L^2-cohomology

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0 Introduction

The purpose of this miniseries consisting of three talks is to give some examples of striking applications of methods from L^2-cohomology based on the theory of finite von Neumann algebras to problems in geometry, manifold theory and group theory. We will not talk about applications to von Neumann algebras themselves but refer for instance to the work and talks of Connes-Shlyakhtenko, Gaboriau and Popa. We have tried to keep the three talks as independent of one another as possible.

The author wants to express is gratitude to all the people involved in the organization of the special activity on non-commutative geometry in Luminy in February 2004, in particular to Anthony Wassermann.

In the sequel ring will always mean associative ring with unit. The letter G denotes a discrete group. The von Neumann algebra of a group will be denoted by $\mathcal{N}(G)$.

1 Ring Properties and Dimension Functions of Finite von Neumann Algebras

In the sequel we fix a finite (complex) von Neumann algebra $\mathcal{A}$ together with a faithful finite normal trace $\operatorname{tr}\colon \mathcal{A} \to \mathbb{C}$.

Definition 1.1 (Finitely generated Hilbert $\mathcal{N}(G)$-module). *A* finitely generated Hilbert $\mathcal{A}$-module *is a Hilbert space V together with a $*$-homomorphism*

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$\mathcal{A} \to \mathcal{B}(V)$ such that there exists an isometric linear $\mathcal{A}$-embedding $V \to l^2(\mathcal{A})^n$ for some integer $n \geq 0$. A morphism of finitely generated Hilbert $\mathcal{A}$-modules is a bounded $\mathcal{A}$-equivariant operator.

Let $\mathcal{H}(\mathcal{A})$ be the $\mathbb{C}$-category with involution (coming from taking adjoint operator) of finitely generated Hilbert $\mathcal{A}$-modules and $\mathcal{P}(\mathcal{A})$ be the $\mathbb{C}$-category with involution (coming from taking dual modules) of finitely generated projective $\mathcal{A}$-modules. Notice that the definition of $\mathcal{P}(\mathcal{A})$ involves only the structure of a $\mathbb{C}$-algebra with involution of $\mathcal{A}$ but not the topology.

Lemma 1.2. *There is an equivalence of $\mathbb{C}$-categories with involution*

$$\nu \colon \mathcal{H}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}).$$

We do not give the full definition of ν and its inverse ν^{-1}. At least we say that ν^{-1} sends an $\mathcal{A}$-homomorphism $\mathcal{A}^m \to \mathcal{A}^n$ to the morphism of finitely generated Hilbert $\mathcal{A}$-modules $l^2(\mathcal{A})^m \to l^2(\mathcal{A})^n$ obtained from f by completion.

Lemma 1.2 allows us to switch hence and forth between the functional analytic category $\mathcal{H}(\mathcal{A})$ and the purely algebraic category $\mathcal{P}(\mathcal{A})$. In the category $\mathcal{H}(\mathcal{A})$ there is the obvious notion of taking the closure of the image of a morphism $f \colon V \to W$ of finitely generated Hilbert $\mathcal{A}$-modules which is again a finitely generated Hilbert $\mathcal{A}$-module. We translate this into a purely algebraic definition in $\mathcal{P}(\mathcal{A})$ as follows.

Definition 1.3. *Let R be a ring. Let M be a R-submodule of N. Define the* closure *of M in N to be the R-submodule of N*

$$\overline{M} \quad = \quad \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* \text{ with } M \subset \ker(f)\}.$$

For a R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \end{aligned}$$

Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It can also be described as the kernel of the canonical map

$$i(M) \colon M \to (M^*)^*$$

which sends $x \in M$ to the map $M^* \to R, \quad f \mapsto f(x)$. Notice that $\mathbf{TP}M = 0$ and that $\mathbf{P}M = 0$ is equivalent to $M^* = 0$.

Assumption 1.4. *We assume that there is a dimension function* dim *which assigns to any finitely generated projective R-module P a non-negative real number*

$$\dim(P) \in [0, \infty)$$

with the following properties:

(i) If P and Q are finitely generated projective R-modules, then

$$P \cong_R Q \;\Rightarrow\; \dim(P) = \dim(Q);$$
$$\dim(P \oplus Q) \;=\; \dim(P) + \dim(Q);$$

(ii) Let $K \subset Q$ be a submodule of the finitely generated projective R-module Q. Then its closure $\overline{K}$ (see Definition 1.3) is a direct summand in Q and

$$\dim(\overline{K}) = \sup\{\dim(P) \mid P \subset K \textit{ finitely generated projective submodule}\}.$$

Next we explain that this dimension function can be extended to all R-modules and implies certain nice ring theoretic properties for R.

