REFERENCES

[tDP] T. tom Dieck and T. Petrie, Geometric modules over the Burnside ring. Inventiones Math. 47 (1978), 273-287.

[DRE] A. Dress, Contributions to the theory of induced representations. Springer Lecture Notes in Mathematics, vol. 342, 1973, 183-240.

[ILL] S. Illman, Equivariant singular homology and cohomology I. Memoirs Amer. Math. Soc. vol. 156, 1975.

[LE1] L. G. Lewis, Jr., The equivariant Hurewicz map. Preprint.

[LE2] L. G. Lewis, Jr. An introduction to Mackey functors (in preparation).

[LMM] L. G. Lewis, Jr., J. P. May, and J. E. McClure, Ordinary RO(G)-graded cohomology. Bull. Amer. Math. Soc. 4 (1981), 208-212.

[LMSM] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics, vol. 1213, 1986.

[LIN] H. Lindner, A remark on Mackey functors. Manuscripta Math. 18 (1976), 273-278.

[LIU] A. Liulevicius, Characters do not lie. Transformation Groups. London Math. Soc. Lecture Notes Series, vol. 26, 1976, 139-146.

[MAT] T. Matumoto, On G-CW complexes and a theorem of J. H. C. Whitehead. J. Fac. Sci. Univ. Tokyo 18 (1971/72), 363-374.

[WIR] K. Wirthmüller, Equivariant homology and duality. Manuscripta Math. 11 (1974), 373-390.

THE EQUIVARIANT DEGREE

by

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0. Introduction

Abstract. In this paper we study the possible values $\deg f^H$, $H \subset G$ for a G-map $f : M \to N$ if M and N are compact smooth G-manifolds and G a compact Lie group. We generalize results about maps between spheres of G-representations. We give applications to one-fixed point actions and G-surgery. We prove that the unstable H-homotopy type of the sphere of the H-normal slice $S\nu(M^H,M)_x$ for $x \in M^H$ is a G-homotopy invariant of M.

Survey. As an illustration we state a consequence of our main result in a very special situation where it is easy to formulate.

Let G be finite. Consider a compact smooth G-manifold M such that M^H is non-empty, connected and orientable for all $H \subset G$. Assume either that G is nilpotent or that $\dim M^H \geq \dim M^K - 2$ holds for $H \underset{\neq}{\subset} K$, $H, K \in Iso(M) = \{G_x \mid x \in M\}$. Here and elsewhere G_x denotes the isotropy group $\{g \in G \mid gx = x\}$ of $x \in M$. The set of finite G-sets S with $Iso(S) \subset Iso(M)$ is an abelian semi-group under disjoint union. Let $A(G,Iso(M))$ be its Grothendieck group. The cartesian product induces the structure of a commutative ring with unit on it. Let $Con(G)$ be the set of conjugacy classes of subgroups of G and $C(G)$ be the ring $\prod_{Con(G)} \mathbb{Z}$. Then $A(G,Iso(M))$ is a subring of $C(G)$ by identifying S with $(\operatorname{card} S^H \mid (H) \in Con(G))$. For a G-selfmap $f : M \to M$ define $DEG(f) \in C(G)$ by $(\deg f^H \mid (H) \in Con(G))$.

Theorem A.

a) $DEG(f) \in A(G,Iso(M) \subset C(G)$.

b) If $H \subset G$ is a p-group then:

$\deg f \equiv \deg f^H \bmod p$.

c) If G has odd order and $\deg f^H \in \{\pm 1\}$ for each $H \subset G$, then we have for all $H \subset G$:

$\deg f = \deg f^H$. □

This theorem is well known for M as the one-point compactification V^c of a G-representation V. The proof for V^c uses the equivariant Lefschetz index and Smith theory. These methods do not suffice for M a G-manifold. Our main tool is quasi-transversality and the notion of a local degree.

The notion of the degree is used to classify G-homotopy classes of G-maps $f : V^c \to W^c$ (see tom Dieck [6], p. 213, Laitinen [14], Tornehave [21]) and G-homotopy types of G-homotopy representations (see tom Dieck-Petrie [8]). It plays also a role in equivariant surgery theory (see for example Dovermann-Petrie [11], Lück-Madsen [17]). We give a survey over the various sections.

In section one we define the fibre transport tp_M of the tangent bundle of a G-manifold and the notion of an $O(G)$-transformation $\varphi : f^* tp_N \to tp_M$ for a G-map $f : M \to N$. Roughly speaking, φ assigns to each point x in M a G-map (not necessarily a G_x-homotopy equivalence) $TN^c_{fx} \to TM^c_x$ such that certain compatibility conditions hold. Using φ we get a one-to-one-correspondence between local orientations of M^H at x and N^H at fx for each $H \subset G$ and $x \in M^H$. This enables us to define the equivariant degree

$DEG(f,\varphi) \in C(N) = \prod_{(H) \in Con(G)} \prod_{\pi_0(N^H)/WH} \mathbb{Z}$ in section two.

In section three the Burnsidering $A(G,\mathfrak{F})$ of a compact Lie group with respect to a family $\mathfrak{F}$ is treated. We identify $[V^c,V^c]^G$ and

$A(G,Iso(V))$ for a G-representation V. We introduce in section four a multiplicative submonoid $End_{tp_N} \subset C(N)$ and prove $DEG(f,\varphi) \in End_{tp_N}$ for any f and φ in section five. We will see that End_{tp_N} does not involve f and φ but depends only on the component structure of N. The main idea of the proof is best explained in the special case where G is finite and all N^H are non-empty and connected. Then $C(N) = C(G) = \prod_{Con(G)} \mathbb{Z}$ and End_{tp_N} is $A(G,Iso(N))$.

Choose y in N^G and make f quasi-transverse to y. Then $f^{-1}(y)$ is finite and for each $x \in f^{-1}(y)$ f looks like a (not necessarily linear) norm-preserving G_x-map $T_xM \to T_yN$ in a G_x-neighbourhood of x. Consider a G-orbit c of $f^{-1}(y)$. For each x in c we obtain G_x-maps $TM^c_x \to TN^c_y$ by f and $TN^c_y \to TM^c_x$ by φ. Their composition $TN^c_y \to TN^c_y$ defines an element in $A(G_x,Iso(TN_y))$. Its image in $A(G,Iso(N))$ under the induction homomorphism for $G_x \subset G$ is independent of the choice of x and denoted by $d(c)$. Let d be the sum $\Sigma d(c)$ running over $c \in f^{-1}(y)/G$. Since the global degree can be computed by local degrees, $DEG(f,\varphi) \in C(G)$ is just $d \in A(G,Iso(N))$. Roughly speaking, we have counted the local degrees orbitwise in the Burnside ring to get the global degree.

Section six contains some examples to illustrate our results. We give an elementary proof of the following known statement (see Atiyah-Bott [1], Browder [4], Ewing-Stong [12]).

<u>Corollary B.</u> There is no closed G-manifold M with $\dim M \geq 1$ such that each M^H is connected and orientable and M^G a single point if G is the product of a p-group and a torus. □

It is of special interest to choose $\varphi : f^* tp_N \to tp_M$ as an $O(G)$-equivalence i. e. all $TN^C_{fx} \to TM^C_x$ are G_x-homotopy equivalences. Then another choice of φ would change the equivariant degree only by a unit. Moreover, we have:

<u>Theorem C.</u> <u>A normal G-map</u> $f : M \to N$ <u>can be changed into a G-homotopy equivalence by equivariant surgery only if there is an</u> $O(G)$-<u>equivalence</u> $\varphi : tp_N \to f^* tp_M$ <u>with</u> $DEG(f,\varphi) = 1$. □

The existence of an $O(G)$-equivalence φ is related to the notion of the equivariant first Stiefel Whitney class w_M of a G-manifold. In section seven we relate tp_M und w_M and show that the existence of an $O(G)$-equivalence $\varphi : f^* tp_N \to tp_M$ is equivalent to $f^* w_N = w_M$. We prove:

<u>Theorem D.</u> <u>If</u> $f : M \to N$ <u>is a G-homotopy equivalence we have</u> $f^* w_N = w_M$. □

This implies the unstable version of the stable result in Kawakubo [13].

<u>Corollary E.</u> <u>If</u> $f : M \to N$ <u>is a G-homotopy equivalence, we get for</u> $x \in M$:

$$TM^C_x \simeq_{G_x} TN^C_{fx} \ . \ \square$$

Our setting and proofs would be much simpler if we supposed that all fixed point sets are non-empty, connected and orientable. Unfortunately, such conditions are unrealistic in the study of G-manifolds. Hence we make no assumptions about the existence of

G-fixed points or about the connectivity or orientability of the fixed point sets and do not demand $\pi_0(f^H)$ being bijective.

Our notion of the equivariant degree using $O(G)$-transformations has some advantages compared with the one using fundamental classes. It is in this generality much easier to state elementary properties like bordism invariance or the computation by local degrees in our language. We have the global choice of φ instead of the various choices of fundamental classes $[M^H]$ and $[N^H]$. Notice that the choice of $[M^H]$ is independent of the one of $[M^K]$ for $(K) \neq (H)$ and $[N^K]$. Hence in the case of fundamental classes the interaction between the various fixed point sets are not taken into account, what is done in our setting. It seems to be difficult, or even impossible, to state some of our results by means of fundamental classes. For example, the statement of example 6.5 makes no sense if it is formulated with fundamental classes and in example 6.3 there must appear signs because we can substitute $[M^H]$ by $-[M^H]$ and thus change the corresponding degree by a sign. The advantages of our approach for the notion of an equivariant normal map is worked out in Lück-Madsen [17]. (see also theorem C above and example 2.8).

<u>Conventions:</u> We denote by G a compact Lie group unless it explicitly is stated differently. Subgroups are assumed to be closed. A G-representation is always real. A G-manifold M is a compact smooth G-manifold with smooth G-action and possibly non-empty boundary. We call a component C of M^H an isotropy component if there is a x in C with isotropy group $G_x = H$. We say that M fullfills condition (*) if it satisfies the conditions

i) and ii) or the conditions i) and iii) below.

i) $C \neq \{point\}$ for all $C \in \pi_o(M^H)$, $H \subset G$.

ii) If $C \in \pi_o(M^H)$ is an isotropy component, $C^{>H}$ is $\{x \in C \mid G_x \neq H\}$ and $H \subset G$ we have $\dim C^{>H} + 2 \leq \dim C^H$.

iii) G is finite and nilpotent.

