L^2-Invariants from the Algebraic Point of View

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Abstract

We give a survey on L^2-invariants such as L^2-Betti numbers and L^2-torsion taking an algebraic point of view. We discuss their basic definitions, properties and applications to problems arising in topology, geometry, group theory and K-theory.

Key words: dimensions theory over finite von Neumann algebras, L^2-Betti numbers, Novikov Shubin invariants, L^2-torsion, Atiyah Conjecture, Singer Conjecture, algebraic K-theory, geometric group theory, measure theory.

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0 Introduction

The purpose of this survey article is to present an algebraic approach to L^2-invariants such as L^2-Betti numbers and L^2-torsion. Originally these were defined analytically in terms of heat kernels. After it was discovered that they have simplicial and homological algebraic counterparts, there have been many applications to various problems in topology, geometry, group theory and algebraic K-theory, which on the first glance do not involve any L^2-notions. Therefore it seems to be useful to give a quick and friendly introduction to these notions in particular for mathematicians who have more algebraic than analytic background. This does not at all mean that the analytic aspects are less important, but for certain applications it is not necessary to know the analytic approach

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and it is possible and easier to focus on the algebraic aspects. Moreover, questions about L^2-invariants of heat kernels such as the Atiyah Conjecture or the zero-in-the-spectrum-Conjecture turn out to be strongly related to algebraic questions about modules over group rings.

The hope of the author is that more people take notice of L^2-invariants and L^2-methods, and may be able to apply them to their favourite problems, which not necessarily come a priori from an L^2-setting. Typical examples of such instances will be discussed in this survey article. There are many open questions and conjectures which have the potential to stimulate further activities.

The author has tried to write this article in a way which makes it possible to quickly pick out specific topics of interest and read them locally without having to study too much of the previous text.

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In the sequel ring will always mean associative ring with unit and R-module will mean left R-module unless explicitly stated differently. The letter G denotes a discrete group. Actions of G on spaces are always from the left.

Contents

1 Group von Neumann Algebras

The integral group ring $\mathbb{Z}G$ plays an important role in topology and geometry, since for a G-space its singular chain complex or for a G-CW-complex its cellular chain complex are $\mathbb{Z}G$-chain complexes. However, this ring is rather complicated and does not have some of the useful properties which other rings such as fields or semisimple rings have. Therefore it is very hard to analyse modules over $\mathbb{Z}G$. Often in algebra one studies a complicated ring by investigating certain localizations or completions of it which do have nice properties. They still contain and focus on useful information about the original ring, which now becomes accessible. Examples are the quotient field of an integral domain, the p-adic completion of the integers or the algebraic closure of a field. In this section we present a kind of completion of the complex group ring $\mathbb{C}G$ given by the group von Neumann algebra and discuss its ring theoretic properties.

1.1 The Definition of the Group von Neumann Algebra

Denote by $l^2(G)$ the Hilbert space $l^2(G)$ consisting of formal sums $\sum_{g\in G} \lambda_g \cdot g$ for complex numbers λ_g such that $\sum_{g\in G} |\lambda_g|^2 < \infty$. The scalar product is defined by

$$\left\langle \sum_{g\in G} \lambda_g \cdot g, \sum_{g\in G} \mu_g \cdot g \right\rangle \quad := \quad \sum_{g\in G} \lambda_g \cdot \overline{\mu_g}.$$

This is the same as the Hilbert space completion of the complex group ring $\mathbb{C}G$ with respect to the pre-Hilbert space structure for which G is an orthonormal basis. Notice that left multiplication with elements in G induces an isometric G-action on $l^2(G)$. Given a Hilbert space H, denote by $\mathcal{B}(H)$ the C^*-algebra of bounded (linear) operators from H to itself, where the norm is the operator norm and the involution is given by taking adjoints.

Definition 1.1 (Group von Neumann algebra). *The* group von Neumann algebra $\mathcal{N}(G)$ *of the group G is defined as the algebra of G-equivariant bounded operators from $l^2(G)$ to $l^2(G)$*

$$\mathcal{N}(G) \quad := \quad \mathcal{B}(l^2(G))^G.$$

In the sequel we will view the complex group ring $\mathbb{C}G$ as a subring of $\mathcal{N}(G)$ by the embedding of $\mathbb{C}$-algebras $\rho_r\colon \mathbb{C}G \to \mathcal{N}(G)$ which sends $g \in G$ to the G-equivariant operator $r_{g^{-1}}\colon l^2(G) \to l^2(G)$ given by right multiplication with g^{-1}.

Remark 1.2 (The general definition of von Neumann algebras). In general a *von Neumann algebra* $\mathcal{A}$ is a sub-$*$-algebra of $\mathcal{B}(H)$ for some Hilbert space H, which is closed in the weak topology and contains $\mathrm{id}\colon H \to H$. Often in the literature the group von Neumann algebra $\mathcal{N}(G)$ is defined as the closure in the weak topology of the complex group ring $\mathbb{C}G$ considered as $*$-subalgebra of $\mathcal{B}(l^2(G))$. This definition and Definition 1.1 agree (see [60, Theorem 6.7.2 on page 434]).

Example 1.3 (The von Neumann algebra of a finite group). If G is finite, then nothing happens, namely $\mathbb{C}G = l^2(G) = \mathcal{N}(G)$.

Example 1.4 (The von Neumann algebra of $\mathbb{Z}^n$). In general there is no concrete model for $\mathcal{N}(G)$. However, for $G = \mathbb{Z}^n$, there is the following illuminating model for the group von Neumann algebra $\mathcal{N}(\mathbb{Z}^n)$. Let $L^2(T^n)$ be the Hilbert space of equivalence classes of L^2-integrable complex-valued functions on the n-dimensional torus T^n, where two such functions are called equivalent if they differ only on a subset of measure zero. Define the ring $L^\infty(T^n)$ by equivalence classes of essentially bounded measurable functions $f\colon T^n \to \mathbb{C}$, where essentially bounded means that there is a constant $C > 0$ such that the set $\{x \in T^n \mid |f(x)| \geq C\}$ has measure zero. An element $(k_1, \ldots, k_n)$ in $\mathbb{Z}^n$ acts isometrically on $L^2(T^n)$ by pointwise multiplication with the function $T^n \to \mathbb{C}$, which maps $(z_1, z_2, \ldots, z_n)$ to $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism $l^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n)$. Hence $\mathcal{N}(\mathbb{Z}^n) = \mathcal{B}(L^2(T^n))^{\mathbb{Z}^n}$. We obtain an isomorphism (of C^*-algebras)

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n)$$

by sending $f \in L^\infty(T^n)$ to the $\mathbb{Z}^n$-equivariant operator

$$M_f\colon L^2(T^n) \to L^2(T^n), \quad g \mapsto g \cdot f,$$

where $g \cdot f(x)$ is defined by $g(x) \cdot f(x)$.

Let $i\colon H \to G$ be an injective group homomorphism. It induces a ring homomorphism $\mathbb{C}i\colon \mathbb{C}H \to \mathbb{C}G$, which extends to a ring homomorphism

$$\mathcal{N}(i)\colon \mathcal{N}(H) \to \mathcal{N}(G) \tag{1.5}$$

as follows. Let $g\colon l^2(H) \to l^2(H)$ be a H-equivariant bounded operator. Then $\mathbb{C}G \otimes_{\mathbb{C}H} l^2(H) \subseteq l^2(G)$ is a dense G-invariant subspace and

$$\mathrm{id}_{\mathbb{C}G} \otimes_{\mathbb{C}H} g\colon \mathbb{C}G \otimes_{\mathbb{C}H} l^2(H) \to \mathbb{C}G \otimes_{\mathbb{C}H} l^2(H)$$

is a G-equivariant linear map, which is bounded with respect to the norm coming from $l^2(G)$. Hence it induces a G-equivariant bounded operator $l^2(G) \to l^2(G)$, which is by definition the image of $g \in \mathcal{N}(H)$ under $\mathcal{N}(i)$.

In the sequel we will ignore the functional analytic aspects of $\mathcal{N}(G)$ and will only consider its algebraic properties as a ring.

1.2 Ring Theoretic Properties of the Group von Neumann Algebra

On the first glance the von Neumann algebra $\mathcal{N}(G)$ looks not very nice as a ring. It is an *integral domain,* i.e. has no non-trivial zero-divisors if and only if G is trivial. It is Noetherian if and only if G is finite (see [80, Exercise 9.11]). It is for instance easy to see that $\mathcal{N}(\mathbb{Z}^n) \cong L^\infty(T^n)$ does contain non-trivial zero-divisors and is not Noetherian. The main advantage of $\mathcal{N}(G)$ is that it contains many more idempotents than $\mathbb{C}G$. This has the effect that $\mathcal{N}(G)$ has the following ring theoretic property. A ring R is called *semihereditary* if every finitely generated submodule of a projective module is again projective. This implies that the category of finitely presented R-modules is an abelian category.

Theorem 1.6 (Von Neumann algebras are semihereditary). *Any von Neumann algebra $\mathcal{A}$ is semihereditary.*

Proof. This follows from the facts that any von Neumann algebra is a Baer $*$-ring and hence in particular a Rickart C^*-algebra [5, Definition 1, Definition 2 and Proposition 9 in Chapter 1.4] and that a C^*-algebra is semihereditary if and only if it is Rickart [1, Corollary 3.7 on page 270]. ☐

Remark 1.7 (Group von Neumann algebras are semihereditary). It is quite useful to study the following elementary proof of Theorem 1.6 in the special case of a group von Neumann algebra $\mathcal{N}(G)$. One easily checks that it suffices to show for a finitely generated submodule $M \subseteq \mathcal{N}(G)^n$ that M is projective. Let $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ be an $\mathcal{N}(G)$-linear map. Choose a matrix $A \in M(m,n;\mathcal{N}(G))$ such that f is given by right multiplication with A. Because of $\mathcal{N}(G) = \mathcal{B}(l^2(G))^G$ we can define a G-equivariant bounded operator

$$\nu(f)\colon l^2(G)^m \to l^2(G)^n, \quad (u_1,\ldots,u_m) \;\mapsto\; \left(\sum_{i=1}^m \overline{a_{i,1}^*(\overline{u_i})},\ldots,\sum_{i=1}^m \overline{a_{i,n}^*(\overline{u_i})}\right),$$

where by definition $\overline{\sum_{g\in G}\lambda_g \cdot g} := \sum_{g\in G}\overline{\lambda_g}\cdot g$ and $a_{i,j}^*$ denotes the adjoint of $a_{i,j}$. With these conventions $\nu(\mathrm{id}) = \mathrm{id}$, $\nu(r\cdot f + s\cdot g) = r\cdot\nu(f) + s\cdot\nu(g)$ and $\nu(g\circ f) = \nu(g)\circ\nu(f)$ for $r,s\in\mathbb{C}$ and $\mathcal{N}(G)$-linear maps f and g. Moreover we have $\nu(f)^* = \nu(f^*)$ for an $\mathcal{N}(G)$-map $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$, where

$f^*\colon \mathcal{N}(G)^n \to \mathcal{N}(G)^m$ is given by right multiplication with the matrix $(a^*_{j,i})$, if f is given by right multiplication with the matrix $(a_{i,j})$, and $\nu(f)^*$ is the adjoint of the operator $\nu(f)$.

Every equivariant bounded operator $l^2(G)^m \to l^2(G)^n$ can be written as $\nu(f)$ for a unique f. Moreover, the sequence $\mathcal{N}(G)^m \xrightarrow{f} \mathcal{N}(G)^n \xrightarrow{g} \mathcal{N}(G)^p$ of $\mathcal{N}(G)$-modules is exact if and only if the sequence of bounded G-equivariant operators $l^2(G)^m \xrightarrow{\nu(f)} l^2(G)^n \xrightarrow{\nu(g)} l^2(G)^p$ is exact. More details and explanations for the last two statements can be found in [80, Section 6.2].

Consider the finitely generated $\mathcal{N}(G)$-submodule $M \subseteq \mathcal{N}(G)^n$. Choose an $\mathcal{N}(G)$-linear map $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ with image M. The kernel of $\nu(f)$ is a closed G-invariant linear subspace of $l^2(G)^m$. Hence there is an $\mathcal{N}(G)$-map $p\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^m$ such that $\nu(p)$ is a G-equivariant projection, whose image is $\ker(\nu(f))$. Now $\nu(p) \circ \nu(p) = \nu(p)$ implies $p \circ p = p$ and $\operatorname{im}(\nu(p)) = \ker(\nu(f))$ implies $\operatorname{im}(p) = \ker(f)$. Hence $\ker(f)$ is a direct summand in $\mathcal{N}(G)^m$ and $\operatorname{im}(f) = M$ is projective.

The point is that in order to get the desired projection p one passes to the interpretation by Hilbert spaces and uses orthogonal projections there. We have enlarged the group ring $\mathbb{C}G$ to the group von Neumann algebra $\mathcal{N}(G)$, which does contain these orthogonal projections in contrast to $\mathbb{C}G$.

1.3 Dimension Theory over the Group von Neumann Algebra

An important feature of the group von Neumann algebra is its trace.

Definition 1.8 (Von Neumann trace). *The* von Neumann trace *on $\mathcal{N}(G)$ is defined by*

$$\operatorname{tr}_{\mathcal{N}(G)}\colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e\rangle_{l^2(G)},$$

where $e \in G \subseteq l^2(G)$ is the unit element.

It enables us to define a dimension for finitely generated projective $\mathcal{N}(G)$-modules.

Definition 1.9 (Von Neumann dimension for finitely generated projective $\mathcal{N}(G)$-modules). *Let P be a finitely generated projective $\mathcal{N}(G)$-module. Choose a matrix $A = (a_{i,j}) \in M(n,n;\mathcal{N}(G))$ with $A^2 = A$ such that the image of the $\mathcal{N}(G)$-linear map $r_A\colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ given by right multiplication with A is $\mathcal{N}(G)$-isomorphic to P. Define the* von Neumann dimension *of P by*

$$\dim_{\mathcal{N}(G)}(P) \quad := \quad \sum_{i=1}^n \operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}) \quad \in [0,\infty).$$

We omit the standard proof that $\dim_{\mathcal{N}(G)}(P)$ depends only on the isomorphism class of P but not on the choice of the matrix A. Obviously

$$\dim_{\mathcal{N}(G)}(P \oplus Q) = \dim_{\mathcal{N}(G)}(P) + \dim_{\mathcal{N}(G)}(Q).$$

It is not hard to show that $\dim_{\mathcal{N}(G)}$ is *faithful*, i.e. $\dim_{\mathcal{N}(G)}(P) = 0 \Leftrightarrow P = 0$ holds for any finitely generated projective $\mathcal{N}(G)$-module P.

