ON THE FARRELL-JONES AND RELATED CONJECTURES

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Abstract. These extended notes are based on a series of six lectures presented at the summer school "Cohomology of groups and algebraic K-theory" which took place in Hangzhou, China from July 1 until July 12 in 2007. They give an introduction to the *Farrell-Jones* and the *Baum-Connes Conjecture*.

Key words: K- and L-groups of group rings and group C^*-algebras, Farrell-Jones Conjecture, Baum-Connes Conjecture,

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0. Introduction

These extended notes are based on a series of six lectures presented at the summer school "Cohomology of groups and algebraic K-theory" which took place in Hangzhou, China from July 1 until July 12 in 2007. They contain an introduction to the *Farrell-Jones* and the *Baum-Connes Conjecture*.

Given a group G, the Farrell-Jones Conjecture and the Baum-Connes Conjecture respectively predict the values of the algebraic K- and L-theory of the group ring RG and of the topological K-theory of the reduced group C^*-algebra respectively. These are very hard to compute directly. These conjectures identify them via assembly maps to much easier to handle equivariant homology groups of certain classifying spaces. This is the computational aspect of these conjectures.

But also the structural aspect is very important. The assembly maps have geometric or analytic interpretations. Hence the Farrell-Jones Conjecture and the Baum-Connes Conjecture imply many very well-known conjectures such as the ones due to Bass, Borel, Kaplansky, and Novikov. The point is that the Farrell-Jones Conjecture and the Baum-Connes Conjecture have been proven for many groups for which the other conjectures were a priori not known.

The prerequisites consist of a solid knowledge of homology theory and CW-complexes and of a basic knowledge of rings, modules, homological algebra, groups, group homology, finite dimensional representation theory of finite groups, group actions, categories, homotopy groups, and manifolds. The challenge for the reader but also the beauty, impact and fascination of these conjectures come from the broad scope of mathematics which they address and which is needed for proofs and applications.

For a more advanced survey on the Farrell-Jones and the Baum-Cones Conjecture we refer to Lück-Reich [126]. There more details are given and more aspects are discussed but it is addressed to a more advanced reader and requires much more previous knowledge.

We fix some notation. Ring will always mean associative ring with unit (which is not necessarily commutative). Examples are the ring of integers $\mathbb{Z}$, the fields of rational numbers $\mathbb{Q}$, of real numbers $\mathbb{R}$ and of complex numbers $\mathbb{C}$, the finite field $\mathbb{F}_p$ of p elements, and the group ring RG for a ring R and a group G. Ring homomorphisms are unital. Modules are understood to be left modules unless

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explicitly stated differently. Groups are understood to be discrete unless explicitly stated differently.

The notes are organized as the six lectures in Hangzhou have been:

Contents

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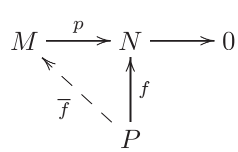
1. The role of lower and middle K-theory in topology

The outline of this section is:

- Introduce the projective class group $K_0(R)$.
- Discuss its algebraic and topological significance (e.g., finiteness obstruction).
- Introduce $K_1(R)$ and the Whitehead group $\mathrm{Wh}(G)$.
- Discuss its algebraic and topological significance (e.g., s-cobordism theorem).
- Introduce negative K-theory and the Bass-Heller-Swan decomposition.

Definition 1.1 (Projective R-module)**.** An R-module P is called *projective* if it satisfies one of the following equivalent conditions:

(1) P is a direct summand in a free R-module;
(2) The following lifting problem has always a solution



(3) If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

Example 1.2 (Principal ideal domains)**.** Over a field or, more generally, over a principal ideal domain every projective module is free. If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree. For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.

Example 1.3 (Product of rings)**.** Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Example 1.4 (Trivial representation of a finite group)**.** Let F be a field of characteristic p for p a prime number or $p = 0$. Let G be a finite group. Then F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G. It is a free FG-module only if G is trivial.

Definition 1.5 (Projective class group)**.** Let R be a ring. Define its *projective class group* $K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

The projective class group $K_0(R)$ is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum. There do exists rings R with $K_0(R) = 0$, e.g., $R = \operatorname{end}(F)$ for a field F.

Definition 1.6 (Reduced projective class group)**.** The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

Remark 1.7 (Stably finitely generated free modules)**.** Let P be a finitely generated projective R-module. It is stably finitely generated free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$. Hence $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.

There exists finitely generated projective R-modules which are stably finitely generated free but not finitely generated free. An example is $R = C(S^2)$ and $P = C(TS^2)$, where $C(S^2)$ is the ring of continuous functions $S^2 \to \mathbb{R}$ and $C(TS^2)$ is the $C(S^2)$-module of sections of the tangent bundle of S^2. However, in most of the applications the relevant question is whether a finitely generated projective R-module is stable finitely generated free and not whether it is finitely generated free.

Remark 1.8 (Universal dimension function)**.** The assignment $P \mapsto [P] \in K_0(R)$ is the *universal additive invariant* or *dimension function* for finitely generated projective R-modules in the following sense. Given an abelian group A and an assignment associating to a finitely generated projective R-module P an element $a(P) \in A$ such that $a(P_0) - a(P_1) + a(P_2) = 0$ holds for any exact sequence of finitely generated projective R-modules $0 \to P_0 \to P_1 \to P_2 \to 0$, there exists precisely one homomorphism of abelian groups $\phi\colon K_0(R) \to A$ satisfying $\phi([P]) = a(P)$ for every finitely generated projective R-module P.

Remark 1.9 (Induction)**.** Let $f\colon R \to S$ be a ring homomorphism. Consider S as a S-R-bimodule via f. Given an R-module M, let f_*M be the S-module $S \otimes_R M$. We obtain a homomorphism of abelian groups

$$f_*\colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P]$$

called *induction* or *change of rings homomorphism*. Thus K_0 becomes a covariant functor from the category of rings to the category of abelian algebras.

Lemma 1.10 (K_0 and products)**.** *Let R and S be rings. Then the two projections from $R \times S$ to R and S induce isomorphisms*

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

Theorem 1.11 (Morita equivalence)**.** *Let R be a ring and $M_n(R)$ be the ring of (n,n)-matrices over R. We can consider R^n as a $M_n(R)$-R-bimodule and as a R-$M_n(R)$-bimodule by scalar and matrix multiplication. Tensoring with these yields mutually inverse isomorphisms*

$$\begin{array}{rclcrcl} K_0(R) & \xrightarrow{\cong} & K_0(M_n(R)), & & [P] & \mapsto & [{}_{M_n(R)}R^n{}_R \otimes_R P]; \\ K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R), & & [Q] & \mapsto & [{}_R R^n{}_{M_n(R)} \otimes_{M_n(R)} Q]. \end{array}$$

Example 1.12 (Principal ideal domains)**.** Let R be a principal ideal domain. Let F be its quotient field. Then we obtain mutually inverse isomorphisms

$$\begin{array}{rclcrcl} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & & n & \mapsto & n \cdot [R]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example 1.13 (Representation ring)**.** Let G be a finite group and let F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$. Taking the character of a representation yields an isomorphism

$$R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} = K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \operatorname{class}(G, \mathbb{C}),$$

where $\operatorname{class}(G; \mathbb{C})$ is the complex vector space of class functions $G \to \mathbb{C}$, i.e., functions, which are constant on conjugacy classes. We refer for instance to the book of Serre [167] for more information about the representation theory of finite groups.

Definition 1.14 (Dedekind domain)**.** A commutative ring R is called *Dedekind domain* if it is an integral domain, i.e., contains no non-trivial zero-divisors, and for every pair of ideals $I \subseteq J$ of R there exists an ideal $K \subseteq R$ with $I = JK$.

A ring is called *hereditary* if every ideal is projective, or, equivalently, if every submodule of a projective R-module is projective.

Theorem 1.15 (Characterization of Dedekind domains)**.** *The following assertions are equivalent for a commutative integral domain with quotient field F:*

(1) R is a Dedekind domain;
(2) R is hereditary;
(3) Every finitely generated torsionfree R-module is projective;
(4) R is Noetherian and integrally closed in its quotient field F and every non-zero prime ideal is maximal.

Proof. This follows from [49, Proposition 4.3 on page 76 and Proposition 4.6 on page 77] and the fact that a finitely generated torsionfree module over an integral domain R can be embedded into R^n for some integer $n \geq 0$ (see Auslander-Buchsbaum [8, Proposition 3.3 in Chapter 9 on page 321]). □

Example 1.16 (Ring of integers)**.** Recall that an *algebraic number field* is a finite algebraic extension of $\mathbb{Q}$ and the *ring of integers* in F is the integral closure of $\mathbb{Z}$ in F. The ring of integers in an algebraic number field is a Dedekind domain. (see [154, Theorem 1.4.18 on page 22]).

Example 1.17 (Dedekind domains)**.** Let R be a Dedekind domain. We call two ideals I and J in R equivalent if there exists non-zero elements r and s in R with $rI = sJ$. The *ideal class group $C(R)$* is the abelian group of equivalence classes of ideals under multiplication of ideals. Then $C(R)$ is finite and we obtain an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

A proof of the claim above can be found for instance in [132, Corollary 11 on page 14] and [154, Theorem 1.4.12 on page 20 and and Theorem 1.4.19 on page 23].

The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p (see [132, Remark 3.4 on page 30]).

Theorem 1.18 (Swan (1960))**.** *If G is finite, then $\widetilde{K}_0(\mathbb{Z}G)$ is finite.*

Proof. See [175, Theorem 8.1 and Proposition 9.1]. □

Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X. This is the zero-th term of a generalized cohomology theory $K^*(X)$ called topological K-theory. It is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\{\bullet\}) = \mathbb{Z}$ and $K^1(\{\bullet\}) = \{0\}$, where $\{\bullet\}$ is the space consisting of one point.

Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Theorem 1.19 (Swan (1962)). *If X is a compact space, then there is an isomorphism*

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Proof. See [176]. □

Definition 1.20 (Finitely dominated). A CW-complex X is called *finitely dominated* if there exists a finite CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

Obviously a finite CW-complex is finitely dominated.

Problem 1.21. Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

A finitely dominated CW-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its *finiteness obstruction* as follows. Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$. Let $C_*(\widetilde{X})$ be the cellular chain complex. It is a free $\mathbb{Z}\pi$-chain complex. Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$. Finite projective means that every P_i is finitely generated projective and $P_i \neq 0$ holds only for finitely many element $i \in \mathbb{Z}$.

Definition 1.22 (Wall's finiteness obstruction). Define

$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

This definition is indeed independent of the choice of P_*.

Theorem 1.23 (Wall (1965)). *A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.*

Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Proof. See [187] and [188]. □

A finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex since $\widetilde{K}_0(\mathbb{Z}) = \{0\}$.

Corollary 1.24 (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$). *The following statements are equivalent for a finitely presented group G:*

(1) *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex;*
(2) $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$.

Conjecture 1.25 (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G). *If G is torsionfree, then*

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

For more information about the finiteness obstruction we refer for instance to [75], [76], [116], [134], [150], [156], [182], [187] and [188].

Definition 1.26 (K_1-group)**.** Define the $K_1(R)$ to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

(1) Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
(2) $[g \circ f] = [f] + [g]$.

Theorem 1.27 ($K_1(R)$ and matrices)**.** *There is a natural isomorphism*

$$K_1(R) \cong GL(R)/[GL(R), GL(R)],$$

where the target is the abelianization of the general linear group $GL(R) = \bigcup_{n \geq 1} GL_n(R)$.

Proof. See [154, Theorem 3.1.7 on page 113]. □

Remark 1.28 ($K_1(R)$ and row and column operations)**.** An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

Remark 1.29 ($K_1(R)$ and determinants)**.** If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

which in general is not bijective.

The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the universal determinant for R, where R is not necessarily commutative. Namely, given any abelian group A together with an assignment which associates to an R-automorphism $f\colon P \to P$ of a finitely generated projective R-module an element $[f]$ such that the obvious analogues of the relations appearing in Definition 1.26 hold, there exists precisely one homomorphism of abelian groups $\phi\colon K_1(R) \to A$ sending $[f]$ to $a(f)$ for every R-automorphism f of a finitely generated projective R-module.

There do exists rings R with $K_1(R) = 0$, e.g. $R = \operatorname{end}(F)$ for a field F.

Remark 1.30 ($K_1(R)$ of principal ideal domains)**.** There exists principal ideal domains R such that $\det\colon K_1(R) \to R^{\times}$ is not bijective. For instance Grayson [81] gives such an example, namely, take $\mathbb{Z}[x]$ and invert x and all polynomials of the shape $x^m - 1$ for $m \geq 1$. Other examples can be found in Ischebeck [94].

Theorem 1.31 (K_1 of ring of integers, Bass-Milnor-Serre (1967))**.** *Let R be the ring of integers in an algebraic number field. Then the determinant induces an isomorphism*

$$\det\colon K_1(R) \xrightarrow{\cong} R^{\times}.$$

Proof. See [23, 4.3]. □

Definition 1.32 (Whitehead group)**.** The *Whitehead group* of a group G is defined to be

$$\operatorname{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma 1.33. *We have* $\operatorname{Wh}(\{1\}) = \{0\}$.

Proof. The ring $\mathbb{Z}$ possesses an Euclidean algorithm. Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1). For every ring such operations do not change the class of a matrix in $K_1(R)$. □

Let G be a finite group. Let F be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Define $r_F(G)$ to be the number of irreducible F-representations of G. This is the same as the number of F-conjugacy classes of elements of G. Here $g_1 \sim_{\mathbb{C}} g_2$ if and only if $g_1 \sim g_2$, i.e., $gg_1g^{-1} = g_2$ for some $g \in G$. We have $g_1 \sim_{\mathbb{R}} g_2$ if and only if $g_1 \sim g_2$ or $g_1 \sim g_2^{-1}$ holds. We have $g_1 \sim_{\mathbb{Q}} g_2$ if and only if $\langle g_1 \rangle$ and $\langle g_1 \rangle$ are conjugated as subgroups of G.

Theorem 1.34 ($\mathrm{Wh}(G)$ for finite groups G)**.**

1. *The Whitehead group* $\mathrm{Wh}(G)$ *is a finitely generated abelian group;*
2. *Its rank is* $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
3. *The torsion subgroup of* $\mathrm{Wh}(G)$ *is the kernel of the map* $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.

In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable (see Oliver [138]).

Definition 1.35 (h-cobordism)**.** An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem 1.36 (s-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)**.**

1. *Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.*

   Then W is homeomorphic (diffeomorpic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion
$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0)).$$
   vanishes;
2. *Let G be a finitely presented group G, n an integer $n \geq 5$ and x an element in* $\mathrm{Wh}(G)$. *Then there exists an n-dimensional h-cobordism $(W; M_0, M_1)$ over M_0 with $\tau(W, M_0) = x$.*

Corollary 1.37 (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)**.** *The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$*

1. *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
2. $\mathrm{Wh}(G) = \{0\}$.

Conjecture 1.38 (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)**.** *If G is torsionfree, then*
$$\mathrm{Wh}(G) = \{0\}.$$

Conjecture 1.39 (Poincaré Conjecture)**.** *Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.*

Then M is homeomorphic to S^n.

Theorem 1.40. *For $n \geq 5$ the Poincaré Conjecture is true.*

Proof. We sketch the proof for $n \geq 6$. Let M be an n-dimensional homotopy sphere. Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism. Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = \partial D_1^n \times \{0\}$. By the Alexander trick, i.e., by coning the homeomorphism of ∂D^n to the cone of ∂D^n which is D^n, we can extend the homeomorphism $f|_{\partial D_2^n}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n = \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_2^n \to D_1^n$. The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n. □

Remark 1.41 (Exotic spheres)**.** The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$. The problem is that the Alexander trick does not work smoothly. Indeed, there exists so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n. For more information about exotic spheres we refer for instance to [101], [110], [113] and [119, Chapter 6].

Remark 1.42 (The Poincaré Conjecture and the s-cobordism theorem in low dimensions)**.** The Poincaré Conjecture has been proved in dimension 4 by Freedman [79] and in dimension 3 by Perelman (see [144], [145] and for more details for instance [103], [137]). It is true in dimensions 1 and 2 for elementary reasons.

The s-cobordism theorem is known to be false (smoothly) for $n = \dim(M_0) = 4$ in general, by the work of Donaldson [55], but it is true for $n = \dim(M_0) = 4$ for so called "good" fundamental groups in the topological category by results of Freedman [79], [80]. The trivial group is an example of a "good" fundamental group. Counterexamples in the case $n = \dim(M_0) = 3$ are constructed by Cappell-Shaneson [41].

Remark 1.43 (Surgery program)**.** The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall. For more information about surgery theory we refer for instance to [33], [38], [39], [73], [74], [98], [104], [148], [173], [172], and [189].

More information about Whitehead torsion and the s-cobordism theorem can be found for instance in [47], [100], [119, Chapter 1], [130], [131], [157, page 87-90].

Definition 1.44 (Bass-Nil-groups)**.** Define for $n = 0, 1$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem 1.45 (Bass-Heller-Swan decomposition for K_1, Bass-Heller-Swan(1964))**.** *There is an isomorphism, natural in R,*

$$K_0(R) \oplus K_1(R) \oplus \mathrm{NK}_1(R) \oplus \mathrm{NK}_1(R) \xrightarrow{\cong} K_1(R[t,t^{-1}]) = K_1(R[\mathbb{Z}]).$$

Proof. See for instance [22] (for regular rings), [19, Chapter XII], [154, Theorem 3.2.22 on page 149]. □

Notice that the Bass-Heller-Swan decomposition for K_1 gives the possibility to define $K_0(R)$ in terms of K_1. This motivates the following definition.

