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Equivariant *L*-Theory I

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Introduction

This paper gives a new definition of G-equivariant surgery groups with better formal properties than previous definitions, and it calculates the equivariant groups in terms of usual L-groups in special cases.

The main new feature in our presentation is the systematic use of groupoids with G-actions, and the concept of fibre transport (of G-bundles). This makes possible a definition of equivariant L-groups much in the spirit of C.T.C. Wall's chapter 9, [21]. The surgery groups are bordism classes of *degree one* G-maps $f: M \to N$ covered by a G-bundle map $\hat{f}$: $TM \oplus \mathbb{R}^k \to \xi$ together with a certain map from N into a *reference* (space) R, which, very roughly, captures the first Stiefel-Whitney classes of ξ and TN.

The normal map $(f, \hat{f})$ defines a triple $(\pi^G N, w_1(\xi), w_1(N))$ as follows. The first term is the collection of fundamental groupoids πN^H of the various fixed sets, considered as a functor on the orbit category. The next two terms are the 'homotopy classes' of the maps

$$w_1(\xi), w_1(N): \pi^G N \to \pi^G B(G, n+k) = \mathbb{B}_{n+k}$$

where $B(G, n+k)$ classifies (locally linear) $G-\mathbb{R}^{n+k}$ bundles. We remark that $w_1(\xi)$ can be different from $w_1(N)$. However, they do agree when pushed into $\mathbb{B}F_{n+k} = \pi^G BF(G, n+k)$ where $BF(G, n+k)$ is the classifying space for (locally linear) $G-S^{n+k}$ fibrations.

One should expect the equivariant L-groups to depend on the above triple, generalizing Wall's $L_n^1(\pi_1 N, w_1)$. However, this is not quite enough. As pointed out in [7; 19], the definition from [21, Chap. 9] contains a bug: conjugation with an element $g \in \pi_1 N$ with $w_1(g) = -1$ induces the identity after Wall's geometric definition, but it should be multiplication with -1 after the algebraic definition of L-groups. The difficulty is one of base points and can be done

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away with by systematically working with orientation covers, or by passing to transports as we have preferred.

Given a G-$\mathbb{R}^{n+k}$ bundle (or G-fibration) ξ over N and a (homotopy class of a) path σ from x to y, there is the corresponding fibre transport $\sigma_*: \xi_x \to \xi_y$. This is a well-defined isomorphism class. The transport information can be collected into an $\mathcal{O}(G)$-functor

$$\mathrm{tp}_\xi: \pi^G N \to \mathbb{B}_{n+k}.$$

The 'homotopy class' of tp_ξ is $w_1(\xi)$. The tangent bundle TN gives another transport

$$\mathrm{tp}_N: \pi^G N \to \mathbb{B}_n \xrightarrow{\Sigma^k} \mathbb{B}_{n+k}.$$

Passing to the fibrewise one point compactifications one gets transports tp_ξ^c and tp_N^c into $\mathbb{B}F_{n+k}$. They are 'homotopic' by the equivariant Freudental suspension theorem, because $(\xi \oplus V)^c \simeq (TN \oplus \mathbb{R}^k \oplus V)^c$ for a suitable representation V. In equivalent category terms, there exists an $\mathcal{O}(G)$-transformation

$$\varphi: \mathrm{tp}_\xi^c \to \Sigma^k \, \mathrm{tp}_N^c.$$

There is no apriori choice for φ, but given one, we have the 4-tuple

$$R(f, \hat{f}, \varphi) = (\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi),$$

which record sufficient information to define L-groups. Note that $\varphi \circ \hat{f}$ identifies the tangent fibres of M and N. This allows the definition of equivariant degree $\mathrm{Deg}(f; \varphi \circ \hat{f})$, and in particular the concept of degree one.

Extracting the relevant properties of $R(f, \hat{f}, \varphi)$ one obtains the concept of a (geometric) reference R and one can consider normal maps $(f, \hat{f}, \varphi)$ together with maps $\rho: R(f, \hat{f}, \varphi) \to R$. Bordism classes of such define the equivariant L-groups $\mathcal{L}_n(R) = \mathcal{L}_n(G; R)$, corresponding to the L^1-groups in [21, Chap. 9]. The use of reference R, independent of the range manifold N, has the usual advantages, namely one gets abelian groups with good functorial properties.

We prove the analog of Wall's $L^1 = L^2$ theorem, namely that every class in $\mathcal{L}_n(R)$ has a representative for which ρ is an 'isomorphism'; this is the $\pi - \pi$ lemma.

Our definition of $\mathcal{L}_n^h(R)$ makes sense in the smooth and in the locally linear PL and $\mathcal{T}op$ categories. However, the $\pi - \pi$ theorem, corresponding to [21, Chap. 4], may only true in the first two categories. The 'correct' topological L-groups could therefore be different from ours. We define $\mathcal{L}_n^s(R)$ whenever the reference R is G-simple. This works both in the smooth and in the PL category.

In the second part of the work we give an exact sequence similar to the Conner-Floyd neighbouring family sequence in equivariant bordism. This facilitates calculations. Indeed for G of odd order we go further and show that

$\mathscr{L}_n^q(R)$ is a direct sum of ordinary L-groups for $q=s, h$. If $R=R(f, \hat{f}, \varphi)$ as above and G is odd then

$$\mathscr{L}_n^q(R) \cong \sum^{\oplus} L^q_{n(x,H)}(\mathbb{Z}[E(x, H)], w).$$

Here $E(x, H)$ are the groups

$$E(x, H) = \pi_1(EWH(x) \times_{WH(x)} N^H(x))$$

where $x \in N^H$, $N^H(x)$ is the component of N^H which contains x and $WH(x) \subset WH$ is the stabilizer of $N^H(x)$ in $\pi_0 N^H$, and $n(x, H) = \dim N^H(x)$.

At many places below our arguments are based on stability results for automorphism groups of representations. We state the relevant results, refering the reader to [15 II, III] for details. Let V be an $\mathbb{R}G$-module with

$$5 \leqq \dim V^H \leqq \dim V^K - 3$$

for isotropy groups $K \subsetneqq H$. We have automorphism groups

$$GL_G(V) \subset PL_G(V) \subset \mathscr{T}\!op_G(V) \subset F_G(V)$$

where $F_G(V)$ consists of the proper G-homotopy equivalences of V. Then

$$\mathrm{Aut}_G(V) \to \mathrm{Aut}_G(V \oplus \mathbb{R})$$

is $(\dim V^G - 1)$ connected except for $\mathrm{Aut}_G = \mathscr{T}\!op_G$ where the map is $(\dim V^G - 3)$-connected.

The paper is divided into sections as follows

1. Preliminary Notions

Equivariant topology is burdened by an involved set of notions, designed to keep track of the combinatorial structure of the components of the various fixed point sets and their fundamental groups. In this section we collect the necessary notions, namely the equivariant fundamental groupoid, the fibre transport and the equivariant degree, refering to [12, 13] for certain details.

A groupoid is a small category in which all morphisms are isomorphisms. The fundamental groupoid πX of a space has objects the points of X, and morphisms $w: x_0 \to x_1$ are homotopy classes of paths from x_1 to x_0.

Given a finite group G, let $\mathcal{O}(G)$ be the orbit category of homogeneous spaces G/H and G-maps.

(1.1) **Definition.** An $\mathcal{O}(G)$-groupoid is a contravariant functor from $\mathcal{O}(G)$ to groupoids. The fundamental $\mathcal{O}(G)$-groupoid $\pi^G X$ of a G-space X is the functor $\pi^G X(G/H) = \pi \operatorname{Hom}_G(G/H, X)$.

Given an $\mathcal{O}(G)$-groupoid $\mathcal{G}$ and a subgroup H of G, let $\mathcal{G}(G/H)^\wedge$ be the isomorphism classes of objects in $\mathcal{G}(G/H)$, considered as an WH-set. For $x \in \mathcal{G}(G/H)$, let $WH(x)$ be the isotropy group of $\hat{x} \in \mathcal{G}(G/H)^\wedge$, and let $\operatorname{Aut}(x)$ denote the automorphisms of the object x.

(1.2) **Definition.** The group $E(x, H)$ consists of pairs (σ, w) with $\sigma \in \operatorname{Aut}_G(G/H)$ and $w: x \to \sigma^* x$ a morphism in $\mathcal{G}(G/H)$. Multiplication is the "semi-direct product",

$$(\sigma_1, w_1) \cdot (\sigma_2, w_2) = (\sigma_1 \circ \sigma_2, \sigma_2^* w_1 \circ w_2).$$

We elaborate on (1.2) in the geometric situation where $\mathcal{G} = \pi^G X$. Then $\mathcal{G}(G/H)^\wedge = \pi_0 X^H$ and $\hat{x} = X^H(x)$ is the component which contains x. If $\sigma: G/H \to G/H$ sends H to gH then $g^{-1} H g = H$ and $\sigma^*(x) = gx$. Hence w is a homotopy class of paths from gx to x; and $\sigma^*(w)$ is the path $g \cdot w$. A point in the universal cover $\tilde{X}^H(x)$ is a homotopy class u of paths ending at x. There is an action of $E(x, H)$ on $\tilde{X}^H(x)$ by $u \cdot (\sigma, w) = \sigma^* u \circ w$. Note that

$$E(x, H) \cong \pi_1(EWH(x) \times_{WH(x)} X^H(x)), \tag{1.3}$$

and the extension

$$1 \to \pi_1(X^H, x) \to E(x, H) \to WH(x) \to 1.$$

An $\mathcal{O}(G)$-*functor* between $\mathcal{O}(G)$-groupoids is a natural transformation $f: \mathcal{G}_0 \to \mathcal{G}_1$. Let I be the category with three objects 0 and 1 and three morphisms, namely the identities and $0 \to 1$. An $\mathcal{O}(G)$-*transformation* $h: f_0 \to f_1$ between two $\mathcal{O}(\mathcal{G})$-functors is an $\mathcal{O}(G)$-functor

$$h: \mathcal{G}_0 \times I \to \mathcal{G}_1 \quad \text{with} \quad h \mid \mathcal{G}_0 \times \{i\} = f_i,$$

where $\mathcal{G}_0 \times I$ sends G/H to $\mathcal{G}_0(G/H) \times I$. We call two $\mathcal{O}(G)$-functors f_0 and f_1 *homotopic* if there is an $\mathcal{O}(G)$-transformation between them. This is an equivalence relation (reflexive since morphisms of $\mathcal{G}_1(G/H)$ are isomorphisms). The set of homotopy classes is denoted $[\mathcal{G}_0, \mathcal{G}_1]^{\mathcal{O}(G)}$. The homotopy class represented by f is denoted $[f]$.

