Lectures on L^2-Betti numbers

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1. Introduction to L^2-Betti Numbers

Let G be a discrete group.

Definition 1.1 *Denote by $\mathbb{Z}G$, $\mathbb{Q}G$ and $\mathbb{C}G$ the integral, rational and complex group ring. An element in $\mathbb{C}G$ is a (formal) sum $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\lambda_g \neq 0$ for only finitely many elements $g \in G$.*

Denote by $l^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$. This is the Hilbert space completion of $\mathbb{C}G$ with respect to the inner product for which G is an orthonormal basis.

Define the **group von Neumann algebra**

$$\mathcal{N}(G) \;:=\; \mathcal{B}(l^2(G))^G$$

to be the algebra of bounded G-equivariant operators $l^2(G) \to l^2(G)$.

The **von Neumann trace** *is defined by*

$$\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{l^2(G)}.$$

Example 1.2 If G is finite, then $\mathbb{C}G = l^2(G) = \mathcal{N}(G)$. The trace $\mathrm{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

Example 1.3 Let G be $\mathbb{Z}^n$. Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$. Let $L^\infty(T^n)$ be the Banach space of essentially bounded functions $f : T^n \to \mathbb{C} \coprod \{\infty\}$. An element $(k_1, \ldots, k_n)$ in $\mathbb{Z}^n$ acts isometrically on $L^2(T^n)$ by pointwise multiplication with the function $T^n \to \mathbb{C}$ which maps $(z_1, z_2, \ldots, z_n)$ to $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism $l^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n)$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n) = \mathcal{B}(L^2(T^n))^{\mathbb{Z}^n}$$

by sending $f \in L^\infty(T^n)$ to the $\mathbb{Z}^n$-operator $M_f : L^2(T^n) \to L^2(T^n) \quad g \mapsto g \cdot f$ Under this identification the trace becomes

$$\mathrm{tr}_{\mathcal{N}(\mathbb{Z}^n)} : L^\infty(T^n) \to \mathbb{C} \qquad f \mapsto \int_{T^n} f d\mu.$$

Definition 1.4 *A* **Hilbert $\mathcal{N}(G)$-module** *V is a Hilbert space V together with a linear isometric G-action such that there exists a Hilbert space H and an isometric linear G-embedding of V into the tensor product of Hilbert spaces $H \otimes l^2(G)$ with the obvious G-action.*

A **map of Hilbert $\mathcal{N}(G)$-modules** *$f : V \to W$ is a bounded G-equivariant operator.*

We call a Hilbert $\mathcal{N}(G)$-module V **finitely generated** *if there is a non-negative integer n and a surjective map $\oplus_{i=1}^n l^2(G) \to V$ of Hilbert $\mathcal{N}(G)$-modules.*

Definition 1.5 *Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p : l^2(G)^n \to l^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$. Define the* **von Neumann dimension** *of V by*

$$\begin{aligned}\dim_{\mathcal{N}(G)}(V) &:= \operatorname{tr}_{\mathcal{N}(G)}(p) \\ &:= \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \ \in [0, \infty).\end{aligned}$$

This notion extends to arbitrary Hilbert $\mathcal{N}(G)$-modules if we allow the value ∞.

Definition 1.6 *We call a sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$* **weakly exact** *at V if the kernel* $\ker(p)$ *of p and the closure* $\operatorname{clos}(\operatorname{im}(i))$ *of the image* $\operatorname{im}(i)$ *of i agree.*

A map of Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a **weak isomorphism** *if it is injective and has dense image.*

Example 1.7 The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad u \mapsto (z-1) \cdot u$$

is a weak isomorphism but not an isomorphism.

Theorem 1.8 *1. We have for a Hilbert $\mathcal{N}(G)$-module V*

$$V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0;$$

2. If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of Hilbert $\mathcal{N}(G)$-modules, then

$$\begin{aligned} \dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) & \\ = \ & \dim_{\mathcal{N}(G)}(V); \end{aligned}$$

3. Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of V, directed by $\subset$. Then

$$\begin{aligned} \dim_{\mathcal{N}(G)}(\operatorname{clos}(\cup_{i\in I} V_i)) & \\ = \ & \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\}; \end{aligned}$$

Example 1.9 If G is finite, then a finitely generated Hilbert $\mathcal{N}(G)$-module is the same a unitary G-representation and

$$\dim_{\mathcal{N}(G)}(V) \ = \ \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example 1.10 Let G be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

Definition 1.11 *A G-CW-**complex** X is a G-space with a G-invariant filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \cup_{n \geq 0} X_n = X$$

such that X carries the colimit topology and X_n is obtained from X_{n-1} by attaching equivariant n-dimensional cells, i.e. there exists a G-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i \in I_n} Q_i"'] & X_n
\end{tikzcd}$$

We call X **finite** *if it is built by finitely many equivariant cells. We call X* **of finite type** *if each skeleton X_n is finite.*

- A G-CW-complex is finite if and only if $G\backslash X$ is compact;

- It is a proper G-space if and only if each isotropy group is finite;

- Let X be a simplicial complex with simplicial G-action. Then its barycentric division X' carries a G-CW-structure;

- If the smooth manifold M carries a smooth proper cocompact group action G, then it admits an equivariant triangulation and hence a G-CW-structure;

- If $X \to Y$ is a regular G-covering, then a CW-structure on Y induces a G-CW-structure on X and vice versa;

Definition 1.12 *Let X be a free G-CW-complex of finite type. Denote by $C_*(X)$ its* **cellular $\mathbb{Z}$-chain complex**. *Define its* **cellular L^2-chain complex** $C_*^{(2)}(X)$ *to be the Hilbert $\mathcal{N}(G)$-chain complex*

$$C_*^{(2)}(X) \ := \ l^2(G) \otimes_{\mathbb{Z}G} C_*(X).$$

Define its **p-th L^2-homology** *to be the finitely generated Hilbert $\mathcal{N}(G)$-module*

$$H_p^{(2)}(X;\mathcal{N}(G)) := \ker(C_p^{(2)})/\overline{\operatorname{im}(c_{p+1}^{(2)})}.$$

Define its **p-th L^2-Betti number**

$$b_p^{(2)}(X;\mathcal{N}(G)) \ = \ \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(X;\mathcal{N}(G))\right).$$

Notice that $C_p(X) = \oplus_{I_p}\mathbb{Z}G$. Hence $C_p^{(2)}(X) = \oplus_{I_p} l^2(G)$. Each differential $c_p^{(2)}$ is a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules since each I_p is finite by assumption.

The p-**th Laplace operator** is defined by

$$\Delta_p := c_{p+1}^{(2)} \circ (c_{p+1}^{(2)})^* + (c_p^{(2)})^* \circ c_p^{(2)} \colon C_p^{(2)}(X) \to C_p^{(2)}(X).$$

Each $H_p^{(2)}(X;\mathcal{N}(G))$ carries the structure of a finitely generated Hilbert $\mathcal{N}(G)$-module since the natural map

$$\begin{aligned}\ker(\Delta_p) = \ker(c_p^{(2)}) \cap \ker(c_{p+1}^{(2)}) \\ \xrightarrow{\cong} H_p^{(2)}(X;\mathcal{N}(G))\end{aligned}$$

is an isometric G-equivariant isomorphism and $\ker(\Delta_p) \subset C_p^{(2)}(X) = \oplus_{I_p} l^2(G)$.

Theorem 1.13 (Cellular L^2-Betti numbers).