Definition 1.46 (Negative K-theory)**.** Define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t,t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem 1.47 (Bass-Heller-Swan decomposition for negative K-theory)**.** *For $n \leq 1$ there is an isomorphism, natural in R,*

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition 1.48 (Regular ring)**.** A ring R is called *regular* if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

Principal ideal domains are regular. In particular $\mathbb{Z}$ and any field are regular. If R is regular, then $R[t]$ and $R[t,t^{-1}] = R[\mathbb{Z}]$ are regular. If R is Noetherian, then RG is not in general Noetherian. Theorem 1.47 implies

Theorem 1.49 (Bass-Heller-Swan decomposition for regular rings)**.** *Suppose that R is regular. Then*

$$\begin{aligned} K_n(R) &= 0 \quad \textit{for } n \leq -1; \\ \mathrm{NK}_n(R) &= 0 \quad \textit{for } n \leq 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition reduces for $n \leq 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]) = K_n(R[\mathbb{Z}]).$$

There are also higher algebraic K-groups $K_n(R)$ for $n \geq 2$ due to Quillen (1973). They are defined as homotopy groups of certain spaces or spectra. We refer to the lectures of Grayson. Most of the well known features of $K_0(R)$ and $K_1(R)$ extend to both negative and higher algebraic K-theory. For instance the Bass-Heller-Swan decomposition holds also for higher algebraic K-theory.

Remark 1.50 (Similarity between K-theory and group homology)**.** Notice the following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$, which look similar:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\{\bullet\}) \oplus \mathcal{H}_{n-1}(\{\bullet\}). \end{aligned}$$

If G and K are groups, then we have the following formulas, which look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G*K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G*K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

Question 1.51 (K-theory of group rings and group homology)**.** Is there a relation between $K_n(RG)$ and group homology of G?

2. The Isomorphism Conjectures in the torsionfree case

The outline of this section is:

- We introduce spectra and how they yield homology theories.
- We state the Farrell-Jones-Conjecture and the Baum-Connes Conjecture for torsionfree groups.
- We discuss applications of these conjectures such as the Kaplansky Conjecture and the Borel Conjecture.
- We explain that the formulations for torsionfree groups cannot extend to arbitrary groups.

Given two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$, their *one-point-union* and their *smash product* are defined to be the pointed spaces

$$\begin{aligned} X \vee Y &:= \{(x, y_0) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\} \subseteq X \times Y; \\ X \wedge Y &:= (X \times Y)/(X \vee Y). \end{aligned}$$

We have $S^{n+1} \cong S^n \wedge S^1$.

Definition 2.1 (Spectrum)**.** A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.

Example 2.2 (Sphere spectrum)**.** The *sphere spectrum* $\mathbf{S}$ has as n-th space S^n and as n-th structure map the homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.

Example 2.3 (Suspension spectrum)**.** Let X be a pointed space. Its *suspension spectrum* $\Sigma^\infty X$ is given by the sequence of spaces $\{X \wedge S^n \mid n \geq 0\}$ with the homeomorphisms $(X \wedge S^n) \wedge S^1 \cong X \wedge S^{n+1}$ as structure maps. We have $\mathbf{S} = \Sigma^\infty S^0$.

Definition 2.4 (Ω-spectrum)**.** Given a spectrum $\mathbf{E}$, we can consider instead of the structure map $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$ its adjoint

$$\sigma'(n)\colon E(n) \to \Omega E(n+1) = \operatorname{map}(S^1, E(n+1)).$$

We call $\mathbf{E}$ an *Ω-spectrum* if each map $\sigma'(n)$ is a weak homotopy equivalence.

Definition 2.5 (Homotopy groups of a spectrum)**.** Given a spectrum $\mathbf{E}$, define for $n \in \mathbb{Z}$ its *n-th homotopy group*

$$\pi_n(\mathbf{E}) := \operatorname*{colim}_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$ with k-th structure map

$$\pi_{k+n}(E(k)) \xrightarrow{\sigma'(k)} \pi_{k+n}(\Omega E(k+1)) = \pi_{k+n+1}(E(k+1)).$$

Notice that a spectrum can have in contrast to a space non-trivial negative homotopy groups. If $\mathbf{E}$ is an Ω-spectrum, then $\pi_n(\mathbf{E}) = \pi_n(E(0))$ for all $n \geq 0$.

Example 2.6 (Eilenberg-MacLane spectrum)**.** Let A be an abelian group. The n-th *Eilenberg-MacLane space $K(A,n)$* associated to A for $n \geq 0$ is a CW-complex with $\pi_m(K(A,n)) = A$ for $m = n$ and $\pi_m(K(A,n)) = \{0\}$ for $m \neq n$.

The associated *Eilenberg-MacLane spectrum* $\mathbf{H}(A)$ has as n-th space $K(A,n)$ and as n-th structure map a homotopy equivalence $K(A,n) \to \Omega K(A,n+1)$.

Example 2.7 (Algebraic K-theory spectrum)**.** For a ring R there is the *algebraic K-theory spectrum* $\mathbf{K}(R)$ with the property

$$\pi_n(\mathbf{K}(R)) = K_n(R) \quad \text{ for } n \in \mathbb{Z}.$$

For its definition see [42], [115], and [143].

Next we state the L-theoretic version. Since we will not focus on L-theory in these lectures, we will use L-theory as a black box and will later explain its relevance when we discuss applications. At least we mention that L-theory may be thought of a kind of K-theory not for finitely generated projective modules and their automorphisms but for quadratic forms over finitely generated projective modules and their automorphisms modulo hyperbolic forms.

Example 2.8 (Algebraic L-theory spectrum)**.** For a ring with involution R there is the *algebraic L-theory spectrum* $\mathbf{L}^{\langle -\infty \rangle}(R)$ with the property

$$\pi_n(\mathbf{L}^{\langle -\infty \rangle}(R)) = L_n^{\langle -\infty \rangle}(R) \quad \text{ for } n \in \mathbb{Z}.$$

For its construction we refer for instance to Quinn [147] and Ranicki [151].

Example 2.9 (Topological K-theory spectrum)**.** By *Bott periodicity* there is a homotopy equivalence

$$\beta\colon BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z}).$$

The *topological K-theory spectrum* $\mathbf{K}^{\mathrm{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$. The structure maps are given in even degrees by the map β and in odd degrees by the identity $\mathrm{id}\colon \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$.

Definition 2.10 (Homology theory)**.** Let Λ be a commutative ring, for instance $\mathbb{Z}$ or $\mathbb{Q}$. A *homology theory* $\mathcal{H}_*$ with values in Λ-modules is a covariant functor from the category of CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

  If (X,A) is a CW-pair and $f\colon A \to B$ is a cellular map , then

  $$\mathcal{H}_n(X,A) \xrightarrow{\cong} \mathcal{H}_n(X \cup_f B, B).$$

- Disjoint union axiom

  $$\bigoplus_{i\in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i\in I} X_i\right).$$

Definition 2.11 (Smash product)**.** Let $\mathbf{E}$ be a spectrum and X be a pointed space. Define the *smash product* $X \wedge \mathbf{E}$ to be the spectrum whose n-th space is $X \wedge E(n)$ and whose n-th structure map is

$$X \wedge E(n) \wedge S^1 \xrightarrow{\mathrm{id}_X \wedge \sigma(n)} X \wedge E(n+1).$$

Theorem 2.12 (Homology theories and spectra)**.** *Let* $\mathbf{E}$ *be a spectrum. Then we obtain a homology theory* $H_*(-;\mathbf{E})$ *by*

$$H_n(X,A;\mathbf{E}) := \pi_n\left((X \cup_A \operatorname{cone}(A)) \wedge \mathbf{E}\right).$$

It satisfies

$$H_n(\{\bullet\};\mathbf{E}) = \pi_n(\mathbf{E}).$$

Example 2.13 (Stable homotopy theory)**.** The homology theory associated to the sphere spectrum $\mathbf{S}$ is *stable homotopy* $\pi_*^s(X)$. The groups $\pi_n^s(\{\bullet\})$ are finite abelian groups for $n \neq 0$ by a result of Serre (1953). Their structure is only known for small n.

Example 2.14 (Singular homology theory with coefficients)**.** The homology theory associated to the Eilenberg-MacLane spectrum $\mathbf{H}(A)$ is singular homology with coefficients in A.

Example 2.15 (Topological K-homology)**.** The homology theory associated to the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ is *K-homology* $K_*(X)$. We have

$$K_n(\{\bullet\}) \cong \begin{cases} \mathbb{Z} & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Next we give the formulation of the Farrell-Jones Conjecture for K- and L-theory and the Baum-Connes Conjecture in the case of a torsionfree group. The general formulations for arbitrary groups will require more prerequisites and will be presented later. We begin with the K-theoretic version. Recall:

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}(R)$ is the (non-connective) algebraic K-theory spectrum of R;
- $H_n(\{\bullet\};\mathbf{K}(R)) \cong \pi_n(\mathbf{K}(R)) \cong K_n(R)$ for $n \in \mathbb{Z}$.
- BG is the classifying space of the group G, i.e., the base space of the universal G-principal G-bundle $G \to EG \to BG$. Equivalently, $BG = K(G,1)$. The space BG is unique up to homotopy.

Conjecture 2.16 (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)**.** *The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map*

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Recall:

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}^{\langle -\infty \rangle}(R)$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(R)) \cong \pi_n(\mathbf{L}^{\langle -\infty \rangle}(R)) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$.

Conjecture 2.17 (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)**.** *The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map*

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Recall:

- $K_n(BG)$ is the topological K-homology of BG, where $K_*(-) = H_*(-; \mathbf{K}^{\text{top}})$ for $\mathbf{K}^{\text{top}}$ the topological K-theory spectrum.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G which is the closure in the norm topology of $\mathbb{C}G$ considered as subalgebra of $\mathcal{B}(l^2(G))$.

Conjecture 2.18 (Baum-Connes Conjecture for torsionfree groups)**.** *The Baum-Connes Conjecture for the torsionfree group G predicts that the assembly map*

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G; \mathbb{R})).$$

In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.

Notation 2.19. Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the K-theoretic and L-theoretic respectively Farrell-Jones Conjecture for the coefficient ring (with involution) R.

Let $\mathcal{BC}$ be the class of groups which satisfy the Baum-Connes Conjecture.

Theorem 2.20 (Lower and middle *K*-theory of group rings in the torsionfree case)**.** *Suppose that G is torsionfree.*

1. *If R is regular and $G \in \mathcal{FJ}_K(R)$, then*
   (a) *$K_n(RG) = 0$ for $n \leq -1$;*
   (b) *The change of rings map $K_0(R) \to K_0(RG)$ is bijective;*
   (c) *In particular $\widetilde{K}_0(RG)$ is trivial if and only if $\widetilde{K}_0(R)$ is trivial.*
2. *If $G \in \mathcal{FJ}_K(\mathbb{Z})$, then the Whitehead group $\operatorname{Wh}(G)$ is trivial.*

Proof. The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence. It converges to $H_n(BG; \mathbf{K}(R))$ which is isomorphic to $K_n(RG)$ by the assumption that G satisfies the Farrell-Jones Conjecture. The E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

(1) Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$. Hence the spectral sequence is a first quadrant spectral sequence. This implies $K_n(RG) \cong H_n(BG; \mathbf{K}(R)) = 0$ for $n \leq -1$ and that the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\{\bullet\}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}(R)) \cong K_0(RG).$$

(2) We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$0 \to H_0(BG; K_1(\mathbb{Z})) = \{\pm 1\} \to H_1(BG; \mathbf{K}(\mathbb{Z})) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G,G] \to 0.$$

This implies $\operatorname{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} = 0$. □

We summarize that we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

(1) $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
(2) $\widetilde{K}_0(\mathbb{Z}G) = 0$;
(3) $\operatorname{Wh}(G) = 0$;
(4) Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
(5) Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;
(6) If G belongs to $\mathcal{FJ}_K(\mathbb{Z})$, then it is of type FF if and only if it is of type FP (Serre's problem).

Conjecture 2.21 (Kaplansky Conjecture)**.** *The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.*

In the next theorem we will use the notion of a *sofic group* that was introduced by Gromov and originally called *subamenable group*. Every residually amenable group is sofic but the converse is not true. The class of sofic groups is closed under taking subgroups, direct products, free amalgamated products, colimits and inverse limits, and, if H is a sofic normal subgroup of G with amenable quotient G/H, then G is sofic. This is a very general notion, e.g., no group is known which is not sofic. For more information about the notion of a sofic group we refer to [60]. The next result is taken from [15, Theorem 0.12].

Theorem 2.22 (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-Lück-Reich(2007))**.** *Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:*

(1) F is commutative and has characteristic zero and G is torsionfree;
(2) G is torsionfree and sofic;
(3) The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.

Then 0 and 1 are the only idempotents in FG.

Proof. Let p be an idempotent in FG. We want to show $p \in \{0,1\}$. Denote by $\epsilon\colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.

Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module. Since $G \in \mathcal{FJ}_K(F)$, we can conclude that

$$i_*\colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective. Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

$$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain

$$FG^m \cong i_*(P) \oplus FG^n.$$

Inserting this in the first equation yields

$$(p)^k \oplus FG^m \cong FG^m.$$

Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$. □

Theorem 2.23 (The Baum-Connes Conjecture and the Kaplansky Conjecture)**.** *Let G be a torsionfree group with $G \in \mathcal{BC}$. Then 0 and 1 are the only idempotents in $C_r^*(G)$ and in particular in $\mathbb{C}G$.*

Proof. There is a trace map

$$\mathrm{tr}\colon C_r^*(G) \to \mathbb{C}$$

which sends $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e\rangle_{l^2(G)}$. The L^2-index theorem due to Atiyah (1976) (see [6]) shows that the composite

$$K_0(BG) \to K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$

coincides with

$$K_0(BG) \xrightarrow{K_0(\mathrm{pr})} K_0(\{\bullet\}) = \mathbb{Z} \to \mathbb{C}.$$

Hence $G \in \mathcal{BC}$ implies $\mathrm{tr}(p) \in \mathbb{Z}$. Since $\mathrm{tr}(1) = 1$, $\mathrm{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\mathrm{tr}(p) \in \mathbb{R}$ and $0 \leq \mathrm{tr}(p) \leq 1$. We conclude $\mathrm{tr}(0) = \mathrm{tr}(p)$ or $\mathrm{tr}(1) = \mathrm{tr}(p)$. Since the trace tr is faithful, this implies already $p = 0$ or $p = 1$. □

The next conjecture is one of the basic conjectures about the classification of topological manifolds.

Conjecture 2.24 (Borel Conjecture)**.** *The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.*

Remark 2.25 (Borel versus Mostow)**.** The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.

Remark 2.26 (The Borel Conjecture fails in the smooth category)**.** The Borel Conjecture is not true in the smooth category by results of Farrell-Jones [63], i.e., there exists aspherical closed manifolds which are homeomorphic but not diffeomorphic.

Remark 2.27 (Topological rigidity for non-aspherical manifolds)**.** There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-Lück [106]).

Theorem 2.28 (The Farrell-Jones Conjecture and the Borel Conjecture)**.** *If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.*

Remark 2.29 (The Borel Conjecture in dimension ≤ 3)**.** Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension three. The Borel Conjecture in dimension one and two is obviously true.

Next we give some explanations about the proof of Theorem 2.28.

Definition 2.30 (Structure set)**.** The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

The next result follows directly from the definitions.

Theorem 2.31. *The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.*

Let $\mathbf{L}\langle 1 \rangle$ be the 1-*connective cover* of the L-theory spectrum $\mathbf{L}$. It is characterized by the following property. There is a natural map of spectra $\mathbf{L}\langle 1 \rangle \to \mathbf{L}$ which induces an isomorphism on the homotopy groups in dimensions $n \geq 1$ and the homotopy groups of $\mathbf{L}\langle 1 \rangle$ vanish in dimensions $n \leq 0$.

Theorem 2.32 (Ranicki (1992))**.** *There is an exact sequence of abelian groups, called algebraic surgery exact sequence, for an n-dimensional closed manifold M*

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; \mathbf{L}\langle 1 \rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M; \mathbf{L}\langle 1 \rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

Proof. See [151, Definition 15.19 on page 169 and Theorem 18.5 on page 198]. □

The K-theoretic version of the Farrell-Jones Conjecture ensures that we do not have to deal with decorations, e.g., it does not matter if we consider $\mathbf{L}$ or $\mathbf{L}^{\langle -\infty \rangle}$. (This follows from the so called *Rothenberg sequences*). The L-theoretic version of the Farrell-Jones Conjecture implies that $H_n(M; \mathbf{L}) \to L_n(\mathbb{Z}\pi_1(M))$ is bijective for all $n \in \mathbb{Z}$. An easy spectral sequence argument shows that $H_k(M; \mathbf{L}\langle 1 \rangle) \to H_k(M; \mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$. For $k = n$ and $k = n+1$ the map A_k is the composite of the map $H_k(M; \mathbf{L}\langle 1 \rangle) \to H_k(M; \mathbf{L})$ with the map $H_k(M; \mathbf{L}) \to L_k(\mathbb{Z}\pi_1(M))$. Hence A_{n+1} is surjective and A_n is injective. Theorem 2.32 implies that $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element. Now Theorem 2.28 follows from Theorem 2.31.

More information on the Borel Conjecture can be found for instance in [62], [63], [64], [67], [68], [72] [77], [105], [119], [126].

Next we explain that the versions of the Farrell-Jones and the Baum-Connes Conjecture above cannot be true if we drop the assumption that G is torsionfree or that R is regular

Example 2.33 (The condition torsionfree is essential)**.** The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial. For instance the version of the Baum-Connes Conjecture above would predict for a finite group G

$$K_0(BG) \cong K_0(C_r^*(G)) \cong R_{\mathbb{C}}(G).$$

However, $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} K_0(\{\bullet\}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ and $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ holds if and only if G is trivial.

