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THE TRANSFER MAPS INDUCED IN THE ALGEBRAIC K_0- AND K_1-GROUPS BY A FIBRATION I

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0. Introduction.

A. ABSTRACT. Given a fibration $F \to E \xrightarrow{p} B$ with B and E connected and F of the homotopy type of a finitely dominated CW-complex, we define algebraic transfer maps $p_n^*: K_n(\mathbf{Z}[\pi_1(B)]) \to K_n(\mathbf{Z}[\pi_1(E)])$ for $n = 0, 1$. If p is a PL-bundle p_1^* reduces to the geometric transfer $p^!: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$ constructed in Anderson [2]. The homomorphism $p^!$ sends the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(B))$ of a homotopy equivalence $f: B_1 \to B$ between finite simplicial complexes to $\tau(\bar{f}) \in \mathrm{Wh}(\pi_1(E))$ if $\bar{f}$ is the map induced by the pull-back-construction applied to f and p:

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow p \\ B_1 & \xrightarrow{f} & B \end{array}$$

The projective class group transfer p_0^* agrees with the geometric transfer $p^!$ constructed in Ehrlich [9] for a fibration $F \to E \to B$ with F and B of the homotopy type of a finitely dominated CW-complex and B and E connected. Given a finitely dominated CW-complex B_1 with Wall obstruction $w(B_1)$ and a map $f: B_1 \to B$, Ehrlich defines: $p^!(f_*(w(B_1))) = \bar{f}_*(w(E_1))$.

One can find algebraic computations of $p^!$ in special cases in Anderson [1], Ehrlich [8], [9], Munkholm [14], Munkholm–Pedersen [15] using the homology of a certain covering of the fibre. By writing down explicit matrices representing elements in the K-groups, an algebraic description of $p^!$ is stated in Munkholm–Pedersen [16], Munkholm–Ranicki [17] if the fibre is the one dimensional sphere S^1. Our goal is to give

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an algebraic description in general. This can be used to calculate the Wall obstruction of the total space from the one of the base space and similarly for the Whitehead torsion.

B. A MOTIVATION AND SURVEY OF THE CONSTRUCTION OF p^*. In order to compute the geometric transfer we have to get some information about the cellular $\mathbb{Z}[\pi_1(E)]$-chain complex $C^c_*(\tilde{E})$ of the universal covering of E in terms of F and B. It suffices to do this in a special situation since an element $\eta \in K_0(\mathbb{Z}[\pi_1(B)])$ respectively $\mathrm{Wh}(\pi_1(B))$ can be geometrically realized in the following simple way. One constructs a space B_1 by attaching cells to B only in dimensions k and $k+1$ for $k \geqq 2$ and a map $r_1: B_1 \to B$ with $r_1|B = \mathrm{ID}$ such that $r_{1_*}(w(B_1)) = \eta$, respectively $\tau(r_1) = \eta$, holds (Theorem 2.1). Given a pair of fibrations

$$F \to (E_1, E) \xrightarrow{(p_1, p)} (B_1, B),$$

we have only to calculate $C^c_*(\tilde{E}_1, \tilde{E})$ in terms of the fibre and the cellular $\mathbb{Z}[\pi_1(B)]$-chain complex $C^c_*(\tilde{B}_1, \tilde{B})$ concentrated in two dimensions.

Let $\hat{p}: \tilde{E} \to B$ be the composition of $p: E \to B$ and the universal covering of E. This is a $\pi_1(E)$-equivariant fibration. The transport of the fibre $\hat{F} = \hat{p}^{-1}(b)$ along paths in B defines a homomorphism

$$u: \pi_1(B) \to [\hat{F}, \hat{F}]_{\pi_1(E)}$$

into the monoid of $\pi_1(E)$-homotopy classes of $\pi_1(E)$-self maps of $\hat{F}$. Over a cell D of (B_1, B) the space $\tilde{E}_1$ looks like $\hat{F} \times D$. It turns out that $C^c_*(\tilde{B}_1, \tilde{B})$ and u determine completely how these pieces $\hat{F} \times D$ are glued together. Namely, we assign to u a functor

$$U: \mathbb{Z}[\pi_1(B)] - \mathrm{BMOD} \to \mathbb{Z}[\pi_1(E)] - \mathrm{CC}$$

from the category of based free $\mathbb{Z}[\pi_1(B)]$-modules into the homotopy category of projective $\mathbb{Z}[\pi_1(E)]$-chain complexes such that for the non-trivial differential d of $C^c_*(\tilde{B}_1, \tilde{B})$ and a representative ε_* of $U(d)$ the mapping cone of ε_* is (simple) homotopy equivalent to $C^c_*(\tilde{E}_1, \tilde{E})$ (Theorem 2.2).

In section 3 we will assign to U a homomorphism

$$U_*: K_n(\mathbb{Z}[\pi_1(B)]) \to K_n(\mathbb{Z}[\pi_1(E)])$$

in a purely algebraic manner such that

$$U_*(w(C^c_*(\tilde{B}_1, \tilde{B}))) = w(C^c_*(\tilde{E}_1, \tilde{E}))$$

respectively $U_*(\tau(C^c_*(\tilde{B}_1, \tilde{B}))) = \tau(C^c_*(\tilde{E}_1, \tilde{E}))$ is valid. This map U_* is the algebraic transfer. (If one is interested only in the algebraic construction it suffices to read the self contained sections 3 and 4.)

C. Review of results. The main result of this paper is that the geometric and algebraic transfer coincide (Theorem 5.4). We explain in section 6 how $u\colon \pi_1(B) \to [\hat{F}, \hat{F}]_{\pi_1(E)}$ is determined by the homotopy operation of $\pi_1(E)$ on the pointed fibre. We will use this algebraic description to make some computations of the transfer in part II (to appear in J. Pure Appl. Algebra). It turns out that all the results for orientable S^1-fibrations in Munkholm–Pedersen [16] can be extended to fibrations which are untwisted i.e. the homotopy action of $\pi_1(E)$ on the pointed fibre is trivial. This includes orientable fibrations with a connected H-space as fibre.

One of the most interesting results will be the following. Let $F \to E \to B$ be an untwisted fibration with $\pi_1(F)$ infinite and $\pi_1(B)$ finite. Then p_n^* is zero for $n = 0,1$ in all cases except the one where $\pi_1(F) \cong \mathbb{Z}$, $\pi_1(E)$ finite and $n = 1$ holds. In this special case p_1^* is $\chi(\tilde{F}) \cdot \beta$ with $\chi(\tilde{F})$ the Euler characteristic of the universal covering of F and β the transfer of an orientable S^1-fibration with the same fundamental group data. The homomorphism β is extensively discussed in Oliver [18] and not zero in general.

D. Further problems. It is possible to write down an algebraic transfer for L-theory and lower K-theory but it is not clear whether there exists a transfer homomorphism for higher K-theory. As the general behaviour of an untwisted fibration with a finite odd dimensional Poincaré complex as fibre corresponds to the one of an orientable S^1-fibration it would be interesting to have an example of such a fibration for which one could prove more easily than for an orientable S^1-fibration in Oliver [18] that the induced transfer on K_1 is not trivial. In this context the question arises how to construct an untwisted fibration $p\colon E \to B$ for which the kernel of $p_*\colon \pi_1(E) \to \pi_1(B)$ is cyclic and the transfer on K_1 is not induced by the transfer of an S^1-fibration with the same fundamental group datas.

This paper essentially is a part of the author's Ph. D. thesis (Göttingen 1984). Some details which are not explained here can be found in it. I would like to thank Prof. Tammo tom Dieck for his help.

The paper is divided into the following sections:

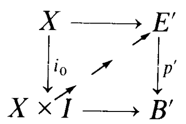
1. The chain homotopy representation associated to a fibration.
2. Computations of cellular chain complexes.
3. Invariants for chain complexes in K_0 and K_1.

4. The algebraic transfer.
5. The algebraic transfer induced by a fibration.
6. The homotopy operation of $\pi_1(E)$ on the pointed fibre determines the chain homotopy representation.
7. Proof of Theorem 2.2.

1. The chain homotopy representation associated to a fibration.

Let $p\colon (E,e) \to (B,b)$ be a fibration with connected E and B and with fibre $F = p^{-1}(b)$. Let $q_E\colon (\tilde{E},\tilde{e}) \to (E,e)$ be the universal covering of E and let $\hat{p}\colon (\tilde{E},\tilde{e}) \to (B,b)$ be the composition $p \circ q_E$ with fibre $\hat{F} = \hat{p}^{-1}(b)$. One easily checks that $\hat{p}$ is a Γ-fibration for $\Gamma = \pi_1(E,e)$, that is:

DEFINITION 1.1. Let Γ be a discrete group. A Γ-fibration is a Γ-equivariant map $p'\colon E' \to B'$ such that Γ acts trivially on B' and p' has the Γ-equivariant homotopy lifting property for any Γ-space X



Here I is the unit interval with the trivial action and the diagram is a diagram of Γ-spaces. We have made the assumption that Γ acts trivially on B', as it is always fulfilled in the cases we will regard. For $\Gamma = 1$ a Γ-fibration is a usual fibration (see Switzer [24, p. 52], Whitehead [28, p. 29]). Given a Γ-fibration $p'\colon E' \to B'$, the Γ-equivariant transport of the fibre F' along paths in the base space defines a homomorphism $\pi_1(B') \to [F',F']_\Gamma$ into the monoid of Γ-homotopy classes of Γ-self maps of F' (see section 8 A, Switzer [24, p. 343], Whitehead [28, p. 186]). Applying this to $\hat{p} = p \circ q_E$, we get a homomorphism $u\colon \pi \to [\hat{F},\hat{F}]_\Gamma$ for $\pi = \pi_1(B,b)$ and $\Gamma = \pi_1(E,e)$. Now we assume that F has the homotopy type of a CW-complex so that we can choose a Γ-complex Y and a Γ-homotopy equivalence $\lambda\colon Y \to \hat{F}$. Let $[C^c_*(Y), C^c_*(Y)]_\Gamma$ be the ring of $\mathbb{Z}[\Gamma]$-homotopy classes of $\mathbb{Z}[\Gamma]$-self maps of the cellular $\mathbb{Z}[\Gamma]$-chain complex of Y. For a ring R we denote by R° the dual ring i.e. the multiplication $r \cdot s$ in R° is given by $s \cdot r$ in R. We define

$$U_{p,e}\colon \mathbb{Z}[\pi] \to [C^c_*(Y), C^c_*(Y)]^\circ_\Gamma$$

by

$$w \mapsto [C^c_*(\lambda^{-1} \circ u(w^{-1}) \circ \lambda)].$$

DEFINITION 1.2. We call $U_{p,e}\colon \mathbb{Z}[\pi] \to [C^c_*(Y), C^c_*(Y)]^\circ_\Gamma$ the chain homotopy representation associated to the fibration $p\colon (E,e) \to (B,b)$.