Two examples will illustrate the concepts. If $\mathcal{G}_i = \Gamma_i$ are groups (= groupoids with one element), and $G = \{1\}$,

$$[\mathcal{G}_0, \mathcal{G}_1]^{\mathcal{O}(G)} = \operatorname{Hom}(\Gamma_1, \Gamma_2)/\operatorname{Inn}(\Gamma_2).$$

If $\mathcal{G}_i = \pi^G X_i$, and $f: X_0 \to X_1$ is a G-map we get $f: \pi^G X_0 \to \pi^G X_1$, and $f_0 \simeq_G f_1$ implies that $[f_0] = [f_1]$ in $[\pi^G X_0, \pi^G X_1]^{\mathcal{O}(G)}$.

An $\mathcal{O}(G)$-functor $f: \mathcal{G}_0 \to \mathcal{G}_1$ is a *homotopy equivalence* if there exists an $\mathcal{O}(G)$-functor $g: \mathcal{G}_1 \to \mathcal{G}_0$ such that the composites are homotopic to the identities. We also need the (definitely) weaker notion:

(1.4) **Definition.** A *weak homotopy equivalence* $f: \mathcal{G}_0 \to \mathcal{G}_1$ is an $\mathcal{O}(G)$-functor such that for each G/H, $f(G/H): \mathcal{G}_0(G/H) \to \mathcal{G}_1(G/H)$ is an equivalence of categories (i.e. full and faithful, and a bijection on isomorphism classes of objects).

Given $\mathcal{G}$ we can compose with the "space of a category" construction [16] to get a functor $|\mathcal{G}|: \mathcal{O}(G) \to \{\text{Spaces}\}$. Construction (C) of [6] gives a $G-CW$ complex $K(\mathcal{G}, 1)$ together with an $\mathcal{O}(G)$-functor

$$\mu: \pi^G K(\mathcal{G}, 1) \to \mathcal{G}$$

such that

(1.5)
(i) μ is a weak homotopy equivalence
(ii) $K(\mathcal{G}, 1)^H \simeq K(\mathcal{G}(G/H), 1)$.

(cf. [18]).

(1.6) **Proposition.** ([12]) *Let Y be a $G-CW$-complex and $\mu: \pi^G Y \to \mathcal{G}$ an $\mathcal{O}(G)$-functor. Then (Y, μ) satisfies condition* (1.5) *if and only if the map $[X, Y]^G \to [\pi^G X, \mathcal{G}]^{\mathcal{O}(G)}$ which sends $[f]$ to $[\mu \circ \pi^G f]$ is bijective for all $G-CW$-complexes X.*

For $G=1$, (1.6) reduces to the isomorphism $[X, K(\pi, 1)] \simeq \operatorname{Hom}(\pi_1 X, \pi)/\operatorname{Inn}(\pi)$.

(1.7) **Corollary.** (i) *An $\mathcal{O}(G)$-functor $f: \mathcal{G}_0 \to \mathcal{G}_1$ is a weak homotopy equivalence if and only if it induces a bijection $f_*: [\pi^G X, \mathcal{G}_0]^{\mathcal{O}(G)} \to [\pi^G X, \mathcal{G}_1]^{\mathcal{O}(G)}$ for all $G-CW$-complexes X.*

(ii) *For $G-CW$-complexes X and Y any weak homotopy equivalence from $\pi^G X$ to $\pi^G Y$ is a homotopy equivalence.*

We next consider the equivariant fibre transport and its associated equivariant first Stiefel Whitney class. We shall work with locally linear $G-\mathbb{R}^n$ bundles, [11, p. 218]. This concept can be taken in the smooth category (G-vector bundles) or in the piece-wise linear or topological categories. We assume in the two latter cases a distinguished zero section, preserved by all bundle maps. When there is no need to separate between the categories we simply speak of $G-\mathbb{R}^n$ bundles or G-bundles when it is unnecessary to specify the dimension.

Write $\mathbb{B}_n(X)$ for the groupoid of $G-\mathbb{R}^n$ bundles over X and isotopy classes of $G-\mathbb{R}^n$ bundle isomorphisms (over id_X). A G-map $f: X \to Y$ induces a functor $f^*: \mathbb{B}_n(Y) \to \mathbb{B}_n(X)$. Letting X vary over the homogenous spaces we get an $\mathcal{O}(G)$-groupoid

$$\mathbb{B}_n: \mathcal{O}(G) \to \{\text{groupoids}\}.$$

Given a $G-\mathbb{R}^n$ bundle $\xi \downarrow X$, let $\mathrm{tp}_\xi(G/H)\colon \pi_0(X^H) \to \mathbb{B}_n(G/H)$ be the functor which maps $x\colon G/H \to X$ to $x^*\xi$. A morphism $\sigma\colon x \to y$, given by a G-homotopy $\sigma\colon G/H \times I \to X$, is sent to the fibre transport $\mathrm{tp}_\xi(\sigma)\colon x^*\xi \to y^*\xi$ defined as the restriction to $G/H \times \{0\}$ of a bundle isomorphism $x^*\xi \times I \to \sigma^*\xi$ which is the identity on $G/H \times \{1\}$. The collection $\mathrm{tp}_\xi(G/H)$ defines an $\mathcal{O}(G)$-functor (the *fibre-transport* of ξ)

$$\mathrm{tp}_\xi\colon \pi^G X \to \mathbb{B}_n. \tag{1.8}$$

Let $[\mathrm{tp}_\xi]\colon X \to K(\mathbb{B}_n, 1)$ be the corresponding G-homotopy class, cf. (1.6). We have the following elementary properties of the fibre transport, where γ is the classifying $G-\mathbb{R}^n$ bundle over the classifying space $B(G, n)$:

(1.9 i) tp_γ is a weak homotopy equivalence.

(1.9 ii) If $f_\xi\colon X \to B(G, n)$ classifies ξ then $[\mathrm{tp}_\gamma] \circ [f_\xi] = [\mathrm{tp}_\xi]$ in $[\pi^G X, \mathbb{B}_n]^{\mathcal{O}(G)}$

(1.9 iii) If X is a 1-dimensional $G-CW$ complex then $\mathbb{B}_n(X) \cong [\pi^G X, \mathbb{B}_n]^{\mathcal{O}(G)}$ by $[\xi] \to [\mathrm{tp}_\xi]$.

The first two properties are obvious from the definitions. The third uses that $i\colon Y \to K(\pi^G Y, 1)$ is 2-connected on all fixed sets, and (1.6).

In the non-equivariant situation, an $\mathbb{R}^n$-bundle over a 1-dimensional space is determined by its underlying spherical fibration since $\pi_1(BO) \cong \pi_1(BPL) \cong \pi_1(BTop) \cong \pi_1(BF)$. The corresponding equivariant statement is false. Indeed $\pi_0(\mathrm{Aut}_G(V)) \not\cong [V^c, V^c]^G$ in general for representations V, cf. [20, p. 101], [15 II]. Thus we must study equivariant spherical fibrations separately.

A $G-S^n$ Hurewicz fibration η over a $G-CW$-complex X is locally linear if for each x in X there exists a G_x-invariant neighbourhood $U_x \subset X$ such that $\eta \mid U_x$ is G_x-fibre homotopy equivalent to a trivial G_x-fibration $U_x \times V_x^c$ for some n-dimensional G_x-representation V_x. Here V_x^c denotes the one-point-compactification. We abbreviate and call such fibrations $G-S^n$ fibrations. They have a classifying space $BF(G, n)$. If ξ is a $G-\mathbb{R}^n$ bundle its fibrewise one-point compactification ξ^c determines a $G-S^n$ fibration.

Let $\mathbb{B}F_n(X)$ be the groupoid of $G-S^n$ fibrations over X with homotopy classes of fibrewise G-homotopy equivalences as morphisms, and let $\mathbb{B}F_n$ be the corresponding $\mathcal{O}(G)$-groupoid. Each $\eta \in \mathbb{B}F_n(X)$ gives a fibre transport $[\mathrm{tp}_\eta] \in [\pi^G X, \mathbb{B}F_n]^{\mathcal{O}(G)}$.

Fibrewise one-point compactification gives a G-map

$$J\colon B(G, n) \to BF(G, n),$$

well-defined up to homotopy, and a corresponding $\mathcal{O}(G)$-functor $j\colon \mathbb{B}_n \to \mathbb{B}F_n$ such that $[\mathrm{tp}_\gamma] \circ [\pi^G J] = [j] \circ [\mathrm{tp}_\gamma]$. For an $\mathcal{O}(G)$-functor $f\colon \pi^G X \to \mathbb{B}_n$, write f^c for $j \circ f$. In particular $\mathrm{tp}_{\xi^c} = \mathrm{tp}_\xi^c$ for a $G-\mathbb{R}^n$ bundle ξ.

(1.10) **Definition.** The equivariant first Stiefel-Whitney class of a $G-S^n$ fibration η is the homotopy class $w(\eta) = [\mathrm{tp}_\eta]$ in $[\pi^G X, \mathbb{B}F_n]^{\mathcal{O}(G)}$.

If ξ is a $G-\mathbb{R}^n$ bundle we abuse notation and write $w(\xi)$ instead of $w(\xi^c)$. For a G-manifold M, we use the abbreviations $\mathrm{tp}_M = \mathrm{tp}_{TM}$, $\mathrm{tp}_M^c = \mathrm{tp}_{TM^c}$ and $w(M) = w(TM)$.

