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Equivariant L-Theory II

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§ 0. Introduction

Equivariant algebraic K-theory decomposes as a sum of ordinary algebraic K-theory of group rings (see [4, 5, 7, 12]). This follows for K_1 because the determinant of a upper triangular matrix is the product of the entries on the diagonal.

For equivariant L-theory the situation is more complicated. We do obtain a set of exact orbit sequences similar to the neighbouring family sequences obtained by Connor and Floyd in [2]. But these exact sequences do not always split in a way analogous to the splitting of equivariant K-theory. There are easy counterexamples for $G=\mathbb{Z}/2$. However, if the transformation group has odd order, then the equivariant L-groups do in fact decompose in the expected fashion, cf. Theorem 2.11 below.

We work in this paper in the smooth and locally linear PL-category simultaneously. A consequence of the existence of the exact orbit sequence is that the equivariant L-groups are equal for these two manifold categories.

The paper is founded upon the definition of equivariant L-theory given in [9]. We refer the reader to that paper for the somewhat cumbersome definitions. A reference (I.?.?) always refers to the first part [9].

It is our hope that the present definitions of equivariant L-groups and the calculational techniques presented here will make further calculations possible. It seems to us to be of some interest to evaluate equivariant L-groups for some of the standard 2-groups, for example, and to determine the equivariant Rothenberg sequence.

The Paper is Subdivided into Sections as Follows:

§ 1. The Orbit Sequence

Let $R=(\mathscr{G}, t_0, t_1, \tau)$ be a geometric reference of ambient dimension n, as defined in (I.2.3) and (I.3.1). A subset $\mathscr{C}\subset \mathrm{Iso}\,(t_1)$ is *admissible* if with the notion of (I.1.16)

$$(x,H)^\wedge \leqq (y,K)^\wedge, \qquad (x,H)^\wedge \in \mathscr{C} \Rightarrow (y,K)^\wedge \in \mathscr{C}$$

A surgery problem $(f, \hat{f}, \varphi, \varrho)$ from M to N with reference map

$$\varrho=(\lambda, \mu_0, \mu_1):(\pi^G N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi)\to(\mathscr{G}, t_0, t_1, \tau)$$

is called *$\mathscr{C}$-restricted* if for each isotropy component $D\in\pi_0(N^H)$ with $\lambda(G/H)(D)\in\mathscr{C}$ there is precisely one component $C\in\pi_0(M^H)$ with $f^H(C)\subset D$, and if $f^H|C:C\to D$ is a [simple] $WH(x)$-homotopy equivalence for $x\in C$. The $\mathscr{C}$-restricted bordism classes of $\mathscr{C}$-restricted surgery problems with reference to R form an abelian group $\mathscr{L}_n(R)[\mathscr{C}]$. In the sequel $\mathscr{L}$ means either $\mathscr{L}^s$ or $\mathscr{L}^h$. Recall that for $\mathscr{L}^s$ we require R to be simple, cf. (I.2.4).

Consider two neighbouring admissible sets $\mathscr{C}_0\subset\mathscr{C}_1\subset \mathrm{Iso}\,(t_1)$

$$\mathscr{C}_1=\mathscr{C}_0\cup\{(x,H)^\wedge\}, \qquad (x,H)^\wedge\notin\mathscr{C}_0$$

Write $\mathscr{L}_n(R)[\mathscr{C}_0,\mathscr{C}_1]$ for the equivalence classes of surgery problems $(f,\hat{f},\varphi)$ with reference R such that f is $\mathscr{C}_0$-restricted and ∂f is $\mathscr{C}_1$-restricted, cf. (I.2.2iii.). For a nullbordism $(F,\partial_0 F,\partial_1 F)$ of $\partial_1 F$ we require F to be $\mathscr{C}_0$-restricted and $\partial_0 F$ to be $\mathscr{C}_1$-restricted.

There is a sequence, infinite to the left,

$$(1.1)\qquad \cdots \xrightarrow{i} \mathscr{L}_{n+1}(\textstyle\sum R)[\mathscr{C}_0] \xrightarrow{j} \mathscr{L}_{n+1}(\textstyle\sum R)[\mathscr{C}_0,\mathscr{C}_1] \xrightarrow{\partial} \mathscr{L}_n(R)[\mathscr{C}_1]$$
$$\xrightarrow{i} \mathscr{L}_n(R)[\mathscr{C}_0] \xrightarrow{j} \mathscr{L}_n(R)[\mathscr{C}_0,\mathscr{C}_1]$$

Here ∂ is restriction to the boundary and i and j are the forgetful maps. The usual argument from [2] gives

(1.2) **Theorem.** *The sequence* (1.1) *is exact.* □

The Weyl group WH acts on $\mathscr{G}(G/H)^\wedge$. For $x\in\mathscr{G}(G/H)$ let $WH(x)$ be the isotropy group of $\hat{x}$. From (I.1.2) we have the group $E=E_{[t_1]}(x,H)$, and the extension

$$\{1\}\to \mathrm{Aut}_{\mathscr{G}(G/H)}(x)\to E\to WH(x)\to\{1\}$$

Define the orientation homomorphism

$$(1.3)\qquad \varepsilon=\varepsilon_{[t_1]}(x,H):E(x,H)\to\{\pm 1\}$$

by sending $(\sigma,w)\in E(x,H)$ to $\deg k^H$ where k is the composition of H-maps

$$k:t_1(G/H)(x)^c_{eH} \xrightarrow{t_1(G/H)(w)^c_{eH}} t_1(G/H)(\sigma^* x)^c_{eH} \xrightarrow{\sigma^*} t_1(G/H)(x)^c_{eH}$$

We will identify the relative groups in (1.1) with the usual algebraic L-groups. More precisely we shall define an isomorphism

$$r:\mathscr{L}_n(R)[\mathscr{C}_0,\mathscr{C}_1]\to L_k(\mathbb{Z}E,\varepsilon), \qquad k=\mathrm{Dim}_{[t_1]}(x,H).$$

Notice when $(\mathscr{G}, t_1) = (\pi^G N, \mathrm{tp}_N)$ that $E = \pi_1(EWH(x) \times_{WH(x)} N^H(x))$, $k = \dim N^H(x)$ and that ε is given by transport of local orientations in $N^H(x)$. The definition of r requires that R and $\sum^2 R$ satisfy the strong gap conditions (I.1.18).

Let $\eta \in \mathscr{L}_n(R)[\mathscr{C}_0, \mathscr{C}_1]$ be represented by $(f, \hat{f}, \varphi, \varrho)$ with $\lambda : \pi^G N \to \mathscr{G}$ a weak $O(G)$-equivalence, cf. (I.1.4), and f^H 2-connected, cf. (I.3.2). Let C and D be the components of $\pi_0(M^H)$ and $\pi_0(N^H)$ with $\lambda(G/H)(D)^\wedge = \hat{x}$ and $f^H(C) \subset D$. The group E acts on the universal coverings $\tilde{C}$ and $\tilde{D}$ and $f^H|C$ lifts to an E-normal map $\tilde{f} : \tilde{C} \to \tilde{D}$ which is an E-homotopy equivalence on the singular E-set.

