Journal of Pure and Applied Algebra 45 (1987) 143–169
North-Holland

THE TRANSFER MAPS INDUCED IN THE ALGEBRAIC K_0- AND K_1-GROUPS BY A FIBRATION II

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Communicated by C.A. Weibel
Received 20 July 1985

0. Introduction

0.1. *Abstract*

In this paper we continue the study of the algebraic transfer $p^*: K_n(\mathbb{Z}\pi_1(B)) \to K_n(\mathbb{Z}\pi_1(E))$ for $n=0,1$ defined in [12] for a fibration $p: E \to B$. The algebraic transfer p^* agrees with the geometric transfers $p^!: K_0(\mathbb{Z}\pi_1(B)) \to K_0(\mathbb{Z}\pi_1(E))$ and $p^!: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$ constructed in [7,8] and [4] respectively. The geometric K_0-transfer sends Wall's finiteness obstruction of B to the one of E. The Whitehead torsion of a homotopy equivalence $f: B_0 \to B$ is mapped by the Whitehead transfer to the one of $\bar{f}: E_0 \to E$ given by the pullback. An algebraic vanishing theorem for p^* is a vanishing theorem for $p^!$ and is thus geometrically meaningful. Such algebraic vanishing theorems are obtained in the last three sections.

0.2. *Survey of the contents*

In Section 1 we give a review of the construction of the algebraic transfer. On the one hand we construct an abelian group $K_0^c(R-S)$ and a pairing $T: K_0^c(R-S) \otimes K_n(R) \to K_n(S)$ for $n=0,1$ and rings R and S. On the other hand we assign to a fibration $p: E \to B$ with a finitely dominated CW-complex as fibre an element $[p] \in K_0^c(\mathbb{Z}\pi_1(B) - \mathbb{Z}\pi_1(E))$.

We explain in Section 2 how $[p]$ and p^* can be computed from homology if the homology possesses finitely generated projective resolutions.

In Section 3 we prove that the algebraic transfer is compatible with the Bass–Heller–Swan homomorphisms. We extend the constructions above to negative K-groups.

We examine in Section 4 the orientation data of a fibration. They consist of the fundamental group sequence and the transport of the fibre resp. pointed fibre along loops in the base resp. total space.

This leads to the notion of a chain complex with a twist in Section 5. Given a nor-

0022-4049/87/$3.50

mal subgroup H of G and a $\mathbb{Z}H$-chain complex, a G-twist is an extension of the H-operation to a G-operation up to homotopy. We can assign to a fibration such a chain complex with a twist using the orientation data. It determines the class $[p]$ in $K_0^c(\mathbb{Z}\pi_1(B)-\mathbb{Z}\pi_1(E))$.

In Section 6 we apply representation theory to compute $[p]$ and p^*. If $\pi_1(E)$ is finite and we use rational coefficients, it turns out that $[p]$ is given by a rational representation. Its character is computed by Lefschetz numbers. The algebraic transfer is given by tensoring with the representation corresponding to $[p]$.

We examine orientable fibrations in Section 7. We show that there is a section s of $p_*: \pi_1(E)\to\pi_1(B)$ such that p^* is given by $s_*: K_n(\mathbb{Z}\pi_1(B))\to K_n(\mathbb{Z}\pi_1(E))$ if the fibre is a finite CW-complex with non-vanishing Euler characteristic. If $\pi_1(F)$ can be written as $\mathbb{Z}\times H$ such that $\mathbb{Z}$ is contained in the kernel of $\pi_1(F)\to\pi_1(E)$, then $[p]$ and p^* are zero.

In Section 8 we treat untwisted fibrations. Untwisted means that the transport of the pointed fibre is trivial. We are interested in vanishing theorems for the transfer. In this context untwisted fibrations are of special interest because for them $p_*\circ p^*$ is always zero (Theorem 8.2). If we further assume a finite fibre, the composition $p^*\circ p_*$ vanishes. If a fibration is not untwisted with a finite fibre, we cannot, in general, expect $p_*\circ p^*=p^*\circ p_*=0$ or even $p^*=0$.

The main result for an untwisted fibration is that p^* can be written as a composition $\beta_1\circ\beta_2\circ\cdots\circ\beta_r\circ q^*$ such that the β_i-s are S^1-transfers and q^* the transfer of an untwisted fibration whose fibre has a finite fundamental group (Theorem 8.1). This leads to some vanishing results (Theorem 8.3).

For explicit calculations it is reasonable to assume F and $\pi_1(B)$ to be finite because one has not much information about $K_n(\mathbb{Z}\pi)$ for infinite π. But then the K_0-transfer is zero (Theorem 8.3(b)). If we further presume that $\pi_1(F)$ is infinite, the K_1-transfer also turns out to be trivial except for the case where $\pi_1(F)$ is $\mathbb{Z}$ and $\pi_1(E)$ is finite. In this special case p^* is $\chi(\tilde{F})\cdot\beta$ where β is the transfer of an orientable S^1-fibration with the same fundamental group data and $\chi(\tilde{F})$ the Euler characteristic of the universal covering of the fibre. The homomorphism β, however, is not zero in general (see [19]).

Section 9 contains the proof that for an orientable fibration with a connected compact Lie group G the transfer p^* is zero if G is not isomorphic to $T^a\times SO(3)^b$ and can always be written as a composition of S^1-transfers.

0.3. Conventions and notations

Given a fibration $F\to E\xrightarrow{p}B$ we always assume that E and B are connected and F is a finitely dominated CW-complex. We write $\Gamma=\pi_1(E)$, $\pi=\pi_1(B)$ and $\Delta=\text{kernel}(p_*:\Gamma\to\pi)$. The epimorphism $\delta:\pi(F)\to\Delta$ is induced by the inclusion $F\subset E$.

We denote by A a commutative ring with unit. For a group G the group ring with A-coefficients is written as AG or $A[G]$.

Module means left module unless a right action is stated explicitly. Chain complexes always consist of projective modules. The functor 'cellular chain complex with A-coefficients' is denoted by $C(?, A)$. If $f: C \to D$ is a chain map, its mapping cone is given by

$$\cdots \to C_{*-1} \oplus D_* \xrightarrow[\begin{bmatrix} -c_{*-1} & 0 \\ f_{*-1} & d_* \end{bmatrix}]{} C_{*-2} \oplus D_{*-1}.$$

1. Review of the algebraic transfer

The purpose of this section is to recall the construction of the algebraic transfer defined in [12]. Namely, given associative rings with unit R and S, we introduce an abelian group $K_0^c(R-S)$ and a pairing ${}^R T^S: K_0^c(R-S) \otimes K_n(R) \to K_n(S)$ for $n = 0, 1$.

A chain homotopy representation (C, U) consists of an S-chain complex C and a ring homomorphism $U: R \to [C, C]_s^\circ$ into the dual ring of homotopy classes of chain maps $C \to C$. A morphism $[f]: (C, U) \to (D, V)$ of chain homotopy representations is a homotopy class $[f]$ of chain maps $f: C \to D$ with $f \circ U(r) \simeq V(r) \circ f$ for all $r \in R$. We call a sequence of morphisms of chain homotopy representations

$$(C^1, U^1) \xrightarrow{[i]} (C^0, U^0) \xrightarrow{[p]} (C^2, U^2)$$

exact if there exists a choice of representatives $i, p, U^j(r)$ for $j = 0, 1, 2$ and $r \in R$ such that the following diagram has exact rows and commutes strictly (not only up to homotopy):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1 & \xrightarrow{i} & C^0 & \xrightarrow{p} & C^2 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle U^1(r)} & & \Big\downarrow{\scriptstyle U^0(r)} & & \Big\downarrow{\scriptstyle U^2(r)} & & \\ 0 & \longrightarrow & C^1 & \xrightarrow[i]{} & C^0 & \xrightarrow[p]{} & C^2 & \longrightarrow & 0 \end{array}$$

Let $K_0^c(R-S)$ be the Grothendieck group of the category of chain homotopy representations. We recall that the Grothendieck group of a small category with exact sequences is the quotient of the free abelian group generated by the isomorphism classes of objects and the subgroup generated by elements $[X] - [Y] + [Z]$ for each exact sequence $X \to Y \to Z$.

Before we define the pairing ${}^R T^S$ we have to introduce two invariants for chain *complexes. Let* $f: C \to C$ be a self-equivalence of a finitely dominated S-chain complex C. Choose a chain equivalence $h: C \to P$ and a homotopy inverse h^{-1} for a finitely generated projective S-chain complex P. If d is the differential and Δ a chain contraction of the algebraic mapping cone D of $h \circ f \circ h^{-1}$, then

$$(\mathrm{d} + \Delta): \bigoplus_{n=0}^{\infty} D_{2n+1} \to \bigoplus_{n=0}^{\infty} D_{2n}$$

is an automorphism of the finitely generated projective S-module $\bigoplus_{n=0}^{\infty} P_n$. Define the torsion $t(f)$ of f in $K_1(S)$ by the class of $\mathrm{d}+\Delta$.

Let C be a finitely dominated S-chain complex and $p: C \to C$ a homotopy projection, i.e. $p \circ p \simeq p$. A split object (D, r, i) for p consists of a chain complex D and chain maps $r: C \to D$ and $i: D \to C$ with $r \circ i \simeq \mathrm{ID}$ and $i \circ r \simeq p$. Such a split object exists uniquely up to homotopy and is the homotopy theoretic summand of C defined by p, namely $D \oplus \mathrm{Cone}(i) \simeq C$. Define the finiteness obstruction $w(p) \in K_0(S)$ of p by Wall's finiteness obstruction $w(D)$ for any split object (D, r, i). We recall that $w(D)$ is given by $\sum (-1)^n [P_n]$ for any finitely generated projective chain complex P with $P \simeq D$ (see [27, p. 138]).

Let $F: \{\text{based free } R\text{-modules}\} \to \mathrm{ho}\{S\text{-chain complexes}\}$ be an additive functor from the category of based free R-modules into the homotopy category of S-chain complexes such that $F(R)$ is finitely dominated. We define homomorphisms $F_n: K_n(R) \to K_n(S)$ for $n = 0, 1$.

Let $p: R^k \to R^k$ be a projection, i.e. $p \circ p = p$, such that its image represents η in $K_0(R)$. Define $F_0(\eta)$ by $w(F(p))$. Given an automorphism $f: R^n \to R^n$ representing $\eta \in K_1(R)$, let $F_1(\eta)$ be $t(F(f))$.

Let (C, U) be a chain homotopy representation with finitely dominated C. We associate to (C, U) an additive functor $F: \{\text{based free } R\text{-modules}\} \to \mathrm{ho}\{S\text{-chain complexes}\}$ which sends $R^n \to R^m$ $x \to xA$ for a matrix $A = (r_{i,j})$ to $\bigoplus_n C \to \bigoplus_m C$ given by $(U(r_{j,i}))$. Hence we can assign to (C, U) a homomorphism $F_n: K_n(R) \to K_n(S)$ for $n = 0, 1$. Since the Grothendieck group $K_0^{\mathrm{c}}(R-S)$ is generated by the isomorphism classes of chain homotopy representations we get a pairing ${}^R T^S: K_0^{\mathrm{c}}(R-S) \otimes K_n(R) \to K_n(S)$ for $n = 0, 1$. The proof that this is well defined can be found in [12].

Given $x \in K_0^{\mathrm{c}}(R-S)$ we write ${}^R T_x^S: K_n(R) \to K_n(S)$ for ${}^R T^S(x, ?)$. Sometimes we abbreviate ${}^R T^S$ and ${}^R T_x^S$ by T and T_x.

Let $F \to E \xrightarrow{p} B$ be a fibration with F a finitely dominated CW-complex and B and E connected. We write $\Gamma = \pi_1(E)$ and $\pi = \pi_1(B)$. We denote by $\hat{p}: \tilde{E} \to B$ the composition of p with the universal covering of E. It is a Γ-equivariant fibration whose fibre $\hat{F}$ is a Γ-space. The equivariant fibre transport defines a homomorphism $u: \pi \to [\hat{F}, \hat{F}]_\Gamma$. Define a ring homomorphism $U: A\pi \to [C(\hat{F}, A), C(\hat{F}, A)]_{A\Gamma}^0$ by $w \to [C(u(w^{-1}), A)]$ so that we get a chain representation $(C(\hat{F}, A), U)$.