Given G-manifolds M and N of the same dimensions and a G-map $f\colon M \to N$, we define an equivariant degree which collects all information about the degrees of the components of the fixed point maps $f^H\colon M^H \to N^H$. This requires that we can compare the local orientations of M and N at corresponding points.

We fix an $\mathscr{C}(G)$-equivalence $\varphi\colon f^* \mathrm{tp}_N^c \to \mathrm{tp}_M^c$, giving G_x-homotopy equivalences $\varphi_x\colon T_{f(x)} N^c \to T_x M^c$, compatible with the action of G and with fibre transports along curves. The degree of $f\colon M \to N$ will depend on φ.

Consider first the non-equivariant case $G = 1$ and suppose M and N are connected. Let $y \in N$ be an (interior) regular value for f. For $x \in f^{-1}(y)$ there is a commutative diagram

$$\begin{array}{ccc} T_x M & \xrightarrow{\hat{f}_x} & T_y N \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ M & \xrightarrow{f} & N \end{array}$$

Here $\hat{f}_x$ is an isomorphism (in the relevant category), i and j are inclusions with $i(0) = x$, $j(0) = y$, and the differentials di_0 and dj_0 are the identity of $T_x M$ and $T_y N$, respectively. Define

(1.11) $$\deg(f; \varphi) = \sum_{x \in f^{-1}(y)} \deg(\varphi_x \circ \hat{f}_x^c).$$

For an $\mathscr{C}(G)$-groupoid $\mathscr{G}$, write

(1.12) $$\mathrm{Conj}(\mathscr{G}) = \coprod_{(H)} \mathscr{G}(G/H)^{\wedge}/WH,$$

where (H) runs over conjugacy classes of subgroups of G and where the circonflex denotes isomorphism classes of objects in the given category. Thus for $\mathscr{G} = \pi^G N$, $\mathscr{G}(G/H)^{\wedge}/WH = \pi_0(N^H)/WH$. The element in $\mathrm{Conj}(\mathscr{G})$ determined by $y \in \mathscr{G}(G/H)$ is denoted $(y, H)^{\wedge}$.

Let $C = N^H(y)$ be the component of N^H, which contains y. Let $C_1, \ldots, C_r$ be the components of M^H which map into C and $f_i\colon C_i \to C$ the restriction of f^H. Then $\dim C_i = \dim C$. The given $\mathscr{C}(G)$-transformation defines

$$\varphi_i\colon f_i^* \mathrm{tp}_C^c \to \mathrm{tp}_{C_i}^c, \qquad (\varphi_i)_x = \varphi_x^H.$$

(1.13) **Definition.** The element $\mathrm{Deg}(f; \varphi) \in \mathrm{Hom}(\mathrm{Conj}(\pi^G N), \mathbb{Z})$ is given by

$$\mathrm{Deg}(f; \varphi)(y, H)^{\wedge} = \sum_{i=1}^{r} \deg(f_i; \varphi_i)$$

on $(y, H)^{\wedge} \in \pi_0(N^H)/WH$.

We list some obvious properties of (1.13). Consider a G-map of triads

$$F: (P; M, M_+) \to (Q; N, N_+)$$

with $F \mid M = f$. Suppose Φ: $F^* \operatorname{tp}_Q^c \to \operatorname{tp}_P^c$ agrees with (the suspension of) φ when we identify $T_x P = T_x M \oplus \mathbb{R}$ for $x \in M$ using the inward pointing normal. The inclusion induces $j: \pi^G N \to \pi^G Q$. Let j^* be the $\mathbb{Z}$-dual. Then

(1.14) $$\operatorname{Deg}(f; \varphi) = j^* \operatorname{Deg}(F; \Phi)$$

As a special case, consider a G-homotopy $h: M \times I \to N$ between f_0 and f_1. Fibre transport defines an $\mathcal{O}(G)$-transformation

$$\psi_h: f_0^* \operatorname{tp}_N^c \to f_1^* \operatorname{tp}_N^c.$$

We get from (1.14) that

$$\operatorname{Deg}(f_0; \varphi \circ \psi_h) = \operatorname{Deg}(f_1; \varphi).$$

Finally, suppose we have G-maps $f: L \to M$, $g: M \to N$ and $\mathcal{O}(G)$-transformations

$$\varphi: f^* \operatorname{tp}_M^c \to \operatorname{tp}_L^c, \qquad \psi: g^* \operatorname{tp}_N^c \to \operatorname{tp}_M^c.$$

If $\pi_0(g^H)$: $\pi_0(M^H) \to \pi_0(N^H)$ is bijective for all $H \subset G$ then

(1.15) $$\operatorname{Deg}(g \circ f; \varphi \circ f^*(\psi)) = \operatorname{Deg}(g; \psi)(g^*)^{-1} \operatorname{Deg}(f; \varphi)$$

as functions on $\operatorname{Conj}(\pi^G N)$.

Consider the partial ordering in $\operatorname{Conj}(\mathcal{G})$ given by

(1.16) $$(x, H)^\wedge \leqq (y, K)^\wedge \Leftrightarrow \sigma^* y^\wedge = x^\wedge \text{ in } \mathcal{G}(G/H)^\wedge$$

for some $\sigma: G/H \to G/K$. For any $\mathcal{O}(G)$-functor $t: \mathcal{G} \to \mathbb{B}F_n$ there is a dimension function

(1.17) $$\operatorname{Dim}_{[t]}: \operatorname{Conj}(\mathcal{G}) \to \mathbb{Z}$$

whose value on $(x, H)^\wedge$ is equal to the dimension of the H-fixed set of the fibre $t(G/H)(x)_{eH}$. For $\mathcal{G} = \pi^G N$ and $t = \operatorname{tp}_M$, $\operatorname{Dim}_{[t]}(X, H)^\wedge$ is the dimension of the component $N^H(x)$ of N^H which contains x.

(1.18) **Definition.** (i) For a fixed $\mathcal{O}(G)$-functor $t: \mathcal{G} \to \mathbb{B}F_n$, $(y, K)^\wedge \in \operatorname{Conj}(\mathcal{G})$ is called an isotropy object if

$$(x, H)^\wedge > (y, K)^\wedge \Rightarrow \operatorname{Dim}_{[t]}(x, H)^\wedge < \operatorname{Dim}_{[t]}(y, K)^\wedge.$$

The set of all isotropy objects is denoted $\operatorname{Iso}(t)$.

(ii) The $\mathcal{O}(G)$-functor t satisfies the weak gap conditions if for each pair of isotropy objects $(x, H) < (y, K)$

$$8 \leqq \operatorname{Dim}_{[t]}(y, K)^\wedge + 3 \leqq \operatorname{Dim}_{[t]}(x, H)^\wedge.$$

(iii) It satisfies the strong gap conditions if

$$10 \leqq 2 \operatorname{Dim}_{[t]}(y, K)^{\wedge} < \operatorname{Dim}_{[t]}(x, H)^{\wedge}$$

A G-manifold N satisfies the gap conditions if tp_N^c does. Consider such G-manifolds and G-maps $f\colon M \to N$ for which $\pi_0 f^H\colon \pi_0 M^H \to \pi_0 N^H$ is a bijection for all $H \subset G$. Under these assumptions we have the following results from [13].

(1.19) **Proposition.** *Suppose for each pair of corresponding components $C \subset M^H$ and $D \subset N^H$*

$$(f^H)^*(w_1(D)) = w_1(C) \quad \textit{and} \quad \deg(f^H\colon C \to D) = 1.$$

Then

(i) *there exists a unique $\mathcal{O}(G)$-transformation $\varphi\colon f^* \mathrm{tp}_N \to \mathrm{tp}_M$ with* $\operatorname{Deg}(f; \varphi) \equiv 1$ *in* $\operatorname{Hom}(\operatorname{Conj}(\pi^G N), \mathbb{Z})$,

(ii) *if $|G|$ is odd, then for any φ,* $\operatorname{Deg}(f; \varphi)$ *is constant either $+1$ or -1.*

2. The Equivariant Surgery Obstruction Group

Our definition of the equivariant surgery obstruction groups is modeled upon C.T.C. Wall's geometric approach in the case of $G=1$, [21, Chap. 9]. It is a variant, and extension, of the surgery groups introduced by T. Petrie and H. Dovermann in [3].

The basic problem in the equivariant setting is which bundle data to use. On the one hand one needs unstable data in order to make surgeries and on the other hand one wants to be able to construct normal maps by equivariant transversality; this corresponds to G-stable bundle data. Only in special cases (e.g. $|G|$ odd and PL category) can one destabilize G-stable bundle data. We first present our definitions and then give a discussion of their applicability.

The definition of $\mathcal{L}^h$ makes sense both in the smooth and in the locally linear PL or $\mathcal{T}\!op$ categories. For $\mathcal{L}^s$ one needs the smooth or the locally linear PL category.

Given G-bundles ξ_1 and ξ_2, an $\mathbb{R}$-*stable* bundle map from ξ_1 to ξ_2 is a G-bundle map

$$\hat{f}\colon \xi_1 \oplus \mathbb{R}^{k_1} \to \xi_2 \oplus \mathbb{R}^{k_2}$$

where G acts trivially on $\mathbb{R}^k$. It will not be necessary to keep track of the dimensions k_i, so we will abbreviate notation and write $\hat{f}\colon \xi_1 \to \xi_2$.

Given an $\mathcal{O}(G)$-groupoid and an $\mathcal{O}(G)$-functor $t\colon \mathcal{G} \to \mathbb{B}F_n$ (or $\mathbb{B}_n$), its k-fold suspension $\Sigma^k t$ is the composite $\mathcal{G} \to \mathbb{B}F_n \xrightarrow{\Sigma^k} \mathbb{B}F_{n+k}$. An $\mathbb{R}$-stable $\mathcal{O}(G)$-transformation between the $\mathcal{O}(G)$-functors t_1 and t_2 is an $\mathcal{O}(G)$-transformation $\varphi\colon \Sigma^{k_1} t_1 \to \Sigma^{k_2} t_2$. Again we will often abbreviate and write $\varphi\colon t_1 \to t_2$. The $\mathcal{O}(G)$-groupoids $\mathbb{B}_n$ and $\mathbb{B}F_n$ were defined in Sect. 1.