We define $r(\eta) \in L_k(\mathbb{Z}E, \varepsilon)$ as follows: After preliminary surgeries the kernel groups $K_*(C)$ are concentrated in the middle dimension. If $\dim C = 2 \cdot k$ only $K_k(C) \neq \{0\}$ and is (stably) free. There is a quadratic form on $K_k(C)$. Indeed, the construction in [13, p. 45] gives a $\mathbb{Z}E$-linear map $K_k(C) \to \mathrm{Imm}(S^k, \tilde{C})$ into the module of regular homotopy classes of immersions and our gap conditions imply that $\mathrm{Imm}(S^k, \tilde{C} \backslash \tilde{C}_s) \to \mathrm{Imm}(S^k, \tilde{C})$ is bijective. The strong gap conditions for $\sum^2 R$ are needed to ensure the inequality $k + 1 + \dim \tilde{C}_s < \dim \tilde{C}$. Now Wall's construction of the quadratic form works verbatim. The case $\dim C = 2 \cdot k + 1$ is similar using [13, Chap. 6] to get a formation.

One can adopt the arguments of [13] to show that r is well defined and injective. The gap conditions guarantee that the necessary surgeries on $M^H(x)$ can be done in the complement of the singular set. The $\mathbb{R}$-stable bundle data give unstable normal bundle data both in the smooth and in the PL category. Thus each $WH(x)$-surgery on $M^H(x)$ can be extended to a G-surgery on M.

(1.4) **Theorem.** *Suppose that R and $\sum^2 R$ satisfy the strong gap conditions* (I.1.18). *Then*

$$r : \mathscr{L}_n(R)[\mathscr{C}_0, \mathscr{C}_1] \to L_k(\mathbb{Z}E, \varepsilon)$$

is an isomorphism.

The proof of (1.4) will be broken up in a sequence of lemmas and constructions, but here is the main idea: by the $\pi-\pi$-lemma (I.3.2) elements of $\mathscr{L}_n(R)$ can be represented by surgery problems over a manifold N with $\pi^G N \to \mathscr{G}$ a weak $O(G)$-equivalence. Elements of $L_k(\mathbb{Z}E, \varepsilon)$ can be realized as surgery problems (over the free part) of $N^H(x)$, and these can be transfered to surgery problems over the (block) disk bundle of the normal bundle $\nu(N^H(x), N)$. This gives $NH(x)$-surgery problems and then G-surgery problems by applying the induction functor $G \times_{NH(x)} -$. This defines the inverse of r.

We begin the more detailed account of (1.4) by describing the relevant constructions on the level of references. Given $R = (\mathscr{G}, t_0, t_1, \tau)$ and $\hat{x} \in \mathscr{G}(G/H)^\wedge$, we define a $WH(x)$-reference R^H_x and $NH(x)$-references R^x_H and R_x all based on the same $O(WH(x))$-groupoid $\mathscr{G}_x$.

Let $\mathscr{G}(G/H) \subset \mathscr{G}(G/H)$ be the subgroupoid of objects y with $\hat{y} = \hat{x}$ and define

$$\mathscr{G}_x(WH(x)/1) = \mathscr{G}(G/H)_x \quad \text{and} \quad \mathscr{G}_x(WH(x)/K) = \emptyset \quad \text{for} \quad K \neq 1 .$$

If $\mathscr{G} = \pi^G N$ then $\mathscr{G}_x = \pi^{WH(x)}(N^H(x) \times EWH(x))$. Define $O(WH(x))$-functors

$$t^H_0 : \mathscr{G}_x \to \mathbb{B}_{k+m}, \quad t^H_1 : \mathscr{G}_x \to \mathbb{B}_k$$

with $k=\dim(t_1(G/H)(x)^H)$ by

$$t_i^H(y)_{eH}=t_i(G/H)(y)_{eH}^H$$

This gives a $WH(x)$-reference $R_x^H=(\mathscr{G}_x, t_0^H, t_1^H, \tau^H)$. The $NH(x)$-reference R_H^x has the underlying $O(NH(x))$-groupoid $p^*\mathscr{G}_x$ for $p: NH(x)\to WH(x)$ the projection (see [I.§4]) and

$$t_{0H}: p^*\mathscr{G}(G/H)_x\to \mathbb{B}_{n-k+m}, \quad t_{1H}: p^*\mathscr{G}(G/H)_x\to \mathbb{B}_{n-k}$$

is defined to be complementary to t_i^H inside t_i. This can be done in the smooth category by taking bundles to mean bundles with a metric and maps to preserve this metric. To define t_{iH} in the PL-category we need the following lemma. Given an $\mathbb{R}H$-module V, write $PL_H(V)$ for its equivariant PL-automorphisms and $F_H(V)$ for its proper self H-homotopy equivalences.

(1.5) **Lemma.** *Let $V=V^H\oplus V_H$ as $\mathbb{R}H$-modules.*

i) *If V satisfies the strong gap conditions* (I.1.18) *then the cartesian product of maps defines an isomorphism*

$$\pi_0 PL(V^H)\times \pi_0 PL_H(V_H)\to \pi_0 PL_H(V).$$

ii) *If V satisfies the weak gap conditions* (I.1.18) *the join defines an isomorphism*

$$\pi_0 F(V^H)\times \pi_0 F_H(V_H)\to \pi_0 F_H(V).$$

Proof. There are split fibrations

$$F_H(V, V^H)\to F_H(V)\to F(V^H)$$
$$PL_H(V, V^H)\to PL_H(V)\to PL(V^H)$$

whose fibres are the subspaces of maps which are the identity on the fixed set. It remains to be shown that the suspension defines isomorphisms

$$\pi_0 F_H(SV_H)\to \pi_0 F_H(V, V^H)$$
$$\pi_0 PL_H(V_H)\to \pi_0 PL_H(V, V^H)$$

The first isomorphism is a special case of the equivariant Freudenthal suspension theorem, the second is more complicated. First, the fibre in

$$\mathscr{P}PL_H(V_H\oplus\mathbb{R}^i, \mathbb{R}^i)\to PL_H(V_H\oplus R^i, \mathbb{R}^i)\to PL_H(V_H\oplus\mathbb{R}^{i+1}, \mathbb{R}^{i+1})$$

is i-connected, essentially by the regular neighborhood theorem of [10]. Second, for the same reason we get

$$\pi_0 PL_H(V_H\oplus\mathbb{R}^{i+1}, \mathbb{R}^{i+1})\cong \pi_0 PL_H(DV_H\times D^{i+1}, D^{i+1})$$

and

$$PL_H(DV_H\times D^{i+1}, D^{i+1})\cong PL_H(S(V_H\times\mathbb{R}^{i+1}), S^i)\cong PL_H(V_H\oplus\mathbb{R}^i, \mathbb{R}^i)$$

by applications of the Alexander trick (cf. [10]). □

From (1.5) and the obvious equivalence between G-bundles over G/H and $\mathbb{R}H$-modules we get the "complementary" $NH(x)$-reference

$$R_H^x=(p^*\mathscr{G}_x, t_{0H}, t_{1H}, \tau_H)$$

Finally we have the $NH(x)$-reference

$$R_x = (p^*\mathcal{G}_x, t_0, t_1, \tau)$$

where we just restrict t_i and τ to $p^*\mathcal{G}_x$ in the obvious way. By definition $t_i^H \oplus t_{iH} = t_i$ and $\tau^H \wedge \tau_H = \tau$ so that

(1.6) $$R_x = p^* R_x^H \oplus R_H^x \quad \text{(as } NH(x)\text{-references)}$$

Since surgery on $WH(x)$-free manifolds is the same as surgery on the orbit spaces we have

(1.7) **Lemma.** $\mathcal{L}_k(R_x^H) = L_k(\mathbb{Z}, \varepsilon)$. □

We want to define the surgery transfer

$$\mathrm{trf}: \mathcal{L}_k(WH(x); R_x^H) \to \mathcal{L}_n(G; R)[\mathcal{C}_0, \mathcal{C}_1]$$

This requires one realizes the reference R_H^x by a *fibre triple* and, for uniqueness, the construction of universal fibre triple. In detail:

(1.8) **Definition.** A fibre triple over a G-space B is a triple $\Theta = (v_0, v_1, h)$ of two $G - \mathbb{R}^n$ bundles and a fibre homotopy equivalence $h: Sv_0 \to Sv_1$ between the associated spherical fibrations.