Definition 1.1. Let $[p] \in K_0^{\mathrm{c}}(A\pi - A\Gamma)$ be the class of $(C(\hat{F}, A), U)$. The *algebraic transfer* of p with A-coefficients $p^*: K_n(A\pi) \to K_n(A\Gamma)$ is defined by $T_{[p]}$ for $n = 0, 1$.

2. Homological computations

In this section we want to calculate the class of a chain homotopy representation (C, U) in $K_0^{\mathrm{c}}(R-S)$ and the homomorphisms $T_{[C, U]}: K_n(R) \to K_n(S)$ by its homology.

We denote by $K_0(R-S)$ the Grothendieck group of S-R-bimodules possessing a finitely generated projective S-resolution regarded only as left S-modules. Given such a module M, the tensor product $M\otimes_R ?$ yields an exact functor from the category of finitely generated projective R-modules into the category of S-modules having a finitely generated projective S-resolution. This induces a pairing ${}_S\otimes_R : K_0(R-S)\otimes K_n(R)\to K_n(S)$ for $n=0,1,2,3,\ldots$ (see [22, pp. 106, 109]).

Let M be an S-R-bimodule and P a finitely generated projective resolution of M regarded as S-module. The right R-module structure can be considered as a ring homomorphism $R\to \mathrm{HOM}_S(M,M)^0$. The map $[P,P]_S\to \mathrm{HOM}_S(M,M)$ sending $[f]$ to $H_0(f)$ is an isomorphism of rings (see [13, p. 87]). This yields a ring homomorphism $U: R\to [P,P]_S^\circ$ so that (P,U) is a chain homotopy representation. Define a map $j: K_0(R-S)\to K_0^c(R-S)$ by $[M]\to[P,U]$.

Theorem 2.1. (a) *The map j is a well-defined homomorphism.*

(b) *The following diagram commutes for $n=0,1$:*

$$\begin{array}{ccc}
K_0(R-S)\otimes K_n(R) & \xrightarrow{\ {}_S\otimes_R\ } & K_n(S) \\
\Big\downarrow{\scriptstyle j\otimes \mathrm{ID}} & & \Big\downarrow{\scriptstyle \mathrm{ID}} \\
K_0^c(R-S)\otimes K_n(R) & \xrightarrow{\ {}^R T^S\ } & K_n(S)
\end{array}$$

(c) *Let (C,U) be a chain homotopy representation such that each $H_n(C)$ possesses a finitely generated projective resolution of left S-modules. Then $\sum_{n=0}^{\infty}(-1)^n[H_n(C)]$ is a well-defined element in $K_0(R-S)$ sent by j to $[C,U]$ in $K_0^c(R-S)$.*

(d) *If S is regular, then j is an isomorphism with inverse map $K_0^c(R-S)\to K_0(R-S)$ sending $[C,U]$ to $\sum_{n=0}^{\infty}(-1)^n[H_n(C)]$.*

We will see in Section 6 that this theorem is a good tool for computations. All the various homological computations of the transfer induced by a fibration in [3, 7, 8, 14, 15] can easily be derived from it. The rest of this section contains the proof of Theorem 2.1 and some remarks at the end.

Proof of Theorem 2.1. (a) The difficult part of the proof consists in showing that j is compatible with the relations in $K_0(R-S)$ given by exact sequences. This is a consequence of the following Lemma 2.2. Its proof is closely related to the proof that K_0 of the category of finitely generated projective modules and K_0 of the category of modules possessing a finitely generated projective resolution are isomorphic [26, 102 ff]. Given an S-R-bimodule M, we call a chain homotopy representation (C,U) an S-R-resolution for M if C is a finitely generated projective S-resolution of M as left S-module and $H_0(C)$ and M are isomorphic as S-R-bimodules.

Lemma 2.2. *Let* $0 \to M^1 \xrightarrow{j} M^0 \xrightarrow{q} M^2 \to 0$ *be an exact sequence of S-R-bimodules and* (C^1, U^1) *resp.* (C^2, U^2) *an S-R-resolution for* M^1 *resp.* M^2. *Then there exists an exact sequence of chain homotopy representations*

$$(C^1, U^1) \xrightarrow{[i]} (C^0, U^0) \xrightarrow{[p]} (C^2, U^2)$$

such that (C^0, U^0) *is an S-R-resolution of* M^0.

Proof. We construct inductively for $n = -1, 0, 1, \dots$ commutative diagrams of S-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1_n & \xrightarrow{i_n} & C^0_n & \xrightarrow{p_n} & C^2_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c^1_n} & & \downarrow{\scriptstyle c^0_n} & & \downarrow{\scriptstyle c^2_n} & & \\
0 & \longrightarrow & C^1_{n-1} & \xrightarrow{i_{n-1}} & C^0_{n-1} & \xrightarrow{p_{n-1}} & C^2_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c^1_{n-1}} & & \downarrow{\scriptstyle c^0_{n-1}} & & \downarrow{\scriptstyle c^2_{n-1}} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle c^1_0} & & \downarrow{\scriptstyle c^0_0} & & \downarrow{\scriptstyle c^2_1} & & \\
0 & \longrightarrow & C^1_{-1} & \xrightarrow{i_{-1}} & C^0_{-1} & \xrightarrow{p_{-1}} & C^2_{-1} & \longrightarrow & 0
\end{array}$$

and maps $U^j(r)_k : C^j_k \to C^j_k$ for $-1 \le k \le n$ and $j = 1, 0, 2$ and $r \in R$ with the following properties:

(1) $0 \to C^1_{-1} \xrightarrow{i_{-1}} C^0_{-1} \xrightarrow{p_{-1}} C^2_{-1} \to 0$ is just $0 \to M^1 \xrightarrow{j} M^0 \xrightarrow{q} M^2 \to 0$ and $U^j_{-1}(r)$ right multiplication with r on M^j.

(2) C^j_k, c^j_k and $U^j(r)_k$ come from the given S-R resolutions (C^j, U^j) of M^j for $j = 1, 2$. We have chosen representatives $U^j(r) : C^j \to C^j$ for $j = 1, 2$.

(3) C^0_k is the direct sum $C^1_k \oplus C^2_k$ and i_k the canonical inclusion and p_k the canonical projection for $k \ge 0$.

(4) The columns and rows are exact.

(5) $$i_{k-1} \circ c^1_k = c^0_k \circ i_k, \quad 0 \le k \le n,$$
$$p_{k-1} \circ c^0_k = c^2_k \circ p_k, \quad 0 \le k \le n,$$
$$U^j(r)_{k-1} \circ c^j_k = c^j_k \circ U^j(r)_k, \quad 0 \le k \le n,\ j = 1, 0, 2.$$

(6) $$U^0(r)_k \circ i_k = i_k \circ U^1(r)_k, \quad -1 \le k \le n,\ j = 1, 0, 2,$$
$$U^2(r)_k \circ p_k = p_k \circ U^0(r)_k, \quad -1 \le k \le n,\ j = 1, 0, 2.$$

These data give us the desired *S-R*-resolution (C^0, U^0) of M^0 and the exact sequence of chain homotopy representations

$$(C^1, U^1) \xrightarrow{[i]} (C^0, U^0) \xrightarrow{[p]} (C^2, U^2) \quad \text{for } n \to \infty.$$

The beginning of the induction $n = -1$ is determined by property (1) so that only the induction step remains to be done. Property (3) determines

$$0 \to C^1_{n+1} \xrightarrow{i_{n+1}} C^0_{n+1} \xrightarrow{p_{n+1}} C^2_{n+1} \to 0.$$

Because of $p_n(\text{kernel}(c^0_n)) = \text{kernel}(c^2_n)$, we can choose $f: C^2_{n+1} \to \text{kernel}(c^0_n)$ with $p_n \circ f = c^2_{n+1}$. Define $c^0_{n+1}: C^0_{n+1} = C^1_{n+1} \oplus C^2_{n+1} \to C^0_n$ by $(i_n \circ c^1_{n+1}, f)$. A short diagram chase proves that $\text{image}(c^0_{n+1}) = \text{kernel}(c^0_n)$. Now $p_n \circ (U^0(r)_n \circ f - f \circ U^2(r)_{n+1}) = 0$ and $c^0_n \circ (U^0(r)_n \circ f - f \circ U^2(r)_{n+1}) = 0$ is valid. Because of $\text{image}(i_n \circ c^1_{n+1}) = \text{kernel}(p_n) \cap \text{kernel}(c^0_n)$ a map $g: C^2_{n+1} \to C^1_{n+1}$ with $i_n \circ c^1_{n+1} \circ g = U^0(r)_n \circ f - f \circ U^2(r)_{n+1}$ can be constructed. Define $U^0(r)_{n+1}: C^0_{n+1} \to C^0_{n+1}$ as the map $C^1_{n+1} \oplus C^2_{n+1} \to C^1_{n+1} \oplus C^2_{n+1}$ given by

$$\begin{bmatrix} U^1(r)_{n+1} & g \\ 0 & U^2(r)_{n+1} \end{bmatrix}.$$

One easily checks that all the properties (1) to (6) are fulfilled. Hence j is well defined. □

(b) We verify only the case $n = 0$. Let $p: R^m \to R^m$ be a projection and $\eta \in K_0(R)$ represented by its image. Let (C, U) be an *S-R*-resolution for the *S-R*-module M so that j sends $[M]$ to $[C, U]$. Choose a split object (D, r, i) for the homotopy projection $F(p)$ if F is the additive functor assigned to (C, U) in Section 1. Because of $H_i(D_*) = \text{image}(H_i(F(p)))$, we have $H_i(D_*) = 0$ for $i > 0$ and $H_0(D_*) = M \otimes_R \text{image}(p)$. If D is homotopy equivalent to the finitely generated projective chain complex P, then P is a resolution of $M \otimes_R \text{image}(p)$. By definition this implies

$$\begin{aligned} T(j([M]), \eta) &= T([C, U], \eta) = w(D) = w(P) = \sum (-1)^n [P_n] \\ &= [M \otimes_R \text{image}(p)] = [M]\ {}_S\otimes_R\ \eta. \end{aligned}$$

(c) We use induction over $m = 0, 1, 2, \ldots$ with $H_i(C) = 0$ for all $i > m$. The case $m = 0$ is just the definition of j and the inductive step is contained in the following lemma:

Lemma 2.3. *Let (C^1, U^1) be a chain homotopy representation with $H_i(C^1) = 0$ for $i > m$. Let (C^2, U^2) be an S-R-resolution for $H_m(C^1)$. Then there exists an exact sequence of chain homotopy representations*

$$(C^1; U^1) \xrightarrow{[i]} (C^0, U^0) \xrightarrow{[p]} \Sigma^{m+1}(C^2, U^2)$$

for Σ^{m+1} *as the* $(m+1)$*-fold suspension such that* $H_i(C^0)=H_i(C^1)$ *for* $i<m$ *and* $H_i(C^0)=0$ *for* $i\geq m$.

Proof. Let $f:\Sigma^m C^2 \to C^1$ be an S-chain map inducing the identity on the mth homology group. Let $h(r):\Sigma^m C^2 \to C^1$ be a chain homotopy $h(r):U^1(r)\circ f\simeq f\circ \Sigma^m U^2(r)$ for $r\in R$. Define C^0 as the mapping cone of f so that there is an exact sequence

$$0\to C^1 \xrightarrow{i} C^0 \xrightarrow{p} \Sigma^{m+1}C^2 \to 0.$$

Define a chain map $U^0(r):C^0\to C^0$ by

$$\begin{bmatrix} (\Sigma^m U^2(r))_{n-1} & 0 \\ h(r)_{n-1} & U^1(r)_n \end{bmatrix} : (\Sigma^m C^2)_{n-1}\oplus C_n^1 \to (\Sigma^m C^2)_{n-1}\oplus C_n^1.$$

Using the long homology sequence of f one shows that $H_i(C^0)=0$ for $i\geq m$ and $H_i(C^0)=H_i(C^1)$ for $i<m$ is valid. To prove that $(C^1,U^1)\xrightarrow{[i]}(C^0,U^0)\xrightarrow{[p]}\Sigma^{m+1}(C^2,U^2)$ is an exact sequence of chain homotopy representations it suffices to verify that $U^0(r_1)\circ U^0(r_2)\simeq U^0(r_1\cdot r_2)$ and $U^0(r_1)+U^0(r_2)\simeq U^0(r_1+r_2)$ holds for $r_1,r_2\in R$. We will do this only for the first relation. Choose for $r_1,r_2\in R$ a homotopy $h^1(r_1,r_2):C^1\to C^1$ between $U^1(r_1)\circ U^1(r_2)$ and $U^1(r_1\cdot r_2)$. Define $h^0(r_1,r_2)_i:C_i^0\to C_{i+1}^0$ for $i<m$ by $h^1(r_1,r_2)_i:C_i^1\to C_{i+1}^1$ and for $i=m$ by $0\oplus h^1(r_1,r_2)_m:C_m^1\to(\Sigma^m C^2)_m\oplus C_{m+1}^1$. If c^0 denotes the differential of C^0 we get

$$c_{i+1}^0\circ h^0(r_1,r_2)_i + h^0(r_1,r_2)_{i-1}\circ c_i^0 = U^0(r_1)_i\circ U^0(r_2)_i - U^0(r_1\cdot r_2)_i$$

for $i\leq m$.