Up to conjugation with a $\mathbb{Z}[\Gamma]$-homotopy equivalence $U_{p,e}$ is uniquely determined by p and e. We will see that $U_{p,e}$ contains the whole information needed to describe the geometric transfer algebraically. If $\mathbb{Z}[\pi]$-BMOD is the category of based $\mathbb{Z}[\pi]$-modules and $\mathbb{Z}[\Gamma]$-CC the homotopy category of $\mathbb{Z}[\Gamma]$-chain complexes, we can interprete $\boldsymbol{U} = U_{p,e}$ as a covariant functor

$$U\colon \mathbb{Z}[\pi]\text{-BMOD} \to \mathbb{Z}[\Gamma]\text{-CC}$$

also denoted by U and compatible with $\oplus$. A morphism $\bigoplus_I \mathbb{Z}[\pi] \to \bigoplus_J \mathbb{Z}[\pi]$ of based (left) modules given by a matrix $(d_{i,j})$ with entries in $\mathbb{Z}[\pi]$ is mapped by U to the homotopy class

$$(U(d_{i,j}))^{\mathrm{tr}}\colon \bigoplus_I C^c_*(Y) \to \bigoplus_J C^c_*(Y).$$

We denote by tr the transposed matrix. One should notice that in the first case $(d_{i,j})$ operates from the right and in the second case $(U(d_{i,j}))^{\mathrm{tr}}$ from the left.

2. Computations of cellular chain complexes.

This chapter contains two theorems which connect geometry and algebra and lead to an algebraic description of the geometric transfer. The first one gives us a very simple geometric realization of elements in $K_0(\mathbb{Z}[\pi])$ and $\mathrm{Wh}(\pi)$ for a connected space B with $\pi = \pi_1(B)$ and universal covering $\widetilde{B}$.

THEOREM 2.1. *Let (D_*, d_*) be a based free $\mathbb{Z}[\pi]$-chain complex concentrated in dimension k and $k+1$ for $k \geqq 2$. Then there exist a relative π-CW-complex $(\widetilde{B}_1, \widetilde{B})$ and a π-map $r_1\colon \widetilde{B}_1 \to \widetilde{B}$ with $r_1 | \widetilde{B}_1 = \mathrm{ID}$ such that $C^c_*(\widetilde{B}_1, \widetilde{B})$ is based isomorphic to (D_*, d_*).*

PROOF. We write (D_*, d_*) as

$$\ldots \to 0 \to \bigoplus_I \mathbb{Z}[\pi] \xrightarrow{(d_{i,j})} \bigoplus_J \mathbb{Z}[\pi] \to 0 \to \ldots .$$

For each $j \in J$ we attach a copy of $\pi \times S^k$ to $\widetilde{B}$ by $\pi \times \{*\} \to \widetilde{B}$ $(w,*) \mapsto w \cdot \widetilde{b}$ for some base points $*$ in S^k and $\widetilde{b}$ in $\widetilde{B}$. We get $(\widetilde{B}_0, \widetilde{B})$ with the obvious retraction $r_0 \colon \widetilde{B}_0 \to \widetilde{B}$. We write $d_{i,j}$ as $\sum_{w \in \pi} a(i,j,w) \cdot w$. Let

$$a(i,j,w) \colon (S^k, *) \to (S^k, *)$$

be a map of degree $a(i,j,w)$. The map

$$\alpha(w,j) \colon (S^k, *) \to (\widetilde{B}_0, w \cdot \widetilde{b})$$

identifies S^k and the part $\{w\} \times S^k$ of the jth cell $\pi \times S^k$ in $\widetilde{B}_0$. Let

$$\beta(i,j,w) \colon (S^k, *) \to (\widetilde{B}_0, \widetilde{b})$$

be a map homotopic to

$$\alpha(w,j) \circ a(i,j,w) \colon (S^k, *) \to (\widetilde{B}_0, w \cdot \widetilde{b})$$

along a path in $\widetilde{B}$ from $\widetilde{b}$ to $w \cdot \widetilde{b}$. For each $i \in I$ choose a representative

$$\gamma_i \colon (S^k, *) \to (\widetilde{B}_0, \widetilde{b})$$

of $\sum \beta(i,j,w)$ in $\pi_k(\widetilde{B}_0, \widetilde{B}, \widetilde{b})$, where the sum is taken over all $(j,w) \in J \times \pi$ with $a(i,j,w) \neq 0$. Now attach a cell $\pi \times D^{k+1}$ to $\widetilde{B}_0$ by $\pi \times S^k \to \widetilde{B}_0$ $(w,x) \mapsto w \cdot \gamma_i(x)$ for each $i \in I$. As

$$r_{0_*} \colon \pi_k(\widetilde{B}_0, \widetilde{b}) \to \pi_k(\widetilde{B}, \widetilde{b})$$

maps $[\gamma_i]$ to zero we can extend r_0 to r_1.

The next theorem contains the computation of the cellular chain complex of the universal covering of the total space in the situation of Theorem 2.1.

The mapping cone $\mathrm{Cone}(f_*)_*$ of a chain map $f_* \colon C_* \to D_*$ is defined as

$$\begin{array}{ccc} \mathrm{Cone}(f_*)_n & & \mathrm{Cone}(f_*)_{n-1} \\ \| & \begin{bmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{bmatrix} & \| \\ C_{n-1} \oplus D_n & \xrightarrow{\hspace{3em}} & C_{n-2} \oplus D_{n-1} \end{array}$$

The suspension ΣC_* is $\mathrm{Cone}(C_* \to 0_*)$ and $\Sigma^k C_*$ is the k-fold suspension.

Let $(p_1, p): (E_1, E) \to (B_1, B)$ be a pair of fibrations (respectively PL-bundles) with

$$\pi = \pi_1(B) = \pi_1(B_1) \quad \text{and} \quad \Gamma = \pi_1(E) = \pi_1(E_1).$$

Let $(g_1, g): (B_1, B) \to (A_1, A)$ be a homotopy equivalence (respectively simple homotopy equivalence) into a pair of CW-complexes such that the cellular chain complex $C^c_*(\tilde{A}_1, \tilde{A})$ of the pair of universal coverings is concentrated in dimension k and $k+1$ for $k \geqq 2$. Then its $(k+1)$th-differential

$$d: \bigoplus_I \mathsf{Z}[\pi] \xrightarrow[(d_{i,j})]{} \bigoplus_J \mathsf{Z}[\pi]$$

defines a morphism in $\mathsf{Z}[\pi]$-BMOD. Let $U: \mathsf{Z}[\pi]\text{-BMOD} \to \mathsf{Z}[\Gamma]\text{-CC}$ be the functor defined by the chain homotopy representation associated to $p: E \to B$ in section 1.

THEOREM 2.2. *Then there exists a pair of Γ-homotopy equivalences (respectively simple homotopy equivalences) $(f_1, f): (\tilde{E}_1, \tilde{E}) \to (X_1, X)$ into a pair of Γ-CW-complexes and a $\mathsf{Z}[\Gamma]$-chain map*

$$\varepsilon_*: \bigoplus_I C^c_*(Y) \to \bigoplus_J C^c_*(Y)$$

such that ε_* represents $U(d)$ and $C^c_*(X_1, X)$ is based isomorphic to $\text{Cone}(\Sigma^k \varepsilon_*)_*$.

The proof of Theorem 2.2 is referred to section 7. The theorem leads to the following construction of the algebraic transfer and explains the role of the chain homotopy representation. Corollary 7.5 is responsible for the appearance of the chain homotopy representation in Theorem 2.2.

3. Invariants for chain complexes in K_0 and K_1.

Let R and S be associative rings with unit. A functor F between the corresponding categories of finitely generated projective modules compatible with $\oplus$ induces a homomorphism $F_*: K_n(R) \to K_n(S)$ for all $n \geqq 0$ (see Quillen [20, p. 95]). We want to generalize this for K_0 and K_1. Namely, we will assign to a functor F: R-BMOD $\to$ S-FDCC from the category of finitely generated based R-modules into the homotopy category of finitely dominated projective S-chain complexes compatible with $\oplus$ a map $F_*: K_n(R) \to K_n(S)$ for $n = 0,1$. The main difficulty lies in the fact that S-FDCC is an additive category but not an exact category in the sense of Quillen [20, p. 92] or a category with cofibrations in the sense of Waldhausen [25]. Namely, we cannot define a kernel or cokernel of a homotopy class of chain maps. We will approximate the notion of a cokernel by taking the mapping cone of some representative of the homotopy class. This will not determine a unique chain complex but a (simple) homotopy class.

Our program to construct F_* can be outlined as follows: We will assign to each self-homotopy equivalence (respectively homotopy projection) $f_*: C_* \to C_*$ in S-FDCC an element $t(f_*) \in K_1(S)$ (respectively $w(f_*) \in K_0(S)$). Then we can define:

Definition 3.1. Given a functor F: R-BMOD $\to$ S-FDCC of additive categories we define $F_*: K_n(R) \to K_n(S)$ for $n = 0,1$ in the following way: Let $\eta \in K_1(R)$ respectively $K_0(R)$ be represented by an automorphism $f: R^m \to R^m$ (respectively the image of a projection $f: R^m \to R^m$ (that is $f \circ f = f$)). Then define $F_*(\eta)$ as $t(F(f))$ respectively $w(F(f))$.

Before we give the definition of t and w we introduce some notation: Modules are always left modules. Each chain complex C_* is positive and projective, i.e. $C_n = 0$ for $n < 0$ and C_n is projective for $n \geqq 0$. We call C_* finite if C_* is finite dimensional and each C_n finitely generated and projective. We write $C_{\text{even}} = \bigoplus_n C_{2n}$ and $C_{\text{odd}} = \bigoplus_n C_{2n+1}$.

