(S(T+\epsilon)^2 S)$$

Suppose S is invertible. Since then S and $T+\epsilon$ are invertible, we have:

$$\det(S(T+\epsilon)^2 S) = \det(S)^2 \det(T+\epsilon)^2$$

and the claim follows from assertion 2.) We get from assertion 1.) that $\det(ST^2S) = \det(TS^2T)$ holds. Hence we have shown the claim provided S and T are positive and one of the operators S and T is invertible. Now suppose that S and T are positive. Since $(T+\epsilon)$ is invertible, we conclude

$$\det(ST^2S) = \lim_{\epsilon \to 0+} \det(S(T+\epsilon)^2 S) = \lim_{\epsilon \to 0+} \det(S)^2 \det(T+\epsilon)^2 = \det(S) \cdot \det(T)$$

and the claim follows for positive S and T.

To prove assertion 4.) we use the polar decomposition $S = UA$ and $T = BV$ where U and V are unitary and A and B are positive. Since $\det(V^{-1}S^2V) = \det(S^2)$ holds for unitary V and invertible positive S, it is true for unitary V and positive S n general because of assertion 2.) We get using the assertions 1.) to 3.) proved above:

$$\det(ST) = \det(UABV) = \sqrt{\det((UABV)^*(UABV))} = \sqrt{\det(V^{-1}BA^2BV)}$$

$$= \sqrt{\det(BA^2B)} = \sqrt{\det(A)^2} \cdot \sqrt{\det(B)^2} = \sqrt{\det(S^*S) \cdot \det(TT^*)} = \det(S) \cdot \det(T)$$

5.) The claim is obvious if R is trivial. Because of assertion 4.) and the equation:

$$\begin{pmatrix} S & R \\ 0 & T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \cdot \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$$

it suffices to prove:

$$\det \begin{pmatrix} 1 & R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

Since this matrix is a commutator, the claim follows. This finishes the proof of Lemma 4.2.
■

In general it is very hard to compute these invariants. However, the operators appearing in geometry come from the cellular chain complexes of certain coverings and live therefore over the integral group ring. This simplifies the computations as explained below.

Let $A \in M(n,m,\mathbb{C}\pi)$ be a (n,m)-matrix over $\mathbb{C}\pi$. It induces by right multiplication a $\mathbb{C}\pi$-homomorphism of left $\mathbb{C}\pi$-modules

$$R_A : \oplus_{i=1}^{n}\mathbb{C}\pi \longrightarrow \oplus_{i=1}^{m}\mathbb{C}\pi \qquad\qquad x \mapsto xA$$

and by completion a bounded π-equivariant operator

$$R_A : \oplus_{i=1}^{n} l^2(\pi) \longrightarrow \oplus_{i=1}^{m} l^2(\pi)$$

both denoted by R_A. Notice for the sequel that $R_{AB} = R_B \circ R_A$ holds. We define an involution of rings on $\mathbb{C}\pi$ by

$$\overline{\sum_{w\in\pi} \lambda_w \cdot w} = \sum_{w\in\pi} \overline{\lambda_w} \cdot w^{-1}$$

Denote by A^* the (m,n)-matrix obtained from A by transposing and applying the involution above to each entry. As the notation suggests, the bounded π-equivariant operator R_{A^*} is the adjoint of the bounded π-equivariant operator R_A. Define the $\mathbb{C}\pi$-*trace* of an element $u = \sum_{w\in\pi} \lambda_w \cdot w \in \mathbb{C}\pi$ by

$$tr_{\mathbb{C}\pi}(u) = \lambda_e \in \mathbb{C}$$

for e the unit element in π. This extends to a square (n,n)-matrix A over $\mathbb{C}\pi$ by

$$tr_{\mathbb{C}\pi}(A) := \sum_{i=1}^{n} \mathrm{tr}_{\mathbb{C}\pi}(a_{i,i})$$

It follows directly from the definitions that the $\mathbb{C}\pi$-trace $tr_{\mathbb{C}\pi}(A)$ agrees with the von Neumann trace $tr(R_A)$ of the bounded π-equivariant operator R_A.