As C^0 is projective and $H_i(C^0)=0$ for $i\geq m$ we can construct maps $h^0(r_1,r_2)_i:C_i^0\to C_{i+1}^0$ for $i>m$ yielding a homotopy $h^0(r_1,r_2):U^0(r_1)\circ U^0(r_2)\simeq U^0(r_1\cdot r_2)$.

This finishes the proof of Lemma 2.3 and therefore the proof of Theorem 2.1(c). □

(d) is a direct consequence of (c), since for a regular ring S each finitely generated module possesses a finitely generated projective resolution and the homology of a finitely dominated chain complex is finitely generated. □

The finiteness obstruction defines for $R=\mathbb{Z}$ an inverse map $w:K_0^c(\mathbb{Z}-S)\to K_0(S)$ of $j:K_0(S)=K_0(\mathbb{Z}-S)\to K_0^c(\mathbb{Z}-S)$. In this case Theorem 2.1(c) reproves the computation of the finiteness obstruction $w(C)$ of a finitely dominated chain complex by $w(C)=\sum(-1)^n[H_n(C)]$ in [21, p. 893] provided that $H_n(C)$ possesses a finitely generated projective resolution.

Let π and Γ be finite groups. A $Q\Gamma$-$\mathbb{Z}\pi$-bimodule M can be interpreted as $Q[\Gamma\times\pi]$-module if $(x,y)\cdot m = xmy^{-1}$ for $x\in\Gamma$, $y\in\pi$, $m\in M$. Then Theorem 2.1 yields an isomorphism between $K_0^c(\mathbb{Z}\pi-Q\Gamma)$ and the rational representation ring of $\Gamma\times\pi$.

Let R be the polynomial ring $\mathbb{Z}[x]$. Then $j: K_0(\mathbb{Z}[x]-S) \to K_0^c(\mathbb{Z}[x]-S)$ is an isomorphism since a ring homomorphism $U: \mathbb{Z}[x] \to [C,C]_S^\circ$ is just a chain homotopy class of self-chain maps $C \to C$. If $H(A)$ resp. $P(A)$ is the category of S-modules which have a finitely generated projective resolution resp. which are finitely generated and projective, $K_0(\mathbb{Z}[x]-S)$ is isomorphic to

$$K_0(\mathrm{End}(P(A))) = K_0(\mathrm{End}(H(A))).$$

These groups were computed in [1] by characteristic polynomials for commutative S. See also [2,10].

3. Transfer and the Bass–Heller–Swan-homomorphisms

The purpose of this section is to show that the pairing $T: K_0^c(R-S) \otimes K_n(R) \to K_n(S)$ for $n=0,1$ is compatible with the Bass–Heller–Swan-homomorphisms. This enables us to define T also for negative n.

Let t be a generator of $\mathbb{Z}$. We can write $R[t,t^{-1}]$ as $R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$. Denote by $l_t: \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]$ the multiplication with t. Let the homomorphism $h: K_0(R) \to K_1(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}])$ send $[P]$ to the class of the automorphism $\mathrm{ID} \otimes_{\mathbb{Z}} l_t$ of $P \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$.

Given an R-module M we write $M \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]^+$ for the R-submodule generated by elements $x \otimes t^n$ with $n \geq 0$, where R operates only on the left factor. Let f be an automorphism of the $R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$-module $R^n \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$. Choose an integer m such that $t^m f$ maps $R^n \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]^+$ to itself. Then the cokernel of the R-module homomorphism $t^m f: R^n \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]^+ \to R^n \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]^+$ is a finitely generated projective R-module. Define a homomorphism $\varphi: K_1(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]) \to K_0(R)$ by $\varphi([f]) = [\mathrm{cokernel}(t^m f)] - [\mathrm{cokernel}(t^m \mathrm{ID})]$. Then φ and h are well-defined homomorphisms with $\varphi \circ h = \mathrm{ID}$ (see [26, p. 227 ff]).

Let $(C, U: R \to [C,C]_S^\circ)$ be a chain homotopy representation. Now $C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$ is a $S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]$-chain complex. Define a ring homomorphism $V: R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}] \to [C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}], C \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]]_{S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]}^\circ$ by $r \otimes t \to [U(r) \otimes_{\mathbb{Z}} l_t]$. Since $(C \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[\mathbb{Z}], V)$ is a chain homotopy representation we get a homomorphism $B: K_0^c(R-S) \to K_0^c(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}] - S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}])$.

Theorem 3.1. (a) *The following diagram commutes:*

$$\begin{array}{ccc}
K_0^c(R-S) \otimes K_0(R) & \xrightarrow{\quad T \quad} & K_0(S) \\
\downarrow {\scriptstyle B \otimes h} & & \downarrow {\scriptstyle h} \\
K_0^c(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}] - S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]) \otimes K_1(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}]) & \xrightarrow{\quad T \quad} & K_1(S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}])
\end{array}$$

(b) *The following diagram commutes:*

$$\begin{array}{ccc}
K_0^c(R-S)\otimes K_0(R) & \xrightarrow{\quad\quad T \quad\quad} & K_0(S) \\
\uparrow \scriptstyle{\mathrm{ID}\otimes\varphi} & & \\
K_0^c(R-S)\otimes K_1(R\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]) & & \downarrow \scriptstyle{\varphi} \\
\downarrow \scriptstyle{B\otimes\mathrm{ID}} & & \\
K_0^c(R\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]-S\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])\otimes K_1(R\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]) & \xrightarrow{T} & K_1(S\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])
\end{array}$$

(c) *Given* $x\in K_0^c(R-S)$, $T_x: K_0(R)\to K_0(S)$ *is the composition of* $\varphi: K_1(S\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])\to K_0(S)$ *and* $T_{B(x)}: K_1(R\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])\to K_1(S\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])$ *and* $h: K_0(R)\to K_1(R\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}])$.

Proof. The verification of this theorem is straightforward if one has a computation of the Bass–Heller–Swan-homomorphisms for chain complexes. This is given by the following lemma whose proof is analogous to the one in [16, pp. 420–421].

Lemma 3.2. (a) *Let* P *be a finitely generated projective* S*-chain complex and* $f: P\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]\to P\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]$ *an* $S\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]$*-chain equivalence. Choose an integer* $m\geq 0$ *such that* $t^m f$ *induces an* S*-chain-map* $t^m f: P\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]^+\to P\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]^+$. *Then the mapping cone* $\mathrm{Cone}(t^m f)$ *is a finitely dominated* S*-chain complex and* φ *sends the torsion* $t(f)$ *to* $w(\mathrm{Cone}(t^m f))-w(\mathrm{Cone}(t^m\mathrm{ID}))$ *with* w *as Wall's finiteness obstruction.*

(b) *Given a finitely dominated* S*-chain complex* C*, the torsion* t *of* $\mathrm{ID}\otimes_{\mathbb{Z}} l_t: C\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]\to C\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]$ *is* $h(w(C))$. □

Let $F\to E\to B$ be a fibration with connected B, and F a finitely dominated CW-complex. Because of Theorem 3.1(c) the following diagram commutes, since B maps $[p]$ to $[p\times\mathrm{ID}_{S^1}]$:

$$\begin{array}{ccc}
K_0(\mathbb{Z}\pi_1(E)) & \xleftarrow{\ \varphi\ } & K_1(\mathbb{Z}[\pi_1(E)\times\mathbb{Z}]) \\
p^*\uparrow & & \uparrow (p\times\mathrm{ID}_{S^1})^* \\
K_0(\mathbb{Z}\pi_1(B)) & \xrightarrow{\ h\ } & K_1(\mathbb{Z}[\pi_1(B)\times\mathbb{Z}])
\end{array}$$

Hence the K_1-transfer determines the K_0-transfer. This follows also geometrically from [16, p. 422].

Using Theorem 3.1 we can define our pairing $T: K_0^c(R-S)\otimes K_n(R)\to K_n(S)$ also for negative n. Given $n\geq 0$ let $f(j): \mathbb{Z}[\mathbb{Z}^n]\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}]\to\mathbb{Z}[\mathbb{Z}^{n+1}]$ be the ring homomorphism sending $(t_1\otimes\cdots\otimes t_n)\otimes t$ to $t_1\otimes\cdots\otimes t_{j-1}\otimes t\otimes t_j\otimes\cdots\otimes t_n$ for $j=$

$1, \ldots, n+1$. It just permutes the variables. This induces a map $f(j)_*$ on the K_1-groups. If $h: K_0(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^n]) \to K_1(R \otimes_{\mathbb{Z}} [\mathbb{Z}^n] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}])$ is the Bass-Heller-Swan-homomorphism for $R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^n]$, one can define $K_{-n}(R)$ as the subgroup $\bigcap_{j=1}^{n+1} \text{image}(f(j)_* \circ h)$ of $K_1(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}])$.

Define $T: K_0^c(R-S) \otimes K_{-n}(R) \to K_{-n}(S)$ as the map for $n \geq 1$ making the following diagram commutative if k denotes the inclusion:

$$\begin{array}{ccc}
K_0^c(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}] - S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}]) \otimes K_1(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}]) & \xrightarrow{T} & K_1(S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}]) \\
\Big\uparrow{\scriptstyle B^{n+1} \otimes k} & & \Big\uparrow{\scriptstyle k} \\
K_0^c(R-S) \otimes K_{-n}(R) & \xrightarrow{T} & K_{-n}(S)
\end{array}$$

In the notation of [12, pp. 14,15] one checks that the maps $f(j)_* \circ B^{n+1}$ and $B^{n+1} \circ f(j)^*: K_0^c(R-S) \to K_0^c(R \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}] - S \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+1}])$ agree. Now apply Lemma 4.3 and Lemma 4.4 in [12, pp. 14,15] and Theorem 3.1 to prove that the definition for negative K-groups makes sense.

4. The orientation data of a fibration

We collect in this section the orientation data of a fibration consisting of the fundamental group sequence and the (pointed) fibre transport and state some elementary but important properties of them.

Let $F \to E \to B$ be a fibration with connected F, E and B. The transport of the fibre along paths in the base space induces a homomorphism $\omega: \pi_1(B) \to [F, F]$ into the monoid of homotopy classes of self-maps of F (see [28, p. 186]). Similarly the transport of the pointed fibre along paths in the total space yields a homomorphism $\sigma: \pi_1(E) \to [F, F]^+$ into the monoid of pointed homotopy classes of pointed self-maps of the pointed fibre. We always suppress the notion of the base-points. The homomorphism $\varrho: \pi_1(F) \to [F, F]^+$ sends the class of a loop w to the class of a pointed self-map of F which is homotopic along w to the identity (see [28, p. 98 ff]). Let $G_1(F)$ be the kernel of ϱ. This group was originally defined in [9]. We denote by $f: [F, F]^+ \to [F, F]$ the forgetful map. One easily checks the following proposition (see [11, p. 3.3]):

Proposition 4.1. (a) *The following sequence is exact:*

$$1 \to G_1(F) \hookrightarrow \pi_1(F) \xrightarrow{\varrho} [F, F]^+ \xrightarrow{f} [F, F] \to 1.$$

(b) $\omega \circ p_* = f \circ \sigma$.
(c) $\sigma \circ i_* = \varrho$ *for the inclusion* $i: F \to E$.
(d) $\text{kernel}(i_*) \subset G_1(F)$.
(e) $i_*(G_1(F)) \subset \text{center}(\pi_1(E))$.