Theorem 1.5. (Dimension function for arbitrary $\mathcal{N}(G)$-modules, L.). *Suppose that $(R, \dim)$ satisfies Assumption 1.4. Then:*

(i) R is semihereditary, *i.e. every finitely generated submodule of a projective module is projective;*

(ii) If $K \subset M$ is a submodule of the finitely generated R-module M, then $M/\overline{K}$ is finitely generated projective and $\overline{K}$ is a direct summand in M;

(iii) If M is a finitely generated R-module, then $\mathbf{P}M$ is finitely generated projective and

$$M \cong \mathbf{P}M \oplus \mathbf{T}M;$$

(iv) There is a dimension function

$$\dim\colon \{R-\textit{modules}\} \;\to\; [0, \infty]$$

defined for all R-modules which has and is uniquely determined by the following properties:

(a) **Extension Property**

If M is a finitely generated projective R-module, then $\dim(M)$ *agrees with the given dimension;*

(b) **Additivity**

If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of R-modules, then

$$\dim(M_1) \;=\; \dim(M_0) + \dim(M_2),$$

where for $r, s \in [0, \infty]$ we define $r + s$ by the ordinary sum of two real numbers if both r and s are not ∞, and by ∞ otherwise;

(c) **Cofinality**

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \bigcup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then

$$\dim(M) \;=\; \sup\{\dim(M_i) \mid i \in I\};$$

(d) **Continuity**

If $K \subset M$ is a submodule of the finitely generated R-module M, then

$$\dim(K) \;=\; \dim(\overline{K});$$

In particular we get for a finitely generated R-module M:

$$\begin{aligned} \dim(M) &= \dim(\mathbf{P}M); \\ \dim(\mathbf{T}M) &= 0. \end{aligned}$$

Since every R-module is the union of the directed system of finitely generated submodules, it is easy to check that the dimension function, if it exists, must be given by

$$\dim(M) \;:=\; \sup\{\dim(P) \mid P \subset M \text{ finitely generated projective submodule}\}.$$

The hard part in the proof is to show that this definition indeed has all the listed properties.

Example 1.6. Let $R = \mathbb{Z}$. Let M be a finitely generated $\mathbb{Z}$-module and $K \subset M$. Then

$$\begin{aligned} \overline{K} &= \{x \in M \mid n \cdot x \in K \text{ for some } n \in \mathbb{Z}\}; \\ \mathbf{T}M &:= \operatorname{tors}(M); \\ \mathbf{P}M &= M/\operatorname{tors}(M). \end{aligned}$$

If we define the dimension of a finitely generated abelian group P by the unique n for which P is $\mathbb{Z}$-isomorphic to $\mathbb{Z}^n$, then $\mathbb{Z}$ together with this dimension function obviously satisfies Assumption 1.4. The dimension function constructed in Theorem 1.5 is explicitly given by

$$\dim(M) \;=\; \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M),$$

where $\dim_{\mathbb{Q}}$ denotes the dimension of a $\mathbb{Q}$-vector space.

Lemma 1.7. *Define for a projective $\mathcal{A}$-module P its dimension by*

$$\dim(P) := \sum_{i=1}^{r} \operatorname{tr}(a_{i,i})$$

for any matrix $A \in M(n,n,\mathcal{A})$ for which $A^2 = A$ and the image of the $\mathcal{A}$-map $\mathcal{A}^n \to \mathcal{A}^n$ given by A is $\mathcal{A}$-isomorphic to P.

Then the pair $(\mathcal{N}(G), \dim)$ satisfies Assumption 1.4.

Proof. The definition of dim above is the so called *Hattori-Stallings rank* of P. It coincides under the identification of the categories $\mathcal{H}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ appearing in Lemma 1.2 to the von Neumann dimension of finitely generated Hilbert $\mathcal{A}$-modules. The condition (i) appearing in Assumption 1.4 is obviously satisfied. The condition (ii) can be easily verified working in $\mathcal{H}(\mathcal{A})$ using the normality of the trace $\operatorname{tr}\colon \mathcal{A} \to \mathbb{C}$. $\square$

We mention that every von Neumann algebra is semihereditary.

Thanks to Theorem 1.5 we can assign to every $\mathcal{A}$-module M its dimension $\dim(M) \in [0,\infty]$ such that properties like Additivity, Cofinality and Continuity hold. This is the basis for the following definitions and the forthcoming applications.

Definition 1.8. (Definition of L^2-Betti numbers for G-spaces and groups). *Let G be a (discrete) group and X be a G-space. Define its p-th* L^2-Betti number *to be*

$$b_p^{(2)}(G) \;=\; \dim\left(H_p(C_*(X) \otimes_{\mathbb{Z}G} \mathcal{N}(G))\right) \quad \in [0,\infty],$$

where $C_*(X)$ is the singular chain complex of X.

Define the p-th L^2-Betti number of G *to be*

$$b_p^{(2)}(G) \;=\; b_p^{(2)}(EG;\mathcal{N}(G)) \quad \in [0,\infty].$$

Remark 1.9 (L^2-Betti numbers for finite von Neumann algebras). Let $HH_p(\mathbb{C}G;\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{\mathrm{op}})$ be the p-th Hochschild homology of $\mathbb{C}G$ with coefficients in the $\mathbb{C}G$-bimodule $\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{\mathrm{op}}$, where $\overline{\otimes}$ means the tensor product of von Neumann algebras. One can show

$$b_p^{(2)}(G) \;=\; \dim_{\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{\mathrm{op}}}\left(HH_p(\mathbb{C}G;\mathcal{N}(G)\overline{\otimes}\mathcal{N}(G)^{\mathrm{op}})\right).$$