Let $A \in M(n,m,\mathbb{C}\pi)$ be a (n,m)-matrix over $\mathbb{C}\pi$. In the sequel let K be any positive real number satisfying

$$K \geq \parallel R_A \parallel_\infty$$

where $\parallel R_A \parallel_\infty$ is the operator norm of the bounded π-equivariant operator R_A. For $u = \sum_{w\in\pi} \lambda_w \cdot w \in \mathbb{C}\pi$ define $\parallel u \parallel_1$ by $\sum_{w\in\pi} \parallel \lambda_w \parallel$. Then a possible choice for K is given by:

$$K = \sqrt{m} \cdot \max\left\{\parallel a_{i,j} \parallel_1 \mid 1 \leq i \leq n, 1 \leq j \leq m\right\}$$

The bounded π-equivariant operator $1 - K^{-2} \cdot R_A^* R_A : \oplus_{i=1}^{n} l^2(\pi) \longrightarrow \oplus_{i=1}^{n} l^2(\pi)$ is positive. Let $(1 - K^{-2} \cdot A^*A)^p$ be the p-fold product of matrices and $(1 - K^{-2} \cdot R_A^* R_A)^p$ the p-fold composition of operators.

Definition 4.3 *The* characteristic sequence *of a matrix* $A \in M(n, m, \mathbb{C}\pi)$ *and a non-negative real number* K *satisfying* $K \geq \parallel R_A \parallel_\infty$ *is the sequence of real numbers*

$$c(A,K)_p := tr_{\mathbb{C}\pi}\left(\left(1 - K^{-2} \cdot AA^*\right)^p\right) = tr\left(\left(1 - K^{-2} \cdot R_A^* R_A\right)^p\right). \qquad \blacksquare$$

Next we want to prove:

Theorem 4.4 *Let* $A \in M(n, m, \mathbb{C}\pi)$ *be a* (n, m)*-matrix over* $\mathbb{C}\pi$*. Denote by* F *the spectral density function of* R_A*. Let* K *be a positive real number satisfying* $K \geq \parallel R_A \parallel_\infty$*. Then:*

1.) The characteristic sequence $c(A,K)_p$ *is a monotone decreasing sequence of non-negative real numbers.*

2.) We have:

$$b(R_A) = \lim_{p\to\infty} c(A,K)_p.$$

3.) Define $\beta(A) \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ *by*

$$\beta(A) := \sup\left\{\beta \in \mathbb{R}^{\geq 0} \mid \lim_{p\to\infty} p^\beta \cdot (c(A,K)_p - b(R_A)) = 0\right\}$$

Then we have:

$$\alpha(R_A) \leq \beta(A).$$

4.) Let K *be any positive real number satisfying* $K \geq \parallel R_A \parallel$*. Then the sum of positive real numbers*

$$\sum_{p=1}^{\infty} \frac{1}{p} \cdot (c(A,K)_p - b(R_A))$$

converges if and only if the integral

$$\int_{0+}^{\infty} \ln(\lambda)\, dF$$

does converge. If both converge then:

$$2 \cdot \ln(\det(R_A)) = 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot (c(A,K)_p - b(R_A)).$$

5.) Suppose $\alpha(R_A) > 0$*. Then* $det(R_A)$ *is a positive real number. Given a real number* α *satisfying* $0 < \alpha < \alpha(A)$*, there is a real number* C *such that we have for all* $L \geq 1$*:*

$$0 \leq c(A,K)_L - b(R_A) \leq \frac{C}{L^\alpha}$$

and

$$0 \leq -2 \cdot \ln(det(R_A)) + 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{p} \cdot (c(A,K)_p - b(R_A)) \leq \frac{C}{L^\alpha}$$

Proof : 1.) The bounded π-equivariant operator

$$1 - K^{-2} \cdot R_A^* R_A : \oplus_{i=1}^n l^2(\pi) \longrightarrow \oplus_{i=1}^n l^2(\pi)$$

is positive and satisfies:

$$0 \leq 1 - K^{-2} \cdot R_A^* R_A \leq 1$$

This implies for $0 \leq p \leq q$:

$$0 \leq \left(1 - K^{-2} \cdot R_A^* R_A\right)^q \leq \left(1 - K^{-2} \cdot R_A^* R_A\right)^p \leq 1$$

and the first assertion follows as the trace is monotone.