Definition 3.2. Let $f_*: C_* \to C_*$ be a self-equivalence of a finitely dominated projective chain complex. Choose a finite projective S-chain complex P_* and homotopy equivalences $h_*: C_* \to P_*$ and $g_*: P_* \to P_*$ with $h_* \circ f_* \simeq g_* \circ h_*$. If c_* is the differential and γ_* a chain contraction of $\text{Cone}(g_*)_*$ define

$$\varphi : \text{Cone}(g_*)_{\text{odd}} \to \text{Cone}(g_*)_{\text{even}}$$

by $(c_* + \gamma_*)$. Then φ is an automorphism of the finitely generated projective module $\bigoplus_n P_n$. Define $t(f_*) \in K_1(S)$ as the class of φ.

One can easily prove using Gersten [11] that t is well defined. Compare also with the definition of the Whitehead torsion in Cohen [5, p. 52] and of the absolute torsion in Ranicki [21] and Ranicki [22].

Now we consider a homotopy projection $p_*: C_* \to C_*$ (that is $p_* \circ p_* \simeq p_*$). We call (D_*, r_*, i_*) a split object for p_* if D_* is a projective chain complex and $r_*: C_* \to D_*$ and $i_*: D_* \to C_*$ chain maps with $i_* \circ r_*$

$\simeq p_*$ and $r_* \circ i_* \simeq \mathrm{ID}_*$. A split object is the homotopy theoretic summand of C_* defined by p_*. Namely, one can easily prove that C_* is homotopy equivalent to $D_* \oplus \mathrm{Cone}(i_*)_*$.

DEFINITION 3.3. Given a homotopy projection $p_*: C_* \to C_*$ in S-FDCC we define $w(p_*) \in K_0(S)$ as the Wall obstruction $w(D_*)$ for any split object (D_*, r_*, i_*).

We recall that the Wall obstruction $w(D_*)$ of a finitely dominated chain complex D_* is defined as $\sum(-1)^n[P_n]$ for any finite projective chain complex P_* homotopy equivalent to D_* (see Wall [27, p. 38], Wall [26]). The Definition 3.3 makes sense because of the following lemma. It is a special case of Freyd [10]. We will, however, give an explicit construction.

LEMMA 3.4. *Each homotopy projection $p_*: C_* \to C_*$ possesses a split object (D_*, r_*, i_*). If (D'_*, r'_*, i'_*) is another one, there exists a homotopy equivalence $f_*: D_* \to D'_*$ with $f_* \circ r_* \simeq r'_*$ and $i'_* \circ f_* \simeq i_*$.*

PROOF. Defining f_* by $r'_* \circ i_*$ and f_*^{-1} by $r_* \circ i'_*$ one shows uniqueness. Hence only the existence remains to be proved. We will construct D_* by a kind of Eilenberg swindle:

Let E^1_* and E^0_* be copies of $\bigoplus_{n=0}^{\infty} C_*$. Let $q_*: E^1_* \to E^0_*$ be the chain map defined by the matrix

$$A = \begin{vmatrix} \mathrm{ID}_* - p_* & 0 & 0 & 0 & \cdots \\ p_* & \mathrm{ID}_* - p_* & 0 & 0 & \cdots \\ 0 & p_* & \mathrm{ID}_* - p_* & 0 & \cdots \\ 0 & 0 & p_* & \mathrm{ID}_* - p_* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

Let $s_*: E^0_* \to E^1_*$ be given by the transposed matrix A^{tr} and $u_*: E^0_* \to C_*$ by $(p_*, 0, \ldots, 0)$ and $v_*: C_* \to E^0_*$ by $(\mathrm{ID}_*, 0, \ldots, 0)^{\mathrm{tr}}$. Now choose a homotopy $h_*: p_* \circ p_* \simeq p_*$ and define chain homotopies $\varphi^1_*: E^1_* \to E^1_{*+1}$ and $\varphi^0_*: E^0_* \to E^1_{*+1}$ and $\psi_*: E^1_* \to C^1_{*+1}$ by:

$$\varphi^0_*: \begin{bmatrix} h_* & -h_* & 0 & 0 & \cdots \\ -h_* & 2h_* & -h_* & 0 & \cdots \\ 0 & -h_* & 2h_* & -h_* & \cdots \\ 0 & 0 & -h_* & 2h_* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\varphi^1_*: \begin{bmatrix} 2h_* & -h_* & 0 & 0 & \cdots \\ -h_* & 2h_* & -h_* & 0 & \cdots \\ 0 & -h_* & 2h_* & -h_* & \cdots \\ 0 & 0 & -h_* & 2h_* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\psi_*: (-h_*, 0, 0, \ldots).$$

One easily checks that $\varphi^0_*: \mathrm{ID}_* \simeq v_* \circ u_* + q_* \circ s_*$ and $\varphi^1_*: s_* \circ q_* \simeq \mathrm{ID}_*$ and $\psi_*: u_* \circ q_* \simeq 0_*$ are valid. Then the following maps are chain maps:

$$r_*: C_* \xrightarrow[(0, v_*)^{tr}]{} \mathrm{Cone}(q_*) = E^0_{*-1} \oplus E^0_*$$

$$i_*: \mathrm{Cone}(q_*)_* \xrightarrow[(\psi_{*-1}, u_*)]{} C_*.$$

Define the chain homotopy $\Phi_*: \mathrm{Cone}(q_*)_* \to \mathrm{Cone}(q_*)_{*+1}$ by

$$\begin{bmatrix} \mathrm{ID}_* & 0 \\ \varepsilon_* & \mathrm{ID}_* \end{bmatrix} \cdot \begin{bmatrix} \varphi^1_{*-1} & s_* \\ 0 & \varphi^0_* \end{bmatrix}$$

for $\varepsilon_* = -v_{*+1} \circ \psi_* - q_{*+1} \circ \varphi^1_* - \varphi^0_* \circ q_*$. Then Φ_* is a homotopy between ID_* and $r_* \circ i_*$ and we have $i_* \circ r_* = p_*$. Therefore $(\mathrm{Cone}(q_*)_*, r_*, i_*)$ is a split object for p_*.

Since we have constructed for a homotopy projection $p_*: C_* \to C_*$ together with a homotopy $h_*: p_* \circ p_* \simeq p_*$ an explicit domination $r_*: C_* \to D_*$, $i_*: D_* \to C_*$, and $\Phi_*: \mathrm{ID}_* \simeq r_* \circ i_*$ one can use the instant Wall obstruction in Ranicki [23] to get a finitely generated projective module defined by a square matrix A with $A^2 = A$ and representing $w(p_*)$.

The next theorem collects the main properties of t and w and ensures that

Definition 3.1 makes sense. For its proof use Gersten [11], Ranicki [21], Ranicki [22], Wall [27, p. 138], Cohen [5, p. 48]. If we write $t(f_*)$ (respectively $w(f_*)$) we assume that $f_*: C_* \to C_*$ is a self equivalence (respectively homotopy projection) of a finitely dominated chain complex C_*.

THEOREM 3.5. a) *Homotopy invariance.*

i) $f_* \simeq g_* \Rightarrow t(f_*) = t(g_*)$ *respectively* $w(f_*) = w(g_*)$.
ii) *Let* $h_*: C_* \to D_*$ *be a homotopy equivalence and* $f_*: C_* \to C_*$ *and* $g_*: D_* \to D_*$ *self maps with* $h_* \circ f_* \simeq g_* \circ h_*$. *Then* $t(f_*) = t(g_*)$ (*respectively* $w(f_*) = w(g_*)$).

b) *Additivity. The following diagram with exact rows commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1_* & \xrightarrow{i_*} & C^0_* & \xrightarrow{p_*} & C^2_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^1_*} & & \downarrow{\scriptstyle f^0_*} & & \downarrow{\scriptstyle f^2_*} & & \\ 0 & \longrightarrow & C^1_* & \xrightarrow[i_*]{} & C^0_* & \xrightarrow[p_*]{} & C^2_* & \longrightarrow & 0 \end{array}$$

Then $t(f^1_*) - t(f^0_*) + t(f^2_*) = 0$ *respectively* $w(f^1_*) - w(f^0_*) + w(f^2_*) = 0$.

c) *Logarithmic property*

$$t(f_* \circ g_*) = t(f_*) + t(g_*).$$

4. The algebraic transfer.

Using Definition 3.1 it is easy to define the algebraic transfer as a pairing: We introduce a category $R-S$-FDCC motivated by Definition 1.2. Objects are pairs (C_*, U) consisting of a finitely dominated projective S-chain complex C_* and a ring homomorphism $U: R \to [C_*, C_*]^\circ_S$. A morphism $[f_*]: (C_*, U) \to (D_*, V)$ is a homotopy class of S-chain maps $f_*: C_* \to D_*$ with $V(r)_* \circ f_* \simeq f_* \circ U(r)_*$ for all $r \in R$. We call

$$(C^1_*, U^1_*) \xrightarrow{[i_*]} (C^0_*, U^0_*) \xrightarrow{[p_*]} (C^2_*, U^2_*)$$

exact, if there exists a choice of representatives $i_*, p_*, U^j(r)_*$ such that for all $r \in R$ the following diagram has exact rows and commutes strictly (not only up to homotopy).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1_* & \xrightarrow{i_*} & C^0_* & \xrightarrow{p_*} & C^2_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle U^1(r)_*} & & \downarrow{\scriptstyle U^0(r)_*} & & \downarrow{\scriptstyle U^2(r)_*} & & \\ 0 & \longrightarrow & C^1_* & \xrightarrow[i_*]{} & C^0_* & \xrightarrow[p_*]{} & C^2_* & \longrightarrow & 0 \end{array}$$

DEFINITION 4.1. The Grothendieck group of $R-S$-FDCC is denoted by $K_0^c(R-S)$.

We recall that the Grothendieck group of a small category with exact sequences is the quotient of the free abelian group generated by the isomorphism classes of objects and the subgroup generated by elements $[X]-[Y]+[Z]$ for all exact sequences $X \to Y \to Z$.