1. *Homotopy invariance*
   Let $f : X \to Y$ be a G-map of free G-CW-complexes of finite type. If f is a weak homotopy equivalence (after forgetting the G-action), then
   $$b_p^{(2)}(X) \;=\; b_p^{(2)}(Y);$$

2. *Euler-Poincaré formula (Atiyah)*

   Let X be free finite G-CW-complex. Let $\chi(G\backslash X)$ be the Euler characteristic of the finite CW-complex. Then

$$\chi(G\backslash X) \;=\; \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(X);$$

3. *Poincaré duality*

   Let M be a cocompact free proper G-manifold of dimension n which is orientable. Then

$$b_p^{(2)}(M) \;=\; b_{n-p}^{(2)}(M, \partial M);$$

4. *Künneth formula (Zucker)*

   Let X be a free G-CW-complex of finite type and Y be a free H-CW-complex of finite type. Then $X \times Y$ is a free $G \times H$-CW-complex of finite type and we get for all $n \geq 0$

$$b_n^{(2)}(X \times Y) \;=\; \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y);$$

5. *Wedges*

Let X_1, X_2, *. . . ,* X_r *be connected* CW*-complexes of finite type and* $X = \vee_{i=1}^r X_i$ *be their wedge. Then*

$$
\begin{aligned}
& b_1^{(2)}(\widetilde{X}) - b_0^{(2)}(\widetilde{X}) \\
&= \; r - 1 + \sum_{j=1}^{r} \left(b_1^{(2)}(\widetilde{X_j}) - b_0^{(2)}(\widetilde{X_j}) \right);
\end{aligned}
$$

and for $2 \leq p$

$$
b_p^{(2)}(\widetilde{X}) \; = \; \sum_{j=1}^{r} b_p^{(2)}(\widetilde{X_j});
$$

6. *Morse inequalities (Novikov-Shubin)*

Let X *be a free* G*-*CW*-complex of finite type. Let* $\beta_p(G\backslash X)$ *be the number of* p*-cells in* $G\backslash X$*. Then we get for* $n \geq 0$

$$
\begin{aligned}
& \sum_{p=0}^{n} (-1)^{n-p} \cdot b_p^{(2)}(X) \\
& \qquad \leq \; \sum_{p=0}^{n} (-1)^{n-p} \cdot \beta_p(G\backslash X);
\end{aligned}
$$

7. *Zero-th L^2-Betti number*

   Let X be a connected free G-CW-complex of finite type. Then

$$b_0^{(2)}(X) \;=\; \frac{1}{|G|},$$

   where $\frac{1}{|G|}$ is to be understood as zero if the order $|G|$ of G is infinite;

8. *Restriction*

   Let X be a free G-CW-complex of finite type and let $H \subset G$ be a subgroup of finite index $[G:H]$. Then $\operatorname{res}_G^H X$ is a free H-CW-complex of finite type and

$$[G:H] \cdot b_p^{(2)}(X;\mathcal{N}(G)) \\ \;=\; b_p^{(2)}(\operatorname{res}_G^H X;\mathcal{N}(H));$$

9. *Induction*

   Let H be a subgroup of G and let X be a free H-CW-complex of finite type. Then $G \times_H X$ is a G-CW-complex of finite type and

$$b_p^{(2)}(G \times_H X;\mathcal{N}(G)) \;=\; b_p^{(2)}(X;\mathcal{N}(H)).$$

Example 1.14 If G is finite and X is a free G-CW-complex of finite type, then $b_p^{(2)}(X)$ is the classical p-th Betti number of X multiplied with $\frac{1}{|G|}$.

Lemma 1.15 *Let X be a free $\mathbb{Z}^n$-CW-complex of finite type. Then*

$$b_p^{(2)}(X) = \dim_{\mathbb{C}[\mathbb{Z}^n]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^n]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} H_p(X)\right),$$

where $\mathbb{C}[\mathbb{Z}^n]^{(0)}$ is the quotient field of $\mathbb{C}[\mathbb{Z}^n]$.

Example 1.16 Let $X \to Y$ be a finite covering with d-sheets of connected CW-complexes of finite type. Then Theorem 1.13 (8) implies

$$b_p^{(2)}(\widetilde{Y}) = d \cdot b_p^{(2)}(\widetilde{X}).$$

In particular we get for a connected CW-complex X of finite type for which there is a selfcovering $X \to X$ with d-sheets for some integer $d \geq 2$ that $b_p^{(2)}(\widetilde{X}) = 0$ for all $p \geq 0$. This implies for any finite CW-complex X of finite type

$$b_p^{(2)}(\widetilde{S^1 \times X}) = 0.$$

Theorem 1.17 (Long weakly exact L^2-homology sequence, Cheeger-Gromov).

Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ *be an exact sequence of Hilbert* $\mathcal{N}(G)$*-chain complexes whose chain modules have finite dimension. Then there is a long weakly exact homology sequence*

$$\ldots \xrightarrow{H_{n+1}^{(2)}(p_*)} H_{n+1}^{(2)}(E_*) \xrightarrow{\partial_{n+1}} H_n^{(2)}(C_*)$$
$$\xrightarrow{H_n^{(2)}(i_*)} H_n^{(2)}(D_*) \xrightarrow{H_n^{(2)}(p_*)} H_n^{(2)}(E_*) \xrightarrow{\partial_n} \ldots$$

Theorem 1.18 (L^2-Betti numbers and S^1-actions).

Let X *be a connected* S^1*-*CW*-complex of finite type, for instance a connected compact manifold with* S^1*-action. Suppose that for one orbit* S^1/H *(and hence for all orbits) the inclusion into* X *induces a map on* π_1 *with infinite image. (In particular the* S^1*-action has no fixed points.) Then we get for all* $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) = 0.$$

Theorem 1.19 (L^2-Betti numbers and aspherical S^1-manifolds).

Let M be an aspherical closed manifold with non-trivial S^1-action. Then the action has no fixed points and the inclusion of any orbit into X induces an injection on the fundamental groups. All L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ are trivial and $\chi(M) = 0$.

Example 1.20 Let F_g be the orientable closed surface. Since $F_0 = S^2$ is simply-connected, we get

$$\begin{aligned} b_p^{(2)}(\widetilde{F_0}) = b_p(S^2) &= 1 \text{ if } p = 0, 2. \\ b_p^{(2)}(\widetilde{F_0}) = b_p(S^2) &= 0 \text{ if } p \neq 0, 2; \end{aligned}$$

If $g \geq 0$, then $\pi_1(F_g)$ is infinite and hence $b_0^{(2)}(\widetilde{F_g}) = 0$. By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$. Since $\dim(F_g) = 2$, we get $b_p^{(2)}(\widetilde{F_g}) = 0$ for $p \geq 3$. Using the Euler-Poincaré formula we get

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\ b_p^{(2)}(\widetilde{F_0}) &= 0 \text{ for } p \neq 1. \end{aligned}$$

Theorem 1.21 (L^2-Betti numbers of 3-manifolds, Lott-Lück).

Let M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j which are non-exceptional. Assume that $\pi_1(M)$ is infinite. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ are given by

$$\begin{aligned}
b_0^{(2)}(\widetilde{M}) &= 0; \\
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\quad + \left| \{C \in \pi_0(\partial M) \mid C \cong S^2\} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\quad + \left| \{C \in \pi_0(\partial M) \mid C \cong S^2\} \right|; \\
b_3^{(2)}(\widetilde{M}) &= 0.
\end{aligned}$$

Theorem 1.22 L^2**-Hodge-de Rham Theorem, Dodziuk).**

Let M be a cocompact free proper G-manifold with G-invariant Riemannian metric and K an equivariant smooth triangulation of M. Suppose that M has no boundary. Let

$$\mathcal{H}^p_{(2)}(M) = \{\omega \in \Omega^p(\widetilde{M}) \mid \Delta_p(M) = 0\}$$

be the space of harmonic L^2-forms on $\widetilde{M}$. Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\mathcal{H}^p_{(2)}(M) \xrightarrow{\cong} H^p_{(2)}(K).$$

Corollary 1.23

$$b_p^{(2)}(M) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\ dvol.$$

where $\mathcal{F}$ is a fundamental domain for the G-action and $e^{-t\Delta_p}(x,y)$ is the heat kernel on $\widetilde{M}$.

Theorem 1.24 *Let M be a hyperbolic closed Riemannian manifold of dimension n. Then:*

$$b_p^{(2)}(\widetilde{M}) \quad \begin{cases} = 0 & \text{, if } 2p \neq n \\ > 0 & \text{, if } 2p = n \end{cases}.$$

Proof: A direct computation shows that $\mathcal{H}^p_{(2)}(\mathcal{H}^n)$ is not zero if and only if $2p = n$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathcal{H}^n$.

Theorem 1.25 *Let M be a hyperbolic closed manifold of dimension n. Then*

1. *If $n = 2m$ is even, then*
$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Proof: (1) We get from the Euler-Poincaré formula and Theorem 1.24

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $n = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathcal{H}^n$ is contractible, M is aspherical. Now apply Theorem 1.19.