The following proposition contains the main properties of $G_1(X)$:

Proposition 4.2. (a) *The center of $\pi_1(X)$ contains $G_1(X)$. A central element w in $\pi_1(X)$ belongs to $G_1(X)$ if and only if $l(w):\tilde{X}\to\tilde{X}$ $x\to wx$ is $\pi_1(X)$-homotopic to the identity.*

(b) *Let X be a finite CW-complex with $G_1(X)\neq\{1\}$. Then its Euler characteristic $\chi(X)$ is zero.*

(c) *Let X be a finitely dominated CW-complex with $G_1(X)\cap[\pi_1(X),\pi_1(X)]\neq G_1(X)$. Then $\chi(X)$ is zero.*

(d) *If X is a H-space, we get $G_1(X)=\pi_1(X)$.*

Proof. (a), (b) and (d) are proved in [9].

(c) Choose an epimorphism $f:\pi_1(X)\to G$ into a finite group such that there is a $g\in G$ with $g\neq 1$, $g\in f(G_1(X))$.

Let $\bar{X}$ be the covering of X with G as group of deck transformations. Since the change of rings induces the zero map $\tilde{K}_0(\mathbb{Z}G)\to\tilde{K}_0(QG)$ [25], there is a finitely generated free QG-chain complex D homotopy equivalent to $C(\bar{X},Q)$. We get from (a) that $l(g):D\to D$ is homotopic to the identity. This implies for the Lefschetz number Λ_Q:

$$0=\Lambda_Q(l(g):D\to D)=\Lambda_Q(\mathrm{ID}:D\to D)=\chi_Q(D)=\chi_{\mathbb{Z}}(C(\bar{X},Q))=\chi(\bar{X})=|G|\cdot\chi(X).$$

Therefore $\chi(X)$ is zero. □

The following proposition is the basic observation for proving that the algebraic transfer for arbitrary fibres can sometimes be expressed by S^1-transfer maps.

Proposition 4.3. *Let X be a CW-complex. There exists a CW-complex Y with $X\simeq Y\times S^1$ if and only if $\pi_1(X)$ can be written as $G\times\mathbb{Z}$ with $\mathbb{Z}\subset G_1(X)$.*

Proof. Since G_1 is compatible with the cartesian product and for a homotopy equivalence $f:X\to Y$ the group $G_1(X)$ is mapped by f_* to $G_1(Y)$ (see [9]), $X\simeq Y\times S^1$ implies $\pi_1(X)=G\times\mathbb{Z}$ with $\mathbb{Z}\subset G_1(X)$. It remains to prove the other implication. Let $w:S^1\to X$ represent the generator of $\mathbb{Z}$. Because of $\mathbb{Z}\subset G_1(X)$ there is a homotopy $h:X\times I\to X$ with $h_0=h_1=\mathrm{ID}$ such that $h(*,?)$ is $w\circ e$ for the obvious identification $e:I\to S^1$. This induces a map $g:X\times S^1\to X$ with $g(*,?)=w$ and $g(?,e(0))=\mathrm{ID}_X$. Let $q:\bar{X}\to X$ be the covering of X with $q_*(\pi_1(\bar{X}))=G$. Then the composition $\bar{X}\times S^1\xrightarrow{q\times\mathrm{ID}}X\times S^1\xrightarrow{g}X$ is a weak homotopy equivalence of CW-complexes since we have $\pi_1(\bar{X}\times S^1)=\pi_1(\bar{X})\times\pi_1(S^1)=G\times\mathbb{Z}=\pi_1(X)$ and $\pi_n(\bar{X}\times S^1=\pi_n(\bar{X})\times\pi_n(S^1)=\pi_n(X)$ for $n>1$. Hence $\bar{X}\times S^1$ and X are homotopy equivalent. □

Definition 4.4. We call a fibration $F\to E\xrightarrow{p}B$ *untwisted* if $\sigma:\pi_1(E)\to[F,F]^+$ *is* trivial and *orientable* if $\omega:\pi_1(B)\to[F,F]$ is trivial.

Proposition 4.1 implies that p is untwisted if and only if p is orientable and $G_1(F)=\pi_1(F)$. A G-principal bundle for a connected topological group is an untwisted fibration.

5. Chain complexes with a twist

In this section we explain how the chain homotopy representation of a fibration defined in Section 1 or [12, pp. 4,5] can be read off from the fibre and the orientation data. This leads to the notion of a chain complex with a twist. It is useful if one studies the algebraic transfer for group rings.

Now we set some notations we will use for the rest of the paper. Let $F \to E \xrightarrow{p} B$ be a fibration of connected spaces and base points $e \in E$ and $b=p(e)$, and $F=p^{-1}(b)$. We write $\Gamma=\pi_1(E,e)$ and $\pi=\pi_1(B,b)$ and $\Delta=\text{kernel}(p_*:\Gamma\to\pi)$. Hence we get an exact sequence $1\to\Delta\to\Gamma\xrightarrow{p_*}\pi\to 1$. The inclusion $F\subset E$ defines an epimorphism $\delta:\pi_1(F,e)\to\Delta$. The corresponding covering is denoted by $q:(\bar{F},\bar{e})\to(F,e)$.

The transport of the pointed fibre along loops in E yields a homomorphism $\sigma:\Gamma\to[F,F]^+$. For $w\in\Gamma$ choose a representative $s(w):(F,e)\to(F,e)$ of $\sigma(w)$.

If $c(w):\Delta\to\Delta$ is the homomorphism $d\to wdw^{-1}$ we get $\delta\circ s(w)_*=c(w)\circ\delta$. Hence there exists a unique lift $L(w):(\bar{F},\bar{e})\to(\bar{F},\bar{e})$ which is a $c(w)$-equivariant map. This defines a free $c(w)$-equivariant homotopy class $[L(w)]$ of $c(w)$-maps $\bar{F}\to\bar{F}$. It depends only on $w\in\Gamma$ and not on the choice of $s(w)$. If $l(d):\Delta\to\Delta$ is the $c(d)$-equivariant map $x\to dx$, Proposition 4.1(c) implies

(i) $\quad L(d)\simeq_{c(d)} l(d)\quad$ for $d\in\Delta$,

(ii) $\quad L(w_1)\circ L(w_2)\simeq_{c(w_1\cdot w_2)} L(w_1\cdot w_2)\quad$ for $w_1,w_2\in\Gamma$.

We can think of the collection $\{[L(w)]\mid w\in\Gamma\}$ as an extension of the Δ-operation to a Γ-operation up to homotopy. This leads to the following definition:

Let H be a normal subgroup of G and $c(g):H\to H$ be $h\to ghg^{-1}$ and A be a commutative ring with unit. For $h\in H$ the left multiplication with h is denoted by $l(h)$.

Definition 5.1. A *G-twist* L for an H-chain complex C is a collection $\{[L(g)]\mid g\in G\}$ of $c(g)$-chain-maps $C\to C$ with

(i) $\quad L(h)\simeq_{c(h)} l(h)\quad$ for $h\in H$,

(ii) $\quad L(g_1)\circ L(g_2)\simeq_{c(g_1\cdot g_2)} L(g_1\cdot g_2)\quad$ for $g_1,g_2\in G$.

A morphism $[f]:(C,L)\to(D,M)$ of AH-chain-complexes with a G-twist is a AH-homotopy class of AH chain maps $f:C\to D$ with $M(g)\circ f\simeq_{c(g)} f\circ L(g)$ for all $g\in G$. We call

$$(C^1,L^1)\xrightarrow{[i]}(C^0,L^0)\xrightarrow{[p]}(C^2,L^2)$$

exact if there is a choice of representatives i, p and $L^j(g)$ for all $j=0,1,2$ and $g\in G$ such that the following diagram has exact rows and commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1 & \xrightarrow{\;i\;} & C^0 & \xrightarrow{\;p\;} & C^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L^1(g)} & & \downarrow{\scriptstyle L^0(g)} & & \downarrow{\scriptstyle L^2(g)} & & \\ 0 & \longrightarrow & C^1 & \xrightarrow{\;i\;} & C^0 & \xrightarrow{\;p\;} & C^2 & \longrightarrow & 0 \end{array}$$

Define $K_0^c(H-G,A)$ as the Grothendieck group of the category of finitely dominated AH-chain complexes with a G-twist.

The collection $\{[L(w)]\mid w\in\Gamma\}$ above induces a Γ-twist on the cellular $A\Delta$-chain complex $C(\bar{F},A)=C(\bar{F})\otimes_{\mathbb{Z}}A$.

Definition 5.2. Define $[p]\in K_0^c(\Delta-\Gamma,A)$ as the class of

$$(C(\bar{F},A),\{[C(L(w),A)]\mid w\in\Gamma\}).$$

Given a AH-chain complex with a G-twist (C,L), we get a chain representation $(AG\otimes_{AH}C,V)$ with

$$V:A[G/H]\to[AG\otimes_{AH}G,AG\otimes_{AH}C]_{AG}^{\circ}$$

sending gH to the homotopy class of

$$AG\otimes_{AH}C\to AG\otimes_{AH}C\quad 1\otimes x\to g\otimes L(g^{-1})(x).$$

This yields a homomorphism

$$\Lambda:K_0^c(H-G,A)\to K_0^c(A[G/H]-AG).$$

One easily checks, using [12, pp. 20,21],

Proposition 5.3. *The homomorphism* $\Lambda:K_0^c(\Delta-\Gamma,A)\to K_0^c(A\pi-A\Gamma)$ *sends* $[p]\in K_0^c(\Delta-\Gamma,A)$ *of Definition* 5.2 *to* $[p]\in K_0^c(A\pi-A\Gamma)$ *of Definition* 1.1.

If one studies the algebraic transfer of a fibration it is often more convenient to work with $K_0^c(\Delta-\Gamma,A)$ than with $K_0^c(A\pi-A\Gamma)$. The main advantage of the approach using chain homotopy representations is that it can be used for arbitrary rings and not only for group rings.

Now we state some definitions and propositions concerning chain complexes with a twist. We omit the proofs because they are very similar to the one of Section 2 and a detailed treatment can be found in [11].

Definition 5.4. Let $\otimes_t$ be the pairing which makes the following diagram commutative for $n=0,1$:

$$\begin{array}{ccc} K_0^c(H-G,A)\otimes K_n(A[G/H]) & \xrightarrow{\otimes_t} & K_n(AG) \\ \Big\downarrow{\scriptstyle A\otimes \mathrm{ID}} & & \Big\downarrow{\scriptstyle \mathrm{ID}} \\ K_0^c(A[G/H]-AG)\otimes K_n(A[G/H]) & \xrightarrow{T} & K_n(AG) \end{array}$$

Proposition 5.3 implies for a fibration p that $[p]\otimes_t ?$ is the algebraic transfer p^* of Definition 1.1.

Let $K_0(H-G,A)$ be the Grothendieck group of AG-modules whose restriction to AH possesses a finitely generated projective AH-resolution. The tensorproduct over A together with the diagonal action induces a pairing $\otimes_A : K_0(H-G,A)\otimes K_n(A[G/H])\to K_n(AG)$. Given an AG-module M and a finitely generated projective AH-resolution P of its restriction to AH, let L be the G-twist on P uniquely defined by the property that $H_0(L(g))$ is left multiplication with g on M. We get a homomorphism $j: K_0(H-G,A)\to K_0^c(H-G,A)$ mapping $[M]$ to $[P,L]$.

Proposition 5.5. (a) *We have* $\otimes_t\circ(j\otimes \mathrm{ID})=\otimes_A$.

(b) *Let* (C,L) *be a* AH*-chain complex with a* G*-twist such that each* $H_n(C)$ *regarded as* AH*-module possesses a finitely generated projective* AH*-resolution. Then* j *maps* $\sum(-1)^n[H_n(C)]\in K_0(H-G,A)$ *to* $[C,L]\in K_0^c(H-G,A)$.

Let K be a normal subgroup of H and G. Given a AH-chain complex C with a G-twist L we get a G/K-twist $\hat{L}$ on the $A[H/K]$-chain complex $A[H/K]\otimes_{AH}C$ by $\hat{L}(gK): A[H/K]\otimes_{AH}C\to A[H/K]\otimes_{AH}C$ sending $hK\otimes x$ to $ghg^{-1}K\otimes L(g)(x)$. This yields a homomorphism $q(K): K_0^c(H-G,A)\to K_0^c(H/K\to G/K,A)$ and corresponds to dividing out a K-operation in geometry.

Proposition 5.6. *Let* $\mathrm{pr}_*: K_n(AG)\to K_n(A[G/K])$ *be induced from the projection. Then* $\mathrm{pr}_*\circ\otimes_t=\otimes_t\circ(q(K)\times \mathrm{ID})$.