Given an object (C_*, U) of $R-S$-FDCC let $U: R$-BMOD $\to S$-FDCC be the corresponding functor (see section 1). Using Definition 3.1 we get a homomorphism $U_*: K_n(R) \to K_n(S)$ for $n = 0,1$:

DEFINITION 4.2. The pairing

$$^RT^S: K_0^c(R-S) \otimes K_n(R) \;\to\; K_n(S)$$

is defined by

$$^RT^S([C_* U], x) := U_*(x) \quad \text{for } n = 0,1.$$

For $y \in K_0^c(R-S)$ let $^RT^S_y: K_n(R) \to K_n(S)$ be given by $^RT^S(y, ?)$.

We sometimes write simply T for $^RT^S$. This pairing is compatible with the additive relations in $K_0^c(R-S)$ because of Theorem 3.5. Given an object (C_*, U) of $R-S$-FDCC, the map $T_{[C_*,U]}$ is the composition of

$$U_*: K_n(R) \to K_n([C_*, C_*]_S^\circ)$$

and a homomorphism $K_n([C_*, C_*]^\circ) \to K_n(S)$ defined by the obvious functor $[C_*, C_*]^\circ -$ BMOD $\to S$-FDCC according to Definition 3.1. If C_* is concentrated in dimension 0 then C_0 is a $S-R$-bimodule and the second homomorphism is the usual Morita homomorphism (Curtis-Rainer [6]) and $T_{[C_*,U]}$ is just given as $C_0 \otimes_R$?.

Now we regard the example of a ring S with a pseudostructure. This is a ring S with an automorphism $s \mapsto s^t$ and an element $\sigma \in S$ such that the relations $\sigma^t = \sigma$ and $s^t\sigma = \sigma s$ hold for each $s \in S$. Then the left ideal (σ) generated by σ is a twosided ideal. Let C_* be the S-chain complex $S \xrightarrow{\sigma} S$ concentrated in dimension 0 and 1. We define a chain map $f(s)_*: C_* \to C_*$ for $s \in S$ by

$$f(s)_1: S \xrightarrow{s^t} S \quad \text{and} \quad f(s)_0: S \xrightarrow{s} S.$$

A nullhomotopy for $f(\sigma)_*$ is given by the identity on S. Hence we get a ring homomorphism $U: S/(\sigma) \to [C_*, C_*]^\circ_S$ and thus homomorphisms

$$T_{[C_*,U]}: K_n(S/(\sigma)) \to K_n(S) \quad \text{for } n = 0,1.$$

One easily verifies that this is the homomorphism stated in Munkholm–Ranicki [17] for $n = 0$ respectively Munkholm–Pedersen [16]for $n = 1$ which describes the transfer induced by a S^1-fibration algebraically. In Oliver [18] one can find a detailed study of the K_1-transfer of an orientable S^1-fibration with finite fundamental groups. It contains examples for which the K_1-transfer is not zero (see also Munkholm–Pedersen [15, p. 423]).

Now we state the natural properties of these pairings. Let F be a functor from the category of projective S-modules into the category of projective S'-modules which is compatible with $\oplus$ and sends S to a finitely generated projective S'-module (for example localization or completion). Then F induces an exact functor $F: R-S\text{-FDCC} \to R-S'\text{-FDCC}$ and therefore homomorphisms $F_*: K_0^c(R-S) \to K_0^c(R-S')$ and $F_*: K_n(S) \to K_n(S')$.

LEMMA 4.3. *The following diagram commutes for* $n = 0,1$:

$$\begin{array}{ccc} K_0^c(R-S) \otimes K_n(R) & \xrightarrow{{}^R T^S} & K_n(S) \\ \downarrow{\scriptstyle F_* \oplus \mathrm{ID}} & & \downarrow{\scriptstyle F_*} \\ K_0^c(R-S') \otimes K_n(R) & \xrightarrow{{}^R T^{S'}} & K_n(S') \end{array}$$

Especially this can be applied to a ring homomorphism $\varphi: S \to S'$ and the induction functor $\mathrm{ind}(\varphi)$. Then we write φ_* instead of $\mathrm{ind}(\varphi)_*$.

A ring homomorphism $\psi: R' \to R$ induces a homomorphism

$$\psi^*: K_0^c(R-S) \to K_0^c(R'-S)$$

by $[C_*, U] \to [C_*, U \circ \psi]$.

LEMMA 4.4. *The following diagram commutes for* $n = 0,1$:

$$\begin{array}{ccc} K_0^c(R'-S) \otimes K_n(R') & \xrightarrow{{}^{R'} T^S} & K_n(S) \\ \uparrow{\scriptstyle \psi^* \otimes \mathrm{ID}} & & \\ K_0^c(R-S) \otimes K_n(R') & & \Big\uparrow{\scriptstyle \mathrm{ID}} \\ \downarrow{\scriptstyle \mathrm{ID} \otimes \psi_*} & & \\ K_0^c(R-S) \otimes K_n(R) & \xrightarrow{{}^R T^S} & K_n(S) \end{array}$$

This yields the formulas $\varphi_* \circ T_y = T_{\varphi_*(y)}$ and $T_y \circ \psi_* = T_{\psi^*(y)}$.

For applications in geometry we define for two groups π and Γ.

DEFINITION 4.5. The map $\theta_1 : K_0^c(\mathbf{Z}[\pi]-\mathbf{Z}[\Gamma]) \to \mathrm{HOM}(\pi, \mathrm{Wh}(\Gamma))$ is defined by

$$\theta_1([C_*, U]) : x \to \mathrm{pr}(t(U(x) : C_* \to C_*))$$

for the canonical projection $\mathrm{pr}: K_1(\mathbf{Z}[\Gamma]) \to \mathrm{Wh}(\Gamma)$.

Let $K_0^s(\mathbf{Z}[\pi]-\mathbf{Z}[\Gamma])$ be the kernel of θ_1. The pairing of Definition 4.2 induces a pairing

$$_{\mathbf{Z}[\pi]}T^{\mathbf{Z}[\Gamma]} : K_0^s(\mathbf{Z}[\pi]-\mathbf{Z}[\Gamma]) \otimes \mathrm{Wh}(\pi) \to \mathrm{Wh}(\Gamma).$$

5. The algebraic transfer induced by a fibration.

Let $p:(E,e) \to (B,b)$ be a fibration of connected spaces whose not necessarily connected fibre has the homotopy type of a finitely dominated CW-complex. Let $K_0^c(p,e)$ be an abbreviation for $K_0^c(\mathbf{Z}[\pi_1(B,b)] - \mathbf{Z}[\pi_1(E,e)])$ and analogously for $K_0^s(p,e)$. The chain homotopy representation $U_{p,e}$ of Definition 1.2 determines an isomorphism class of objects in $\mathbf{Z}[\pi_1(B,b)]-\mathbf{Z}[\pi_1(E,e)]-\mathrm{FDCC}$ and thus a class $U(p,e) \in K_0^c(p,e)$. Define

$$\theta_1(p,e) : K_0^c(p,e) \to H^1(B, \mathrm{Wh}(\pi_1(E)))$$

as the composition of the homomorphism θ_1 of Definition 4.5 and the isomorphism

$$\mathrm{HOM}(\pi_1(B,b), \mathrm{Wh}(\pi_1(E,e))) \to \mathrm{HOM}(H_1(B), \mathrm{Wh}(\pi_1(E))) \to H^1(B, \mathrm{Wh}(\pi_1(E))).$$

DEFINITION 5.1. For $n = 0,1$

$$p^* : K_n(\mathbf{Z}[\pi_1(B,b)]) \to K_n(\mathbf{Z}[\pi_1(E,e)])$$

denotes the homomorphism $T_{U(p,e)}$ of Definition 4.2. If $\theta_1(p,e)(U(p,e))$ vanishes, then $U(p,e)$ is an element of $K_0^s(p,e)$ and defines

$$p^* : \mathrm{Wh}(\pi_1(B,b)) \to \mathrm{Wh}(\pi_1(E,e))$$

by Definition 4.5.

Now we want to get rid of the base points. The problem of the choice of the base points and the models of the universal coverings is extensively discussed in Cohen [5, pp. 63–65] for Whitehead groups.

Let w be a path in E from e_1 to e_0 and w^- the inverse path. Conjugation with w, respectively $p \circ w^-$, defines a homomorphism

$$c_w : \pi_1(E,e_0) \to \pi_1(E,e_1),$$

respectively $c_{p \circ w^-} : \pi_1(B,b_1) \to \pi_1(B,b_0)$. For a group homomorphism φ we denote by $Z[\varphi]$ the induced map on the group rings. Now one verifies that

$$Z[c_w]_* \circ Z[c_{p \circ w^-}]^* : K_0^c(p,e_0) \to K_0^c(p,e_1)$$

maps $U(p,e_0)$ to $U(p,e_1)$ and that

$$\theta_1(p,e_1) \circ Z[c_w]_* \circ Z[c_{p \circ w^-}]^* = \theta_1(p,e_0)$$

is valid. Hence we get a well-defined cohomology class $V_1(p) \in H^1(B, \mathrm{Wh}(\pi_1(E)))$ by $\theta_1(p,e)(U(p,e))$. Because of Lemmata 4.3 and 4.4 the following definition makes sense.

Definition 5.2. Let

$$p^* : K_0(Z[\pi_1(B)]) \to K_0(Z[\pi_1(E)])$$

be induced by the collection of homomorphisms of Definition 5.1 for $e \in E$.

We call p simple if $V_1(p)$ vanishes. For simple p let

$$p^* : \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$$

be induced by the homomorphism of Definition 5.1.

A consequence of Lemma 4.3 and 4.4 is:

Corollary 5.3. *The algebraic transfer is compatible with pull backs. Namely, for a pull back*

$$\begin{array}{ccc} E_0 & \xrightarrow{\bar{f}} & E \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ B_0 & \xrightarrow{f} & B \end{array}$$

the following diagram commutes

$$\begin{array}{ccc} K_0(Z[\pi_1(E_0)]) & \xrightarrow{\bar{f}_*} & K_0(Z[\pi_1(E)]) \\ {\scriptstyle p_0^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ K_0(Z[\pi_1(B_0)]) & \xrightarrow{f_*} & K_0(Z[\pi_1(B)]). \end{array}$$

If p and p_0 are simple the same is true for Whitehead groups.