Connes and Shlyakhtenko propose the following definition of the L^2-Betti number of a finite von Neumann algebra $\mathcal{A}$:

$$b_p^{(2)}(\mathcal{A}) \;=\; \dim_{\mathcal{A}\overline{\otimes}\mathcal{A}^{\mathrm{op}}}\left(HH_p(\mathcal{A};\mathcal{A}\overline{\otimes}\mathcal{A}^{\mathrm{op}})\right).$$

Notice that for a free group of rank n we have $b_1^{(2)}(F_n) = n-1$. If one could show for a group G, that

$$b_1^{(2)}(G) \;=\; b_1^{(2)}(\mathcal{N}(G)),$$

then one would know that $\mathcal{N}(F_m)$ and $\mathcal{N}(F_n)$ are isomorphic if and only if $m = n$.

2 Rigidity for the Passage from $\mathbb{Z}$ to $\mathcal{N}(G)$

Definition 2.1 (Algebraic middle K-and G-theory of a ring). *Let R be a ring. Define the* projective class group $K_0(R)$ *to be the abelian group whose generators $[P]$ are isomorphism classes of finitely generated projective R-modules and whose relations are $[P_1] = [P_0] + [P_2]$ for every exact sequence of $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.*

Define $G_0(R)$ analogously but replace finitely generated projective by finitely generated everywhere.

Let $GL(R)$ be the colimit of the directed system of groups

$$GL(1,R) \subseteq GL(2,R) \subseteq GL(3,R) \subseteq \ldots,$$

where the various inclusions are given by taking the block sum with the $(1,1)$ *identity matrix. Define*

$$K_1(R) \;=\; GL(R)/[GL(R), GL(R)].$$

Next we summarize what is known about the middle algebraic K- and G-theory and about the L-theory of von Neumann algebras. The following results are due to Murray-von Neumann for K_0 and to Lück-Rœrdam for K_1.

Theorem 2.2. Middle algebraic K-theory of a von Neumann algebra). *Let $\mathcal{A}$ be a von Neumann algebra. Let*

$$\mathcal{A} \;=\; \mathcal{A}_{I_f} \times \mathcal{A}_{I_\infty} \times \mathcal{A}\mathcal{A}_{II_1} \times \mathcal{A}_{II_\infty} \times \mathcal{A}_{III}$$

be its canonical decomposition. Then

(i) We have for $n = 0, 1$ natural isomorphisms

$$K_n(\mathcal{A}) \;=\; K_n(\mathcal{A}_{I_f}) \times K_n(\mathcal{A}_{I_\infty}) \times K_n(\mathcal{A}_{II_1}) \times K_n(\mathcal{A}_{II_\infty}) \times K_n(\mathcal{A}_{III});$$

(ii) We have for $n = 0, 1$

$$K_n(\mathcal{A}_{I_\infty}) \;=\; K_n(\mathcal{A}_{II_\infty}) \;=\; K_n(\mathcal{A}_{III}) \;=\; 0.$$

(iii) The center-valued universal trace induces an injection

$$K_0(\mathcal{A}_{I_f}) \to \mathcal{Z}(\mathcal{A})^{\mathbb{Z}/2},$$

and a bijection

$$K_0(\mathcal{A}_{II_1}) \xrightarrow{\cong} \mathcal{Z}(\mathcal{A})^{\mathbb{Z}/2},$$

where $\mathcal{Z}(\mathcal{A})$ is the center of $\mathcal{A}$ with the $\mathbb{Z}/2$-operation coming from taking the adjoint and the group structure on $\mathcal{Z}(\mathcal{A})^{\mathbb{Z}/2}$ comes from the addition;

(iv) There are isomorphisms

$$K_1(\mathcal{A}_{I_f}) \xrightarrow{\cong} \mathcal{Z}(\mathcal{A})^{\mathrm{inv}},$$

and

$$K_0(\mathcal{A}_{II_1}) \xrightarrow{\cong} \mathcal{Z}(\mathcal{A})^{+},$$

where $\mathcal{Z}(\mathcal{A})^{\mathrm{inv}}$ is the multiplicative group of units in the center $\mathcal{Z}(\mathcal{A})$ and $\mathcal{Z}(\mathcal{A})^{+}$ is the multiplicative group $\{aa^* \mid a \in \mathcal{Z}(\mathcal{A})^{\mathrm{inv}}\}$.

Recall that the topological K-theory of a von Neumann algebra agrees with the projective class group $K_0(\mathcal{A})$ in even degrees and vanishes in odd degrees.

One can also compute the algebraic L-theory. We only state the result for the projective versions.