Recall that the dual M^* of a left or right R-module M is the right or left R-module $\hom_R(M, R)$ respectively, where the R-multiplication is given by $(fr)(x) = f(x)r$ or $(rf)(x) = rf(x)$ respectively for $f \in M^*$, $x \in M$ and $r \in R$.

Definition 1.10 (Closure of a submodule). *Let M be an R-submodule of N. Define the* closure *of M in N to be the R-submodule of N*

$$\overline{M} \;:=\; \{x \in N \;\mid\; f(x) = 0 \textit{ for all } f \in N^* \textit{ with } M \subseteq \ker(f)\}.$$

For an R-module M define the R-submodule $\mathbf{T}M$ and the quotient R-module $\mathbf{P}M$ by

$$\begin{array}{rcl}\mathbf{T}M & := & \{x \in M \;\mid\; f(x) = 0 \textit{ for all } f \in M^*\}; \\ \mathbf{P}M & := & M/\mathbf{T}M.\end{array}$$

Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It can also be described as the kernel of the canonical map $i(M)\colon M \to (M^*)^*$, which sends $x \in M$ to the map $M^* \to R, \quad f \mapsto f(x)$. Notice that $\mathbf{TP}M = 0$, $\mathbf{PP}M = \mathbf{P}M$, $M^* = (\mathbf{P}M)^*$ and that $\mathbf{P}M = 0$ is equivalent to $M^* = 0$.

The next result is the key ingredient in the definition of L^2-Betti numbers for G-spaces. Its proof can be found in [76, Theorem 0.6], [80, Theorem 6.7].

Theorem 1.11. (Dimension function for arbitrary $\mathcal{N}(G)$-modules).

(i) *If $K \subseteq M$ is a submodule of the finitely generated $\mathcal{N}(G)$-module M, then $M/\overline{K}$ is finitely generated projective and $\overline{K}$ is a direct summand in M;*

(ii) *If M is a finitely generated $\mathcal{N}(G)$-module, then $\mathbf{P}M$ is finitely generated projective, there is an exact sequence $0 \to \mathcal{N}(G)^n \to \mathcal{N}(G)^n \to \mathbf{T}M \to 0$ and*

$$M \cong \mathbf{P}M \oplus \mathbf{T}M;$$

(iii) *There exists precisely one dimension function*

$$\dim_{\mathcal{N}(G)}\colon \{\mathcal{N}(G)\textit{-modules}\} \;\to\; [0, \infty] := \{r \in \mathbb{R} \mid r \geq 0\} \amalg \{\infty\}$$

which satisfies:

(a) *Extension Property*
If M is a finitely generated projective $\mathcal{N}(G)$-module, then $\dim_{\mathcal{N}(G)}(M)$ agrees with the expression introduced in Definition 1.9;

(b) *Additivity*
If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of $\mathcal{N}(G)$-modules, then

$$\dim_{\mathcal{N}(G)}(M_1) \;=\; \dim_{\mathcal{N}(G)}(M_0) + \dim_{\mathcal{N}(G)}(M_2),$$

where for $r, s \in [0, \infty]$ we define $r + s$ by the ordinary sum of two real numbers if both r and s are not ∞, and by ∞ otherwise;

(c) *Cofinality*

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \bigcup_{i\in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subseteq M_k$. Then

$$\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(M_i) \mid i \in I\};$$

(d) *Continuity*

If $K \subseteq M$ is a submodule of the finitely generated $\mathcal{N}(G)$-module M, then

$$\dim_{\mathcal{N}(G)}(K) = \dim_{\mathcal{N}(G)}(\overline{K}).$$

Definition 1.12 (Von Neumann dimension for arbitrary $\mathcal{N}(G)$-modules). *In the sequel we mean for an (arbitrary) $\mathcal{N}(G)$-module M by* $\dim_{\mathcal{N}(G)}(M)$ *the value of the dimension function appearing in Theorem 1.11 and call it the* von Neumann dimension *of M.*

Remark 1.13 (Uniqueness of the dimension function). There is only one possible definition for the dimension function appearing in Theorem 1.11, namely one must have

$$\dim_{\mathcal{N}(G)}(M) := \sup\{\dim_{\mathcal{N}(G)}(P) \mid P \subseteq M \text{ finitely generated projective submodule}\} \quad \in [0, \infty].$$

Namely, consider the directed system of finitely generated $\mathcal{N}(G)$-submodules $\{M_i \mid i \in I\}$ of M which is directed by inclusion. By Cofinality

$$\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(M_i) \mid i \in I\}.$$

From Additivity and Theorem 1.11 (ii) we conclude

$$\dim_{\mathcal{N}(G)}(M_i) = \dim_{\mathcal{N}(G)}(\mathbf{P}M_i)$$

and that $\mathbf{P}M_i$ is finitely generated projective. This shows uniqueness of $\dim_{\mathcal{N}(G)}$. The hard part in the proof of Theorem 1.11 (iii) is to show that the definition above does have all the desired properties.

We also see what $\dim_{\mathcal{N}(G)}(M) = 0$ means. It is equivalent to the condition that M contains no non-trivial projective $\mathcal{N}(G)$-submodule, or, equivalently, no non-trivial finitely generated projective $\mathcal{N}(G)$-submodule.

Example 1.14 (The von Neumann dimension for finite groups). If G is finite, then $\mathcal{N}(G) = \mathbb{C}G$ and $\operatorname{tr}_{\mathcal{N}(G)}\left(\sum_{g\in G} \lambda_g \cdot g\right)$ is the coefficient λ_e of the unit element $e \in G$. For an $\mathcal{N}(G)$-module M its von Neumann dimension $\dim_{\mathcal{N}(G)}(V)$ is $\frac{1}{|G|}$-times the complex dimension of the underlying complex vector space M.

The next example implies that $\dim_{\mathcal{N}(G)}(P)$ for a finitely generated projective $\mathcal{N}(G)$-module can be any non-negative real number.

Example 1.15 (The von Neumann dimension for $\mathbb{Z}^n$). Consider $G = \mathbb{Z}^n$. Recall that $\mathcal{N}(\mathbb{Z}^n) = L^\infty(T^n)$. Under this identification we get for the von Neumann trace

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^n)} \colon \mathcal{N}(\mathbb{Z}^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu,$$

where μ is the standard Lebesgue measure on T^n.

Let $X \subseteq T^n$ be any measurable set and $\chi_X \in L^\infty(T^n)$ be its characteristic function. Denote by $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image P is a finitely generated projective $\mathcal{N}(\mathbb{Z}^n)$-module, whose von Neumann dimension $\dim_{\mathcal{N}(\mathbb{Z}^n)}(P)$ is the volume $\mu(X)$ of X.

In view of the results above the following slogan makes sense.

Slogan 1.16. The group von Neumann algebra $\mathcal{N}(G)$ behaves like the ring of integers $\mathbb{Z}$ provided one ignores the properties integral domain and Noetherian.

Namely, Theorem 1.11 (ii) corresponds to the statement that a finitely generated $\mathbb{Z}$-module M decomposes into $M = M/\operatorname{tors}(M) \oplus \operatorname{tors}(M)$ and that there exists an exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z}^n \to \mathbb{Z}^n \to \operatorname{tors}(M) \to 0$, where $\operatorname{tors}(M)$ is the $\mathbb{Z}$-module consisting of torsion elements. One obtains the obvious analog of Theorem 1.11 (iii) if one considers

$$\{\mathbb{Z}\text{-modules}\} \to [0,\infty], \quad M \mapsto \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M).$$

One basic difference between the case $\mathbb{Z}$ and $\mathcal{N}(G)$ is that there exist projective $\mathcal{N}(G)$-modules with finite dimension which are not finitely generated, which is not true over $\mathbb{Z}$. For instance take the direct sum $P = \bigoplus_{i=1}^\infty P_i$ of $\mathcal{N}(\mathbb{Z}^n)$-modules P_i appearing in Example 1.15 with $\dim_{\mathcal{N}(\mathbb{Z}^n)}(P_i) = 2^{-i}$. Then P is projective but not finitely generated and satisfies $\dim_{\mathcal{N}(\mathbb{Z}^n)}(P) = 1$.

The proof of the following two results is given in [80, Theorem 6.13 and Theorem 6.39].

Theorem 1.17 (Dimension and colimits). *Let $\{M_i \mid i \in I\}$ be a directed system of $\mathcal{N}(G)$-modules over the directed set I. For $i \le j$ let $\phi_{i,j} \colon M_i \to M_j$ be the associated morphism of $\mathcal{N}(G)$-modules. For $i \in I$ let $\psi_i \colon M_i \to \operatorname{colim}_{i\in I} M_i$ be the canonical morphism of $\mathcal{N}(G)$-modules. Then:*

(i) We get for the dimension of the $\mathcal{N}(G)$-module given by the colimit

$$\dim_{\mathcal{N}(G)}\left(\operatorname{colim}_{i\in I} M_i\right) = \sup\left\{\dim_{\mathcal{N}(G)}(\operatorname{im}(\psi_i)) \mid i \in I\right\};$$

(ii) Suppose for each $i \in I$ that there exists $i_0 \in I$ with $i \le i_0$ such that $\dim_{\mathcal{N}(G)}(\operatorname{im}(\phi_{i,i_0})) < \infty$ holds. Then

$$\begin{aligned}&\dim_{\mathcal{N}(G)}\left(\operatorname{colim}_{i\in I} M_i\right)\\ &\quad = \sup\left\{\inf\left\{\dim_{\mathcal{N}(G)}(\operatorname{im}(\phi_{i,j} \colon M_i \to M_j)) \mid j \in I, i \le j\right\} \mid i \in I\right\}.\end{aligned}$$

Theorem 1.18 (Induction and dimension). *Let $i\colon H \to G$ be an injective group homomorphism. Then*

(i) Induction with $\mathcal{N}(i)\colon \mathcal{N}(H) \to \mathcal{N}(G)$ is a faithfully flat functor $M \mapsto i_*M := \mathcal{N}(G) \otimes_{\mathcal{N}(i)} M$ from the category of $\mathcal{N}(H)$-modules to the category of $\mathcal{N}(G)$-modules, i.e. a sequence of $\mathcal{N}(H)$-modules $M_0 \to M_1 \to M_2$ is exact at M_1 if and only if the induced sequence of $\mathcal{N}(G)$-modules $i_*M_0 \to i_*M_1 \to i_*M_2$ is exact at i_*M_1;

(ii) For any $\mathcal{N}(H)$-module M we have:

$$\dim_{\mathcal{N}(H)}(M) \;=\; \dim_{\mathcal{N}(G)}(i_*M).$$

Example 1.19 (The von Neumann dimension and $\mathbb{C}[\mathbb{Z}^n]$-modules). Consider the case $G = \mathbb{Z}^n$. Then $\mathbb{C}[\mathbb{Z}^n]$ is a commutative integral domain and hence has a quotient field $\mathbb{C}[\mathbb{Z}^n]_{(0)}$. Let $\dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}$ denote the usual dimension for vector spaces over $\mathbb{C}[\mathbb{Z}^n]_{(0)}$. Let M be a $\mathbb{C}[\mathbb{Z}^n]$-module. Then

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}\left(\mathcal{N}(\mathbb{Z}^n) \otimes_{\mathbb{C}[\mathbb{Z}^n]} M\right) \;=\; \dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}\left(\mathbb{C}[\mathbb{Z}^n]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}^n]} M\right). \tag{1.20}$$

This follows from the following considerations. Let $\{M_i \mid i \in I\}$ be the directed system of finitely generated submodules of M. Then $M = \operatorname{colim}_{i\in I} M_i$. Since the tensor product has a right adjoint, it is compatible with colimits. This implies together with Theorem 1.17

$$\begin{aligned}
\dim_{\mathcal{N}(\mathbb{Z}^n)}\left(\mathcal{N}(\mathbb{Z}^n) \otimes_{\mathbb{C}[\mathbb{Z}^n]} M\right) &= \sup\left\{\dim_{\mathcal{N}(\mathbb{Z}^n)}\left(\mathcal{N}(\mathbb{Z}^n) \otimes_{\mathbb{C}[\mathbb{Z}^n]} M_i\right)\right\};\\
\dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}\left(\mathbb{C}[\mathbb{Z}^n]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}^n]} M\right) &= \sup\left\{\dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}\left(\mathbb{C}[\mathbb{Z}^n]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}^n]} M_i\right)\right\}.
\end{aligned}$$

Hence it suffices to prove the claim for a finitely generated $\mathbb{C}[\mathbb{Z}^n]$-module N. The case $n = 1$ is easy. Then $\mathbb{C}[\mathbb{Z}]$ is a principal integral domain and we can write

$$N \;=\; \mathbb{C}[\mathbb{Z}]^r \;\oplus\; \bigoplus_{i=1}^{k} \mathbb{C}[\mathbb{Z}]/(u_i)$$

for non-trivial elements $u_i \in \mathbb{C}[\mathbb{Z}]$ and some non-negative integers k and r. One easily checks that there is an exact $\mathcal{N}(\mathbb{Z})$-sequence

$$0 \to \mathcal{N}(\mathbb{Z}) \xrightarrow{r_{u_i}} \mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}) \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}[\mathbb{Z}]/(u_i) \to 0$$

using the identification $\mathcal{N}(\mathbb{Z}) = L^\infty(S^1)$ from Example 1.4 to show injectivity of the map r_{u_i} given by multiplication with u_i. This implies

$$\dim_{\mathcal{N}(\mathbb{Z})}\left(\mathcal{N}(\mathbb{Z}) \otimes_{\mathbb{C}[\mathbb{Z}]} N\right) \;=\; r \;=\; \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\left(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} N\right).$$

In the general case $n \geq 1$ one knows that there exists a finite free $\mathbb{C}[\mathbb{Z}^n]$-resolution of N. Now the claim follows from [80, Lemma 1.34].

This example is the commutative version of a general setup for arbitrary groups, which will be discussed in Subsection 4.2.

A center-valued dimension function for finitely generated projective modules will be introduced in Definition 7.3. It can be used to classify finitely generated projective $\mathcal{N}(G)$-modules (see Theorem 7.5) and shows that the representation theory of finite dimensional representations over a finite group extends to infinite groups if one works with $\mathcal{N}(G)$ (see Remark 7.6).