A fibre triple Θ induces a reference $R(\Theta)$

(1.9) $$R(\Theta) = (\pi^G B, \mathrm{tp}_{v_0}, \mathrm{tp}_{v_1}, \textstyle\sum \mathrm{tp}_h)$$

where $\sum$ is the suspension. Conversely, given a reference $R = (\mathcal{G}, t_0, t_1, \tau)$ of dimension n we construct a universal fibre triple $\Theta(R)$ as follows. First, by (1.5) we can desuspend t_0 so that both t_0 and t_1 are $O(G)$-functors with range $\mathbb{B}_n$. Consider the homotopy pull back

$$\begin{array}{ccc} \Phi & \longrightarrow & B(G,n) \\ \downarrow & & \downarrow{\scriptstyle S} \\ B(G,n) & \xrightarrow{S} & BF(G,n-1) \end{array}$$

of the classifying spaces for $G - \mathbb{R}^n$ bundles (smooth or PL) and $G - S^{n-1}$ fibrations. There is a canonical fibre triple (γ_0, γ_1, h) over Φ. The equivariant Eilenberg-MacLane space $K(\pi^G\Phi, 1)$ is the homotopy pull back of

$$K(\pi^G B(G,n), 1) \xrightarrow{S_*} K(\pi^G BF(G, n-1), 1) \xleftarrow{S_*} K(\pi^G B(G,n), 1).$$

The given spherical reference induces maps $t_j: K(\mathcal{G}, 1) \to K(B_n, 1)$ and an homotopy $\sigma: S_* \circ t_0 \to S_* \circ t_1$. Thus it gives a map $T: K(\mathcal{G}, 1) \to K(\pi^G\Phi, 1)$. We define $B(R)$ to be the homotopy pull back in

$$\begin{array}{ccc} B(R) & \xrightarrow{i_0} & \Phi \\ \downarrow{\scriptstyle i_1} & & \downarrow \\ K(\mathcal{G}, 1) & \xrightarrow{T} & K(\pi^G\Phi, 1) \end{array}$$

and get a fibre-triple

$$\Theta(R) = (i_0^*(\gamma_0), i_0^*(\gamma_1), i_0^*(h)) \tag{1.10}$$

over $B(R)$. The composition $\mu: \pi^G B(R) \to \pi^G K(\mathcal{G}, 1) \to \mathcal{G}$ is a weak $O(G)$-equivalence and extends to a map of references $\mu: R(\Theta(R)) \to R$. Now $\Theta(R)$ is universal by the easy

(1.11) **Lemma.** *Given a fibre triple Θ over B and a map of spherical references $\varrho: R(\Theta) \to R'$, there is an up to homotopy unique map of fibre-triples $g: \Theta \to \Theta(R')$ with $\mu \circ R(g) \simeq \varrho$.* □

In the smooth category the bundles $v_i = t_i^*(\gamma_i)$ can be given a metric and the spherical fibre space $v_i \setminus \{\text{zero section}\}$ can be realized as the sphere bundle Sv_i. The corresponding statement is false in the PL-category: one has to use block bundles instead. For our applications below we need *spherical fibre triples* $S\Theta = (Sv_0, Sv_1, h)$ in both categories.

Let $\tilde{B}(G, S^{n-1})$ resp. $\tilde{B}(G, n)$ be the classifying space for locally linear PL block $G - S^{n-1}$ bundles resp. PL block $G - \mathbb{R}^n$ bundles.

(1.12) **Lemma.** *There is a weak $O(G)$-homotopy equivalence*

$$\pi^G \tilde{B}(G, S^{n-1}) \simeq \pi^G \tilde{B}(G, n)$$

Proof. Consider the $O(G)$-transformations

$$\pi^G \tilde{B}(G, S^{n-1}) \to \pi^G \tilde{B}(G, n) \leftarrow \pi^G B(G, n)$$

both induced from the maps of classifying spaces. They turn out to be weak $O(G)$-homotopy equivalences. Namely, we have

$$\tilde{B}(G, S^{n-1})^H = \sqcup BPL_{\tilde{H}}(SW)$$

$$\tilde{B}(G, n)^H = \sqcup BPL_{\tilde{H}}(W)$$

$$B(G, n)^H = \sqcup BPL_H(W)$$

where in all cases the disjoint union $\sqcup$ varies over isomorphism classes of $\mathbb{R}H$-modules W of dimension n. For each such W

$$\pi_1 BPL_{\tilde{H}}(SW) \to \pi_1 BPL_{\tilde{H}}(W) \leftarrow \pi_1 BPL_H(W)$$

are isomorphisms, cf. [10, §2]. □

The construction (1.9) can be carried out with $B(G, n)$ replaced by $\tilde{B}(G, S^{n-1})$ and gives a spherical fibre triple

$$S\Theta(R) = (Sv_0, Sv_1, h) \tag{1.13}$$

of two locally linear PL block $G - S^{n-1}$ bundles and a fibre homotopy equivalence h. By (1.12) $S\Theta(R)$ induces a reference $R(S\Theta(R))$, weakly homotopy equivalent to R, and the obvious analogue of (1.11) remains valid.

We can now define the surgery transfer. Consider the following set of *transfer data* $(*)$ associated with (1.6):

(∗) i) B is a free $WH(x)$ space and $\lambda_x^H : \pi^{WH(x)}B \to \mathscr{G}_x^H$ is a weak $O(WH(x))$-equivalence.

ii) $S\Theta = (Sv_0, Sv_1, h)$ is a spherical $NH(x)$-fibre triple (smooth or PL) over B.

iii) $\varrho_H^x : R(S\Theta) \to R_H^x$ is a map of $NH(x)$-references with $p^*\lambda_x^H$ as underlying $O(NH(x))$-transformation for $p: NH(x) \to WH(x)$ the projection.

For example, we can take B to be the base space of the spherical fibre triple $S\Theta(R_H^x)$ in (1.13) and let ϱ_H^x be the map μ from (1.11). This is the universal situation which any other set of transfer data map into. For computations it is sometimes useful to use transfer data different from the universal ones.