Theorem 1.26 Vanishing of L^2-Betti numbers of mapping tori).

Let $f : X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{T_f}) \;=\; 0.$$

Proof: There is a d-sheeted covering $T_{f^d} \to T_f$. Hence

$$b_p^{(2)}(\widetilde{T_f}) \;=\; \frac{b_p^{(2)}(\widetilde{T_{f^d}})}{d}.$$

If $\beta_p(X)$ is the number of p-cells, then there is up to homotopy equivalence a CW-structure on T_{f^d} with $\beta(T_{f^d}) = \beta_p(X) + \beta_{p-1}(X)$. We have

$$\begin{aligned} b_p^{(2)}(\widetilde{T_{f^d}}) \;&=\; \dim_{\mathcal{N}(G)}(H_p^{(2)}(C_p^{(2)}(\widetilde{T_{f^d}})) \\ &\leq\; \dim_{\mathcal{N}(G)}\left(C_p^{(2)}(\widetilde{T_{f^d}})\right) \;=\; \beta_p(T_{f^d}). \end{aligned}$$

This implies for all $d \geq 1$

$$b_p^{(2)}(\widetilde{T_f}) \;\leq\; \frac{\beta_p(X) + \beta_{p-1}(X)}{d}.$$

Taking the limit for $d \to \infty$ yields the claim.

Example 1.27 The following examples show that in general there are hardly any relations between the ordinary Betti numbers $b_p(X)$ and the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type.

Given a group G such that BG is of finite type, define its p-th L^2-Betti number and its p-th Betti number by

$$\begin{aligned} b_p^{(2)}(G) &:= b_p^{(2)}(EG;\mathcal{N}(G)); \\ b_p(G) &:= b_p(BG). \end{aligned}$$

We get from Theorem 1.13 (4), (5) and (8) for $r \geq 2$ and non-trivial groups G_1, G_2, ..., G_r whose classifying spaces BG_i

are of finite type

$$\begin{aligned}
b_1^{(2)}(*_{i=1}^r G_i) &= r - 1 + \sum_{i=1}^r \left(b_1^{(2)}(G_i) - \frac{1}{|G_i|}\right); \\
b_0^{(2)}(*_{i=1}^r G_i) &= 0; \\
b_p^{(2)}(*_{i=1}^r G_i) &= \sum_{i=1}^r b_p^{(2)}(G_i) \quad \text{for } p \geq 2; \\
b_p(*_{i=1}^r G_i) &= \sum_{i=1}^r b_p(G_i) \quad \text{for } p \geq 1; \\
b_0^{(2)}(\mathbb{Z}/n) &= \frac{1}{n}; \\
b_p^{(2)}(\mathbb{Z}/n) &= 0 \quad \text{for } p \geq 1; \\
b_p(\mathbb{Z}/n) &= 0 \quad \text{for } p \geq 1; \\
b_p^{(2)}(G_1 \times G_2) &= \sum_{i=0}^p b_i^{(2)}(G_1) \cdot b_{p-i}^{(2)}(G_2); \\
b_p(G_1 \times G_2) &= \sum_{i=0}^p b_i(G_1) \cdot b_{p-i}(G_2).
\end{aligned}$$

From this one easily verifies for any integers $m \geq 0$, $n \geq 1$ and $i \geq 1$ that for the group

$$G_i(m,n) = \mathbb{Z}/n \times \left(*_{k=1}^{2m+2} \mathbb{Z}/2\right) \times \left(\prod_{j=1}^{i-1} *_{l=1}^4 \mathbb{Z}/2\right)$$

its classifying space $BG_i(m,n)$ is of finite type and

$$\begin{aligned} b_i^{(2)}(G_i(m,n)) &= \frac{m}{n}; \\ b_p^{(2)}(G_i(m,n)) &= 0 \qquad \text{for } p \neq i; \\ b_p(G_i(m,n)) &= 0 \qquad \text{for } p \geq 1. \end{aligned}$$

Given an integer $l \geq 1$ and a sequence r_1, r_2, ..., r_l of non-negative rational numbers, we can construct a group G such that BG is of finite type and

$$\begin{aligned} b_p^{(2)}(G) &= r_p \qquad \text{for } 1 \leq p \leq l; \\ b_p^{(2)}(G) &= 0 \qquad \text{for } l+1 \leq p; \\ b_p(G) &= 0 \qquad \text{for } p \geq 1. \end{aligned}$$

Namely, take

$$G = \mathbb{Z}/n \times *_{i=2}^{k} G_i(m_i, n_i).$$

On the other hand we can construct for any sequence n_1, n_2, ... of non-negative integers a CW-complex X of finite type such that $b_p(X) = n_p$ and $b_p^{(2)}(\widetilde{X}) = 0$ holds for $p \geq 1$, namely take

$$X = B(\mathbb{Z}/2 * \mathbb{Z}/2) \times \vee_{p=1}^{\infty} \left(\vee_{i=1}^{n_p} S^p \right).$$

Theorem 1.28 Proportionality Principle for L^2-invariants

Let M be a simply connected Riemannian manifold. Then there are constants $B_p^{(2)}(M)$ for $p \geq 0$ depending only on the Riemannian manifold M such that for any discrete group G with a free proper cocompact action on M by isometries the following holds

$$b_p^{(2)}(M; \mathcal{N}(G)) = B_p^{(2)}(M) \cdot \mathrm{vol}(G \backslash M).$$

2. The Generalized Dimension function

Remark 2.1 Recall that by definition

$$\begin{aligned}\mathcal{N}(G) &:= \mathcal{B}(l^2(G), l^2(G))^G \\ &= \operatorname{mor}_{\mathcal{N}(G)}(l^2(G), l^2(G)).\end{aligned}$$

This induces a bijection of $\mathbb{C}$-vector spaces

$$M(m, n, \mathcal{N}(G)) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{N}(G)}(l^2(G)^m, l^2(G)^n).$$

It is compatible with multiplication of matrices and composition of morphisms. This extends to finitely generated Hilbert $\mathcal{N}(G)$-modules and finitely projective $\mathcal{N}(G)$-modules.

Theorem 2.2 (Modules over $\mathcal{N}(G)$ and Hilbert $\mathcal{N}(G)$-modules).
We obtain an equivalence of $\mathbb{C}$-categories

$$\begin{aligned}\nu\colon &\{\textit{fin. gen. proj. } \mathcal{N}(G)\textit{-mod.}\} \\ &\to \{\textit{fin. gen. Hilb. } \mathcal{N}(G)\textit{-mod.}\}.\end{aligned}$$

Definition 2.3 *Let R be a ring. Let M be a R-submodule of N. Define the* **closure** *of M in N to be the R-submodule of N*

$$\begin{aligned}\overline{M} &= \{x \in N \mid f(x) = 0 \text{ for all} \\ &\qquad f \in N^* \text{ with } M \subset \ker(f)\}.\end{aligned}$$

For a R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\begin{aligned}\mathbf{T}M &:= \{x \in M \mid f(x) = 0 \\ &\qquad \text{for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M.\end{aligned}$$

We call a sequence of R-modules $L \xrightarrow{i} M \xrightarrow{q} N$ **weakly exact** *if $\overline{\operatorname{im}(i)} = \ker(q)$.*

Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It can also be described as the kernel of the canonical map

$$i(M) : M \to (M^*)^*$$

which sends $x \in M$ to the map $M^* \to R\ f \mapsto f(x)^*$. Notice that $\mathbf{TP}M = 0$ and that $\mathbf{P}M = 0$ is equivalent to $M^* = 0$.

Example 2.4 *Let $R = \mathbb{Z}$. Let M be a finitely generated $\mathbb{Z}$-module and $K \subset M$. Then*

$$\begin{aligned} \overline{K} &= \{x \in M \mid n \cdot x \in K \text{ for some } n \in \mathbb{Z}\}; \\ \mathbf{T}M &:= \operatorname{tors}(M); \\ \mathbf{P}M &= M/\operatorname{tors}(M). \end{aligned}$$

A sequence $M_0 \to M_1 \to M_2$ of finitely generated $\mathbb{Z}$-modules is weakly exact if and only if it is exact after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$.