(2.1) **Definition.** A G-normal map $(f, \hat{f}, \varphi)$ of triads consists of

(i) a G-map

$$(f; \partial_1 f, \partial_0 f)\colon (M; \partial_1 M, \partial_0 M) \to (N; \partial_1 N, \partial_0 N)$$

of G-manifolds with $\partial M=\partial_0 M\cup\partial_1 M$, $\partial(\partial_1 M)=\partial_1 M\cap\partial_0 M=\partial(\partial_0 M)$ and $\partial_i f=f\,|\,\partial_i M$,

(ii) a $G-\mathbb{R}^m$-bundle ξ over N and an $\mathbb{R}$-stable G-bundle map $\hat{f}$: TM $\to\xi$ over f,

(iii) an $\mathbb{R}$-stable $\mathcal{O}(G)$-transformation φ: $\mathrm{tp}^c_\xi\to\mathrm{tp}^c_N$.

If $\partial_1 M=\emptyset$ we call $(f,\hat{f},\varphi)$ a G-normal map of pairs.

A G-normal map gives the $\mathbb{R}$-stable $\mathcal{O}(G)$-transformation

$$\varphi\circ\hat{f}\colon \mathrm{tp}^c_M\to f^*\,\mathrm{tp}^c_\xi\to f^*\,\mathrm{tp}^c_N$$

so we have the function $\mathrm{Deg}(f;(\varphi\circ\hat{f})^{-1})$: $\mathrm{Conj}(\pi^G N)\to\mathbb{Z}$. It does not matter that $\varphi\circ\hat{f}$ is only given $\mathbb{R}$-stably.

In (1.18) we defined the concept of isotropy object in $\mathrm{Conj}(\pi^G N)$ w.r.t. tp_N; $(x,H)^\wedge\in\pi_0 N^H/NH$ is isotropic if the component $C(x)$ of N^H determined by x contains an element with isotropy group H. The subset of all isotropy objects is denoted $\mathrm{Iso}(\mathrm{tp}_N)$. For a G-map $f\colon M\to N$, $f_*\colon\mathrm{Conj}(\pi^G M)\to\mathrm{Conj}(\pi^G N)$ maps $(x,H)^\wedge$ to $(f(x),H)^\wedge$. A G-homotopy equivalence $f\colon X\to Y$ between $G-CW$ complexes defines an equivariant torsion $wh^G(f)\in Wh^G(\pi^G Y)$ in the equivariant Whitehead group, [5; 10; 14]. The map f is *G-simple* if $wh^G(f)=0$.

(2.2) **Definition.** A [G-simple] G-surgery problem is a G-normal map which satisfies the extra conditions

(i) $\mathrm{Deg}(f;(\varphi\circ\hat{f})^{-1})\equiv 1$,

(ii) $f_*\colon\mathrm{Conj}(\pi^G M)\to\mathrm{Conj}(\pi^G N)$ has $\mathrm{Iso}(\mathrm{tp}_M)=f_*^{-1}\,\mathrm{Iso}(\mathrm{tp}_N)$,

(iii) $\partial_0 f\colon\partial_0 M\to\partial_0 N$ is a [G-simple] G-homotopy equivalence.

(2.3) **Definition.** A reference $R=(\mathcal{G},t_0,t_1,\tau)$ of ambient dimension n consists of an $\mathcal{O}(G)$-groupoid $\mathcal{G}$, two $\mathcal{O}(G)$-functors

$$t_0\colon\mathcal{G}\to\mathbb{B}_{n+k},\ t_1\colon\mathcal{G}\to\mathbb{B}_n$$

and an $\mathbb{R}$-stable $\mathcal{O}(G)$-equivalence $\tau\colon t_0^c\to t_1^c$ between the associated functors $t_i^c\colon\mathcal{G}\to\mathbb{B}F_{n+k}$. The suspension $\sum R$ is the $(n+1)$-dimensional reference $\sum R=(\mathcal{G},\Sigma t_0,\Sigma t_1,\Sigma\tau)$.

(2.4) **Definition.** A reference $R=(\mathcal{G},t_0,t_1,\tau)$ is called G-simple if for each pair (x,H) with $x\in\mathcal{G}(G/H)$, the homotopy equivalence

$$\tau\colon t_0^c(G/H)(x)_{eH}\to t_1^c(G/H)(x)_{eH}$$

is H-simple.

A map of [G-simple] references of the same ambient dimension

$$\rho\colon(\mathcal{G},t_0,t_1,\tau)\to(\mathcal{G}',t_0',t_1',\tau') \tag{2.5}$$

is a triple $\rho=(\lambda,\mu_0,\mu_1)$ consisting of an $\mathcal{O}(G)$-functor $\lambda\colon\mathcal{G}\to\mathcal{G}'$ and $\mathbb{R}$-stable [G-simple] $\mathcal{O}(G)$-transformations $\mu_i\colon t_i\to\lambda^* t_i'$ such that

(i) $\tau'\circ\mu_0^c=\mu_1^c\circ\tau$

(ii) $\lambda_*^{-1}\,\mathrm{Iso}(t_1')=\mathrm{Iso}(t_1)$.

The opposite of a reference map $\rho=(\lambda, \mu_0, \mu_1)$ is defined to be

$$-\rho=(\lambda, -\mu_0, -\mu_1)$$

where $-\mu_i=\mu_i \oplus(-\mathrm{id})\colon t_i \oplus \mathrm{tp}_{\mathbb{R}} \to \lambda^* t_i' \oplus \mathrm{tp}_{\mathbb{R}}$.

Given a G-normal map $(f, \hat{f}, \varphi)$ we get an associated reference $(\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi)$.

(2.6) **Definition.** A G-normal map with reference $R=(\mathscr{G}_0, t_0, t_1, \tau)$ is a G-normal map together with a map of references $\rho\colon (\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi) \to (\mathscr{G}, t_0, t_1, \tau)$.

Given an n-dimensional reference R, we can now imitate the geometric definition of L-groups given in [21, Chap. 9] as certain bordism groups. Let

$$\begin{aligned} &f\colon (M, \partial M) \to (N, \partial N),\ \partial f \text{ [G-simple] homotopy equivalence,} \\ &\hat{f}\colon TM \to \xi, \\ &\varphi\colon \mathrm{tp}_\xi^c \to \mathrm{tp}_N^c, \\ &\rho\colon (\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi) \to R \end{aligned}$$

be a G-surgery problem of pairs with $\dim M=n$ and with reference R.

A [G-simple] null bordism of $(f, \hat{f}, \varphi, \rho)$ is a G-surgery problem of triads of one dimension higher $(F, \hat{F}, \Phi, \rho)$,

$$\begin{aligned} &F\colon (P, \partial_0 P, \partial_1 P) \to (Q, \partial_0 Q, \partial_1 Q),\ \partial_1 F=f \\ &\hat{F}\colon \mathrm{TP} \to \eta,\ \partial_1 \hat{F}=\hat{f},\ \partial_0 F \text{ a [G-simple] homotopy equivalence} \\ &\phi\colon \mathrm{tp}_\eta^c \to \mathrm{tp}_Q^c, \quad \rho\colon (\pi^G Q, \mathrm{tp}_\eta, \mathrm{tp}_Q, \phi) \to \textstyle\sum R \end{aligned}$$

such that there are isomorphisms $u_0\colon \partial_0 P \to M$, $v_0\colon \partial_0 Q \to N$ and an $\mathbb{R}$-stable bundle map $\hat{v}_0\colon \eta \mid \partial_0 Q \to \xi$ for which all the obvious compatibility conditions hold.

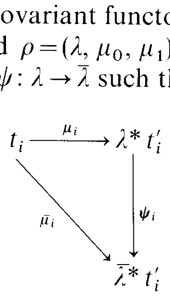
More generally, two G-surgery problems of pairs over R are bordant if there is a null bordism of the disjoint union $(f_0, \hat{f}_0, \varphi_0, \rho_0)+(f_1, \hat{f}_1, \varphi_1, -\rho_1)$.

(2.7) **Definition.** The bordism classes of G-surgery problems of n-dimensional pairs with reference $R=(\mathscr{G}, t_0, t_1, \tau)$ is denoted $\mathscr{L}_n^h(\mathscr{G}, t_0, t_1, \tau)$. If R is G-simple the corresponding G-simple bordism classes of G-simple surgery problems is denoted $\mathscr{L}_n^s(\mathscr{G}, t_0, t_1, \tau)$.

The sets $\mathscr{L}_n^h(\mathscr{G}, t_0, t_1, \tau)$ and $\mathscr{L}_n^s(\mathscr{G}, t_0, t_1, \tau)$ are groups under disjoint union of bordism classes. The zero element is the empty $\varnothing \to \varnothing$ and $-[f, \hat{f}, \varphi, \rho]=[f, \hat{f}, \varphi, -\rho]$.

A map $\rho\colon (\mathscr{G}, t_0, t_1, \tau) \to (\mathscr{G}', t_0', t_1', \tau)$ of references induces a map $\rho_*\colon \mathscr{L}_n(R) \to \mathscr{L}_n(R')$, so our $\mathscr{L}$-groups are covariant functors.

Two maps $\bar{\rho}=(\bar{\lambda}, \bar{\mu}_0, \bar{\mu}_1)$ and $\rho=(\lambda, \mu_0, \mu_1)$ from R to R' are *homotopic* of there is an $\mathscr{O}(G)$-transformation $\psi\colon \lambda \to \bar{\lambda}$ such that



commutes for $i=0, 1$ where ψ_i is induced from ψ in the obvious way,

$$\psi_i(G/H)(x)=t_i'(G/H)(\psi(G/H)(x)).$$

(2.8) **Lemma.** Homotopic maps of references induce the same map of $\mathscr{L}$-groups.