A G-map $f : M \to N$ respects always the boundary and we assume $\dim C = \dim D$ for all $C \in M_o(M^H)$, $D \in M_o(N^H)$, $H \subset G$ with $f^H(C) \subset D$.

Acknowledgement. The author wishes to thank the topologists at Århus for their hospitality and support during 1985 - 1986 when the main part of this paper was written. The author is indebted to Ib Madsen and Erkki Laitinen for their useful comments.

1. The fibre transport.

We organize the book-keeping of the components of the various fixed point sets and their fundamental groups for a G-space as follows. We recall that an object of the fundamental groupoid $\Pi(Y)$ of a space Y is a point in y and a morphism $y_o \to y_1$ is a homotopy class of paths from y_1 to y_o. The orbit category O(G) has the homogenous spaces G/H as objects and G-maps as morphisms.

Definition 1.1. The fundamental O(G)-groupoid $\Pi^G X$ of a G-space X is the contravariant functor $\Pi^G X : O(G) \to \{groupoids\}$ sending G/H to $\Pi(X^H) = \Pi(map(G/H,X)^G)$. □

In general an O(G)-category resp. O(G)-groupoid is a contravariant

functor from O(G) into the category of small categories resp. groupoids. We recall that a groupoid is a category whose morphisms are all isomorphisms. An O(G)-functor $F : C \to D$ between O(G)-categories is a natural transformation. Let I be the category of two objects 0 and 1 and three morphisms $ID : 0 \to 0$, $ID : 1 \to 1$ and $u : 0 \to 1$. We define an O(G)-transformation $\varphi : F_o \to F_1$ between O(G)-functors F_o and $F_1 : C \to D$ as an O(G)-functor $\varphi : C \times I \to D$ with $C \mid i = F_i$. Given a second O(G)-transformation $\psi : F_1 \to F_2$, let the composition $\psi \circ \varphi : F_o \to F_2$ be determined by $\psi \circ \varphi(id,u) = \psi(id,u) \circ \varphi(id,u) : (x,0) \to (x,1)$ for all $x \in C$. One should think of an O(G)-functor $F : C \to D$ as a collection of functors $F(G/H) : C(G/H) \to D(G/H)$ and of an O(G)-transformation $\varphi : F_o \to F_1$ as a collection of natural transformations $\varphi(G/H) : F_o(G/H) \to F_1(G/H)$ fitting nicely together. An O(G)-transformation $\varphi : F_o \to F_1$ is called an O(G)-equivalence if there is an O(G)-transformation $\psi : F_1 \to F_o$ with both compositions the identity.

A G-map $f : X \to Y$ induces an O(G)-functor $\Pi^G f : \Pi^G X \to \Pi^G Y$ whereas a G-homotopy $h : X \times I \to Y$ between f and g determines an O(G)-equivalence $\Pi^G f \to \Pi^G g$.

A G-S^n-Hurewicz-fibration $\eta \downarrow X$ is called locally linear if there exists a G_x-neighbourhood U_x for each x in X such that U_x is G_x-fibre homotopy equivalent to $U_x \times SV_x$ for some G_x-representation V_x. We call a locally linear G-S^n-Hurewicz-fibration briefly a G-S^n-fibration. An example is the fibrewise one-point compactification ξ^c of a G-$\mathbb{R}^n$-bundle ξ. Denote by $bf_{G,n}(X)$ the category of G-S^n-fibrations over X with G-fibre homotopy classes of fibrewise G-maps as morphisms. We obtain an $O(G)$-category $bf_{G,n}$ by letting X vary over all homogenous spaces. One should notice that $bf_{G,n}(G/H)$ is equivalent to the category with spheres of H-representations and H-homotopy classes of H-maps as morphisms. We prefer $bf_{G,n}(G/H)$ because of its better transformation behaviour in view of $O(G)$.

The fibre transport of a G-S^n-fibration $\eta \downarrow X$ defines an $O(G)$-functor $tp_\eta : \Pi^G X \to bf_{G,n}$ analogously to the non-equivariant case (see [19], p. 343). The functor $tp(G/H) : \Pi(X^H) \to bf_{G,n}(G/H)$ sends a point in X^H given by $x : G/H \to X$ to $x^*\eta$. Let $h : G/H \times I \to X$ be a G-homotopy from y to x representing a morphism $x \to y$. Choose a solution $\bar{h}$ of the G-homotopy lifting problem

$$\begin{array}{ccc} x^*\eta \times 1 & \longrightarrow & \eta \\ \downarrow & \nearrow_{\bar{h}} & \downarrow \\ x^*\eta \times I & \xrightarrow{h\circ(p\times id)} & X \end{array}$$

Define $x^*\eta \to y^*\eta$ by the pull-back property and $\bar{h}_o$.

<u>Definition 1.2. We call $tp_\eta : \pi^G M \to bf_{G,n}$ the fibre transport of $\eta \downarrow X$. The fibre transport tp_M of a G-manifold M is $tp_{TM}c$.</u>

<u>2. The equivariant degree.</u>

We consider a G-map $f : M \to N$ between G-manifolds an an $O(G)$-

transformation $\varphi : f^*tp_N \to tp_M$ with $f^*tp_N := tp_N \circ \pi^G f$. We want to define its equivariant degree $DEG(f,\varphi)$ lying in a certain ring $C(N)$.

We consider the case $G = 1$ and both M and N connected first. Recall that we always assume $\dim M = \dim N$. Suppose that $\varphi(x) : TN^c_{fx} \to TM^c_x$ is not nullhomotopic for one (and hence all) $x \in M$. Otherwise define $DEG(f,\varphi) \in \mathbb{Z}$ to be zero. Let u be any loop in M at x. By functoriality of φ we get $\varphi(x) \circ tp_N(f \circ u) \simeq tp_M(u) \circ \varphi(x)$. Since the first Stiefel-Whitney class $w_1(M) \in H^1(M,\mathbb{Z}/2) = HOM(\pi_1(M)$ sends u to $\deg tp_M(u)$ we have $f^*w_1(N) = w_1(M)$. Let $p : \hat{M} \to M$ be the orientation covering if $w_1(M)$ is non-trivial and the identity $\hat{M} = M \to M$ otherwise and define $p : \hat{N} \to N$ analogously. Then $\hat{M}$ and $\hat{N}$ are orientable connected manifolds and we can choose a lift $\hat{f} : \hat{M} \to \hat{N}$. If $f(\hat{M}) \subset \partial\hat{N}$ let $DEG(f,\varphi)$ be zero. Otherwise choose a point $\hat{x} \in \hat{M} \setminus \partial\hat{M}$ with $\hat{f}\hat{x} \in \hat{N} \setminus \partial\hat{N}$. Write $x = p\hat{x}$. Let $c : \hat{M} \to T\hat{M}^c_x$ and $c : \hat{N} \to T\hat{N}^c_{fx}$ be the collaps maps uniquely determined up to homotopy by the property that the differentials at x and fx are the identity. Let d be the degree of the following endomorphism of $\mathbb{Z}$.

$$\begin{array}{ccc} \mathbb{Z} = H_n(\hat{M},\partial\hat{M}) & \xrightarrow{\hat{f}_*} & H_n(\hat{N},\partial\hat{N}) \\ \cong\downarrow c_* & & \cong\downarrow c_* \\ H_n(T\hat{M}^c_{\hat{x}}) & & H_n(T\hat{N}^c_{\hat{f}\hat{x}}) \\ \cong\downarrow (Tp_{\hat{x}})_* & & \cong\downarrow (Tp_{\hat{f}\hat{x}})_* \\ H_n(TM^c_x) & \xleftarrow{\varphi(x)_*} & H_n(TN^c_{fx}) \end{array}$$

A straightforward calculation shows that d is independent of the choices of $\hat{f}$ and $\hat{x}$. Now define $DEG(f,\varphi)$ as 2d if $w_1(M) = 0$ and $w_1(N) \neq 0$, and as d otherwise. The factor 2 in the case $w_1(M) = 0$ and $w_1(N) \neq 0$ is due to the fact that then $\hat{M}$ is only one of the two components of the pullback of the orientation covering of N.

The global degree has an easy description by local degrees. Let y be a point in $N \setminus \partial N$. Assume that $f^{-1}(y)$ is finite and f looks in a neighbourhood of x like a proper map $k(x) : (TM_x, 0) \to (TN_y, 0)$ with $k(x)^{-1}(0) = 0$ if we identify the tangent space with neighbourhoods by an exponential map. Then:

Proposition 2.1

$$DEG(f,\varphi) = \sum_{x \in f^{-1}(y)} \deg(k(x)^c \circ \varphi(x)^c : TN_y^c \to TN_y^c)$$

Proof. Use [9], p. 267. □

As an illustration consider the example of a n-fold covering $p : M \to N$ between connected manifolds. Its differential induces an O(1)-transformation $\varphi_p : p^* tp_N \to tp_M$. By proposition 2.1 $DEG(p,\varphi_p)$ is n. This applies in particular to $p : S^{2m} \to \mathbb{R}P^{2m}$. Notice that S^{2m} is orientable but $\mathbb{R}P^{2m}$ not.

Now we treat the general case. Let Con(G) be the set of conjugacy classes of subgroups of G. The set of isomorphism classes $\bar{x}$ of objects x in a category C is denoted by $\bar{C}$. Given an O(G)-groupoid $\mathcal{G}$, we write $CON(\mathcal{G})$ for $\prod_{(H) \in Con(G)} \overline{\mathcal{G}(G/H)/WH}$ and $C(\mathcal{G})$ for the ring of functions $CON(\mathcal{G}) \to \mathbb{Z}$. Let CON(X) and C(X) be $CON(\Pi^G X)$ and $C(\Pi^G X)$ for a G-space X.

We will define $DEG(f,\varphi)$ in C(N) by specifying integers $DEG(f,\varphi)(D,H)$ for all $H \subset G$ and $D \subset \pi_0(N^H)$. Let $C_1,\dots,C_r$ be the components of M^H with $f^H(C_i) \subset D$ and $f_i : C_i \to D$ be the map induced by f^H. Because of $(TM \mid M^H)^H = T(M^H)$ we obtain from φ non-equivariant transformations $\varphi_i : f_i^* tp_D \to tp_{C_i}$ by restriction and taking the H-fixed point sets. We have introduced $DEG(f_i,\varphi_i)$ above. Define:

$$DEG(f,\varphi)(D,H) = \sum_{i=1}^{r} DEG(f_i,\varphi_i).$$

The sum shall be zero for $r = 0$.