PROOF. For $e_0 \in E_0$ and $e \in E$ with $\bar{f}(e_0) = e$ one verifies:

$$\mathbf{Z}[\pi_1(\bar{f})]_*(U(p_0, e_0)) = \mathbf{Z}[\pi_1(f)]^*(U(p,e)).$$

The class $V_1(p) \in H^1(B, \mathrm{Wh}(\pi_1(E))) = \mathrm{HOM}(H_1(B), \mathrm{Wh}(\pi_1(E)))$ can be described in the following more familiar way if there exists a homotopy equivalence $\lambda: Y \to F_b$ from a finite CW-complex into the fibre of p over b. If τ is the Whitehead torsion, $\omega: \pi_1(B,b) \to [F_b, F_b]$ the homotopy operation of $\pi_1(B,b)$ on the fibre, $i: F_b \to E$ the inclusion and c_λ conjugation with λ, one gets the following commutative diagram

$$\begin{array}{ccccccc}
\pi_1(B,b) & \xrightarrow{\omega} & [F_b, F_b] & \xrightarrow{c_\lambda} & [Y,Y] & \xrightarrow{\tau} & \mathrm{Wh}(\pi_1(Y)) \\
\Big\downarrow & & & & & & \Big\downarrow \lambda_* \\
 & & & & & & \mathrm{Wh}(\pi_1(F_b)) \\
 & & & & & & \Big\downarrow i_* \\
H_1(B) & & & \xrightarrow{V_1(p)} & & & \mathrm{Wh}(\pi_1(E)).
\end{array}$$

Obviously $V_1(p)$ vanishes for an orientable fibration (that is $\omega = 0$) and for a fibration with $\mathrm{Wh}(\pi_1(F_b)) = 0$. Each PL-bundle and each local trivial fibre bundle with a finite CW-complex as fibre is simple as the fibre transport is given by homeomorphisms which are simple homotopy equivalences (Chapman [4], Cohen [5, p. 102]).

Now we can state the main result.

THEOREM 5.4. *Whenever the geometric transfer $p^!$ for a fibration respectively* PL*-bundle $p: E \to B$ is defined the same is true for the algebraic transfer p^* and $p^!$ and p^* coincide.*

PROOF. This is a consequence of the definitions of p^* and $p^!$ and of Theorems 2.1 and 2.2. In the K_0-case one can assume $w(B) = w(E) = 0$ since p^*, respectively $p^!$, are compatible with pull-back by Corollary 5.3, respectively definition. Given a projection $p: R^n \to R^n$ with image P one gets an explicit free resolution of P by the Eilenberg swindle. This corresponds to the explicit construction of a split object in Lemma 3.4.

One should notice that for a covering $p: E \to B$ with a finite set as fibre the transfer p^* is just the classical transfer induced by restriction with

$$\mathbf{Z}[p_*]: \mathbf{Z}[\pi_1(E)] \to \mathbf{Z}[\pi_1(B)].$$

Each fibration $E \xrightarrow{p} B$ can be written as a composition

$$E \xrightarrow{p_1} B_0 \xrightarrow{p_0} B$$

of a fibration p_1 with a connected fibre and a covering p_0. Because of $p^! = p_1^! \circ p_0^!$ it suffices to study fibrations with connected fibres.

6. The homotopy operation of $\pi_1(E)$ on the pointed fibre determines the chain homotopy representation.

Let $p: (E,e) \to (B,b)$ be a fibration with fibre $F = p^{-1}(b)$ and connected spaces E and B. Although it is not necessary we assume for simplicity in this section that F is connected. We write $\pi = \pi_1(B,b)$, $\Gamma = \pi_1(E,e)$, and Δ for the kernel of $p_*: \Gamma \to \pi$. The epimorphism $\partial: \pi_1(F,e) \to \Delta$ is induced by the inclusion $F \subset E$ and $q_F: \bar{F} \to F$ is the covering corresponding to ∂. Let $q_E: (\tilde{E},\tilde{e}) \to (E,e)$ and $q_B: (\tilde{B},\tilde{b}) \to (B,b)$ be universal coverings. Define $\hat{p}: (\tilde{E},\tilde{e}) \to (B,b)$ as $p \circ q_E$ and $\tilde{p}: (\tilde{E},\tilde{e}) \to (\tilde{B},\tilde{b})$ by $q_B \circ \tilde{p} = \hat{p}$. Let $\hat{F} = \hat{p}^{-1}(b)$ be the fibre of the Γ-fibration $\hat{p}$.

We can identify $\bar{F}$ with the fibre $\tilde{p}^{-1}(b)$ of the Δ-fibration $\tilde{p}$. We get the following diagrams:

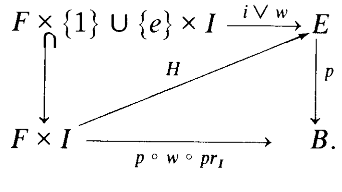
$$\begin{array}{ccccc} \Delta & \longrightarrow & \bar{F} & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \tilde{E} & \xrightarrow{q_E} & E \\ \downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \pi & \longrightarrow & \tilde{B} & \xrightarrow{q_b} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \bar{F} & \xrightarrow{\subset} & \hat{F} \\ \downarrow & & \downarrow \\ \tilde{E} & \xrightarrow{\mathrm{ID}} & \tilde{E} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\ \tilde{B} & \xrightarrow{q_B} & B . \end{array}$$

Obviously $\bar{F}$ is contained in $\hat{F}$ as a Δ-space and

$$h: \Gamma \times_\Delta \bar{F} \to \hat{F}(c,x) \to cx$$

is a Γ-homeomorphism.

Now $\omega: \pi \to [F,F]$ (respectively $\sigma: \Gamma \to [(F,e),(F,e)]^+$) is the homomorphism into the monoid of (pointed) homotopy classes of (pointed) self-maps of the fibre. Given a loop w in (E,e) a representative of $\sigma(w)$ is defined by H_0 for a solution H of the homotopy lifting problem



Analogous for ω (see Whitehead [28, pp. 35+185+186], Switzer [24, p. 343]). In Whitehead [28, pp. 98–100]) an operation

$$\bar{\rho}: \pi_1(F,e) \times [(F,e),(F,e)]^+ \to [(F,e),(F,e)]^+$$

is defined. We denote by

$$\rho: \pi_1(F,e) \to [(F,e),(F,e)]^+$$

the homomorphism given by the evaluation of $\bar{\rho}$ on the identity on (F,e). Let $G_1(F,e)$ be the kernel of ρ (compare Gottlieb [12, p. 842]).

We have defined $u: \pi \to [\hat{F},\hat{F}]_\Gamma$ in section 1.

Now $\sigma: \Gamma \to [(F,e),(F,e)]^+$ induces a map $\hat{\sigma}: \Gamma \to [\hat{F},\hat{F}]_\Gamma$ in the following way. Choose a representative $s(w): (F,e) \to (F,e)$ of $\sigma(w)$ for $w \in \Gamma$. Conjugation with w induces a homomorphism $c_w: \Delta \to \Delta$. Because of $\partial \circ s(w)_* = c_w \circ \partial$ there exists a unique lift

$$\bar{s}(w): (\bar{F},\tilde{e}) \to (\bar{F},\tilde{e}),$$

which is automatically c_w-equivariant.

Hence we can define

$$\hat{s}(w): \Gamma \times_\Delta \bar{F} \to \Gamma \times_\Delta \bar{F}$$

by $(c,x) \to (cw, \bar{s}(w^{-1})(x))$. Let $\hat{\sigma}(w)$ be given by the Γ-homotopy class of $h \circ \hat{s}(w) \circ h^{-1}$.

THEOREM 6.1.

a) *The map* $\hat{\sigma}: \Gamma \to [\hat{F},\hat{F}]_\Gamma$ *is a well-defined homomorphism.*

b) *The homotopy operation u is determined by* $\hat{\sigma}$ *and thus by* σ. *Namely,* $\hat{\sigma}$ *factorizes in* $u \circ p_*$.

c) i) *The forgetful map* $f: [(F,e),(F,e)]^+ \to [F,F]$ *is the projection onto the orbit space of the operation* $\bar{\rho}$.

ii) $\omega \circ p_* = f \circ \sigma$.

iii) $\sigma(\partial(v)\cdot w) = \bar{\rho}(v,\sigma(w))$ *for* $v \in \pi_1(F_b,e)$ *and* $w \in \Gamma$. *Especially:* $\sigma \circ \partial = \rho$.

The proof of this theorem is straightforward. It shows in combination with Theorem 5.4 and the definition of p^* that the geometric transfer induced by a fibration depends only on the so called fundamental group data, i.e., the fibre F, the homotopy operation σ and the sequence $\pi_1(F) \to \pi_1(E) \to \pi_1(B)$. This result has been proved geometrically by universal fibrations respecting this data in Pedersen [19].

Definition 6.2. We call p untwisted if σ is trivial and orientable if ω is trivial.

"Untwisted" implies "orientable" but the converse implication is not valid in general. Namely, p is untwisted if and only if $G_1(F) = \pi_1(F)$ and p is orientable. Therefore each orientable fibration with a connected H-space as fibre is untwisted. This includes principal G-bundles for connected topological group G.

7. Proof of Theorem 2.2.

The proof consists of four parts. In part A we give some information about Γ-fibrations we need in part B to explain the construction of $(f_1,f)\colon (\tilde{E}_1,\tilde{E}) \to (X_1,X)$. We prove in part C that in the PL-case (f_1,f) can be chosen as a pair of simple homotopy equivalences and $C^c_*(X_1,X)$ is computed in part D.

A. *Γ-fibrations.* The definition of a Γ-fibration was given in section 1. The usual definitions of fibre map, fibre homotopy, etc. can be translated directly (Switzer [24, p. 342], Whitehead [28, p. 38]). A map $(\bar{f},f)\colon p \to p'$ of Γ-fibrations $p\colon E \to B$ and $p'\colon E' \to B'$ consists of Γ-maps $\bar{f}\colon E \to E'$ and $f\colon B \to B'$ with $p' \circ \bar{f} = f \circ p$. A Γ-homotopy $h\colon Z \times I \to E$ is a Γ-fibre homotopy if $p \circ h$ is stationary. Two maps $f_0, f_1 : Z \to E$ are Γ-fibre homotopic ($f_0 \simeq_p f_1$) if there exists a Γ-fibre homotopy h with $h_0 = f_0$ and $h_1 = f_1$. This implies $p \circ f_0 = p \circ f_1$. For two Γ-fibrations $p\colon E \to B$ and $p'\colon E' \to B$ over the same space B a Γ-fibre map $(\bar{f}, \mathrm{ID})\colon p \to p'$ is a Γ-fibre equivalence if there exists a Γ-fibre map $(\bar{g}, \mathrm{ID})\colon p' \to p$ with $\bar{f} \circ \bar{g} \simeq_{p'} \mathrm{ID}$ and $\bar{g} \circ \bar{f} \simeq_p \mathrm{ID}$. The fibre $F = p^{-1}(b)$ is a Γ-subspace of E.