2 Definition and Basic Properties of L^2-Betti Numbers

In this section we define L^2-Betti numbers for arbitrary G-spaces and study their basic properties. Our general algebraic definition is very general and is very flexible. This allows to apply standard techniques such as spectral sequences and Mayer-Vietoris arguments directly. The original analytic definition for free proper smooth G-manifolds with G-invariant Riemannian metrics is due to Atiyah and will be briefly discussed in Subsection 2.3.

2.1 The Definition of L^2-Betti Numbers

Definition 2.1 (L^2-Betti numbers of G-spaces). *Let X be a (left) G-space. Equip $\mathcal{N}(G)$ with the obvious $\mathcal{N}(G)$-$\mathbb{Z}G$-bimodule structure. The* singular homology $H_p^G(X;\mathcal{N}(G))$ of X with coefficients in $\mathcal{N}(G)$ *is the homology of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X)$, where $C_*^{\mathrm{sing}}(X)$ is the singular chain complex of X with the induced $\mathbb{Z}G$-structure. Define the* p-th L^2-Betti number of X *by*

$$b_p^{(2)}(X;\mathcal{N}(G)) \ := \ \dim_{\mathcal{N}(G)} \left(H_p^G(X;\mathcal{N}(G))\right) \quad \in [0,\infty],$$

where $\dim_{\mathcal{N}(G)}$ *is the dimension function of Definition 1.12.*

If G and its action on X are clear from the context, we often omit $\mathcal{N}(G)$ in the notation above. For instance, for a connected CW-complex X we denote by $b_p^{(2)}(\widetilde{X})$ the L^2-Betti number $b_p^{(2)}(\widetilde{X};\mathcal{N}(\pi_1(X)))$ of its universal covering $\widetilde{X}$ with respect to the obvious $\pi_1(X)$-action.

Notice that we have *no* assumptions on the G-action or on the topology on X, we do *not* need to require that the operation is free, proper, simplicial or cocompact. Thus we can apply this definition to the *classifying space for free proper G-actions EG*, which is a free G-CW-complex which is contractible (after forgetting the group action). Recall that EG is unique up to G-homotopy. Its quotient $BG = G\backslash EG$ is a connected CW-complex, which is up to homotopy uniquely determined by the property that $\pi_n(BG) = \{1\}$ for $n \geq 2$ and $\pi_1(BG) \cong G$ holds, and called *classifying space of G*. Moreover, $G \to EG \to BG$ is the universal G-principal bundle.

Definition 2.2 (L^2-Betti numbers of groups). *Define for any (discrete) group G its* p-th L^2-Betti number *by*

$$b_p^{(2)}(G) \ := \ b_p^{(2)}(EG,\mathcal{N}(G)).$$

Remark 2.3 (Comparison with the approach by Cheeger and Gromov). A detailed comparison of our approach with the one by Cheeger and Gromov [15, section 2] can be found in [80, Remark 6.76]. Cheeger and Gromov [15, Section 2] define L^2-cohomology and L^2-Betti numbers of a G-space X by considering the category whose objects are G-maps $f\colon Y \to X$ for a simplicial complex Y with cocompact free simplicial G-action and then using inverse limits to extend the classical notions for finite free G-CW-complexes such as Y to X. Their approach is technically more complicated because for instance they work with cohomology instead of homology and therefore have to deal with inverse limits instead of directed limits. Our approach is closer to standard notions, the only non-standard part is the verification of the properties of the extended dimension function (Theorem 1.11).

Remark 2.4 (L^2-Betti numbers for von Neumann algebras). The algebraic approach to L^2-Betti numbers of groups as

$$b_p^{(2)}(G) = \dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_p^{\mathbb{C}G}(\mathbb{C}, \mathcal{N}(G))\right)$$

based on the dimension function for arbitrary modules and homological algebra plays a role in the definition of L^2-Betti numbers for certain von Neumann algebras by Connes-Shlyakhtenko [18]. The point of their construction is to introduce invariants which depend on the group von Neumann algebra $\mathcal{N}(G)$ only. If one could show that their invariants applied to $\mathcal{N}(G)$ agree with the L^2-Betti numbers of G, one would get a positive answer to the open problem, whether the von Neumann algebras of two finitely generated free groups F_1 and F_2 are isomorphic as von Neumann algebras if and only if the groups F_1 and F_2 are isomorphic.

Definition 2.5 (G-CW-complex). *A G-CW-complex X is a G-space together with a G-invariant filtration*

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration (i.e. a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed in X_n for all $n \geq 0$) and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e. there exists a G-pushout

$$\begin{array}{ccc}
\coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i} & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow[\coprod_{i \in I_n} Q_i]{} & X_n
\end{array}$$

The space X_n is called the *n-skeleton* of X. A G-CW-complex X is *proper* if and only if all its isotropy groups are finite. A G-space is called *cocompact* if $G\backslash X$ is compact. A G-CW-complex X is *finite* if X has only finitely many equivariant cells. A G-CW-complex is finite if and only if it is cocompact. A G-CW-complex X is *of finite type* if each n-skeleton is finite. It is called *of dimension* $\leq n$ if $X = X_n$ and *finite dimensional* if it is of dimension $\leq n$ for some integer n. A free G-CW-complex X is the same as a regular covering $X \to Y$ of a CW-complex Y with G as group of deck transformations.

Notice that Definition 2.5 also makes sense in the case where G is a topological group. Every proper smooth cocompact G-manifold is a proper G-CW-complex by means of an equivariant triangulation.

For a G-CW-complex one can use the cellular $\mathbb{Z}G$-chain complex instead of the singular chain complex in the definition of L^2-Betti numbers by the next result. Its proof can be found in [76, Lemma 4.2]. For more information about G-CW-complexes we refer for instance to [104, Sections II.1 and II.2], [71, Sections 1 and 2], [80, Subsection 1.2.1].

Lemma 2.6. *Let X be a G-CW-complex. Let $C_*^c(X)$ be its cellular $\mathbb{Z}G$-chain complex. Then there is a $\mathbb{Z}G$-chain homotopy equivalence $C_*^{\mathrm{sing}}(X) \to C_*^c(X)$ and we get*

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; \dim_{\mathcal{N}(G)}\left(H_p\left(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^c(X)\right)\right).$$

The definition of $b_p^{(2)}(X;\mathcal{N}(G))$ and the above lemma extend in the obvious way to pairs (X, A).

2.2 Basic Properties of L^2-Betti Numbers

The basic properties of L^2-Betti numbers are summarized in the following theorem. Its proof can be found in [80, Theorem 1.35 and Theorem 6.54] except for assertion (viii) which follows from [80, Lemma 13.45].

Theorem 2.7 (L^2-Betti numbers for arbitrary spaces).

(i) Homology invariance

We have for a G-map $f\colon X \to Y$:

(a) Suppose for $n \geq 1$ that for each subgroup $H \subseteq G$ the induced map $f^H\colon X^H \to Y^H$ is $\mathbb{C}$-homologically n-connected, i.e. the map

$$H_p^{\mathrm{sing}}(f^H;\mathbb{C})\colon H_p^{\mathrm{sing}}(X^H;\mathbb{C}) \to H_p^{\mathrm{sing}}(Y^H;\mathbb{C})$$

induced by f^H on singular homology with complex coefficients is bijective for $p < n$ and surjective for $p = n$. Then

$$\begin{aligned} b_p^{(2)}(X) &= b_p^{(2)}(Y) && \text{for } p < n;\\ b_p^{(2)}(X) &\geq b_p^{(2)}(Y) && \text{for } p = n; \end{aligned}$$

(b) *Suppose that for each subgroup $H \subseteq G$ the induced map $f^H\colon X^H \to Y^H$ is a $\mathbb{C}$-homology equivalence, i.e. $H_p^{\mathrm{sing}}(f^H;\mathbb{C})$ is bijective for $p \geq 0$. Then*

$$b_p^{(2)}(X) \;=\; b_p^{(2)}(Y) \qquad \text{for } p \geq 0;$$

(ii) *Comparison with the Borel construction*
Let X be a G-CW-complex. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_p^{(2)}(G_x) = 0$ for all $p \geq 0$. Then

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; b_p^{(2)}(EG \times X;\mathcal{N}(G)) \qquad \text{for } p \geq 0,$$

where G acts diagonally on $EG \times X$;

(iii) *Invariance under non-equivariant $\mathbb{C}$-homology equivalences*
Suppose that $f\colon X \to Y$ is a G-equivariant map of G-CW-complexes such that the induced map $H_p^{\mathrm{sing}}(f;\mathbb{C})$ on singular homology with complex coefficients is bijective for all p. Suppose that for all $x \in X$ the isotropy group G_x is finite or satisfies $b_p^{(2)}(G_x) = 0$ for all $p \geq 0$, and analogously for all $y \in Y$. Then we have for all $p \geq 0$

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; b_p^{(2)}(Y;\mathcal{N}(G));$$

(iv) *Independence of equivariant cells with infinite isotropy*
Let X be a G-CW-complex. Let $X[\infty]$ be the G-CW-subcomplex consisting of those points whose isotropy subgroups are infinite. Then we get for all $p \geq 0$

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; b_p^{(2)}(X, X[\infty];\mathcal{N}(G));$$

(v) *Künneth formula*
Let X be a G-space and Y be an H-space. Then $X \times Y$ is a $G \times H$-space and we get for all $n \geq 0$

$$b_n^{(2)}(X \times Y) \;=\; \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y),$$

where we use the convention that $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ for $r \in (0,\infty]$ and $r + \infty = \infty$ for $r \in [0,\infty]$;

(vi) *Induction*
Let $i\colon H \to G$ be an inclusion of groups and let X be an H-space. Let $\mathcal{N}(i)\colon \mathcal{N}(H) \to \mathcal{N}(G)$ be the induced ring homomorphism (see (1.5)). Then:

$$\begin{aligned} H_p^G(G \times_H X;\mathcal{N}(G)) &= \mathcal{N}(G) \otimes_{\mathcal{N}(i)} H_p^H(X;\mathcal{N}(H)); \\ b_p^{(2)}(G \times_H X;\mathcal{N}(G)) &= b_p^{(2)}(X;\mathcal{N}(H)); \end{aligned}$$

(vii) *Restriction to subgroups of finite index*

Let $H \subseteq G$ be a subgroup of finite index $[G : H]$. Let X be a G-space and let $\operatorname{res}_G^H X$ be the H-space obtained from X by restriction. Then

$$b_p^{(2)}(\operatorname{res}_G^H X; \mathcal{N}(H)) \;=\; [G : H] \cdot b_p^{(2)}(X; \mathcal{N}(G));$$

(viii) *Restriction with epimorphisms with finite kernel*

Let $p\colon G \to Q$ be an epimorphism of groups with finite kernel K. Let X be a Q-space. Let p^*X be the G-space obtained from X using p. Then

$$b_p^{(2)}(p^*X; \mathcal{N}(G)) \quad = \quad \frac{1}{|K|} \cdot b_p^{(2)}(X; \mathcal{N}(Q));$$

(ix) *Zero-th homology and L^2-Betti number*

Let X be a path-connected G-space. Then:

(a) *There is an $\mathcal{N}(G)$-isomorphism $H_0^G(X; \mathcal{N}(G)) \xrightarrow{\cong} \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}$;*

(b) *$b_0^{(2)}(X; \mathcal{N}(G)) = |G|^{-1}$, where $|G|^{-1}$ is defined to be zero if the order $|G|$ of G is infinite;*

(x) *Euler-Poincaré formula*

Let X be a free finite G-CW-complex. Let $\chi(G\backslash X)$ be the Euler characteristic of the finite CW-complex $G\backslash X$, i.e.

$$\chi(G\backslash X) \quad := \quad \sum_{p \geq 0} (-1)^p \cdot |I_p(G\backslash X)| \qquad \in \mathbb{Z},$$

where $|I_p(G\backslash X)|$ is the number of p-cells of $G\backslash X$. Then

$$\chi(G\backslash X) \quad = \quad \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(X);$$

(xi) *Morse inequalities*

Let X be a free G-CW-complex of finite type. Then we get for $n \geq 0$

$$\sum_{p=0}^{n} (-1)^{n-p} \cdot b_p^{(2)}(X) \quad \leq \quad \sum_{p=0}^{n} (-1)^{n-p} \cdot |I_p(G\backslash X)|;$$

(xii) *Poincaré duality*

Let M be a cocompact free proper G-manifold of dimension n which is orientable. Then

$$b_p^{(2)}(M) \quad = \quad b_{n-p}^{(2)}(M, \partial M);$$

(xiii) Wedges

Let $X_1, X_2, \ldots, X_r$ be connected (pointed) CW-complexes of finite type and $X = \bigvee_{i=1}^r X_i$ be their wedge. Then

$$\begin{aligned} b_1^{(2)}(\widetilde{X}) - b_0^{(2)}(\widetilde{X}) &= r - 1 + \sum_{j=1}^r \left(b_1^{(2)}(\widetilde{X_j}) - b_0^{(2)}(\widetilde{X_j}) \right); \\ b_p^{(2)}(\widetilde{X}) &= \sum_{j=1}^r b_p^{(2)}(\widetilde{X_j}) \qquad \text{for } 2 \leq p; \end{aligned}$$

(xiv) Connected sums

Let $M_1, M_2, \ldots, M_r$ be compact connected m-dimensional manifolds for $m \geq 3$. Let M be their connected sum $M_1 \# \ldots \# M_r$. Then

$$\begin{aligned} b_1^{(2)}(\widetilde{M}) - b_0^{(2)}(\widetilde{M}) &= r - 1 + \sum_{j=1}^r \left(b_1^{(2)}(\widetilde{M_j}) - b_0^{(2)}(\widetilde{M_j}) \right); \\ b_p^{(2)}(\widetilde{M}) &= \sum_{j=1}^r b_p^{(2)}(\widetilde{M_j}) \qquad \text{for } 2 \leq p \leq m-2. \end{aligned}$$

Example 2.8. If G is finite, then $b_p^{(2)}(X; \mathcal{N}(G))$ reduces to the classical Betti number $b_p(X)$ multiplied with the factor $|G|^{-1}$.