Example 2.34 (The condition regular is essential)**.** If G is torsionfree, then the version of the K-theoretic Farrell-Jones Conjecture predicts

$$\begin{aligned} H_n(B\mathbb{Z};\mathbf{K}(R)) = H_n(S^1;\mathbf{K}(R)) &= H_n(\{\bullet\};\mathbf{K}(R)) \oplus H_{n-1}(\{\bullet\};\mathbf{K}(R)) \\ &= K_n(R) \oplus K_{n-1}(R) \cong K_n(R\mathbb{Z}). \end{aligned}$$

In view of the Bass-Heller-Swan decomposition this is only possible if $\mathrm{NK}_n(R)$ vanishes which is true for regular rings R but not for general rings R.

Next we want to discuss what we may have to take into account if we want to give a formulation of the Farrell-Jones and the Baum-Connes Conjecture which may have a chance to be true for all groups.

Remark 2.35 (Assembly)**.** For a field F of characteristic zero and some groups G one knows that there is an isomorphism

$$\operatorname*{colim}_{\substack{H\subseteq G\\ |H|<\infty}} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

This indicates that one has at least to take into account the values for all finite subgroups to assemble $K_n(FG)$.

Remark 2.36 (Degree Mixing)**.** The Bass-Heller-Swan decomposition shows that the K-theory of finite subgroups in degree $m \le n$ can affect the K-theory in degree n and that at least in the Farrell-Jones setting finite subgroups are not enough.

Remark 2.37 (No Nil-phenomena occur in the Baum-Connes setting)**.** In the Baum-Connes setting Nil-phenomena do not appear. Namely, a special case of a result due to Pimsner-Voiculescu [146] says

$$K_n(C_r^*(G\times\mathbb{Z})) \cong K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

Remark 2.38 (Homological behavior)**.** There is still a lot of homological behavior known for $K_*(C_r^*(G))$. For instance there exists a long exact Mayer-Vietoris sequence associated to amalgamated products $G_1 *_{G_0} G_2$ by Pimsner-Voiculescu [146].

$$\begin{aligned} \cdots \to K_n(C_r^*(G_0)) \to K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \to K_n(C_r^*(G)) \\ \to K_{n-1}(C_r^*(G_0)) \to K_{n-1}(C_r^*(G_1)) \oplus K_{n-1}(C_r^*(G_2)) \to \cdots \end{aligned}$$

This is very similar to the corresponding Mayer-Vietoris sequence in group homology theory

$$\begin{aligned} \cdots \to H_n(G_0) \to H_n(G_1)) \oplus H_n(G_2) \to H_n(G) \\ \to H_{n-1}(G_0) \to H_{n-1}(G_1) \oplus H_{n-1}G_2) \to \cdots \end{aligned}$$

It comes from the fact that there is a model for BG which contains BG_0, BG_1 and BG_2 as CW-subcomplexes such that $BG = BG_1 \cup BG_2$ and $BG_0 = BG_1 \cap BG_2$.

An analogous similarity exists for the Wang-sequence associated to a semi-direct product $G \rtimes \mathbb{Z}$

Similar versions of the Mayer-Vietoris sequence and the Wang sequence in algebraic K-and L-theory of group rings are due to Cappell (1974) and Waldhausen (1978) provided one makes certain assumptions on R or ignores certain Nil-phenomena.

Question 2.39 (Classifying spaces for families)**.** Is there a version $E_{\mathcal{F}}(G)$ of the classifying space EG which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?

Question 2.40 (Equivariant homology theories)**.** Can one define appropriate G-homology theories $\mathcal{H}_*^G$ that are in some sense computable and yield when applied to $E_{\mathcal{F}}(G)$ a term which potentially is isomorphic to the groups $K_n(RG)$, $L_n^{-\langle\infty\rangle}(RG)$ or $K_n(C_r^*(G))$?

In the torsionfree case they should reduce to $H_n(BG;\mathbf{K}(R))$, $H_n(BG;\mathbf{L}^{-\langle\infty\rangle})$ and $K_n(BG)$.

3. Classifying spaces for families

The outline of this section is:

- We introduce the notion of the classifying space of a family $\mathcal{F}$ of subgroups $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$.
- In the case, where $\mathcal{F}$ is the family $\mathcal{COM}$ of compact subgroups, we present some nice geometric models for $E_{\mathcal{F}}(G)$ and explain $E_{\mathcal{F}}(G) \simeq J_{\mathcal{F}}(G)$.
- We discuss finiteness properties of these classifying spaces.

The material of this section is an extract of the survey article by Lück [125], where more information and proofs of the results stated below are given.

In this section group means locally compact Hausdorff topological group with a countable basis for its topology, unless explicitly stated differently.

Definition 3.1 (G-CW-complex)**.** A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n\geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i\in I_n} Q_i^n} & X_n \end{array}$$

Example 3.2 (Simplicial actions)**.** Let X be a (geometric) simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and all isotropy groups are open and closed. The G-space X' inherits the structure of a G-CW-complex.

Definition 3.3 (Proper G-action)**.** A G-space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is compact.

Lemma 3.4. *(1) A proper G-space has always compact isotropy groups.*
(2) A G-CW-complex X is proper if and only if all its isotropy groups are compact.

Proof. See [117, Theorem 1.23 on page 19]. □

Example 3.5 (Smooth actions)**.** Let G be a Lie group acting properly and smoothly on a smooth manifold M. Then M inherits the structure of G-CW-complex (see Illman [93]).

Definition 3.6 (Family of subgroups)**.** A *family $\mathcal{F}$ of subgroups* of G is a set of (closed) subgroups of G which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

$\mathcal{TR}$ = {trivial subgroup};
$\mathcal{FIN}$ = {finite subgroups};
$\mathcal{VCYC}$ = {virtually cyclic subgroups};
$\mathcal{COM}$ = {compact subgroups};
$\mathcal{COMOP}$ = {compact open subgroups};
$\mathcal{ALL}$ = {all subgroups}.

Definition 3.7 (Classifying G-CW-complex for a family of subgroups)**.** Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

(1) All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
(2) For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

We abbreviate $\underline{E}G := E_{\mathcal{COM}}(G)$ and call it the *universal G-CW-complex for proper G-actions*. We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem 3.8 (Homotopy characterization of $E_{\mathcal{F}}(G)$)**.** *Let $\mathcal{F}$ be a family of subgroups.*

(1) There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
(2) A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is weakly contractible.

Example 3.9 ($E_{\mathcal{ALL}}(G)$)**.** A model for $E_{\mathcal{ALL}}(G)$ is G/G;

Example 3.10 (Universal principal G-bundle)**.** The projection $EG \to BG := G\backslash EG$ is the universal G-principal bundle for G-CW-complexes.

Example 3.11 (Infinite dihedral group)**.** Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group. A model for ED_∞ is the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$. A model for $\underline{E}D_\infty$ is $\mathbb{R}$ with the obvious D_∞-action. Notice that every model for ED_∞ or BD_∞ must be infinite-dimensional, whereas there exists a cocompact 1-dimensional model for $\underline{E}D_\infty$.

Lemma 3.12. *If G is totally disconnected, then $E_{\mathcal{COMOP}}(G) = \underline{E}G$.*

Definition 3.13 ($\mathcal{F}$-numerable G-space)**.** An *$\mathcal{F}$-numerable G-space* is a G-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by G-subspaces satisfying:

(1) For each $i \in I$ there exists a G-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$;
(2) There is a locally finite partition of unity $\{e_i \mid i \in I\}$ subordinate to $\{U_i \mid i \in I\}$ by G-invariant functions $e_i \colon X \to [0,1]$.

Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable G-space belong to $\mathcal{F}$.

If $f \colon X \to Y$ is a G-map and Y is $\mathcal{F}$-numerable, then X is also $\mathcal{F}$-numerable.

Lemma 3.14. *A G-CW-complex is $\mathcal{F}$-numerable if and only if each isotropy group appears as a subgroup of an element in $\mathcal{F}$.*

Definition 3.15 (Classifying numerable G-space for a family of subgroups)**.** Let $\mathcal{F}$ be a family of subgroups of G. A model $J_{\mathcal{F}}(G)$ for the *classifying numerable G-space for the family of subgroups $\mathcal{F}$* is a G-space which has the following properties:

(1) $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable;
(2) For any $\mathcal{F}$-numerable G-space X there is up to G-homotopy precisely one G-map $X \to J_{\mathcal{F}}(G)$.

We abbreviate $\underline{J}G := J_{\mathcal{COM}}(G)$ and call it the *universal numerable G-space for proper G-actions* or briefly *the universal space for proper G-actions.* We also write $JG = J_{\mathcal{TR}}(G)$.

Theorem 3.16 (Homotopy characterization of $J_{\mathcal{F}}(G)$)**.** *Let $\mathcal{F}$ be a family of subgroups.*

(1) For any family $\mathcal{F}$ there exists a model for $J_{\mathcal{F}}(G)$ whose isotropy groups belong to $\mathcal{F}$;
(2) Let X be an $\mathcal{F}$-numerable G-space. Equip $X \times X$ with the diagonal action and let $\mathrm{pr}_i \colon X \times X \to X$ be the projection onto the i-th factor for $i = 1, 2$. Then X is a model for $J_{\mathcal{F}}(G)$ if and only if for each $H \in \mathcal{F}$ there is $x \in X$ with $H \subseteq G_x$ and pr_1 and pr_2 are G-homotopic.
(3) For $H \in \mathcal{F}$ the H-fixed point set $J_{\mathcal{F}}(G)^H$ is contractible.

Proof. See [125, Theorem 2.5]. □

Example 3.17 (Universal G-principal bundle)**.** The projection $JG \to G\backslash JG$ is the universal G-principal bundle for numerable free proper G-spaces.

Theorem 3.18 (Comparison of $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$, Lück (2005))**.**

(1) There is up to G-homotopy precisely one G-map

$$\phi \colon E_{\mathcal{F}}(G) \to J_{\mathcal{F}}(G);$$

(2) It is a G-homotopy equivalence if one of the following conditions is satisfied:
  (a) Each element in $\mathcal{F}$ is open and closed;
  (b) G is discrete;
  (c) $\mathcal{F}$ is $\mathcal{COM}$;
(3) Let G be totally disconnected. Then $EG \to JG$ is a G-homotopy equivalence if and only if G is discrete.

Proof. See [125, Theorem 3.7]. □

Next we want to illustrate that the space $\underline{E}G = \underline{J}G$ often has very nice geometric models and appear naturally in many interesting situations.

Let $C_0(G)$ be the Banach space of complex valued functions of G vanishing at infinity with the supremum-norm. The group G acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$. Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions f such that f is not identically zero and has non-negative real numbers as values.

Theorem 3.19 (Operator theoretic model, Abels (1978))**.** *The G-space $PC_0(G)$ is a model for $\underline{J}G$.*

Proof. See [1, Theorem 2.4]. □

Theorem 3.20. *Let G be discrete. A model for $\underline{J}G$ is the space*

$$X_G = \left\{ f \colon G \to [0,1] \;\middle|\; f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm.

Theorem 3.21 (Simplicial Model)**.** *Let G be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements g_i in G.*

Then $P_\infty(G)$ is a model for $\underline{E}G$.

Remark 3.22 (Comparison of X_G and $P_\infty(G)$)**.** The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies. The identity map induces a G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism.

Theorem 3.23 (Almost connected groups, Abels (1978).)**.** *Suppose that G is almost connected, i.e., the group G/G^0 is compact for G^0 the component of the identity element.*

Then G contains a maximal compact subgroup K which is unique up to conjugation, and the G-space G/K is a model for $\underline{J}G$.

Proof. See [1, Corollary 4.14]. □

As a special case we get:

Theorem 3.24 (Discrete subgroups of almost connected Lie groups)**.** *Let L be a Lie group with finitely many path components.*

Then L contains a maximal compact subgroup K which is unique up to conjugation, and the L-space L/K is a model for $\underline{E}L$.

If $G \subseteq L$ is a discrete subgroup of L, then L/K with the obvious left G-action is a finite dimensional G-CW-model for $\underline{E}G$.

Theorem 3.25 (Actions on CAT(0)-spaces)**.** *Let G be a (locally compact second countable Hausdorff) topological group. Let X be a proper G-CW-complex. Suppose that X has the structure of a complete simply connected* CAT(0)*-space for which G acts by isometries.*

Then X is a model for $\underline{E}G$.

Proof. By [31, Corollary II.2.8 on page 179] the K-fixed point set of X is a non-empty convex subset of X and hence contractible for any compact subgroup $K \subset G$. □

Remark 3.26. The result above contains as special case isometric G actions on simply-connected complete Riemannian manifolds with non-positive sectional curvature and G-actions on trees.

Let Σ be an *affine building* sometimes also called *Euclidean building*. This is a simplicial complex together with a system of subcomplexes called *apartments* satisfying the following axioms:

(1) Each apartment is isomorphic to an affine Coxeter complex;
(2) Any two simplices of Σ are contained in some common apartment;
(3) If two apartments both contain two simplices A and B of Σ, then there is an isomorphism of one apartment onto the other which fixes the two simplices A and B pointwise.

The precise definition of an *affine Coxeter complex*, which is sometimes called also *Euclidean Coxeter complex*, can be found in [35, Section 2 in Chapter VI], where also more information about affine buildings is given. An affine building comes with metric $d\colon \Sigma \times \Sigma \to [0,\infty)$ which is non-positively curved and complete. The building with this metric is a CAT(0)-space. A simplicial automorphism of Σ is always an isometry with respect to d. For two points x, y in the affine building there is a unique line segment $[x, y]$ joining x and y. It is the set of points $\{z \in \Sigma \mid d(x,y) = d(x,z) + d(z,y)\}$. For $x, y \in \Sigma$ and $t \in [0,1]$ let $tx + (1-t)y$ be the point $z \in \Sigma$ uniquely determined by the property that $d(x,z) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$. Then the map

$$r\colon \Sigma \times \Sigma \times [0,1] \to \Sigma, \qquad (x,y,t) \mapsto tx + (1-t)y$$

is continuous. This implies that Σ is contractible. All these facts are taken from [35, Section 3 in Chapter VI] and [31, Theorem 10A.4 on page 344].

Suppose that the group G acts on Σ by isometries. If G maps a non-empty bounded subset A of Σ to itself, then the G-action has a fixed point (see [35, Theorem 1 in Section 4 in Chapter VI on page 157]). Moreover the G-fixed point set must be contractible since for two points $x, y \in \Sigma^G$ also the segment $[x, y]$ must lie in Σ^G and hence the map r above induces a continuous map $\Sigma^G \times \Sigma^G \times [0,1] \to \Sigma^G$. This implies together with Example 3.2, Theorem 3.8 (2), Lemma 3.12 and Theorem 3.18

Theorem 3.27 (Affine buildings)**.** *Let G be a topological (locally compact second countable Hausdorff) group. Suppose that G acts on the affine building by simplicial automorphisms such that each isotropy group is compact. Then each isotropy group is compact open, Σ is a model for $J_{\mathcal{COMOP}}(G)$ and the barycentric subdivision Σ' is a model for both $J_{\mathcal{COMOP}}(G)$ and $E_{\mathcal{COMOP}}(G)$. If we additionally assume that G is totally disconnected, then Σ is a model for both $\underline{J}G$ and $\underline{E}G$.*

Example 3.28 (Bruhat-Tits building)**.** An important example is the case of a reductive p-adic algebraic group G and its associated affine Bruhat-Tits building $\beta(G)$ (see [179], [180]). Then $\beta(G)$ is a model for $\underline{J}G$ and $\beta(G)'$ is a model for $\underline{E}G$ by Theorem 3.27.

For more information about buildings we refer to the lectures of Abramenko.

The Rips complex $P_d(G, S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realization of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.

The obvious G-action by simplicial automorphisms on $P_d(G, S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G, S)'$.

Theorem 3.29 (Rips complex, Meintrup-Schick (2002))**.** *Let G be a discrete group with a finite symmetric set of generators. Suppose that (G, S) is δ-hyperbolic for the real number $\delta \geq 0$. Let d be a natural number with $d \geq 16\delta + 8$.*

Then the barycentric subdivision of the Rips complex $P_d(G, S)'$ is a finite G-CW-model for $\underline{E}G$.

Proof. See [129]. □

Arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group possess finite models for $\underline{E}G$. Namely, let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$ and let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup. If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left A-action is a model for $\underline{E}A$ as already explained above. However, the A-space $G(\mathbb{R})/K$ is not necessarily cocompact. But there is a finite model for $\underline{E}A$ by the following result.

Theorem 3.30 (Borel-Serre compactification)**.** *The Borel-Serre compactification (see [29], [168]) of $G(\mathbb{R})/K$ is a finite A-CW-model for $\underline{E}A$.*

Proof. This is pointed out in Adem-Ruan [2, Remark 5.8], where a private communication with Borel and Prasad is mentioned. A detailed proof is given by Ji [97]. □

For more information about arithmetic groups we refer to the lectures of Abramenko.

Let $\Gamma^s_{g,r}$ be the mapping class group of an orientable compact surface F of genus g with s punctures and r boundary components. We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative. It is well-known that the associated Teichmüller space $\mathcal{T}^s_{g,r}$ is a contractible space on which $\Gamma^s_{g,r}$ acts properly.

We could not find a clear reference in the literature for the to experts known statement that there exist a finite $\Gamma^s_{g,r}$-CW-model for $\underline{E}\Gamma^s_{g,r}$. The work of Harer [85] on the existence of a spine and the construction of the spaces $T_S(\epsilon)^H$ due to Ivanov [95, Theorem 5.4.A] seem to lead to such models. However, a detailed proof can be found in a manuscript by Mislin [135].