Proof. There is an obvious normal bordism, by crossing a normal map with reference R with I. □

(2.9) *Remark.* Let $G=1$ and let $\mathscr{G}=\pi$ be a group. An $\mathcal{O}(1)$-functor $t: \pi \to \mathbb{B}_n$ is an n-dimensional vector space together with a homomorphism $w: \pi \to \pi_0 \operatorname{Aut}(V) = \{\pm 1\}$. Consider the reference $R=(\pi, t, t, \mathrm{id})$. Conjugation with a fixed group element $g\in\pi$ defines an automorphism $\rho=(c(g), \mathrm{id}, \mathrm{id})$ of R. This is homotopic to the automorphism $\sigma=(\mathrm{id}, w(g)\cdot \mathrm{id}, w(g)\cdot \mathrm{id})$. Indeed, $\psi: c(g)\to \mathrm{id}$ is the $\mathcal{O}(1)$-transformation induced by the morphism g^{-1}. By (2.8), $\rho_*=\sigma_*$. By definition of the group structure in $\mathscr{L}_n(R)$, $\sigma_*=-\mathrm{id}$ if $w(g)=-1$; cf. the discussion and slight correction of Wall's definition of L-groups given in [7].

In the next section we prove under mild restrictions on R that each element of $\mathscr{L}_n^?(R)$ can be represented by a G-surgery problem with $\rho: (\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi) \to R$ such that $\lambda: \pi^G N \to \mathscr{G}$ is a weak homotopy equivalence.

Our $\mathbb{R}$-stable bundle data used above are more restrictive than one would like them to be, so it is in order to make two points. First, they look more restrictive than the bundle data used in [3, Chap. 4], but actually they are not, see the appendix. Second, for G of odd order and in the locally linear PL category the natural G-stable bundle data, $\hat{f}: TM \oplus V \to \xi$ (V an arbitrary $\mathbb{R}G$-module can be desuspended to the $\mathbb{R}$-stable ones provided M (and N) satisfies the strong gap conditions, cf. [15 II].

(2.10) **Lemma.** *Let $f: (M, \partial M)\to (N, \partial N)$ be a map covered by an $\mathbb{R}$-stable bundle map $\hat{f}: TM \to \xi$. Then there exists a representation W and a fibrewise G-map*

$$\Phi: (\xi \oplus W)^c \to (TN \oplus W)^c.$$

Proof. Let V be a representation in which M can be embedded. Then the composition $M \to N \to \xi \oplus V$ is homotopic to a G-embedding, say $f_0: M\to \xi\oplus V$, and

$$f_0^*(T(\xi\oplus V)) \cong f^*(\xi\oplus V)\oplus f^*(TN) \cong TM\oplus V \oplus f^*(TN).$$

It follows that $\nu(f_0)\oplus TM \cong V\oplus f^*(TN)\oplus TM$ and (after a change of V) we may assume

$$\nu(f_0)\cong V\oplus f^*(TN).$$

We now collapse onto a tubular neighbourhood of $f_0(M)$ in $\xi\oplus V$ and compose with the bundle map $f^*(TN)\to TN$ to get a G-map of Thom spaces

$$\bar{\Phi}: N^{\xi\oplus V} \to M^{f^*(TN)\oplus V} \to N^{TN\oplus V}.$$

This is induced by a fibrewise G-map

$$\Phi\colon (\xi \oplus W)^c \to (TN \oplus W)^c$$

for some representation W which contains V. Indeed, embed N in some representation U and add $\nu(N, U)$ to Φ; this gives

$$N^{\xi \oplus V \oplus \nu} \to N^{U \oplus V} = N_+ \wedge (U \oplus V)^c.$$

Since the Thom space is the quotient of the fibrewise one point compactification by the section at ∞, we obtain

$$(\xi \oplus V \oplus \nu)^c \to (U \oplus V)^c$$

and we can add TN again. This gives (2.8) with $W = U \oplus V$. □

The converse of (2.8) is not in general true. Given G-bundles ξ and η over N, and a fibrewise G-map $\Phi\colon \xi^c \to \eta^c$. Suppose Φ can be deformed to a G-map f which is G-transverse to $N \subset \eta$. Set $M = f^{-1}(N)$. Then $f\colon M \to N$ is covered by a bundle map $\hat{f}\colon TM \oplus \hat{f}^*(\eta) \to TN \oplus \xi$. Thus we pick a complement $\eta \oplus \zeta \cong N \times U$ to get

$$\begin{array}{ccc} TM \oplus U & \xrightarrow{\hat{f}} & TN \oplus (\xi \oplus \zeta) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N. \end{array}$$

This is not a normal map in the sense of (3.1 (ii)) unless U is the trivial representation; we need to desuspend $\hat{f}$. Second, in general Φ cannot be made G-transverse, see [15 I] for a discussion. If G has odd order, and N satisfies the strong gap conditions then the transversality and desuspension can always be achieved in the locally linear PL category.

The next lemma is an easy consequence of (1.13) and (1.18)

(2.11) **Lemma.** *Let $(F, f_+, f)\colon (P, M_+, M) \to (Q, N_+, N)$ be a G-map between triads satisfying the weak gap conditions. Suppose the inclusions $M_+ \subset P$ and $N_+ \subset Q$ are G simply connected and that f_+ is a G-homotopy equivalence. Then there exists an $\mathcal{O}(G)$-transformation $\varphi\colon f^* \operatorname{tp}_N \to \operatorname{tp}_M$ such that* $\operatorname{Deg}(f; \varphi) \equiv 1$.

Hence a G-map which satisfies 2.1 (i), (ii) can be normally cobordant to a G-homotopy equivalence only if (2.1 iii)), (2.2 i)) and (2.2 (ii)) are satisfied.

3. The $\pi - \pi$ Results

In this section we prove (under mild restrictions) that each bordism class of G-surgery problems contains a representative whose reference map ρ is an equivalence. This is the G-quivariant version of [21, Theorem 9.4]: $L_n^1 = L_n^2$. Also the equivariant $\pi - \pi$ theorem is valid provided we work in the smooth or in the PL categories, cf. [4], or [21, Theorem 3.3].

First, we use the ordering of (1.16) to introduce a necessary restriction on the reference $R = (\mathcal{G}, t_0, t_1, \tau)$.

(3.1) **Definition.** An $\mathcal{O}(G)$-functor $t\colon \mathcal{G} \to \mathbb{B}F_n$ is called geometric if for each $(x, H)^\wedge \in \operatorname{Conj}(\mathcal{G})$ there exists a maximal isotropy object $(x_0, H_0)^\wedge \in \operatorname{Iso}(t)$, larger than $(x, H)^\wedge$ with the same dimension, i.e.

(i) $(x, H)^\wedge \leqq (x_0, H_0)^\wedge$
(ii) $(x, H)^\wedge \leqq (y, K)^\wedge$ and $(y, K)^\wedge \in \operatorname{Iso}(t) \Rightarrow (x_0, H_0)^\wedge \leqq (y, K)^\wedge$
(iii) $\operatorname{Dim}_{[t]}(x, H)^\wedge = \operatorname{Dim}_{[t]}(x_0, H_0)^\wedge$
(iv) For $\sigma \in \operatorname{Hom}_G(G/H), G/H_0)$, $\sigma^*\colon \operatorname{Aut}_{\mathcal{G}(G/H_0)}(x_0) \xrightarrow{\cong} \operatorname{Aut}_{\mathcal{G}(G/H)}(\sigma^* x_0)$.

The standard example $(\mathcal{G}, t) = (\pi^G N, \operatorname{tp}_N^c)$ is geometric. Indeed, for each component C of N^H there is a unique $K \supset H$ with $C \subset N^K$ and $K = G_x$ for some $x \in C$.

A reference $R = (\mathcal{G}, t_0, t_1, \tau)$ is called geometric if $(\mathcal{G}, t_1^c)$ or equivalently $(\mathcal{G}, t_0^c)$ is geometric. Our main result is the following

(3.2) **Theorem** ($\pi - \pi$ *lemma*). *Let R be a geometric reference of ambient dimension n which satisfies the weak gap conditions. Then any element $\omega \in \mathcal{L}_n^h(R)$ (resp. $\mathcal{L}_n^s(R)$) contains a representative $(f, \hat{f}, \varphi, \rho)$, with $\lambda\colon \pi^G N \to \mathcal{G}$ a weak $\mathcal{O}(G)$-equivalence.*

Before we begin the proof of (3.2) we do a preliminary modification on $(f, \hat{f}, \varphi, \rho)$ to obtain the following:

(3.3) (i) There is a 2-dimensional $G-CW$ complex K with $\pi^G K = \mathcal{G}$, and a G-map $\lambda\colon N^{(2)} \to K$ from the 2-skeleton $N^{(2)}$ of N inducing $\lambda\colon \pi^G N \to \mathcal{G}$
(ii) $f^H\colon M^H \to N^H$ is 2-connected for $H \subset G$
(iii) $\rho_*\colon \operatorname{Conj}(\pi^G N) \to \operatorname{Conj}(\mathcal{G})$ maps $\operatorname{Iso}(\operatorname{tp}_N)$ *onto* $\operatorname{Iso}(t_0)$.

We may take K to be the 2-skeleton of $K(\mathcal{G}, 1)$. By (1.7), there is a map of references

$$i\colon (\pi^G K, i^* t_0, i^* t_1, i^* \tau) \to (\mathcal{G}, t_0, t_1, \tau),$$

inducing an isomorphism between the corresponding L-groups, and $\lambda\colon \pi^G N \to \mathcal{G}$ can be realized by $\lambda\colon N^{(2)} \to K$. We get (ii) by doing zero and one-dimensional surgeries.