To a given set of transfer data, we get a $WH(x)$-reference

$$R(*) = (\pi^{WH(x)}B, (\lambda_x^H)^* t_0^H, (\lambda_x^H)^* t_1^H, (\lambda_x^H)^* \tau)$$

weakly $O(WH(x))$-homotopy equivalent to R_x^H, and hence

$$\mathscr{L}_k(WH(x); R(*)) \cong \mathscr{L}_k(WH(x); R_x^H) \cong L_k(\mathbb{Z}E, \varepsilon)$$

where the last isomorphism is from (1.7). Define neighbouring families $\mathscr{C}_0' \subset \mathscr{C}_1'$ to be

$$\mathscr{C}_i' = \{L \subset NH(x) | (x, L)^\wedge \in \mathscr{C}_1\}$$

with $\mathscr{C}_i$ from (1.4). Given the transfer data (∗), we define (cf. 1.6)

$$\mathrm{trf}(*) : \mathscr{L}_k(WH(x); R(*)) \to \mathscr{L}_n(NH(x); R_x)[\mathscr{C}_0', \mathscr{C}_1'] \tag{1.14}$$

as follows: Consider an element in $\mathscr{L}_k(WH(x); R(*))$ represented by a $WH(x)$-surgery problem $(f, \bar{f}, \varphi)$ with $f: M \to N$ and reference map

$$\varrho : (\pi^{WH(x)}N, \mathrm{tp}_\xi, \mathrm{tp}_N, \varphi) \to R(*)$$

whose underlying $O(WH(x))$-transformation is induced from a $WH(x)$-map $\lambda : N \to B$. Let

$$Dh : Dv_0 \to Dv_1$$

be obtained from $h : Sv_0 \to Sv_1$ by coning. The pull back construction defines a $NH(x)$-surgery problem $(F, \hat{F}, \Phi)$ with

$$F : f^*\lambda^* Dv_0 \xrightarrow{\ \bar{f}\ } \lambda^* Dv_0 \xrightarrow{\ \lambda^* Dh\ } \lambda^* Dv_1$$

$$\hat{F} : T(f^*\lambda^* Dv_0) = p_0^* TM \oplus p_0^* f^* \lambda^* v_0 \xrightarrow{\ p_0^* \hat{f} \oplus \bar{f}\ } p_0^* \xi \oplus p_0^* \lambda^* v_0 .$$

The map trf(∗) is independent of the choice of (∗) by (1.11). Using induction one obtains an homomorphism, cf. (I. §4)

$$i_* : \mathscr{L}_n(WH(x); R_x^H)[\mathscr{C}_0', \mathscr{C}_1'] \to \mathscr{L}_n(G; R)[\mathscr{C}_0, \mathscr{C}_1] \tag{1.15}$$

Define

$$q : L_k(\mathbb{Z}E, \varepsilon) \to \mathscr{L}_n(G; R)[\mathscr{C}_0, \mathscr{C}_1] \tag{1.16}$$

by the composition $i_* \circ \mathrm{trf}(*)$. The composition of q with $r : \mathscr{L}_n(R)[\mathscr{C}_0, \mathscr{C}_1] \to L_k(\mathbb{Z}E, \varepsilon)$ is the identity as $Dv_0^H = Dv_1^H = B$ for the fibre-triple $\Theta = (v_0, v_1, h)$ over B. This finishes the proof of Theorem 1.4. □

Combining (1.2) and (1.4) we get the so called orbit sequence.

(1.17) **Theorem.** *Suppose that R and $\sum^{m+2} R$ satisfy the strong gap conditions* (I.1.18) *for $m \geqq 0$. Then there is an exact sequence*

$$\mathscr{L}_{n+m}(\textstyle\sum^{n+m} R)[\mathscr{C}_1] \xrightarrow{i} \mathscr{L}_{n+m}(\textstyle\sum^{n+m} R)[\mathscr{C}_0] \xrightarrow{j} L_{k+m}(\mathbb{Z}E, \varepsilon) \xrightarrow{\partial} \cdots$$

$$\mathscr{L}_n(R)[\mathscr{C}_1] \xrightarrow{i} \mathscr{L}_n(R)[\mathscr{C}_0] \xrightarrow{j} L_k(\mathbb{Z}E, \varepsilon) \quad \square$$

We give a simple example to show that the equivariant L-groups do not always decompose as a direct sum of the expected ordinary L-groups.

(1.18) *Example.* Let G be $\mathbb{Z}/2$ and V be a free $\mathbb{R}G$-module of dimension $n-k$. Suppose $k \geqq 6$ and $2 \cdot k + 3 < n$. Consider the n-dimensional reference $R = (\mathscr{G}, t, t, \mathrm{id})$ with

$$\mathscr{G}(G/G) = \mathscr{G}(G/1) = \{x\}$$

$$t(G/G)(x) = V \oplus \mathbb{R}^k \cdot \qquad t(G/1)(x) = G \times (V \oplus \mathbb{R}^k)$$

From (1.17) we get an exact sequence

$$L^h_{k+1}(\mathbb{Z}) \xrightarrow{\partial} L^h_n(\mathbb{Z}G, \varepsilon) \xrightarrow{i} \mathscr{L}^h_n(R) \xrightarrow{j} L^h_k(\mathbb{Z}) \xrightarrow{\partial} L^h_{n-1}(\mathbb{Z}G, \varepsilon)$$

Here ε is trivial if and only if $n \equiv k \bmod 2$. The boundary ∂ corresponds geometrically to crossing with $\mathbb{R}P^{n-k-1}$.

For $n = 0, 2$ and the non-trivial orientation ε the L-groups $L^h_n(\mathbb{Z}G, \varepsilon) = \mathbb{Z}/2$ and $L^h_2(\mathbb{Z}) = \mathbb{Z}/2$ are detected by the Arf-invariant c. If f is a normal map and X a manifold, we have Sullivan's product formula

$$c(f \times X) = c(f) \cdot \chi(X)$$

Hence $\partial : L^h_{k+1}(\mathbb{Z}) \to L^h_n(\mathbb{Z}G, \varepsilon)$ is an isomorphism for $k = 1$ and $n = 0, 2$ (modulo 4). For $k = 3$ and $n = 0, 2$ ∂ is zero since the Arf-invariant is zero on $L_0(\mathbb{Z})$. Since ∂ is bijective for $k = 1$ and $n = 3$ the groups $\mathscr{L}^h_n(R)$ can be tabulated as follows:

	$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$k \equiv 0$	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}$
$k \equiv 1$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$k \equiv 2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\{0\}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\{0\}$
$k \equiv 3$	$\mathbb{Z}/2$	$\{0\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Indeed the only sticky point is to argue that the exact sequence

$$0 \to L^h_n(\mathbb{Z}[G]) \to \mathscr{L}^h_n(R) \to L^h_k(\mathbb{Z}) \to 0$$

is split when $n \equiv k \equiv 2 \pmod 4$. This can be seen as follows.

We suppress all bundle structures. Let $f: K \to S^k$ be the Kervaire normal map which represents the non-trivial element in $L^h_k(\mathbb{Z})$. Choose a normal cobordism

$$F: V \to S^k \times \mathbb{R}P^{n-k-1} \times I$$

with $\partial_0 F = f \times \mathbb{R}P^{n-k-1}$ and with $\partial_1 F$ a homotopy equivalence. Let D_-^{n-k} be the $(n-k)$-disk with antipodal G-action. Then

$$M = \tilde{V} \cup_{\partial_0} K \times D_-^{n-k} \xrightarrow{H} S^k \times D_-^{n-k}, \qquad H = F \cup f \times D_-^{n-k}$$

represents an element in $\mathscr{L}_n^h(R)$ which maps non-trivially under j, since $(M^G, H^G) = (K, f)$. We must show it has order 2.