Given a homomorphism $f: A\to B$ of commutative rings with unit, we get a change of ring homomorphisms for $K_n(AG)$ and $K_0^c(H-G,A)$, always denoted by f_*. If B is a flat A-module we get also f_* for $K_0(H-G,A)$. All the constructions above are compatible with change of rings provided that f_* is defined.

6. Transfer and representation theory

In this section we want to relate $K_0^c(H-G,A)$ and the algebraic transfer to the representation ring $\mathrm{Rep}_A(G)$ and its operation on the K-theory of AG. The representation ring $\mathrm{Rep}_A(G)$ is the Grothendieck group of AG-modules which are finitely generated and projective over A. We make the following assumption:

The trivial AH-module A possesses a finitely generated projective AH-resolution. (*)

Then we can define $i: \mathrm{Rep}_A(G) \to K_0(H-G, A)$ by $[M] \to [M]$. Let $k: \mathrm{Rep}_A(G) \to K_0^c(H-G, A)$ be the composition $j \circ i$. The tensor-product over A together with the diagonal action induces $\otimes_A: \mathrm{Rep}_A(G) \otimes K_n(AG) \to K_n(AG)$. Under the assumption (*) the AG-module $A[G/H]$ has a finitely generated projective AG-resolution. Let $\mathrm{trf}: K_n(A[G/H]) \to K_n(AG)$ be the transfer map defined by restriction with $AG \to A[G/H]$ in [22, p. 111]. Proposition 5.5 implies

Proposition 6.1. *Assume that* (*) *holds. Then*

(a) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Rep}_A(G) \otimes K_n(A[G/H]) & \xrightarrow{\mathrm{ID} \otimes \mathrm{trf}} & \mathrm{Rep}_A(G) \otimes K_n(AG) \\ \downarrow{\scriptstyle k \otimes \mathrm{ID}} & & \downarrow{\scriptstyle \otimes_A} \\ K_0^c(H-G, A) \otimes K_n(A[G/H]) & \xrightarrow{\otimes_t} & K_n(AG) \end{array}$$

(b) *If (C, L) is a AH-chain complex with a G-twist such that $H_n(C)$ is finitely generated and projective over A, we get $k(\sum (-1)^n [H_n(C)]) = [C, L]$.*

Let $\mathrm{Rep}'_{\mathbb{Z}}(G)$ be the Grothendieck group of $\mathbb{Z}G$-modules which are finitely generated as abelian groups. Define $e: \mathrm{Rep}_{\mathbb{Z}}(G) \to \mathrm{Rep}'_{\mathbb{Z}}(G)$ by $[M] \to [M]$. Then e is an isomorphism. An inverse e^{-1} is given by the following construction [21, p. 890]: Given a $\mathbb{Z}G$-module M which is finitely generated over $\mathbb{Z}$, choose an exact sequence of $\mathbb{Z}G$-modules $0 \to F_1 \to F_0 \to M \to 0$ such that F_0 and F_1 are finitely generated and free as abelian groups. Define $e^{-1}([M]) = [F_0] - [F_1]$. Let $i': Rep'_{\mathbb{Z}}(G) \to K_0(H-G, \mathbb{Z})$ be given by $[M] \to [M]$ and $k': \mathrm{Rep}'_{\mathbb{Z}}(G) \to K_0^c(H-G, \mathbb{Z})$ by $j \circ i'$ provided that (*) holds for $A = \mathbb{Z}$. Then we get

Proposition 6.2. (a) *The map $e: \mathrm{Rep}_{\mathbb{Z}}(G) \to \mathrm{Rep}'_{\mathbb{Z}}(G)$ is an isomorphism with $i' \circ e = i$ and $k' \circ e = k$.*

(b) *If (C, L) is a $\mathbb{Z}H$-chain complex with a G-twist such that $H*(C)$ is finitely generated over $\mathbb{Z}$ and* (*) *is valid, then $k'(\sum (-1)^n [H_n(C)]) = [C, L]$.*

We apply this to a fibration $F \to E \to B$ using the notation of Section 5. Let K be a normal subgroup of Δ and Γ. The Γ-twist L on $\bar{F}$ induces a $A[\Gamma/K]$-structure on $H_n(\bar{F}/K, A)$. Denote by $\mathrm{pr}: \Gamma \to \Gamma/K$ the projection.

Theorem 6.3. (a) *Assume that $H_n(\bar{F}/K, A)$ has a finitely generated projective $A[\Delta/K]$-resolution for all n. Then $q(K): K_0^c(\Delta - \Gamma, A) \to K_0^c(\Delta/K - \Gamma/K, A)$ sends $[p]$ to the image of $\sum (-1)^n [H_n(\bar{F}/K, A)]$ under $j: K_0(\Delta/K - \Gamma/K, A) \to$*

$K_0^c(\Delta/K-\Gamma/K, A)$. The composition $\mathrm{pr}_* \circ p^*: K_n(A\pi) \to K_n(A[\Gamma/K])$ is given by $\sum(-1)^n[H_n(\bar{F}/K, A)]$ in $K_0(\Delta/K-\Gamma/K, A)$ and the pairing $\otimes_A$.

(b) *Assume that $(*)$ holds for Δ/K and $A=\mathbb{Z}$ and that $H_n(\bar{F}/K)$ is finitely generated over $\mathbb{Z}$. Then $q(K): K_0^c(\Delta-\Gamma, \mathbb{Z}) \to K_0^c(\Delta/K-\Gamma/K, \mathbb{Z})$ sends $[p]$ to the image of $\sum(-1)^n[H_n(\bar{F}/K)]$ under $k': \mathrm{Rep}'_{\mathbb{Z}}(\Gamma/K) \to K_0^c(\Delta/K-\Gamma/K, \mathbb{Z})$. The composition $\mathrm{pr}_* \circ p^*: K_n(\mathbb{Z}\pi) \to K_n(\mathbb{Z}[\Gamma/K])$ is given by the image of $\sum(-1)^n[H_n(\bar{F}/K)]$ under $e^{-1}: \mathrm{Rep}'_{\mathbb{Z}}(\Gamma/K) \to \mathrm{Rep}_{\mathbb{Z}}(\Gamma/K)$ and the pairing $\otimes_{\mathbb{Z}}: \mathrm{Rep}_{\mathbb{Z}}(\Gamma/K) \otimes K_n(\mathbb{Z}[\Gamma/K]) \to K_n(\mathbb{Z}[\Gamma/K])$ and the transfer $\mathrm{trf}: K_n(\mathbb{Z}\pi) \to K_n(\mathbb{Z}[\Gamma/K])$.*

Theorem 6.3 was already proved in [14,15] using spectral sequences. We get a computation of $p_* \circ p^*$ from it.

Corollary 6.4. *The transport of the fibre $\omega: \pi \to [F, F]$ defines a $\mathbb{Z}\pi$-structure on $H_*(F)$. Then $p_* \circ p^*: K_n(A\pi) \to K_n(A\pi)$ is given by the image of $\sum(-1)^n[H_n(F)]$ under $\mathrm{Rep}'_{\mathbb{Z}}(\pi) \xrightarrow{e^{-1}} \mathrm{Rep}_{\mathbb{Z}}(\pi) \xrightarrow{f_*} \mathrm{Rep}_A(\pi)$ for $f: \mathbb{Z} \to A$ and the pairing $\otimes_A$.*

As an illustration we consider the case that the fibre is a finitely dominated Eilenberg-MacLane-space and $\pi_1(F) \to \pi_1(E)$ injective. The cellular chain complex of the universal covering is a finitely generated projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi_1(F) = \mathbb{Z}\Delta$. Hence $(*)$ holds and by Theorem 6.3 the transfer p^* is just the classical transfer $\mathrm{trf}: K_n(\mathbb{Z}\pi) \to K_n(\mathbb{Z}\Gamma)$.

For a finite group G the ring QG is semi-simple. Proposition 5.5 implies

Proposition 6.5. *Let G be a finite group with a normal subgroup H. Assume either $H=\{1\}$ and $A=\mathbb{Z}$ or $A=Q$. Then $k: \mathrm{Rep}_A(G) \to K_0^c(H-G, A)$ is an isomorphism and the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Rep}_A(G) \otimes K_n(A[G/H]) & \xrightarrow{\otimes_A} & K_n(AG) \\ \cong \big\downarrow k \otimes \mathrm{ID} & & \big\downarrow \mathrm{ID} \\ K_0^c(H-G, A) \otimes K_n(A[G/H]) & \xrightarrow{\otimes_t} & K_n(AG) \end{array}$$

Representations of finite groups with rational coefficients are uniquely determined by their characters. Let $F \to E \xrightarrow{p} B$ be a fibration with finitely dominated fibre and connected E and B. If Γ is finite, $\bar{F}$ is also a finitely dominated CW-complex. We have defined $[L(w): \bar{F} \to \bar{F}]$ for $w \in \Gamma$ at the beginning of Section 5. Denote by Λ the Lefschetz number of a self-map of a finite CW-complex.

Theorem 6.6. *Let Γ be finite. Then the isomorphism $e^{-1}: K_0^c(\Delta-\Gamma, Q) \to \mathrm{Rep}_Q(\Gamma)$ sends $[p]$ to the representation with character $w \to \Lambda(L(w))$ for $w \in \Gamma$.*

One easily checks that the following statements are equivalent for finite Γ:

(i) $[p] \in K_0^c(\Delta - \Gamma, Q)$ is zero;
(ii) $\Lambda(L(w)) = 0$ for all $w \in \Gamma$;
(iii) $p^*: K_0(Q\pi) \to K_0(Q\Gamma)$ is zero;
(iv) $p^*: K_1(Q\pi) \to K_1(Q\Gamma)$ is zero.

7. Orientable fibrations

Now we want to analyse the transfer of an orientable fibration. We will point out that this can be described easily if the fibre is a finite CW-complex with non-vanishing Euler characteristic, and what meaning the S^1-transfer has.

Let $F \to E \xrightarrow{p} B$ be a fibration of connected spaces F, E and B with F a finitely dominated CW-complex. We make use of the notation introduced in Section 5, e.g. $\Gamma = \pi_1(E)$ and $\pi = \pi_1(B)$. We always assume in this section that p is orientable (Definition 4.4). Let A be a commutative ring with unit. We get from Corollary 6.4

Theorem 7.1. *The composition $p_* \circ p^*: K_n(A\pi) \to K_n(A\pi)$ is multiplication with the Euler characteristic $\chi(F)$.*

In Section 5 we have defined for $w \in \Gamma$ a $c(w)$-equivariant homotopy class of $c(w)$-maps $L(w): \bar{F} \to \bar{F}$. Since $\omega: \pi \to [F, F]$ is trivial we can regard the transport of the pointed fibre as a homomorphism $\sigma: \Gamma \to \text{image}(\varrho: \pi_1(F) \to [F, F]^+)$ and have $\sigma(\Delta) = \text{image}(\varrho)$ because of Proposition 4.1. Hence $L(w): \bar{F} \to \bar{F}$ is the left multiplication with d for some $d \in \Delta$ with $\sigma(d) = \sigma(w)$ so that the Γ-twist on $C(\bar{F})$ is given by the Δ-operation.