Theorem 2.3. Projective algebraic L-theory of a von Neumann algebra). *Let $\mathcal{A}$ be a von Neumann algebra. Let*

$$\mathcal{A} \;=\; \mathcal{A}_{I_f} \times \mathcal{A}_{I_\infty} \times \mathcal{A}_{II_1} \times \mathcal{A}_{II_\infty} \times \mathcal{A}_{III}$$

be its canonical decomposition. Then

(i) We have for $n \in \mathbb{Z}$ natural isomorphisms

$$L_p^n(\mathcal{A}) \;=\; L_p^n(\mathcal{A}_{I_f}) \times L_p^n(\mathcal{A}_{I_\infty}) \times L_p^n(\mathcal{A}_{II_1}) \times L_p^n(\mathcal{A}_{II_\infty}) \times L_p^n(\mathcal{A}_{III});$$

(ii) We have for $n = 0, 1$

$$L_p^n(\mathcal{A}_{I_\infty}) \;=\; L_p^n(\mathcal{A}_{II_\infty}) \;=\; L_p^n(\mathcal{A}_{III}) \;=\; 0.$$

(iii) The L^2-signature induces an isomorphism

$$L_p^0(\mathcal{A}) \xrightarrow{\cong} K_0(\mathcal{A}),$$

(iv) We have

$$L_p^1(\mathcal{A}) \;=\; 0;$$

(v) The L-groups are 2-periodic, i.e. $L_p^n(\mathcal{A}) \cong L_p^{n+2}(\mathcal{A})$ for all $n \in \mathbb{Z}$;

(vi) The quadratic and symmetric L-groups agree, i.e. the symmetrization map

$$L_n^p(\mathcal{A}) \xrightarrow{\cong} L_p^n(\mathcal{A})$$

is bijective for $n \in \mathbb{Z}$.

These computations are useful for detecting elements in the K-or L-theory of more complicated rings than $\mathcal{A}$, namely of integral group rings. We mention the following result.

Theorem 2.4 (L.-Rœrdam). *Let G be a group with a finite normal subgroup $H \subset G$. The map induced by induction $K_1(\mathbb{Z}H) \to K_1(\mathbb{Z}G)$ induces a homomorphism*

$$\alpha\colon \mathbb{Q} \otimes_{\mathbb{Z}G} K_1(\mathbb{Z}H) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_1(\mathbb{Z}G),$$

where $\mathbb{Q}$ is equipped with the trivial G-action and the G-action on $K_1(\mathbb{Z}H)$ comes from the conjugation action of G on H.

Then the map α is injective.

This result is predicted by the Farrell-Jones Conjecture for algebraic K-theory. This conjecture is still open. The point is that the conclusion above holds for all groups G.

The proof of the following theorem is based essentially on work due to Linnell. The reduced K-groups $\widetilde{K}_0(R)$ are defined to be the kernel of the obvious homomorphism $K_n(\mathbb{Z}) \to K_n(R)$. Equivalently, $\widetilde{K}_0(R)$ is obtained from $K_0(R)$ by dividing out the subgroup generated by the class $[R]$.

Theorem 2.5. (Change of rings homomorphism from $\mathbb{Z}G$ to $\mathcal{N}(G)$ for $\widetilde{K}_0$). *The change of rings homomorphism*

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathcal{N}(G))$$

is trivial.

Conjecture 2.6 ($K_1(\mathbb{Z}G)$ and the Fuglede-Kadison determinant). *The composition*

$$K_1(\mathbb{Z}G) \to K_1(\mathcal{N}(G)) \to \mathbb{R}$$

is trivial, where the first map is the obvious change of rings homomorphism and the second map to the additive group of real numbers is given by the logarithm of the Fuglede-Kadison determinant.

This conjecture is for instance known for residually amenable groups by a result of Schick which is based on approximations techniques due to Dodziuk, Matthey and Lück. It is not true that $K_1(\mathbb{Z}G) \to K_1(\mathcal{N}(G))$ is trivial.

Conjecture 2.7 (Atiyah Conjecture). *Let G be a torsionfree group and $A \in M(m,n,\mathbb{Z}G)$ be a matrix. It induces a $\mathcal{N}(G)$-homomorphism*

$$r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n.$$

Then

$$\dim(\ker(r_A)) \ \in \ \mathbb{Z}.$$

Recall that $\dim(\ker(r_A))$ is the same as the von Neumann dimension of the kernel of the map of finitely generated Hilbert modules $r_A^{(2)} \colon l^2(G)^m \to l^2(G)^n$ induced by A.

The Atiyah-Conjecture is true for instance for residually torsionfree amenable group by a result of Schick which is based on a deep theorem of Linnell and approximations techniques due to Lück. It implies the version of the Kaplansky Conjecture that for a torsionfree group G the rational group ring $\mathbb{Q}G$ has no non-trivial zero divisors. The Atiyah-Conjecture has also a formulation for groups with torsion with an upper bound on the orders of its finite subgroups.

Theorem 2.8 (L.). *Let G be a torsionfree group such that BG is of finite type and the Atiyah Conjecture is true for it. Let $H_p^{(2)}(EG;l^2(G))$ be its (reduced) L^2-cohomology as defined in Definition 3.1.*

Then for every p there is a integer $n(p) \geq 0$ such that

$$H_p^{(2)}(EG;\mathcal{N}(G)) \ \cong \ l^2(G)^{n(p)}$$

holds as Hilbert $\mathcal{N}(G)$-modules.

A module R-module M is called *flat* if and only if taking the tensor product with M sends short exact sequences to short exact sequences. This equivalent to the condition that $\mathrm{Tor}_p^R(V,M) = 0$ for all $p \geq 1$ and every R-module V.