We can represent $2\{M, H\} \in \mathscr{L}_n^h(R)$ by the double (DM, DH). Its fixed point problem is $K \coprod K \xrightarrow{f \coprod f} S^k \coprod S^k$, which we want to cobord away. Let $g: N \to S^1 \times_{\mathbb{Z}/2} S^k$ be the normal map over the generalized Klein bottle which represents the non-trivial element in $L_3(\mathbb{Z}[\mathbb{Z}], \varepsilon)$, $\varepsilon \neq 1$. We can take $N = S^1 \times_{\mathbb{Z}/2} K$; (N, g) is a normal map with $\mathbb{Z}/2$ coefficients whose associated Bockstein is the Kervaire normal map (K, f), cf. [15, §3]. Cut open N along K to get

$$g_0: (N_0, K \coprod K) \to (S^k \times I, S^k \coprod S^k)$$

Let $M_0 = \tilde{V} \cup (N_0 \times S_-^{n-k-1}) \cup \tilde{V}$; it has two boundary components, each equal to $S^k \times S_-^{n-k-1}$, which we identify to get the G-manifold M_1, with Bockstein $S^k \times S_-^{n-k-1}$. There is an obvious G-normal map

$$H_1: M_1 \to (S^1 \times_{\mathbb{Z}/2} S^k) \times S_-^{n-k-1}$$

and $\{M_1, H_1\} = 2\{M, H\}$ in $\mathscr{L}_n^h(R)$. We must check that $\{M_1/G, H_1/G\}$ has trivial Arf-invariant. But as a normal map (of manifolds with $\mathbb{Z}/2$ coefficients) $(M_1/G, H_1/G)$ is cobordant to $(N \times \mathbb{R}P^{n-k-1}, g \times \mathbb{R}P^{n-k-1})$. Thus they have the same Arf-invariant, cf. [14, p. 91]. Finally the product formula shows that $c(g \times \mathbb{R}P^{n-k-1}) = 0$ □

(1.19) **Corollary.** *The equivariant surgery groups in the smooth and PL-category agree.*

Proof. This is well-known in the non-equivariant case and follows inductively from (1.17) in the equivariant setting. □

§ 2. Decomposition of Equivariant L-Theory

In this section we show for G of *odd* order that the orbit sequence of § 1 (under mild restrictions) reduces to split short exact sequences. Throughout the section we work in the smooth category, cf. (1.19).

Consider an $O(G)$-functor $t: \mathscr{G} \to \mathbb{B}_n$. For each (x, H) we have the G-$\mathbb{R}^n$ bundle $t(G/H)(x)$ over G/H and the H-module $t(G/H)(x)_{eH}$. Its isomorphism class depends only on $\hat{x} \in \mathscr{G}(G/H)$ and is invariant under the action of $NH(x)$. This is a necessary but not sufficient condition for the H-module $t(G/H)(x)_{eH}$ to extend to a $NH(x)$-module.

(2.1) **Definition.** An $O(G)$-functor $t: \mathscr{G} \to \mathbb{B}_n$ has the *slice extension property* if for each $(x, H)^\wedge$ the H-module $t(G/H)(x)_{eH}$ extends to an $NH(x)$-module.

Let $R = (\mathscr{G}, t_0, t_1, \tau)$ be a geometric reference of ambient dimension n (see I.3.1) which satisfies the following additional properties:

(2.2) i) R is simple, cf. (I.2.4)

ii) $(\mathscr{G}, t_0)$ has the slice extension property (2.1).

iii) R and $\sum^3 R$ satisfy the strong gap conditions (I.1.18).

Notice from (2.2i.) that the H-modules $t(G/H)(x)_{eH}$ and $t_1(G/H)(x)_{eH}$ are linearly isomorphic by de Rham's theorem, cf. [12, ch. 4].

Lemma 2.3. *Suppose that G has odd order. Let t_0 and $t_1 : \mathscr{G} \to \mathbb{B}_n$ be two $O(G)$-functors which satisfy the weak gap conditions* (I.1.18). *Then there is an $O(G)$-transformation $\varphi : t_0 \to t_1$ if and only if*

i) *$t_0(G/H)(x)_{eH}$ and $t_1(G/H)(x)_{eH}$ are (abstactly) linearly H-isomorphic for $x \in \mathscr{G}(G/H)$ and $H \subset G$.*

ii) $\varepsilon_{[t_0]}(x, 1) = \varepsilon_{[t_1]}(x, 1)$ *for* $x \in \mathscr{G}(G/1)$.

Proof. We can assume that $\mathscr{G}(G/1)^{\wedge}$ consists of a single element $\hat{x}_0$. Choose an isomorphism $\varphi_0 : t(G/1)(x_0) \to t_1(G/1)(x_0)$ of G-$\mathbb{R}^n$ bundles. The forgetful map

$$\pi_0 \operatorname{Iso}_H (t_0(G/H)(x_0)_{eH}, t_1(G/H)(x_0)_{eH}) \to \pi_0 \operatorname{Iso} (t_0(G/H)(x_0)_{eH}, t_1(G/H)(x_0)_{eH})$$

is bijective since G has odd order. Hence there are precisely two isotopy classes of bundle isomorphism from $t_0(G/H)(x)$ to $t_1(G/H)(x)$. We use φ_0 to pick one. Indeed, let $\sigma : G/1 \to G/H$ be the projection and let $w : \sigma^* x \to x_0$ be any morphism in $\mathscr{G}(G/1)$. Choose the isomorphism $\varphi(G/H)(x) : t_0(G/H)(x) \to t_1(G/H)(x)$ such that the following diagram of G-$\mathbb{R}^n$ bundles commutes up to isotopy

$$\begin{tikzcd}
\sigma^* t_0{}^{(\prime\prime}/H)(x) \arrow[r] \arrow[d, "\sigma^*\varphi(G/H)(x)"] & t_0(G/1))\sigma^* x) \arrow[r, "w_*"] & t_0(G/1)(x_0) \arrow[d, "\varphi_0"] \\
\sigma^* t_1(G/H)(x) \arrow[r] & t_1(G/1)(\sigma^* x) \arrow[r, "w_*"] & t_1(G/1)(x_0)
\end{tikzcd}$$

This determines the isotopy class of $\varphi(G/H)(x)$ by the assumption (ii), and we get the desired isomorphism $\varphi : t_0 \to t_1$ (compare [8]). $\square$

Notice in particular that under the conditions (2.2) we get $(\mathscr{G}, t_0, t_1, \tau) \cong (\mathscr{G}, t_0, t_0, id)$, provided that the order of G is odd. We continue with a lemma from bordism theory. A smooth G-manifold M is called almost complex if its $\mathbb{R}$-stable tangent bundle $TM \oplus \mathbb{R}^k$ has the structure of a complex G-vector bundle.

(2.4) **Lemma.** *Assume that G has odd order. Let M be a closed almost complex G-manifold of odd dimension. Then for some r, $|G|^r$ copies of M bounds an almost complex G-manifold Q such that $\pi_0(\partial Q^H) \to \pi_0(Q^H)$ is surjective for all $H \subset G$.*

Proof. Let H be a maximal isotropy group for M and $C \subset M^H$ a component. The Weyl group WH acts freely on M^H and the subgroup $WH(x)$ acts freely on C when $x \in C$. Let $v : C \to BU(NH(x), k)^H$ classify the normal bundle of C in M. We get a map

$$f : C/WH(x) \to EWH(x) \times_{WH(x)} BU(NH(x), k)^H$$

which represents a unitrary bordism element $\{C/WH(x), f\}$. Since $BU(NH(x), k)^H = BU(H, k)^H$ is a product of copies of classifying spaces $BU(k_i)$ (see[6]), a simple argument with the Atiyah-Hirzebruch spectral sequence shows that

$$\Omega^U_{\mathrm{odd}}(EWH(x)\times_{WH(x)}BU(NH(x),k)^H)\otimes_{\mathbb{Z}}\mathbb{Z}[1/|WH(x)|]=0$$