Write Δ_0 for $\delta(G_1(F))$. Now $\varrho: \pi_1(F) \to [F, F]^+$ induces an isomorphism $\bar{\varrho}: \pi_1(F)/G_1(F) \to \text{image}(\sigma)$, $\sigma: \Gamma \to \text{image}(\sigma)$ an epimorphism $\bar{\sigma}: \Gamma/\Delta_0 \to \text{image}(\sigma)$ and $\delta: \pi_1(F) \to \Delta$ an isomorphism $\bar{\delta}: \pi_1(F)/G_1(F) \to \Delta/\Delta_0$. Then an isomorphism $\Phi: \Gamma/\Delta_0 \to \Delta/\Delta_0 \times \pi$ is given by $(\bar{\delta} \circ \bar{\varrho}^{-1} \circ \bar{\sigma}) \times p_*$. We get a Γ/Δ_0-twist $\hat{L}$ on the $A[\Delta/\Delta_0]$-chain complex $C(\bar{F}/\Delta_0, A)$ by assuming that $\hat{L}(x)$ is the left multiplication with $\bar{\delta} \circ \bar{\varrho}^{-1} \circ \bar{\sigma}(x)$ for $x \in \Gamma/\Delta_0$. One easily checks that $q(\Delta_0): K_0^c(\Delta - \Gamma, A) \to K_0^c(\Delta/\Delta_0 - \Gamma/\Delta_0, A)$ sends $[p]$ to $[C(\bar{F}/\Delta_0, A), \hat{L}]$. Let $\text{pr}: \Delta \to \Delta/\Delta_0$ be the projection. Then the homomorphism $T_{[C(\bar{F}/\Delta_0, A), \hat{L}]}: K_n(A\pi) \to K_n(A[\Gamma/\Delta_0])$ agrees with the one given by $\text{pr}_* \circ \delta_*(w(F)) \in K_0(A[\Delta/\Delta_0])$ and the pairing $K_0(A[\Delta/\Delta_0]) \otimes K_n(A\pi) \xrightarrow{\otimes_A} K_n(A[\Delta/\Delta_0 \times \pi]) \xrightarrow{\Phi_*^{-1}} K_n(A[\Gamma/\Delta_0])$. If $q: \Gamma \to \Gamma/\Delta_0$ is the projection, Proposition 5.6 implies

Theorem 7.2. *The composition $q_* \circ p^*: K_n(A\pi) \to K_n(A[\Gamma/\Delta_0])$ is the homomorphism defined by $\text{pr}_* \circ \delta_*(w(F)) \in K_0(A[\Delta/\Delta_0])$ and the pairing $K_0(A[\Delta/\Delta_0]) \otimes K_n(A\pi) \xrightarrow{\otimes_A} K_n(A[\Delta/\Delta_0 \times \pi]) \xrightarrow{\Phi_*^{-1}} K_n(A[\Gamma/\Delta_0])$.*

The following corollary of Theorem 7.2 and Proposition 4.2(b) was already proved in [7]:

Corollary 7.3. *Let $F \to E \xrightarrow{p} B$ be an orientable fibration of connected spaces. Assume that F is a finite CW-complex with Euler characteristic $\chi(F) \neq 0$. Then there is an isomorphism $\Phi : \Gamma \to \pi_1(F) \times \pi$ such that for the corresponding section $s : \pi \to \Gamma$ of p_* the transfer $p^* : K_n(A\pi) \to K_n(A\Gamma)$ is given by $\chi(F) \cdot s_*$.*

Now we take a look at the S^1-transfer.

We denote by $\mathbb{Z}_m$ the cyclic group of order m for $m = 1, 2, 3, \ldots$ and by $\mathbb{Z}_0$ the infinite cyclic group $\mathbb{Z}$. Let $t \in \mathbb{Z}_m$ be the generator represented by $1 \in \mathbb{Z}$. Given a group G with a subgroup $\mathbb{Z}_m$ of its center, let $S(\mathbb{Z}_m - G, A) \in K_0^c(\mathbb{Z}_m - G, A)$ be the class of the one-dimensional $A\mathbb{Z}_m$-chain complex $A\mathbb{Z}_m \xrightarrow{t-1} A\mathbb{Z}_m$ with the trivial G-twist L, i.e. $L(g) = [\mathrm{ID}]$ for all $g \in G$. We denote by $\beta(\mathbb{Z}_m - G, A) : K_n(A[G/\mathbb{Z}_m]) \to K_n(AG)$ the map $T_{S(\mathbb{Z}_m - G, A)}$. Sometimes we write S resp. β for $S(\mathbb{Z}_m - G, A)$ resp. $\beta(\mathbb{Z}_m - G, A)$. One should notice that for an orientable S^1-fibration $S^1 \to E \xrightarrow{p} B$ the class $S(\mathbb{Z}_m - G, A)$ is just $[p]$, if $t \in \mathbb{Z}_m = \Delta$ corresponds to the image of $[\mathrm{ID} : S^1 \to S^1] \in \pi_1(S^1)$ under δ.

The transfer of an S^1-fibration was described algebraically for K_0 in [17] and K_1 in [16] by writing down matrices representing elements in the algebraic K-groups. These homomorphisms agree with the maps β. A detailed study of β for finite Γ can be found in [19].

A lot of our results are consequences of the following lemma:

Lemma 7.4. *Let $\mathbb{Z}_m$ be central in G.*

(a) *If $\mathbb{Z}_m \cap [G, G]$ is trivial, $S(\mathbb{Z}_m - G, A)$ and $\beta(\mathbb{Z}_m - G, A)$ are zero.*

(b) *If m is not zero and invertible in A, $S(\mathbb{Z}_m - G, A)$ and $\beta(\mathbb{Z}_m - G, A)$ are zero.*

(c) *If $\mathbb{Z}_m$ is infinite, $\beta(\mathbb{Z}_m - G, A)$ is the transfer* $\mathrm{trf} : K_n(A[G/\mathbb{Z}_m]) \to K_n(AG)$ *defined by restriction in* [22, pp. 111].

Proof. (a) Because of $\mathbb{Z}_m \cap [G, G] = \{1\}$ the projection $\mathrm{pr} : G \to G/[G, G]$ is injective on $\mathbb{Z}_m$ so that we can also regard $\mathbb{Z}_m$ as a subgroup of $G/[G, G]$. Restriction with pr defines a homomorphism $\mathrm{pr}^* : K_0^c(\mathbb{Z}_m - G/[G, G], A) \to K_0^c(\mathbb{Z}_m - G, A)$. Construct an epimorphism of abelian groups $q : G' \to G/[G{:}G]$ with a subgroup $\mathbb{Z} \subset G$ such that the kernel K of q is contained in $\mathbb{Z}$ and q maps $1 \in \mathbb{Z}$ to $t \in \mathbb{Z}_m$. In Section 5 we have defined homomorphisms $q(K) : K_0^c(\mathbb{Z} - G', A) \to K_0^c(\mathbb{Z}_m - G/[G, G], A]$ and $j : K_0(\mathbb{Z} - G', A) \to K_0^c(\mathbb{Z} - G', A)$. Choose a homomorphism $f : G' \to \mathbb{Z}$ such that $f | \mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$ is an injection. If $K_0(\mathbb{Z} - \mathbb{Z}, A)$ corresponds to $\mathbb{Z}$ as a subgroup of itself, restriction with f defines a homomorphism $f^* : K_0(\mathbb{Z} - \mathbb{Z}, A) \to K_0(\mathbb{Z} - G', A)$. The composition $\mathrm{pr}^* \circ q(K) \circ j \circ f^* : K_0(\mathbb{Z} - \mathbb{Z}, A) \to K_0^c(\mathbb{Z}_m - G, A)$ sends the class $[A]$ of the trivial $A\mathbb{Z}$-module A to $S(\mathbb{Z}_m - G, A)$. Since $0 \to A\mathbb{Z} \xrightarrow{t-1} A\mathbb{Z} \to A \to 0$ is exact, $[A]$ vanishes in $K_0(\mathbb{Z} - \mathbb{Z}, A)$.

(b) If m is invertible in A, the trivial $A\mathbb{Z}_m$-module A is a direct summand in $A\mathbb{Z}_m$. Proposition 5.5 implies $S(\mathbb{Z}_m - G, A) = j([A] - [A]) = 0$.

(c) Proposition 5.5 implies for the trivial $A\mathbb{Z}_m$-module $A : S(\mathbb{Z}_m - G, A) = j([A])$. □

In the following theorem we decompose the transfer into the transfer of another orientable fibration and an S^1-fibration. Then the lemma above gives us vanishing theorems for the transfer.

Let H_0 and H_1 be normal subgroups of G such that $h_0 \cdot h_1 = h_1 \cdot h_0$ is valid for all $h_0 \in H_0$ and $h_1 \in H$. We get a homomorphism $\mathrm{pr}: H_0 \times H_1 \to G$ sending h_0, h_1 to $h_0 \cdot h_1$. Its image H is a normal subgroup of G. Let (C, L) resp. (D, M) be an AH_0-resp. AH_1-chain complex with a G-twist. We define an $A[H_0 \times H_1]$-structure on $C \otimes_A D$ in the obvious way. We can equip the AH-chain complex $AH \otimes_{\mathrm{pr}} (C \otimes_A D)$ with a G-twist N by $N(g): h \otimes x \otimes y \to ghg^{-1} \otimes L(g)(x) \otimes M(g)(y)$. This yields a pairing

$$P: K_0^c(H_0 - G, A) \otimes K_0^c(H_1 - G, A) \to K_0^c(H - G, A).$$

Now assume that $\pi_1(F)$ contains subgroups H and $\mathbb{Z}$ with $\pi_1(F) = H \times \mathbb{Z}$ and $\mathbb{Z} \subseteq G_1(F)$. If K is the kernel of $\delta | H: H \to \delta(H)$ and $\tilde{F}$ the universal covering of F, let F' be $\tilde{F}/K$ regarded as $\delta(H)$-space. Given $w \in \Gamma$, define $L'(w): C(F', A) \to C(F', A)$ as the left multiplication with $\delta | H \circ \mathrm{pr}_H(x)$ for any $x \in \pi_1(F)$ with $\sigma(w) = \varrho(x)$. This yields a Γ-twist L' on the $A[\delta(H)]$-chain complex $C(F', A)$. If F'' is the $\delta(H)/\delta(H) \cap \delta(\mathbb{Z})$-space $F'/\delta(H) \cap \delta(\mathbb{Z})$, define a $\Gamma/\sigma(\mathbb{Z})$-twist L'' on $C(F'', A)$ analogously.

The following theorem shows the importance of the S^1-transfer:

Theorem 7.5. *Let p be orientable and assume $\pi_1(F) = H \times \mathbb{Z}$ with $\mathbb{Z} \subseteq G_1(F)$.*

(a) *The pairing defined above*

$$P: K_0^c(\delta(H) - \Gamma, A) \otimes K_0^c(\delta(\mathbb{Z}) - \Gamma, A) \to K_0^c(\Delta - \Gamma, A)$$

sends $[C(F', A), L'] \otimes S(\delta(Z) - \Gamma, A)$ to $[p]$.

(b) *If $\delta(\mathbb{Z}) \cap [\Gamma, \Gamma]$ is trivial, $[p]$ and p^* vanish.*

(c) *The algebraic transfer p^* is the composition of $\beta: K_n(A[\Gamma/\delta(\mathbb{Z})]) \to K_n(A\Gamma)$ and $T_{[C(F'', A), L'']}: K_n(A\pi) \to K_n(A[\Gamma/\delta(\mathbb{Z})])$.*

Proof. (a) is a direct consequence of Proposition 4.1, Proposition 4.3 and Definition 5.2. Then (b) follows from Lemma 7.4(a). We will prove (c) only in the K_1-case since the K_0-case can be proved similarly using the instant Wall obstruction in [23] or can be derived from the K_1-case using Theorem 3.1(c).

The main problem lies in the fact that in the definition of the algebraic transfer $p^*: K_1(A\pi) \to K_1(A\Gamma)$ elements in $K_1(A\pi)$ are represented by automorphisms of modules and in $K_1(A\Gamma)$ by the torsion t of a self-chain equivalence defined in Section 1. This causes difficulties in writing down the composition of two algebraic transfer maps.