Before we can continue with the proof we need the following lemma.

Lemma 4.5 *If $F(\lambda)$ is the spectral density function of R_A or $A \in M(n,m,\mathbb{C}\pi)$ and K satisfies $K \geq \| R_A \|_\infty$, then we get for all $\lambda \in [0,1]$:*

$$(1-\lambda)^p \cdot \left(F(K^2 \cdot \lambda) - F(0)\right) \ \leq \ c(A,K)_p - b(R_A) \ \leq \ F(K^2 \cdot \lambda) - F(0) + (1-\lambda)^p \cdot (n - F(0))$$

Proof : We have for $\mu \in [0, \| R_A \|_\infty^2]$:

$$(1-\lambda)^p \cdot \chi_{[0,\lambda]}(K^{-2} \cdot \mu) \ \leq \ (1 - \ K^{-2} \cdot \mu)^p \ \leq \ \chi_{[0,\lambda]}(K^{-2} \cdot \mu) + (1-\lambda)^p$$

Hence we get by integrating over μ:

$$\begin{aligned}
\int_0^{\|R_A\|_\infty^2} (1-\lambda)^p \cdot \chi_{[0,\lambda]}(K^{-2} \cdot \mu) \, dF \ &\leq \ \int_0^{\|R_A\|_\infty^2} (1 - K^{-2} \cdot \mu)^p \, dF \\
&\leq \ \int_0^{\|R_A\|_\infty^2} \chi_{[0,\lambda]}(K^{-2} \cdot \mu) + (1-\lambda)^p \, dF
\end{aligned}$$

Now we have:

$$\begin{aligned}
\int_0^{\|R_A\|_\infty^2} (1-\lambda)^p \cdot \chi_{[0,\lambda]}(K^{-2} \cdot \mu) \, dF \ &= \ (1-\lambda)^p \cdot \left(F(K^2 \cdot \lambda) - F(0)\right) \\
\int_0^{\|R_A\|_\infty^2} (1 - K^{-2} \cdot \mu)^p \, dF \ &= \ tr\left(\left(1 - K^{-2} \cdot R_A^* R_A\right)^p\right) - \dim(\ker(R_A^* R_A)) \\
\int_0^{\|R_A\|_\infty^2} \chi_{[0,\lambda]}(K^{-2} \cdot \mu) + (1-\lambda)^p \, dF \ &= \ F(K^2 \cdot \lambda) - F(0) \\
&\quad + (1-\lambda)^p \cdot (F(\| R_A \|_\infty^2) - F(0))
\end{aligned}$$

This finishes the proof of Lemma 4.5. ■

2.) If we apply Lemma 4.5 to the value $\lambda = 1 - \sqrt[p]{\frac{1}{p}}$ we obtain for all positive integers p:

$$0 \leq c(A,K)_p - b(R_A) \leq F\left(K^2 \cdot \left(1 - \sqrt[p]{\frac{1}{p}}\right)\right) - F(0) + \frac{n - F(0)}{p}$$

We get $\lim_{x \to 0+} x \cdot \ln(x) = 0$ from l'Hospital's rule. This implies:

$$\lim_{p \to \infty} 1 - \sqrt[p]{\frac{1}{p}} = 0$$

Since the spectral density function is right continuous assertion 2.) of Theorem 4.4 follows.