Theorem 3.31 (Teichmüller space)**.** *The $\Gamma^s_{g,r}$-space $\mathcal{T}^s_{g,r}$ is a model for $\underline{E}\Gamma^s_{g,r}$.*

Let F_n be the free group of rank n. Denote by $\operatorname{Out}(F_n)$ the group of outer automorphisms of F_n, i.e., the quotient of the group of all automorphisms of F_n by the normal subgroup of inner automorphisms. Culler-Vogtmann (see [48], [183]) have constructed a space X_n called *outer space* on which $\operatorname{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Fix a graph R_n with one vertex v and n-edges and identify F_n with $\pi_1(R_n, v)$. A *marked metric graph* (g, Γ) consists of a graph Γ with all vertices of valence at least three, a homotopy equivalence $g\colon R_n \to \Gamma$ called marking and to every edge of Γ there is assigned a positive length which makes Γ into a metric space by the path metric. We call two marked metric graphs (g, Γ) and (g', Γ') equivalent if there is a homothety $h\colon \Gamma \to \Gamma'$ such that $g \circ h$ and h' are homotopic. Homothety means that there is a constant $\lambda > 0$ with $d(h(x), h(y)) = \lambda \cdot d(x, y)$ for all x, y. Elements in outer space X_n are equivalence classes of marked graphs. The main result in [48] is that X is contractible. Actually, for each finite subgroup $H \subseteq \operatorname{Out}(F_n)$ the H-fixed point set X_n^H is contractible (see [107, Proposition 3.3 and Theorem 8.1], [190, Theorem 5.1]).

The space X_n contains a *spine* K_n which is an $\operatorname{Out}(F_n)$-equivariant deformation retraction. This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\operatorname{Out}(F_n)$-action is by simplicial automorphisms and cocompact. Actually the group of simplicial automorphisms of K_n is $\operatorname{Out}(F_n)$ (see Bridson-Vogtmann [32]). We conclude

Theorem 3.32 (Spine of outer space)**.** *The barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\operatorname{Out}(F_n)$.*

Example 3.33 ($SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$)**.** In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$.

Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. We will use either the upper half-plane model or the Poincaré disk model. The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations, i.e., a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by sending a complex number z with positive imaginary part to $\frac{az+b}{cz+d}$. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Since $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant -1, the $SL_2(\mathbb{R})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$ by Remark 3.26.

One easily checks that $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup. Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{R})$ by Theorem 3.24. Since the $SL_2(\mathbb{R})$-action on $\mathbb{H}^2$ is transitive and $SO(2)$ is the isotropy group at $i \in \mathbb{H}^2$, we see that the $SL_2(\mathbb{R})$-manifolds $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{R})$-diffeomorphic.

Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $\underline{E}SL_2(\mathbb{Z})$ (see Theorem 3.24).

The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that $SL_2(\mathbb{Z})$ acts on a tree T which consists of two 0-dimensional equivariant cells with isotropy groups $\mathbb{Z}/4$ and $\mathbb{Z}/6$ and one 1-dimensional equivariant cell with isotropy group $\mathbb{Z}/2$. From Remark 3.26 we conclude that a model for $\underline{E}SL_2(\mathbb{Z})$ is given by the following $SL_2(\mathbb{Z})$-pushout

$$\begin{array}{ccc} SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times \{-1,1\} & \xrightarrow{F_{-1} \coprod F_1} & SL_2(\mathbb{Z})/(\mathbb{Z}/4) \coprod SL_2(\mathbb{Z})/(\mathbb{Z}/6) \\ \downarrow & & \downarrow \\ SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times [-1,1] & \longrightarrow & T = \underline{E}SL_2(\mathbb{Z}) \end{array}$$

where F_{-1} and F_1 are the obvious projections. This model for $\underline{E}SL_2(\mathbb{Z})$ is a tree, which has alternately two and three edges emanating from each vertex. The other model $\mathbb{H}^2$ is a manifold. These two models must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.

Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action. This is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.

The tree T above can be identified with the Bruhat-Tits building of $SL_2(\widehat{\mathbb{Q}_p})$ and hence is a model for $\underline{E}SL_2(\widehat{\mathbb{Q}_p})$ (see [35, page 134]). Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\widehat{\mathbb{Q}_p})$, we get another reason why this tree is a model for $\underline{E}SL_2(\mathbb{Z})$.

Definition 3.34 (Cohomological dimension)**.** Let Λ be a commutative ring. The cohomological dimension $\mathrm{cd}_\Lambda(G)$ of a group G over Λ is defined to be the infimum over all integers d for which there exist a d-dimensional projective ΛG-resolution of the trivial ΛG-module Λ. If $\Lambda = \mathbb{Z}$, we abbreviate $\mathrm{cd}(G) = \mathrm{cd}_\mathbb{Z}(G)$.

By definition $\mathrm{cd}_\Lambda(G) = \infty$ if there is no finite-dimensional projective ΛG-resolution of the trivial ΛG-module Λ.

Example 3.35. If G is a non-trivial finite group, then $\mathrm{cd}(G) = \infty$ and $\mathrm{cd}_\mathbb{Q}(G) = 0$. We conclude that a group G with $\mathrm{cd}(G) < \infty$ must be torsionfree.

Definition 3.36 (Virtual cohomological dimension)**.** A group G is called virtually torsionfree if it contains a torsionfree subgroup $\Delta \subset G$ with finite index $[G : \Delta]$.

Let Λ be a commutative ring. Define the virtual cohomological dimension of a virtually torsionfree group G over Λ by

$$\mathrm{vcd}_\Lambda(G) = \mathrm{cd}_\Lambda(\Delta)$$

for any torsionfree subgroup $\Delta \subset G$ with finite index $[G : \Delta]$.

If $\Lambda = \mathbb{Z}$, we abbreviate $\mathrm{vcd}(G) = \mathrm{vcd}_\mathbb{Z}(G)$.

This definition is indeed independent of the choice of $\Delta \subseteq G$.

Next we investigate the relation between the minimal dimension of a model $\underline{E}G$ with the virtual cohomological dimension provided that G is virtually torsionfree.

Theorem 3.37 (Discrete subgroups of Lie groups)**.** *Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup K. Let $G \subseteq L$ be a discrete subgroup of L. Then L/K with the left G-action is a model for $\underline{E}G$.*

Suppose additionally that G is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index.

Then we have for its virtual cohomological dimension

$$\operatorname{vcd}(G) \leq \dim(L/K).$$

Equality holds if and only if $G\backslash L$ is compact.

Proof. We have already mentioned in Theorem 3.24 that L/K is a model for $\underline{E}G$. The restriction of $\underline{E}G$ to Δ is a Δ-CW-model for $\underline{E}\Delta$ and hence $\Delta\backslash\underline{E}G$ is a CW-model for $B\Delta$. This implies $\operatorname{vcd}(G) := \operatorname{cd}(\Delta) \leq \dim(L/K)$. Obviously $\Delta\backslash L/K$ is a manifold without boundary. Suppose that $\Delta\backslash L/K$ is compact. Then $\Delta\backslash L/K$ is a closed manifold and hence its homology with $\mathbb{Z}/2$-coefficients in the top dimension is non-trivial. This implies $\operatorname{cd}(\Delta) \geq \dim(\Delta\backslash L/K)$ and hence $\operatorname{vcd}(G) = \dim(L/K)$. If $\Delta\backslash L/K$ is not compact, it contains a CW-complex $X \subseteq \Delta\backslash L/K$ of dimension smaller than $\Delta\backslash L/K$ such that the inclusion of X into $\Delta\backslash L/K$ is a homotopy equivalence. Hence X is another model for $B\Delta$. This implies $\operatorname{cd}(\Delta) < \dim(L/K)$ and hence $\operatorname{vcd}(G) < \dim(L/K)$. □

Theorem 3.38 (A criterion for 1-dimensional models for BG, Stallings (1968), Swan (1969))**.** *Let G be a discrete group.*

The following statements are equivalent:

- *There exists a 1-dimensional model for EG;*
- *There exists a 1-dimensional model for BG;*
- *The cohomological dimension of G is less or equal to one;*
- *G is a free group.*

Proof. See [171] and [177]. □

Theorem 3.39 (A criterion for 1-dimensional models for $\underline{E}G$, Dunwoody (1979))**.** *Let G be a discrete group. Then there exists a 1-dimensional model for $\underline{E}G$ if and only if the cohomological dimension of G over the rationals $\mathbb{Q}$ is less or equal to one.*

Proof. See Dunwoody [56, Theorem 1.1]. □

Theorem 3.40 (Virtual cohomological dimension and $\dim(\underline{E}G)$, Lück (2000))**.** *Let G be a discrete group which is virtually torsionfree.*

1. *Then*
$$\operatorname{vcd}(G) \leq \dim(\underline{E}G)$$
for any model for $\underline{E}G$.
2. *Let $l \geq 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \leq l$.*

   Then there exists a model for $\underline{E}G$ whose dimension is
$$\max\{3, \operatorname{vcd}(G)\} + l.$$

Proof. See Lück [118, Theorem 6.4]. □

The following problem has been stated by Brown [34, page 32] and has created a lot of activities.

Problem 3.41. For which discrete groups G, which are virtually torsionfree, does there exist a G-CW-model for $\underline{E}G$ of dimension $\operatorname{vcd}(G)$?

Remark 3.42. The results above give some evidence for the hope that the problem above has a positive answer for every discrete group. However, Leary-Nucinkis [112] have constructed virtually torsionfree groups G for which the answer is negative, i.e., for which the dimension of any model for $\underline{E}G$ is different from $\operatorname{vcd}(G)$.

The following result shows that in general one can say nothing about the quotient $G\backslash\underline{E}G$ although in many interesting cases there do exist small models for it.

Theorem 3.43 (Leary-Nucinkis (2001)). *Let X be a CW-complex. Then there exists a group G with $X \simeq G\backslash \underline{E}G$.*

Proof. See [111]. □

Question 3.44 (Homological Computations based on nice models for $\underline{E}G$). Can nice geometric models for $\underline{E}G$ be used to compute the group homology and more general homology and cohomology theories of a group G?

Question 3.45 (K-theory of group rings and group homology). Is there a relation between $K_n(RG)$ and the group homology of G?

Question 3.46 (Isomorphism Conjectures and classifying spaces of families). Can classifying spaces of families be used to formulate a version of the Farrell-Jones Conjecture and the Baum-Connes Conjecture which may hold for all group G and all rings?

4. Equivariant homology theories

The outline of this section is:

- We introduce the notion of an equivariant homology theory.
- We present the general formulation of the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- We discuss equivariant Chern characters.
- We present some explicit computations of equivariant topological K-groups and of homology groups associated to classifying spaces of groups.

Definition 4.1 (G-homology theory).
A *G-homology theory $\mathcal{H}_*^G$* is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n^G(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

The following definition is taken from [120, Section 1].

Definition 4.2 (Equivariant homology theory). An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X,A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \quad \to \quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X,A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

We have the following examples of equivariant homology theories.

Example 4.3 (Borel homology)**.** Given a non-equivariant homology theory $\mathcal{K}_*$, put

$$\begin{array}{lcl} \mathcal{H}_*^G(X) & := & \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) & := & \mathcal{K}_*(EG \times_G X) \quad \text{(Borel homology)}. \end{array}$$

Example 4.4 (Equivariant bordism)**.** Equivariant bordism $\Omega_*^?(X)$ based on proper cocompact equivariant smooth manifolds with reference map to the G-space X;

Example 4.5 (Equivariant topological K-theory)**.** Equivariant topological K-theory $K_*^?(X)$ defined for proper equivariant CW-complexes has the property that for any finite subgroup $H \subseteq G$ we get

$$K_n^H(\{\bullet\}) \cong K_0^G(G/H) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ 0 & n \text{ odd.} \end{cases}$$

Theorem 4.6 (Lück-Reich (2005))**.** *Given a functor* $\mathbf{E}\colon \mathsf{Groupoids} \to \mathsf{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(\{\bullet\}) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Proof. See [126, Proposition 6.4 on page 738]. □

Theorem 4.7 (Equivariant homology theories associated to K and L-theory, Davis-Lück (1998))**.** *Let R be a ring (with involution). There exist covariant functors*

$$\begin{array}{rcl} \mathbf{K}_R\colon \mathsf{Groupoids} & \to & \mathsf{Spectra}; \\ \mathbf{L}_R^{\langle\infty\rangle}\colon \mathsf{Groupoids} & \to & \mathsf{Spectra}; \\ \mathbf{K}^{\mathrm{top}}\colon \mathsf{Groupoids}^{\mathrm{inj}} & \to & \mathsf{Spectra} \end{array}$$

with the following properties:

- *They send equivalences of groupoids to weak equivalences of spectra;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{array}{rcl} \pi_n(\mathbf{K}_R(G)) & \cong & K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle-\infty\rangle}(G)) & \cong & L_n^{\langle-\infty\rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}(G)) & \cong & K_n(C_r^*(G)). \end{array}$$

Proof. See [52, Section 2]. □

Combining the last two theorems we get

Example 4.8 (Equivariant homology theories associated to K and L-theory)**.** We get equivariant homology theories

$$\begin{array}{c} H_*^?(-;\mathbf{K}_R); \\ H_*^?(-;\mathbf{L}_R^{\langle-\infty\rangle}); \\ H_*^?(-;\mathbf{K}^{\mathrm{top}}), \end{array}$$

satisfying for $H \subseteq G$

$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\{\bullet\};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle-\infty\rangle}) & \cong & H_n^H(\{\bullet\};\mathbf{L}_R^{\langle-\infty\rangle}) & \cong & L_n^{\langle-\infty\rangle}(RH); \\ H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\{\bullet\};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_r^*(H)). \end{array}$$

Now we are ready to give the general formulation of the Farrell-Jones and the Baum-Connes Conjecture.

Conjecture 4.9 (K-theoretic Farrell-Jones-Conjecture)**.** *The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map*

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(\{\bullet\}, \mathbf{K}_R) = K_n(RG),$$

which is the map induced by the projection $E_{\mathcal{VCYC}}(G) \to \{\bullet\}$, is bijective for all $n \in \mathbb{Z}$.

Conjecture 4.10 (L-theoretic Farrell-Jones-Conjecture)**.** *The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map*

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(\{\bullet\}, \mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG),$$

which is the map induced by the projection $E_{\mathcal{VCYC}}(G) \to \{\bullet\}$, is bijective for all $n \in \mathbb{Z}$.

Conjecture 4.11 (Baum-Connes Conjecture)**.** *The Baum-Connes Conjecture predicts that the assembly map*

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(\{\bullet\}, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

which is the map induced by the projection $E_{\mathcal{FIN}}(G) \to \{\bullet\}$, is bijective for all $n \in \mathbb{Z}$.

Remark 4.12 (Original sources for the Farrell-Jones and the Baum-Connes Conjecture)**.** These conjectures were stated in Farrell-Jones [66, 1.6 on page 257] and Baum-Connes-Higson [24, Conjecture 3.15 on page 254]. Our formulations differ from the original ones, but are equivalent (see [10, Section 6], [52, Section 6], and [84]). In the case of the Farrell-Jones Conjecture we slightly generalize the original conjecture by allowing arbitrary coefficient rings instead of $\mathbb{Z}$.

We will discuss these conjectures and their applications in the next section. We will now continue with equivariant homology theories.

Let $\mathcal{H}_*$ be a (non-equivariant) homology theory. There is the Atiyah-Hirzebruch spectral sequence which converges to $\mathcal{H}_{p+q}(X)$ and has as E^2-term

$$E^2_{p,q} = H_p(X; \mathcal{H}_q(\{\bullet\})).$$

Rationally it collapses completely by the following result.

Theorem 4.13 (Non-equivariant Chern character, Dold (1962))**.** *Let $\mathcal{H}_*$ be a homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$.*

Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X; \Lambda) \otimes_\Lambda \mathcal{H}_q(\{\bullet\}) \xrightarrow{\cong} \mathcal{H}_n(X),$$

where $H_p(X;\Lambda)$ is the singular or cellular homology of X with coefficients in Λ.

Proof. At least we give the definition of Dold's Chern character for a CW-complex X, for more details we refer to Dold [54]. It is given by the following composite:

$$\mathrm{ch}_n\colon \bigoplus_{p+q=n} H_p(X; \mathcal{H}_q(*)) \xrightarrow{\alpha^{-1}} \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$

$$\xrightarrow{\bigoplus_{p+q=n}(\mathrm{hur}\otimes \mathrm{id})^{-1}} \bigoplus_{p+q=n} \pi_p^s(X_+, *) \otimes_{\mathbb{Z}} \mathcal{H}_q(*) \xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X),$$

Here the canonical map α is bijective, since any Λ-module is flat over $\mathbb{Z}$ because of the assumption $\mathbb{Q} \subset \Lambda$, The map hur is the Hurewicz homomorphism which is bijective because of Serre's Theorem (see [166], [102]) which says

$$\pi_m^s \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for } m = 0, \\ 0 & \text{else.} \end{cases}$$

The map $D_{p,q}$ sends $[f\colon (S^{p+k}, \{\bullet\}) \to (S^k \wedge X_+, \{\bullet\})] \otimes \eta$ to the image of η under the composite

$$\mathcal{H}_q(*) \cong \mathcal{H}_{p+k+q}(S^{p+k}, \{\bullet\}) \xrightarrow{\mathcal{H}_{p+k+q}(f)} \mathcal{H}_{p+k+q}(S^k \wedge X_+, \{\bullet\}) \cong \mathcal{H}_{p+q}(X).$$

□

We want to extend this to the equivariant setting. This requires an extra structure on the coefficients of an equivariant homology theory $\mathcal{H}_*^?$.

We define a covariant functor called induction

$$\text{ind}\colon \mathcal{FGI} \to \Lambda\text{-Mod}$$

from the category $\mathcal{FGI}$ of finite groups with injective group homomorphisms as morphisms to the category of Λ-modules as follows. It sends G to $\mathcal{H}_n^G(\{\bullet\})$ and an injection of finite groups $\alpha\colon H \to G$ to the morphism given by the induction structure

$$\mathcal{H}_n^H(\{\bullet\}) \xrightarrow{\text{ind}_\alpha} \mathcal{H}_n^G(\text{ind}_\alpha\{\bullet\}) \xrightarrow{\mathcal{H}_n^G(\text{pr})} \mathcal{H}_n^G(\{\bullet\}).$$

Definition 4.14 (Mackey extension)**.** We say that $\mathcal{H}_*^?$ has a Mackey extension if for every $n \in \mathbb{Z}$ there is a contravariant functor called restriction

$$\text{res}\colon \mathcal{FGI} \to \Lambda\text{-Mod}$$

such that the two functors ind and res agree on objects and satisfy the double coset formula, i.e., we have for two subgroups $H, K \subset G$ of the finite group G

$$\text{res}_G^K \circ \text{ind}_H^G = \sum_{KgH \in K\backslash G/H} \text{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \text{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$.