Definition 2.5 *Let P be a finitely generated projective $\mathcal{N}(G)$-module. Choose a matrix $A \in M_n(\mathcal{N}(G))$ with $A^2 = A$ such that the image of $r_A : \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ is $\mathcal{N}(G)$-isomorphic to P. Define*

$$\dim_{\mathcal{N}(G)}(P) := \operatorname{tr}_{\mathcal{N}(G)}(A) \quad [0, \infty).$$

Theorem 2.6 *1. The functors ν and ν^{-1} preserve exact sequences and weakly exact sequences;*

2. If P is a finitely generated projective $\mathcal{N}(G)$-module, then

$$\dim_{\mathcal{N}(G)}(P) = \dim_{\mathcal{N}(G)}(\nu(P)).$$

Remark 2.7 $\mathcal{N}(G)$ is Noetherian if and only if G is finite. It contains zero-divisors if G is non-trivial.

Definition 2.8 *A ring R is called* **semi-hereditary** *if any finitely generated sub-module of a projective module is projective.*

Lemma 2.9 $\mathcal{N}(G)$ *is semihereditary.*

Proof: It suffices to prove for a finitely generated $\mathcal{N}(G)$-submodule $M \subset \mathcal{N}(G)^n$ that it is projective. Choose a $\mathcal{N}(G)$-map $f : \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ whose image is M. Let $\nu(f) : l^2(G)^m \to l^2(G)^n$ be the morphism corresponding to f under ν. Choose a projection $\mathrm{pr} : l^2(G)^m \to l^2(G)^m$ with image $\ker(\nu(f))$. Then

$$l^2(G)^m \xrightarrow{\mathrm{pr}} l^2(G)^m \xrightarrow{\nu^{-1}(f)} l^2(G)^n$$

is exact. Hence

$$\mathcal{N}(G)^m \xrightarrow{\nu^{-1}(\mathrm{pr})} \mathcal{N}(G)^m \xrightarrow{f} \mathcal{N}(G)^n$$

is exact and $\nu^{-1}(\mathrm{pr})^2 = \nu^{-1}(\mathrm{pr})$. Hence $\ker(f) \subset \mathcal{N}(G)^m$ is a direct summand and $M = \mathrm{im}(f)$ is projective.

Remark 2.10 The following results and definitions can be understood by the slogan that $\mathcal{N}(G)$ behaves like $\mathbb{Z}$ if one forgets that $\mathbb{Z}$ is Noetherian and has no-zero-divisors. In this sense all properties of $\mathbb{Z}$ carry over to $\mathcal{N}(G)$.

Lemma 2.11 *Let M be a finitely generated $\mathcal{N}(G)$-module. Then*

1. *Let $K \subset M$ be a submodule. Then $\overline{K} \subset M$ is a direct summand and M/K is finitely generated projective;*

2. *$\mathbf{P}M$ is a finitely generated projective $\mathcal{N}(G)$-module and we get a splitting*

$$M \cong \mathbf{T}M \oplus \mathbf{P}M;$$

3. *If M is finitely presented, then there is an exact sequence*

$$0 \to \mathcal{N}(G)^n \to \mathcal{N}(G)^n \to \mathbf{T}M \to 0.$$

Theorem 2.12 (Dimension function for arbitrary $\mathcal{N}(G)$-modules).

There is precisely one dimension function

$$\dim : \{\mathcal{N}(G) - \textit{modules}\} \;\to\; [0, \infty]$$

which has the following properties;

1. *Extension Property*

   If M is a finitely generated projective R-module, then $\dim(M)$ *agrees with the previously defined notion;*

2. *Additivity*

   If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ *is an exact sequence of R-modules, then*

   $$\dim(M_1) \;=\; \dim(M_0) + \dim(M_2);$$

3. *Cofinality*

   Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \cup_{i \in I} M_i$

and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then

$$\dim(M) \ = \ \sup\{\dim(M_i) \mid i \in I\};$$

4. *Continuity*

   If $K \subset M$ is a submodule of the finitely generated R-module M, then

$$\dim(K) \ = \ \dim(\overline{K});$$

5. *If M is a finitely generated R-module, then*

$$\begin{aligned} \dim(M) \ &= \ \dim(\mathbf{P}M); \\ \dim(\mathbf{T}M) \ &= \ 0; \end{aligned}$$

Proof: We give the proof of uniqueness which leads to the definition of dim. Any $\mathcal{N}(G)$-module M is the colimit over the directed system of its finitely generated submodules $\{M_i \mid i \in I\}$. Hence by Cofinality

$$\dim(M) \ = \sup\{\dim(M_i) \mid i \in I\}.$$

We get for each M_i from Additivity

$$\dim(M_i) \;=\; \dim(\mathbf{P}M_i).$$

Hence we get

$$\begin{aligned}\dim(M) \;=\; & \sup\{\dim(P) \mid P \subset M \\ & \qquad \text{finitely generated projective}\}.\end{aligned}$$

The hard part is now to show that with this definition all the properties are satisfied.

Theorem 2.13 *Let $\{M_i \mid i \in I\}$ be a direct system of R-modules over the directed set I. For $i \leq j$ let $\phi_{i,j} : M_i \to M_j$ be the associated morphism of R-modules. Suppose for each $i \in I$ that there is $i_0 \in I$ with $i \leq i_0$ such that $\dim(\operatorname{im}(\phi_{i,i_0})) < \infty$ holds. Then*

$$\begin{aligned}&\dim\left(\operatorname{colim}_{i \in I} M_i\right) \\ &\quad = \sup\Big\{\inf\Big\{\dim(\operatorname{im}(\phi_{i,j} : M_i \to M_j)) \mid \\ &\qquad\qquad\qquad j \in I, i \leq j\big\} \mid i \in I\big\}.\end{aligned}$$

Remark 2.14 The results above are motivated by the following observations for $R = \mathbb{Z}$. If M is a finitely generated $\mathbb{Z}$-module, then $M/\operatorname{tors}(M)$ is finitely generated free and

$$M = \operatorname{tors}(M) \oplus M/\operatorname{tors}(M).$$

We get a dimension function for all $\mathbb{Z}$-modules by

$$\dim(M) \;:=\; \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M).$$

The difference between $\mathbb{Z}$ and $\mathcal{N}(G)$ is that for a projective $\mathbb{Z}$-module P we have $\dim(P) < \infty$ if and only if P is finitely generated. This is not true for $\mathcal{N}(G)$.

Definition 2.15 *Let X be a (left) G-space. Its* **homology with coefficients in $\mathcal{N}(G)$** *is*

$$H_p^G(X;\mathcal{N}(G)) \;=\; H_p\left(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)\right).$$

Define the **p-th L^2-Betti number of X** *by*

$$b_p^{(2)}(X;\mathcal{N}(G)) \;:=\; \dim_{\mathcal{N}(G)}(H_p^G(X;\mathcal{N}(G))) \in [0,\infty].$$

Lemma 2.16 *Let X be a G-CW-complex of finite type. Then Definition 2.15 of L^2-Betti numbers $b_p^{(2)}(X;\mathcal{N}(G))$ agrees with the previous one.*

Definition 2.17 *The* **p-th L^2-Betti number** *of a group G is*

$$b_p^{(2)}(G) \ := \ b_p^{(2)}(EG,\mathcal{N}(G)).$$

Remark 2.18 Notice that we work with homology. This is very convenient since homology transforms colimits into colimits in general whereas cohomology transforms colimits into exact sequences involving inverse limits and higher inverse limits. Moreover, the dimension function behaves well under colimits but its behaviour under inverse limits is much more complicated.

Theorem 2.19 L^2-Betti numbers for arbitrary spaces).