For (iii), we add appropriate null-bordant G-surgery problems with R-reference to $(f, \hat{f}, \varphi, \rho)$ as follows. Let $(x, H)^\wedge \in \pi^G K$ be an isotropy object. There are the $G - S^{n+k}$ (resp. $G - S^n$) bundles $t_0^c(G/H)(x)$ (resp. $t_1^c(G/H)(x)$) over G/H. We can destabilize the first one, and write

$$\eta_0^c \oplus (\mathbb{R}^k)^c = t_0^c(G/H)(x), \qquad \eta_1^c = t_1^c(G/H)(x).$$

The $\mathbb{R}$-stable homotopy equivalence τ destabilizes to an equivariant homotopy equivalence $\tau\colon \eta_0^c \to \eta_1^c$. The tangent bundles are $T\eta_i^c = p_i^*(\eta_i^c)$ where p_i is the projection for η_i^c. Consider the surgery problem $(\tau, \hat{\tau}, \varphi, \rho)$

$$\begin{array}{ccc} T\eta_0^c & \xrightarrow{\hat{\tau}} & p_1^* \eta_0 \\ \downarrow & & \downarrow \\ \eta_0^c & \xrightarrow{\tau} & \eta_1^c, \end{array}$$

with $\varphi = p_1^* \tau$, and map of references $\rho = (\lambda, \mathrm{id}, \mathrm{id})$, $\lambda = \pi^G(x \circ p_1)$. Here $x \in \mathcal{G}(G/H) = \mathrm{Map}_G(G/H, K)$. We leave for the reader to check that λ satisfies (2.5(ii)); at this point one uses that R is geometric.

Given $(f, \hat{f}, \varphi, \rho)$ which satisfies (3.3), we shall do zero and one-dimensional G-surgeries on $\lambda: N \to K$ to make λ^H 2-connected, and simultaneously do surgery on $f: M \to N$.

One can do surgery only on isotropy components of N, but this suffices since we have

(3.4) **Lemma.** Given a map $(\lambda, \mu_0, \mu_1): R \to R'$ of geometric references. Then $\lambda: \mathcal{G} \to \mathcal{G}'$ is a weak $\mathcal{O}(G)$-equivalence if and only if $\lambda_*: \mathrm{Conj}(\mathcal{G}) \to \mathrm{Conj}(\mathcal{G}')$ maps $\mathrm{Iso}(t_0)$ bijectively to $\mathrm{Iso}(t_0')$ and $\lambda(G/H): \mathcal{G}(G/H) \to \mathcal{G}'(G/H)$ induces an isomorphism between $\mathrm{Aut}_{\mathcal{G}(G/H)}(x)$ and $\mathrm{Aut}_{\mathcal{G}'(G/H)}(\lambda(x))$ for all isotropy objects (x, H). □

Proof of Theorem 3.2. Consider an element in $\pi_{\ell+1}(\lambda^H)$, $\ell = 0, 1$ given by a G-diagram

$$\begin{array}{ccc} G/H \times S^\ell & \xrightarrow{\;j\;} & N^{(2)} \\ \big\downarrow & & \big\downarrow \lambda \\ G/H \times D^{\ell+1} & \xrightarrow{\;k\;} & K \end{array} \tag{3.5}$$

We want to do surgeries on N and on M simultaneously to kill (j, k). This requires that f induces a G-isomorphism from $f^{-1}(G/H \times S^l)$ to $G/H \times S^l$, and in particular that $f^{-1}(eH \times S^l) \subset M^H$.

We only treat the case $\ell = 1$; $\ell = 0$ is similar but easier. Assume for convenience that M^H and N^H are connected (and isotropic), in general one works with connected components separately.

Step 1. Let V and W be the H-modules given by the fibres of ξ and TN over the point $k(eH, 0)$. We have bundles $\zeta_0 = G \times_H V$ and $\zeta_1 = G \times_H W$ over $G/H \times D^2$, and, since D^2 is contractible unique $\mathcal{O}(G)$-equivalences $k^* t_s \to \mathrm{tp}_{\zeta_s}$, $s = 0, 1$. Using μ_s from the reference map $\rho = (\lambda, \mu_0, \mu_1)$ we get over $G/H \times S^1$

$$\mu_0': j^* \mathrm{tp}_\xi \to \mathrm{tp}_{\zeta_0} \quad \text{and} \quad \mu_1': j^* TN \to \mathrm{tp}_{\zeta_1}.$$

Since $G/H \times S^1$ is 1-dimensional we can choose ($\mathbb{R}$-stable) G-bundle isomorphisms $b_0: j^* \xi \to \zeta_0 \oplus \mathbb{R}^k$ and $b_1: j^* \mathrm{TN} \to \zeta_1$ with

$$\mathrm{tp}_{b_s} = \mu_s'. \tag{3.6}$$

We remark that the isotopy classes of b_s are not determined by (3.6), and that $j^* \xi$ and b_0 destabilizes k times.

By the immersion classification theorems, [8], [9], we may suppose that j is an embedding of S^1 in $N_H = \{x \in N \mid G_x = H\}$ associated to the bundle isomorphism $b_1^H: j^* TN^H \to \zeta_1^H$. As WH acts freely on N_H we can G-homotop f to obtain that $f^H: M^H \to N^H$ is transverse to $eH \times S^1 \subset N_H$, and induces a diffeomorphism (resp. PL-homeomorphism or homeomorphism)

$$(f^H)^{-1}(eH \times S^1) \to eH \times S^1$$

(cf. [21, p. 90]). After a further isotopy, we get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ G/H \times S^1 & \xrightarrow{\text{id}} & G/H \times S^1. \end{array}$$

We want to thicken i and j, and begin by destabilizing the G-bundle isomorphisms

$$c: i^* TM \xrightarrow{\hat{f}} j^* \xi \xrightarrow{b_0} \zeta_0, \qquad d = b_1: j^* TN \to \zeta_1.$$

It suffices to consider the corresponding H-bundles over $S^1 = eH \times S^1$.

Let V be an open H-neighbourhood of S^1 in M which H-deforms onto S^1. Since $TM|S^1$ is trivial so is TU, and by G-smoothing theory [11], U is a smooth G-manifold. Let $\nu(S^1, M)$ be the H-normal bundle of S^1 in U. Then $\nu(S^1, M) \oplus \mathbb{R} = \mathrm{TM}|S^1$. Write $V = V_0 \oplus \mathbb{R}^{k+1}$ and $W = W_0 \oplus \mathbb{R}$. Since $\mathrm{Aut}_H(V_0) \to \mathrm{Aut}_H(V)$ is 1-connected in all three categories the maps c and d desuspend to H-bundle isomorphisms

(3.7) $$\bar{c}: \nu(S^1, M) \to V_0, \qquad \bar{d}: \nu(S^1, N) \to W_0.$$

Step 2. We next deform f in a neighbourhood of M^H. Let $e_0: V_0 \to W_0$ be a norm preserving H-equivariant map such that the diagram below of $\mathbb{R}$-stable maps commutes up to H-homotopy

$$\begin{array}{ccc} (V_0 \oplus \mathbb{R})^c & \xrightarrow{e_0 \oplus 1} & (W_0 \oplus \mathbb{R})^c \\ \uparrow{\scriptstyle \bar{d}_x} & & \uparrow{\scriptstyle \bar{c}_x} \\ TM_x^c & \xrightarrow{\psi} & TN_{f(x)} \end{array}$$

($\psi = \varphi \circ \hat{f}$, cf. 2.1 (iii)). By a cofibration argument and because $\mathrm{Deg}(f; \psi^{-1})(x, H)^\wedge = 1$ we may change f up to G-homotopy so that

$$f = \bar{d}^{-1} \circ e_0 \circ \bar{c}: \nu(S^1, M) \to \nu(S^1, N)$$

in a neighbourhood of S^1. Thus there is an open G-set U in M with

(3.8) (i) $M^H \subset U$
(ii) $f^{-1}(S^1) \cap U = S^1$
(iii) $T_x M^c \xrightarrow{\psi} T_{f(x)} N^c \xrightarrow{df_x^{-1}} T_x M^c$ has degree 1
on all fixed sets for $x \in S^1$, (cf. 1.13).

Step 3. We finish the proof under the additional assumption that $U = M$ in (3.8). Indeed, one obtains the desired bordism $F: P \to Q$ by the G-push-outs:

$$\begin{array}{ccccc} G \times_H (D^2 \times DV_0) & \longleftarrow & G \times_H (S^1 \times DV_0) & \xrightarrow{G \times_H \bar{c}} & M \times I \\ \downarrow {\scriptstyle G \times_H (\mathrm{id} \times e_0)} & & \downarrow {\scriptstyle G \times_H (\mathrm{id} \times e_0)} & & \downarrow {\scriptstyle f \times \mathrm{id}} \\ G \times_H (D^2 \times DW_0) & \longleftarrow & G \times_H (S^1 \times DW_0) & \xrightarrow{G \times_H \bar{d}} & N \times I, \end{array}$$

if we identify $v(S^1, M)$ and $v(S^1, N)$ with neighbourhoods in M and N. The assumption $U = M$ and (3.8(ii)) ensure that F respects boundaries. There is an obvious extension of $\lambda: N \to K$ to $\Lambda: Q \to K$. The bundle data extend by construction and the reference map by (3.6). The map F has degree one by (1.14).

Step 4. It remains to obtain $U = M$. We prove inductively that U contains M^K for more and more K, beginning with $K = H$, (3.8). So suppose $M^{>K} \subset U$; we attempt to enlarge U to U_0 with $U_0 \supset M^K$ and without destroying (3.8(ii)).

By possibly shrinking U we can find a G-neighbourhood U' of $\mathrm{cls}(U)$ with $f^{-1}(S^1) \cap U' \setminus U = \varnothing$. Let $M' = M^K \setminus (U \cap M^K)$ and write $f' = f^K | M': M' \to N^K$. Note that WK acts freely on M'.

We can change

$$f' \times \mathrm{id}/WK: M'/WK \to N^K \times M'/WK$$

relative to $U' \cap M'/WK$ to make it transverse to Y/WK, where $Y = (NK/NK \cap H \times S^1) \times M' \subset N^K \times M'$. The preimage is a one-dimensional submanifold of M'/WK. Two of its components can be connected by a path whose image under $f' \times \mathrm{id}/WK$ is homotopic to a path in Y/WK (use (3.3(ii)). We homotop $f' \times \mathrm{id}/WK$ relative to $M' \cap U'/WK$ keeping it transverse to Y/WK, and such that the preimage has one component less. In the end the preimage consists of a single circle.