Remark 4.15 (Existence of Mackey extensions)**.** In every case we will consider such a Mackey extension does exist and is given by an actual restriction. For instance for $H_0^?(-;\mathbf{K}^{\text{top}})$ induction is the functor complex representation ring $R_{\mathbb{C}}$ with respect to induction of representations. The restriction part is given by the restriction of representations.

We need some notation. Consider a subgroup $H \subseteq G$. Denote by C_GH the centralizer and by N_GH the normalizer of $H \subseteq G$. Put

$$W_GH := N_GH/H \cdot C_GH.$$

This is always a finite group. Define for an equivariant homology theory $\mathcal{H}_*^?$

$$S_H\left(\mathcal{H}_q^H(*)\right) := \text{cok}\left(\bigoplus_{\substack{K\subset H\\K\neq H}} \text{ind}_K^H : \bigoplus_{\substack{K\subset H\\K\neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right).$$

Theorem 4.16 (Equivariant Chern character, Lück (2002))**.** *Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}_*^?$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.*

Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}_n^G \colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H;\Lambda) \otimes_{\Lambda[W_GH]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X).$$

Actually $\mathrm{ch}_*^?$ is an equivalence of equivariant homology theories.

Proof. See [120, Theorem 0.2] □

Recall the following basic result from complex representation theory of finite groups.

Theorem 4.17 (Artin's Theorem)**.** *Let G be finite. Then the map*

$$\bigoplus_{C\subset G} \mathrm{ind}_C^G \colon \bigoplus_{C\subset G} R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(G)$$

is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of G.

Proof. See for instance [167, Theorem 17 in 9.2 on page 70]. □

Let C be a finite cyclic group. The Artin defect is the cokernel of the map

$$\bigoplus_{D\subset C, D\neq C} \mathrm{ind}_D^C \colon \bigoplus_{D\subset C, D\neq C} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C).$$

For an appropriate idempotent $\theta_C \in R_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$ the Artin defect is after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

by [120, Lemma 7.4].

Example 4.18 (An improvement of Artin's Theorem)**.** Let $K_*^G = H_*^?(-;\mathbf{K}^{\mathrm{top}})$ be equivariant topological K-theory. We get for a finite subgroup $H \subseteq G$

$$K_n^G(G/H) = K_n^H(\{\bullet\}) = \begin{cases} R_{\mathbb{C}}(H) & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

Hence $S_H\left(K_q^H(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, if H is not cyclic and q is even or if q is odd, and we have $S_C\left(K_q^C(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$, if C is finite cyclic and q is even.

Let G be finite, $X = \{*\}$ and $\mathcal{H}_*^? = K_*^?$. In this very special case Theorem 4.16 yields already something new, namely, an improvement of Artin's theorem, i.e., the equivariant Chern character induces an isomorphism

$$\mathrm{ch}_0^G(\{\bullet\}) \colon \bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right] \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right]$$

where (C) runs over the conjugacy classes of finite cyclic subgroups. (Theorem 4.16 yields only a statement after applying $-\otimes_{\mathbb{Z}} \mathbb{Q}$ but the statement above, where we only invert the order of the group G is proved in [122, Theorem 0.7]).

Theorem 4.19 (Rational computation of $K_*^G(\underline{E}G)$)**.** *For every group G and every $n \in \mathbb{Z}$ we obtain an isomorphism*

$$\bigoplus_{(C)} \bigoplus_{k} H_{p+2k}(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proof. This follows from Theorem 4.16 applied to the case $X = \underline{E}G$ and $\mathcal{H}_*^? = K_*^?$ using the following facts.

- $\underline{E}G^C$ is a contractible proper C_GC- space. Hence the canonical map $BC_GC \to C_GC\backslash\underline{E}G^C$ induces an isomorphism

$$H_p(BC_GC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_p(C_GC\backslash\underline{E}G^C) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- $S_H\left(K_q^H(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if H is not cyclic and q is even or if q is odd.
- $S_C\left(K_q^C(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ if C is finite cyclic and q is even.

□

Remark 4.20 (Rational computation of $K_*(C_r^*(G))$)**.** If the Baum-Connes Conjecture holds for G, Theorem 4.19 yields an isomorphism

$$\bigoplus_{(C)}\bigoplus_{k} H_{p+2k}(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Next we introduce some notation. For a prime p denote by $r(p) = |\operatorname{con}_p(G)|$ the number of conjugacy classes (g) of elements $g \neq 1$ in G of p-power order. Let $\mathbb{I}_G$ is the augmentation ideal of $R_{\mathbb{C}}(G)$. Denote by $\mathbb{I}_p(G)$ the image of the restriction homomorphism $\mathbb{I}(G) \to \mathbb{I}(G_p)$ for the inclusion of the p-Sylow subgroup $G_p \to G$.

Theorem 4.21 (Completion Theorem, Atiyah-Segal (1969))**.** *Let G be a finite group. Then there are isomorphisms of abelian groups*

$$\begin{aligned} K^0(BG) &\cong R_{\mathbb{C}}(G)\widehat{_{\mathbb{I}_G}} \\ &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}_p} \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\widehat{\mathbb{Z}_p})^{r(p)}; \\ K^1(BG) &\cong 0. \end{aligned}$$

Proof. See [7] and for the explicit formula for instance [96, page 125] or [124, Theorem 3.5]. □

Theorem 4.22 (Lück (2005))**.** *Let G be a discrete group. Denote by $K^*(BG)$ the topological (complex) K-theory of its classifying space BG. Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions.*

Then there is a $\mathbb{Q}$-isomorphism

$$\overline{\operatorname{ch}}_G^n\colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g)\in\operatorname{con}_p(G)} \left(\prod_{i\in\mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right).$$

Proof. See [124]. □

Remark 4.23 (Multiplicative structure)**.** The multiplicative structure is also determined in [124].

Remark 4.24 (Finiteness condition about $\underline{E}G$)**.** We have presented in the previous section many groups for which a cocompact G-CW-model for $\underline{E}G$ exists, e.g., hyperbolic groups. Notice that this condition appears in Theorem 4.22 although the conclusion in Theorem 4.22 is about BG and not about $\underline{E}G$ or $G\backslash\underline{E}G$.

Example 4.25 ($SL_3(\mathbb{Z})$)**.** It is well-known that its rational cohomology satisfies $\widetilde{H}^n(BSL_3(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \in \mathbb{Z}$. Actually, by a result of Soulé [170, Corollary on page 8] the quotient space $SL_3(\mathbb{Z})\backslash\underline{E}SL_3(\mathbb{Z})$ is contractible and compact. From the classification of finite subgroups of $SL_3(\mathbb{Z})$ we see that $SL_3(\mathbb{Z})$ contains up to conjugacy two elements of order 2, two elements of order 4 and two elements of order 3 and no further conjugacy classes of non-trivial elements of prime power order. The rational homology of each of the centralizers of elements in $\mathrm{con}_2(G)$ and $\mathrm{con}_3(G)$ agrees with the one of the trivial group. Hence we get

$$\begin{array}{lcl} K^0(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} & \cong & \mathbb{Q} \times (\widehat{\mathbb{Q}_2})^4 \times (\widehat{\mathbb{Q}_3})^2; \\ K^1(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} & \cong & 0. \end{array}$$

The identification of $K^0(BSL_3(\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q}$ above is compatible with the multiplicative structures.

Actually the computation using Brown-Petersen cohomology and the Conner-Floyd relation by Tezuka-Yagita [178] gives the integral computation

$$\begin{array}{lcl} K^0(BSL_3(\mathbb{Z})) & \cong & \mathbb{Z} \times (\widehat{\mathbb{Z}_2})^4 \times (\widehat{\mathbb{Z}_3})^2; \\ K^1(BSL_3(\mathbb{Z})) & \cong & 0. \end{array}$$

Soulé [170] has computed the integral cohomology of $SL_3(\mathbb{Z})$.

Let G be a discrete group. Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following conditions about G:

(M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

(NM) If $M \in \mathcal{MFIN}, M \neq \{1\}$, then $N_G M = M$.

Example 4.26 (Groups satisfying (M) and (NM))**.** By Davis-Lück [53, page 101-102] the following groups satisfy conditions (M) and (NM):

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
- Fuchsian groups;
- One-relator groups G.

For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let

$$\{(M_i) \mid i \in I\}$$

be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$. By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$. Define the G-CW-complex X as the G-pushout

(4.27)
$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle f_1} \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & X \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

Theorem 4.28 (Model for $\underline{E}G$ for groups satisfying (M) and (NM))**.** *Let G be a group satisfying conditions (M) and (NM). Then the G-CW-complex X defined by the G-pushout* (4.27) *is a model for $\underline{E}G$.*

Proof. The isotropy groups of X are all finite. We have to show for $H \subseteq G$ finite that X^H contractible. We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$. We get

$$\left(\coprod_{i \in I} G/M_i\right)^H = (G/M_{i_0})^H = \{\bullet\}.$$

Hence $X^H = \{\bullet\}$. It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence X is contractible. □

Example 4.29 (The homology of groups satisfying (M) and (NM))**.** Let G be a group satisfying conditions (M) and (NM). Because of Theorem 4.28 we obtain the following pushout by taking the G-quotient of the G-pushout (4.27)

$$\begin{array}{ccc} \coprod_{i\in I} BM_i & \longrightarrow & BG \\ \big\downarrow & & \big\downarrow \\ \coprod_{i\in I}\{\bullet\} & \longrightarrow & G\backslash \underline{E}G \end{array}$$

The associated long exact Mayer-Vietoris sequence yields

$$\cdots \to \widetilde{H}_{n+1}(G\backslash \underline{E}G) \to \bigoplus_{i\in I} \widetilde{H}_n(BM_i) \to \widetilde{H}_n(BG) \to \widetilde{H}_n(G\backslash \underline{E}G) \to \cdots .$$

In particular we obtain an isomorphism for $n \ge \dim(\underline{E}G) + 2$

$$\bigoplus_{i\in I} H_n(BM_i) \xrightarrow{\cong} H_n(BG).$$

So we get an explicit computation of $H_n(BG)$ for large n and it is obvious why it is useful to have models for $\underline{E}G$ of as small as possible dimension. Computations for low values of n can sometimes be carried out by spectral sequence arguments or specific arguments.

The following identifications follow from the definition of the *Whitehead groups* $\mathrm{Wh}_n(G)$ for $n \ge 0$ due to Waldhausen [184, Definition 15.6 on page 228 and Proposition 15.7 on page 229] which also makes sense for all $n \in \mathbb{Z}$ if we use the non-connective K-theory spectrum

$$\begin{aligned} \mathrm{Wh}(G) &= \mathrm{Wh}_1(G); \\ \widetilde{K}_0(\mathbb{Z}G) &= \mathrm{Wh}_0(G); \\ K_n(\mathbb{Z}G) &= \mathrm{Wh}_n(G) \qquad \text{for } n \le -1. \end{aligned}$$

For a finite group H define $\widetilde{R}_{\mathbb{C}}(H)$ as the kernel of the ring homomorphism $R_{\mathbb{C}}(H) \to \mathbb{Z}$ sending $[V]$ to $\dim_{\mathbb{C}}(V)$.

Theorem 4.30 (Davis-Lück (2003))**.** *Let G be a discrete group which satisfies the conditions (M) and (NM) above.*

(1) Then there is an isomorphism

$$K_1^G(\underline{E}G) \xrightarrow{\cong} K_1(G\backslash \underline{E}G),$$

and a short exact sequence

$$0 \to \bigoplus_{i\in I} \widetilde{R}_{\mathbb{C}}(M_i) \to K_0(\underline{E}G) \to K_0(G\backslash \underline{E}G) \to 0.$$

(2) The short exact sequence above splits if we invert the orders of all finite subgroups of G;

(3) Suppose that G belongs to $\mathcal{BC}$. (This is the case for the groups appearing in Example 4.26). Then

$$K_n(C_r^*(G)) \cong K_n^G(\underline{E}G);$$

(4) *Suppose that G belongs to $\mathcal{FJ}_K(\mathbb{Z})$. Then there is for $n \in \mathbb{Z}$ an isomorphism of Whitehead groups*

$$\bigoplus_{i \in I} \mathrm{Wh}_n(M_i) \xrightarrow{\cong} \mathrm{Wh}_n(G),$$

where $\mathrm{Wh}_n(M_i) \to \mathrm{Wh}_n(G)$ is induced by the inclusion $M_i \to G$.

Proof. See [53, Theorem 5.1]. □

Remark 4.31 (Small models for $\underline{E}G$ and computations)**.** We see that for computations of group homology or of K- and L-groups of group rings and group C^*-algebras it is important to understand the spaces $G\backslash\underline{E}G$. Often geometric input ensures that $G\backslash\underline{E}G$ is a fairly small CW-complex.

On the other hand recall from Theorem 3.43 that for any CW-complex X there exists a group G with $X \simeq G\backslash\underline{E}G$.

Question 4.32 (Consequences)**.** What are the consequences of the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

5. The Isomorphism Conjectures for arbitrary groups

The outline of this section is:

- We discuss the difference between the families $\mathcal{FIN}$ and $\mathcal{VCYC}$.
- We discuss consequences of the Farrell-Jones and the Baum-Connes Conjecture.

Throughout this section G will always be a discrete group. We have introduced the following notions and conjectures:

- Families of subgroups, e.g., the families $\mathcal{FIN}$ and $\mathcal{VCYC}$ of finite and virtually cyclic subgroups (see Definition 3.6);
- Classifying G-CW-complex $E_{\mathcal{F}}(G)$ for a family of subgroups $\mathcal{F}$ (see Definition 3.7).
- Equivariant homology theories $\mathcal{H}_*^?$ (see Definition 4.2);
- Specific examples of equivariant homology theories associated to K- and L-theory (see Example 4.8)

$$H_*^?(-;\mathbf{K}_R);$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle});$$
$$H_*^?(-;\mathbf{K}^{\mathrm{top}}),$$

  satisfying for $H \subseteq G$

$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\{\bullet\};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH); \\ H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\{\bullet\};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_r^*(H)); \end{array}$$

- The Farrell-Jones Conjecture for algebraic K-theory which predicts the bijectivity of the assembly map induced by the projection $E_{\mathcal{VCYC}}(G) \to \{\bullet\}$

$$H_n^G(E_{\mathcal{VCYC}}(G),\mathbf{K}_R) \to H_n^G(\{\bullet\},\mathbf{K}_R) = K_n(RG)$$

  for all $n \in \mathbb{Z}$ (see Conjecture 4.9);
- The Farrell-Jones Conjecture for algebraic L-theory which predicts the bijectivity of the assembly map induced by the projection $E_{\mathcal{VCYC}}(G) \to \{\bullet\}$

$$H_n^G(E_{\mathcal{VCYC}}(G),\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(\{\bullet\},\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG)$$

  for all $n \in \mathbb{Z}$ (see Conjecture 4.10);

- The Baum-Cones Conjecture which predicts the bijectivity of the assembly map induced by the projection $E_{\mathcal{FIN}}(G) = \underline{E}G \to \{\bullet\}$
$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(\{\bullet\}, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$
for all $n \in \mathbb{Z}$ (see Conjecture 4.11).

Remark 5.1 (The Isomorphism Conjectures interpreted as induction theorems)**.** These Conjecture can be thought of a kind of generalized induction theorem. They allow to compute the value of a functor such as $K_n(RG)$ on G in terms of its values $K_m(RH)$ for all $m \leq n$ and all virtually cyclic subgroups subgroups H of G.

Next we want to investigate, whether one can pass to smaller or larger families in the formulations of the Conjectures. The point is to find the family as small as possible.

Theorem 5.2 (Transitivity Principle)**.** *Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Assume that for every element $H \in \mathcal{G}$ and $n \in \mathbb{Z}$ the assembly map*
$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(\{\bullet\})$$
is bijective, where $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.

Then the relative assembly map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$
$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$
is bijective for all $n \in \mathbb{Z}$.

Proof. See [12, Theorem 1.4]. □

Example 5.3 (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Baum-Connes Conjecture)**.** The Baum-Connes Conjecture 4.11 is known to be true for virtually cyclic groups. The Transitivity Principle 5.2 implies that the relative assembly
$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(E_{\mathcal{VCYC}}(G))$$
is bijective for all $n \in \mathbb{Z}$.

Hence it does not matter in the context of the Baum-Connes Conjecture whether we consider the family $\mathcal{FIN}$ or $\mathcal{VCYC}$.

Example 5.4 (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Farrell-Jones Conjecture)**.** In general the relative assembly maps
$$\begin{aligned} H_n^G(\underline{E}G; \mathbf{K}_R) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}), \end{aligned}$$
are not bijective. Hence in the Farrell-Jones setting one has to pass to $\mathcal{VCYC}$ and cannot use the easier to handle family $\mathcal{FIN}$.

Example 5.5 (The Farrell-Jones Conjecture for algebraic K-theory for the group $\mathbb{Z}$)**.** The Farrell-Jones Conjecture 4.9 for algebraic K-theory for the group $\mathbb{Z}$ is true for trivial reasons since $\mathbb{Z}$ is virtually cyclic and hence the projection $E_{\mathcal{VCYC}}(\mathbb{Z}) \to \{\bullet\}$ is a homotopy equivalence.