Let α be an automorphism of $\bigoplus_k A\pi$ representing $[\alpha] \in K_1(A\pi)$. Since $\delta(\mathbb{Z})$ is central in Γ by Proposition 4.2(a), the homomorphism $\mathrm{pr}: \delta(H) \times \delta(\mathbb{Z}) \to \Delta$ $x, y \to x \cdot y$ is well defined. Let S be the one-dimensional $A[\delta(\mathbb{Z})]$-chain complex $A[\delta(\mathbb{Z})] \xrightarrow{t-1} A[\delta(\mathbb{Z})]$ for t as $\delta(1)$ for the generator $1 \in \mathbb{Z}$. There exists a natural isomorphism of $A\Gamma$-chain complexes between $A\Gamma \otimes_{A\Delta} A\Delta \otimes_{A\mathrm{pr}} (C(F', A) \otimes_A S)$ and $(A\Gamma \otimes_{A[\delta(H)]} C(F', A)) \otimes_{A[\delta(\mathbb{Z})]} S$. By construction there is a self-chain map f of the $A\Gamma$-chain complex $\bigoplus_k A\Gamma \otimes_{A[\delta(H)]} C(F', A)$ such that $f \otimes_{A[\delta(\mathbb{Z})]} \mathrm{ID}_S$ and $A[\Gamma/\delta(\mathbb{Z})] \otimes_A f$ are self-equivalences and $T_{[p]} : K_1(A\pi) \to K_1(A\Gamma)$ sends $[\alpha]$ to $t(f \otimes_{A[\delta(\mathbb{Z})]} \mathrm{ID}_S)$ and $T_{[C(F''), L'']} : K_1(A\pi) \to K_1(A[\Gamma/\delta(\mathbb{Z})])$ maps $[\alpha]$ to $t(A[\Gamma/\delta(\mathbb{Z})] \otimes_A f)$. Hence it suffices to prove:

Let $\mathbb{Z}_m$ be a cyclic subgroup with generator t of the center of G. Let $f : D \to D$ be an AG-chain map such that $f \otimes_{A\mathbb{Z}_m} \mathrm{ID}_S$ is an AG-self-equivalence of $D \otimes_{A\mathbb{Z}_m} S$ and $A[G/\mathbb{Z}_m] \otimes_{AG} f$ an $A[G/\mathbb{Z}_m]$-chain equivalence of $A[G/\mathbb{Z}_m] \otimes_{AG} D$. Then $\beta : K_1(A[G/\mathbb{Z}_m]) \to K_1(AG)$ sends $t(A[G/\mathbb{Z}_m] \otimes_{AG} f)$ to $t(f \otimes_{A\mathbb{Z}_m} \mathrm{ID}_S)$. Let C be the mapping cone of f with differential c. Then $\mathrm{Cone}(f \otimes_{A\mathbb{Z}_m} \mathrm{ID}_S)$ is isomorphic to the mapping cone E of $C \xrightarrow{t-1} C$ given by

$$\cdots \to C_{*-1} \oplus C_* \xrightarrow[\begin{bmatrix} -c & 0 \\ t-1 & c \end{bmatrix}]{} C_{*-2} \oplus C_{*-1} \to \cdots .$$

Choose a chain contraction of E

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix} : C_{*-1} \oplus C_* \to C_* \oplus C_{*+1}.$$

We get the relations $c \circ u + u \circ c = v(t-1) - \mathrm{ID}$ and $c \circ v = v \circ c$ so that another chain contraction h is given by

$$\begin{bmatrix} u & v \\ 0 & -u \end{bmatrix}.$$

Then $t(f \otimes_{A\mathbb{Z}_m} \mathrm{ID}_S)$ is represented by the automorphism $(e+h) : E_{\mathrm{odd}} \to E_{\mathrm{even}}$ of $E_{\mathrm{odd}} = E_{\mathrm{even}} = \bigoplus_{i=0}^{\infty} C_i$ with e as the differential of E. This automorphism is conjugated to the automorphism

$$\begin{bmatrix} v & -c+u \\ c-u & t-1 \end{bmatrix}$$

of $C_{\mathrm{odd}} \oplus C_{\mathrm{even}}$. Because of $C_{\mathrm{odd}} = C_{\mathrm{even}} = \bigoplus_{i=0}^{\infty} D_i$ we can represent $t(f \otimes_{A\mathbb{Z}_m} \mathrm{ID}_S)$ also by

$$\begin{bmatrix} c-u & t-1 \\ -v & c-u \end{bmatrix} : C_{\mathrm{odd}} \oplus C_{\mathrm{even}} \to C_{\mathrm{even}} \oplus C_{\mathrm{odd}}.$$

Since $A[G/\mathbb{Z}_m] \otimes_{AG} (-u)$ is a chain contraction of

$$A[G/\mathbb{Z}_m] \otimes_{AG} C \cong \mathrm{Cone}(A[G/\mathbb{Z}_m] \otimes_{AG} f),$$

the torsion $t(A[G/\mathbb{Z}_m] \otimes_{AG} f)$ is given by

$$A[G/\mathbb{Z}_m] \otimes_{AG} (c-u) : A[G/\mathbb{Z}_m] \otimes_{AG} C_{\mathrm{odd}} \to A[G/\mathbb{Z}_m] \otimes_{AG} C_{\mathrm{even}}.$$

The map $(1+u^2) : C_{\mathrm{even}} \to C_{\mathrm{even}}$ is an automorphism representing zero in $K_1(AG)$.

The composition of

$$(-(1+u^2)^{-1}\circ v, (1+u^2)^{-1}\circ(c-u)) : C_{\text{odd}}\oplus C_{\text{even}}\to C_{\text{odd}}$$

and

$$\begin{bmatrix} -(t-1) \\ c-u \end{bmatrix} : C_{\text{odd}}\to C_{\text{odd}}\oplus C_{\text{even}}$$

is the identity on C_{odd}. Hence $\beta(t(A[G/\mathbb{Z}_m]\otimes_{AG} f))$ is representated by

$$\begin{bmatrix} c-u & t-1 \\ -(1+u^2)^{-1}\circ v & (1+u^2)^{-1}\circ(c-u) \end{bmatrix} : C_{\text{odd}}\oplus C_{\text{even}}\to C_{\text{even}}\oplus C_{\text{odd}}.$$

This implies $\beta(t(A[G/\mathbb{Z}_m]\otimes_{AG} f)) = t(f\otimes_{A\mathbb{Z}_m}\mathrm{ID}_S)$. □

Corollary 7.6. *Assume that $\pi_1(F)$ contains subgroups $\mathbb{Z}$ and H with $\pi_1(F)=\mathbb{Z}\times H$ and $\mathbb{Z}\subset\operatorname{kernel}(\pi_1(F)\to\Gamma)$. Then $[p]$ and p^* vanish for orientable p.*

8. Untwisted fibrations

In this section we regard a fibration $F\to E\xrightarrow{p} B$ of connected spaces with F a finitely dominated connected CW-complex. We always suppose p to be untwisted (Definition 4.4) and use the notation of Section 5, e.g. $\Gamma=\pi_1(E)$ and $\pi=\pi_1(B)$.

The following theorem is a consequence of Theorem 7.5. It shows that the algebraic transfer of an untwisted fibration can be written as a composition of S^1-transfers and the transfer of an untwisted fibration whose fibre has a finite fundamental group.

Theorem 8.1. *Let p be untwisted . Write $\pi_1(F)=H_1\times H_2\times\cdots\times H_r\times G$ for $H_i\cong\mathbb{Z}$. If K is the kernel of $\delta|G: G\to\delta(G)$ and $\tilde F$ the universal covering, let F' be $\tilde F/K$ regarded as $\delta(G)=G/K$-space. Denote by F'' the $\delta(G)/\delta(H_1\times\cdots\times H_r)\cap\delta(G)$-space $F'/\delta(H_1\times\cdots\times H_r)\cap\delta(G)$. Let 1 be the trivial twist. Then*

(a) *The pairing defined in Section 7*

$$P: K_0^c(\delta(H_1)-\Gamma,A)\otimes\cdots\otimes K_0^c(\delta(H_r)-\Gamma,A)\otimes K_0^c(\delta(G)-\Gamma,A)\to K_0^c(\Delta-\Gamma,A)$$

sends

$$S(\delta(H_1)-\Gamma,A)\otimes\cdots\otimes S(\delta(H_r)-\Gamma,A)\otimes[C(F',A),1]$$

to $[p]$.

(b) *If $\beta_i: K_n(A[\Gamma/\delta(H_1\times\cdots\times H_i)])\to K_n(A[\Gamma/\delta(H_1\times\cdots\times H_{i-1})])$ is the homomorphism $\beta(\delta(H_1\times\cdots\times H_i)/\delta(H_1\times\cdots\times H_{i-1})-\Gamma/\delta(H_1\times\cdots\times H_{i-1}),A)$ for $i=1,2,\ldots,r$ and $q^*: K_n(A\pi)\to K_n(A[\Gamma/\delta(H_1\times\cdots\times H_r)])$ is $T_{[C(F'',A),1]}$, then $p^*: K_n(A\pi)\to K_n(A\Gamma)$ is the composition $\beta_1\circ\beta_2\cdots\beta_r\circ q^*$.*

Now we make some computations of $p_*\circ p^*$ and $p^*\circ p_*$. If $\pi_1(F)$ is trivial, then

p^* is, because of Theorem 7.1, given by $\chi(F)\cdot p_*^{-1}: K_n(A\pi)\to K_n(A\Gamma)$, so it suffices to treat $\pi_1(F)\neq\{1\}$ only.

Theorem 8.2. *Assume $\pi_1(F)\neq\{1\}$. Then*

(a) $\chi(F)=0$;

(b) $p_*\circ p^*: K_n(A\pi)\to K_n(A\pi)$ *is zero*;

(c) *If F is a finite CW-complex, then* $p^*\circ p_*: K_n(A\Gamma)\to K_n(A\Gamma)$ *vanishes*;

(d) *If A is a field or A is finite or A is the ring of p-adic integers $\hat{\mathbb{Z}}_p$ for any prime,* $p^*\circ p_*$ *is zero*;

(e) *The composition* $p^*\circ p_*$ *is given by* $f_*\circ\delta_*(w(F))\in K_0(A\Delta)$ *for* f_* *as the change of rings with* $f:\mathbb{Z}\to A$ *and the pairing* $\otimes_{A\Delta}: K_0(A\Delta)\otimes K_n(A\Gamma)\to K_n(A\Gamma)$.

Proof. (a) Proposition 4.2(c).

(b) Theorem 7.1.

(c) Follows from (e).

(d) Because of (e), it suffices to check that $f_*\circ\delta_*(w(F))$ is zero. If $\pi_1(F)$ is infinite, Proposition 4.3 implies $w(F)=0$. Therefore we have only to prove that $f_*: K_0(\mathbb{Z}\Delta)\to K_0(A\Delta)$ is zero for finite Δ. As for any field A the map $\mathbb{Z}\to A$ factorizes over $\mathbb{Z}\to Q$ or $\mathbb{Z}\to\mathbb{Z}_p$, we can assume $A=Q$, $A=\hat{\mathbb{Z}}_p$ or A finite. Then $A\Delta$ is semi-local and $K_0(A\Delta)$ a free abelian group [24, p. 28]. But it follows from [25] that $K_0(\mathbb{Z}\Delta)$ is finite.

(e) One should notice that Δ is central in Γ so that $\otimes_{A\Delta}$ is well defined. In this proof we work with $[p]\in K_0^c(A\pi-A\Gamma)$ defined by chain representations in Definition 1.1. If $A[p_*]: A\Gamma\to A\pi$ is induced by $p_*:\Gamma\to\pi$, it suffices to compute the image of $[p]$ under $A[p_*]^*: K_0^c(A\pi-A\Gamma)\to K_0^c(A\Gamma-\Gamma)$ (see [12, p. 15]). Let $\bar{F}$ be the covering of F with Δ as group of deck transformations. Choose a finitely generated projective $\mathbb{Z}\Delta$-chain complex P equivalent to $C(\bar{F})$. Let L be the Γ-twist on $A\Gamma\otimes_{\mathbb{Z}\Delta}P$ with $L(w): A\Gamma\otimes_{A\Delta}P\to A\Gamma\otimes_{A\Delta}P \quad v\otimes x\to vw\otimes x$ for $w\in\Gamma$ and similarly L' for $A\Gamma\otimes_{\mathbb{Z}\Delta}C(\bar{F})$. We have defined $j: K_0(A\Gamma-A\Gamma)\to K_0^c(A\Gamma-A\Gamma)$ in Section 2. Using the additive relation in K_0^c one gets $A[p_*]^*([p])=[A\Gamma\otimes_{\mathbb{Z}\Delta}C(\bar{F}),L']=[A\Gamma\otimes_{A\Delta}P,L]=j(\sum(-1)^n[A\Gamma\otimes_{A\Delta}P_n])$. Since $\delta_*(w(F))\in K_0(\mathbb{Z}\Delta)$ is $\sum(-1)^n[P_n]$, an application of Theorem 2.1 finishes the proof. □

The next theorem contains some conditions implying that p^* is zero.

Theorem 8.3. *Let p be untwisted.*

(a) *If one of the following conditions is fulfilled,* $p^*: K_n(A\pi)\to K_n(A\Gamma)$ *vanishes for* $n=0,1$ *and any A.*

(i) *There is a direct summand* $\mathbb{Z}$ *in* $\pi_1(F)$ *with* $\delta(\mathbb{Z})\cap[\Gamma,\Gamma]=\{0\}$.

(ii) $\mathrm{rk}_{\mathbb{Z}}(\Delta\cap[\Gamma,\Gamma])<\mathrm{rk}_{\mathbb{Z}}(\Delta)$.