Definition 2.2. We call $DEG(f,\varphi)$ in C(N) the equivariant degree of f with respect to φ. □

Finally we state the elementary properties. Consider a G-map of triads $(F,f,f_+) : (P,M,M_+) \to (Q,N,N_+)$ and O(G)-transformations $\varphi : f^* tp_N \to tp_M$ and $\Phi : F^* tp_Q \to tp_P$. Identifying $TP_x = TM_x \oplus$ using the inward normal we get $tp_P \mid M = tp_{(TM \oplus \mathbb{R})}c$ and analogou $tp_Q \mid N = t_{p(TN \oplus \mathbb{R})}c$. We assume that $\Phi \mid N$ and φ fit together under these identifications

Let $j^* : C(Q) \to C(N)$ be the ring homomorphism given by composition with the obvious map $CON(N) \to CON(Q)$. Th the equivariant degree turns out to be a bordism invariant.

2.3 $$DEG(f,\varphi) = j^* DEG(F,\Phi)$$

The equivariant degree is a homotopy invariant in the followin sense. Given a G-homotopy $h : M \times I \to N$ between f and g we ge an $O(G)$-equivalence $\psi_h : g^*tp_N \to f^*tp_N$ by the fibre transport. Then:

2.4 $$DEG(f,\varphi) = DEG(g,\varphi \circ \psi_h)$$

Consider G-maps $f : L \to M$ and $g : M \to N$ and $O(G)$-equivalences $\varphi : f^*tp_M \to tp_L$ and $\psi : g^*tp_N \to tp_M$. Provided that $\pi_o(g^H) : \pi_o(M^H) \to \pi_o(N^H)$ is bijective for all $H \subset G$, we obtain the <u>composition formula:</u>

2.5 $$DEG(g \circ f, \varphi \circ f^*\psi) = DEG(g,\psi) \cdot (g^*)^{-1}(DEG(f,\varphi))$$

The following examples illustrate our definitions.

<u>Example 2.6.</u> Let $f : V^c \to W^c$ be a G-map for two G-representations V and W with dim V^G, dim $W^G \geq 1$. Any G-map $\Phi : W^c \to V^c$ can be interpreted as an $O(G)$-transformation $\varphi : f^*tp_{V^c} \to tp_{W^c}$ using the facts that $TV^c \oplus (V^c \times \mathbb{R}) = V^c \times (V \oplus \mathbb{R})$ holds and the suspension $[V^c, W^c]^G \to [(V \oplus \mathbb{R})^c, (W \oplus \mathbb{R})^c]^G$ is bijective. Then $DEG(f,\varphi)$ lies in $C(W^c) = C(G)$ and $DEG(f,\varphi)(H)$ is just $\deg(\Phi^H \circ f^H)$ for $(H) \in Con(G)$. □

<u>Example 2.7.</u> Let M be a G-manifold such that the components of M^H are orientable for all $H \subset G$. If $f : M \to M$ is a G-map with $\pi_o(f^H) : \pi_o(M^H) \to \pi_o(M^H)$ the identity for all $H \subset G$ we can define its degree $DEG(f) \in C(M)$ by the collection $(\deg(f^H|C : C \to C)$ $C \in \pi_o(M^H)$, $H \subset G)$. The orientability condition ensures that we get a well-defined $O(G)$-equivalence $\varphi : f^*tp_M \to tp_M$ uniquely determined by the property that $\varphi(G/H)(x)_{eH} : tp_M(G/H)(fx)_{eH} \to tp_M(G/H)(x)_{eH}$ is given by the fibre transport of the H-bundle $TM|M^H$ along any path in M^H from x to fx. One easily checks $DEG(f) = DEG(f,\varphi)$. □

The following non-equivariant example indicates the advantage of our notion of the degree with the one using fundamental classes for surgery.

<u>Example</u> 2.8. Let M and N be closed orientable connected manifolds with fundamental classes [M] and [N] and let M^- be M with $-[M]$. Consider a normal map $f: M \longrightarrow N$, $\hat{f}: TM \oplus \mathbb{R}^k \longrightarrow \xi$ of degree one taken with respect to the fundamental classes. If M_o is $M + M^- + M$ disjoint union gives a normal map of degree one $g = f + f + f$: $M_o \longrightarrow N$, $\hat{g} = \hat{f} + \hat{f} + \hat{f}$. The reader should figure out by himself that it is impossible with these bundle data and orientations to convert f by surgery into a normal map $f_+ : M_+ \longrightarrow N$ of degree one with connected M_+. We can see this using our degree as follows. Fix an $O(1)$ - equivalence $\varphi : tp_\xi$ —— $tp_{TN} \oplus \mathbb{R}^k$. Let $\psi_{\hat{g}} : g^*tp_N \longrightarrow tp_M$ be the $O(1)$ - equivalence uniquely determined by the property that its suspension is $(\varphi \circ tp_{\hat{g}})^{-1} : f^*tp_{TN} \oplus \mathbb{R}^k \longrightarrow tp_{TM} \oplus \mathbb{R}^k$. Since "normally bordant" includes the bundle data, $DEG(g,\psi_{\hat{g}})$ is a normal bordism invariant. But $DEG(g\,\psi_{\hat{g}})$ is ± 3 by Proposition 2.1.

3. The Burnside ring of a compact Lie group.

The Burnside ring of a compact Lie group G was introduced and examined by tom Dieck [5] and [6], p. 103 ff. Since we need some modifications of this material and want to keep the paper self-contained we make some remarks about it in this section.

A *prefamily* $\mathfrak{F}$ is a subset of $\mathfrak{S}(G) = \{H | H \subset G\}$ closed under conjugation. We call $\mathfrak{F}$ a *family* if it is also closed under intersection and *finite* if $\{(H) \in Con(G) \mid H \in \mathfrak{F}\}$ is finite. The set of isotropy groups $Iso(X) = \{G_x \mid x \in X)$ of a finite G-CW-complex X is a finite prefamily. If X is a G-manifold with connected fixed point sets, Iso(X) is a finite family for finite G, but not in general. A counterexample is the sphere in the SO(3)-representation $\mathbb{R}^3 \oplus \mathbb{R}^3$ if SO(3) acts in the obvious way on both summands. If $\mathfrak{F}$ is a prefamily and χ denotes the Euler characteristic let $A(G,\mathfrak{F})$ be the set of equivalence classes of finite G-CW-complexes X with $Iso(X) \subset \mathfrak{F}$ under the equivalence relation $X \sim Y \Leftrightarrow \chi(X^H) = \chi(Y^H)$ for all $H \subset G$. The disjoint union defines an abelian group structure. Moreover, the cartesian product induces the structure of an associative commutative ring with unit if $\mathfrak{F}$ is a family containing G. We can identify $A(G) := A(G,S(G))$ with the Burnside ring in [6] p. 103. Let $C(G,\mathfrak{F})$ be the ring of functions $\{(H) \in Con(G) \mid H \in \mathfrak{F}\} \to \mathbb{Z}$ and $C(G) = C(G,S(G))$. For each $K \subset G$ we obtain a ring homomorphism

$$ch_K : A(G,\mathfrak{F}) \longrightarrow \mathbb{Z} \qquad [X] \longmapsto \chi(X^H).$$

Since WH acts freely on G/H^K and WH contains a circle for infinite WH we get $ch_K(G/H) = 0$ for all K if WH is infinite. For any prefamily $\mathfrak{F}$ let $\mathfrak{F}_f$ be $\{H \in \mathfrak{F} \mid WH \text{ finite}\}$. Using the ideas in [6] p. 3, 4, 104, 119 one proves that ch is given by the product of the ch_K-s:

Proposition 3.1. Let $\mathfrak{F}$ be a finite prefamily. Then $\{[G/H] | H \in \mathfrak{F}_f\}$ is a $\mathbb{Z}$-base of $A(G,\mathfrak{F})$. *The homomorphism*

$$ch : A(G,\mathfrak{F}) \to C(G,\mathfrak{F}_f)$$

is injective with a finite cokernel of order $\prod_{\{(H) \mid H \in \mathfrak{F}_f\}} |WH|$.

Moreover, each ch(G/H) *is divisible by* |WH| *and* $\{\frac{1}{|WH|} ch(G/H) \mid H \in \mathfrak{F}_f\}$ is a $\mathbb{Z}$-base for $C(G,\mathfrak{F}_f)$. □

Now we introduce the equivariant Lefschetz index following [14], chapter 1 to produce a bijection $[V^c,V^c]^G \to A(G,Iso(V))$ for an appropriate G-representation V.

Consider a G-self map $f : X \to X$ of a finite G-CW-complex X. Let $L(f^H,f^{>H})$ be the Lefschetz index of the self map $(f^H,f^{>H})$ of the pair of CW-complexes $(X^H,X^{>H})$.

Definition 3.2. The equivariant Lefschetz index $L^G(f)$ *in* A(G,Iso(X *is defined as*

$$L^G(f) = \sum_{\{(H) | H \in Iso(X)_f\}} \frac{1}{|WH|} \cdot L(f^H,f^{>H}) \cdot [G/H] \;.□$$

Since $(X^H,X^{>H})$ is WH-free, $L(f^H,f^{>H})$ is divisible by |WH|. Proposition 1.8 in [14] extends to compact Lie groups:

Lemma 3.3. $ch_K(L^G(f)) = L(f^K)$ for $K \subset G$.