Example 5.6 (The Farrell-Jones Conjecture for algebraic K-theory for the group $\mathbb{Z}$ and the family $\mathcal{FIN}$)**.** One may wonder what happens if we insert the family of finite subgroups, i.e., whether the map induced by the projection $E_{\mathcal{FIN}}(\mathbb{Z}) = \underline{E}\mathbb{Z} \to \{\bullet\}$

(5.7) $$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}, \mathbf{K}_R) \to H_n^{\mathbb{Z}}(\{\bullet\}, \mathbf{K}_R) = K_n(R[\mathbb{Z}])$$

is bijective. Since $\mathbb{Z}$ is torsionfree, $\underline{E}\mathbb{Z}$ is the same as $E\mathbb{Z}$ and the induction structure yields an isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}, \mathbf{K}_R) = H_n(BZ, \mathbf{K}_R) = H_n(S^1, \mathbf{K}_R) = K_n(R[\mathbb{Z}]) \oplus K_{n-1}(R[\mathbb{Z}]).$$

Hence the map (5.7) can be identified with the map

$$K_n(R) \oplus K_{n-1}(R) \to K_n(R[\mathbb{Z}]).$$

However, by the Bass-Heller Swan decomposition we have the isomorphism

$$K_n(R) \oplus K_{n-1}(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)) \xrightarrow{\cong} K_n(R[t, t^{-1}]) \cong K_n(R[\mathbb{Z}]).$$

Hence the map (5.7) is bijective if and only if $\mathrm{NK}_n(R) = 0$. We have $\mathrm{NK}_n(R) = 0$ under the assumption that R is regular. This is the reason why we have required R to be regular in the version of the Farrell-Jones Conjecture for torsionfree groups 2.16.

Definition 5.8 (Types of virtually cyclic groups)**.** An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II if and only if admits an epimorphism onto D_∞. Let $\mathcal{VCYC}_I$ be the family of virtually cyclic subgroups which are either finite or of type I.

An infinite virtually cyclic group is either of type I or of type II. An infinite subgroups of a virtually cyclic subgroup of type I is again of type I.

Theorem 5.9 (Lück (2004), Quinn (2007), Reich (2007))**.** *The following maps are bijective for all $n \in \mathbb{Z}$*

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCYC}_I}(G); \mathbf{K}_R) & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_n^G(E_{\mathcal{VCYC}_I}(G); \mathbf{L}_R^{\langle -\infty \rangle}). \end{array}$$

Proof. See [123, Lemma 4.2] and [153]. □

Theorem 5.10 (Cappell (1973), Grunewald (2005), Waldhausen (1978))**.**

(1) The following maps are bijective for all $n \in \mathbb{Z}$.

$$\begin{array}{rcl} H_n^G(\underline{E}G; \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}; \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right] & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right]; \end{array}$$

(2) If R is regular and $\mathbb{Q} \subseteq R$, then for all $n \in \mathbb{Z}$ we get a bijection

$$H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R).$$

Proof. See [40], [82, Theorem 5.6], [126, Proposition 2.6 on page 686, Proposition 2.9 and Proposition 2.10 on page 688]. □

Theorem 5.11 (Bartels (2003))**.** *For every $n \in \mathbb{Z}$ the two maps*

$$\begin{array}{rcl} H_n^G(\underline{E}G; \mathbf{K}_R) & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}), \end{array}$$

are split injective.

Proof. See [18]. □

Hence we get (natural) isomorphisms

$$H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R) \cong H_n^G(\underline{E}G; \mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VCYC}}(G), \underline{E}G; \mathbf{K}_R); \tag{5.12}$$

and

$$H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \cong H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus H_n^G(E_{\mathcal{VCYC}}(G), \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}).$$

The analysis of the terms $H_n^G(E_{\mathcal{VCYC}}(G), \underline{E}G; \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCYC}}(G), \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle})$ boils down to investigating Nil-terms and UNil-terms in the sense of Waldhausen and Cappell. The analysis of the terms $H_n^G(\underline{E}G; \mathbf{K}_R)$ and $H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle})$ is using the methods of the previous lecture (e.g., equivariant Chern characters).

Remark 5.13 (Relating the torsionfree versions to the general versions)**.** Obviously the general version of the Baum-Connes Conjecture 4.11 reduces in the torsionfree case to the version 2.18 since for torsionfree G we have $EG = \underline{E}G$.

The general version of the Farrell-Jones Conjecture 4.9 for K-theory reduces in the torsionfree case to the version 2.16 because of the Transitivity Principal 5.2 since a torsionfree virtually cyclic group is isomorphic to $\mathbb{Z}$ and for a regular ring R the Bass-Heller-Swan decomposition shows that the map $H_n^{IZ}(\underline{E}\mathbb{Z}; \mathbf{K}_R) \to K_n(R[\mathbb{Z}])$ is bijective (as explained in Example 5.6).

The general version of the Farrell-Jones Conjecture 4.10 for L-theory reduces in the torsionfree case to the version 2.17 because of the Transitivity Principal 5.2 since a torsionfree virtually cyclic group is isomorphic to $\mathbb{Z}$ and the map $H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R[\mathbb{Z}])$ is bijective by Theorem 5.9

Next we explain in the case $G = SL_2(\mathbb{Z})$ how computations are made possible by the Farrell-Jones and the Baum-Connes Conjecture.

Example 5.14 (K-theory of $C_r^*(SL_2(\mathbb{Z}))$ and of $\mathbb{Z}[SL_2(\mathbb{Z})]$)**.** From Example 3.33 we obtain a $SL_2(\mathbb{Z})$-pushout

$$\begin{array}{ccc} SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times \{-1,1\} & \xrightarrow{F_{-1} \coprod F_1} & SL_2(\mathbb{Z})/(\mathbb{Z}/4) \coprod SL_2(\mathbb{Z})/(\mathbb{Z}/6) \\ \downarrow & & \downarrow \\ SL_2(\mathbb{Z})/(\mathbb{Z}/2) \times [-1,1] & \longrightarrow & T = \underline{E}SL_2(\mathbb{Z}) \end{array}$$

Let $\mathcal{H}_*^?$ be an equivariant homology theory. Then the Mayer-Vietoris sequence applied to the $SL_2(\mathbb{Z})$-pushout above together with the induction structure yields a long exact sequence

$$(5.15) \quad \cdots \to \mathcal{H}_n^{\mathbb{Z}/2}(\{\bullet\}) \to \mathcal{H}_n^{\mathbb{Z}/4}(\{\bullet\}) \oplus \mathcal{H}_n^{\mathbb{Z}/6}(\{\bullet\}) \to \mathcal{H}_n^{SL_2(\mathbb{Z})}(\underline{E}SL_2(\mathbb{Z})) \\ \to \mathcal{H}_{n-1}^{\mathbb{Z}/2}(\{\bullet\}) \to \mathcal{H}_{n-1}^{\mathbb{Z}/4}(\{\bullet\}) \oplus \mathcal{H}_{n-1}^{\mathbb{Z}/6}(\{\bullet\}) \to \cdots .$$

The Baum-Connes Conjecture 4.11 is known to be true for $SL_2(\mathbb{Z})$ (see for instance [89]). Hence in the case, where $\mathcal{H}_*^?$ is equivariant topological K-theory, the long exact sequence (5.15) reduces to the exact sequences

$$0 \to K_1(C_r^*(SL_2(\mathbb{Z}))) \to R_{\mathbb{C}}(\mathbb{Z}/2) \to R_{\mathbb{C}}(\mathbb{Z}/4) \oplus R_{\mathbb{C}}(\mathbb{Z}/6) \\ \to K_0(C_r^*(SL_2(\mathbb{Z}))) \to 0,$$

where the map between the representation rings are induced by the obvious inclusions of groups. Since the inclusion $\mathbb{Z}/2 \to \mathbb{Z}/6$ is split injective and $R_{\mathbb{C}}(\mathbb{Z}/2) \cong \mathbb{Z}^2$, $R_{\mathbb{C}}(\mathbb{Z}/4) \cong \mathbb{Z}^4$ and $R_{\mathbb{C}}(\mathbb{Z}/6) \cong \mathbb{Z}^6$, we conclude

$$K_n(C_r^*(SL_2(\mathbb{Z}))) \cong \begin{cases} \mathbb{Z}^8 & n \text{ even;} \\ 0 & n \text{ odd.} \end{cases}$$

The Farrell-Jones Conjecture for K-theory is known to be true for $SL_2(\mathbb{Z})$ for any coefficient ring R by [14] since it contains a finitely generated free subgroup of finite index and is hence a hyperbolic group. Because of Theorem 5.9 and Theorem 5.11

we obtain an isomorphism

$$(5.16)\quad K_n(R[SL_2(\mathbb{Z})]) \cong H_n^{SL_2(\mathbb{Z})}(E_{\mathcal{FIN}}(SL_2(\mathbb{Z}));\mathbf{K}_R) \oplus H_n^{SL_2(\mathbb{Z})}(E_{\mathcal{VCY}_I}(SL_2(\mathbb{Z})),E_{\mathcal{FIN}}(SL_2(\mathbb{Z}));\mathbf{K}_R).$$

Let $V \subseteq SL_2(\mathbb{Z})$ be a virtually cyclic subgroup of type I, i.e., there is an exact sequence $1\colon F \to V \to \mathbb{Z} \to 1$ for a finite subgroup $F \subseteq V$. Since $SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$, F is conjugated to $\mathbb{Z}/4$, $\mathbb{Z}/6$ or the subgroup $\mathbb{Z}/2$. Since the normalizers of $\mathbb{Z}/6$ and $\mathbb{Z}/4$ are finite, F must be subconjugated to $\mathbb{Z}/2$. Since $\mathbb{Z}/2$ is the center of $SL_2(\mathbb{Z})$, the group V is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}$. The group $\mathrm{NK}_n(\mathbb{Z}[\mathbb{Z}/2])$ vanishes for $n \le 1$. Using the Bass-Heller-Swan decomposition we see that $H_n^V(E_{\mathcal{VCY}_I}(V),E_{\mathcal{FIN}}(V);\mathbf{K}_{\mathbb{Z}}) = 0$ for $n \le 1$. An obvious modification of the Transitivity Principal 5.2 (see [12, Theorem 1.4]) implies that for $n \le 1$ the group $H_n^{SL_2(\mathbb{Z})}(E_{\mathcal{VCY}_I}(SL_2(\mathbb{Z})),E_{\mathcal{FIN}}(SL_2(\mathbb{Z}));\mathbf{K}_{\mathbb{Z}})$ vanishes. Thus from (5.16) we obtain an isomorphism for $n \le 1$.

$$K_n(\mathbb{Z}[SL_2(\mathbb{Z})]) \cong H_n^{SL_2(\mathbb{Z})}(E_{\mathcal{FIN}}(SL_2(\mathbb{Z}));\mathbf{K}_{\mathbb{Z}})$$

Hence the long exact sequence (5.15) yields the long exact sequence

$$\begin{aligned} K_1(\mathbb{Z}[\mathbb{Z}/2]) &\to K_1(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/6]) \to K_1(\mathbb{Z}[SL_2(\mathbb{Z})]) \to K_0(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to K_0(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/6]) \to K_0(\mathbb{Z}[SL_2(\mathbb{Z})]) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to K_{-1}(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to K_{-1}(\mathbb{Z}[SL_2(\mathbb{Z})]) \to K_{-2}(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to K_{-2}(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_{-2}(\mathbb{Z}[\mathbb{Z}/6]) \to K_{-2}(\mathbb{Z}[SL_2(\mathbb{Z})]) \to \cdots \end{aligned}$$

It induces the long exact sequence

$$\begin{aligned} \mathrm{Wh}(\mathbb{Z}/2) &\to \mathrm{Wh}(\mathbb{Z}/4) \oplus \mathrm{Wh}(\mathbb{Z}/6) \to \mathrm{Wh}(SL_2(\mathbb{Z})) \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/4]) \oplus \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) \to \widetilde{K}_0(\mathbb{Z}[SL_2(\mathbb{Z})]) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to K_{-1}(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to K_{-1}(\mathbb{Z}[SL_2(\mathbb{Z})]) \to K_{-2}(\mathbb{Z}[\mathbb{Z}/2]) \\ &\to K_{-2}(\mathbb{Z}[\mathbb{Z}/4]) \oplus K_{-2}(\mathbb{Z}[\mathbb{Z}/6]) \to K_{-2}(\mathbb{Z}[SL_2(\mathbb{Z})]) \to \cdots \end{aligned}$$

The groups $\mathrm{Wh}(\mathbb{Z}/2)$, $\mathrm{Wh}(\mathbb{Z}/4)$, $\mathrm{Wh}(\mathbb{Z}/6)$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2])$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/4])$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6])$, $\widetilde{K}_{-1}(\mathbb{Z}[\mathbb{Z}/2])$, $\widetilde{K}_{-1}(\mathbb{Z}[\mathbb{Z}/4])$ vanish, whereas $\widetilde{K}_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$ (see Bass [19, Theorem 10.6 on page 695], Carter [44], Cassou-Noguès [45], Curtis-Rainer [50, Corollary 50.17 on page 253]), Oliver [138, Theorem 14.1 on page 328]. The groups $K_n(\mathbb{Z}[H])$ vanish for all $n \ge -2$ and all finite groups H (see Carter [44]). Hence we get

$$\begin{array}{lcll} \mathrm{Wh}(SL_2(\mathbb{Z})) & \cong & 0; & \\ \widetilde{K}_0(\mathbb{Z}[SL_2(\mathbb{Z})]) & \cong & 0; & \\ K_{-1}(\mathbb{Z}[SL_2(\mathbb{Z})]) & \cong & \mathbb{Z}: & \\ K_n(\mathbb{Z}[SL_2(\mathbb{Z})]) & \cong & 0 & \text{for } n \le -2. \end{array}$$

Next we show that the Farrell-Jones Conjecture and the Baum-Conjecture imply certain other well-known conjectures.

Conjecture 5.17 (Novikov Conjecture)**.** *The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature*

$$\mathrm{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_0 \to M_1$ with $f_1 \circ g \simeq f_2$ we have

$$\mathrm{sign}_x(M_0,f_0) = \mathrm{sign}_x(M_1,f_1).$$

Theorem 5.18 (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)**.** *Suppose that one of the following assembly maps*

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_n^G(\{\bullet\}, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG); \\ K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) & \to & H_n^G(\{\bullet\}, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)), \end{array}$$

is rationally injective.

Then the Novikov Conjecture holds for the group G.

Proof. See for instance [126, Proposition 3.1 and Proposition 3.5 on page 699] and [152, Proposition 6.6 on page 300]. □

For more information about the Novikov Conjecture we refer for instance to [42], [43], [51], [72], [77], [105], [151] and [155].

Theorem 5.19 (Induction from finite subgroups, Bartels-Lück-Reich (2007))**.**

(1) *Let R be a regular ring such that the order of any finite subgroup of G is invertible in R. Suppose $G \in \mathcal{FJ}_K(R)$. Then the map given by induction from finite subgroups of G*

$$\operatorname{colim}_{\mathsf{Or}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

is bijective;

(2) *Let F be a field of characteristic p for a prime number p. Suppose that $G \in \mathcal{FJ}_K(F)$. Then the map*

$$\operatorname{colim}_{\mathsf{Or}_{\mathcal{FIN}}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

is bijective;

(3) *If $G \in \mathcal{F}_K(\mathbb{Z})$, then the canonical map*

$$\operatorname{colim}_{\mathsf{Or}_{\mathcal{FIN}}(G)} K_{-1}(\mathbb{Z}H) \to K_{-1}(\mathbb{Z}G)$$

is bijective;

(4) *If $G \in \mathcal{F}_K(\mathbb{Z})$, then*

$$K_n(\mathbb{Z}G) = 0 \text{ for } n \leq -2.$$

Proof. See Bartels-Lück-Reich [15, Theorem 0.5], [66, 1.65 on page 260], and Lück-Reich [126, Section 3.1.1 on page 690]. □

Theorem 5.20 (Permutation Modules, Bartels-Lück-Reich (2007))**.** *Suppose that $G \in \mathcal{FJ}_K(\mathbb{Q})$. Then for every finitely generated projective $\mathbb{Q}[G]$-module P there exists integers $k \geq 1$ and $l \geq 0$ and finitely many finite subgroups H_1, H_2, ..., H_r such that*

$$P^k \oplus \mathbb{Q}[G]^l \cong_{\mathbb{Q}[G]} \mathbb{Q}[G/H_1] \oplus \mathbb{Q}[G/H_2] \oplus \cdots \oplus \mathbb{Q}[G/H_r].$$

Proof. Because of [15, Lemma 4.3 and Lemma 4.4] it suffices to prove the claim in the case, where G is finite cyclic. This special case follows from Segal [165]. □

Next we introduce some notation. R be commutative ring and let G be a group. Let $\mathrm{class}(G,R)$ be the R-module of class functions $G \to R$, i.e., functions $G \to R$ which are constant on conjugacy classes. Let $\mathrm{tr}_{RG} \colon RG \to \mathrm{class}(G,R)$ be the R-homomorphism which sends $g \in G$ to the class function which takes the value one on the conjugacy class of g and the value zero otherwise. It extends to a map

$$\mathrm{tr}_{RG} \colon M_n(RG) \to \mathrm{class}(G,R)$$

by taking the sums of the values of the diagonal entries.

Let P be a finitely generated RG-module. Choose a finitely generated projective RG-module Q and an isomorphism $\phi\colon RG^n \xrightarrow{\cong} P \oplus Q$. Let $A \in M_n(RG)$ be a matrix such that $\phi^{-1} \circ (\mathrm{id}_P \oplus 0) \circ \phi\colon RG^n \to RG^n$ is given by A.

Definition 5.21 (Hattori-Stallings rank)**.** Define the Hattori-Stallings rank of P to be the class function

$$\mathrm{HS}_{RG}(P) := \mathrm{tr}_{RG}(A).$$

This definition is independent of the choice of Q and ϕ. Let G be a finite group and let F be a field of characteristic zero. Then a finitely generated RG-module P is the same as a finite dimensional G-representation over F and the Hattori-Stallings rank can be identified with the character of the G-representation (see (5.27)).

Conjecture 5.22 (Bass Conjecture)**.** *Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.*

Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

If G is finite, the Bass Conjecture 5.22 reduces to a theorem of Swan (1960) (see [175, Theorem 8.1], [21, Corollary 4.2]).