Remark 2.9 (Reading off L^2-Betti numbers from $H_p(X;\mathbb{C})$). If $f\colon X \to Y$ is a G-map of free G-CW-complexes which induces isomorphisms $H_p^{\mathrm{sing}}(f;\mathbb{C})$ for all $p \geq 0$, then Theorem 2.7 (i) implies

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; b_p^{(2)}(Y;\mathcal{N}(G)).$$

This does not necessarily mean that one can read off $b_p^{(2)}(X;\mathcal{N}(G))$ from the singular homology $H_p(X;\mathbb{C})$ regarded as a $\mathbb{C}G$-module in general. In general there is for a free G-CW-complex X a spectral sequence converging to $H_{p+q}^G(X;\mathcal{N}(G))$, whose E^2-term is

$$E_{p,q}^2 \;=\; \mathrm{Tor}_p^{\mathbb{C}G}(H_q(X;\mathbb{C}), \mathcal{N}(G)).$$

There is no reason why the equality of the dimension of the E^2-term for two free G-CW-complexes X and Y implies that the dimension of $H_{p+q}^G(X;\mathcal{N}(G))$ and $H_{p+q}^G(Y;\mathcal{N}(G))$ agree. However, this is the case if the spectral sequence collapses from the dimension point of view. For instance, if we make the assumption $\dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_p^{\mathbb{C}G}(M,\mathcal{N}(G))\right) = 0$ for all $\mathbb{C}G$-modules M and $p \geq 2$, Additivity and Cofinality of $\dim_{\mathcal{N}(G)}$ (see Theorem 1.11) imply

$$\begin{aligned} & b_p^{(2)}(X;\mathcal{N}(G)) \;= \\ & \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_p(X;\mathbb{C})) + \dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_1^{\mathbb{C}G}(H_{p-1}(X;\mathbb{C}), \mathcal{N}(G))\right). \end{aligned}$$

The assumption above is satisfied if G is amenable (see Theorem 5.1) or G has cohomological dimension ≤ 1 over $\mathbb{C}$, for instance, if G is virtually free.

Remark 2.10 (L^2-Betti numbers ignore infinite isotropy). Theorem 2.7 (iv) says that the L^2-Betti numbers do not see the part of a G-space X whose isotropy groups are infinite. In particular $b_p^{(2)}(X;\mathcal{N}(G)) = 0$ if X is a G-CW-complex whose isotropy groups are all infinite. This follows from the fact that for a subgroup $H \subseteq G$

$$\dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}[G/H]\right) \quad = \quad \begin{cases} \frac{1}{|H|} & \text{if } |H| < \infty; \\ 0 & \text{if } |H| = \infty. \end{cases}$$

Remark 2.11 (L^2-Betti numbers often vanish). An important phenomenon is that the L^2-Betti numbers of universal coverings of spaces and of groups tend to vanish more often than the classical Betti numbers. This allows to draw interesting conclusions as we will see later.

2.3 Comparison with Other Definitions

In this subsection we give a short overview of the previous definitions of L^2-Betti numbers. Originally they were defined in terms of heat kernels. Their analytic aspects are important, but we will only focus on their algebraic aspects in this survey article. So a reader may skip the brief explanations below.

The notion of L^2-Betti numbers is due to Atiyah [2]. He defined for a smooth Riemannian manifold with a free proper cocompact G-action by isometries its *analytic p-th L^2-Betti number* by the following expression in terms of the *heat kernel* $e^{-t\Delta_p}(x,y)$ of the p-th Laplacian Δ_p

$$b_p^{(2)}(M) \quad = \quad \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\, dvol_x, \tag{2.12}$$

where $\mathcal{F}$ is a fundamental domain for the G-action and $\operatorname{tr}_{\mathbb{C}}$ denotes the trace of an endomorphism of a finite-dimensional vector space. The L^2-Betti numbers are invariants of the large times asymptotic of the heat kernel.

A *finitely generated Hilbert $\mathcal{N}(G)$-module* is a Hilbert space V together with a linear G-action by isometries such that there exists a linear isometric G-embedding into $l^2(G)^n$ for some $n \geq 0$. One can assign to it its *von Neumann dimension* by

$$\dim_{\mathcal{N}(G)}(V) \;:=\; \operatorname{tr}_{\mathcal{N}(G)}(A) \quad \in [0,\infty),$$

where A is any idempotent matrix $A \in M(n,n;\mathcal{N}(G))$ such that the image of the G-equivariant operator $l^2(G)^n \to l^2(G)^n$ induced by A is isometrically linearly G-isomorphic to V.

The expression in (2.12) can be interpreted as the von Neumann dimension of the *space $\mathcal{H}^p_{(2)}(M)$ of square-integrable harmonic p-forms* on M, which is a finitely generated Hilbert $\mathcal{N}(G)$-module (see [2, Proposition 4.16 on page 63])

$$\lim_{t\to\infty}\int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\ dvol_x \quad = \quad \dim_{\mathcal{N}(G)}\left(\mathcal{H}^p_{(2)}(M)\right). \tag{2.13}$$

Given a cocompact free G-CW-complex X, one obtains a chain complex of finitely generated Hilbert $\mathcal{N}(G)$-modules $C_*^{(2)}(X) \ := \ C_*^c(X)\otimes_{\mathbb{Z}G} l^2(G)$. Its *reduced p-th L^2-homology* is the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_p^{(2)}(X;l^2(G)) \quad = \quad \ker(c_p^{(2)})/\overline{\mathrm{im}(c_{p+1}^{(2)})}. \tag{2.14}$$

Notice that we divide out the closure of the image of the $(p+1)$-th differential $c_{p+1}^{(2)}$ of $C_*^{(2)}(X)$ in order to ensure that we obtain a Hilbert space. Then by a result of Dodziuk [24] there is an isometric bijective G-operator

$$\mathcal{H}^p_{(2)}(M) \quad \xrightarrow{\cong} \quad H_p^{(2)}(K;l^2(G)), \tag{2.15}$$

where K is an equivariant triangulation of M. Finally one can show [74, Theorem 6.1]

$$b_p^{(2)}(K;\mathcal{N}(G)) \quad = \quad \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(K;l^2(G))\right), \tag{2.16}$$

where $b_p^{(2)}(K;\mathcal{N}(G))$ is the p-th L^2-Betti number in the sense of Definition 2.1.

All in all we see that our Definition 2.1 of L^2-Betti numbers for arbitrary G-spaces extends the heat kernel definition of (2.12) for smooth Riemannian manifolds with a free proper cocompact G-action by isometries. More details of all these definitions and of their identifications can be found in [80, Chapter 1].

2.4 L^2-Euler Characteristic

In this section we introduce the notion of L^2-Euler characteristic.

If X is a G-CW-complex, denote by $I(X)$ the set of its equivariant cells. For a cell $c\in I(X)$ let (G_c) be the conjugacy class of subgroups of G given by its orbit type and let $\dim(c)$ be its dimension. Denote by $|G_c|^{-1}$ the inverse of the order of any representative of (G_c), where $|G_c|^{-1}$ is to be understood to be zero if the order is infinite.

Definition 2.17 (L^2-Euler characteristic). *Let G be a group and let X be a G-space. Define*

$$\begin{array}{lcl}
h^{(2)}(X;\mathcal{N}(G)) & := & \sum_{p\geq 0} b_p^{(2)}(X;\mathcal{N}(G)) \ \in [0,\infty]; \\
\chi^{(2)}(X;\mathcal{N}(G)) & := & \sum_{p\geq 0} (-1)^p\cdot b_p^{(2)}(X;\mathcal{N}(G)) \ \in \mathbb{R}, \ \textit{if } h^{(2)}(X;\mathcal{N}(G))<\infty; \\
m(X;G) & := & \sum_{c\in I(X)} |G_c|^{-1} \ \in [0,\infty], \quad \textit{if } X \textit{ is a } G\textit{-}CW\textit{-complex}; \\
h^{(2)}(G) & := & h^{(2)}(EG;\mathcal{N}(G)) \ \in [0,\infty]; \\
\chi^{(2)}(G) & := & \chi^{(2)}(EG;\mathcal{N}(G)) \ \in \mathbb{R}, \quad \textit{if } h^{(2)}(G)<\infty.
\end{array}$$

We call $\chi^{(2)}(X;\mathcal{N}(G))$ and $\chi^{(2)}(G)$ the L^2-Euler characteristic *of X and G.*

The condition $h^{(2)}(X;\mathcal{N}(G)) < \infty$ ensures that the sum which appears in the definition of $\chi^{(2)}(X;\mathcal{N}(G))$ converges absolutely and that the following results are true. The reader should compare the next theorem with [15, Theorem 0.3 on page 191]. It essentially follows from Theorem 2.7. Details of its proof can be found in [80, Theorem 6.80].

Theorem 2.18 (L^2-Euler characteristic).

(i) *Generalized Euler-Poincaré formula*
Let X be a G-CW-complex with $m(X;G) < \infty$. Then

$$\begin{array}{rcl} h^{(2)}(X;\mathcal{N}(G)) & < & \infty; \\ \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |G_c|^{-1} & = & \chi^{(2)}(X;\mathcal{N}(G)); \end{array}$$

(ii) *Sum formula*
Consider the following G-pushout

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X \end{array}$$

such that i_1 is a G-cofibration. Suppose that $h^{(2)}(X_i;\mathcal{N}(G)) < \infty$ for $i = 0, 1, 2$. Then

$$\begin{array}{rcl} h^{(2)}(X;\mathcal{N}(G)) & < & \infty; \\ \chi^{(2)}(X;\mathcal{N}(G)) & = & \chi^{(2)}(X_1;\mathcal{N}(G)) + \chi^{(2)}(X_2;\mathcal{N}(G)) - \chi^{(2)}(X_0;\mathcal{N}(G)); \end{array}$$

(iii) *Comparison with the Borel construction*
Let X be a G-CW-complex. If for all $c \in I(X)$ the group G_c is finite or $b_p^{(2)}(G_c) = 0$ for all $p \geq 0$, then

$$\begin{array}{rcl} b_p^{(2)}(X;\mathcal{N}(G)) & = & b_p^{(2)}(EG \times X;\mathcal{N}(G)) \qquad \text{for } p \geq 0; \\ h^{(2)}(X;\mathcal{N}(G)) & = & h^{(2)}(EG \times X;\mathcal{N}(G)); \\ \chi^{(2)}(X;\mathcal{N}(G)) & = & \chi^{(2)}(EG \times X;\mathcal{N}(G)), \text{if } h^{(2)}(X;\mathcal{N}(G)) < \infty; \\ \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |G_c|^{-1} & = & \chi^{(2)}(EG \times X;\mathcal{N}(G)), \text{ if } m(X;G) < \infty; \end{array}$$

(iv) *Invariance under non-equivariant $\mathbb{C}$-homology equivalences*
Suppose that $f\colon X \to Y$ is a G-equivariant map of G-CW-complexes with $m(X;G) < \infty$ and $m(Y;G) < \infty$, such that the induced map $H_p(f;\mathbb{C})$ on homology with complex coefficients is bijective for all $p \geq 0$. Suppose that

for all $c \in I(X)$ the group G_c is finite or $b_p^{(2)}(G_c) = 0$ for all $p \geq 0$, and analogously for all $d \in I(Y)$. Then

$$\begin{aligned}
\chi^{(2)}(X;\mathcal{N}(G)) &= \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |G_c|^{-1} \\
&= \sum_{d \in I(Y)} (-1)^{\dim(d)} \cdot |G_d|^{-1} \\
&= \chi^{(2)}(Y;\mathcal{N}(G));
\end{aligned}$$

(v) *Künneth formula*

Let X be a G-CW-complex and Y be an H-CW-complex. Then we get for the $G \times H$-CW-complex $X \times Y$

$$\begin{aligned}
m(X \times Y; G \times H) &= m(X;G) \cdot m(Y;H); \\
h^{(2)}(X \times Y;\mathcal{N}(G \times H)) &= h^{(2)}(X;\mathcal{N}(G)) \cdot h^{(2)}(Y;\mathcal{N}(H)); \\
\chi^{(2)}(X \times Y;\mathcal{N}(G \times H)) &= \chi^{(2)}(X;\mathcal{N}(G)) \cdot \chi^{(2)}(Y;\mathcal{N}(H)), \\
&\qquad \textit{if } h^{(2)}(X;\mathcal{N}(G)), h^{(2)}(Y;\mathcal{N}(H)) < \infty,
\end{aligned}$$

where we use the convention that $0 \cdot \infty = 0$ and $r \cdot \infty = \infty$ for $r \in (0, \infty]$;

(vi) *Induction*

Let $H \subseteq G$ be a subgroup and let X be an H-space. Then

$$\begin{aligned}
m(G \times_H X; G) &= m(X;H); \\
h^{(2)}(G \times_H X;\mathcal{N}(G)) &= h^{(2)}(X;\mathcal{N}(H)); \\
\chi^{(2)}(G \times_H X;\mathcal{N}(G)) &= \chi^{(2)}(X;\mathcal{N}(H)), \qquad \textit{if } h^{(2)}(X;\mathcal{N}(H)) < \infty;
\end{aligned}$$

(vii) *Restriction to subgroups of finite index*

Let $H \subseteq G$ be a subgroup of finite index $[G : H]$. Let X be a G-space and let $\operatorname{res}_G^H X$ be the H-space obtained from X by restriction. Then

$$\begin{aligned}
m(\operatorname{res}_G^H X; H) &= [G:H] \cdot m(X;G); \\
h^{(2)}(\operatorname{res}_G^H X;\mathcal{N}(H)) &= [G:H] \cdot h^{(2)}(X;\mathcal{N}(G)); \\
\chi^{(2)}(\operatorname{res}_G^H X;\mathcal{N}(H)) &= [G:H] \cdot \chi^{(2)}(X;\mathcal{N}(G)), \ \textit{if } h^{(2)}(X;\mathcal{N}(G)) < \infty,
\end{aligned}$$

where $[G : H] \cdot \infty$ is understood to be ∞;

(viii) *Restriction with epimorphisms with finite kernel*

Let $p \colon G \to Q$ be an epimorphism of groups with finite kernel K. Let X be a Q-space. Let p^*X be the G-space obtained from X using p. Then

$$\begin{aligned}
m(p^*X; G) &= |K|^{-1} \cdot m(X;Q); \\
h^{(2)}(p^*X;\mathcal{N}(G)) &= |K|^{-1} \cdot h^{(2)}(X;\mathcal{N}(Q)); \\
\chi^{(2)}(p^*X;\mathcal{N}(G)) &= |K|^{-1} \cdot \chi^{(2)}(X;\mathcal{N}(Q)), \quad \textit{if } h^{(2)}(X;\mathcal{N}(Q)) < \infty.
\end{aligned}$$

Remark 2.19 (L^2-Euler characteristic and virtual Euler characteristic). The L^2-Euler characteristic generalizes the notion of the virtual Euler characteristic. Let X be a CW-complex which is *virtually homotopy finite*, i.e. there is a d-sheeted covering $p\colon \overline{X} \to X$ for some positive integer d such that $\overline{X}$ is homotopy equivalent to a finite CW-complex. Define the *virtual Euler characteristic* following Wall [105]

$$\chi_{\mathrm{virt}}(X) \;:=\; \frac{\chi(\overline{X})}{d}.$$

One easily checks that this is independent of the choice of $p\colon \overline{X} \to X$ since the classical Euler characteristic is multiplicative under finite coverings. Moreover, we conclude from Theorem 2.18 (i) and (vii) that for virtually homotopy finite X

$$\begin{aligned} m(\widetilde{X};\pi_1(X)) &< \infty; \\ \chi^{(2)}(\widetilde{X};\mathcal{N}(\pi_1(X))) &= \chi_{\mathrm{virt}}(X). \end{aligned}$$

Remark 2.20 (L^2-Euler characteristic and orbifold Euler characteristic). If X is a finite G-CW-complex, then $\sum_{c\in I(X)}(-1)^{\dim(c)} \cdot |G_c|^{-1}$ is also called *orbifold Euler characteristic* and agrees with the L^2-Euler characteristic by Theorem 2.18 (i).