Proof. Since the Lefschetz index is additive ([9], p. 213) one can reduce the problem by induction over the orbit bundles and dimensions to the case $X = \coprod_r G/H \times D^n / \coprod_r G/H \times S^{n-1}$ where one has to show with $*$ the obvious base-point:

$$L(f^K,*) = \begin{cases} \frac{1}{|WH|} \cdot L(f^H,*) \cdot \chi(G/H^K) & \text{if } WH \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

The second case follows from the fact that WH acts freely relative $*$ on X and X^K and contains a circle. The canonical inclusions and projections of the wedge X yield a pair of inverse isomorphisms between $H_*(X,*)$ and $\bigoplus_r H_*((G/H \times S^n)/(G/H \times *),*)$ where $*$ denotes the various base points. Now an easy homological computation reduces the proof of the first case to $X = (G/H \times S^n)/(G/H \times *)$ with WH finite. Then f^H is a self-map of $(WH \times S^n)/(WH \times *)$. The Künneth formula and the obvious map $G/H \times (WH \times S^n)/(WH \times *) \to X$ induce a chain homotopy equivalence such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} C(G/H^K) \otimes_{\mathbb{Z}WH} C(WH \times S^n/WH \times *,*) & \xrightarrow{\simeq} & C(X^K,*) \\ \downarrow \mathrm{id} \otimes_{\mathbb{Z}WH} C(f^H,*) & & \downarrow C(f^K,*) \\ C(G/H^K) \otimes_{\mathbb{Z}WH} C(WH \times S^n/WH \times *,*) & \xrightarrow{\simeq} & C(X^K,*) \end{array}$$

Notice that $C(WH \times S^n/WH \times *,*)$ is concentrated in dimension n and is $\mathbb{Z}WH$ there. Let $\Sigma a_w \cdot w \in \mathbb{Z}WH$ be the element determined by $C(f^H,*)$.

Then $L(f^H,*)$ is $|WH| \cdot a_1$ and $L(f^K,*)$ is $\chi(G/H^K) \cdot a_1$ since $C(G/H^K)$ is $\mathbb{Z}WH$-free. This finishes the proof. □

A G-homotopy representation X of G is a finite-dimensional G-complex of finite orbit type such that for each subgroup H of G the fixed point set X^H is an n(H)-dimensional CW-complex homotopy equivalent to $S^{n(H)}$. If $\dim X^G \geq 1$ and Iso(X) is a family, we equip $[X,X]^G$ and A(G,Iso(X)) with the monoid structure given by composition and multiplication. If 1 denotes [G/G] and $\chi^G(X) := L^G(id_X)$ we have the unit $\chi^G(X) - 1$ in A(G,Iso(X)) and maps

$$\lambda : [X,X]^G \longrightarrow A(G,Iso(X)) \qquad [f] \longrightarrow (L^G(f) - 1) \cdot (\chi^G(X) - 1$$

$$DEG : [X,X]^G \longrightarrow C(G) \qquad [f] \longrightarrow \{\deg f^H \mid (H) \in Con(G)\}$$

The main result of this section is:

Theorem 3.4. Let X be a G-homotopy representation with $\dim X^G \geq 1$ satisfying condition (*) defined in the introduction.

a) $L^G - 1 : [X,X]^G \to A(G,Iso(X))$ is bijective.

b) If Iso(X) is a family the monoid map $\lambda : [X,X]^G \to A(G,Iso(X))$ is bijective and $ch \circ \lambda = DEG$. □

Theorem 3.4 follows from proposition 3.1, lemma 3.3 and the equivariant Hopf theorem 3.5 below. For its proof and further explanations we refer to [6] p. 213, [7] II.4., [14], [18] and [21].

Theorem 3.5. Let X and Y be G-homotopy representations with $\dim X^H = \dim Y^H$ for all $H \subset G$ satisfying condition (*). Choose fundamental classes for X^H and Y^H such that deg f^H for a G-map $f : X \to Y$ is defined.

Then $[X,Y]^G$ is non-empty. Elements $[f]$ are determined by the set $\{\deg f^H \mid H \in Iso(Y)_f\}$. The degree deg f^H is modulo $|WH|$ determined by the deg f^K, $K \supsetneq H$, and fixing these degrees deg f^K the possible deg f^H fill the whole residue class mod $|WH|$. □

We end with some remarks about induction and restriction for an inclusion $j : H \longrightarrow G$ of compact Lie groups.

Let $\mathcal{F}$ be a prefamily for H. Then $j_*\mathcal{F} = \{g^{-1}j(K)g \mid g \in G, K \in \mathcal{F}\}$ is a prefamily for G. We want to define an abelian group homomorphism

$$ind_j : A(H,\mathcal{F}) \to A(G,j_*\mathcal{F})$$

by sending $[X]$ to $[G \times_j X]$. The following formula and proposition 3.1 show that this is well-defined.

3.6 $$\chi((G\times_j X)^K) = \sum_{gH \in G/H^K} \chi(X^{gKg^{-1}}) \quad \text{for } K \subset G,\ WK \text{ finite.}$$

Notice that G/H^K has only finitely many WK-orbits ([2], p. 87) and is therefore finite if WK is finite.

Given a prefamily $\mathcal{F}$ for G, we have the prefamily $j^*\mathcal{F} = \{j^{-1}(K) \mid K \in \mathcal{F}\}$ for H. We obtain an abelian group homomorphism

$$res_j : A(G,\mathcal{F}) \to A(H,j^*\mathcal{F})$$

by restriction: $[X] \to [res_j X]$. If $\mathcal{F}$ is a family containing H and G then $j^*\mathcal{F}$ is a family with $H \in j^*\mathcal{F}$ and res_j is a ring homomorphism.

4. The monoid of endomorphisms of the fibre transport.

If we want to examine the dependency of $DEG(f,\varphi)$ on φ we have to compute in view of the composition formula 2.5 the O(G)-transformations $\varphi : tp_N \to tp_N$ and the possible values $DEG(ID,\varphi)$ in C(N).

More generally we consider the monoid End(tp) of O(G)-transformations $\varphi : tp \to tp$ of any O(G)-functor $tp : \mathcal{G} \to bf_{G,n}$. The group of invertible elements $End(tp)^*$ consists of the O(G)-equivalences $\varphi : tp \to tp$.

Consider $C(\mathcal{G})$ as monoid by its multiplicative structure. The monoid map

$$DEG : End(tp) \to C(\mathcal{G})$$

maps φ to $DEG(\varphi)$ specified by the following function $CON(\mathcal{G}) \to$ Z. For $H \subset G$ and x in $\mathcal{G}(G/H)$ we get a G-fibre map $\varphi(G/H)(x)$. Let $DEG(\varphi)(x,H)$ be the degree of the induced self map on the H-fixed point set $tp(G/H)(x)^H_{eH}$ of the fibre over eH. Recall that $tp(G/H)(x)$ is H-homotopic to SV for some H-representation V. We want to show that $DEG : End(tp) \to C(\mathcal{G})$ is an embedding of monoids and describe its image.

We say that an $O(G)$-transformation $tp : \mathcal{G} \to bf_{G,n}$ <u>satisfies condition</u> $(*)$ if for any $H \subset G$ and $x \in \mathcal{G}(G/H)$ $tp(G/H)(x)_{eH}$ does and has an H-fixed point. If furthermore $Iso(tp(G/H)(x)_{eH})$ is a family we call tp <u>admissible</u>. Consider a G-manifold N satisfying condition $(*)$. Then tp_N satisfies condition $(*)$ and is even admissible if G is finite. If G is finite nilpotent and N a G-manifold such that no component of N^H is a point for $H \subset G$ then tp_N is admissible.

We recall the notion of the homotopy colimit $\Gamma(\mathcal{G})$ (see [20] p. 1625). Objects are pairs (x,H) with $x \in \mathcal{G}(G/H)$ and $H \subset G$. A morphism $(\sigma,u) : (x,H) \to (y,K)$ consists of a G-map $\sigma : G/H \to G/K$ and a morphism $u : x \to \sigma^* y$ with $\sigma^* = \mathcal{G}(\sigma) : \mathcal{G}(G/K) \to \mathcal{G}(G/H)$. Composition is defined by the "semi-direct product formula" $(\tau,v) \circ (\sigma,u) = (\tau \circ \sigma, \sigma^* v \circ u)$. Notice that $\overline{\Gamma(\mathcal{G})}$ is $Con(\mathcal{G})$ (see section 2). The fundamental group category of a G-space X appearing in [7] p. 57 and [15] is $\Gamma(\pi^G X)$. We now introduce contravariant functors A_{tp}, $C_{\mathcal{G}}$ and E_{tp} and relate their inverse limits to End(tp) and $C(\mathcal{G})$. The contravariant functor into the category of monoids

$$E_{tp} : \Gamma(\mathcal{G}) \to MONO$$

maps (x,H) to $[tp(G/H)(x)_{eH}, tp(G/H)(x)_{eH}]^H$. Given a morphism $(\sigma,u) : (x,H) \to (y,K)$ choose g in G with $\sigma(eH) = gK$ so that we obtain a group homomorphism $c(g) : H \to K \quad h \to g^{-1}hg$. If $l(g^{-1})$ is multiplication with g^{-1} we get a H-homotopy equivalence $\alpha : tp(G/H)(x)_{eH} \to res_{c(g)} tp(G/K)(y)_{eK}$ by $l(g^{-1}) \circ tp(G/H)(u)_{eH}$.

Define $E_{tp}(\sigma,u) : [tp(G/K)(y)_{eK}, tp(G/K)(y)_{eK}]^K \to [tp(G/H)(x)_{eH}, tp(G/H)(x)_{eH}]^H$ by restriction with $c(g)$ and conjugation with α. This is well defined since conjugation with an H-self-equivalence induces the identity on $[X,X]^H$ for a G-homotopy representation X (theorem 3.4).

The contravariant functors

$$A_{tp} : \Gamma(\mathcal{G}) \to MONO$$

$$C_{\mathcal{G}} : \Gamma(\mathcal{G}) \to MONO$$

send (x,H) to $A(H, Iso(tp(G/H)(x)_{eH}))$ and $C(H)$. Given a morphism $(\sigma,u) : (x,H) \to (y,K)$ let $g \in G$ and $c(g) : H \to K$ be as above. Define $A_{tp}(\sigma,u)$ and $C_{\mathcal{G}}(\sigma,u)$ as the restriction with $c(g)$.

Let the transformation

$$D : E_{tp} \to C_{\mathcal{G}}$$

$$\Lambda : E_{tp} \to A_{tp}$$

$$CH : A_{tp} \to C_{\mathcal{G}}$$

be induced by the degree and the maps of section three

$$\lambda : [tp(G/H)(x)_{eH}, tp(G/H)(x)_{eH}]^H \to A(H, Iso(tp(G/H)(x)_{eH}))$$

$$ch : A(H, Iso(tp(G/H)(x)_{eH})) \to C(H)$$

The <u>inverse limit</u> of a contravariant functor $F : C \to MONO$ is the submonoid inv F of $\prod_{x \in C} F(x)$ consisting of those elements

$(a_x \mid x \in C)$ such that $F(f)(a_x) = a_y$ holds for any morphism $f : y \to x$.