It would be very convenient if $\mathcal{N}(G)$ were flat as $\mathbb{C}G$-module because then the natural map

$$H_p(C_*(X)) \otimes_{\mathbb{Z}G} \mathcal{N}(G) \to H_p\left(C_*(X) \otimes_{\mathbb{Z}G} \mathcal{N}(G)\right)$$

is bijective for all $p \geq 0$. But this assumption is very unrealistic because of the following conjecture which is known to be true for many groups.

Conjecture 2.9 (Flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$). *The von Neumann algebra $\mathcal{N}(G)$ is flat as $\mathbb{C}G$-module if and only if G is virtually cyclic.*

For dealing with L^2-Betti numbers the following weaker flatness condition is sufficient.

Definition 2.10 (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$). *The von Neumann algebra $\mathcal{N}(G)$ is called* dimension-flat over $\mathbb{C}G$ *if for every $\mathbb{C}G$-module and $p \geq 1$ we have*

$$\dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_p^{\mathbb{C}G}(M; \mathcal{N}(G))\right) = 0.$$

Conjecture 2.11 (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ and amenability). *The von Neumann algebra $\mathcal{N}(G)$ is dimension flat over $\mathbb{C}G$ if and only if G is amenable.*

Theorem 2.12. *(i) If G is amenable, then $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$;*

(ii) If G contains $\mathbb{Z} * \mathbb{Z}$ as a subgroup, then $\mathcal{N}(G)$ is not dimension-flat over $\mathbb{C}G$.

Using an easy spectral sequence argument together with Additivity and Cofinality of the dimension function one can reprove the following result

Theorem 2.13 (Cheeger-Gromov). *All the L^2-Betti numbers of a group which contains an infinite normal amenable subgroup vanish.*

Conjecture 2.14 ($G_0(\mathbb{C}G)$ and amenability). *The following assertions are equivalent:*

(i) $G_0(\mathbb{C}G) \neq 0$;

(ii) $[\mathbb{C}G] \neq 0$ *in* $G_0(\mathbb{C}G)$;

(iii) $[\mathbb{C}G]$ *generates an infinite cyclic subgroup in* $G_0(\mathbb{C}G)$;

(iv) G *is amenable.*

Theorem 2.15. (The class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ and amenability).

(i) Let G be amenable. Then we obtain a well-defined homomorphism

$$d\colon G_0(\mathbb{C}G) \to \mathbb{R}, \quad [M] \mapsto \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} M).$$

It sends $[\mathbb{C}G]$ to 1 and hence $[\mathbb{C}G]$ generates an infinite cyclic subgroup in $G_0(\mathbb{C}G)$;

(ii) Suppose that G contains $\mathbb{Z} * \mathbb{Z}$. Then $[\mathbb{C}G] = 0$ holds in $G_0(\mathbb{C}G)$.

Proof. (i) Everything is obvious except the fact that d is well-defined. One has to check that for an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of finitely generated $\mathbb{C}G$-modules $d([M_1]) = d([M_0]) + d([M_2])$ holds. This follows from the dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$.

(i) Induction with the inclusion $\mathbb{Z} * \mathbb{Z} \to G$ induces a homomorphism

$$G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}]) \to G_0(\mathbb{C}G)$$

which sends $[\mathbb{C}[\mathbb{Z} * \mathbb{Z}]]$ to $[\mathbb{C}G]$. Hence it suffices to show $[\mathbb{C}[\mathbb{Z} * \mathbb{Z}]] = 0$ in $G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$. The cellular chain complex of the universal covering of $S^1 \vee S^1$ yields an exact sequence of $\mathbb{C}[\mathbb{Z}*\mathbb{Z}]$-modules $0 \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}]^2 \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}] \to \mathbb{C} \to 0$, where $\mathbb{C}$ is equipped with the trivial $\mathbb{Z}*\mathbb{Z}$-action. This implies $[\mathbb{C}[\mathbb{Z}*\mathbb{Z}]] = -[\mathbb{C}]$ in $G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$. Hence it suffices to show $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}])$. Choose an epimorphism $f \colon \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$. Restriction with f defines a homomorphism

$$G_0(\mathbb{C}[\mathbb{Z}]) \to G_0(\mathbb{C}[\mathbb{Z} * \mathbb{Z}]).$$

It sends the class of $\mathbb{C}$ viewed as trivial $\mathbb{C}[\mathbb{Z}]$-module to the class of $\mathbb{C}$ viewed as trivial $R[\mathbb{Z} * \mathbb{Z}]$-module. Hence it remains to show $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}[\mathbb{Z}])$. This follows from the exact sequence $0 \to \mathbb{C}[\mathbb{Z}] \xrightarrow{s-1} \mathbb{C}[\mathbb{Z}] \to \mathbb{C} \to 0$ for s a generator of $\mathbb{Z}$ which comes from the cellular $\mathbb{C}[\mathbb{Z}]$-chain complex of $\widetilde{S^1}$. □

3 Applications to Geometry and Group Theory

Definition 3.1. (L^2-Betti numbers of universal coverings of CW-complexes of finite type). *Let X be a connected CW-complex of finite type, i.e. all its skeleta are finite but X may possibly be infinite-dimensional. Let π be its fundamental group and let $\widetilde{X} \to X$ be its universal covering. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.*