The next results follows from the argument in[70, Section 5].

Theorem 5.23 (Linnell-Farrell (2003))**.** *Let G be a group. Suppose that*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic. (This is true if $G \in \mathcal{FJ}_K(F)$ for every field F of prime characteristic).

Then the Bass Conjecture is satisfied for every integral domain R.

Remark 5.24 (Geometric interpretation of the Bass Conjecture)**.** The Bass Conjecture 5.22 can be interpreted topologically. Namely, the Bass Conjecture 5.22 is true for a finitely presented group G in the case $R = \mathbb{Z}$ if and only if every homotopy idempotent selfmap of an oriented smooth closed manifold whose dimension is greater than 2 and whose fundamental group is isomorphic to G is homotopic to a selfmap which has precisely one fixed point (see Berrick-Chatterji-Mislin [28]).

The Bass Conjecture 5.22 for G in the case $R = \mathbb{Z}$ (or $R = \mathbb{C}$) also implies for a finitely dominated CW-complex with fundamental group G that its Euler characteristic agrees with the L^2-Euler characteristic of its universal covering (see Eckmann [59]).

Next we present another version of the Bass Conjecture. Let F be a field of characteristic zero. Fix an integer $m \geq 1$. Let $F(\zeta_m) \supset F$ be the Galois extension given by adjoining the primitive m-th root of unity ζ_m to F. Denote by $\Gamma(m, F)$ the Galois group of this extension of fields, i.e., the group of automorphisms $\sigma\colon F(\zeta_m) \to F(\zeta_m)$ which induce the identity on F. It can be identified with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds. Let g_1 and g_2 be two elements of G of finite order. We call them *F-conjugate* if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$ there exists an element σ in the Galois group $\Gamma(m, F)$ with the property that $g_1^{u(\sigma)}$ and g_2 are conjugate. Two elements g_1 and g_2 are F-conjugate for $F = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ respectively if the cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are conjugate, if g_1 and g_2 or g_1 and g_2^{-1} are conjugate, or if g_1 and g_2 are conjugate respectively.

Denote by $\mathrm{con}_F(G)_f$ the set of F-conjugacy classes $(g)_F$ of elements $g \in G$ of finite order. Let $\mathrm{class}_F(G)_f$ be the F-vector space with the set $\mathrm{con}_F(G)_f$ as basis, or, equivalently, the F-vector space of functions $\mathrm{con}_F(G)_f \to F$ with finite support.

Conjecture 5.25 (Bass Conjecture for fields of characteristic zero as coefficients)**.** *Let F be a field of characteristic zero and let G be a group. The Hattori-Stallings (see Definition 5.21) induces an isomorphism*

$$\mathrm{HS}_{FG} \colon K_0(FG) \otimes_{\mathbb{Z}} F \to \mathrm{class}_F(G)_f.$$

Lemma 5.26. *Suppose that F is a field of characteristic zero and G is a finite group. Then Conjecture 5.25 is true.*

Proof. Since G is finite, an FG-module is a finitely generated projective FG-module if and only if it is a (finite-dimensional) G-representation with coefficients in F and $K_0(FG)$ is the same as the representation ring $R_F(G)$. The Hattori-Stallings rank $\mathrm{HS}_{FH}(V)$ and the character χ_V of a G-representation V with coefficients in F are related by the formula

$$\chi_V(g) \quad = \quad |Z_G\langle g\rangle| \cdot \mathrm{HS}_{FG}(V)(g) \tag{5.27}$$

for $g \in G$, where $Z_G\langle g \rangle$ is the centralizer of g in G. Hence Lemma 5.26 follows from representation theory, see for instance [169, Corollary 1 on page 96]. □

Here is a conjecture related to the Bass Conjecture

Conjecture 5.28. *Let R be an integral domain with quotient field F. Suppose that no prime divisor of the order of a finite subgroup of G is a unit in R. Then the change of rings homomorphism*

$$K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

factorizes as

$$K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Theorem 5.29 (Bartels-Lück-Reich (2007))**.** *Let R be an integral domain with quotient field F. Suppose that no prime divisor of the order of a finite subgroup of G is a unit in R. Suppose that G belongs to $\mathcal{F}_K(R)$.*

Then Conjecture 5.28 is true for G and R.

Proof. See [15, Theorem 0.10]. □

More information and further references about the Bass Conjecture can be found for instance in [20], [27, Section 7], [36], [58], [59], [70], [114] [121, Subsection 9.5.2], and [136, page 66ff].

Conjecture 5.30 (Vanishing of Bass-Nil-groups)**.** *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups G and all $n \in \mathbb{Z}$ that*

$$\mathrm{NK}_n(RG) = 0.$$

The relation of this conjecture to the Farrell-Jones Conjecture is discussed in [15, Section 6.3].

Next we discuss some connections of the Farrell-Jones Conjecture to L^2-invariants. For more information and some explanations about L^2-invariants we refer for instance to Lück [121].

The L^2-torsion of a closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to a (non-vanishing) dimension constant its volume (see [87]).

Conjecture 5.31 (Homotopy invariance of L^2-torsion)**.** *Let X and Y be det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent.*

Then their L^2-torsion agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

The conjecture above allows to extend the notion of volume to hyperbolic groups whose L^2-Betti numbers all vanish.

Theorem 5.32 (Lück (2002))**.** *Suppose that $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then G satisfies the Conjecture above.*

Proof. See [15, Theorem 0.14]. □

Remark 5.33 (p-adic Fuglede-Kadison determinant)**.** Deninger defines a p-adic Fuglede-Kadison determinant for a group G and relates it to p-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.

Remark 5.34 (Atiyah Conjecture)**.** The surjectivity of the map

$$\operatorname*{colim}_{\mathsf{Or}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

plays a role (33 %) in a program to prove the Atiyah Conjecture. It says that for a closed Riemannian manifold with torsionfree fundamental group the L^2-Betti numbers of its universal covering are all integers.

The Atiyah Conjecture is rather surprising in view of the analytic definition of the L^2-Betti numbers by

$$b_p^{(2)}(M) := \lim_{t\to\infty} \int_F e^{-t\widetilde{\Delta}_p}(\widetilde{x},\widetilde{x})dvol_{\widetilde{M}},$$

where F is a fundamental domain for the $\pi_1(M)$-action on $\widetilde{M}$.

Next we explain the relation of the Baum-Connes Conjecture to the Gromov-Lawson-Rosenberg Conjecture.

Definition 5.35 (Bott manifold)**.** A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8.

We fix a choice of a Bott manifold. (The particular choice does not matter.) Notice that the index defined in terms of the Dirac operator $\mathrm{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$. In particular

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \quad = \quad \mathrm{ind}_{C_r^*(\pi_1(M\times B);\mathbb{R})}(M\times B),$$

if we identify $KO_n(C_r^*(\pi_1(M);\mathbb{R})) = KO_{n+8}(C_r^*(\pi_1(M);\mathbb{R}))$ via Bott periodicity. If M carries a Riemannian metric with positive scalar curvature, then the index

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \in KO_n(C_r^*(\pi_1(M);\mathbb{R})),$$

which is defined in terms of the Dirac operator on the universal covering, must vanish by the Bochner-Lichnerowicz formula.

Conjecture 5.36 ((Stable) Gromov-Lawson-Rosenberg Conjecture)**.** *Let M be a closed connected* Spin*-manifold of dimension $n \geq 5$.*

Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0 \qquad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

Theorem 5.37 (Stolz (2002))**.** *Suppose that the assembly map for the real version of the Baum-Connes Conjecture*

$$H_n^G(\underline{E}G;\mathbf{KO}^{\mathrm{top}}) \to KO_n(C_r^*(G;\mathbb{R}))$$

is injective for the group G.

Then the Stable Gromov-Lawson-Rosenberg Conjecture is true for all closed Spin*-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

Proof. See [174]. □

The requirement $\dim(M) \geq 5$ in Theorem 5.37 is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.

Since the Baum-Connes Conjecture is true for finite groups (for the trivial reason that $\underline{E}G = \{\bullet\}$ for finite groups G), the Stable Gromov-Lawson Conjecture holds for finite fundamental groups by Theorem 5.37.

Remark 5.38 (The unstable version of the Gromov-Lawson-Rosenberg Conjecture)**.** The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if the index $\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M)$ vanishes. Schick(1998) [163] has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau. It is not known whether the unstable version is true or false for finite fundamental groups.

Question 5.39 (Status)**.** For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true?

What are open interesting cases?

Question 5.40 (Methods of proof)**.** What are the methods of proof?

Question 5.41 (Relations)**.** What are the relations between the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

6. Summary, status and outlook

The outline of this section is:

- We present other versions of the Isomorphism Conjecture;
- We give a summary about the status of the Farrell-Jones and the Baum-Connes Conjecture:
- We discuss open questions and problems.

Conjecture 6.1 (Isomorphism Conjecture)**.** *Let $\mathcal{H}_*^?$ be an equivariant homology theory. It satisfies the Isomorphism Conjecture for the group G and the family $\mathcal{F}$ if the projection $E_{\mathcal{F}}(G) \to \{\bullet\}$ induces for all $n \in \mathbb{Z}$ a bijection*

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\{\bullet\}).$$

Example 6.2 (The Farrell-Jones and the Baum-Connes Conjecture as special cases of the Isomorphism Conjecture)**.** The Farrell-Jones Conjecture for K-theory or L-theory respectively with coefficients in R (see 4.9 and 4.10 respectively) is the Isomorphism Conjecture 6.1 for $\mathcal{H}_*^? = H_*^?(-;\mathbf{K}_R)$ or $\mathcal{H}_*^? = H_*^?(-;\mathbf{L}_R^{\langle -\infty \rangle})$ respectively and $\mathcal{F} = \mathcal{VCYC}$.

The Baum-Connes Conjecture 4.11 is the Isomorphism Conjecture 6.1 for $\mathcal{H}_*^? = K_*^? = H_*^?(-;\mathbf{K}^{\text{top}})$ and $\mathcal{F} = \mathcal{FIN}$.

There are functors $\mathcal{P}$ and A which assign to a space X the space of pseudo-isotopies and its A-theory. Composing it with the functor sending a groupoid to its classifying space yields functors $\mathbf{P}$ and $\mathbf{A}$ from $\mathsf{Groupoids}$ to $\mathsf{Spectra}$. Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$. They satisfy $H_n^G(G/H;\mathbf{P}) = \pi_n(\mathcal{P}(BH))$ and $H_n^G(G/H;\mathbf{A}) = \pi_n(A(BH))$.

Pseudo-isotopy and A-theory are important theories. In particular they are closely related to the space of selfhomeomorphisms and the space of selfdiffeomorphisms of closed manifolds. For more information about A-theory and pseudoisotopy we refer for instance to [37], [57, Section 9]), [86], [92], [185], [186].

Conjecture 6.3 (The Farrell-Jones Conjecture for pseudo-isotopies and A-theory)**.** *The Farrell-Jones Conjecture for pseudo-isotopies and A-theory respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$ respectively for the family $\mathcal{VCYC}$.*

Theorem 6.4 (Relating pseudo-isotopy and K-theory)**.** *The rational versions of the K-theoretic Farrell-Jones Conjecture for coefficients in $\mathbb{Z}$ and of the Farrell-Jones Conjecture for Pseudoisotopies are equivalent.*

In degree $n \leq 1$ this is even true integrally.

Proof. See [66, 1.6.7 on page 261]. □

There are functors **THH** and **TC** which assign to a ring (or more generally to an **S**-algebra) a spectrum describing its topological Hochschild homology and its topological cyclic homology. These functors play an important role in K-theoretic computations. Composing it with the functor sending a groupoid to a kind of group ring yields functors $\mathbf{THH}_R$ and $\mathbf{TC}_R$ from $\mathsf{Groupoids}$ to $\mathsf{Spectra}$. Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{THH}_R)$ and $H_*^?(-;\mathbf{TC}_R)$. They satisfy $H_n^G(G/H;\mathbf{THH}_R) = \mathbf{THH}_n(RH)$ and $H_n^G(G/H;\mathbf{TC}_R) = \mathbf{TC}_n(RH)$.

Conjecture 6.5 (The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology)**.** *The Farrell-Jones Conjecture for topological Hochschild homology and for topological cyclic homology respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{THH})$ and $H_*^?(-;\mathbf{TC})$ respectively for the family $\mathcal{CYC}$ of cyclic subgroups.*

We can apply the functor topological K-theory also to Banach algebras such that $l^1(G)$. Let $\mathbf{K}_{l^1}^{\mathrm{top}}$ be the functor from $\mathsf{Groupoids}$ to $\mathsf{Spectra}$ which assign to a groupoid the topological K-theory spectrum of its l^1-algebra. We obtain an equivariant homology theory $H_*^?(-;\mathbf{K}_{l^1}^{\mathrm{top}})$. It satisfies $H_n^G(G/H,\mathbf{K}_{l^1}^{\mathrm{top}}) = K_n(l^1(H))$.

Conjecture 6.6 (Bost Conjecture)**.** *The Bost Conjecture is the Isomorphism Conjecture for $H_*^?(-;\mathbf{K}_{l^1}^{\mathrm{top}})$ and the family $\mathcal{FIN}$.*

Remark 6.7 (Relating the Baum-Connes Conjecture and the Bost Conjecture)**.** The assembly map appearing in the Bost Conjecture 6.6

$$H_n^G(\underline{E}G;\mathbf{K}_{l^1}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}_{l^1}^{\mathrm{top}}) = K_n(l^1(G))$$

composed with the change of algebras homomorphism

$$K_n(l^1(G)) \to K_n(C_r^*(G))$$

is precisely the assembly map appearing in the Baum-Connes Conjecture 4.11

$$H_n^G(\underline{E}G;\mathbf{K}^{\mathrm{top}}) = H_n^G(\underline{E}G;\mathbf{K}_{l^1}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)).$$

Remark 6.8 (Relating the Farrell-Jones Conjecture for L-theory and the Baum–Connes Conjecture)**.** We discuss the relation between the Farrell-Jones Conjecture for L-theory and the Baum-Connes Conjecture. Mainly these come from the sequence of inclusions of rings

$$\mathbb{Z}G \to \mathbb{R}G \to C_r^*(G;\mathbb{R}) \to C_r^*(G)$$

and the change of theories from algebraic to topological K-theory and from algebraic L-theory to topological K-theory for C^*-algebras. Namely, we obtain the following commutative diagram

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}^p_{\mathbb{Z}})[1/2] & \longrightarrow & L^p_n(\mathbb{Z}G)[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}^p_{\mathbb{Q}})[1/2] & \longrightarrow & L^p_n(\mathbb{Q}G)[1/2] \\
\downarrow \cong & & \downarrow \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}^p_{\mathbb{R}})[1/2] & \longrightarrow & L^p_n(\mathbb{R}G)[1/2] \\
\downarrow \cong & & \downarrow \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}^p_{C^*_r(?;\mathbb{R})})[1/2] & \longrightarrow & L^p_n(C^*_r(G;\mathbb{R}))[1/2] \\
\downarrow \cong & & \downarrow \cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{R}})[1/2] & \longrightarrow & K_n(C^*_r(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}})[1/2] & \longrightarrow & K_n(C^*_r(G))[1/2]
\end{array}$$

The arrows marked with $\cong$ are known to be bijective (see [149, page 376], [151, Proposition 22.34 on page 252], [155]). If G satisfies the Farrell-Jones Conjecture 4.10 for L-theory for $R = \mathbb{Z}$ and $R = \mathbb{R}$ and the Baum-Connes Conjecture 4.11 for both the real and the complex case, then all horizontal arrows are bijective and hence all arrows except the two lowest vertical ones are isomorphisms. The Baum-Connes Conjecture for the complex case does imply the Baum-Connes Conjecture for the real case (see Baum-Karoubi [25]).

Theorem 6.9 (Rational computations of K-groups, Lück (2002))**.** *Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order.*

Then there is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C^*_r(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

Proof. See [120, Theorem 0.5]. □

The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture 4.9 and the Baum-Connes Conjecture 4.11 by the equivariant Chern character. In particular they are isomorphisms if these conjecture hold for G.

Remark 6.10 (Splitting principle.)**.** The calculation of the relevant K-and L-groups often split into a universal group homology part which is independent of the theory, and a second part which essentially depends on the theory in question and the coefficients.

Remark 6.11 (Integral Computations)**.** In contrast to general rational computations such as the one appearing in Theorem 6.9, complete integral computations of $K_n(\mathbb{Z}G)$, $L_n(\mathbb{Z}G)$ or $K_n(C^*_r(G))$ seem to be possible only in special cases. Here are some examples, where some of these groups are computed. They are always based on the assumption that the Farrell-Jones Conjecture for algebraic K or L-theory or the Baum-Connes Conjecture is true what is in most cases known to be true.

Three-dimensional Heisenberg group and finite extensions	Lück [123]
2-dimensional crystallographic groups and more general cocompact NEC-groups	Lück-Stamm [128], Pearson [142])
Three-dimensional crystallographic groups	Alves-Ontaneda [3]
Fuchsian groups	Berkhove-Juan-Pineda-Pearson [26], Davis-Lück [53], Lück-Stamm [128],
Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F and free conjugation action of F in $\mathbb{Z}^n$	Davis-Lück [53], Lück-Stamm [128]
One relator groups	Davis-Lück [53]
$SL_3(\mathbb{Z})$	Sanchez-Garcia [161], Upadhyay [181]
(Pure) braid groups	Aravinda-Farrell-Roushon [5], Farrell-Roushon [71]
fundamental groups of knot and link complements	Aravinda-Farrell-Roushon [4]
Certain Coxeter groups	Lafont-Ortiz [109], Sanchez-Garcia [162]

Next we discuss the status of the various Conjectures such as the Farrell-Jones Conjecture and the Baum-Connes Conjecture.