3 Computations of L^2-Betti Numbers

In this section we state some cases where the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ for certain compact manifolds or finite CW-complexes X can explicitly be computed. These computations give evidence for certain conjectures such as the Atiyah Conjecture 4.1 for $(G,d,\mathbb{Q})$ and the Singer Conjecture 9.1 which we will discuss later. Sometimes we will also make a few comments on their proofs in order to give some insight into the methods. Besides analytic methods, which will not be discussed, standard techniques from topology and algebra such as spectral sequences and Mayer-Vietoris sequences will play a role. With our algebraic setup and the nice properties of the dimension function such as Additivity and Cofinality these tools are directly available, whereas in the original settings, which we have briefly discussed in Subsection 2.3, these methods do not apply directly and, if at all, only after some considerable technical efforts.

3.1 Abelian Groups

Let X be a $\mathbb{Z}^n$-space. Then we get from (1.20)

$$b_p^{(2)}(X;\mathcal{N}(\mathbb{Z}^n)) \;=\; \dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}\left(\mathbb{C}[\mathbb{Z}^n]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}^n]} H_p^{\mathrm{sing}}(X;\mathbb{C})\right). \qquad (3.1)$$

Notice that $b_p^{(2)}(X;\mathcal{N}(\mathbb{Z}^n))$ is always an integer or ∞.

3.2 Finite Coverings

Let $p\colon X \to Y$ be a finite covering with d-sheets. Then we conclude from Theorem 2.7 (vii)

$$b_p^{(2)}(\widetilde{X}) \quad = \quad d \cdot b_p^{(2)}(\widetilde{Y}). \tag{3.2}$$

This implies for every connected CW-complex X which admits a selfcovering $X \to X$ with d-sheets for $d \geq 2$ that $b_p^{(2)}(\widetilde{X}) = 0$ for all $p \in \mathbb{Z}$. In particular

$$b_p^{(2)}(\widetilde{S^1}) \; = \; 0 \qquad \text{for all } p \in \mathbb{Z}. \tag{3.3}$$

3.3 Surfaces

Let F_g^d be the orientable closed surface of genus g with d embedded 2-disks removed. (As any non-orientable compact surface is finitely covered by an orientable surface, it suffices to handle the orientable case by (3.2).) From the value of the zero-th L^2-Betti number, the Euler-Poincaré formula and Poincaré duality (see Theorem 2.7 (ix), (x) and (xii)) and from the fact that a compact surface with boundary is homotopy equivalent to a bouquet of circles, we conclude

$$\begin{aligned}
b_0^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 1 & \text{if } g = 0, d = 0, 1; \\ 0 & \text{otherwise;} \end{cases} \\
b_1^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 0 & \text{if } g = 0, d = 0, 1; \\ d + 2 \cdot (g-1) & \text{otherwise;} \end{cases} \\
b_2^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 1 & \text{if } g = 0, d = 0; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Of course $b_p^{(2)}(\widetilde{F_g^d}) = 0$ for $p \geq 3$.

3.4 Three-Dimensional Manifolds

In this subsection we state the values of the L^2-Betti numbers of compact orientable 3-manifolds.

We begin with collecting some basic notations and facts about 3-manifolds. In the sequel 3-manifold means connected compact orientable 3-manifold, possibly with boundary. A 3-manifold M is *prime* if for any decomposition of M as a connected sum $M_1 \# M_2$, M_1 or M_2 is homeomorphic to S^3. It is *irreducible* if every embedded 2-sphere bounds an embedded 3-disk. Every prime 3-manifold is either irreducible or is homeomorphic to $S^1 \times S^2$ [50, Lemma 3.13]. A 3-manifold M has a prime decomposition, i.e. one can write M as a connected sum

$$M \quad = \quad M_1 \# M_2 \# \ldots \# M_r,$$

where each M_j is prime, and this prime decomposition is unique up to renumbering and orientation preserving homeomorphism [50, Theorems 3.15, 3.21].

Recall that a connected CW-complex is called *aspherical* if $\pi_n(X) = 0$ for $n \geq 2$, or, equivalently, if $\widetilde{X}$ is contractible. Any aspherical 3-manifold is homotopy equivalent to an irreducible 3-manifold with infinite fundamental group or to a 3-disk. By the Sphere Theorem [50, Theorem 4.3], an irreducible 3-manifold is aspherical if and only if it is a 3-disk or has infinite fundamental group.

Let us say that a prime 3-manifold is *exceptional* if it is closed and no finite covering of it is homotopy equivalent to a Haken, Seifert or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known. Both Thurston's Geometrization Conjecture and Waldhausen's Conjecture that any 3-manifold is finitely covered by a Haken manifold imply that there are none.

Details of the proof of the following theorem can be found in [69, Sections 5 and 6]. The proof is quite interesting since it uses both topological and analytic tools and relies on Thurston's Geometrization.

Theorem 3.4 (L^2-Betti numbers of 3-manifolds). *Let M be the connected sum $M_1 \# \ldots \# M_r$ of (compact connected orientable) prime 3-manifolds M_j which are non-exceptional. Assume that $\pi_1(M)$ is infinite. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ are given by*

$$
\begin{aligned}
b_0^{(2)}(\widetilde{M}) &= 0; \\
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right| - \chi(M); \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\
b_3^{(2)}(\widetilde{M}) &= 0.
\end{aligned}
$$

Notice that in the situation of Theorem 3.4 the p-th L^2-Betti number $b_p(\widetilde{M})$ is a rational number. It is an integer, if $\pi_1(M)$ is torsion-free, and vanishes, if M is aspherical.

3.5 Symmetric Spaces

Let L be a connected semisimple Lie group with finite center such that its Lie algebra has no compact ideal. Let $K \subseteq L$ be a maximal compact subgroup. Then the manifold $M := L/K$ equipped with a left L-invariant Riemannian metric is a symmetric space of non-compact type with $L = \operatorname{Isom}(M)^0$ and $K = \operatorname{Isom}(M)^0_x$, where $\operatorname{Isom}(M)^0$ is the identity component of the group of isometries $\operatorname{Isom}(M)$ and $\operatorname{Isom}(M)^0_x$ is the isotropy group of some point $x \in M$ under the $\operatorname{Isom}(M)^0$-action. Every symmetric space M of non-compact type can be written in this way. The space M is diffeomorphic to $\mathbb{R}^n$. Define its *fundamental rank*

$$\text{f-rk}(M) := \operatorname{rk}_{\mathbb{C}}(L) - \operatorname{rk}_{\mathbb{C}}(K),$$

where $\operatorname{rk}_{\mathbb{C}}(L)$ and $\operatorname{rk}_{\mathbb{C}}(K)$ denotes the so called complex rank of the Lie algebra of L and K respectively (see [62, page 128f]). For a compact Lie group K this is the same as the dimension of a maximal torus. The proof of the next result is due to Borel [6].

Theorem 3.5 (L^2-Betti numbers of symmetric spaces of non-compact type). *Let M be a closed Riemannian manifold whose universal covering $\widetilde{M}$ is a symmetric space of non-compact type.*

Then $b_p^{(2)}(\widetilde{M}) \neq 0$ if and only if $\text{f-rk}(\widetilde{M}) = 0$ *and* $2p = \dim(M)$. *If* $\text{f-rk}(\widetilde{M}) = 0$, *then* $\dim(M)$ *is even and for* $2p = \dim(M)$ *we get*

$$0 \;<\; b_p^{(2)}(\widetilde{M}) \;=\; (-1)^p \cdot \chi(M).$$

This applies in particular to a hyperbolic manifold and thus we get the result of Dodziuk [25].

Theorem 3.6. *Let M be a hyperbolic closed Riemannian manifold of dimension n. Then*

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2p \neq n \\ > 0 & \text{if } 2p = n \end{cases}.$$

If n is even, then

$$(-1)^{n/2} \cdot \chi(M) \quad > \quad 0.$$

The strategy of the proof of Theorem 3.6 is the following. Because of the Euler-Poincaré formula (see Theorem 2.7 (x)) it suffices to show that $b_p^{(2)}(\widetilde{M}) = 0$ for $2p \neq n$ and $b_p^{(2)}(\widetilde{M}) > 0$ for $2p = n$. Because of the Hodge-deRham Theorem (see (2.15)) and the facts that the von Neumann dimension is faithful and $\widetilde{M}$ is isometrically diffeomorphic to the hyperbolic space $\mathbb{H}^n$, it remains to show that the space of harmonic L^2-integrable forms $\mathcal{H}^p_{(2)}(\mathbb{H}^n)$ is trivial for $2p \neq n$ and non-trivial for $2p = n$. Notice that this question is independent of M or the $\pi_1(M)$-action. Using the rotational symmetry of $\mathbb{H}^n$, this question is answered positively by Dodziuk [25].

More generally one has the following so called Proportionality Principle (see [80, Theorem 3.183].)

Theorem 3.7 (Proportionality Principle for L^2-Betti numbers). *Let M be a simply connected Riemannian manifold. Then there are constants $B_p^{(2)}(M)$ for $p \geq 0$ depending only on the Riemannian manifold M with the following property: For every discrete group G with a cocompact free proper action on M by isometries the following holds*

$$b_p^{(2)}(M; \mathcal{N}(G)) \quad = \quad B_p^{(2)}(M) \cdot \operatorname{vol}(G \backslash M).$$

3.6 Spaces with S^1-Action

The next two theorems are taken from [80, Corollary 1.43 and Theorem 6.65].

Theorem 3.8. **(L^2-Betti numbers and S^1-actions).** *Let X be a connected S^1-CW-complex. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.)*

Then we get

$$\begin{aligned} b_p^{(2)}(\widetilde{X}) &= 0 \quad \textit{for } p \in \mathbb{Z}; \\ \chi(X) &= 0. \end{aligned}$$

Proof. We give an outline of the idea of the proof in the case where X is a cocompact S^1-CW-complex, because it is a very illuminating example. The proof in the general case is given in [80, Theorem 6.65]. It is useful to show the following slightly more general statement that for any finite S^1-CW-complex Y and S^1-map $f\colon Y \to X$ we get $b_p^{(2)}(f^*\widetilde{X};\mathcal{N}(\pi_1(X))) = 0$ for all $p \geq 0$, where $f^*\widetilde{X} \to Y$ is the pullback of the universal covering $\widetilde{X} \to X$ with f. We prove the latter statement by induction over the dimension and the number of S^1-equivariant cells in top dimension of Y. In the induction step we can assume that Y is an S^1-pushout

$$\begin{array}{ccc} S^1/H \times S^{n-1} & \xrightarrow{q} & Z \\ \downarrow & & \downarrow j \\ S^1/H \times D^n & \xrightarrow[Q]{} & Y \end{array}$$

for $n = \dim(Y)$. It induces a pushout of free finite $\pi_1(X)$-CW-complexes

$$\begin{array}{ccc} q^*j^*f^*\widetilde{X} & \longrightarrow & j^*f^*\widetilde{X} \\ \downarrow & & \downarrow \\ Q^*f^*\widetilde{X} & \longrightarrow & f^*\widetilde{X} \end{array}$$

The associated long exact Mayer-Vietoris sequence looks like

$$\begin{aligned} \ldots H_p(q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \\ \to H_p(Q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \oplus H_p(j^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \\ \to H_p(f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \to H_{p-1}(q^*j^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \\ \to H_{p-1}(Q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \oplus H_{p-1}(j^*f^*\widetilde{X};\mathcal{N}(\pi_1(X))) \to \ldots \end{aligned}$$

Because of the Additivity of the dimension (see Theorem 1.11 (iii)b) it suffices to prove for all $p \in \mathbb{Z}$

$$\begin{aligned} \dim_{\mathcal{N}(\pi_1(X))}\left(H_p(j^*f^*\widetilde{X};\mathcal{N}(\pi_1(X)))\right) &= 0; \\ \dim_{\mathcal{N}(\pi_1(X))}\left(H_p(q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X)))\right) &= 0; \\ \dim_{\mathcal{N}(\pi_1(X))}\left(H_p(Q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X)))\right) &= 0. \end{aligned}$$

The induction hypothesis applies to $f \circ j\colon Z \to X$ and $f \circ j \circ q\colon S^1/H \times S^{n-1} \to X$. Hence it remains to show

$$\dim_{\mathcal{N}(\pi_1(X))}\Big(H_p(Q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X)))\Big) \;=\; 0.$$

By elementary covering theory $Q^*f^*\widetilde{X}$ is $\pi_1(X)$-homeomorphic to $\pi_1(X) \times_j \widetilde{S^1/H} \times D^n$ for the injective group homomorphism $j\colon \pi_1(S^1/H) \to \pi_1(X)$ induced by $f \circ Q$. We conclude from the Künneth formula and the compatibility of dimension and induction (see Theorem 2.7 (v) and (vi))

$$\dim_{\mathcal{N}(\pi_1(X))}\Big(H_p(Q^*f^*\widetilde{X};\mathcal{N}(\pi_1(X)))\Big) \;=\; b_p^{(2)}(\widetilde{S^1/H}).$$

Since S^1/H is homeomorphic to S^1, we get $b_p^{(2)}(\widetilde{S^1/H}) = 0$ from (3.3). $\square$

The next result is taken from [80, Corollary 1.43].