For a Γ-fibration $p\colon E \to B$ and a map $f\colon Z \to B$ we use the following notation for the pull-back:

$$\begin{array}{ccc} f^*E & \xrightarrow{\ \bar{f}\ } & E \\ {\scriptstyle p_f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ f\ } & B \end{array}$$

Definition 7.1. Let $h\colon Z \times I \to B$ be a homotopy between f_0 and f_1 and H a solution of the Γ-homotopy lifting problem:

$$\begin{array}{ccc} f_0^*E & \xrightarrow{\ \bar{f}_0\ } & E \\ \cap\big\downarrow{\scriptstyle i_0} & \nearrow{\scriptstyle H} & \big\downarrow{\scriptstyle p} \\ f_0^*E \times I & \xrightarrow{\ h \circ (p_{f_0} \times \mathrm{ID})\ } & B \end{array}$$

Define $\alpha_h: f_0^* E \to f_1^* E$ by H_1 and $f_1 \circ p_{f_0}$ using the pull back property.

Of course α_h depends on H but we will see that the Γ-fibre homotopy class of α_h depends only on the homotopy class of h. If h and g are homotopies $Z \times I \to B$ with $h_0 = g_0$ and $h_1 = g_1$ we call them homotopic ($h \simeq g$) if they are homotopic as maps relative to $Z \times \{0,1\}$. Given homotopies $h: f_0 \simeq f_1$ and $g: f_1 \simeq f_2$, let $h * g$ be the obvious homotopy $h * g: f_0 \simeq f_2$. Using Switzer [24, p. 342] one can easily prove:

PROPOSITION 7.2.

a) α_h *is a* Γ*-fibre equivalence.*
b) H *is a* Γ*-homotopy* $\overline{f}_0 \simeq \overline{f}_1 \circ \alpha_h$ *over* h.
c) $h \simeq g \Rightarrow \alpha_h \simeq_{p_{f_1}} \alpha_g$.
d) $\alpha_{h*g} \simeq_{p_{f_2}} \alpha_g \circ \alpha_h$.

Applying this to $Z = \{*\}$ one gets the functor "Γ-equivariant transport of the fibre along paths in B" $\mu: \pi(B) \to \Gamma\text{-TOP}_{\text{HO}}$ from the fundamental groupoid of B into the homotopy category of Γ-spaces (compare Switzer [24, p. 343]).

Let (D, d) be a pointed contractible space and $f: D \to B$ a map. Given a morphism $\xi: f(d) \to b$ in $\pi(B)$, i.e. a homotopy class relative to $\{0,1\}$ of paths from b to $f(d)$ in B, we define:

DEFINITION 7.3. Let h be any homotopy between the constant map $c_b: D \to \{b\} \subset B$ and f such that $h(d, \cdot)$ represents ξ. Define

$$T_\xi: F_b \times D \to f^* E$$

by α_h and

$$T(f, \xi): F_b \times D \to E$$

by $\overline{f} \circ T_\xi$.

Because of Proposition 7.2 and the following Lemma 7.4 the Γ-fibre homotopy class of T_ξ depends only on ξ.

LEMMA 7.4. *Let* h *and* g *be homotopies between* f_0 *and* $f_1: D \to B$ *for contractible* (D, d). *Then* $h \simeq g$ *is valid if and only if the paths* $h(d, \cdot)$ *and* $g(d, \cdot)$ *are homotopic relative to* $\{0,1\}$.

PROOF. Choose a homotopy $\zeta: D \times I \to D$ between ID_D and the constant map c_d relative to $\{d\}$. Now define $\psi^h: D \times I \times I \to B$

$$\psi^h(x,t,s) = \begin{cases} h(\zeta(x,3t),0) & 0 \leqq t \leqq \frac{1}{3}s \\ h\left(\zeta(x,s), \dfrac{3t-s}{3-2s}\right) & \frac{1}{3}s \leqq t \leqq 1-\frac{1}{3}s \\ h(\zeta(x,3-3t),1) & 1-\frac{1}{3}s \leqq t \leqq 1. \end{cases}$$

Then ψ^h is a homotopy between h and

$$\psi_1^h(x,t) = \begin{cases} f_0 \circ \zeta(x,3t) & 0 \leqq t \leqq \frac{1}{3} \\ h(d,3t-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3} \\ f_1 \circ \zeta(x,3-3t) & \frac{2}{3} \leqq t \leqq 1. \end{cases}$$

Obviously $\psi_1^h \simeq \psi_1^g$ is valid, if $h(d,\cdot) \simeq g(d,\cdot)$ relative to $\{0,1\}$.

We will later apply this to the characteristic map of a cell of the base space. Now we get the main result of part A. It is responsible for the appearance of the chain homotopy representation in Theorem 2.2.

COROLLARY 7.5. *Let D be a contractible space and $h\colon (D,S)\times I \to (B,A)$ be a homotopy between the maps of pairs f_0 and f_1 and $p\colon E \to B$ a Γ-fibration. Let ξ_i be a homotopy class of paths relative $\{0,1\}$ from $b \in B$ to $f_i(d)$ for $d \in S$ and ξ be the class of $h(d,\cdot)$.*

Then the following diagram commutes up to homotopy of Γ-pairs

$$\begin{array}{ccc} F_b \times (D,S) & \xrightarrow{T(\xi_0,f_0)} & \\ \Big\downarrow{\scriptstyle \mu(\xi_1 * \xi^- * \xi_0^-)\times \mathrm{ID}} & & (E,E|A) \\ F_b \times (D,S) & \xrightarrow{T(\xi_1,f_1)} & \end{array}$$

B. *The construction of* (f_1,f). We construct (f_1,f) now for (p_1,p) a pair of fibrations. Without loss of generality we can assume $(B_1,B) = (A_1,A)$ as the pullback with a homotopy equivalence gives also a homotopy equivalence between the total spaces. Let B_1 (respectively B_0) be the $(k+1)$- (respectively k-)skeleton of the relative CW-complex (B_1,B). The characteristic map of the ith (respectively jth) cell of dimension $k+1$ (respectively k) is denoted by

$$(P(i),\, p(i))\colon (D^{k+1},S^k) \;\to\; (B_1,B_0)$$

(respectively $(Q(j), q(j))\colon (D^k, S^{k-1}) \to (B_0, B)$. Let ξ_i (respectively η_j) be a homotopy class of paths relative to $\{0,1\}$ from b to $P(i)(*)$ (respectively $Q(j)(*)$) for a base point $*$ in S^k (respectively S^{k-1}). Choose trivializations

$$T(i)\colon \hat{F} \times D^{k+1} \to P(i)^* \tilde{E}_1$$

(respectively $S(j)\colon \hat{F} \times D^k \to Q(j)^* \tilde{E}_1$) using ξ_i (respectively η_j) according to Definition 7.3. The restriction of $T(i)$ (respectively $S(j)$) to S^k (respectively S^{k-1}) is denoted by $t(i)$ (respectively $s(j)$). Since we are working in the category of compactly generated spaces the following diagrams are push-outs and the inclusions cofibrations, Whitehead [28, p. 33]. Let $\tilde{E}_0$ be the restriction of $\tilde{E}_1$ to B_0.

$$\begin{array}{ccc} \sum_J q(j)^* \tilde{E}_0 & \xrightarrow{\;\sum_J \overline{q(j)}\;} & \tilde{E} \\ \cap\big\downarrow & & \cap\big\downarrow \\ \sum_J Q(j)^* \tilde{E}_0 & \xrightarrow{\;\sum_J \overline{Q(j)}\;} & \tilde{E}_0 \end{array}$$

and

$$\begin{array}{ccc} \sum_I p(i)^* \tilde{E}_1 & \xrightarrow{\;\sum_I \overline{p(i)}\;} & \tilde{E}_0 \\ \cap\big\downarrow & & \cap\big\downarrow \\ \sum_I P(i)^* \tilde{E}_1 & \xrightarrow[\;\sum_I \overline{P(i)}\;]{} & \tilde{E}_1. \end{array}$$

We recall that we have chosen a Γ-homotopy equivalence $\lambda\colon Y \to \hat{F}$ with Y a Γ-CW-complex in section 1. Now choose a Γ-homotopy equivalence $f\colon \tilde{E} \to X$ into a Γ-CW-complex X and a homotopy $h(j)$ between $f \circ \overline{q(j)} \circ s(j) \circ (\lambda \times \mathrm{ID})$ and a cellular map $\beta(j)\colon Y \times S^k \to X$. The following diagram of push-outs defines a Γ-homotopy equivalence $(f_0, f)\colon (\tilde{E}_0, \tilde{E}) \to (X_0, X)$ into a pair of Γ-CW-complexes (see Brown [3 p. 249] or tom Dieck [7, p. 161]. The sum is taken over J.

$$
\begin{array}{ccccc}
\sum Q(j)^* \tilde{E}_0 & \hookleftarrow & \sum q(j)^* \tilde{E}_0 & \xrightarrow{\sum \overline{q(j)}} & \tilde{E} \\
\uparrow{\scriptstyle \sum S(j) \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \sum s(j) \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^k & \hookleftarrow & \sum Y \times S^{k-1} & \xrightarrow{\sum \overline{q(j)} \circ s(j) \circ (\lambda \times \mathrm{ID})} & \tilde{E} \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle f} \\
\sum Y \times D^k \times I & \hookleftarrow & \sum Y \times S^{k-1} \times I & \xrightarrow{\sum h(j)} & X \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^k & \hookleftarrow & \sum Y \times S^{k-1} & \xrightarrow{\sum \beta(j)} & X.
\end{array}
$$

The second step is completely analogous. Choose a homotopy of pairs

$$g(i): Y \times (S^k, *) \times I \to (X_0, X)$$

between $f_0 \circ \overline{p(i)} \circ t(i) \circ (\lambda \times \mathrm{ID})$ and a cellular map $\gamma(i)$ and define

$$(f_1, f_0, f): (\tilde{E}_1, \tilde{E}_0, \tilde{E}) \to (X_1, X_0, X)$$

by the following diagram. Here the sum is taken over I.