(iii) $\pi_1(F)=\mathbb{Z}^r\times G$ *for* $G\neq 1$ *and* $\delta(\mathbb{Z}^r)=\Delta$.

(iv) π *is finite and* Γ *infinite.*

(b) *If* π *is finite and F a finite CW-complex with* $\pi_1(F)\neq\{1\}$, *then* $p^*: K_0(\mathbb{Z}\pi)\to K_0(\mathbb{Z}\Gamma)$ *is zero.*

Proof. (a) (i) Theorem 8.1 and Lemma 7.4.

(ii) $\mathrm{rk}_{\mathbb{Z}}(\Delta \cap [\Gamma, \Gamma]) < \mathrm{rk}_{\mathbb{Z}}(\Delta)$ implies condition (i).

(iii) In the notion of Theorem 8.1(b) it suffices to show that $[C(F'', A), 1] \in K_0^c(\{1\} - \pi, A)$ vanishes. The composition $K_0(\{1\} - \pi, \mathbb{Z}) \xrightarrow{j} K_0^c(\{1\} - \pi, \mathbb{Z}) \xrightarrow{f_*} K_0^c(\{1\} - \pi, A)$ for $f : \mathbb{Z} \to A$ sends $\chi(F'') \cdot [\mathbb{Z}]$ to $[C(F'', A), 1]$. Because of Proposition 4.3 the CW-complex F'' is finitely dominated with $\pi_1(F'') = G_1(F'') = G \neq \{1\}$. Proposition 4.2 implies $\chi(F'') = 0$.

(iv) Since Δ is central and π finite, $[\Gamma, \Gamma]$ is finite. Hence (iv)$\Rightarrow$(ii).

(b) Because of Theorem 8.2(c) and (a) above it suffices to prove that $p_* : K_0(\mathbb{Z}\Gamma) \to K_0(\mathbb{Z}\pi)$ is surjective for finite Γ. This follows from the Mayer–Vietoris sequence [24, p. 162] of the Cartesian square

$$\begin{array}{ccc} \mathbb{Z}\Gamma & \longrightarrow & \mathbb{Z}\pi \\ \downarrow & & \downarrow \\ \mathbb{Z}\Gamma/(\Sigma_\Delta) & \longrightarrow & \mathbb{Z}_{|\Delta|}\pi \end{array}$$

and the facts that $K_0(\mathbb{Z}\pi)$ is finite [25] and $K_0(\mathbb{Z}_{|\Delta|}\pi)$ is a free abelian group, as the finite ring $\mathbb{Z}_{|\Delta|}\pi$ is semi-local [24, p. 28]. □

Now we examine the behaviour of $[p]$ for change of rings $f : \mathbb{Z} \to A$.

Theorem 8.4. *Denote by Δ^t the subgroup of torsion elements in Δ. If Δ^t is not $\{1\}$ and the order $|\Delta^t|$ is invertible in A or if Δ^t is $\{1\}$ and $\delta : \pi_1(F) \to \Delta$ no isomorphism, $p^* : K_n(A\pi) \to K_n(A\Gamma)$ vanishes.*

Proof. If Δ^t is $\{1\}$ and δ no isomorphism, $[p] = 0$ follows from Theorem 8.3(a)(iii). For $\Delta^t \neq \{1\}$ the map $\mathbb{Z} \to A$ factorizes through $\mathbb{Z}[1/\Delta^t]$. Hence we only have to prove $[p] = 0$ in $K_0^c(\Delta - \Gamma, A)$ for $\Delta^t \neq \{1\}$ and $A = \mathbb{Z}[1/\Delta^t]$. Then A is flat over $\mathbb{Z}$. In the notion of Theorem 8.1(a) for finite G it suffices to check that $S(\delta(H_i) - \Gamma, A)$ or $[C(F', A), 1]$ vanishes. If G is trivial, one of the $\delta(H_i)$ must be finite and $S(\delta(H_i) - \Gamma, A)$ is zero because of Lemma 7.4(b). Assume that G is a finite nontrivial group. Proposition 4.3 implies that $\tilde{F}/G = F'/\delta(G)$ is a finitely dominated CW-complex with $\pi_1(\tilde{F}/G) = G_1(\tilde{F}/G) = G \neq \{1\}$ so that $\chi(F'/\delta(G))$ and hence $\chi(F')$ is zero because of Proposition 4.2(c). As $|\delta(G)|$ is invertible in A the trivial $A[\delta(G)]$-module A is a direct summand in $A[\delta(G)]$. The homology $H_n(F')$ is finitely generated as abelian group and $\delta(G)$ acts trivially. The homomorphism $j : K_0(\delta(G) - \Gamma, A) \to K_0^c(\delta(G) - \Gamma, A)$ sends $\chi(F') \cdot [A]$ for the trivial $A\Gamma$-module A to $[C(F'), 1]$ so that $[C(F', A), 1]$ is zero. □

Finally we treat the K_1-transfer for finite Γ. In the K_0-case for finite Γ we already know $p^* = 0$ from Theorem 8.3(b).

Theorem 8.5. *Let p be untwisted with $\pi_1(F)\neq 1$ and Γ be finite. Then $p^*: K_1(A\pi)\to K_1(A\Gamma)$ is trivial if $|\Delta|$ is invertible in A, if A is finite or if A is $\hat{\mathbb{Z}}_p$ for any prime.*

Proof. If $|\Delta|$ is invertible in A, the result follows from Theorem 8.4. In the other cases $A\Gamma\to A\pi$ is a surjection of semi-local rings so that $p_*: K_1(A\Gamma)\to K_1(A\pi)$ is surjective [5, p. 87]. But Theorem 8.2(d) implies $p^*\circ p_* = 0$. □

The subgroup $\mathrm{CL}_1(\mathbb{Z}\Gamma)$ of $K_1(\mathbb{Z}\Gamma)$ is defined as the kernel of the map $K_1(\mathbb{Z}\Gamma)\to K_1(Q\Gamma)\oplus\bigoplus_p K_1(\hat{\mathbb{Z}}_p\Gamma)$.

Corollary 8.6. *Let p be untwisted with $\pi_1(F)\neq\{1\}$ and Γ finite. Then*
(a) $\mathrm{image}(p^*)\subset \mathrm{CL}_1(\mathbb{Z}\Gamma)$;
(b) *If F is finite,* $\mathrm{CL}_1(\mathbb{Z}\pi)\subset \mathrm{kernel}(p^*)$.

Proof. (a) follows from Theorem 8.5 and the fact that p^* is compatible with change of rings.

(b) The composition $p^*\circ p_*$ is zero by Theorem 8.2(c). But $p_*(\mathrm{CL}_1(\mathbb{Z}\Gamma)) = \mathrm{CL}_1(\mathbb{Z}\pi)$ is proved in [18, p. 184]. □

Theorem 8.7. *Let p be untwisted, $\pi_1(F)$ infinite and π finite.*
(a) *If $\pi_1(F)$ is not isomorphic to $\mathbb{Z}$, p^* vanishes.*
(b) *For $\pi_1(F)=\mathbb{Z}$ we get $p^*=\chi(\tilde{F})\cdot\beta(\Delta-\Gamma,\mathbb{Z})$ with $\chi(\tilde{F})$ the Euler characteristic of the universal covering of $\tilde{F}$.*

Proof. Because of Theorem 8.3 it suffices to treat the K_1-transfer for finite Γ. Decompose $\pi_1(F)$ in $\mathbb{Z}\times G$. Theorem 8.1 implies that we can write p^* as $\beta(\delta(\mathbb{Z})-\Gamma,\mathbb{Z})\circ T_{[C(F''),1]}$ for $[C(F''),1]\in K_0^c(\delta(G)/\delta(G)\cap\delta(\mathbb{Z})-\Gamma/\delta(\mathbb{Z}),\mathbb{Z})$. For $G\neq 1$ we get from Corollary 8.6 $\mathrm{CL}_1(\mathbb{Z}[\Gamma/\delta(\mathbb{Z})])\subset\mathrm{kernel}(\beta(\delta(\mathbb{Z})-\Gamma,\mathbb{Z}))$ and $\mathrm{image}(T_{[C(F''),1]})\subset\mathrm{CL}_1(\mathbb{Z}[\Gamma/\delta(\mathbb{Z})])$ so that p^* is zero. For $G=\{1\}$ we have $F=S^1\times\tilde{F}$ (Proposition 4.3) and hence $[p]=\chi(\tilde{F})\cdot S(\Delta-\Gamma,\mathbb{Z})$ in $K_0^c(\Delta-\Gamma,\mathbb{Z})$. □

With this theorem we have computed the algebraic transfer of an untwisted fibration for finite π and infinite $\pi_1(F)$ completely. The only non-trivial case is the K_1-transfer of an orientable S^1-fibration with Γ finite. This case is extensively studied in [19].

9. Orientable fibrations with a connected compact Lie group as fibre

We want to prove that the transfer p^* is zero for an orientable fibration with a connected compact Lie group G. See also [16, pp. 429,430].

Theorem 9.1. *Let $G\to E\xrightarrow{p} B$ be an orientable fibration.*

(a) *If G is not isomorphic to* $T^a \times SO(3)^b$, *then* $[p]$ *and* p^* *vanish.*

(b) *If* G *is isomorphic to* $T^a \times SO(3)^b$, *then* $p^*: K_n(A\pi) \to K_n(A\Gamma)$ *is* $2b \cdot \beta_1 \circ \beta_2 \circ \cdots \circ \beta_{a+b}$ *for appropriate* $\beta_i = \beta(\Delta_i - \Gamma_i, A)$ *and* $n = 0, 1$.

(c) *Let* π *be finite. If* G *is not* $\{1\}, S^1$ *or* SO(3), *the transfer* $p^*: K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\Gamma)$ *vanishes for* $n = 0, 1$. *For* $G = S^1$ *resp.* SO(3) *the map* $p^*: K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\Gamma)$ *is* $\beta(\mathbb{Z}_m - \Gamma, \mathbb{Z})$ *resp.* $2 \cdot \beta(\mathbb{Z}_m - \Gamma, \mathbb{Z})$. *The transfer* $p^*: K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}\Gamma)$ *is zero.*

Proof. Let $T^n \subset G$ be the maximal torus in G. Then $\pi_1(T^n) \to \pi_1(G)$ is surjective [6, p. 223]. Let $\overline{T^n}$ be the covering of T^n belonging to $\pi_1(T^n) \to \pi_1(G) \xrightarrow{\delta} \Delta$ and $\bar{G} \to G$ the covering corresponding to δ. Then $\bar{G}$ is a free $\bar{T}^n$-CW-complex and Δ operates on $\bar{G}$ by the inclusion $\Delta \subset \overline{T^n} \subset \bar{G}$. Using the additive relation in $K_0^c(\Delta - \Gamma, A)$ one shows $[p] = \chi(G/T^n) \cdot [C(\overline{T^n}, 1)]$ where 1 denotes the trivial Γ-twist. One should notice that p is untwisted. Writing $\pi_1(T^n) = \mathbb{Z}^n$ let $\mathbb{Z}_{m_i}$ be the image of the ith summand under $\delta: \pi_1(T^n) \to \Delta$. Then the pairing defined before Theorem 7.5 can be iterated yielding a map $P: K_0^c(\mathbb{Z}_{m_1} - \Gamma, A) \otimes \cdots \otimes K_0^c(\mathbb{Z}_{m_n} - \Gamma, A) \to K_0^c(\Delta - \Gamma, A)$ with $[p] = \chi(G/T^n) \cdot P(S(\mathbb{Z}_{m_1} - \Gamma, A) \otimes \cdots \otimes S(\mathbb{Z}_{m_n} - \Gamma, A))$.

(a) If G is not isomorphic to $T^a \times SO(3)^b$ there is a subgroup S^3 in G [20, p. 221]. Hence we can find a maximal torus $T^n = S^1 \times \cdots \times S^1$ such that the first factor is contained in S^3 and is therefore nullhomotopic in G. This implies $\mathbb{Z}_{m_1} = \{1\}$. Because of Lemma 7.4(a) the classes $S(\mathbb{Z}_{m_1} - \Gamma, A)$ and $[p]$ vanish.

(b) For $G = T^a \times SO(3)^b$ we have $\chi(G/T^n) = \chi((S^2)^b) = 2b$. Apply Theorem 7.5(c).

(c) For $a + b > 1$ we get $p^* = 0$. This follows from Theorem 8.3 (a) (iv) and Corollary 8.6 in the K_1-case. For K_0, apply Theorem 8.3(b). □

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