We define a monoid map

$$\beta : \text{inv lim } C_{\mathcal{G}} \to C(\mathcal{G}) = \prod_{\overline{\Gamma(\mathcal{G})}} \mathbb{Z}$$

as follows. Let $pr_H : C(H) \to \mathbb{Z}$ be the projection onto the factor belonging to $(H) \in Con(H)$. An element in the inverse limit given by $\{u(x,H) \in C(H) \mid (x,H) \in \Gamma(\mathcal{G})\}_*$ is sent to $\{pr_H(u(x,H)) \in \mathbb{Z} \mid \overline{(x,H)} \in \overline{\Gamma(\mathcal{G})}\}$.

Let $\alpha(x,H) : End(tp) \to E_{tp}(x,H)$ be the monoid map sending φ to $\varphi(G/H)(x)_{eH}$. We obtain a homomorphism of monoids

$$\alpha : End(tp) \to \text{inv lim } E_{tp}$$

Theorem 4.1.

a) If $tp : \mathcal{G} \to bf_{G,n}$ fullfills condition (*), the following diagram of monoids commutes. All maps are injective and α is bijective.

$$\begin{array}{ccc} End(tp) & \xrightarrow[\cong]{\alpha} & \text{inv lim } E_{tp} \\ {\scriptstyle DEG}\downarrow & & \downarrow {\scriptstyle \text{inv lim } D} \\ C(\mathcal{G}) & \xleftarrow[\beta]{} & \text{inv lim } C_{\mathcal{G}} \end{array}$$

b) If tp is admissible the following diagram of monoids commutes. All maps are injective and inv lim Λ is bijective.

$$\begin{array}{ccc} \text{inv lim } E_{tp} & \xrightarrow[\cong]{\text{inv lim } \Lambda} & \text{inv lim } A_{tp} \\ {\scriptstyle \text{inv lim } D}\downarrow & \swarrow {\scriptstyle \text{inv lim } CH} & \\ \text{inv lim } C_{\mathcal{G}} & & \end{array}$$

Proof. Everything follows directly from theorem 3.4 and the definitions. □

Let $* : MONO \to GROUPS$ be the functor "invertible elements". Since the inverse limit is compatible with $*$ and $End(tp)^*$ is the group $Aut(tp)$ of $O(G)$-equivalences $tp \to tp$ we conclude:

Corollary 4.2. For admissible tp the following diagram of abelian groups commutes. The maps α^* and inv lim Λ^* are bijective the others injective.

$$\begin{array}{ccc}
Aut(tp) & \xrightarrow[\alpha^*]{\cong} & \text{inv lim } E^*_{tp} \\
\big\downarrow DEG^* & & \big\downarrow \cong \ \text{inv lim } \Lambda^* \\
 & & \text{inv lim } A^*_{tp} \\
 & & \big\downarrow \text{inv lim } CH \\
C(\mathfrak{g})^* & \xleftarrow[\beta^*]{} & \text{inv lim } C^*_{\mathfrak{g}}
\end{array} \qquad \square$$

Corollary 4.3. Let N be a connected G-manifold satisfying condition (*).

a) If N^H is connected and non-empty for all $H \subset G$ and Iso(N) a family then:

$$End(tp_N) = A(G, Iso(N)) \subset C(N) = C(G)$$

b) Let G be a torus. Assume that any component of N^H contains a G-fixed point for $H \subset G$. Then we have for $y \in N$ the bijection

$$End(tp_N) \to \mathbb{Z} \quad \varphi \to deg(\varphi(G/1)(y))$$

c) If G is finite of odd order we get for $y \in N$ an isomorphism.

$$Aut(tp_N)^* \to \{\pm 1\} \quad \varphi \to deg(\varphi(G/1)(y))$$

Proof:

a) If x is a G-fixed point, we have for any object (y,H) in $\Gamma(\pi^G X)$ a morphism $(y,H) \to (x,G)$. Two such morphisms define the same

map $E_{tp_N}(x,G) \to E_{tp_N}(y,H)$ and inv lim $E_{tp_N}(x,G)$ is $End(tp_N)$.

Hence $End(tp_N) = E_{tp_N}(x,G) = A(G, Iso(N))$.

b) If X is a G-homotopy representation of the torus G with $\dim X^G \geq 1$ then $[X,X]^G \to \mathbb{Z}$ $[f] \to deg\, f$ is bijective by proposition 3.1 and theorem 3.4.

c) $ch_1^* = A(G)^* \to \{\pm 1\}$ is a isomorphism by [6] p. 8 if G has odd order. □

If N is a G-manifold and tp_N is admissible, $End(tp_N) \cong$ inv lim A_{tp} depends only on the component structure of N and the sets $Iso(TN_x)_f$ for all $x \in N$ which can be read off from the dimension function.

5. The degree relations.

In this section we state the central result of this paper. In the following we identify $End(tp_N)$ with its image in C(N) under the embedding DEG.

Theorem 5.1. Let $f : M \to N$ be a G-map of n-dimensional G-manifolds satisfying condition (*) and $\varphi : f^* tp_N \to tp_M$ be an O(G)-transformation. Then:

$$DEG(f,\varphi) \in End(tp_N) \subset C(N) \quad \square$$

The rest of this section deals with its proof. Examples to illustrate its meaning are given in the next section. The most important ingredients are the concept of quasi-transversality which we will extend to compact Lie groups (see [10], chapter 3 for finite G) and local degrees.

We call the G-map $f : M \to N$ of G-manifolds quasi-transverse to y in N if the following is true.

i) The preimage $f^{-1}(y)$ consists of finitely many orbits G_y/H with all $W_{G_y}H$ finite.

ii) Equip the G-normal bundle $\nu(f^{-1}(G/G_y),M)$ and $\nu(G/G_y,N)$ with equivariant metrics. There is a norm preserving G-fibre map

$$\begin{array}{ccc} (f^{-1}(G/G_y),M) & \xrightarrow{\ k\ } & (G/G_y,N) \\ \downarrow & & \downarrow \\ f^{-1}(G/G_y) & \xrightarrow{\ f|\ } & G/G_y \end{array}$$

such that f looks like k in a tubular neighbourhood.

Lemma 5.2. We can change f up to G-homotopy such that f is quasi-transverse to y. (see also [10, ch. 3]).

Proof. Let $K_1, K_2, \ldots, K_r$ be a complete system of representatives of conjugacy classes (K) of subgroups of G with K occuring as isotropy group in M and $K \subset G_y$. We construct inductively an open G-set U_i containing $M^{K_1}, \ldots, M^{K_i}$ such that i) and ii) hold if one substitutes $f^{-1}(y)$ and $f^{-1}(G/G_y)$ by their intersections with U_i. We can assume $(K_i) \subset (K_j) \Rightarrow i \geq j$.

The induction begin $i = 0$ is trivial: $U = \emptyset$. In the induction step from $i-1$ to i write $U = U_{i-1}$, $K = K_i$. By possibly shrinking U we can suppose the existence of a closed G-set V with $\mathrm{int}(V) \supset \mathrm{clos}(U)$

and $f^{-1}(G/G_y) \cap V \setminus U = \emptyset$. Let M_o be $M^K \setminus (U \cap M^K)$. By induction hypothesis WK acts freely on M_o. If f_o is $f^K|M_o$ consider the non-equivariant map

$$(f_o \times id)/WK : M_o/WK \to (N^K \times M_o)/WK$$

We can change it homotopically relative $V \cap M_o/WK$ into f_1 such that f_1 is transverse to $(G/G_y^K \times M_o)/WK$. By a cofibration argument we can assume $(f_o \times id)/WK = f_1$. Now G/G_y^K is a finite disjoint union of WK-orbits $\coprod_{i=1}^{r} WK\cdot(g_iG_y)$ (see [2], p. 87). One easily checks $\dim(f_o/WK)^{-1}(WK\cdot g_iG_y) = \dim(f_o \times id/WK)^{-1}(WK\cdot g_iG_y \times M_o/WK) = -\dim NK \cap G_{g_iy}/K$. Hence $(f_o/WK)^{-1}(WK\cdot g_iG_y)$ consists of finitely many points if $NK \cap g_iG_y/K$ is finite and is empty otherwise. In other words $f^{-1}(y) \cap GM_o$ consists of finitely many orbits G_y/K_j such that $W_{G_y}K_j = NK_j \cap G_y/K_j$ is finite. We can treat any such orbit separately.

Consider any x in GM_o with $f(x) = y$ so that f maps $G/G_x \to G/G_y$ by the projection. We identify $\nu(G/G_x,M)$ with a tubular G-neighbourhood of G/G_x and analogously for y. We have $\dim \nu(G/G_x,M)_x^L \leq \dim \nu(G/G_y,N)_y^L$ for all $L \subset G_x$ so that we can extend any non-equivariant map $S\nu(G/G_x,M)_x^{G_x} \to S\nu(G/G_y,N)_y^{G_x}$ to a G_x-map $S\nu(G/G_x,M)_x \to S\nu(G/G_y,N)_y$. Since $\nu(G/G_x,M) = G \times_{G_x} \nu(G/G_x,M)_x$, $\nu(G/G_y,N) = G \times_{G_y} \nu(G/G_y,N)_y$ and $(K) = (G_x)$ holds we can construct a norm preserving fibre map

$$\begin{array}{ccc} \nu(G/G_x,M) & \xrightarrow{\ k\ } & \nu(G/G_y,N) \\ \downarrow & & \downarrow \\ G/G_x & \longrightarrow & G/G_y \end{array}$$

such that the restriction of k to the K-fixed point set agrees with f^K. By a cofibration argument we can change f in a small G-neighbourhood of G/G_x relative to M^K such that k coincides with f on all $\nu(G/G_x,M)$. Now one easily enlarges U to the desired U_i. This finishes the proof of lemma 5.2. □

Proof of theorem 5.1. We have to construct $\Delta \in \mathrm{End}(tp_N)$ such that $\mathrm{DEG} : \mathrm{End}(tp_N) \to C(\Pi^G N)$ sends Δ to $\mathrm{DEG}(f,\varphi)$. Let $\varepsilon(y,H) \in C_{\Pi^G N}(y,H) := C(H)$ for $(y,H) \in \Gamma(\Pi^G N)$ be defined by $\mathrm{Con}(H) \longrightarrow \mathbb{Z}$ (K) $\longrightarrow \mathrm{DEG}(f,\varphi)(y,K)$. One checks directly that $\{\varepsilon(y,H) \mid (y,H) \in \Gamma(\Pi^G N)\}$ determines an element in inv lim $C_{\Pi^G N}$ mapped by β : inv lim $C_{\Pi^G N} \to C(\Pi^G N)$ to $\mathrm{DEG}(f,\varphi)$. Suppose that we can construct for each $(y,H) \in \Gamma(\Pi^G N)$ a H-self-map $\delta(y,H)$ of $tp_N(G/H)(y)_{eH} = TN_y^c$ such that $D(y,H) : E_{tp_N}(y,H) \to C_{\Pi^G N}(y,H)$ sends $\delta(y,H)$ to $\varepsilon(y,H)$. Then $\{\delta(y,H) \mid (y,H) \in \Gamma(\Pi^G N)\}$ defines an element Δ' in inv lim E_{tp_N}. By theorem 4.1 there is $\Delta \in \mathrm{End}(tp_N)$ such that $\alpha : \mathrm{End}(tp_N) \longrightarrow$ inv lim E_{tp_N} maps Δ to Δ', and Δ has the desired property.