$$
\begin{array}{ccccc}
\sum P(i)^* \tilde{E}_0 & \hookleftarrow & \sum p(i)^* \tilde{E}_0 & \xrightarrow{\sum \overline{p(i)}} & \tilde{E}_0 \\
\uparrow{\scriptstyle \sum T(i) \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \sum t(i) \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^{k+1} & \hookleftarrow & \sum Y \times S^k & \xrightarrow{\sum \overline{p(i)} \circ t(i) \circ (\lambda \times \mathrm{ID})} & \tilde{E}_0 \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle f_0} \\
\sum Y \times D^{k+1} \times I & \hookleftarrow & \sum Y \times S^k \times I & \xrightarrow{\sum g(i)} & X_0 \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^{k+1} & \hookleftarrow & \sum Y \times S^k & \xrightarrow{\sum \gamma(i)} & X_0.
\end{array}
$$

C. *In the* PL-*case* (f_1, f) *can be chosen as a pair of simple homotopy equivalences.* The difficulty here lies in the technical point that the characteristic maps $(Q(j), q(j))$ and $(P(i), p(i))$ can only be simplicial if $q(j)$ *and* $p(i)$ are injective because $Q(j)$ and $P(i)$ are injective on the interior. But only if they are simplicial we get a simplicial structure on $P(i)^* \tilde{E}_0$ and $Q(j)^* \tilde{E}$. We will solve this problem by shrinking $Q(j)$ respectively $P(i)$ to PL-embeddings $Q(j)'$ and $P(i)'$.

Without loss of generality we can assume that (A_1, A) is equal to (B_1, B) and that the attaching maps $q(j)$ and $p(i)$ are simplicial for a triangulation of B_1 for which B and B_0 are simplicial subcomplexes (use Anderson [2, p. 174], Cohen [5, p. 24]). Further we can suppose that there exists PL-embeddings

$$(Q(j)', q(j)')\colon (D^k, S^{k-1}) \to (B_0, B) \quad \text{for each } j \in J$$

whose image is a simplex contained in the interior of the jth cell. Choose a path ζ_j in the jth cell in B_0 from $Q(j)(*)$ to $Q(j)'(*)$. Define η'_j as $\eta_j * \zeta_j$. By definition of a PL-bundle (see Anderson [1, p. 181]) there exists a map $S(j)'\colon \hat{F} \times D^k \to Q(j)'^* \tilde{E}$ corresponding to η'_j and Definition 7.3 such that it is a PL-homeomorphism and especially a simple homotopy equivalence. Its restriction to S^{k-1} is denoted by $s(j)'$. Let B'_0 be $B_0 \setminus \sum_J$ image $(Q(j)')^\circ$. There exists a strong deformation retraction $r_0\colon B'_0 \to B_0$ covered by a PL-bundle map $\overline{r_0}\colon \tilde{E}_0|B'_0 \to \tilde{E}$ such that both r_0 and $\overline{r_0}$ are simple homotopy equivalences. As $Q(j)$ and $r_0 \circ Q(j)'$ are homotopic along ζ_j the map $\overline{r_0} \circ \overline{q(j)'} \circ s(j)'$ and $\overline{q(j)} \circ s(j)$ are homotopic (Corollary 7.5). Because $\overline{r_0} \circ \overline{q(j)'} \circ s(j)'$ is already simplicial there exists a cellular Γ-homotopy $h(j)'$ between $f \circ \overline{r_0} \circ \overline{q(j)'} \circ s(j)' \circ (\lambda \times \mathrm{ID})$ and the map $\beta(j)$ of part B. Now define a Γ-homotopy equivalence

$$(f_0, f)\colon (\tilde{E}_0, \tilde{E}) \to (X_0, X)$$

by the following diagram of push-outs. The sum is taken over J.

$$\begin{array}{ccccc}
\sum Q(j)'^* \tilde{E}_0 & \longleftarrow & \sum q(j)'^* \tilde{E}_0 & \xrightarrow{\ \sum \overline{q(j)'}\ } & \tilde{E}'_0 \\
\uparrow{\scriptstyle \sum S(j)' \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \sum s(j)' \circ (\lambda \times \mathrm{ID})} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^k & \longleftarrow & \sum Y \times S^{k-1} & \xrightarrow{\ \sum \overline{q(j)'} \circ s(j)' \circ (\lambda \times \mathrm{ID})\ } & \tilde{E}'_0 \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle f \circ \overline{r_0}} \\
\sum Y \times D^k \times I & \longleftarrow & \sum Y \times S^{k-1} \times I & \xrightarrow{\ \sum h(j)'\ } & X \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle \mathrm{ID}} \\
\sum Y \times D^k & \longleftarrow & \sum Y \times S^{k-1} & \xrightarrow{\ \sum \beta(j)\ } & X.
\end{array}$$

Using the topological invariance of the Whitehead torsion (Chapman [4], Cohen [5, p. 102]), the sum and product formula Cohen [5, pp. 76 + 77] and the fact that $S(j)'$, $s(j)'$, λ, f, and $\overline{r_0}$ are simple homotopy equivalences one proves that f_0 is a simple homotopy equivalence. One should notice that in each step the push-out space has at least a CW-structure and that (X_0, X) constructed here is the same as in part B. The same argument shows that f_1 can also be constructed as a simple homotopy equivalence.

D. *Computation of* $C_*^c(X_1, X)$. We have defined X_1 as the push-out of

$$\sum_I Y \times D^{k+1} \hookleftarrow \sum_I Y \times S^k \xrightarrow[\sum_I \gamma(i)]{} X_0$$

for a map $\gamma(i): Y \times (S^k, *) \to (X_0, X)$. Let $\delta(i)$ be the restriction of $\gamma(i)$ to $Y \times \{*\}$. Now the cone of

$$C_*^c(\gamma(i), \delta(i)): C_*^c(Y \times (S^k, *)) \to C_*^c(X_0, X)$$

is the cellular chain complex of the relative Γ-CW-complex $(\mathrm{Cone}(\gamma(i)), \mathrm{Cone}(\delta(i)))$. There exist relative Γ-homeomorphisms

$$(X_1, \mathrm{Cylinder}(\delta(i)) \to (X_1, X_0)$$

and

$$(X_1, \mathrm{Cylinder}(\delta(i)) \to (\mathrm{Cone}(\gamma(i)), \mathrm{Cone}(\delta(i)))$$

inducing based isomorphisms between the cellular chain complexes. Therefore $C_*^c(X_1, X)$ is based isomorphic to $\mathrm{Cone}(C_*^c(\gamma(i), \delta(i)))_*$. Since X_0 has been defined as the push-out of

$$\sum_J Y \times D^k \hookleftarrow \sum_J Y \times S^{k-1} \xrightarrow{\sum_J \beta(j)} X,$$

we have for the characteristic map

$$(R(j), \beta(j)): Y \times (D^k, S^{k-1}) \to (X_0, X)$$

a based isomorphism

$$\bigoplus_J C_*^c(R(j), \beta(j)): \bigoplus_J C_*^c(Y \times (D^k, S^{k-1})) \to C_*^c(X_0, X).$$

Now it remains to show that using this isomorphism and the identification

$$C_*^c(Y \times (D^k, S^{k-1})) \cong C_*^c(Y \times (S^k, *)) \cong \Sigma^k C_*^c(Y)$$

the chain map $C_*^c(\gamma(i), \delta(i))$ is for any $i \in I$ a representative of the chain homotopy class of

$$\Sigma^k C_*^c(Y) \xrightarrow[(\Sigma^k C_*^c(U(d_{i,j})))_j^{tr}]{} \bigoplus_J \Sigma^k C_*^c(Y).$$

We recall that the matrix $(d_{i,j})$ describes the nontrivial differential d of $C^c_*(\tilde{B}_1, \tilde{B})$. Now we fix $i \in I$ and write $d_{i,j}$ as $\sum_{w \in \pi} a(j,w) \cdot w$. We denote by K the finite index set

$$K = \{(j,w) \in J \times \pi | a(j,w) \neq 0\}.$$

Let

$$\Omega : (D^k, S^{k-1}, *) \to (S^k, *, *)$$

and

$$\psi : \sum_K Y \times (D^k, S^{k-1}) \to Y \times \bigvee_K (D^k, S^{k-1})$$

be the obvious projections. Choose a map

$$\nabla : (D^k, S^{k-1}) \to \bigvee_K (D^k, S^{k-1})$$

inducing the diagonal map

$$\Delta : H_k(D^k, S^{k-1}) \to H_k\left(\bigvee_K (D^k, S^{k-1})\right) \cong \bigoplus_K H_k(D^k, S^{k-1}).$$

Given a homotopy class ξ of paths relative to $\{0,1\}$ from b_1 to b_0 and $[\Phi, \varphi] \in \pi_k(B_0, B, b_0)$ we define $\xi \cdot [\Phi, \varphi] \in \pi_k(B_0, B, b_1)$ by the homomorphism induced by ξ (see Whitehead [28, p. 101]).

The Hurewicz homomorphism and the universal covering $(\tilde{B}_1, \tilde{B}_0, \tilde{B}) \to (B_1, B_0, B)$ induce an isomorphism between $C^c_*(\tilde{B}_1, \tilde{B})$ and the $\mathbb{Z}[\pi]$-chain complex given by the connecting homomorphism

$$\partial : \pi_{k+1}(B_1, B_0, b) \to \pi_k(B_0, B, b)$$

of the triple (B_1, B_0, B) (see Whitehead [28, p. 289]). We can assume that the cellular $\mathbb{Z}[\pi]$-base of $C^c_*(\tilde{B}_1, \tilde{B})$ corresponds to the elements $\xi_i \cdot [P(i), p(i)]$ and $\eta_j \cdot [Q(j), q(j)]$. Therefore we get in $\pi_k(B_0, B, b)$:

$$\partial(\xi_i \cdot [P(i), p(i)]) = \sum_J d_{i,j} \cdot \eta_j \cdot [Q(j), q(j)]$$

$$= \sum_K a(j,w) \cdot w \cdot \eta_j \cdot [Q(j), q(j)].$$

Let

$$\hat{Q}(j,w) : (D^k, S^{k-1}, *) \to (B_1, B, b)$$

be a representative of the homotopy class given by the composition of $w \cdot \eta_j \cdot [Q(j), q(j)]$ and a map $(D^k, S^{k-1}, *) \to (D^k, S^{k-1}, *)$ of degree $a(j, w)$.