Theorem 3.9. *Let M be an aspherical closed manifold with non-trivial S^1-action. (Non-trivial means that $sx \neq x$ holds for at least one element $s \in S^1$ and one element $x \in M$). Then the action has no fixed points and the inclusion of any orbit into X induces an injection on the fundamental groups. All L^2-Betti numbers $b_p^{(2)}(\widetilde{M})$ are trivial and $\chi(M) = 0$.*

3.7 Mapping Tori

Let $f\colon X \to X$ be a selfmap. Its *mapping torus T_f* is obtained from the cylinder $X \times [0,1]$ by glueing the bottom to the top by the identification $(x,1) = (f(x),0)$. There is a canonical map $p\colon T_f \to S^1$ which sends (x,t) to $\exp(2\pi i t)$. It induces a canonical epimorphism $\pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$ if X is path-connected.

The following result is taken from [80, Theorem 6.63].

Theorem 3.10 (Vanishing of L^2-Betti numbers of mapping tori). *Let $f\colon X \to X$ be a cellular selfmap of a connected CW-complex X and let $\pi_1(T_f) \xrightarrow{\phi} G \xrightarrow{\psi} \mathbb{Z}$ be a factorization of the canonical epimorphism into epimorphisms ϕ and ψ. Suppose for given $p \geq 0$ that $b_p^{(2)}(G \times_{\phi \circ i} \widetilde{X};\mathcal{N}(G)) < \infty$ and $b_{p-1}^{(2)}(G \times_{\phi \circ i} \widetilde{X};\mathcal{N}(G)) < \infty$ holds, where $i\colon \pi_1(X) \to \pi_1(T_f)$ is the map induced by the obvious inclusion of X into T_f. Let $\overline{T_f}$ be the covering of T_f associated to ϕ, which is a free G-CW-complex. Then we get*

$$b_p^{(2)}(\overline{T_f};\mathcal{N}(G)) \quad = \quad 0.$$

Proof. We give the proof in the special case where X is a connected finite CW-complex and $\phi = \mathrm{id}$, i.e. we show for a connected finite CW-complex X that $b_p^{(2)}(\widetilde{T_f}) = 0$ for all $p \geq 0$. For each positive integer d there is a finite d-sheeted covering $\overline{T_f} \to T_f$ associated to the subgroup of index d in $\pi_1(T_f)$ which is the preimage of $d\mathbb{Z} \subseteq \mathbb{Z}$ under the canonical homomorphism $\pi_1(T_f) \to \mathbb{Z}$. There is a homotopy equivalence $T_{f^d} \to \overline{T_f}$. We conclude from (3.2) and homotopy invariance of L^2-Betti numbers (see Theorem 2.7 (i))

$$b_p^{(2)}(\widetilde{T_f}) \;=\; \frac{b_p^{(2)}(\widetilde{T_{f^d}})}{d}.$$

There is a CW-complex structure on T_{f^d} with $\beta_p(X)+\beta_{p-1}(X)$ p-cells, if $\beta_p(X)$ is the number of p-cells in X. We conclude from Additivity of the dimension function (see Theorem 1.11 (iii)b)

$$\begin{aligned} b_p^{(2)}(\widetilde{T_{f^d}}) \;\leq\; & \dim_{\mathcal{N}(\pi_1(T_{f^d}))}\left(\mathcal{N}(\pi_1(T_{f^d})) \otimes_{\mathbb{Z}\pi_1(T_{f^d})} C_p(\widetilde{T_{f^d}})\right) \\ & \qquad\qquad = \;\beta_p(X) + \beta_{p-1}(X). \end{aligned}$$

This implies for all positive integers d

$$0 \;\leq\; b_p^{(2)}(\widetilde{T_f}) \;\leq\; \frac{\beta_p(X) + \beta_{p-1}(X)}{d}.$$

Taking the limit for $d \to \infty$ implies $b_p^{(2)}(\widetilde{T_f}) = 0$. □

3.8 Fibrations

The next result is proved in [80, Lemma 6.6. and Theorem 6.67]. The proof is based on standard spectral sequence arguments and the fact that the dimension function is defined for arbitrary $\mathcal{N}(G)$-modules.

Theorem 3.11 (L^2-Betti numbers and fibrations).

(i) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected CW-complexes. Consider a factorization $p_*\colon \pi_1(E) \xrightarrow{\phi} G \xrightarrow{\psi} \pi_1(B)$ of the map induced by p into epimorphisms ϕ and ψ. Let $i_*\colon \pi_1(F) \to \pi_1(E)$ be the homomorphism induced by the inclusion i. Suppose for a given integer $d \geq 1$ that $b_p^{(2)}(G \times_{\phi \circ i_*} \widetilde{F}; \mathcal{N}(G)) = 0$ for $p \leq d-1$ and $b_d^{(2)}(G \times_{\phi \circ i_*} \widetilde{F}; \mathcal{N}(G)) < \infty$ holds. Suppose that $\pi_1(B)$ contains an element of infinite order or finite subgroups of arbitrarily large order. Then $b_p^{(2)}(G \times_\phi \widetilde{E}; \mathcal{N}(G)) = 0$ for $p \leq d$;

(ii) Let $F \xrightarrow{i} E \to B$ be a fibration of connected CW-complexes. Consider a group homomorphism $\phi\colon \pi_1(E) \to G$. Let $i_*\colon \pi_1(F) \to \pi_1(E)$ be the homomorphism induced by the inclusion i. Suppose that for a given integer $d \geq 0$ the L^2-Betti number $b_p^{(2)}(G \times_{\phi \circ i_*} \widetilde{F}; \mathcal{N}(G))$ vanishes for all $p \leq d$. Then the L^2-Betti number $b_p^{(2)}(G \times_\phi \widetilde{E}; \mathcal{N}(G))$ vanishes for all $p \leq d$.

4 The Atiyah Conjecture

In this section we discuss the Atiyah Conjecture

Conjecture 4.1 (Atiyah Conjecture). *Let G be a discrete group with an upper bound on the orders of its finite subgroups. Consider $d \in \mathbb{Z}$, $d \geq 1$ such that the order of every finite subgroup of G divides d. Let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. The* Atiyah Conjecture *for (G, d, F) says that for any finitely presented FG-module M we have*

$$d \cdot \dim_{\mathcal{N}(G)} (\mathcal{N}(G) \otimes_{FG} M) \ \in \ \mathbb{Z}.$$

4.1 Reformulations of the Atiyah Conjecture

We present equivalent reformulations of the Atiyah Conjecture 4.1.

Theorem 4.2 (Reformulations of the Atiyah Conjecture). *Let G be a discrete group. Suppose that there exists $d \in \mathbb{Z}$, $d \geq 1$ such that the order of every finite subgroup of G divides d. Let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then the following assertions are equivalent:*

(i) The Atiyah Conjecture 4.1 is true for (G, d, F), i.e. for every finitely presented FG-module M we have

$$d \cdot \dim_{\mathcal{N}(G)} (\mathcal{N}(G) \otimes_{FG} M) \ \in \ \mathbb{Z};$$

(ii) For every FG-module M we have

$$d \cdot \dim_{\mathcal{N}(G)} (\mathcal{N}(G) \otimes_{FG} M) \ \in \ \mathbb{Z} \amalg \{\infty\}.$$

Proof. See [80, Lemma 10.7 and Remark 10.11]. □

We mention that the Atiyah Conjecture 4.1 is true for (G, d, F) if and only if for any finitely generated subgroup $H \subseteq G$ the Atiyah Conjecture 4.1 is true for (H, d, F) (see [80, Lemma 10.4]).

The next result explains that the Atiyah Conjecture 4.1 for $(G, d, \mathbb{Q})$ for a finitely generated group G is a statement about the possible values of L^2-Betti numbers.

Theorem 4.3 (Reformulations of the Atiyah Conjecture for $F = \mathbb{Q}$). *Let G be a finitely generated group with an upper bound $d \in \mathbb{Z}, d \geq 1$ on the orders of its finite subgroups. Then the following assertions are equivalent:*

(i) The Atiyah Conjecture 4.1 is true for $(G, d, \mathbb{Q})$;

(ii) For every free proper smooth cocompact G-manifold M without boundary and $p \in \mathbb{Z}$ we have

$$d \cdot b_p^{(2)}(M; \mathcal{N}(G)) \ \in \ \mathbb{Z};$$

(iii) For every finite free G-CW-complex X and $p \in \mathbb{Z}$ we have

$$d \cdot b_p^{(2)}(X; \mathcal{N}(G)) \;\in\; \mathbb{Z};$$

(iv) For every G-space X and $p \in \mathbb{Z}$ we have

$$d \cdot b_p^{(2)}(X; \mathcal{N}(G)) \;\in\; \mathbb{Z} \amalg \{\infty\}.$$

Proof. This follows from [80, Lemma 10.5] and Theorem 4.2. □

We mention that all the explicit computations presented in Section 3 are compatible with the Atiyah Conjecture 4.1.

4.2 The Ring Theoretic Version of the Atiyah Conjecture

In this subsection we consider the following *fundamental square of ring extensions*

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{i} & \mathcal{N}(G) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ \mathcal{D}(G) & \xrightarrow[l]{} & \mathcal{U}(G) \end{array} \tag{4.4}$$

which we explain next.

As before $\mathbb{C}G$ is the complex group ring and $\mathcal{N}(G)$ is the group von Neumann algebra.

By $\mathcal{U}(G)$ we denote the algebra of affiliated operators. Instead of its functional analytic definition we describe it algebraically, namely, it is the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-trivial zero-divisors in $\mathcal{N}(G)$. The proof that this multiplicative subset satisfies the Ore condition and basic definitions and properties of Ore localization and of $\mathcal{U}(G)$ can be found for instance in [80, Sections 8.1 and 8.2]. In particular $\mathcal{U}(G)$ is flat when regarded as an $\mathcal{N}(G)$-module. Moreover, the ring $\mathcal{U}(G)$ is a *von Neumann regular ring*, i.e. every finitely generated submodule of a projective module is a direct summand. This is a stronger condition than being semihereditary.

Given a finitely generated projective $\mathcal{U}(G)$-module Q, there is a finitely generated projective $\mathcal{N}(G)$-module P such that $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} P$ and Q are $\mathcal{U}(G)$-isomorphic. If P_0 and P_1 are two finitely generated projective $\mathcal{N}(G)$-modules, then $P_0 \cong_{\mathcal{N}(G)} P_1 \Leftrightarrow \mathcal{U}(G) \otimes_{\mathcal{N}(G)} P_0 \cong_{\mathcal{U}(G)} \mathcal{U}(G) \otimes_{\mathcal{N}(G)} P_1$. This enables us to define a dimension function for $\dim_{\mathcal{U}(G)}$ with properties analogous to $\dim_{\mathcal{N}(G)}$ (see [80, Section 8.3], [98] or [99]).

Theorem 4.5. (Dimension function for arbitrary $\mathcal{U}(G)$-modules).
There exists precisely one dimension function

$$\dim_{\mathcal{U}(G)} : \{\mathcal{U}(G)\text{-}modules\} \;\to\; [0, \infty]$$

which satisfies:

(i) Extension Property

If M is an $\mathcal{N}(G)$-module, then

$$\dim_{\mathcal{U}(G)}\big(\mathcal{U}(G)\otimes_{\mathcal{N}(G)} M\big) \;=\; \dim_{\mathcal{N}(G)}(M);$$

(ii) Additivity

If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of $\mathcal{U}(G)$-modules, then

$$\dim_{\mathcal{U}(G)}(M_1) \;=\; \dim_{\mathcal{U}(G)}(M_0) + \dim_{\mathcal{U}(G)}(M_2);$$

(iii) Cofinality

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M. Then

$$\dim_{\mathcal{U}(G)}(M) \;=\; \sup\{\dim_{\mathcal{U}(G)}(M_i) \mid i \in I\};$$

(iv) Continuity

If $K \subseteq M$ is a submodule of the finitely generated $\mathcal{U}(G)$-module M, then

$$\dim_{\mathcal{U}(G)}(K) \;=\; \dim_{\mathcal{U}(G)}(\overline{K}).$$

Remark 4.6 (Comparing $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathcal{N}(G) \subseteq \mathcal{U}(G)$). Recall the Slogan 1.16 that the group von Neumann algebra $\mathcal{N}(G)$ behaves like the ring of integers $\mathbb{Z}$, provided one ignores the properties integral domain and Noetherian. This is supported by the construction and properties of $\mathcal{U}(G)$. Obviously $\mathcal{U}(G)$ plays the same role for $\mathcal{N}(G)$ as $\mathbb{Q}$ plays for $\mathbb{Z}$ as the definition of $\mathcal{U}(G)$ as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-zero-divisors and Theorem 4.5 show.

A subring $R \subseteq S$ is called *division closed* if each element in R, which is invertible in S, is already invertible in R. It is called *rationally closed* if each square matrix over R, which is invertible over S, is already invertible over R. Notice that the intersection of division closed subrings of S is again division closed, and analogously for rationally closed subrings. Hence the following definition makes sense.

Definition 4.7 (Division and rational closure). *Let S be a ring with subring $R \subseteq S$. The* division closure $\mathcal{D}(R \subseteq S)$ *or* rational closure $\mathcal{R}(R \subseteq S)$ *respectively is the smallest subring of S which contains R and is division closed or rationally closed respectively.*

The ring $\mathcal{D}(G)$ appearing in the fundamental square (4.4) is the rational closure of $\mathbb{C}G$ in $\mathcal{U}(G)$.

Conjecture 4.8 (Ring theoretic version of the Atiyah Conjecture). *Let G be a group for which there exists an upper bound on the orders of its finite subgroups. Then:*

(R) *The ring $\mathcal{D}(G)$ is semisimple;*

(K) *The composition*

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(\mathbb{C}H) \xrightarrow{a} K_0(\mathbb{C}G) \xrightarrow{j} K_0(\mathcal{D}(G))$$

is surjective, where a is induced by the various inclusions $H \to G$.