1. *Homotopy invariance*

   Let $f : X \to Y$ be a G-map. Suppose such that for each subgroup $H \subset G$ the induced map $f^H : X^H \to Y^H$ is a $\mathbb{C}$-homology equivalence, i.e. $H_p^{\mathrm{sing}}(f^H;\mathbb{C}) : H_p^{\mathrm{sing}}(X^H;\mathbb{C}) \to H_p^{\mathrm{sing}}(Y^H;\mathbb{C})$ is bijective for $p \geq 0$. Then for all $p \geq 0$ the induced map $f_* : H_p^G(X;\mathcal{N}(G)) \to H_p^G(Y;\mathcal{N}(G))$ is bijective and we get

   $$b_p^{(2)}(X) = b_p^{(2)}(Y) \qquad \text{for } p \geq 0;$$

2. *Comparison with the Borel construction*

   Let X be a G-CW-complex. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_p^{(2)}(G_x) = 0$ for all $p \geq 0$. Then for $p \geq 0$

   $$b_p^{(2)}(X;\mathcal{N}(G)) = b_p^{(2)}(EG \times X;\mathcal{N}(G));$$

3. *Independence of equivariant cells with infinite isotropy*

   Let X be a G-CW-complex. Let $X[\infty]$ be the G-CW-subcomplex consisting of those points whose isotropy subgroups are infinite. Then we get for all $p \geq 0$

   $$b_p^{(2)}(X; \mathcal{N}(G)) = b_p^{(2)}(X, X[\infty]; \mathcal{N}(G));$$

4. *Künneth formula*

   Let X be a G-space and Y be a H-space. Then $X \times Y$ is a $G \times H$-space and we get for all $n \geq 0$

   $$b_n^{(2)}(X \times Y) = \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y),$$

   where we use the convention that $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ for $r \in (0, \infty]$ and $r + \infty = \infty$ for $r \in [0, \infty]$;

5. *Induction*

   Let $i : H \to G$ be an inclusion of groups and let X be a H-space. Let $i : \mathcal{N}(H) \to$

$\mathcal{N}(G)$ be the induced ring homomorphism. Then

$$H_p^G(G \times_H X; \mathcal{N}(G)) \;=\; i_* H_p^H(X; \mathcal{N}(H));$$
$$b_p^{(2)}(G \times_H X; \mathcal{N}(G)) \;=\; b_p^{(2)}(X; \mathcal{N}(H));$$

6. *Restriction*

   Let $H \subset G$ be a subgroup of finite index $[G : H]$. Let X be a G-space and let $\operatorname{res}(X)$ *be the H-space obtained from X by restriction. Then*

$$b_p^{(2)}(\operatorname{res}(X); \mathcal{N}(H)) \;=\; [G : H] \cdot b_p^{(2)}(X; \mathcal{N}(G));$$

7. *Zero-th homology and L^2-Betti number*

   Let X be a path-connected G-space. Then

$$b_0^{(2)}(X; \mathcal{N}(G)) = |G|^{-1}.$$

   Moreover $H_0^G(X; \mathcal{N}(G))$ is trivial if and only if G is non-amenable.

Definition 2.20 *A group G is called* **amenable** *if there is a (left) G-invariant linear operator $\mu : l^\infty(G,\mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ which satisfies*

$$\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\} \quad \text{for all } f \in l^\infty(G,\mathbb{R}).$$

The **class of elementary amenable** *groups is defined as the smallest class of groups which has the following properties:*

1. *It contains all finite and all abelian groups;*
2. *It is closed under taking subgroups;*
3. *It is closed under taking quotient groups;*
4. *It is closed under extensions;*
5. *It is closed under directed unions,*

Remark 2.21 The class of amenable groups is closed under the operations above. Hence it contains the class of elementary amenable groups. A group which contains $\mathbb{Z} * \mathbb{Z}$ is not amenable.

Corollary 2.22 (Brooks).
Let M be a closed Riemannian manifold. Then the Laplace operator acting on functions on $\widetilde{M}$ has zero in its spectrum if and only if $\pi_1(M)$ is amenable.

Theorem 2.23 (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ for amenable G).
Let G be amenable and M be a $\mathbb{C}G$-module. Then for $p \geq 1$

$$\dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G), M)\right) = 0.$$

Theorem 2.24 *Let G be an amenable group and X be a G-space. Then*

$$b_p^{(2)}(X; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_p^{\mathrm{sing}}(X; \mathbb{C})\right).$$

Corollary 2.25 (Cheeger-Gromov).
Let G be a group which contains an infinite normal amenable subgroup. Then for $p \geq 0$

$$b_p^{(2)}(G; \mathcal{N}(G)) \;=\; 0.$$

If there is a finite model for BG, then

$$\chi(G) := \chi(BG) \;=\; 0.$$

Proof: If G is amenable, this follows from $H_p(EG; \mathbb{C}) = 0$ for $p \geq 1$. In the general case one uses a spectral sequence argument.

Definition 2.26 *Let R be an (associative) ring (with unit). Define its* **projective class group** *$K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules. Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.*

Theorem 2.27 *Let G be an amenable group. Then we get a well-defined map*

$$\begin{aligned} \dim : G_0(\mathbb{C}G) &\to \mathbb{R}, \\ [M] &\mapsto \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M). \end{aligned}$$

In particular $[\mathbb{C}G]$ generates an infinite cyclic subgroup in $G_0(\mathbb{C}G)$.

Lemma 2.28 *If G contains $\mathbb{Z} * \mathbb{Z}$ as subgroup, then*

$$[\mathbb{C}G] = 0 \quad \in G_0(\mathbb{C}G).$$

Conjecture 2.29 *G is amenable if and only if*

$$[\mathbb{C}G] \neq 0 \quad \in G_0(\mathbb{C}G).$$

Theorem 2.30 (L^2-Betti numbers and S^1-actions).
Let X be a connected S^1-CW-complex. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Let $\widetilde{X}$ be the universal covering of X with the canonical $\pi_1(X)$-action. Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) = 0.$$

Theorem 2.31 L^2-Betti numbers and fibrations
Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected CW-complexes. Suppose that $\pi_1(F) \to \pi_1(E)$ is injective. Suppose for a given integer $d \geq 1$ that $b_p^{(2)}(\widetilde{F}) = 0$ for $p \leq d-1$ and $b_d^{(2)}(\widetilde{F}) < \infty$ holds. Suppose that $\pi_1(B)$ contains an element of infinite order or finite subgroups of arbitrary large order. Then $b_p^{(2)}(\widetilde{E}) = 0$ for $p \leq d$.

Definition 2.32 *Let G be a finitely presented group. Define its* **deficiency** $\operatorname{def}(G)$ *to be the maximum $g(P) - r(P)$, where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.*

Lemma 2.33 *Let G be a finitely presented group. Then*

$$\operatorname{def}(G) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof We have to show for any presentation P that

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\begin{aligned} \chi(X) &= 1 - g(P) + r(P) \\ &= b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}). \end{aligned}$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{aligned} b_p^{(2)}(\widetilde{X}) &= b_p^{(2)}(G) \quad \text{for } p = 0, 1; \\ b_2^{(2)}(\widetilde{X}) &\geq b_2^{(2)}(G). \end{aligned}$$

Example 2.34 The free group F_g has the obvious presentation $\langle s_1, s_2, \ldots s_g \mid \emptyset\rangle$ and its deficiency is realized by this presentation, namely def$(F_g) = g$.

If G is a finite group, def$(G) \leq 0$ by Lemma 2.33 because of $b_0^{(2)}(G) = |G|^{-1}$ and $b_1^{(2)}(G) = 0$.

The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n\rangle$ realizes the deficiency.

The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is -1, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t]\rangle$ realizes the deficiency.

Example 2.35 One may expect that the deficiency is additive under free products. This is not true by the following example due to Hog, Lustig and Metzler(1985). The group $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

and one may think that its deficiency is -2. However, it turns out that its deficiency is -1. For instance, there is the following presentation, which looks on the first glance to be the presentation above with one relation missing

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

The following calculation shows that, from the five relations appearing in the presentation above, the relation $t_0^2 = 1$ follows which shows that the presentation above indeed one of of $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$.