We have achieved that $f': M' \to N^K$ is WK-transverse to $NK/NK \cap H \times S^1$ and that $(f')^{-1}(NK/NK \cap H \times S^1)$ is the preimage of a circle $S^1 \subset M'/WH$ under the WK-principal bundle $M' \to M'/WK$. Let $v: S^1 \to S^1$ be the map induced by f^K from $S^1 \subset M'/WK$ to the circle $S^1 \subset N^K/WK$ coming from $NK/NK \cap H \times S^1 \subset N^K$. Consider the $\mathcal{O}(G)$-transformation

$$\psi = (\varphi \circ \hat{f})^{-1}: f^* \mathrm{tp}_N^c \to \mathrm{tp}_M^c$$

and let ψ^K be the induced transformation on M^K. Choose a regular value $z \in S^1 \subset N^K$ of f^K. Notice that $(f^K)^{-1}(S^1)$ consists of a part in M^H which is a single point $\{y\}$ and a part in M'. One easily checks that

(3.9) $\quad \mathrm{Deg}(f^K, \psi^K)(z, K) = \deg(\psi^K(y)^{-1} \circ T_y f^K) + |NK/NK \cap H| \cdot \deg v$

Since $\mathrm{Deg}(f^K, \psi^K)(z, K)$ and $\deg(\psi^K(y)) \circ T_y f^K)$ are both equal to one, $\deg v = 0$. By the weak gap conditions (1.18) and (3.3 (ii)), $M_K/WK \to N^K/WK$ is 2-connected so that $S^1 \subset M_K/WK$ is null-homotopic. Hence $(f')^{-1}(NK/NK \cap H \times S^1)$ is equal to $WK \times S^1$ and f^K restricted to $eK \times S^1$ is null-homotopic.

We can assume f^K, in a tubular neighbourhood of $S^1 = eK \times S^1 \subset WK \times S^1$, looks like $c \times \mathrm{id}: S^1 \times D^{m-1} \to \{z\} \times D^{m-1} = v_1(S^1, N^K)$. The obvious extension $c \times \mathrm{id}: D^2 \times D^{m-1} \to \{z\} \times D^{m-1}$ has the property that its restriction to $D^2 \times S^{m-2}$ does not meet $S^1 \times N^K$. Hence we can do surgery,

$$\begin{array}{ccc} S^1 & \longrightarrow & M^K \\ \downarrow & & \downarrow{\scriptstyle f^K} \\ D^2 & \xrightarrow{\;c\;} & N^K. \end{array}$$

The new normal map $f_+: M_+ \to N$ is equal to f on U, and $f_+^{-1}(S^1) \cap (M^K \setminus (U \cap M^K)) = \varnothing$. We can enlarge U to $U_0 \supset M^K$ such that $f^{-1}(S^1) \cap U_0 = S^1$. This completes the proof. □

Note in particular that (3.2), applied to the normal map $\varnothing \to \varnothing$, gives a normal map $M \to N$ with the prescribed 'fundamental groupoid data'. The next theorem was proved in [4] when $\pi_1(M^H) = 0$ and in [21, Chap. 3] when $G = 1$.

(3.10) **Theorem** (*$\pi - \pi$ theorem*). *Let $f: (M, \partial_0 M, \partial_1 M) \to (N, \partial_0 N, \partial_1 N)$ be a [G-simple] G-surgery problem in the smooth or PL* category. *Suppose $\pi^G(\partial_1 N) \to \pi^G(N)$ is a weak homotopy equivalence and that N satisfies the strong gap conditions* (1.18). *Then F is G-normally cobordant (rel $\partial_0 N$) to a surgery problem $(f_+, \partial_0 f_+, \partial_1 f_+)$ which is a [simple] G-homotopy equivalence of triads.* □

We shall not elaborate on the proof of (3.10). It is similar to the proof presented in [4], but two comments are in order. Firstly, there are certain minor errors in [4], which the reader can overcome by using some of the material presented in the proof of (3.2) above. Secondly, in the extension of the $\pi - \pi$ theorem to the case of non-simply connected fixed sets one uses the group extension.

$$1 \to \pi_1(M^H, x) \to E(x, H) \to WH(x) \to 1$$

and the $E(x, H)$-action on the universal covering $M^H(x)^\sim$ over the component $M^H(x)$ of M^H which contains x. The surgery is done in the regular part of $M^H(x)^\sim$. In the G-simple case one needs to compare the equivariant Whitehead torsions $wh(f)$ and $wh(f, \partial f)$ for G-homotopy equivalences

$$(f, \partial f): (M, \partial M) \to (N, \partial N), \; f: M \to N$$

This is done [1], where an involution

$$*: Wh(\pi^G N) \to Wh(\pi^G N)$$

is constructed (by reversing h-cobordisms) such that

(3.11) $$wh(f) = -*wh(f, \partial f).$$

(Here we use the assumption that R be G-simple.)

In particular, $wh(f)=0 \Leftrightarrow wh(f, \partial f)=0$. See [4, Chap. 5] for a discussion when $\pi_1(N^H, x)=0$ for all $H \subset G$, $x \in N^H$.

Finally, one defines the equivariant surgery obstruction of an n-dimensional normal map $(f, \hat{f}, \varphi)$ to be the class $\lambda^G(f, \hat{f}, \varphi)$ it represents in $\mathscr{L}_n(\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi)$. From (3.2) and (3.10) one gets

(3.12) **Theorem** *If $N \times I$ satisfies the strong gap conditions in* (1.18), *then a* (G-*simple*) *normal map* $(f, \hat{f}, \varphi)$ *is normally cobordant to a* (G-*simple*) G-*homotopy equivalence if and only if* $\lambda^G(f, \hat{f}, \varphi)=0$.

4. Functorial Properties

This section compares the G-equivariant L-groups for varying G. Let $i: \Gamma \to G$ be a homomorphism of groups. It induces a functor $i: \mathcal{O}(\Gamma) \to \mathcal{O}(G)$, $i(X) = G \times_\Gamma X$. For any category $\mathscr{C}$ consider the functor categories $[\mathcal{O}(\Gamma), \mathscr{C}]^{\mathrm{op}}$ of contravariant functors. Then i induces

$$i^*: [\mathcal{O}(G), \mathscr{C}]^{\mathrm{op}} \to [\mathcal{O}(\Gamma), \mathscr{C}]^{\mathrm{op}}.$$

Under mild restrictions on $\mathscr{C}$, i^* has a left adjoint

$$i_*: [\mathcal{O}(\Gamma), \mathscr{C}]^{\mathrm{op}} \to [\mathcal{O}(G), \mathscr{C}]^{\mathrm{op}}$$

(see e.g. [17 II, § 17]). In particular, for each $\mathscr{G} \in [\mathcal{O}(G), \mathscr{C}]^{\mathrm{op}}$ there is the adjunction

(4.1) $$\psi(\mathscr{G}): i_* i^* \mathscr{G} \to \mathscr{G}.$$

We spell out the definition of $i_*(\mathscr{G})$ when $\mathscr{C}$ is the category of groupoids. Given an $\mathcal{O}(\Gamma)$-groupoid $\mathscr{G}$, and $G/L \in \mathcal{O}(G)$ let

(4.2) $$\mathscr{G}(G/L)' = \coprod_{\Gamma/K \in \mathcal{O}(\Gamma)} \mathscr{G}(\Gamma/K) \times \mathrm{Hom}_G(G/L, i(\Gamma/K)).$$

There is an equivalence relation on $\mathscr{G}(G/L)'$, defined as follows. For a Γ-map $f: \Gamma/K \to \Gamma/H$ and object

$$(u, v) \in \mathscr{G}(\Gamma/H) \times \mathrm{Hom}_G(G/L, i(\Gamma/K))$$

we set

$$(\mathscr{G}(f)\, u, v) \sim (u, i(f)\, v).$$

Similarly for a morphism $\varphi: u_0 \to u_1$ in $\mathscr{G}(\Gamma/H)$,

$$(\mathscr{G}(f)(\varphi), v) \sim (\varphi, i(f)(v)).$$

This generates an equivalence relation and $(i_* \mathscr{G})(G/L)$ is the groupoid of equivalence classes. Letting L vary over the subgroups of G we obtain the $\mathcal{O}(G)$-groupoid $i_*(\mathscr{G})$.

If X is a Γ-space and Y a G-space then

$$i_* \pi^\Gamma(X) = \pi^G(G \times_\Gamma X), \qquad i^* \pi^G(Y) = \pi^\Gamma(Y).$$

Let $\mathbb{B}_n^G$ be the groupoid of $G-\mathbb{R}^n$ bundles over the orbits G/H considered in §1. It is obvious that $i^* \mathbb{B}_n^G = \mathbb{B}_n^\Gamma$, and an $\mathcal{O}(G)$-functor $t: \mathscr{G} \to \mathbb{B}_n^G$ induces an $\mathcal{O}(\Gamma)$-functor $i^* t: i^* \mathscr{G} \to \mathbb{B}_n^\Gamma$. An $\mathcal{O}(\Gamma)$-functor $t: \mathscr{G} \to \mathbb{B}_n^\Gamma$ induces an $\mathcal{O}(G)$ functor by setting

$$i_*(t): i_*(\mathscr{G}) \to i_* \mathbb{B}_n^\Gamma = i_* i^* \mathbb{B}_n^G \xrightarrow{\psi} \mathbb{B}_n^G$$

with ψ from (4.1). If $t = \mathrm{tp}_\xi$ is the transport of a Γ-bundle over X then $i_*(t)$ is the transport of the G-bundle $G \times_\Gamma \xi$ over $G \times_\Gamma X$.