Define its cellular L^2-chain complex $C_*^{(2)}(X)$ *to be the Hilbert $\mathcal{N}(G)$-chain complex*

$$C_*^{(2)}(X) \ := \ l^2(G) \otimes_{\mathbb{Z}G} C_*(X).$$

It looks like

$$\ldots \xrightarrow{c_{p+1}^{(2)}} \bigoplus_{i=1}^{\beta_p} l^2(\pi) \xrightarrow{c_p^{(2)}} \bigoplus_{i=1}^{\beta_{p-1}} l^2(\pi) \xrightarrow{c_{p-1}^{(2)}} \ldots,$$

where β_p is the number of p-cells in X and each differential is a bounded π-equivariant operator.

Define its p-th (reduced) L^2-homology *to be the finitely generated Hilbert $\mathcal{N}(G)$-module*

$$H_p^{(2)}(\widetilde{X}; l^2(\pi)) := \ker(c_p^{(2)})/\overline{\operatorname{im}(c_{p+1}^{(2)})}.$$

Define its p-th L^2-Betti number *to be the von Neumann dimension of* $H_p^{(2)}(\widetilde{X})$

$$b_p^{(2)}(\widetilde{X}) \ = \ \dim_{\mathcal{N}(\pi)}\left(H_p^{(2)}(\widetilde{X}; l^2(\pi))\right) \quad \in [0, \infty).$$

This agrees with the more general definition of L^2-Betti numbers presented in the first talk.

Theorem 3.2. (Basic Properties of L^2-Betti numbers).

(i) **Homotopy invariance**

If $f\colon X \to Y$ be homotopy equivalence of connected CW-complexes of finite type, then we get for all $p \geq 0$

$$b_p^{(2)}(X) \;=\; b_p^{(2)}(Y);$$

(ii) **Euler-Poincaré formula (Atiyah)**

Let X be a connected finite CW-complex with Euler characteristic $\chi(X)$. Then

$$\chi(X) \;=\; \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{X});$$

(iii) **Poincaré duality**

Let M be a connected closed manifold of dimension n. Then we get for every $p \geq 0$

$$b_p^{(2)}(\widetilde{M}) \;=\; b_{n-p}^{(2)}(\widetilde{M});$$

(iv) **Künneth formula (Zucker)**

Let X and Y be connected CW-complexes of finite type. Then we get for all $n \geq 0$

$$b_n^{(2)}(\widetilde{X \times Y}) \;=\; \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

(v) **Morse inequalities (Novikov-Shubin)**

Let X be a connected CW-complex of finite type. Let $\beta_p(X)$ be the number of p-cells in X. Then we get for $n \geq 0$

$$\sum_{p=0}^{n} (-1)^{n-p} \cdot b_p^{(2)}(\widetilde{X}) \;\leq\; \sum_{p=0}^{n} (-1)^{n-p} \cdot \beta_p(X);$$

(vi) **Wedges**

Let $X_1, X_2, \ldots, X_r$ be connected CW-complexes of finite type and $X = \vee_{i=1}^{r} X_i$ be their wedge. Then

$$b_1^{(2)}(\widetilde{X}) - b_0^{(2)}(\widetilde{X}) \;=\; r - 1 + \sum_{j=1}^{r} \left(b_1^{(2)}(\widetilde{X_j}) - b_0^{(2)}(\widetilde{X_j}) \right);$$

and for $p \geq 2$

$$b_p^{(2)}(\widetilde{X}) \;=\; \sum_{j=1}^{r} b_p^{(2)}(\widetilde{X_j});$$

(vii) **Zero-th L^2-Betti number**

Let X be a connected CW-complex of finite type. Then

$$b_0^{(2)}(X) \;=\; \frac{1}{|\pi|};$$

(viii) **Finite coverings**

Let $p\colon X \to Y$ be a covering of connected CW-complexes of finite type with a finite number n of sheets. Then we get for all $p \geq 0$

$$n \cdot b_p^{(2)}(\widetilde{X}) \;=\; b_p^{(2)}(\widetilde{Y}).$$

Notice that the assertions (i), (ii), (iii), (iv), (v) and (vi) appearing in Theorem 3.2 have obvious analogues for the classical Betti numbers, whereas assertions (vii) and (viii) mark a basic difference between L^2-Betti numbers and Betti numbers.

Remark 3.3 (Relation between Betti numbers and L^2-Betti numbers). One can show that the only general relation between the Betti numbers and L^2-Betti numbers for a connected finite CW-complex X is given by the Euler-Poincaré formula

$$\sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\widetilde{X}) \;=\; \sum_{p\geq 0}(-1)^p \cdot b_p(X).$$

On the other hand L^2-Betti numbers are in the following sense asymptotic Betti numbers. Namely, if π admits a sequence of nested normal subgroups of finite index

$$\pi \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \ldots$$

with $\bigcap_{n=1}^{\infty} \Gamma_n$, then by a result of Lück

$$b_p^{(2)}(\widetilde{X}) \;=\; \lim_{n\to\infty} \frac{b_p(X_n)}{[\pi : \Gamma_n]},$$

where $X_n \to X$ is the finite sheeted covering associated to $\Gamma_n \subseteq \pi$.