3.) Let β and α be any real number satisfying

$$0 < \beta < \alpha < \alpha(R_A).$$

Choose a real number γ satisfying

$$\frac{\beta}{\alpha} < \gamma < 1.$$

We conclude from Lemma 4.5 for $\lambda = p^{-\gamma}$

$$0 \leq c(A,K)_p - b(R_A) \leq F(K^2 \cdot p^{-\gamma}) - F(0) + (1 - p^{-\gamma})^p \cdot (n - F(0))$$

By the definition of $\alpha(R_A)$ there is $\delta > 0$ such that we have for $0 < \lambda < \delta$:

$$F(\lambda) - F(0) \leq \lambda^\alpha$$

The last two inequalities imply for p satisfying $p^{-\gamma} < \delta$:

$$0 \leq p^\beta \cdot (c(A,K)_p - b(R_A)) \leq p^\beta \cdot \left((K^2 \cdot p^{-\gamma})^\alpha + (1 - p^{-\gamma})^p \cdot (n - F(0))\right)$$

We get using l'Hospital's rule:

$$\begin{aligned}
\lim_{x \to \infty} x \cdot \ln(1 - x^{-\gamma}) &= -\infty \\
\lim_{x \to \infty} \frac{\ln(x)}{x \ln(1 - x^{-\gamma})} &= 0 \\
\lim_{x \to \infty} \left(\frac{\beta \ln(x)}{x \ln(1 - x^{-\gamma})} + 1\right) &= 1 \\
\lim_{x \to \infty} \beta \ln(x) + x \ln(1 - x^{-\gamma}) &= -\infty \\
\lim_{x \to \infty} x^\beta (1 - x^{-\gamma})^x &= 0
\end{aligned}$$

Since $\beta - \gamma\alpha < 0$ holds we have:

$$\lim_{x \to \infty} (K^2 \cdot x)^{\beta - \gamma\alpha} = 0$$

Hence we get:

$$\lim_{p\to\infty} p^{\beta} \cdot \big((K^2 \cdot p^{-\gamma})^{\alpha} + (1-p^{-\gamma})^p \cdot (n - F(0))\big) = 0$$

This implies using the inequality above:

$$\lim_{p\to\infty} p^{\beta} \cdot (c(A,K)_p - b(R_A)) = 0$$

We have shown $\beta \leq \beta(A)$. Since β was an arbitrary number satisfying $0 < \beta < \alpha(R_A)$, assertion 3.) of Theorem 4.4 follows.

4.) We get the following chain of equations where a sum or integral is put to be $-\infty$ if and only if it does not converge.

$$\begin{aligned}
& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot (c(A,K)_p - b(R_A)) \\
= \;& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot \big(\mathrm{tr}_{\mathbb{C}\pi}\left(\left(1 - K^{-2} \cdot AA^*\right)^p\right) - b(R_A)\big) \\
= \;& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot \big(\mathrm{tr}\left(\left(1 - K^{-2} \cdot R_A^* R_A\right)^p\right) - b(R_A)\big) \\
= \;& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot \mathrm{tr}\left(\int_0^{\|R_A\|_\infty^2} \left(1 - K^{-2} \cdot \lambda\right)^p \, dE_\lambda\right)
\end{aligned}$$

Since the trace is linear, monotone and ultra-weakly continuous, we get further:

$$\begin{aligned}
= \;& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot \int_0^{\|R_A\|_\infty^2} \left(1 - K^{-2} \cdot \lambda\right)^p \, d\mathrm{tr}(E_\lambda) \\
= \;& 2 \cdot (n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{\infty} \frac{1}{p} \cdot \int_{0+}^{\|R_A\|_\infty^2} \left(1 - K^{-2} \cdot \lambda\right)^p \, dF
\end{aligned}$$

We can put the sum under the integral sign because of Levi's theorem of monotone convergence since $(1 - K^{-2} \cdot \lambda)^p$ is non-negative for $0 < \lambda \leq \| R_A \|_\infty^2 \leq K^2$:

$$= 2 \cdot (n - b(R_A)) \cdot \ln(K) - \int_{0+}^{\|R_A\|_\infty^2} \sum_{p=1}^{\infty} \frac{1}{p} \cdot \left(1 - K^{-2} \cdot \lambda\right)^p \, dF$$