Hence $|WH(x)|^r\cdot\{C/WH(x),f\}=0$ for suitable r. We get an almost complex $NH(x)$-manifold P with $\partial P=|G|^r\cdot C$ and an $NH(x)$-bundle η over P which restricts to $|G|^r\cdot\nu(C,M)$ on the boundary. We extend the nullbordism P of $|G|^r\cdot C$ to a bordism Q of M by $(G\times_{NH(x)}D\eta)\cup(|G|^r\cdot M\times[0,1])$ where we glue along $G\times_{NH(x)}D\eta|\partial P$. Doing this simultaneously for all $C\in\pi_0(M^H)/WH$ we obtain a bordism Q from $|G|^r\cdot M$ to an almost complex manifold N such that $\mathrm{Iso}(N)$ is strictly smaller than $\mathrm{Iso}(M)$ and $\pi_0((|G|^r\cdot M)^K)\to\pi_0(Q^K)$ is surjective for $K\subset G$. After a finite number of steps we get the desired nullbordism. □

(2.5) **Proposition.** *Suppose that G has odd order and R satisfies* (2.2i.) *and* (2.2ii.) *and that R and* $\sum^2 R$ *satisfy the strong gap conditions* (I.1.18). *Then the map of Theorem* 1.17, *localized at* 2,

$$j\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}:\mathscr{L}_n(R)[\mathscr{C}_1]\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}\to L_k(\mathbb{Z}E(x,H),\varepsilon(x,H))\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}$$

is split surjective.

Proof. We shall construct an homorphism

$$s:L_{k(x,H)}(\mathbb{Z}E(x,H),\varepsilon(x,H))\to\mathscr{L}_n(R)[\mathscr{C}_0]$$

which splits $j\otimes\mathbb{Z}_{(2)}$. By (2.2.ii) we can pick a $NH(x)$-module V and a linear H-isomorphism $\Omega:t_1(G/H)(x)^H_{eH}\oplus V\to t_1(G/H)(x)_{eH}$. We apply Lemma 2.4 to get a $NH(x)$-manifold Q with $\partial Q=m\cdot SV$ for some odd m and $\pi_0(\partial Q^K)\to\pi_0(Q^K)$ surjective for $K\subset G$. Let $P=Q\cup_{\partial Q}m\cdot DV$. The map s is essentially given by crossing with P. Namely, let R^H_x be the $WH(x)$-reference associated to R in § 1. Each element in $L_{k(x,H)}(\mathbb{Z}E(x,H),\varepsilon(x,H))$ can be represented by a normal $WH(x)$-map $(f,\hat{f},\varphi,\varrho)$ with

$$f:M\to N,\ \hat{f}:TM\to\xi,\ \varphi:\mathrm{tp}^c_\xi\to\mathrm{tp}^c_N\quad\text{and}\quad\varrho:(\pi^G N,\mathrm{tp}_\xi,\mathrm{tp}_N,\varphi)\to R^H_x$$

Suppose for simplicity $NH(x)=G$, otherwise one further has to apply $G\times_{NH(x)}(-)$. Consider the normal G-map $(f\times\mathrm{id}_P,\hat{f}\times\mathrm{id}_{TP},\varphi\times\mathrm{id},\hat{\varrho})$ with

$$\hat{\varrho}:(\pi^G N\times P,\ \mathrm{tp}_{\xi\times PT},\ \mathrm{tp}_{TN\times P},\ \varphi\wedge\mathrm{id})\to R$$

defined as follows. First, the projection $\mathrm{pr}:N\times P\to N$ induces a unique reference map

$$(\pi^G N\times P,\mathrm{tp}_{\xi\times TP},\mathrm{tp}_{TN\times TP},\ \varphi\wedge\mathrm{id})\to(\pi^G N,\mathrm{tp}_\xi\oplus\mathrm{tp}_{\underline{V}},\mathrm{tp}_N\oplus\mathrm{tp}_{\underline{V}},\varphi\wedge\mathrm{id})$$

This uses that $\pi_0(\partial Q^K)\to\pi_0(Q^K)$ is onto and Lemma 2.3. Now compose with

$$(\pi^G N,\mathrm{tp}_\xi\oplus\mathrm{tp}_{\underline{V}},\mathrm{tp}_N\oplus\mathrm{tp}_{\underline{V}},\varphi\wedge\mathrm{id})\to(p^*\mathscr{G}_x,t^H_0\oplus\mathrm{tp}_{\underline{V}},t^H_1\oplus\mathrm{tp}_{\underline{V}},\tau^H\wedge\mathrm{id})\to R$$

where $p:G=NH(x)\to WH(x)$ is the projection and the second map comes from Ω using Lemma 2.3 again.

Finally, observe that $j\circ s=m\cdot\mathrm{id}$ for odd m since P^H consists of m points. □

Recall from [11, 3.6.4] that

(2.6) $$L_n(\mathbb{Z}E,\varepsilon)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]\xrightarrow{\cong}L_n(\mathbb{Q}E,\varepsilon)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

is an isomorphism for any (E,ε). It does not matter which kind of L-groups we use since the relative term in the Rothenberg-sequence is 2-torsion. The symmetrization map induces an isomorphism

(2.7) $$1+T: L_n(\mathbb{Q}E,\varepsilon)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]\to L^n(\mathbb{Q}E,\varepsilon)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

Now we use the Mishchenko-Ranicki theory of symmetric chain complexes to construct an homomorphism

(2.8) $$\sum\nolimits^{\oplus}\sigma(x,H):\mathscr{L}_n(R)\to\sum\nolimits^{\oplus}L^{k(x,H)}(\mathbb{Q}E(x,H),\varepsilon(x,H))$$

where the sums run over $(x,H)^\wedge\in\operatorname{Iso}(t_1)$ and $k(x,H)=\dim t_1(G/H)^H_{eH}$. Let $\omega\in\mathscr{L}_n(R)$ be represented by a normal G-map $(f,\bar f,\varphi)$ with a reference map $(\lambda,\mu_0,\mu_1):(\pi^G N,\operatorname{tp}_\xi,\operatorname{tp}_N,\varphi)\to R$ such that $\pi^G f$ and λ are weak $O(G)$-equivalences, cf. (I.3.2).

Let $C\in\pi_0(M^H)$ and $D\in\pi_0(N^H)$ be the components such that $f^H(C)\subset D$ and $\lambda(G/H)(D)=\hat x$. The group $E(x,H)$ acts on the universal converings $\tilde C$ and $\tilde D$, and f lifts to a $E(x,H)$-equivariant map $\tilde f:\tilde C\to\tilde D$. The chain complexes $C_*(\tilde C,\mathbb{Q})$ and $C_*(\tilde D,\mathbb{Q})$ consists of finitely generated projective $\mathbb{Q}E(x,H)$-modules since they are finitely generated free over the group $\pi_1(C)=\pi_1(D)$ which has finite index in $E(x,H)$. By the relative symmetric construction in [11, ch 1] we obtain pairs of symmetric $\mathbb{Q}E(x,H)$-Poincaré chain complexes $C_*(\partial\tilde C,\mathbb{Q})\subset C_*(\tilde C,\mathbb{Q})$ and $C_*(\partial\tilde D,\mathbb{Q})\subset C_*(\tilde D,\mathbb{Q})$. Glueing them together with the $\mathbb{Q}E(x,H)$-homotopy equivalence $C(\partial\tilde f)$ yields a symmetric $\mathbb{Q}E(x,H)$-Poincaré chain complex. Its cobordism class is the element $\sigma(x,H)(\omega)\in L^{k(x,H)}(\mathbb{Q}E(x,H),\varepsilon(x,H))$ we seek.