Theorem 6.12 (Bartels-Lück-Reich (2007))**.** *Let R be a ring. Then every subgroup of a hyperbolic group belongs to $\mathcal{FJ}_K(R)$.*

Proof. See [14]. □

Theorem 6.13 (Bartels-Lück)**.** *Let R be a ring with involution. Then every subgroup of a hyperbolic group belongs to $\mathcal{FJ}_L(R)$.*

Proof. The proof will appear in the paper [13] which is in preparation. □

Theorem 6.14 (Bartels-Echterhoff-Reich (2007))**.** *Let R be a ring (with involution). Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). Let G be a subgroup of the colimit $\operatorname{colim}_{i \in I} G_i$.*

1. *Suppose that for all $i \in I$ and every subgroup $H \subseteq G_i$ we have $H \in \mathcal{F}_K(R)$. Then $G \in \mathcal{FJ}_K(R)$;*
2. *Suppose that for all $i \in I$ and every subgroup $H \subseteq G_i$ we have $H \in \mathcal{F}_L(R)$. Then $G \in \mathcal{FJ}_L(R)$;*
3. *Suppose that for all $i \in I$ and every subgroup $H \subseteq G_i$ the Bost Conjecture holds for H. Then the Bost Conjecture holds for G.*

Proof. See [9, Theorem 0.8]. □

Corollary 6.15. *Let $\{G_i \mid i \in I\}$ be a directed system of hyperbolic groups (with not necessarily injective structure maps). Let G be the colimit $\operatorname{colim}_{i \in I} G_i$. Let $H \subseteq G$ be any subgroup of G. Let R be a ring (with involution).*

Then $H \in \mathcal{FJ}_K(R)$, $H \in \mathcal{FJ}_L(R)$ and H satisfies the Bost Conjecture 6.6.

Proof. In the case $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ one has just to combine Theorem 6.12, Theorem 6.13, and Theorem 6.14. In the case of the Bost Conjecture one has to use Theorem 6.14 and the results of Lafforgue [108]). Details can be found in [9, Theorem 0.9]. □

Example 6.16 (Groups satisfying the hypothesis of Corollary 6.15)**.** The groups appearing in Theorem 6.12 are certainly wild in Bridson's universe of groups (see [30]). Many recent constructions of groups with exotic properties are given by colimits of directed systems of hyperbolic groups. Examples are

- groups with expanders in the sense of Gromov;
- Lacunary hyperbolic groups in the sense of Olshanskii-Osin-Sapir [139];
- Tarski monsters, i.e., infinite groups whose proper subgroups are all finite cyclic of p-power order for a given prime p;

Notice that Gromov's groups with expanders belong to $\mathcal{FJ}_K(R)$ for all R, whereas the Baum-Connes Conjecture with coefficients is not true for them by Higson-Lafforgue-Skandalis [90].

Remark 6.17 (Twisted coefficients)**.** The results above do extend to the more general case, where one allows twisted group rings or more general crossed product rings $R * G$ in the setting of the Farrell-Jones Conjecture and coefficients in a G-C^*-algebra in the setting of the Bost Conjecture. There are also so called fibered versions of the Farrell-Jones Conjecture and of an Isomorphism Conjecture in general (see for instance [12, Definition 1.2], [11, Definition 1.1], [66, 1.7]. In these more advanced settings with coefficients or the fibered setting one has that the class of groups for which the conjectures with coefficients or the fibered version are true is closed under taking finite direct products and taking subgroups.

Proofs of these claims can be found in [12, Lemma 1.3], [11, Lemma 1.2], [46, Theorem 2.5 and Theorem 3.17], [66, Theorem A.8 on page 289], [126, 5.5.4], [140, Corollary 7.12].

Example 6.18 (Torsionfree hyperbolic groups)**.** If G is a torsionfree hyperbolic group and R any ring, then we get from Theorem 6.12 as explained in [14, page 2] an isomorphism

$$H_n(BG;\mathbf{K}(R)) \oplus \left(\bigoplus_{\substack{(C), C\subseteq G, C\neq 1 \\ C \text{ maximal cyclic}}} \mathrm{NK}_n(R) \right) \xrightarrow{\cong} K_n(RG).$$

Remark 6.19 (Program for CAT(0)-groups)**.** Bartels and Lück have a program to prove $G \in \mathcal{FJ}_K(R)$ and $G \in \mathcal{FJ}_L(R)$ if G acts properly and cocompactly on a simply connected CAT(0)-space. This would imply $G \in \mathcal{FJ}_K(R)$ and $G \in \mathcal{FJ}_L(R)$ for all subgroups G of cocompact lattices in almost connected Lie groups and for all limit groups G.

Theorem 6.20 (Mineyev-Yu (2002))**.** *Every subgroup of a hyperbolic group belongs to $\mathcal{BC}$.*

Proof. See [133]. □

Definition 6.21 (a-T-menable group)**.** A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space.

The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products. It is not closed under semi-direct products. Examples of a-T-menable groups are:

- countable amenable groups;

- countable free groups;
- discrete subgroups of $SO(n,1)$ and $SU(n,1)$;
- Coxeter groups;
- countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes.

A group G has *Kazhdan's property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n,\mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Theorem 6.22 (Higson-Kasparov(2001))**.** *A group G which is a-T-menable satisfies the Baum Connes Conjecture (with coefficients).*

Proof. See [89]. □

Theorem 6.23 (Farrell-Jones (1993))**.** *Let G be a subgroup of a cocompact lattice in an almost connected Lie group. Then the Farrell-Jones Conjecture for pseudo-isotopy is true for G.*

Proof. See [66, Theorem 2.1 on page 263]. □

Theorem 6.24 (Lück-Reich-Rognes-Varisco (2007))**.** *The Farrell-Jones Conjecture for topological Hochschild homology is true for all groups.*

Proof. See [127]. □

For more information about the theorems above and further results we refer to the talks by Bartels, Rosenthal and Varisco.

Remark 6.25 (Borel-Conjecture)**.** Recall that the Borel Conjecture 2.24 is true for a closed n-dimensional manifold M with fundamental group G if G belongs to both $\mathcal{FJ}_K(\mathbb{Z})$ and $\mathcal{FJ}_K(\mathbb{Z})$ and $n \geq 5$ (see Theorem 2.28). Recall from Corollary 6.15 that any subgroup of a colimit over a directed system of hyperbolic groups (with not necessarily injective structure maps) satisfy this assumption and that very exotic groups occur in this way (see Example 6.16).

Here are other groups for which the Borel Conjecture has been proved.

Theorem 6.26 (Farrell-Jones)**.** *The Borel Conjecture and the L-theoretic Farrell-Jones Conjecture with coefficients in $\mathbb{Z}$ are true for a group G if one of the following conditions are satisfied:*

- *G is the fundamental group of a closed Riemannian manifold with non-positive curvature;*
- *G is the fundamental group of a complete Riemannian manifold with pinched negative curvature;*
- *G is a torsionfree subgroup of $GL(n,\mathbb{R})$.*

Proof. See [67], [68]. □

For more information we refer to [126, Section 5],

In the following table we list prominent classes of groups and state whether they are known to satisfy the Farrell-Jones Conjectures 4.9 and 4.10 and the Baum-Connes Conjecture 4.11 or versions of them. Some of the classes are redundant. A question mark means that the author does not know about a corresponding result. A phrase like injectivity or after inverting 2 is true means that the corresponding assembly map is injective or is bijective after inverting 2. The reader should keep in mind that there may exist results of which the authors are not aware. The following table is an updated version of the one appearing in [126, 5.3].

type of group	Baum-Connes Conjecture 4.11	Farrell-Jones Conjecture 4.9 for K-theory	Farrell-Jones Conjecture 4.10 for L-theory
a-T-menable groups	true with coefficients (see Theorem 6.22)	?	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)
amenable groups	true with coefficients (see Theorem 6.22)	?	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)
elementary amenable groups	true with coefficients (see Theorem 6.22)	true fibered for a ring with finite characteristic N after inverting N (see [15, Theorem 0.3])	true fibered after inverting 2 for $R = \mathbb{Z}$ (see [15, Lemma 1.12 and Lemma 7.1] or see [69, Theorem 5.2])
virtually poly-cyclic	true with coefficients (see Theorem 6.22)	true rationally for $R = \mathbb{Z}$, true fibered for $R = \mathbb{Z}$ in the range $n \leq 1$ (see [126, Remark 5.3].	true fibered after inverting 2 for $R = \mathbb{Z}$ (see [15, Lemma 1.12 and Lemma 7.1] or see [69, Theorem 5.2])
torsion free virtually solvable subgroups of $GL(n,\mathbb{C})$	true with coefficients (see Theorem 6.22)	true in the range ≤ 1 [69, Theorem 1.1]	true fibered after inverting 2 [69, Theorem 5.2]
discrete subgroups of Lie groups with finitely many path components	injectivity true (see [126, Theorem 5.9 and Remark 5.11 on page 718])	injectivity is true for the family $\mathcal{FIN}$ and all rings R (see [17])	injectivity is true for the family $\mathcal{FIN}$ and rings R with vanishing $K_n(RH)$ for $n \leq -2$ and $H \subseteq G$ finite (see [17])
subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components	injectivity is true (see [126, Theorem 5.9 and Remark 5.11 on page 718])	true rationally, true fibered in the range $n \leq 1$ (see [66, 1.6.7 on page 261 and Theorem 2.1 on page 263].)	injectivity is true for the family $\mathcal{FIN}$ and rings R with vanishing $K_n(RH)$ for $n \leq -2$ and $H \subseteq G$ finite (see [17])
linear groups	injectivity is true (see [83])	?	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)
finitely generated subgroup of $GL_n(k)$ for a global field k	injectivity is true (see [83])	injectivity is true for $R = \mathbb{Z}$ (see [97])	injectivity is true for $R = \mathbb{Z}$ (see [97])
torsion free discrete subgroups of $GL(n,\mathbb{R})$	injectivity is true (see [83])	true in the range $n \leq 1$ (see [68] and also [126, Theorem 5.5 on page 722])	true for $R = \mathbb{Z}$ (see [68] and also [126, Theorem 5.5 on page 722])

type of group	Baum-Connes Conjecture 4.11	Farrell-Jones Conjecture 4.9 for K-theory	Farrell-Jones Conjecture 4.10 for L-theory
Groups with finite $\underline{E}G$ and finite asymptotic dimension	injectivity is true with coefficients (see [88, Theorem 1.1], [91, Theorem 1,1 and Lemma 4.3],	injectivity is true for the family $\mathcal{FIN}$ and all rings R (see [17])	injectivity is true for the family $\mathcal{FIN}$ and rings R with vanishing $K_n(RH)$ for $n \leq -2$ and $H \subseteq G$ finite (see [17])
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see [99])	?	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see [99])	?	injectivity is true for $R = \mathbb{Z}$ (see [78, Corollary 2.3]
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular	rational injectivity is true (see [99])	true in the range $n \leq 1$ for $R = \mathbb{Z}$ (see [68, Proposition 0.10 and Lemma 0.12])	true for $R = \mathbb{Z}$ (see [68])
$\pi_1(M)$ for a complete Riemannian manifold M with pinched negative sectional curvature	rational injectivity is true (see [99])	true in the range $n \leq 1$ and true rationally for $= \mathbb{Z}$ (see [68, Proposition 0.10 and Lemma 0.12 and page 216])	true for $R = \mathbb{Z}$ (see and also [126, Theorem 5.5 on page 722])
$\pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see [99])	true fibered in the range $n \leq 1$, true rationally for $R = \mathbb{Z}$ (see [65]).	true for $R = \mathbb{Z}$ (see [68] and also [126, Theorem 5.5 on page 722])
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups (see [133])	true for all coefficients R (see [16])	true for $R = \mathbb{Z}$ (see [68] and also [126, Theorem 5.5 on page 722])

type of group	Baum-Connes Conjecture 4.11	Farrell-Jones Conjecture 4.9 for K-theory	Farrell-Jones Conjecture 4.10 for L-theory
subgroups of directed colimits of word hyperbolic groups	?	true for all R (see [14] and [9, Theorem 0.9])	true for all R (see [13] and [9, Theorem 0.9])
subgroups of word hyperbolic groups	true (see [133])	true for all R (see [14])	true for all R (see [13])
one-relator groups	true with coefficients (see [141])	rational injectivity is true for $R = \mathbb{Z}$ or for regular R with $\mathbb{Q} \subseteq R$ (see [11])	true after inverting 2 for all R (see [11, Proposition 0.9 and Theorem 0.13]), true after inverting 2 for $R = \mathbb{Z}$ fibered (see [160])
torsion free one-relator groups	true with coefficients (see [141])	true for R regular [184, Theorem 19.4 on page 249 and Theorem 19.5 on page 250]	true after inverting 2 for all R (see [40, Corollary 8], [11, Proposition 0.9 and Theorem 0.13]), true after inverting 2 for $R = \mathbb{Z}$ fibered (see [160])
3-manifold groups	?	true fibered for $R = \mathbb{Z}$ in the range $n \leq 1$ (see [158, Corollary 4.2] and [159, Corollary 1.1.5])	?
Haken 3-manifold groups (in particular knot groups)	true with coefficients (see [136, Theorem 5.23])	true for R regular (see [184, Theorem 19.4 on page 249 and Theorem 19.5 on page 250])	true after inverting 2 for all R (see [40, Corollary 8])
$SL(n, \mathbb{Z}), n \geq 3$	injectivity is true (see [83])	injecivity is true for the family $\mathcal{FIN}$ and $R = \mathbb{Z}$ (see [97])	injecivity is true for the family $\mathcal{FIN}$ and $R = \mathbb{Z}$ (see [97])
Artin's braid group B_n	true with coefficients (see [136, Theorem 5.25], [164])	true for $R = \mathbb{Z}$ fibered in the range $n \leq 1$, true for $R = \mathbb{Z}$ rationally (see [71])	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)
pure braid group C_n	true with coefficients (see [136, Theorem 5.25], [164])	true for $R = \mathbb{Z}$ in the range $n \leq 1$ (see [5])	true after inverting 2 for all R (see [11, Proposition 0.9 and Theorem 0.13])
Thompson's group F	true with coefficients [61]	?	injectivity is true after inverting 2 for $R = \mathbb{Z}$ (see Remark 6.8)

Remark 6.27 (Open cases)**.** We mention some interesting groups or classes of groups for which the Conjectures are still open.

- The Farrell-Jones Conjecture for K-theory 4.9 and for L-theory 4.10 and the Baum-Connes Conjecture 4.11 are to the authors's knowledge open for $SL_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups and $\operatorname{Out}(F_n)$;
- The Farrell-Jones Conjecture for K-theory 4.9 and for L-theory 4.10 are to the author's knowledge open for solvable groups and one-relator groups, whereas the Baum-Connes Conjecture 4.11 is known for these groups.
- There are certain groups with expanders for which the Baum-Connes Conjecture 4.11 is to the author's knowledge open and the version with coefficients is actually false (see Higson-Lafforgue-Skandalis [90]). The Farrell-Jones Conjecture for K-theory 4.9 and for L-theory 4.10 are known for these groups since they are examples of directed colimits of hyperbolic groups.

Remark 6.28 (Possible candidates for counterexamples)**.** It is not known whether there are counterexamples to the Farrell-Jones Conjecture or the Baum-Connes Conjecture. There seems to be no promising candidate of a group G which is a potential counterexample to the K- or L-theoretic Farrell-Jones Conjecture or the Bost Conjecture. We cannot name a property or a lack of a certain property of a group which may be a reason for this group to be counterexample. There are many groups with rather exotic properties for which these Conjectures are known to be true.

Remark 6.29 (The suspicious Baum-Connes Conjecture)**.** The Baum-Connes Conjecture is the one for which it is most likely that there may exist a counterexample. One reason is the existence of counterexamples to the version with coefficients (see Higson-Lafforgue-Skandalis [90]). Another reason is that $K_n(C_r^*(G))$ has certain failures concerning functoriality which do not occur for $K_n^G(\underline{E}G)$. For instance $K_n(C_r^*(G))$ is not known to be functorial for arbitrary group homomorphisms since the reduced group C^*-algebra is not functorial for arbitrary group homomorphisms. These failures are not present for the Farrell-Jones and the Bost Conjecture, i.e., for $K_n(RG)$, $L^{\langle -\infty \rangle}(RG)$ and $K_n(l^1(G))$.

Remark 6.30 (Methods of proof)**.** Most of the proofs of the Farrell-Jones Conjecture use methods from controlled topology. Roughly speaking, controlled topology means that one considers free modules with a basis and thinks of these basis elements as sitting in a metric space. Then a map between such modules can be visualized by arrows between these basis elements. Control means that these arrows are small. Our homological approach to the assembly map is good for structural investigations but not for proofs. For proofs of the Farrell-Jones Conjecture or the Baum-Connes Conjecture it is often helpful to get some geometric input. In the Farrell-Jones setting the door to geometry is opened by interpreting the assembly map as a forget control map. The task to show for instance surjectivity is to manipulate a representative of the K-or L-theory class such that its class is unchanged but one has gained control. This is done by geometric constructions which yield contracting maps. These constructions are possible if some geometry connected to the group is around, such as negative curvature. We refer to the lectures of Bartels for such controlled methods.

The approach using topological cyclic homology goes back to Böckstedt-Hsiang-Madsen. It is of homotopy theoretic nature. We refer to the lecture of Varisco for more information about that approach.

The methods of proof for the Baum-Connes Conjecture are of analytic nature. The most prominent one is the Dirac-Dual-Dirac method based on KK-theory due to Kasparov. KK-theory is a bivariant theory together with a product. The assembly map is given by multiplying with a certain element in a certain KK-group. The essential idea is to construct another element in a dual KK-group which implements the inverse of the assembly map.

The analytic methods for the proof of the Baum-Connes Conjecture do not seem to be applicable to the Farrell-Jones setting. One would hope for a transfer of methods from the Farrell-Jones setting to the Baum-Connes Conjecture. So far not much has happened in this direction.

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