The Taylor series $-\sum_{p=1}^{\infty} \frac{1}{p} \cdot (1-\mu)^p$ of $\ln(\mu)$ about 1 converges for $|1-\mu| < 1$.

$$
\begin{aligned}
&= 2 \cdot (n - b(R_A)) \cdot \ln(K) +_{0+}^{\|R_A\|_\infty^2} \ln(K^{-2} \cdot \lambda)\ dF \\
&= 2 \cdot (n - b(R_A)) \cdot \ln(K) + \int_{0+}^{\|R_A\|_\infty^2} \ln(\lambda)\ dF - \int_{0+}^{\|A\|_\infty^2} \ln(K^2)\ dF \\
&= 2 \cdot (n - b(R_A)) \cdot \ln(K) + \int_{0+}^{\|R_A\|_\infty^2} \ln(\lambda)\ dF - \ln(K^2) \cdot (F(\| A \|_\infty^2) - F(0)) \\
&= 2 \cdot (n - b(R_A)) \cdot \ln(K) + \int_{0+}^{\|R_A\|_\infty^2} \ln(\lambda)\ dF - \ln(K^2) \cdot (n - b(R_A)) \\
&= \int_{0+}^{\|R_A\|_\infty^2} \ln(\lambda)\ dF \\
&= \int_{0+}^{\infty} \ln(\lambda)\ dF
\end{aligned}
$$

5.) Let α be any number satisfying $\alpha < \alpha(R_A)$. Then we conclude from assertion 3.)

$$\lim_{p \to \infty} p^\alpha \left(c(A,K)_p - b(R_A)\right) = 0.$$

Let C be any positive number such that for all p:

$$p^\alpha \left(c(A,K)_p - b(R_A)\right) \le C$$

We conclude:

$$0 \le c(A,K)_p - b(R_A) \le \frac{C}{p^\alpha}$$

Next we estimate

$$
\begin{aligned}
0 \le -2 \cdot \ln(det(A)) + 2(n - b(R_A)) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{p} \cdot (c(A,K)_p - b(R_A)) &= \\
\sum_{p=L+1}^{\infty} \frac{1}{p} \cdot (c(A,K)_p - b(R_A)) &= \\
\sum_{p=L+1}^{\infty} p^{-1-\alpha} \cdot (p^\alpha \cdot (c(A,K)_p - b(R_A))) &\le \\
C \cdot \sum_{p=L+1}^{\infty} p^{-1-\alpha} &\le \\
C \cdot \int_L^\infty x^{-1-\alpha} dx &= \frac{C}{\alpha} \cdot L^{-\alpha}
\end{aligned}
$$

This finishes the proof of Theorem 4.4. ■

Remark 4.6 We conjecture that the inequality $\alpha(R_A) \leq \beta(A)$ of Theorem 4.4.3 is a equality. Define a number

$$\overline{\beta}(A) := \inf \left\{ \beta \in \mathbb{R}^{\geq 0} \mid \lim_{p \to \infty} p^{\beta} \cdot (c(A,K)_p - b(R_A)) = \infty \right\}$$

If F denotes the spectral density function of R_A, define the number $\overline{\alpha}(R_A)$ to be

$$\overline{\alpha}(R_A) = \limsup_{\lambda \to 0+} \frac{\ln(F(\lambda) - F(0))}{\ln(\lambda)}$$

provided $F(\lambda) > b(R_A)$ holds for $\lambda > 0$. Otherwise put $\overline{\alpha}(R_A) = \infty$. Then one can check analogously as in the proof of Theorem 4.4.3 that

$$\alpha(R_A) \leq \beta(A) \leq \overline{\beta}(A) \leq \overline{\alpha}(R_A)$$

holds. We conjecture that all these four numbers are the same for a (n,m)-matrix A over $\mathbb{C}\pi$. Notice hat there are no known counterexamples to the equality of α and $\overline{\alpha}$ if one considers the large time behaviour of the heat kernel of the universal covering of a compact Riemannian manifold (see [17]). em ■

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