Let $ch' : A(H,\mathrm{Iso}(TN_y)) \to C(H)$ be the composition of the inclusion $A(H,\mathrm{Iso}(TN_y)) \to A(H)$, multiplication with the unit $\chi^H(TN_y^c)-1 : A(H) \to A(H)$ and $ch : A(H) \to C(H)$. The map $L^H-1 : [TN_y^c, TN_y^c]^H \to A(H,\mathrm{Iso}(TN_y))$ is a bijection (theorem 3.4.) and $ch' \circ (L^H-1) = \mathrm{DEG}$. Hence theorem 5.1. is true if we can construct for any $(y,H) \in \Gamma(\Pi^G N)$ an element $d \in A(H,\mathrm{Iso}(TN_y))$ satisfying:

5.3. $ch'_K(d) = \mathrm{DEG}(f,\varphi)(y,K)$ for all $K \subset H$.

Now we construct d. We can assume in view of 2.4 and lemma 5.2 that f is quasi-transverse to y. Furthermore we can suppose $H = G_y$. We want to assign to each H-orbit c in $f^{-1}(y)$ an element d(c) in $A(H,\mathrm{Iso}(TN_y))$. Choose x in c. Then $TM_x = (TG/G_x)_x \oplus \nu(G/G_x,M)_x$ and $TN_y = (TG/G_y)_y \oplus \nu(G/G_y,N)_y$. Split $Tp_x : (TG/G_x)_x \to (TG/G_y)_y$ induced from the projection p as $0 \oplus q_x : (TG_y/G_x)_x \oplus V \to (TG/G_y)_y$ with q_x a G_x-linear isomorphism. Checking the dimensions and using elementary obstruction theory we can extend the G_x-map $k_x : \nu(G/G_x,M)_x \to \nu(G/G_y,N)$ coming from k appearing in the definition of quasi-transverse to a norm preserving G_x-map $k'_x : \nu(G/G_x,M)_x \oplus (TG_y/G_x)_x \to \nu(G/G$ Since $k'_x \oplus q_x : TM_x \to TN_y$ is norm-preserving we obtain a G_x-se map $1_x : TN_y^c \to TN_y^c$ by $(k'_x \oplus q_x)^c \circ \varphi(G/G_x)_{eG_x}$. As H/G_x^K contair only finitely many $W_H K$-orbits (see [2], p. 87) H/G_x^K is finite ar $(TG_y/G_x)_x^K = \{0\}$ for $K \subset H$ with finite $W_H K$. Hence

5.4. $1_x^K = (Tp_x^K \oplus k_x^K)^c \circ \varphi(G/G_x)(x)_{eG_x}^K$ if $W_H K$ is finite.

Denote the image of 1_x under $L^{G_x}-1 : [TN_y^c, TN_y^c]^{G_x} \to A(G_x,\mathrm{Iso}($ by d(x) and the image of d(x) under $\mathrm{ind}_{G_x}^H : A(G_x,\mathrm{Iso}(TN_y)) \to$ $A(H,\mathrm{Iso}(TN_y))$ by d(c). For $u \in A(H)$ and $v \in A(G_x)$ one easily ch $\mathrm{ind}_{G_x}^H(\mathrm{res}_{G_x}^H(u)\cdot v) = u \cdot \mathrm{ind}_{G_x}^H(v)$. We obtain from 3.6. and 5.4. ar $\mathrm{res}_{G_x}^H(\chi^H(TN_y^c)-1) = \chi^{G_x}(TN_y^c)-1$

5.5. $ch'_K \circ \mathrm{ind}_{G_x}^H(d(x)) =$

$$\sum_{hG_x \in H/G_x^K} \deg((Tp_x^{hKh^{-1}} \oplus k_x^{hKh^{-1}})^c \circ \varphi(G/G_x)(x)_{eG_x}^{hKh^{-1}}) =$$

$$\sum_{z \in c^K} \deg((Tp_z^K \oplus k_z^K)^c \circ \varphi(G/G_z)(z)_{eG_z}^K) \text{ if } W_H K \text{ is finite.}$$

This shows in particular using proposition 3.1. that $d(c)$ does not depend on the choice of x. Define $d = \sum_{c \in f^{-1}(y)/H} d(c)$.

If $f^K : M^K \to N^K$ and $\varphi^K : f^{K*}tp_N K \to tp_M K$ are induced by f and $DEG(f^K,\varphi^K)$ is the non-equivariant degree we have by definition $DEG(f,\varphi)(y,K) = DEG(f^K,\varphi^K)(y)$ and by proposition 2.1.:

5.6. $$DEG(f,\varphi)(y,K) = \sum_{z \in (f^K)^{-1}(y)} \deg((Tp_z^K \oplus k_z^K) \circ \varphi(G/G_z)(z)^K_{eG_z}$$

if $W_H K$ is finite.

Combining 5.5. and 5.6. gives

5.7. $ch'_K(d) = DEG(f,\varphi)(y,K)$ if $W_H K$ is finite.

Let $K \subset H$ be any subgroup. We can find a bigger subgroup K' with $K \subset K' \subset H$ such that K'/K is a torus T, $W_H K'$ is finite and $ch_K = ch_{K'}$, and hence $ch'_K = ch'_{K'}$ holds (see [6] p. 113). If we can show $DEG(f,\varphi)(y,K) = DEG(f,\varphi)(y,K')$ the assertion 5.3 is a consequence of 5.7. Since T acts on N^K with fixed point set $N^{K'}$ this follows from:

<u>Lemma 5.9.</u> <u>Let</u> $g : P \to Q$ <u>be a</u> T-<u>map between</u> T-<u>manifolds and</u> $\psi : g^*tp_Q \to tp_P$ an O(T)-<u>transformation</u>. <u>Then we get</u> $DEG(g,\psi)(y,T) = DEG(g,\psi)(y,1)$ <u>for a</u> T-<u>fixed point</u> y.

<u>Proof.</u> We can assume that g is quasi-transverse to y. Since WL for $L \subset T$ is finite only for $L = T$ the preimage $f^{-1}(y)$ is a finite set of T-fixed points $x_1,..x_r$. By proposition 2.1 we obtain for certain T-self maps $l_i : TN_y \to TN_y$ that $DEG(f,\psi)(y,T) = \Sigma \deg(l_i^T)$ and $DEG(f,\psi)(y,1) = \Sigma \deg(l_i)$. Now apply proposition 3.1 and theorem 3.4. This finishes the proof of lemma 5.9 and of theorem 5.1. □

6. Some examples

The theorem 5.1 is very general so that it is necessary to give some examples to explain its meaning. The general problem is to calculate inv lim A_{tp} as a subring of $C(\mathcal{G})$. This can be done in special cases where $\Gamma(\mathcal{G})$ is rather simple or $A(H,\mathfrak{F}) \subset C(H)$ is well understood for all subgroups H of G. We recall that $DEG(f,\varphi)$ lies in $C(N) = \prod_{Con(G)} \pi_0(N^H)/WH$ for a G-map $f : M \to N$ and a O(G)-transformation $\varphi : f^*tp_N \to tp_M$ and $DEG(f,\varphi)(z,H)$ is the integer belonging to the component of N^H containing $z \in N^H$.

In the following we always assume that N fulfills condition (*) and is connected.

<u>Example 6.1.</u> Assume that N^H is non-empty and connected for all $H \subset G$. Suppose that Iso(N) is a family. This follows already from our assumption if G is finite. Then we have $C(N) = C(G)$ and by corollary 4.3 and theorem 5.1.

$$DEG(f,\varphi) \in A(G,Iso(N)) \subset C(G)$$

Hence we obtain the same relations as in the special case $M = N = V^c$ with V a G-representation (see theorem 3.4.). The assumption $N^G \neq \emptyset$ is essential. If N is connected and free we get $End(tp_N) = A(1) = C(N) = \mathbb{Z}$. Indeed, each integer d can be realized as the degree of a self-map of some connected free orientable G-manifold N. Take any connected free orientable G-manifold N_o and a map $f : S^1 \to S^1$ of degree d then $id \times f : N_o \times S^1 \to N_o \times S^1$ is an example. However, if N is the sphere of a free G-representation the degree of f is 1 modulo $|G|$ for finite G and 1 for infinite G. One explanation for this phenomenon is that the suspension of a manifold is not a manifold in general but the suspension of a homotopy representation is again a homotopy representation. □

Example 6.2. Let G be a torus T^n and assume that each component of N^H contains a G-fixed point for $H \subset G$.