We start by proving inductively for $k = 1, 2, \ldots$ the relation $s_i^k t_i s_i^{-k} = t_i^{r_i^k}$ for $i =$

$0, 1$ where $r_0 = 3$ and $r_1 = 4$. The beginning of the induction is obvious, the induction step follows from the calculation

$$s_i^{k+1} t_i s_i^{-(k+1)} = s_i s_i^k t_i s_i^{-k} s_i^{-1}$$
$$= s_i t_i^{r_i^k} s_i^{-1} = \left(s_i t_i s_i^{-1}\right)^{r_i^k} = \left(t_i^{r_i}\right)^{r_i^k} = t_i^{r_i^{k+1}}.$$

This implies, for $k = 2$, $i = 0$ and $k = 3$,$i = 1$

$$\begin{aligned} t_0 &= t_0^{3^2}; \\ t_1 &= t_1^{4^3}. \end{aligned}$$

Since $t_0^2 = t_1^3$, we conclude

$$\begin{aligned} (t_0^2)^4 &= 1; \\ (t_0^2)^{21} &= 1. \end{aligned}$$

As 4 and 21 are prime, we get $t_0^2 = 1$ and the claim follows.

Theorem 2.36 *Let* $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and one of the following conditions is satisfied.*

1. $b_1^{(2)}(H) < \infty$;

2. *The ordinary first Betti number of* H *satisfies* $b_1(H) < \infty$ *and* $b_1^{(2)}(K) = 0$;

Then:

(i) $\mathrm{def}(G) \leq 1$;

(ii) Let M *be a closed oriented 4-manifold with* G *as fundamental group. Then*

$$|\mathrm{sign}(M)| \ \leq \ \chi(M);$$

Remark 2.37 Next we compare our approach with the one of Cheeger and Gromov. We only consider the case of a countable simplicial complex X with free simplicial G-action. Then for any exhaustion $X_0 \subset X_1 \subset X_2 \subset \ldots \subset X$ by G-equivariant simplicial subcomplexes for which $G\backslash X$ is compact, the p-th L^2-Betti number in the sense and notation of Cheeger-Gromov is given by

$$b_p^{(2)}(X:G) = \lim_{j\to\infty} \lim_{k\to\infty} \dim_{\mathcal{N}(G)} \left(\operatorname{im}\left(\overline{H}^p_{(2)}(X_k:G) \xrightarrow{i^*_{j,k}} \overline{H}^p_{(2)}(X_j:G)\right)\right),$$

where $i_{j,k} : X_j \to X_k$ is the inclusion for $j \leq k$. There is an identification

$$\overline{H}^p_{(2)}(X_j:G) = H^p_{(2)}(X_j;\mathcal{N}(G)).$$

Notice that for a G-map $f : Y \to Z$ of G-CW-complexes of finite type $H_p^{(2)}(Y;\mathcal{N}(G))$ can be identified with $H^p_{(2)}(Y;\mathcal{N}(G))$ and analogously for Z and that under these identifications $H^p_{(2)}(f) = (H_p^{(2)}(f))^*$. We conclude from Additivity

$$\begin{aligned}
&\dim_{\mathcal{N}(G)}\left(\operatorname{im}\left(H^p_{(2)}(f)\right)\right) \\
&\qquad\qquad = \dim_{\mathcal{N}(G)}\left(\operatorname{im}\left(H_p^{(2)}(f)\right)\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\dim_{\mathcal{N}(G)}\left(\operatorname{im}\left(\overline{H}^p_{(2)}(X_k : G) \xrightarrow{i^*_{j,k}} \overline{H}^p_{(2)}(X_j : G)\right)\right) \\
&\quad = \dim_{\mathcal{N}(G)}\left(\operatorname{im}\left(H_p^G(X_j; \mathcal{N}(G))\right.\right. \\
&\qquad\qquad \left.\left.\xrightarrow{(i_{j,k})_*} H_p^G(X_k; \mathcal{N}(G))\right)\right).
\end{aligned}$$

Hence we conclude from Theorem 2.13 that the definitions in Cheeger-Gromov(1986) and in Definition 2.15 for a countable free simplicial complex X with free simplicial G-action agree.

3. Survey on Further Results and Conjectures

Theorem 3.1 (Approximation Theorem) *Let X be a free G-CW-complex of finite type. Suppose that G is residually finite, i.e. there is a nested sequence*

$$G = G_0 \supset G_1 \supset G_2 \supset \dots$$

of normal subgroups of finite index. Then $G_n \backslash X$ is a CW-complex of finite type and for any such sequence $(G_n)_{n \geq 1}$

$$b_p^{(2)}(X; \mathcal{N}(G)) \;=\; \lim_{n \to \infty} \frac{b_p(G_n \backslash X)}{[G : G_n]}.$$

Remark 3.2 We have already seen in the first lecture that there are no relations between $b_p^{(2)}(X; \mathcal{N}(G))$ and $b_p(G \backslash X)$ for a finite G-CW-complex X except for the Euler-Poincaré formula

$$\begin{aligned} \chi(G \backslash X) &= \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(X; \mathcal{N}(G)) \\ &= \sum_{p \geq 0} (-1)^p \cdot b_p(G \backslash X). \end{aligned}$$

One decisive difference between the ordinary Betti numbers and L^2-Betti numbers is that the ordinary ones are not multiplicative under finite coverings, whereas the L^2-Betti numbers are, i.e. for a d-sheeted covering $p : X \to Y$ we get

$$b_p^{(2)}(\widetilde{X}; \mathcal{N}(\pi_1(X)) \;=\; d \cdot b_p^{(2)}(\tilde{Y}; \mathcal{N}(\pi_1(Y)).$$

With the expression $\lim_{n\to\infty} \frac{b_p(G\backslash X)}{[G:G_n]}$ we try to force the Betti numbers to be multiplicative by a limit process.

Theorem 3.1 says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

Example 3.3 Consider S^1 and the nested sequence

$$\pi_1(S^1) = \mathbb{Z} \supset 2^1 \cdot \mathbb{Z} \supset 2^2 \cdot \mathbb{Z} \supset 2^3 \cdot \mathbb{Z} \supset \ldots .$$

Then

$$\begin{aligned} b_p^{(2)}(\widetilde{S^1}; \mathcal{N}(\mathbb{Z})) &= \lim_{n\to\infty} \frac{b_p((2^n \cdot \mathbb{Z})\backslash \widetilde{S^1})}{[\pi_1(S^1) : 2^n \cdot \mathbb{Z}]} \\ &= \lim_{n\to\infty} \frac{b_p(S^1)}{2^n} \\ &= 0. \end{aligned}$$

Theorem 3.4 (Schick-Lück)
Let (X, A) be a pair of finite free G-CW-complexes. Suppose that $G\backslash(X, A)$ is a Poincaré pair of dimension $4m$. Suppose that G is residually finite, i.e. there is a nested sequence

$$G = G_0 \supset G_1 \supset G_2 \supset \dots$$

of normal subgroups of finite index. Then $G_i\backslash(X, A)$ is a finite Poincaré pair of dimension $4m$ and for any such sequence $(G_n)_{n\geq 1}$

$$\mathrm{sign}^{(2)}(X, A; \mathcal{N}(G)) = \lim_{n\to\infty} \frac{\mathrm{sign}(G\backslash(X, A))}{[G : G_i]}.$$

Remark 3.5 In the case that $Y = \emptyset$ and $G\backslash X = M$ for a closed orientable manifold M, Theorem 3.4 follows from the index theorem of Atiyah which says

$$\mathrm{sign}^{(2)}(X, A; \mathcal{N}(G)) = \frac{\mathrm{sign}(G_n\backslash(X, A))}{[G : G_n]}.$$

In particular the signature is multiplicative under finite coverings.

If $G\backslash(X,Y) = (M,\partial M)$ for a compact manifold with non-empty boundary or if $Y = \emptyset$ and X is a Poincaré complex (which is not necessarily a closed orientable manifold), then the equation above is not true and the signature is not multiplicative under finite coverings.

Given a group G, let $\mathcal{FIN}(G)$ be the set of finite subgroups of G. Denote by

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \ \subset \ \mathbb{Q}$$

the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $\{\frac{1}{|H|} \mid H \in \mathcal{FIN}(G)\}$.