Given a Γ-reference $R = (\mathscr{G}, t_0, t_1, \tau)$ one gets a G-reference $i_* R = (i_* \mathscr{G}, i_* t_0, i_* t_1, i_* \tau)$ which is geometric if R is. Similarly, a G-reference S gives a Γ-reference $i^* S$, geometric if S is.

If $f: M \to N$ etc. is a Γ-surgery problem with reference R then we can apply the functor $G \times_\Gamma(-)$ to obtain a G-surgery problem with reference $i_* R$. Similarly, a G-surgery problem with reference S can be composed with $i: \Gamma \to G$ to define a Γ-surgery problem with reference $i^* S$. This defines homorphisms

(4.3) $$i_*: \mathscr{L}_n(\Gamma, R) \to \mathscr{L}_n(G, i_* R), \qquad i^*: \mathscr{L}_n(G; S) \to \mathscr{L}_n(\Gamma, i^* S)$$

We next define a Mackey-functor over G for a given geometric reference R of ambient dimension n. On objects,

(4.4) $$G/H \to \mathscr{L}_n(H, i(H)^* R)$$

where $i(H): H \to G$ is the inclusion. Consider a G-map $\sigma: G/H \to G/K$. Choose $g \in G$ with $\sigma(eH) = g^{-1} K$. Let $c(g)$ be conjugation with g, i.e. $c(g)(\bar{g}) = g \bar{g} g^{-1}$. We obtain a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{c(g)} & G \\ {\scriptstyle i(H)} \uparrow & & \uparrow {\scriptstyle i(K)} \\ H & \xrightarrow{c(g)} & K \end{array}$$

Consider the functors id and $c(g): \mathcal{O}(G) \to \mathcal{O}(G)$ induced by induction with the corresponding group homomorphism. For any object G/H in $\mathcal{O}(G)$ we have a morphism $r(g): G/gHg^{-1} \to G/H$ which sends $\bar{g}(gHg^{-1})$ to $\bar{g}gH$, and get a natural transformation $r(g): c(g) \to \mathrm{id}$. It induces an isomorphism of references $\rho(g): R \to c(g)^* R$. The adjunction between i_* and i^* carries over to references to give a bijection

$$\mathrm{ad}: \mathrm{Hom}(c(g)_* i(H)^* R, i(K)^* R) \to \mathrm{Hom}(i(H)^* R, c(g)^* i(K)^* R).$$

Since $c(g)^* i(K)^* R = i(H)^* c(g)^* R$ we get an element $i(H)^*(\rho(g))$ in $\operatorname{Hom}(i(H)^* R, c(g)^* i(K)^* R)$. Let $\mu(g)\colon c(g)_* i(H)^* R \to i(K)^* R$ be its preimage under ad. Now define

$$\sigma_*\colon \mathscr{L}_n(H, i(H)^* R) \to \mathscr{L}_n(K, i(K)^* R) \tag{4.5}$$

as the composition

$$\mathscr{L}_n(H, i(H)^* R) \xrightarrow{c(g)_*} \mathscr{L}_n(K, c(g)_* i(H)^* R) \xrightarrow{\mathscr{L}_n(K,\mu(g))} \mathscr{L}_n(K, i(K)^* R).$$

Similarly, let

$$\sigma^*\colon \mathscr{L}_n(K, i(K)^* R) \to \mathscr{L}_n(H, i(H)^* R) \tag{4.6}$$

be the composition

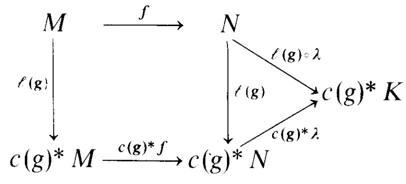
$$\mathscr{L}_n(K, i(K)^* R) \xrightarrow{c(g)^*} \mathscr{L}_n(H, c(g)^* i(K)^* R) \xrightarrow{\text{id}} \mathscr{L}_n(H, i(H)^* c(g)^* R)$$

$$\xrightarrow{i(H)^* \rho(g)^{-1}} \mathscr{L}_n(H, i(H)^* R).$$

We have to show that these definitions are independent of the choice of which g we pick with $\sigma(eH) = g^{-1} K$. This follows from

(4.7) **Lemma.** *The homomorphisms $c(g)^*$ and $\mathscr{L}_n(G, \rho(g))$ from $\mathscr{L}_n(G, R)$ to $\mathscr{L}_n(G, c(g)^* R)$ are equal.*

Proof. Suppose that $\mathscr{G} = \pi^G K$. An element in $\mathscr{L}_n(G, R)$ is represented by a surgery problem $f\colon M \to N$ and a reference map $\lambda\colon N \to K$, suppressing the bundle data. Then $c(g)^*$ applied to this element is represented by $c(g)^* f\colon c(g)^* M \to c(g)^* N$ and $c(g)^* \lambda\colon c(g)^* N \to c(g)^* K$. Multiplication $\ell(g)\colon K \to c(g)^* K$ is a G-map, and $\mathscr{L}_n(G, \rho(g))$ sends the given surgery problem to $f\colon M \to N$, $\ell(g) \circ \lambda\colon N \to c(g)^* K$. These two elements agree in $\mathscr{L}_n(G, c(g)^* R)$ since the following diagram commutes:



The reader can easily verify that the maps in (4.5) and (4.6) give the functor in (4.4) the structure of a Mackey functor. In particular we have

(4.8) **Corollary.** *The equivariant L-group $\mathscr{L}_n(G, R)$ is a module over the Burnside ring $A(G)$.*

Appendix

Comparison With Other Definitions (Smooth Category)

In the original source for G-surgery groups [3] the bundle data required for a G-normal map is somewhat different from ours (2.1 (ii)). We examine the difference.

For any G-bundle η over a G-space X, let η^H be the fixed point bundle over X^H and write η_H for the quotient bundle, $(\eta \mid X^H)/\eta^H$, with a similar notation for maps. Note that $\eta \mid X^H \cong \eta^H \oplus \eta_H$, but with no preferred isomorphism (unless we have an invariant inner metric on η).

Dovermann and Petrie's bundle data for a normal map over $f: M \to N$ consists of bundle isomorphisms

(A1)
$$a: TM \oplus V \to f^*(\xi) \oplus V \qquad (G\text{-map})$$
$$b_H: TM_H \to f^*(\xi_H) \qquad (NH\text{-map})$$

such that $a_H = b_H \oplus V_H$ and such that the b_H satisfy obvious compatibility conditions. Here V is any (large) $\mathbb{R}G$-module with $V^G \neq 0$.

(A2) **Proposition.** *Suppose $f: M \to N$ is a map between G-manifolds covered by a bundle map a. If there exist bundle maps b_H satisfying* (A1) *then there exists a bundle map $\hat{f}$: $TM \oplus V^G \to f^*(\xi) \oplus V^G$* so that $a = \hat{f} \oplus V_G$.

Before we give the easy proof of (A2) let us remark that a bundle map $\hat{f}: \eta_1 \to \eta_2$ between two bundles over M is equivalent to a section of the G-bundle $\operatorname{Iso}(\eta_1, \eta_2)$ over M whose fibre over $x \in M$ is the set of isomorphisms $\operatorname{Iso}(\eta_{1x}, \eta_{2x})$ with the induced action of G_x.

Proof of (A2). The proof will be by induction over the orbit types. This procedure is described for example in [2, § 8.1]. It reduces the proof to the following special case. Consider

$$a \mid M^H: TM \mid M^H \oplus V \to f^*(\xi \mid M^H) \oplus V$$

as an NH-isomorphism. Let $(M^H)^s$ be the singular set in M^H consisting of points with larger isotropy groups. Suppose

$$a \mid (M^H)^s = a_1 \oplus V_G$$

for some NH-bundle map

$$a_1: TM \mid (M^H)^s \oplus V^G \to f^* \xi \mid (M^H)^s \oplus V^G.$$

We want to show that there exists an NH-bundle map

$$a_2: TM \mid M^H \oplus V^G \to f^* \xi \mid M^H \oplus V^G$$

so that $a \mid M^H = a_2 \oplus V_G$.

The assumptions (A1) implies that there are decompositions $TM|M^H \cong TM^H \oplus TM_H$, $f^*(\xi) \cong f^*(\xi^H) \oplus f^*(\xi_H)$ so that

$$a|M^H = a^H \oplus b_H \oplus V_H \qquad (V_H \oplus V_G^H = V_G).$$

Hence it suffices to show that $a^H = a_0 \oplus V_G^H$ for some NH-bundle map

$$a_0 : TM^H \oplus V^G \to f^*(\xi^H) \oplus V^G$$

We do have a_0 defined over $(M^H)^s$, namely by setting $a_0|(M^H)^s = a_1^H$. The NH-space M^H is build up from $(M^H)^s$ by attaching free NH/H-cells, and we can attempt to extend a_1^H to a_0, cell by cell. Suppose a_0 already defined over X and let $Y = X \cup NH/H \times D^{i+1}$. The obstruction to extend a_0 over Y such that $a_0 \oplus V_G^H = a^H$ lie in

$$\pi_{i+1}(GL_H(T_x M^H \oplus V^G) \xrightarrow{\varphi} GL_H(T_x M^H \oplus V^H))$$

where φ adds the identity along V_G^H. There is no H-action on $T_x M^H \oplus V^H$, so

$$GL_H(T_x M^H \oplus V^H) \simeq O(\dim M^H + \dim V^H).$$

Hence φ is $(\dim M^H + \dim V^G - 1)$-connected, and since $\dim V^G \neq 0$ the stated homotopy group vanishes. This proves the existence of a_0 and completes the inductive step. □

We have used the definition (3.5′) in [4] as (A1). There is a weaker definition (3.5) in [4] (see also [3, Sect. 3]), where a: $TM \oplus V \to f^* \xi$ for some G-bundle ξ and b_H: $TM_H \to \eta_H$ for some bundle functors η_H such that certain compatibility conditions hold. However, the argument above shows that one can destabilize ξ to $\xi' \oplus V_G$ for appropriate ξ' and that one can destabilize a to $\hat{f}$: $TM \oplus V^G \to \xi'$ also in this case.

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