One may also say that L^2-Betti numbers are obtained from the classical Betti numbers by forcing multiplicativity for finite sheeted coverings to be true (see Theorem 3.2 (viii)).

Example 3.4 (L^2-Betti numbers of CW-complexes covering themselves). Consider a connected CW-complex X of finite type for which there is a self-covering $X \to X$ with d-sheets for some integer $d \geq 2$. Then we get from Theorem 3.2 (viii) for all $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) \;=\; 0.$$

This implies for every finite CW-complex X of finite type and all $p \geq 0$

$$b_p^{(2)}(\widetilde{S^1 \times X}) \;=\; 0.$$

Example 3.5. Let F_g be the orientable closed surface. Since $F_0 = S^2$ is simply-connected, we get

$$\begin{array}{lclcll} b_p^{(2)}(\widetilde{F_0}) & = & b_p(S^2) & = & 1 & \text{if } p = 0,2; \\ b_p^{(2)}(\widetilde{F_0}) & = & b_p(S^2) & = & 0 & \text{if } p \neq 0,2. \end{array}$$

If $g \geq 0$, then $\pi_1(F_g)$ is infinite and hence $b_0^{(2)}(\widetilde{F_g}) = 0$. By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$. Since $\dim(F_g) = 2$, we get $b_p^{(2)}(\widetilde{F_g}) = 0$ for $p \geq 3$. Using the Euler-Poincaré formula we get

$$\begin{array}{lcl} b_1^{(2)}(\widetilde{F_g}) & = & -\chi(F_g) \;=\; 2g-2; \\ b_p^{(2)}(\widetilde{F_0}) & = & 0 \quad \text{for } p \neq 1. \end{array}$$

Theorem 3.6. (L^2-Betti numbers of 3-manifolds, Lott-L.)
Let M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of compact connected orientable prime 3-manifolds M_j which are Haken oder satisfy Thurston's Geometrization Conjecture. Assume that $\pi_1(M)$ is infinite. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ are given by

$$\begin{array}{lcl} b_p^{(2)}(\widetilde{M}) & = & 0 \quad \text{for } p \neq 1,2; \\ b_1^{(2)}(\widetilde{M}) & = & (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right| - \chi(M); \\ b_2^{(2)}(\widetilde{M}) & = & (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|. \end{array}$$

Theorem 3.7. (L^2-Betti numbers and S^1-actions, L.).
Let X be a connected S^1-CW-complex of finite type, for instance a connected compact manifold with S^1-action. Suppose that for one orbit $S^1 \cdot x$ (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) \;=\; 0.$$

Theorem 3.8. (L^2-Betti numbers and aspherical S^1-manifolds, L.).
Let M be an aspherical closed manifold with non-trivial S^1-action. Then the action has no fixed points and the inclusion of any orbit into X induces an injection on the fundamental groups. All L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ are trivial and $\chi(M) = 0$.

Theorem 3.9. Vanishing of L^2-Betti numbers of mapping tori, L.).
Let $f \colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{T_f}) \;=\; 0.$$

Proof. There is a d-sheeted covering $p\colon E \to T_f$ such that E and T_{f^d} are homotopy equivalent. Hence

$$b_p^{(2)}(\widetilde{T_f}) \;=\; \frac{b_p^{(2)}(\widetilde{T_{f^d}})}{d}.$$

If $\beta_p(X)$ is the number of p-cells, then there is a CW-structure on T_{f^d} with $\beta(T_{f^d}) = \beta_p(X) + \beta_{p-1}(X)$. We have

$$b_p^{(2)}(\widetilde{T_{f^d}}) \;=\; \dim\left(H_p^{(2)}(C_p^{(2)}(\widetilde{T_{f^d}}))\right) \;\leq\; \dim\left(C_p^{(2)}(\widetilde{T_{f^d}})\right) \;=\; \beta_p(T_{f^d}).$$

This implies for all $d \geq 1$

$$0 \;\leq\; b_p^{(2)}(\widetilde{T_f}) \;\leq\; \frac{\beta_p(X) + \beta_{p-1}(X)}{d}.$$

Taking the limit for $d \to \infty$ yields the claim. □

Theorem 3.10. **L^2-Hodge-de Rham Theorem, Dodziuk).** *Let M be a connected closed Riemannian manifold. Let*

$$\mathcal{H}_{(2)}^p(\widetilde{M}) = \left\{\omega \in \Omega^p(\widetilde{M}) \;\middle|\; \Delta_p(\omega) = 0, \int_{\widetilde{M}} \omega \wedge *\omega \;<\; \infty\right\}$$

be the space of harmonic L^2-forms on $\widetilde{M}$. Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\mathcal{H}_{(2)}^p(\widetilde{M}) \;\xrightarrow{\cong}\; H_{(2)}^p(\widetilde{M}, l^2(\pi)).$$

Moreover we get

$$b_p^{(2)}(M) \;=\; \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}\left(e^{-t\Delta_p}(x,x)\right)\, dvol.$$

where $\mathcal{F}$ is a fundamental domain for the G-action and $e^{-t\Delta_p}(x,y)$ is the heat kernel on $\widetilde{M}$.