We get from corollary 4.3 and theorem 5.1

$$DEG(f,\varphi)(z,H) = DEG(f,\varphi)(y,1)$$

for all $H \subset G$ and $z \in N^H$. □

Example 6.3. If H is a p-group the homomorphism ch_1 and $ch_H : A(H) \to \mathbb{Z}$ fulfill $ch_1 \equiv ch_H$ mod p. If H is a torus ch_1 and ch_H agree. Hence we get for each (z,H) by theorem 5.1 (see 5.3).

$$DEG(f,\varphi)(z,H) = DEG(f,\varphi)(z,1) \bmod p, \text{ if H is a p-group}$$

$$DEG(f,\varphi)(z,H) = DEG(f,\varphi)(z,1) \quad , \text{ if H is a torus}$$

If G is itself a p-group we obtain for all (z,H) (see also [3], [11] p. 10).

$$DEG(f,\varphi)(z,H) \equiv DEG(f,\varphi)(z,1) \bmod p \quad □$$

Remark 5.4. Now we give the proof of corollary B stated in the introduction. Assume the existence of M. Since M has finite orbit type (see [6] p. 121) we can find a finite p-group $L \subset$ G with $M^G = M^L$. Hence we can suppose that G itself is a finite p-group. We use induction over $|G|$. The induction begin $G = Z/p$ is done in [1] or by the following argument reflecting the re-sults of example 6.3. Let $c : M \to TM_x^C$ be the collaps map. I- ∞ is the point at infinity $c^{-1}(\infty) \cap M^G$ is empty. Since $c^{-1}(\infty)$ is contained in the free part of M we can use non-equivariant transversality to change c up to G-homotopy such that c is transverse to ∞ in the non-equivariant sense and still $c^{-1}(\infty) \cap M^G = \emptyset$ holds. We can assume that G acts orientation preserving, otherwise consider $M \times M$. Hence the local degree c at x and gx for $x \in c^{-1}(\infty)$ and $g \in G$ agree. Each orbit in finite set $c^{-1}(\infty)$ consists of p elements. Therefore the degr- of c must be divisible by p. A contradiction, since computin- deg c by its local degrees at $0 \in TM_y^C$ yields one. In the ind- step choose a central subgroup C in G with $C = Z/p$. If $M^C \neq$ we get a contradiction to the induction hypothesis. Namely, sider the G/C-action on M^C. But $M^C = M^G$ is impossible by th- duction begin applied to the C-action on M. This finishes t- proof of corollary B.

If one drops the assumption in corollary B that all M^H are connected the result remains true for G an abelian p-group is false for G a non-abelian p-group provided that p is od- (see [4], [12]). A complete classification of compact Lie with one-fixed point actions on (orientable) closed G-manifolds is given in [12]. □

Example 6.5. Let G be a finite group of odd order. If DEG(f,φ) lies in $C(N)^*$ we get from corollary 4.3. and theorem 5.1.

$$DEG(f,\varphi)(z,H) = DEG(f,\varphi)(y,1)$$

for all (z,H). □

Remark 6.6. All the relations we get for DEG(f,φ) **also hold** in the case of an endomorphism $f : M \to M$ of a G-manifold with $\pi_o(f^H) : \pi_o(M^H) \to \pi_o(M^H)$ the identity and all components of M^H orientable for $H \subset G$. Then we can define DEG(f) without specifying φ (see theorem A in the introduction and example 2.7.). □

Finally we mention a consequence of theorem 4.1. and theorem 5.1. and the composition formula 2.5.

Corollary 6.7. If DEG(f,φ) lies in $C(N)^*$ for some O(G)-transformation φ then there is an O(G)-equivalence ψ with

$$DEG(f,\psi) \equiv 1 \quad \square$$

7. The fibre transport and the first equivariant Stiefel-Whitney class.

In this section we analyze the fibre transport from a bundle theoretic point of view. We relate it and the question when an O(G)-equivalence $\varphi : f^*tp_N \to tp_M$ exists to the equivariant

analogue of the first Stiefel-Whitney class.

We have introduced the notion of an O(G)-groupoid, O(G)-functor and O(G)-transformation in section one. We call two O(G)-functors F_o and $F_1 : \mathcal{G}_o \to \mathcal{G}_1$ homotopic if there exists an O(G)-equivalence $\varphi : F_o \to F_1$. Let $[\mathcal{G}_o, \mathcal{G}_1]^{O(G)}$ be the set of homotopy classes of O(G)-functors $\mathcal{G}_o \to \mathcal{G}_1$. A G-map $f : X \to Y$ induces an O(G)-functor $\pi^G f : \pi^G X \to \pi^G Y$. A G-homotopy $h : X \times I \to Y$ defines an O(G)-equivalence $\pi^G h_o \to \pi^G h_1$. Hence we get a well-defined map $[X,Y]^G \to [\pi^G X, \pi^G Y]^{O(G)}$ $[f] \to [\pi^G f]$.

Let $\eta = \eta(G,n) \downarrow BF(G,n)$ be the classifying $G\text{-}S^n$-fibration. It is characterized by the property that the map $[X,BF(G,n)]^G \to bf_{G,n}$ sending [f] to the G-fibre homotopy class of $f^*\eta$ is bijective.

Definition 7.1. Let $\xi \downarrow X$ be a $G\text{-}S^n$-fibration and $f_\xi : X \to BF(G,n)$ be a classifying map. We call $w_\xi = [\pi^G f] \in [\pi^G X, \pi^G BF(G,n)]^{O(G)}$ the first equivariant Stiefel-Whitney class of ξ. Let w_M be w_{TM^c} for a G-manifold M. □

This notion reduces for G = 1 to the ordinary definition of the first Stiefel-Whitney class $w_1(M) \in H^1(M,Z/2) = Hom(\pi_1(M), \pi_1(BF(n)))$. It is related to the fibre transport by:

Proposition 7.2. Let $tp : \pi^G B(G,n) \to bf_{G,n}$ be the fibre transport of the universal $G\text{-}S^n$-fibration.

a) For each $H \subset G$ $tp_\eta(G/H) : \pi(B(G,n)^H) \to bf_{G,n}(G/H)$ is an equivalence of categories.

b) For any G-complex X we get a bijection

$$(tp_\eta)_* : [\pi^G X, \pi^G B(G,n)]^{O(G)} \rightarrow [\pi^G X, bf_{G,n}]^{O(G)}.$$

c) If $\xi \downarrow X$ is a G-S^n-fibration $(tp_\eta)_*$ sends w_ξ to $[tp_\xi]$.

d) Let ξ_1 and ξ_2 be G-S^n-fibrations over the same one-dimensional G-complex X. Then ξ_1 and ξ_2 are G-fibre homotopy equivalent if and only if $w_{\xi_1} = w_{\xi_2}$ holds.

Proof:

a) We must show that $tp_\eta(G/H)$ induces a bijection between the sets of isomorphism classes of objects and for any object $x \in \pi(B(G,n)^H)$ an isomorphism $Aut(x) \rightarrow Aut(tp_\eta(G/H)(x))$ $u \rightarrow tp_\eta(G/H)(u)$. The first assertion follows directly from the universal property. The second follows from the observation that H-fibre homotopy classes of H-S^n-fibrations over S^1 equipped with the trivial H-action are in one to one correspondence to H-homotopy classes of self H-maps of the fibre by the fibre transport. b) and d) follow directly from [16] whereas c) is obvious. □

Given two G-S^n-fibrations ξ and η over X we want to analyze when $w_\xi = w_\eta$ holds. If $w_1(\xi^H)$ and $w_1(\eta^H)$ are the (non-equivariant) first Stiefel-Whitney classes in $H^1(X^H, Z/2)$ we have the following obvious conditions for $w_\xi = w_\eta$:

i) $\xi_x \simeq_{G_x} \eta_x$ for each $x \in X$.

ii) $w_1(\xi^H) = w_1(\eta^H)$ for $H \subset G$.

The following example shows that they are not sufficient in general. If $\mathbb{R}$ is the trivial and $\mathbb{R}_-$ the non-trivial $\mathbb{Z}/2$ representation and

$\mathbb{Z}/2$ acts freely on S^1, consider $\xi = S^1 \times \mathbb{R}^c$ and $\eta = S^1 \times \mathbb{R}^c_-$.

Under certain conditions, however, i) and ii) are sufficient.

Theorem 7.3. Let ξ and η be G-S^n-fibrations over X. Then $w_\xi = w_\eta$ holds if one of the following conditions is satisfied.

i) X^H is connected and $w_1(\xi^H) = w_1(\eta^H)$ for all $H \subset G$. There is a x in X^G with $\xi_x \simeq_{G_x} \eta_x$.

ii) The group G is finite of odd order and $\xi_x \simeq_{G_x} \eta_x$ for all $x \in M$. We have after forgetting the group action $w_1(\xi) = w_1(\eta)$.

Proof: We have to specify for each $H \subset G$ and $x \in X^H$ a G-fibre homotopy equivalence $\varphi(G/H)(x) : tp\ (G/H)(x) \rightarrow tp\ (G/H)(x)$. We do this by determining a H-homotopy equivalence $\varphi(G/H)(x)_{eH}$: $\xi_x \rightarrow \eta_x$ between the fibres over eH. The independence of the choice of the path u below follows from the assumptions about the first Stiefel-Whitney classes.

i) Fix y in X^G and a G-homotopy equivalence $\Phi : \xi_y \rightarrow \eta_y$. Define $\varphi(G/H)(x)_{eH}$ by requiring that the following diagram commutes up to H-homotopy for a path u from y to x in X^H.

$$\begin{array}{ccc} \xi_x & \xrightarrow{\varphi(G/H)(x)_{eH}} & \eta_x \\ {\scriptstyle tp\ (G/H)(u)_{eH}}\downarrow & & \downarrow{\scriptstyle tp\ (G/H)(u)_{eH}} \\ \xi_y & \xrightarrow{\Phi} & \eta_y \end{array}$$

ii) Without loss of generality we can suppose that X is connected. Fix a point y in X and a non-equivariant homotopy equivalence $\Phi : \xi_y \rightarrow \eta_y$. By assumption there is a H-homotopy equivalence

$\xi_x \to \eta_x$ and there are only two up to homotopy because of $A(H)^* = A(H, ISO(\eta_x))^* = \{\pm 1\}$ (see [6], p. 8) and theorem 3.4. Let $\varphi(G/H)(x)_{eH}$ be the one making the following non-equivariant diagram commutative up to homotopy for a path u between y and x.

$$\begin{array}{ccc} \xi_x & \xrightarrow{\varphi(G/H)(x)_{eH}} & \eta_x \\ {\scriptstyle tp(G/1)(u)_e}\downarrow & & \downarrow{\scriptstyle tp(G/1)(u)_e} \\ \xi_y & \xrightarrow{\quad\Phi\quad} & \eta_y \end{array}$$

□

Now we examine whether w_M is a homotopy invariant. If $f : M \to N$ is a G-homotopy equivalence $f^*(TN \oplus V)^c$ and $(TM \oplus V)^c$ are G-fibre homotopy equivalent for appropriate V by [13], theorem 2.3. If this could be destabilized to $f\ TN^c \simeq_G TM^c$ we would in particular obtain $f^* w_N = w_M$. Unfortunately, this is not possible in general, theorem 3.4. gives counterexamples over a point. Now we prove the unstable result $f^* w_N = w_M$.

