K- and L-theory of group rings

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Outline

- Basics about groups rings and K-theory
- The statement of the Farrell-Jones Conjecture
- Some prominent conjectures
- The status of the Farrell-Jones Conjecture
- Open problems
- Very brief remarks about proofs

Group rings

- Let R be a ring and let G be a group.

Definition (Group ring RG)

The *group ring RG* is the R-algebra whose underlying R-module is the free R-module generated by the set G and whose multiplication comes from the multiplication in G.

- Group rings are in general very complicated but very interesting rings.

- Representation theory
  Let M be an R-module. Suppose that G acts on M by R-linear maps. Then these data determine the structure of an RG-module on M. The converse is also true.
- Topology
  The cellular chain complex $C_*(\widetilde{X})$ of the universal covering $\widetilde{X}$ of a space X is a $\mathbb{Z}[\pi_1(X)]$-chain complex. This allows to enrich classical invariants over $\mathbb{Z}$ to more complicated invariants over $\mathbb{Z}[\pi_1(X)]$.

Definition (Projective class group $K_0(R)$)

Define the projective class group of a ring R

$$K_0(R)$$

to be the abelian group defined by:

Generators: Isomorphism classes $[P]$ of finitely generated projective R-modules P.

Relations: We get $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules.
- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R) := GL(R)/[GL(R), GL(R)].$$

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\}.$$

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

- Bass and Quillen have defined $K_n(R)$ for all $n \in \mathbb{Z}$.
- K-theory has some basic feature such as compatibility with products, Morita equivalence, Bass-Heller-Swan decomposition and localization sequences.
- L-groups $L_n(R)$ are defined in a similar way but now for quadratic forms.
- In contrast to K-theory the L-groups are four-periodic, i.e., $L_n(R) = L_{n+4}(R)$
- In general algebraic K-and L-theory are very hard to compute but of high significance.

The statement of the Farrell-Jones Conjecture

- Let $\mathcal{H}_*$ be a (generalized) homology theory. It satisfies:
- Suspension

$$\mathcal{H}_n(B\mathbb{Z}) = \mathcal{H}_n(S^1) \cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}) = \mathcal{H}_n(B\{1\}) \oplus \mathcal{H}_{n-1}(B\{1\}).$$

- Mayer-Vietoris-sequence
  If $G = G_1 *_{G_0} G_2$, then we get a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{H}_n(BG_0) \to \mathcal{H}_n(BG_1) \oplus \mathcal{H}_n(BG_2) \to \mathcal{H}_n(BG) \\ \to \mathcal{H}_{n-1}(BG_0) \to \mathcal{H}_{n-1}(BG_1) \oplus \mathcal{H}_{n-1}(BG_2) \to \cdots \end{aligned}$$

- Let R be a regular ring. Then:
- Bass-Heller-Swan have shown:

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) = K_n(R[\{1\}]) \oplus K_{n-1}(R[\{1\}]).$$

- If $G = G_1 *_{G_0} G_2$ and G_0, G_1 and G_2 are torsionfree and belong to a certain class CL, then Waldhausen has established the exact sequence

$$\begin{aligned} \cdots \to K_n(R[G_0]) \to K_n(R[G_1]) \oplus K_n(R[G_2]) \to K_n(R[G]) \\ \to K_{n-1}(R[G_0]) \to K_{n-1}(R[G_1]) \oplus K_{n-1}(R[G_2]) \to \cdots \end{aligned}$$

- This raises the question: Is there a generalized homology theory $\mathcal{H}_*$ satisfying

$$\mathcal{H}_n(BG) \cong K_n(RG)$$

for all torsionfree groups G and $n \in \mathbb{Z}$, where R is a fixed regular ring?

- If yes, we must have for all $n \in \mathbb{Z}$

$$\mathcal{H}_n(\mathrm{pt}) = K_n(R).$$

- Hence our candidate for $\mathcal{H}_*$ is $H_*(-; \mathbf{K}_R)$, the generalized homology theory associated to the (non-connective) K-theory spectrum $\mathbf{K}_R$ of the ring R.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The version of the Farrell-Jones Conjecture above is not true for finite groups unless the group is trivial.
- For instance we get for a finite group G and $R = \mathbb{C}$:

$$\begin{aligned} K_0(\mathbb{C}G) &= R_{\mathbb{C}}(G); \\ H_0(BG; \mathbf{K}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q} &= \mathbb{Q}. \end{aligned}$$

- Also the condition regular is needed in general.
- Namely, we have

$$\begin{aligned} K_n(R[\mathbb{Z}]) &= K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R); \\ H_n(B\mathbb{Z}; \mathbf{K}_R) &= K_n(R) \oplus K_{n-1}(R). \end{aligned}$$

- There is a more complicated version of the Farrell-Jones Conjecture which may be true for all groups G and rings R and makes also sense for twisted group rings.
- In the sequel we will refer to this general version.
- All of the statements above have analogues for L-theory.

Some prominent conjectures

- Construction of idempotents in RG

  Suppose that $g \in G$ has finite order $|g|$. Put $N = \sum_{i=1}^{|g|} g^i$. Then

  $$N \cdot N = |g| \cdot N.$$

  If $|g|$ is invertible in R and different from 1, then RG contains a non-trivial idempotent, namely $\frac{N}{|g|}$.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and a field F that 0 *and* 1 *are the only idempotents in FG.*

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

- If $f\colon M \to N$ is a homotopy equivalence of closed aspherical manifolds, then the Novikov Conjecture predicts

$$f_*\mathcal{L}(M) = \mathcal{L}(N).$$

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Wall and Farrell-Jones.

Conjecture (Gromov)

If G is a torsionfree hyperbolic group whose boundary is a standard sphere, then there is a closed aspherical manifold M with $G = \pi_1(M)$.

- There are further interesting prominent conjectures by Bass and by Moody and conjectures about L^2-invariants and about Poincaré duality groups, which we do not state.
- One of the basic features of the Farrell-Jones Conjecture is that it implies all the conjectures mentioned above, where in some cases one has to assume $\dim \geq 5$.

Status of the Farrell-Jones Conjectures

Theorem (Main Theorem (Bartels-Echterhoff-Farrell-Lück-Reich (2008-2011)))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds. It has the following properties:

- *Hyperbolic groups belong to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*

Theorem (continued)

- CAT(0)-*groups belong to* $\mathcal{FJ}$ *;*
- *Virtually poly-cyclic groups belong to* $\mathcal{FJ}$ *;*
- *Cocompact lattices in almost connected Lie groups belong to* $\mathcal{FJ}$ *;*
- *All* 3*-manifold groups belong to* $\mathcal{FJ}$ *.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture and hence also the other conjectures mentioned above.

- Davis- Januszkiewics have constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

Open problems

- What are candidates for groups or closed aspherical manifolds for which the conjectures due to Farrell-Jones, Novikov or Borel may be false?
- There are still many interesting groups for which the Farrell-Jones Conjecture is open.
- Examples are:
  - Solvable groups;
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.

- There is an analogue of the Farrell-Jones Conjecture for the topological K-theory of group C^*-algebras, the Baum-Connes Conjecture. Can methods of proof be transferred from one setting to the other?

About the proof

- One needs to interpret the assembly map which is easiest described in terms of homotopy theory as a forget control homomorphism.
- Then the task is to show how to get control.
- This is achieved for hyperbolic and CAT(0)-groups by constructing flow spaces which mimic the geodesic flow on a Riemannian manifold with negative or non-positive sectional curvature.
- The proof of the inheritance results is of homotopy theoretic nature.
- Poly-cyclic groups are handled by transfer methods.