LEMMA 7.6. *The maps* $\xi_i \cdot (p(i) \circ \Omega)$ *and*

$$\bigvee_K \hat{Q}(j, w) \circ \nabla : (D^k, S^{k-1}, *) \to (B_0, B, b)$$

are homotopic.

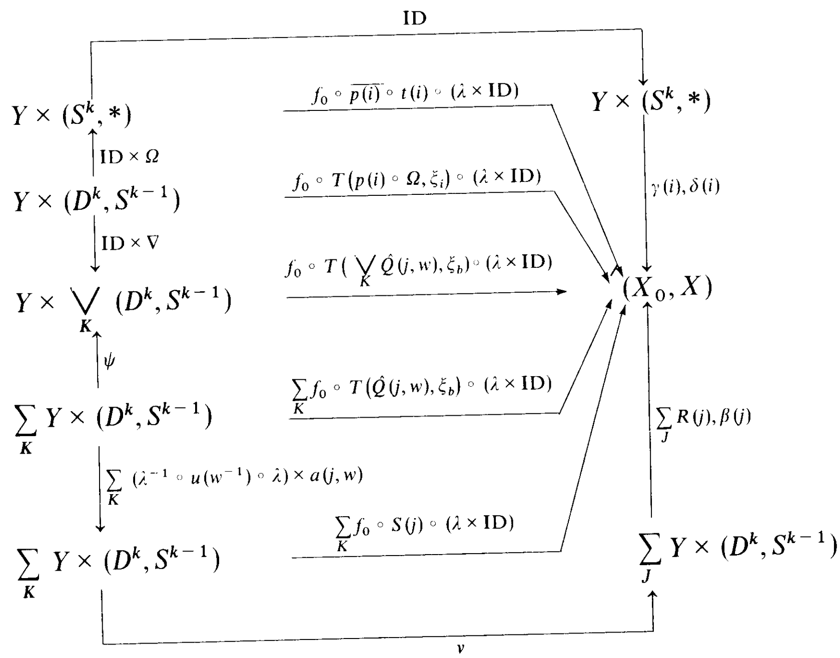
PROOF. We get in $\pi_k(B_0, B, b)$:

$$\partial(\xi_i \cdot [P(i), p(i)]) = \xi_i [p(i) \circ \Omega]$$

and

$$[\bigvee_K \hat{Q}(j, w) \circ \nabla] = \sum_K a(j, w) \cdot w \cdot \eta_j \cdot [Q(j), q(j)].$$

Applying Lemma 7.6 and Corollary 7.5 one proves that the following diagram of pairs of Γ-spaces commutes up to homotopy. The map v sends the summand corresponding to (j, w) identically to the one of j. The class of the constant path in b is denoted by ξ_b. The maps $t(i)$ and $S(j)$ have been defined in part B and the maps $T(p(i), \xi_b)$, etc. in Definition 7.3. The homomorphism $u: \pi \to [F, F]_\Gamma$ is induced by the fibre transport μ defined in section 7A and was already introduced in section 1.



The following diagram of chain complexes commutes where $\cong$ denotes a based isomorphism. Let diag be the diagonal $K \times K$-matrix with $a(j, w) \cdot C^c_*(\lambda^{-1} \circ u(w^{-1}) \circ \lambda)$ as entry on the diagonal corresponding to $(j, w) \in K$. The map v' sends the summand corresponding to $(j, w) \in K$ identically to the one of j and Δ is the diagonal map:

$$\begin{array}{ccc}
\Sigma^k C^c_*(Y) & \cong & C^c_*(Y \times (S^k, *)) \\
\Big\uparrow{\scriptstyle \mathrm{ID}} & & \Big\uparrow{\scriptstyle C^c_*(\mathrm{ID} \times \Omega)} \\
\Sigma^k C^c_*(Y) & \cong & C^c_*(Y \times (D^k, S^{k-1})) \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle C^c_*(\mathrm{ID} \times \nabla)} \\
\bigoplus_K \Sigma^k C^c_*(Y) & \cong & C^c_*(Y \times \bigvee_K (D^k, S^{k-1})) \\
\Big\uparrow{\scriptstyle \mathrm{ID}} & & \Big\uparrow{\scriptstyle C^c_*(\psi)} \\
\bigoplus_K \Sigma^k C^c_*(Y) & \cong & C^c_*(\sum_K Y \times (D^k, S^{k-1})) \\
\Big\downarrow{\scriptstyle \mathrm{diag}} & & \Big\downarrow{\scriptstyle C^c_*(\sum_K (\lambda^{-1} \circ u(w^{-1}) \circ \lambda \times a(j,w))} \\
\bigoplus_K \Sigma^k C^c_*(Y) & \cong & C^c_*(\sum_K Y \times (D^k, S^{k-1})) \\
\Big\downarrow{\scriptstyle v'} & & \Big\downarrow{\scriptstyle C^c_*(v)} \\
\bigoplus_J \Sigma^k C^c_*(Y) & \cong & C^c_*(\sum_J Y \times (D^k, S^{k-1})).
\end{array}$$

Now $v' \circ \mathrm{diag} \circ \Delta$ is just

$$\Sigma^k C^c_*(Y) \xrightarrow[(\Sigma^k C^c_*(U(d_{i,j})))^{\mathrm{tr}}_j]{} \bigoplus_J \Sigma^k C^c_*(Y) \,.$$

This finishes the proof of Theorem 2.2.

REFERENCES

1. D. R. Anderson, *The Whitehead torsion of the total space of a fibre bundle*, Topology 11 (1972), 179–194.
2. D. R. Anderson, *The Whitehead torsion of a fibre homotopy equivalence*, Michigan Math. J. 21 (1974), 171–180.
3. R. Brown, *Elements of modern topology*, McGraw-Hill, London, 1968.
4. T. A. Chapman, *Topological invariance of the Whitehead torsion*, Amer. J. Math. 96 (1974), 488–497.
5. M. M. Cohen, *A course in simple homotopy theory* (Graduate Texts in Math. 10), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
6. C. W. Curtis and I. Reiner, *Methods of representation theory I: With applications to finite groups and orders*, J. Wiley and Sons, New York, 1981.
7. T. tom Dieck, *Partition of unity in homotopy theory*, Compositio Math. 23 (1971), 159–167.

8. K. Ehrlich, *Finiteness obstructions of fibre spaces*, Ph. D. thesis, Cornell University, 1977.
9. K. Ehrlich, *Fibrations and a transfer map in algebraic K-theory*, J. Pure Appl. Algebra 14 (1979), 131–139.
10. P. Freyd, *Splitting homotopy idempotents* (Proc. Conf. categorical algebra, La Jolla, Calif., 1965), pp. 173–176, Springer-Verlag, Berlin - Heidelberg - New York, 1966.
11. S. Gersten, *The torsion of a self-equivalence*, Topology 6 (1967), 411–414.
12. D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. 87 (1965), 840–856.
13. S. MacLane, *Homology* (Grundlehren Math. Wiss. 114), Springer-Verlag, Berlin - Heidelberg - New York, 1963.
14. H. J. Munkholm, *Transfer on algebraic K-theory and Whitehead torsion for* PL-*fibrations*, J. Pure Appl. Algebra 20 (1981), 195–225.
15. H. J. Munkholm and E. K. Pedersen, *On the Wall finiteness obstruction for the total space of certain fibrations*, Trans. Amer. Math. Soc. 261 (1980), 529–545.
16. H. J. Munkholm and E. K. Pedersen, *Whitehead transfer for S^1-bundles, an algebraic description*, Comment. Math. Helv. 56 (1981), 404–430.
17. H. J. Munkholm and A. A. Ranicki, *The projective class group transfer induced by an S^1-bundle*, (Proc. Conf., London, Ontario, 1981), eds. R. M. Kane, S. O. Kochman, P. S. Selick, V. P. Snaith. (Canad. Math. Soc. Conf. Proc. 2) part 2, pp. 461–484. Amer. Math. Soc., Providence, R. I., 1982.
18. B. Oliver, *The Whitehead transfer homomorphism for oriented S^1-bundles*, Math. Scand. 57 (1985), 51–104.
19. E. K. Pedersen, *Universal geometric examples for maps in algebraic K- and L-theory*, J. Pure Appl. Algebra (1981), 179–192.
20. D. Quillen, *Higher algebraic K-theory* I in *Algebraic K-theory* I, *Higher K-theories* (Proc. Seattle, 1972) ed. H. Bass (Lecture Notes in Math. 341), pp. 85–147. Springer-Verlag, Berlin - Heidelberg - New York, 1973.
21. A. A. Ranicki, *The algebraic theory of torsion* I, in Algebraic and Geometric Topology (Proc. Conf. Rutgers Univ., New Brunswick, N.J. 1983), eds. A. Ranicki, N. Levitt, F. Quinn (Lecture Notes in Math. 1126) pp. 199–237, Springer-Verlag, Berlin - Heidelberg - New York, 1985.
22. A. A. Ranicki, *The algebraic theory of torsion* II, Preprint, Edinburgh, 1984.
23. A. A. Ranicki, *The algebraic theory of finiteness obstruction*, Math. Scand. 57 (1985), 105–126.
24. R. M. Switzer, *Algebraic topology – homotopy and homology* (Grundlehren Math. Wiss. 212), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
25. F. Waldhausen, *Algebraic K-theoy of spaces*, (Proc. Conf. Rutgers Univ., New Brunswick, N.J. 1983), eds. A. Ranicki, N. Levitt, F. Quinn (Lecture Notes in Math. 1126) pp. 318–419, Springer-Verlag, Berlin - Heidelberg - New York, 1985.
26. C. T. C. Wall, *Finiteness conditions for* CW-*complexes* I, Ann. of Math. 81 (1965), 56–69.
27. C. T. C. Wall, *Finiteness conditions for* CW-*complexes* II, Proc. Roy. Soc. Sér. A 295 (1966), 129–139.
28. G. W. Whitehead, *Elements of homotopy theory* (Graduate Texts in Math. 61), Springer-Verlag, Berlin - Heidelberg - New York, 1978.

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