<u>Theorem 7.4.</u> <u>Let</u> $f : M \to N$ <u>be a G-map between G-manifolds such that</u> $\pi_o(f^H) : \pi_o(M^H) \to \pi_o(N^H)$ <u>is bijective for all</u> $H \subset G$. <u>Suppose for</u> $C \in \pi_o(M^H)$ <u>and</u> $D \in \pi_o(N^H)$ <u>with</u> $f^H(C) \subset D$ <u>that</u> $(f^H|C)^* w_1(D)$ <u>is</u> $w_1(C)$. <u>Then the (non-equivariant) degree</u> $\deg(f^H|C : C \to D) \in \mathbb{Z}/\{\pm 1\}$ <u>is defined. Assume</u> $\deg\ (f^H|C) = \pm 1$ <u>and that M and N fulfill condition</u> (*).

<u>Then there is an O(G)-equivalence</u> $\omega: f^* tp_N \to tp_M$ <u>uniquely determined by the property that</u> $DEG(f, \omega) \equiv 1$.

<u>Proof.</u> Denote by $M^H(x)$ the component of M^H containing x for $H \subset G$, $x \in M^H$. Let $\hat{M}^H(x)$ be $M^H(x)$ if $w_1(M^H(x))$ is zero and the orientation covering otherwise so that $\hat{M}^H(x)$ is an orientable connected manifold. Choose a lift $\hat{x} \in M^H(x)$ of x and a lift $\hat{f}^H : \hat{M}^H(x) \to \hat{N}^H(fx)$ of $f|M^H(x) : M^H(x) \to N^H(fx)$. Write $\hat{y} = \hat{f}^H(\hat{x})$ and $y = f^H(x)$. Let $\psi(x,H) : (TN^c_y)^H \to (TM^c_x)^H$ be a (non-equivariant) map for $x \in M^H$, $H \subset G$ making the following diagram commutative where c denotes the collaps map and p the projection and n is dim $M^H(x) = \dim N^H(fx)$.

$$\begin{array}{ccc} H_n(\hat{M}^H(x)), \partial\hat{M}^H(x)) & \xrightarrow{\hat{f}^H_*} & H_n(\hat{N}^H(y)), \partial\hat{N}^H(y)) \\ {\scriptstyle c_*}\downarrow\cong & & \cong\downarrow{\scriptstyle c_*} \\ H_n(T\hat{M}^H(x)^c_{\hat{x}})_* & & H_n(T\hat{N}^H(y)^c_{\hat{y}})_* \\ {\scriptstyle (Tp^c_{\hat{x}})}\downarrow\cong & & \cong\downarrow{\scriptstyle (Tp^c_{\hat{y}})} \\ H_n((TM^c_x)^H) & \xleftarrow{\psi(x,H)_*} & H_n((TN^c_y)^H) \end{array}$$

One easily checks that the homotopy class of $\psi(x,H)$ depends only on (x,H) but not on the choice of $\hat{x}$ and $\hat{f}^H$.

<u>Lemma 7.5.</u> <u>There is up to</u> G_x<u>-homotopy exactly one</u> G_x<u>-map</u> $\Psi(x) : TN^c_{fx} \to TM^c_x$ <u>for each x in M such that</u> $\Psi(x)^H$ <u>and</u> $\psi(x,H)$ <u>are non-equivariantly homotopic for all</u> $H \subset G_x$. <u>Each</u> $\Psi(x)$ <u>is a</u> G_x<u>-homotopy equivalence</u>.

Let m be the product $\pi|W_{G_x}H|$ running over $\{(H) \in Con(G_x) \mid H \in ISO(T$ $W_{G_x}H$ finite$\}$. We get from [7] p. 173 + 174 the existence of G-maps

$\omega : TN_y \to TM_x$ and $\omega' : TM_x \to TN_y$ with $\deg((\omega' \circ \omega)^H) = 1 \mod m$ for all $H \subset G$. Because of the equivariant Hopf theorem 3.5. and theorem 3.4. lemma 7.5. follows from:

7.6. The element $d = \{\deg(\omega^H \circ \psi(x,H)^{-1}) \mid (H) \in Con(G_x)\} \in C(G_x)$ lies in the image of $DEG : [TN_y, TN_y]^{G_x} \to C(G_x)$.

We firstly give the proof of 7.6. under the assumption that N^H is connected for all $H \subset G_x$. Consider $f : M \to N$ as a G_x-map so that x is a G_x-fixed point in M. As in the proof of theorem 7.3. ii) we get an $O(G_x)$-transformation $\varphi : f^* tp_N \to tp_M$ uniquely determined by the property that $\varphi(G_x/G_x)(x)$ is just ω. Note that for any $z \in M^H$, $H \subset G$ the case $w_1(M^H(z)) = 0$ and $w_1(N^H(z)) \neq 0$ never occurs because of $\deg(f^H|M^H(z)) = \pm 1$. Now one checks directly that $DEG(f,\varphi) \in C(G_x)$ is just d. By theorem 5.1. we have $d \in image(DEG : [TN_y^c, TN_y^c]^{G_x} \to C(G_x))$.

In the general case one has the problem that ω does not determine an $O(G_x)$-transformation $\varphi : f^* tp_N \to tp_M$ if there is a non-connected N^H for some $H \subset G_x$. But we can restrict everything to the $O(G_x)$-subgroupoid $\Pi_x^{G_x} M$ of $\Pi^{G_x} M$ with $\Pi_x^{G_x} M(G_x/H) := \Pi(M^H(x))$ so that we consider only the component of M^H containing x. Then we get an $O(G_x)$-transformation $\varphi : f^* tp_N \mid \Pi_x^{G_x} M \to tp_M \mid \Pi_x^{G_x} M$ by ω as before. As in section two we can at least define $DEG(f,\varphi)$ in $C(\Pi_x^{G_x} N) = C(G_x)$ and get $d = DEG(f,\varphi)$. The same argument as in the proof of theorem 5.1. gives $d \in$ image $(DEG : [TN_y^c, TN_y^c]^{G_x} \to C(G_x))$. This shows 7.6. and finishes the proof of Lemma 7.5.

Let $\varphi : f^* tp_N \to tp_M$ be defined by the property that $\varphi(G/H)(x)_{eH} : TN_{fx}^c \to TM_x^c$ is the restriction from G_x to H of $\psi(x)$. We leave it to the reader to check that φ is an well-defined O(G)-equivalence. By construction $DEG(f,\varphi) \equiv 1$ holds. The uniqueness follows from theorem 4.1. This finishes the proof of theorem 7.4. □

We obtain as a corollary the homotopy invariance of the first equivariant Stiefel-Whitney class and the unstable version of the result in [13], corollary 2.4.

<u>Corollary</u> 7.7. <u>Let</u> $f : M \longrightarrow N$ <u>be a G-map satisfying the assumpt</u>
<u>of</u> Theorem 7.4. <u>Then we have</u> $f^* w_N = w_M$ <u>and the spheres of the n</u>
<u>slices of</u> M <u>at</u> x <u>and</u> N <u>at</u> fx <u>are</u> G_x - <u>homotopy equivalent for al</u>
$x \in M$.

<u>Proof</u> We derive $f^* w_N = w_M$ and $TM_x^c \simeq_{G_x} TN_{fx}$ from theorem 7.4. Now apply theorem 3.5. □

If G_x is connected TM_x and TN_{fx} are even isomorphic as G_x-repres
tations (see [22]). For finite G there are non-isomorphic G-rep
tations V and W with $V^c \simeq_G W^c$ (see [6], p. 249).

Now we give a necessary condition for converting a G-map
$f : M \to N$ between G-manifolds into a G-homotopy equivalence b
surgery. Notice that this would imply the existence of a bordis
appearing below. Theorem 7.8. motivates the approach to equivariant surgery given in [17].

[16] Lück, W.: Equivariant Eilenberg-MacLane spaces $K(\mathcal{G},\mu,1)$ with possibly non-connected or empty fixed point sets, manuscr. math. 58, 67 - 75 (1987)

[17] Lück, W. and Madsen, I.: Equivariant L-theory, Aarhus preprint, (1988).

[18] Rubinsztein, R. L.: On the equivariant homotopy of spheres, preprint, Polish Academy of Science (1973).

[19] Switzer, R. M.: Algebraic topology - homology and homotopy, Springer Verlag, Berlin-Heidelberg-New York (1975).

[20] Thomason, R. W.: First quadrant spectral sequences in algebraic K-theory via homotopy colimit, Comm. in Algebra 10 (15), 1589 - 1668 (1982).

[21] Tornehave, J.: Equivariant maps of spheres with conjugate orthogonal actions, Can. Math. Soc. Conf. Proc., Vol. 2 part 2, 275-301, (1982).

[22] Traczyk, P.: On the G-homotopy equivalences of spheres of representations, Math. Zeitschrift 161, 257 - 261 (1978).

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SURGERY TRANSFER

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Introduction

Given a Hurewicz fibration $F \longrightarrow E \xrightarrow{p} B$ with fib[r] n-dimensional geometric Poincaré complex F we cons[t] algebraic transfer maps in the Wall surgery obstru[ction] groups

$$p^! : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geqslant 0)$$

and prove that they agree with the geometri[c] defined transfer maps. In subsequent work we obtain specific computations of the composites $p_! p^!$ with $p_! : L_m(\mathbb{Z}[\pi_1(E)]) \longrightarrow L_m(\mathbb{Z}[\pi_1(B)])$ the chan[ge of] rings maps, and some vanishing results.

The construction of $p^!$ is most straightforwa[rd in] the case when F is finite, with L_* the free L-[groups] L^h_*. In §9 we shall extend the definition of finitely dominated F and the projective L-groups well as to simple F and the simple L-groups L^s_* also to the intermediate cases.

There are two main sources of applications surgery transfer. The equivariant surgery obstr[uction] groups of Browder and Quinn [1] were defined in of the geometric surgery transfer maps of the sphere bundles of the fixed point sets. An alg[ebraic] version will necessarily involve the algebraic transfer maps. (In this connection see Lück and [8].) The recent work of Hambleton, Milgram, Tay[lor and] Williams [3] on the evaluation of the obstructions of normal maps of closed manifol[ds with] finite fundamental group depends on the factor[ization] of the assembly map by twisted product formula[e which] are closely related to the algebraic surgery tra[nsfer].

Our construction of the quadratic [L-theory] transfer maps is by a combination of the a[lgebraic]