Conjecture 3.6 (Strong Atiyah Conjecture)
A group G satisfies the **strong Atiyah Conjecture** *if for any matrix $A \in M(m,n,\mathbb{Q}G)$ the von Neumann dimension of the kernel of the G-equivariant bounded operator $r_A^{(2)}\colon l^2(G)^m \to l^2(G)^n$, $x \mapsto xA$ satisfies*

$$\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{(2)}\colon l^2(G)^m \to l^2(G)^n\right)\right) \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

Remark 3.7 If G is torsionfree, then

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} = \mathbb{Z}.$$

Lemma 3.8 *Let G be a group. Then the following statements are equivalent:*

1. *For any cocompact free proper G-manifold M without boundary we have*

$$b_p^{(2)}(M;\mathcal{N}(G)) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z};$$

2. *For any cocompact free proper G-CW-complex X we have*

$$b_p^{(2)}(X;\mathcal{N}(G)) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z};$$

3. *The Atiyah Conjecture 3.6 is true for G.*

Remark 3.9 Atiyah asked originally the following question. Let $G \to \overline{M} \to M$ be a G-covering of a closed Riemannian manifold M. Is then

$$b_p^{(2)}(M) \;=\; \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x)) \; dvol$$

a rational number?

Lemma 3.10 *A group G satisfies the strong Atiyah Conjecture if and only if for any finitely presented $\mathbb{Q}G$-module M*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} M) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

Proof: Given a matrix $A \in M(m,n,\mathbb{Q}G)$, let

$$r_A \colon \mathbb{Q}G^m \to \mathbb{Q}G^n$$

resp.

$$r_A^{\mathcal{N}(G)} \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$$

be the associated $\mathbb{Q}G$- resp. $\mathcal{N}(G)$-map given by right multiplication with A. Since the tensor product $\mathcal{N}(G) \otimes_{\mathbb{Q}G} -$ is right exact, coker $\left(r_A^{\mathcal{N}(G)}\right)$ is $\mathcal{N}(G)$-isomorphic to $\mathcal{N}(G)\otimes_{\mathbb{Q}G}\text{coker}(r_A)$. We conclude from Additivity

$$\begin{aligned}&\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{\mathcal{N}(G)}\right)\right)\\ =\ & m-n+\dim_{\mathcal{N}(G)}\left(\mathcal{N}(G)\otimes_{\mathbb{Q}G}\text{coker}(r_A)\right).\end{aligned}$$

As $\mathcal{N}(G)$ is semihereditary, $\ker\left(r_A^{\mathcal{N}(G)}\right)$ is finitely generated projective. We have

$$\begin{aligned}&\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{\mathcal{N}(G)}\right)\right)\\ &\qquad\qquad=\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{(2)}\right)\right).\end{aligned}$$

Therefore

$$\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{(2)}\right)\right)\ \in\ \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$$

if and only if

$$\dim_{\mathcal{N}(G)}\left(\mathcal{N}(G)\otimes_{\mathbb{Q}G}\text{coker}(r_A)\right)\ \in\ \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

Conjecture 3.11 (Kaplanski Conjecture) *The Kaplanski Conjecture for a torsionfree group G and a field F says that the group ring FG has no non-trivial zero-divisors.*

Lemma 3.12 *The Kaplanski Conjecture holds for G and the field $\mathbb{Q}$ if the strong Atiyah Conjecture 3.6 holds for G.*

Proof: Let $x \in \mathbb{Q}G$ be a zero-divisor. Let $r_x^{(2)} \colon l^2(G) \to l^2(G)$ be given by right multiplication with x. We get

$$0 < \dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) \leq 1$$

Since by assumption $\dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) \in \mathbb{Z}$, we conclude $\dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) = 1$. Since $\ker(r_x^{(2)})$ is closed in $l^2(G)$, we conclude $\ker(r_x^{(2)}) = l^2(G)$ and hence $x = 0$.

Definition 3.13 *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed union and extensions with elementary amenable quotients.*

Theorem 3.14 (Linnell) *Let G be a group such that there is an upper bound on the orders of finite subgroups and G belongs to $\mathcal{C}$. Then the strong Atiyah Conjecture 3.6 holds for G.*

Definition 3.15 *Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following operations:*

1. *Amenable quotient*
   Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{G}$ and the quotient G/H is amenable. Then $G \in \mathcal{G}$;

2. *Colimits*
   If $G = \operatorname{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{G}$, then G belongs to $\mathcal{G}$;

3. *Inverse limits*
   If $G = \lim_{i\in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{G}$, then G belongs to $\mathcal{G}$;

4. *Subgroups*
   If H is isomorphic to a subgroup of the group G with $G \in \mathcal{G}$, then $H \in \mathcal{G}$;

5. *Quotients with finite kernel*
   Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. If K is finite and G belongs to $\mathcal{G}$, then Q belongs to $\mathcal{G}$.

Theorem 3.16 (Schick)
Let $\{G_i \mid i \in I\}$ be a directed system of groups such that each G_i belongs to the class $\mathcal{G}$ and satisfies the strong Atiyah Conjecture 3.6. Then both its colimit and its inverse limit satisfy the strong Atiyah Conjecture 3.6.

The **lamplighter group** L is defined by the semidirect product

$$L := \oplus_{n\in\mathbb{Z}}\mathbb{Z}/2 \rtimes \mathbb{Z}$$

with respect to the shift automorphism of $\oplus_{n\in\mathbb{Z}}\mathbb{Z}/2$, which sends $(x_n)_{n\in\mathbb{Z}}$ to $(x_{n-1})_{n\in\mathbb{Z}}$. Let $e_0 \in \oplus_{n\in\mathbb{Z}}\mathbb{Z}/2$ be the element whose entries are all zero except the entry at 0. Denote by $t \in \mathbb{Z}$ the standard generator of $\mathbb{Z}$.Then $\{e_0t, t\}$ is a set of generators for L. The associate **Markov operator** $M\colon l^2(G) \to l^2(G)$ is given by right multiplication with $\frac{1}{4}\cdot(e_0t + t + (e_0t)^{-1} + t^{-1})$. It is related to the Laplace operator $\Delta_0\colon l^2(G) \to l^2(G)$ of the Cayley graph of G by $\Delta_0 = 4\cdot\mathrm{id} - 4\cdot M$.

Theorem 3.17 (Grigorchuk-Zuk)
The von Neumann dimension of the kernel of the Markov operator M of the lamplighter group L associated to the set of generators $\{e_0t, t\}$ is $1/3$. In particular L does not satisfy the strong Atiyah Conjecture 3.6.

Remark 3.18 No counterexample to the strong Atiyah Conjecture 3.6 is known if one replaces $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ by $\mathbb{Q}$ or if one assumes that there is a bound on the orders of finite subgroups of G.

Conjecture 3.19 (Singer Conjecture) *If M is an aspherical closed manifold, then*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \quad \text{if } 2p \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2p \neq \dim(M); \\ > 0 & \text{if } 2p = \dim(M). \end{cases}$$

Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{M})$$

the Singer Conjecture 3.19 implies the following conjecture in the cases where M is aspherical or has negative sectional curvature.

Conjecture 3.20 (Hopf Conjecture) *If M is an aspherical closed manifold of even dimension, then*

$$(-1)^{\mathrm{dim}(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\mathrm{sec}(M)$, *then for $\epsilon = (-1)^{\mathrm{dim}(M)/2}$*

$$\begin{array}{rcrlrcl}
\epsilon \cdot \chi(M) & > & 0 & \text{if} & \mathrm{sec}(M) & < & 0; \\
\epsilon \cdot \chi(M) & \geq & 0 & \text{if} & \mathrm{sec}(M) & \leq & 0; \\
\chi(M) & = & 0 & \text{if} & \mathrm{sec}(M) & = & 0; \\
\chi(M) & \geq & 0 & \text{if} & \mathrm{sec}(M) & \geq & 0; \\
\chi(M) & > & 0 & \text{if} & \mathrm{sec}(M) & > & 0.
\end{array}$$

Theorem 3.21 (Jost-Xin)

Let M be a closed connected Riemannian manifold of dimension $\mathrm{dim}(M) \geq 3$. *Suppose that there are real numbers $a > 0$ and $b > 0$ such that the sectional curvature satisfies $-a^2 \leq \mathrm{sec}(M) \leq 0$ and the Ricci curvature is bounded from above by $-b^2$. If the non-negative integer p satisfies* $2p \neq \mathrm{dim}(M)$ *and $2pa \leq b$, then*

$$b_p^{(2)}(\widetilde{M}) = 0.$$

Theorem 3.22 (Ballmann-Brüning) *Let M be a closed connected Riemannian manifold. Suppose that there are real numbers $a > 0$ and $b > 0$ such that the sectional curvature satisfies $-a^2 \leq \sec(M) \leq -b^2$. If the non-negative integer p satisfies*

$$\begin{aligned} 2p &< \dim(M) - 1; \\ p \cdot a &< (\dim(M) - 1 - p) \cdot b, \end{aligned}$$

then

$$b_p^{(2)}(\widetilde{M}) = 0.$$

Remark 3.23 Direct computations show that the Singer Conjecture 3.19 holds for a closed Riemannian manifold M if $\dim(M) \leq 3$ (assuming Thurston's Geometrization) or if M is a locally symmetric space or if M carries an S^1-action.