Lemma 4.9. *Let G be a group. Suppose that there exists $d \in \mathbb{Z}$, $d \geq 1$ such that the order of every finite subgroup of G divides d. If the group G satisfies the ring theoretic version of the Atiyah Conjecture 4.8, then the Atiyah Conjecture 4.1 for $(G, d, \mathbb{C})$ is true.*

Proof. Let M be a finitely presented $\mathbb{C}G$-module. Then $\mathcal{D}(G) \otimes_{\mathbb{C}G} M$ is a finitely generated projective $\mathcal{D}(G)$-module since $\mathcal{D}(G)$ is semisimple by assumption. We obtain a well-defined homomorphism of abelian groups

$$D \colon K_0(\mathcal{D}(G)) \to \mathbb{R}, \quad [P] \mapsto \dim_{\mathcal{U}(G)} \left(\mathcal{U}(G) \otimes_{\mathcal{D}(G)} P\right).$$

Because of the fundamental square (4.4) and Theorem 4.5 (i) we have

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M) \ = \ D([\mathcal{D}(G) \otimes_{\mathbb{C}G} M]).$$

Hence it suffices to show that $d \cdot \operatorname{im}(D)$ is contained in $\mathbb{Z}$. Because of assumption (**K**) it suffices to check for each finite subgroup $H \subseteq G$ and each finitely generated projective $\mathbb{C}H$-module P

$$d \cdot \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{\mathbb{C}H} P) \ \in \ \mathbb{Z}.$$

Example 1.14 and Theorem 1.18 imply

$$\begin{aligned}
\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{\mathbb{C}H} P) &= \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{\mathbb{C}H} P) \\
&= \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathcal{N}(H)} P) \\
&= \dim_{\mathcal{N}(H)}(P) \\
&= \frac{\dim_{\mathbb{C}}(P)}{|H|}.
\end{aligned}$$

Obviously $d \cdot \frac{\dim_{\mathbb{C}}(P)}{|H|} \ \in \ \mathbb{Z}$. □

4.3 The Atiyah Conjecture for Torsion-Free Groups

Remark 4.10 (The Atiyah Conjecture in the torsion-free case). Let G be a torsion-free group. Then we can choose $d = 1$ in the Atiyah Conjecture 4.1. The Atiyah Conjecture 4.1 for $(G, 1, F)$ says that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M) \ \in \ \mathbb{Z}$ holds for every finitely presented FG-module M and Theorem 4.2 says that then this holds automatically for all FG-modules M with $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M) < \infty$. In the case, where $F = \mathbb{Q}$ and G is a torsion-free finitely generated group G, Theorem 4.3 implies that the Atiyah Conjecture 4.1 for $(G, 1, F)$ is equivalent to the statement that $b_p^{(2)}(X; \mathcal{N}(G)) \ \in \ \mathbb{Z}$ is true for all G-spaces X.

Remark 4.11 (The ring theoretic version of the Atiyah Conjecture in the torsion-free case). Let G be a torsion-free group. Then the ring theoretic version of the Atiyah Conjecture 4.8 reduces to the statement that $\mathcal{D}(G)$ is a skewfield. In this case we can assign to every $\mathcal{D}(G)$-module N its dimension $\dim_{\mathcal{D}(G)}(N) \in \mathbb{Z} \amalg \{\infty\}$ in the usual way and we get for every $\mathbb{C}G$-module M

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M) \;=\; \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathbb{C}G} M) \;=\; \dim_{\mathcal{D}(G)}(\mathcal{D}(G) \otimes_{\mathbb{C}G} M).$$

Example 4.12 (The case $G = \mathbb{Z}^n$). In the case $G = \mathbb{Z}^n$ the fundamental square of ring extensions (4.4) can be identified with

$$\begin{array}{ccc} \mathbb{C}[\mathbb{Z}^n] & \longrightarrow & L^\infty(T^n) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}[\mathbb{Z}^n]_{(0)} & \longrightarrow & MF(T^n) \end{array}$$

where $MF(T^n)$ the ring of equivalence classes of measurable functions $T^n \to \mathbb{C}$. We have already proved

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(\mathcal{N}(\mathbb{Z}^n) \otimes_{\mathbb{C}[\mathbb{Z}^n]} M) \;=\; \dim_{\mathbb{C}[\mathbb{Z}^n]_{(0)}}(\mathbb{C}[\mathbb{Z}^n]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}^n]} M)$$

in Example 1.19.

4.4 The Atiyah Conjecture Implies the Kaplanski Conjecture

The following conjecture is a well-known conjecture about group rings.

Conjecture 4.13 (Kaplanski Conjecture). *Let F be a field and let G be a torsion-free group. Then FG contains no non-trivial zero-divisors.*

Theorem 4.14 (The Atiyah and the Kaplanski Conjecture). *Let G be a torsion-free group and let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Then the Atiyah Conjecture 4.1 for $(G, 1, F)$ implies the Kaplanski Conjecture 4.13 for F and G.*

Proof. Let $u \in FG$ be a zero-divisor. Then the kernel of the $\mathcal{N}(G)$-map $r_u \colon \mathcal{N}(G) \to \mathcal{N}(G)$ given by right multiplication with u is non-trivial. Since $\mathcal{N}(G)$ is semihereditary, the image of r_u is projective. Hence both $\ker(r_u)$ and $\mathcal{N}(G)/\ker(r_u)$ are finitely generated projective $\mathcal{N}(G)$-modules. Additivity of $\dim_{\mathcal{N}(G)}$ implies

$$0 < \dim_{\mathcal{N}(G)}(\ker(r_u)) \leq \dim_{\mathcal{N}(G)}(\mathcal{N}(G)) = 1.$$

We conclude from Remark 4.10 that $\dim_{\mathcal{N}(G)}(\ker(r_u))$ is an integer. Additivity of $\dim_{\mathcal{N}(G)}$ implies

$$\dim_{\mathcal{N}(G)}\left(\mathcal{N}(G)/\ker(r_u)\right) \;=\; 0.$$

We conclude $\mathcal{N}(G)/\ker(r_u) = 0$ and hence $u = 0$. $\square$

4.5 The Status of the Atiyah Conjecture

Let $l^\infty(G,\mathbb{R})$ be the space of equivalence classes of bounded functions from G to $\mathbb{R}$ with the supremum norm. Denote by 1 the constant function with value 1.

Definition 4.15 (Amenable group). *A group G is called* amenable*, if there is a (left) G-invariant linear operator $\mu\colon l^\infty(G,\mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$, which satisfies for all $f \in l^\infty(G,\mathbb{R})$*

$$\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\}.$$

The latter condition is equivalent to the condition that μ is bounded and $\mu(f) \geq 0$ if $f(g) \geq 0$ for all $g \in G$.

Definition 4.16 (Elementary amenable group). *The* class of elementary amenable groups *$\mathcal{EAM}$ is defined as the smallest class of groups which has the following properties:*

(i) It contains all finite and all abelian groups;

(ii) It is closed under taking subgroups;

(iii) It is closed under taking quotient groups;

(iv) It is closed under extensions, i.e. if $1 \to H \to G \to K \to 1$ is an exact sequence of groups and H and K belong to $\mathcal{EAM}$, then also $G \in \mathcal{EAM}$;

(v) It is closed under directed unions*, i.e. if $\{G_i \mid i \in I\}$ is a directed system of subgroups such that $G = \bigcup_{i\in I} G_i$ and each G_i belongs to $\mathcal{EAM}$, then $G \in \mathcal{EAM}$. (Directed means that for two indices i and j there is a third index k with $G_i, G_j \subseteq G_k$.)*

The class of amenable groups satisfies all the conditions appearing in Definition 4.16. Hence every elementary amenable group is amenable. The converse is not true.

Definition 4.17 (Linnell's class of groups $\mathcal{C}$). *Let $\mathcal{C}$ be the smallest class of groups, which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients.*

The next result is due to Linnell [65].

Theorem 4.18 (Linnell's Theorem). *Let G be a group in $\mathcal{C}$. Suppose that there exists $d \in \mathbb{Z}$, $d \geq 1$ such that the order of every finite subgroup of G divides d. Then the ring theoretic version of the Atiyah Conjecture 4.8 for G and hence the Atiyah Conjecture 4.1 for $(G, d, \mathbb{C})$ are true.*

The next definition and the next theorem are due to Schick [101].

Definition 4.19. *Let $\mathcal{D}$ be the smallest non-empty class of groups such that*

(i) *If $p\colon G \to A$ is an epimorphism of a torsion-free group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}$ for every finite group $B \subseteq A$, then $G \in \mathcal{D}$;*

(ii) *$\mathcal{D}$ is closed under taking subgroups;*

(iii) *$\mathcal{D}$ is closed under colimits and inverse limits over directed systems.*

Theorem 4.20. (i) *If the group G belongs to $\mathcal{D}$, then G is torsion-free and the Atiyah Conjecture 4.1 for $(G, 1, \mathbb{Q})$ is true for G;*

(ii) *The class $\mathcal{D}$ is closed under direct sums, direct products and free products. Every residually torsion-free elementary amenable group belongs to $\mathcal{D}$.*

More information about the status of the Atiyah Conjecture 4.1 can be found for instance in [80, Subsection 10.1.3].

4.6 Groups Without Bound on the Order of Its Finite Subgroups

Given a group G, let $\mathcal{FIN}(G)$ be the set of finite subgroups of G. Denote by

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \quad \subseteq \quad \mathbb{Q} \tag{4.21}$$

the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $\{\frac{1}{|H|} \mid H \in \mathcal{FIN}(G)\}$.

There is the following formulation of the Atiyah Conjecture for arbitrary groups in the literature.

Conjecture 4.22 (Atiyah Conjecture for arbitrary groups G). *A group G satisfies the* Atiyah Conjecture *if for every finitely presented $\mathbb{C}G$-module M we have*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M) \quad \in \quad \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

There do exist counterexamples to this conjecture. The *lamplighter group L* is defined by the semidirect product

$$L := \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2\right) \rtimes \mathbb{Z}$$

with respect to the shift automorphism of $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2$, which sends $(x_n)_{n\in\mathbb{Z}}$ to $(x_{n-1})_{n\in\mathbb{Z}}$. Let $e_0 \in \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2$ be the element whose entries are all zero except the entry at 0. Denote by $t \in \mathbb{Z}$ the standard generator of $\mathbb{Z}$ which we will also view as an element of L. Then $\{e_0t, t\}$ is a set of generators for L. The associated *Markov operator* $M\colon l^2(G) \to l^2(G)$ is given by right multiplication with $\frac{1}{4}\cdot(e_0t+t+(e_0t)^{-1}+t^{-1})$. It is related to the Laplace operator $\Delta_0\colon l^2(G) \to$

$l^2(G)$ of the Cayley graph of G by $\Delta_0 = 4 \cdot \mathrm{id} - 4 \cdot M$. The following result is a special case of the main result in the paper of Grigorchuk and Żuk [41, Theorem 1 and Corollary 3] (see also [40]). An elementary proof can be found in [22].

Theorem 4.23 (Counterexample to the Atiyah Conjecture for arbitrary groups). *The von Neumann dimension of the kernel of the Markov operator M of the lamplighter group L associated to the set of generators $\{e_0 t, t\}$ is $1/3$. In particular L does not satisfy the Atiyah Conjecture 4.22.*

To the author's knowledge there is no example of a group G for which there is a finitely presented $\mathbb{C}G$-module M such that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} M)$ is irrational.

Let $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$. Because this group is locally finite, it satisfies the Atiyah Conjecture for arbitrary groups 4.22, i.e. $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}A} M) \in \mathbb{Z}[1/2]$ for every finitely presented $\mathbb{C}A$-module M. On the other hand, each non-negative real number r can be realized as $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}A} M)$ for a finitely generated $\mathbb{C}A$-module (see [80, Example 10.13]). Notice that there is no upper bound on the orders of finite subgroups of A, so that this is no contradiction to Theorem 4.2.

5 Flatness Properties of the Group von Neumann Algebra

The proof of next result can be found in [77, Theorem 5.1] or [80, Theorem 6.37].

Theorem 5.1. (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ for amenable G). *Let G be amenable and M be a $\mathbb{C}G$-module. Then*

$$\dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G), M)\right) = 0 \qquad \text{for } p \geq 1,$$

where we consider $\mathcal{N}(G)$ as an $\mathcal{N}(G)$-$\mathbb{C}G$-bimodule.

It implies using an easy spectral sequence argument

Theorem 5.2 (L^2-Betti numbers and homology in the amenable case). *Let G be an amenable group and X be a G-space. Then*

(i) $b_p^{(2)}(X; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_p^{\mathrm{sing}}(X; \mathbb{C})\right);$

(ii) Suppose that X is a G-CW-complex with $m(X; G) < \infty$. Then

$$\begin{aligned}\chi^{(2)}(X) &= \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |G_c|^{-1} \\ &= \sum_{p \geq 0} (-1)^p \cdot \dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_p(X; \mathbb{C})\right).\end{aligned}$$

Further applications of Theorem 5.1 will be discussed in Section 6 and Section 7.

Conjecture 5.3. (Amenability and dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$). *A group G is amenable if and only if for every $\mathbb{C}G$-module M*

$$\dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G), M)\right) = 0 \qquad \text{for } p \geq 1$$

holds.

Remark 5.4 (Evidence for Conjecture 5.3). Theorem 5.1 proves the "only if"-statement of Conjecture 5.3. Some evidence for the "if"-statement of Conjecture 5.3 comes from the following fact. Notice that a group which contains a non-abelian free group as a subgroup, cannot be amenable.

Suppose that G contains a free group $\mathbb{Z} * \mathbb{Z}$ of rank 2 as a subgroup. Notice that $S^1 \vee S^1$ is a model for $B(\mathbb{Z} * \mathbb{Z})$. Its cellular $\mathbb{C}[\mathbb{Z} * \mathbb{Z}]$-chain complex yields an exact sequence $0 \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}]^2 \to \mathbb{C}[\mathbb{Z}*\mathbb{Z}] \to \mathbb{C} \to 0$, where $\mathbb{C}$ is equipped with the trivial $\mathbb{Z} * \mathbb{Z}$-action. One easily checks $b_1^{(2)}(\widetilde{S^1 \vee S^1}) = -\chi(S^1 \vee S^1) = 1$. This implies

$$\dim_{\mathcal{N}(\mathbb{Z}*\mathbb{Z})}\left(\operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}*\mathbb{Z}]}(\mathcal{N}(\mathbb{Z} * \mathbb{Z}), \mathbb{C})\right) = 1.$$

We conclude from Theorem 1.18 (i)

$$\mathcal{N}(G) \otimes_{\mathcal{N}(\mathbb{Z}*\mathbb{Z})} \operatorname{Tor}_1^{\mathbb{C}[\mathbb{Z}*\mathbb{Z}]}(\mathcal{N}(\mathbb{Z} * \mathbb{Z}), \mathbb{C}) = \operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}G \otimes_{\mathbb{C}[\mathbb{Z}*\mathbb{Z}]} \mathbb{C}).$$

Theorem 1.18 (ii) implies

$$\dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}G \otimes_{\mathbb{C}[\mathbb{Z}*\mathbb{Z}]} \mathbb{C})\right) = 1.$$

One may ask for which groups the von Neumann algebra $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module. This is true if G is virtually cyclic, i.e. G is finite or contains $\mathbb{Z}$ as a normal subgroup of finite index. There is some evidence for the following conjecture (see [77, Remark 5.15]).