(2.10) **Proposition.** *Suppose that R satisfies* (2.2). *Then the maps $\sigma(x,H)$ in* (2.8) *induce an isomorphism*

$$\sum\nolimits^{\oplus}\sigma(x,H)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]:\mathscr{L}_n(R)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$
$$\to\sum\nolimits^{\oplus}L_{k(x,H)}(\mathbb{Z}E(x,H),\varepsilon(x,H))\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

where the sum runs over $(x,H)^\wedge\in\operatorname{Iso}(t_1)$.

Proof. This follows from Theorem 1.17 since the composition

$$\mathscr{L}_n(R)[\mathscr{C}_0]\xrightarrow{j}L_k(\mathbb{Z}E,\varepsilon)\to L^k(\mathbb{Q}E,\varepsilon)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$$

is just $\sigma(x,H)\otimes_{\mathbb{Z}}\mathbb{Z}[1/2]$ composed with $\mathscr{L}_n(R)[\mathscr{C}_0]\to\mathscr{L}_n(R)$. □

(2.11) **Theorem.** *Let R be a geometric reference of ambient dimension n, cf.* (I.3.1), *satisfying* (2.2). *If G has odd order, then there is an isomorphism*:

$$\mathscr{L}_n(R)\cong\sum\nolimits^{\oplus}L_{k(x,H)}(\mathbb{Z}E(x,H),\varepsilon(x,H))$$

where $k(x,H)=\operatorname{Dim}_{[t_1]}(x,H)$, cf. (I.1.17), *and $\varepsilon(x,H)=\varepsilon_{[t_1]}(x,H)$, cf.* (1.3), *and the sum runs over $(x,H)^\wedge\in\operatorname{Iso}(t_1)$, cf.* (I.1.18). □

§ 3. The Equivariant Rothenberg Sequence

We want to compare $\mathscr{L}_n^h(R)$ and $\mathscr{L}_n^s(R)$ for a geometric reference R of ambient dimension n. Define a relative group $\mathscr{L}_n^{h,s}(R)$ to be the bordism classes of surgery problems $(f, \hat{f}, \varphi, \varrho)$ with reference R such that f is a G-homotopy equivalence and ∂f is a simple G-homotopy equivalence. A nullbordism $(F, \partial_0 R, \partial_1 F)$ of $\partial_1 F$ requires F to be a G-homotopy equivalence and $\partial_0 F$ to be a simple G-homotopy equivalence. Consider the sequence, infinite to the left,

$$(3.1)\quad \cdots \xrightarrow{i} \mathscr{L}_{n+1}^s(\textstyle\sum R) \xrightarrow{j} \mathscr{L}_{n+1}^h(\textstyle\sum R) \xrightarrow{\partial} \mathscr{L}_n^{h,s}(R) \xrightarrow{i} \mathscr{L}_n^s(R) \xrightarrow{j} \mathscr{L}_n^h(R)$$

Here ∂ restricts to the boundary and i and j are the forgetful maps. One easily checks.

(3.2) **Theorem.** *The sequence* (3.1) *is exact.* □

In order to express $\mathscr{L}_n^{h,s}(R)$ algebraically some preparations are needed. Given a geometric pair $(\mathscr{G}, t_1)$ of ambient dimension n, cf. (I.3.1), we define

$$(3.3)\qquad Wh^G(\mathscr{G}, t_1) = \sum_{(x,H)^\wedge \in \operatorname{Iso}(t_1)}^{\oplus} Wh(\mathbb{Z}E(x,H))$$

Notice that this depends only on $\operatorname{Dim}_{[t_1]}$. If N is a G-manifold it is customary to write $Wh_\varrho^G(N)$ for $Wh^G(\pi^G N, \operatorname{tp}_N)$. Now suppose that N satisfies the weak gap conditions (I.1.18). One may define an involution

$$(3.4)\qquad * : Wh_\varrho^G(N) \to Wh_\varrho^G(N)$$

by reversing h-cobordisms over N (see [1]). Consider two G-manifolds M and N together with an $O(G)$-equivalence $\lambda: \pi^G M \to \pi^G N$ (see [I.1.4]) such that $\lambda^* w_N = w_M$. See (I.1.10) for the definition of the equivariant first Stiefel-Whitney class w_N. Then λ induces an isomorphism $\lambda_*: Wh_\varrho^G(M) \to Wh_\varrho^G(N)$ compatible with the involutions (see [1, 1.10 and 2.13]). It follows from the π-π-lemma (I.3.2) applied to the empty normal map that there is a G-manifold N together with an $O(G)$-equivalence $\lambda: \pi^G N \to \mathscr{G}$ satisfying $w_N = \lambda^*[t_1]$. Thus we can define an involution

$$(3.5)\qquad * : Wh^G(\mathscr{G}, t_1) \to Wh^G(\mathscr{G}, t_1)$$

depending only on $\mathscr{G}$ and t_1 by requiring that $\lambda_*: Wh_\varrho^G(N) \to Wh^G(\mathscr{G}, t_1)$ respects the involutions.

Next we consider the Tate cohomology group $\hat{H}^{k+1}(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1))$. We define an homomorphism

$$(3.6)\qquad \Theta: \mathscr{L}_{n+k}^{h,s}(R) \to \hat{H}^{k+1}(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1))$$

by sending a $(n+k)$-dimensional normal map $(f, \hat{f}, \varphi, \varrho)$ to the image of the equivariant Whitehead torsion $wh^G(f) \in Wh_\varrho^G(N)$ under λ_*. As R is simple and $wh^G(\partial f) = \{0\}$ we get $wh^G(f) + (-1)^k \cdot *(wh^G(f)) = 0$ in $Wh_\varrho^G(N)$ (see [1, 4.3]). Thus $wh^G(f)$ defines an element in $\hat{H}^{k+1}(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1))$. Given a nullbordism for $(f, \hat{f}, \varphi, \varrho)$ with underlying map $(F, f_+, f): (P, M_+, M) \to (Q, N_+, N)$ of triads, we get $wh^G(f) = -wh^G(F) - (-1)^k \cdot *(wh^G(F))$ if all torsion elements are mapped to $Wh^G(\mathscr{G}, t_1)$ (cf. [1, 4.3]). Hence $wh^G(f)$ represents zero when $(f, \hat{f}, \varphi, \varrho)$ is nullbordant. The map Θ is obviously compatible with the addition.

(3.7) **Theorem.** *Suppose that R is a simple geometric reference of ambient dimension n satisfying the weak gap conditions* (I.1.18). *Then*

$$\Theta : \mathscr{L}^{h,s}_{n+k}(R) \to \hat{H}^{k+1}(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1))$$

is injective for $k \geqq 0$ and bijective for $k \geqq 1$.

Proof. We start with injectivity. Assume that $\Theta(f, \hat{f}, \varphi, \varrho)$ is zero and that $\lambda : \pi^G N \to \mathscr{G}$ is an $O(G)$-equivalence (cf. I.3.2). Then $wh^G(f) \in \hat{H}^{k+1}(\mathbb{Z}/2; Wh^G_\varrho(N))$ vanishes so $wh^G(f) = u - (-1)^k * (u)$ for some u. Let (P, M_+, M) be an h-cobordism with $wh^G(M \subset P) = wh^G(f) + u$. Let $(F, f_+, f) : (P, M_+, M) \to (N \times I, N \times \{0\}, \partial N \times I \cup N \times \{1\})$ be a G-homotopy equivalence. Then $wh^G(F) = -u$ and we get

$$\begin{aligned} wh^G(f_+) &= wh^G(\partial F) - wh^G(f) \\ &= -(-1)^{k+1} \cdot * (wh^G(F)) - wh^G(F) - u + (-1)^k * (u) = 0 \end{aligned}$$

Hence $(f, \hat{f}, \varphi, \varrho)$ is nullbordant; the necessary bundle data are easily constructed as F is a G-homotopy equivalence.