Theorem 3.11. **(L^2-Betti numbers of hyperbolic manifolds , Dodziuk).** *Let M be a hyperbolic closed Riemannian manifold of dimension n. Then:*

$$b_p^{(2)}(\widetilde{M}) \qquad \begin{cases} = 0 & , \text{ if } 2p \neq n; \\ > 0 & , \text{ if } 2p = n. \end{cases}$$

Proof. A direct computation shows that $\mathcal{H}_{(2)}^p(\mathcal{H}^n)$ is not zero if and only if $2p = n$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathcal{H}^n$. □

Corollary 3.12 **(S^1-actions on hyperbolic manifolds).** *Let M be a closed hyperbolic manifold. Then it carries no non-trivial S^1-action. If its dimension is even, it cannot fiber over a circle, or more generally, cannot be homotopy equivalent to a mapping torus of an selfhomotopy equivalence of a finite connected CW-complex.*

Proof. If the dimension of M is even, this follows from Theorem 3.8, Theorem 3.9 and Theorem 3.11. In dimension odd one has to use L^2-torsion to get the result. $\square$

Definition 3.13 (Deficiency). *Let G be a finitely presented group. Define its* deficiency $\operatorname{def}(G)$ *to be the maximum $g(P) - r(P)$, where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.*

Example 3.14 (The deficiency of some elementary groups). The free group F_g has the obvious presentation $\langle s_1, s_2, \ldots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\operatorname{def}(F_g) = g$.

The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n = 1 \rangle$ realizes the deficiency.

The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is -1, the obvious presentation $\langle s, t \mid s^n = t^n = [s,t] = 1 \rangle$ realizes the deficiency.

If G is a finite group, $\operatorname{def}(G) \leq 0$.

Example 3.15 (Non-additivity of deficieny). The deficiency is not additive under free products by the following example due to Hog, Lustig and Metzler. The group $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle.$$

One may think that its deficiency is -2. However, it turns out that its deficiency is -1 and is realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

Lemma 3.16. *Let G be a finitely presented group. Then*

$$\operatorname{def}(G) \ \leq \ 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof. We have to show for any presentation P that

$$g(P) - r(P) \ \leq \ 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Its fundamental group is G. It has 1 zero-cell, $g(P)$ one-cells and $r(P)$ two-cells and no further cells. Hence

$$\chi(X) \ = \ 1 - g(P) + r(P) \ = \ b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{aligned} b_p^{(2)}(\widetilde{X}) &= b_p^{(2)}(G) \qquad \text{for } p = 0, 1; \\ b_2^{(2)}(\widetilde{X}) &\geq b_2^{(2)}(G). \end{aligned}$$

$\square$

Theorem 3.17 (Deficiency and extensions, L.). *Let* $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented,* H *is finitely generated and* K *is infinite. Then:*

(i) $b_1^{(2)}(G) = 0$*;*

(ii) $\operatorname{def}(G) \leq 1$*;*

(iii) Let M *be a closed oriented* 4*-manifold with* G *as fundamental group. Then*

$$|\operatorname{sign}(M)| \;\leq\; \chi(M).$$

Proof. Assertion (i) is the hard part. It implies Assertion (ii) by Lemma 3.16 and assertion (iii) by the L^2-index theorem of Atiyah. □

We mention the following further two applications of L^2-invariants, namely of the L^2-signature and L^2-ρ-invariants, combined with the computations of algebraic K- and L-groups of finite von Neumann algebras.

Theorem 3.18 (Cochran-Orr-Teichner). *There are non-slice knots in* 3*-space whose Casson-Gordon invariants are all trivial.*

Theorem 3.19 (Chang-Weinberger). *Let* M^{4k+3} *be a closed oriented smooth manifold for* $k \geq 1$ *whose fundamental group has torsion. Then there are infinitely many smooth manifolds which are homotopy equivalent to M (and even simply and tangentially homotopy equivalent to M) but not homeomorphic to M.*

Finally we state two open conjectures which have gained a lot of attention during the last years and certainly will create further work in the future. Some of the results give evidence or prove them in special cases and they are known for certain classes of groups or manifolds. The first conjecture is the space version of the previous Atiyah Conjecture 2.7.

Conjecture 3.20 (Atiyah Conjecture). *If* X *is a connected CW-complex of finite type with torsionfree fundamental group, then all its* L^2*-Betti numbers* $b_p^{(2)}(\widetilde{X})$ *are integral.*

Conjecture 3.21 (Singer Conjecture). *Let* M *be a closed aspherical manifold of dimension* n*. Then*

$$b_p^{(2)}(\widetilde{M};\mathcal{N}(G)) \;=\; 0 \quad \textit{if } 2p \neq n.$$

If the dimension $n = 2k$ *is even, then*

$$(-1)^k \cdot \chi(M) \geq 0.$$

If the dimension $n = 2k$ *is even and* M *carries a Riemannian metric with negative sectional curvature, then*

$$\begin{aligned} b_k^{(2)}(\widetilde{M}) &\;>\; 0; \\ (-1)^k \cdot \chi(M) &\;>\; 0. \end{aligned}$$