Definition 3.24 *A* **Kähler hyperbolic manifold** *is a closed connected Kähler manifold M whose fundamental form ω is $\widetilde{d}$(bounded), i.e. its lift $\widetilde{\omega} \in \Omega^p(\widetilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded $(p-1)$-form $\eta \in \Omega^{p-1}(\widetilde{M})$.*

Example 3.25 The closed manifold M is Kähler hyperbolic if it satisfies one of the following conditions:

1. M is a closed Kähler manifold which is homotopy equivalent to a Riemannian manifold with negative sectional curvature;

2. M is a closed Kähler manifold such that $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial;

3. $\widetilde{M}$ is a symmetric Hermitian space of non-compact type;

4. M is a complex submanifold of a Kähler hyperbolic manifold;

5. M is a product of two Kähler hyperbolic manifolds.

Theorem 3.26 (Gromov)
Let M be a closed Kähler hyperbolic manifold of complex dimension m and real dimension $n = 2m$. Then

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad \text{if } p \neq m; \\ b_m^{(2)}(\widetilde{M}) &> 0; \\ (-1)^m \cdot \chi(M) &> 0; \end{aligned}$$

Theorem 3.27 *Let M be a closed Kähler hyperbolic manifold of complex dimension m and real dimension $n = 2m$.*

1. *The canonical line bundle $L = \Lambda^m T^* M$ is quasiample, i.e. its Kodaira dimension is m;*

2. *M satisfies all of the following four assertions (which are equivalent for closed Kähler manifolds):*

   (a) *M is Moishezon, i.e. the transcendental degree of the field $\mathcal{M}(X)$ of* meromorphic functions is equal to m;

(b) M is Hodge, i.e. the Kähler form represents a class in $H^2(M;\mathbb{C})$ which lies in the image of $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{C})$;

(c) M can be holomorphically embedded into $\mathbb{CP}^N$ for some N;

(d) M is a projective algebraic variety;

3. The fundamental group is an infinite non-amenable group of deficiency ≤ 1. It cannot be a non-trivial free product.

Let X be a topological space and let $C_*^{\text{sing}}(X;\mathbb{R})$ be its singular chain complex with real coefficients. Let $S_p(X)$ be the set of all singular p-simplices. Then $C_p(X;\mathbb{R})$ is the real vector space with $S_p(X)$ as basis. Define the L^1-**norm** of an element $x \in C_p(X)$, which is given by the (finite) sum $\sum_{\sigma \in S_p(X)} \lambda_\sigma \cdot \sigma$, by

$$||x||_1 \;\; := \;\; \sum_\sigma |\lambda_\sigma|.$$

Define the L^1-**seminorm** of an element y in the p-th singular homology $H_p^{\text{sing}}(X;\mathbb{R}) := H_p(C_*^{\text{sing}}(X;\mathbb{R}))$ by

$$\begin{aligned} ||y||_1 \;\; := \;\; \inf\{||x||_1 \mid x \in C_p^{\text{sing}}(X;\mathbb{R}), \\ \partial_p(x) = 0, y = [x]\}. \end{aligned}$$

Notice that $||y||_1$ defines only a semi-norm on $H_p^{\text{sing}}(X;\mathbb{R})$, it is possible that $||y||_1 = 0$ but $y \neq 0$. The next definition is due to Gromov and Thurston.

Definition 3.28 *Let M be a closed connected orientable manifold of dimension n. Define its* **simplicial volume** *to be the non-negative real number*

$$||M|| := ||j([M])||_1 \quad \in [0, \infty)$$

for any choice of fundamental class $[M] \in H_n^{\mathrm{sing}}(M, \mathbb{Z})$ and $j : H_n^{\mathrm{sing}}(M; \mathbb{Z}) \to H_n^{\mathrm{sing}}(M; \mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$.

Theorem 3.29 (Simplical volume of hyperbolic manifolds)
Let M be a closed hyperbolic orientable manifold of dimension n. Let v_n be the volume of the regular ideal simplex in $\overline{\mathbb{H}^n}$. Then

$$||M|| = \frac{\mathrm{vol}(M)}{v_n}.$$

Example 3.30 We have $||S^2|| = ||T^2|| = 0$. Let F_g be the closed connected orientable surface of genus $g \geq 1$. Then

$$||F_g|| = 2 \cdot |\chi(F_g)| = 4g - 4.$$

Definition 3.31 *Let M be a smooth manifold. Define its* **minimal volume** $\text{minvol}(M)$ *to be the infimum over all volumes* $\text{vol}(M, g)$, *where g runs though all complete Riemannian metrics on M, for which the sectional curvature satisfies $|\sec(M, g)| \leq 1$.*

Example 3.32 Obviously any closed flat Riemannian manifold has vanshing minimal volume. Hence we get

$$\text{minvol}(T^n) = ||T^n|| = 0.$$

Let F_g be the closed orientable surface of genus g, then

$$\begin{aligned} \text{minvol}(F_g) &= 2\pi \cdot |\chi(F_g)| = 2\pi \cdot |2 - 2g| \\ &= \pi \cdot ||F_g|| \end{aligned}$$

by the following argument. The Gauss-Bonnet formula implies for any Riemannian metric on F_g whose sectional curvature satisfies $|\sec| \leq 1$

$$\begin{aligned} \text{vol}(F_g) &\geq \int_{F_g} |\sec| dvol \geq \left| \int_{F_g} \sec dvol \right| \\ &= |2\pi \cdot \chi(F_g)|. \end{aligned}$$

If $g \neq 1$ and we take the Riemannian metric whose sectional curvature is constant 1 or -1, then the Gauss-Bonnet Theorem shows

$$|2\pi \cdot \chi(F_g)| \;=\; \left|\int_{F_g} \sec dvol\right| \;=\; \mathrm{vol}(F_g).$$

Now the claim follows.

Notice that $||S^2|| = 0$ and $\mathrm{minvol}(S^2) \neq 0$.

We have

$$\begin{aligned} \mathrm{minvol}(\mathbb{R}^2) &= 2\pi(1+\sqrt{2}); \\ \mathrm{minvol}(\mathbb{R}^n) &= 0 \qquad \text{for } n \geq 3. \end{aligned}$$

Theorem 3.33 (Gromov-Thurston)
Let M be a closed connected orientable Riemannian manifold of dimension n. Then

$$||M|| \leq (n-1)^n \cdot n! \cdot \mathrm{minvol}(M).$$

Conjecture 3.34 (Simplical volume and L^2-invariants)
Let M be an aspherical closed oriented manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \quad \textit{for } p \geq 0.$$

Example 3.35 Let M be an aspherical closed orientable manifold. Then Conjecture 3.34 is true in following cases:

1. Suppose that M carries an S^1-action. Then minvol$(M) = 0$, $||M|| = 0$ and $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$;

2. Let $H \subset \pi_1(M)$ be a normal infinite subgroup. If H is amenable, then $||M|| = 0$. If H is elementary amenable subgroup, then $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$;

3. If there is a selfmap $f : M \to M$ of degree deg(f) different from -1, 0, and 1, then $||M|| = 0$.

If any normal subgroup of finite index of $\pi_1(M)$ is Hopfian and there is a self-map $f : M \to M$ of degree $\deg(f)$ different from -1, 0, and 1, then $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$;

4. Supose that M has dimension 3 (and satisfies Thurston's Geometrization Conjecture). Then $||M|| = 0$ implies that $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$;

5. If the minimal volume $\mathrm{minvol}(M)$ of M is zero, then $||M|| = 0$ and $b_p^{(2)}(\widetilde{M}) = 0$ for all $p \geq 0$.