Now suppose $k \geqq 1$. Consider $u \in Wh^G(\mathscr{G}, t_1)$ with $u + (-1)^k * (u) = 0$. By the π-π-lemma (see I.3.2.) applied to the empty normal map we obtain a G-surgery problem $(f, \hat{f}, \varphi, \varrho)$ with reference map to $\sum^{k-1} R$ such that $\lambda : \pi^G N \to \mathscr{G}$ is an $O(G)$-equivalence and $wh^G(f) = \{0\}$. Construct $(F, f_+, f) : (P, M_+, M) \to (N \times I, N \times \{0\}, \partial N \times I \cup N \times \{1\})$ such that (P, M_+, M) is an h-cobordism and $wh^G(F) = u$. Then we have

$$\begin{aligned} wh^G(f_+) &= wh^G(f_+) + wh^G(f) = wh^G(\partial F) \\ &= -wh^G(F) - (-1)^k \cdot * (wh^G(F)) = -u - (-1)^k * (u) = 0 \end{aligned}$$

One easily extends (F, f_+, f) to a G-surgery problem with the required properties. □

(3.8) **Corollary.** *Under the assumptions of* (3.7) *there is an exact sequence*

$$\begin{aligned} \cdots \to \mathscr{L}^s_{n+k}(\textstyle\sum^k R) \to \mathscr{L}^h_{n+k}(\textstyle\sum^k R) \to \hat{H}^k(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1)) \to \mathscr{L}^s_{n+k-1}(\textstyle\sum^{k-1} R) \\ \to \mathscr{L}^h_{n+k-1}(\textstyle\sum^{k-1} R) \to \cdots \to \hat{H}^0(\mathbb{Z}/2; Wh^G(\mathscr{G}, t_1)). \quad \square \end{aligned}$$

§ 4. The Equivariant Surgery Sequence

For a (compact) G-manifold N it is a basic problem to classify its (simple) homotopy G-structures of pairs (M, f) with M a G-manifold and $f : M \to N$ a (simple) G-homotopy equivalence (rel ∂). We can take manifold to mean either smooth or locally linear PL.

The local tangent representations for M and N need not to agree, but we fix them. To this end we pick an $O(G)$-functor

(4.1) $$t_0 : \pi^G N \to \mathbb{B}_n$$

and consider triples (M, f, ω) of a G-manifold M, a G-homotopy equivalence $f : M \to N$ with ∂f an isomorphism and an $O(G)$-transformation $\omega : \mathrm{tp}_M \to f^* t_0$.

Given (M, f, ω), there exists an $(O(G)$-transformation $\tau: t_0^c \to \mathrm{tp}_N^c$ (of $(O(G)$-functors with range $\mathbb{B}F_n$) uniquely determined by

(4.2) $$\operatorname{Deg}(f, (f^* \tau \circ \omega^c)^{-1}) = 1$$

(cf. [8]). We want to fix also τ and let T be the triple $T = (t_0, \tau, \kappa)$ where $\kappa = s$ or h indicates if we work with simple G-homotopy equivalences or not: if $\kappa = s$ both f and τ should be simple, cf. (I.2.4).

Define the set of T-restricted homotopy G-structures $\mathscr{S}_G^T(N, \partial N)$ to be the equivalence classes of triples (M, f, ω) which satisfy (4.2) and where f is simple if $\kappa = s$. Two G-structures (M_0, f_0, ω_0) and (M_1, f_1, ω_1) are equivalent if there is a (simple) G-homotopy equivalence of triads

$$F: (P, \partial P_0, \partial P_1) \to (N \times I, N \times 0, N \times 1 \cup \partial N \times I),$$

and an $O(G)$-transformation

$$\Omega: \mathrm{tp}_P \to F^* pr_1^* t_0,$$

and G-isomorphisms (diffeomorphisms or PL-isomorphisms)

$$u_i: M_i \to \partial_i P$$

such that the obvious compatibility conditions are satisfied.

There is a similar definition of T-restricted normal maps $\mathscr{N}_G^T(N, \partial N)$ where we drop that f is a (simple) G-homotopy equivalence but retain the bundle data via a bundle map $\hat{f}: TM \oplus \mathbb{R}^k \to \xi$, give an $O(G)$-transformation $\mu: \mathrm{tp}_\xi \to \sum^k t_0$, and still require degree 1.

With these definitions it is easy to construct an action

$$\alpha: \mathscr{L}_{n+1}^\kappa(\pi^G N, \textstyle\sum t_0, \sum \mathrm{tp}_N, \sum \tau) \times \mathscr{S}_G^T(N, \partial N) \to \mathscr{S}_G^T(N, \partial N)$$

and a map

$$\eta: \mathscr{S}_G^T(N, \partial N) \to \mathscr{N}_G^T(N, \partial N)$$

which is constant on orbits under α and a map given by the surgery obstruction, cf. (I.§ 3)

$$\lambda: \mathscr{N}_G^T(N, \partial N) \to \mathscr{L}_n^\kappa(\pi^G N, t_0, \mathrm{tp}_N, \tau).$$

One simply imitates the definitions from [13, ch. 10] using the π-π-theorem in the definition of α. This gives:

(4.3) **Theorem.** *If both N and $N \times I$ satisfy the strong gap conditions* (I.1.18) *then there is an exact sequence*

$$\mathscr{L}_{n+1}^\kappa(\pi^G N, \textstyle\sum t_0, \sum \mathrm{tp}_N, \sum \tau) \xrightarrow{\alpha} \mathscr{S}_G^T(N, \partial N) \xrightarrow{\eta} \mathscr{N}_G^T(N, \partial N)$$
$$\xrightarrow{\lambda} \mathscr{L}_n^\kappa(\pi^G N, t_0, \mathrm{tp}_N, \tau)$$

Thus $\lambda^{-1}(0) = \operatorname{Im}(\eta)$ and two elements $s_1, s_2 \in \mathscr{S}_G^T(N, \partial N)$ agree under η if and only if they belong to the same orbit. $\square$

If $\mathscr{S}_G^T(N, \partial N)$ contains an isomorphism or, equivalently, if $t_0 \cong \mathrm{tp}_N$ then the sequence in 4.3 can be continued to the left to give a long exact sequence. Set $R = (\pi^G N, \mathrm{tp}_N, \mathrm{tp}_N, \mathrm{id})$ and suppose $N \times D^{n+1}$ and N satisfy the strong gap conditions I.1.18. Then the sequence

$$\begin{aligned}&\mathscr{L}^{\kappa}_{n+m+1}(\textstyle\sum^{m+1}R) \xrightarrow{\alpha} \mathscr{S}^{T}_{G}(N\times D^{m},\partial) \xrightarrow{\eta} \mathscr{N}^{T}_{G}(N\times D^{m},\partial)\\ (4.4)\qquad &\xrightarrow{\lambda} \mathscr{L}^{\kappa}_{n+m}(\textstyle\sum^{m}R)\to\cdots\end{aligned}$$

is exact if $T=(\mathrm{tp}_N, \mathrm{id}, \kappa)$. The normal invariants are independent of $\kappa=s$ or h, and can, in favourable circumstances be given by an homotopy functor, see [3] and [10].

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