Conjecture 5.5 (Flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$). *The group von Neumann algebra $\mathcal{N}(G)$ is flat over $\mathbb{C}G$ if and only if G is virtually cyclic.*

6 Applications to Group Theory

Recall the Definition 2.1 of the L^2-Betti numbers of a group G by $b_p^{(2)}(G) := b_p^{(2)}(EG; \mathcal{N}(G))$. In this section we present tools for and examples of computations of the L^2-Betti numbers and discuss applications to group theory. We will explain in Remark 7.8 that for a torsion-free group with a model of finite type for BG the knowledge of $b_p^{(2)}(G; \mathcal{N}(G))$ is the same as the knowledge of the reduced L^2-homology $H_p^{(2)}(EG, l^2(G))$, or, equivalently, of $\mathbf{P}H_p^G(EG; \mathcal{N}(G))$ if G satisfies the Atiyah Conjecture 4.1 for $(G, 1, \mathbb{Q})$.

6.1 L^2-Betti Numbers of Groups

Theorem 2.7 implies:

Theorem 6.1 (L^2-Betti numbers and Betti numbers of groups). *In the sequel we use the conventions $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ for $r \in (0, \infty]$ and $r + \infty = \infty$ for $r \in [0, \infty]$ and put $|G|^{-1} = 0$ for $|G| = \infty$. Let $G_1, G_2, \ldots$ be a sequence of non-trivial groups.*

(i) Free amalgamated products
For $r \in \{2, 3, \ldots\} \amalg \{\infty\}$ we get

$$\begin{aligned} b_0^{(2)}(*_{i=1}^r G_i) &= 0; \\ b_1^{(2)}(*_{i=1}^r G_i) &= \begin{cases} r - 1 + \sum_{i=1}^r \left(b_1^{(2)}(G_i) - \frac{1}{|G_i|} \right) & , \text{ if } r < \infty; \\ \infty & , \text{ if } r = \infty; \end{cases} \\ b_p^{(2)}(*_{i=1}^r G_i) &= \sum_{i=1}^r b_p^{(2)}(G_i) \qquad \text{for } p \geq 2; \\ b_p(*_{i=1}^r G_i) &= \sum_{i=1}^r b_p(G_i) \qquad \text{for } p \geq 1; \end{aligned}$$

(ii) Künneth formula

$$\begin{aligned} b_p^{(2)}(G_1 \times G_2) &= \sum_{i=0}^p b_i^{(2)}(G_1) \cdot b_{p-i}^{(2)}(G_2); \\ b_p(G_1 \times G_2) &= \sum_{i=0}^p b_i(G_1) \cdot b_{p-i}(G_2); \end{aligned}$$

(iii) Restriction to subgroups of finite index
For a subgroup $H \subseteq G$ of finite index $[G : H]$ we get

$$b_p^{(2)}(H) = [G : H] \cdot b_p^{(2)}(G);$$

(iv) Extensions with finite kernel
Let $1 \to H \to G \to Q \to 1$ be an extension of groups with finite H. Then

$$b_p^{(2)}(Q) = |H| \cdot b_p^{(2)}(G);$$

(v) Zero-th L^2-Betti number
We have $b_0^{(2)}(G) = 0$ for $|G| = \infty$ and $b_0^{(2)}(G) = |G|^{-1}$ for $|G| < \infty$.

Example 6.2 (Independence of L^2-Betti numbers and Betti numbers). Given an integer $l \geq 1$ and a sequence $r_1, r_2, \ldots, r_l$ of non-negative rational numbers, we can construct a group G such that BG is of finite type and

$$\begin{array}{rcl} b_p^{(2)}(G) & = & \begin{cases} r_p & \text{for } 1 \leq p \leq l; \\ 0 & \text{for } l+1 \leq p; \end{cases} \\ b_p(G) & = & 0 \qquad\qquad \text{for } p \geq 1, \end{array}$$

holds as follows.

For integers $m \geq 0$, $n \geq 1$ and $i \geq 1$ define

$$G_i(m,n) = \mathbb{Z}/n \times \left(*_{k=1}^{2m+2}\mathbb{Z}/2\right) \times \left(\prod_{j=1}^{i-1} *_{l=1}^{4}\mathbb{Z}/2\right)$$

One easily checks using Theorem 6.1.

$$\begin{array}{rcl} b_i^{(2)}(G_i(m,n)) & = & \frac{m}{n}; \\ b_p^{(2)}(G_i(m,n)) & = & 0 \qquad\qquad \text{for } p \neq i; \\ b_p(G_i(m,n)) & = & 0 \qquad\qquad \text{for } p \geq 1. \end{array}$$

Define the desired group G as follows. For $l = 1$ put $G = G_1(m,n)$ if $r_1 = m/n$. It remains to treat the case $l \geq 2$. Choose integers $n \geq 1$ and $k \geq l$ with $r_1 = \frac{k-2}{n}$. Fix for $i = 2, 3, \ldots, k$ integers $m_i \geq 0$ and $n_i \geq 1$ such that $\frac{m_i}{n \cdot n_i} = r_i$ holds for $1 \leq i \leq l$ and $m_i = 0$ holds for $i > l$. Put

$$G \;=\; \mathbb{Z}/n \times *_{i=2}^{k} G_i(m_i, n_i).$$

One easily checks using Theorem 6.1 that G has the prescribed L^2-Betti numbers and Betti numbers and a model for BG of finite type.

On the other hand we can construct for any sequence $n_1, n_2, \ldots$ of non-negative integers a CW-complex X of finite type such that $b_p(X) = n_p$ and $b_p^{(2)}(\widetilde{X}) = 0$ holds for $p \geq 1$, namely take

$$X = B(\mathbb{Z}/2 * \mathbb{Z}/2) \times \bigvee_{p=1}^{\infty} \left(\bigvee_{i=1}^{n_p} S^p\right).$$

This example shows by considering the $(l+1)$-skeleton that for a finite connected CW-complex X the only general relation between the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ of its universal covering $\widetilde{X}$ and the Betti numbers $b_p(X)$ of X is given by the Euler-Poincaré formula (see Theorem 2.7 (x))

$$\sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{X}) \;=\; \chi(X) \;=\; \sum_{p \geq 0} (-1)^p \cdot b_p(X).$$

6.2 Vanishing of L^2-Betti Numbers of Groups

Let d be a non-negative integer or $d = \infty$. In this subsection we want to investigate the class of groups

$$\mathcal{B}_d \quad := \quad \{G \mid b_p^{(2)}(G) = 0 \text{ for } 0 \leq p \leq d\}. \tag{6.3}$$

Notice that $\mathcal{B}_0$ is the class of infinite groups by Theorem 6.1 (v).

Theorem 6.4. *Let d be a non-negative integer or $d = \infty$. Then:*

(i) The class $\mathcal{B}_\infty$ contains all infinite amenable groups;

(ii) If G contains a normal subgroup H with $H \in \mathcal{B}_d$, then $G \in \mathcal{B}_d$;

(iii) If G is the union of a directed system of subgroups $\{G_i \mid i \in I\}$ such that each G_i belongs to $\mathcal{B}_d$, then $G \in \mathcal{B}_d$;

(iv) Suppose that there are groups G_1 and G_2 and group homomorphisms $\phi_i \colon G_0 \to G_i$ for $i = 1, 2$ such that ϕ_1 and ϕ_2 are injective, G_0 belongs to $\mathcal{B}_{d-1}$, G_1 and G_2 belong to $\mathcal{B}_d$ and G is the amalgamated product $G_1 *_{G_0} G_2$ with respect to ϕ_1 and ϕ_2. Then G belongs to $\mathcal{B}_d$;

(v) Let $1 \to H \to G \to K \to 1$ be an exact sequence of groups such that $b_p^{(2)}(H)$ is finite for all $p \leq d$. Suppose that K is infinite amenable or suppose that BK has finite d-skeleton and there is an injective endomorphism $j \colon K \to K$ whose image has finite index, but is not equal to K. Then $G \in \mathcal{B}_d$;

(vi) Let $1 \to H \to G \to K \to 1$ be an exact sequence of groups such that $H \in \mathcal{B}_{d-1}$, $b_d^{(2)}(H) < \infty$ and K contains an element of infinite order or finite subgroups of arbitrary large order. Then $G \in \mathcal{B}_d$;

(vii) Let $1 \to H \to G \to K \to 1$ be an exact sequence of infinite countable groups such that $b_1^{(2)}(H) < \infty$. Then $G \in \mathcal{B}_1$.

Proof. (i) We get $b_p^{(2)}(G) = 0$ for $p = 0$ from Theorem 6.1 (v). The case $p \geq 1$ follows from Theorem 5.2 (i) since $H_p^{\text{sing}}(EG; \mathbb{C}) = 0$ for $p \geq 1$.

(ii) Apply Theorem 3.11 (ii) to the fibration $BH \to BG \to B(G/H)$.

(iii) The proof is based on a colimit argument. See [80, Theorem 7.2 (3)].

(iv) The proof is based on a Mayer-Vietoris argument. See [80, Theorem 7.2 (4)].

(v) See [80, Theorem 7.2 (5)].

(vi) This follows from Theorem 3.11 (i) applied to the fibration $BH \to BG \to B(G/H)$.

(vii) This is proved by Gaboriau [38, Theorem 6.8]. □

More information about the vanishing of the first L^2-Betti number can be found for instance in [4]. Obviously the following is true

Lemma 6.5. *If G belongs to $\mathcal{B}_\infty$, then $\chi^{(2)}(G) = 0$.*

Remark 6.6 (The Theorem of Cheeger and Gromov). We rediscover from Theorem 6.4 the result of Cheeger and Gromov [15] that all the L^2-Betti numbers of an infinite amenable group G vanish. A detailed comparison of our approach and the approach by Cheeger and Gromov to L^2-Betti numbers can be found in [80, Remark 6.76].

Remark 6.7 (Advantage of the general definition of L^2-Betti numbers). Recall that we have given criterions for $G \in \mathcal{B}_\infty$ in Theorem 6.4. Now it becomes clear why it is worth while to extend the classical notion of the Euler characteristic $\chi(G) := \chi(BG)$ for groups G with finite BG to arbitrary groups. For instance it may very well happen for a group G with finite BG that G contains a normal group H which is not even finitely generated and has in particular no finite model for BH and which belongs to $\mathcal{B}_\infty$ (for instance, H is amenable). Then the classical Euler characteristic is not defined any more for H, but we can still conclude that the classical Euler characteristic of G vanishes by Remark 2.19, Theorem 6.4 and Lemma 6.5.

6.3 L^2-Betti Numbers of Some Specific Groups

Example 6.8 (Thompson's group). Next we explain the following observation about *Thompson's group F*. It is the group of orientation preserving dyadic PL-automorphisms of $[0, 1]$, where dyadic means that all slopes are integral powers of 2 and the break points are contained in $\mathbb{Z}[1/2]$. It has the presentation

$$F = \langle x_0, x_1, x_2, \ldots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.$$

This group has some very interesting properties. Its classifying space BF is of finite type [8] but is not homotopy equivalent to a finite dimensional CW-complex since F contains $\mathbb{Z}^n$ as a subgroup for all $n \geq 0$ [8, Proposition 1.8]. It is not elementary amenable and does not contain a subgroup which is free on two generators [7], [10]. Hence it is a very interesting question whether F is amenable or not. We conclude from Theorem 6.4 (i) that a necessary condition for F to be amenable is that $b_p^{(2)}(F)$ vanishes for all $p \geq 0$. By [80, Theorem 7.10] this condition is satisfied.

Example 6.9 (Artin groups). Davis and Leary [19] compute for every Artin group A the reduced L^2-cohomology and thus the L^2-Betti numbers of the universal covering $\widetilde{S_A}$ of its *Salvetti complex* S_A. The Salvetti complex S_A is a CW-complex which is conjectured to be a model for the classifying space BA of A. This conjecture is known to be true in many cases and implies that the L^2-Betti numbers of A are given by the L^2-Betti numbers of $\widetilde{S_A}$.

Example 6.10 (Right angled Coxeter groups). The L^2-homology and the L^2-Betti numbers of right angled Coxeter groups are treated by Davis and Okun [20]. More details will be given in Remark 9.6.

Example 6.11 (Fundamental groups of surfaces and 3-manifolds). Let G be the fundamental group of a compact orientable surface F_g^d of genus g with d boundary components. Suppose that G is non-trivial which is equivalent to the condition that $d \geq 1$ or $g \geq 1$. Then F_g^d is a model for BG and we have computed $b_p^{(2)}(G) = b_p^{(2)}(\widetilde{F_g^d})$ in Subsection 3.3.

Let G be the fundamental group of a compact orientable 3-manifold M. The case $|G| < \infty$ is clear, since then the universal covering is homotopy equivalent to a sphere or contractible. So let us assume $|G| = \infty$. Under the condition that M in non-exceptional, we have computed $b_p^{(2)}(\widetilde{M})$ in Theorem 3.4. If M is prime, then either $M = S^1 \times S^2$ and $G = \mathbb{Z}$ and $b_p^{(2)}(G) = 0$ for all $p \geq 0$ or M is irreducible, in which case M is aspherical and $b_p^{(2)}(G) = b_p^{(2)}(\widetilde{M})$.

Suppose that M is not prime. Then still $b_1^{(2)}(G) = b_1^{(2)}(\widetilde{M})$ by Theorem 2.7 (i)a since the classifying map $M \to BG$ is 2-connected. Suppose the prime decomposition of M looks like $M = \#_{i=1}^r M_i$. Then $G = *_{i=1}^r G_i$ for $G_i = \pi_1(M_i)$. We know $b_p^{(2)}(G_i)$ for each i if each M_i is non-exceptional and we get $b_p^{(2)}(G) = \sum_{i=1}^r b_p^{(2)}(G_i)$ for $p \geq 2$ from Theorem 6.1 (i).

Example 6.12 (One relator groups). Let $G = \langle g_1, g_2, \ldots g_s \mid R\rangle$ be a torsion-free one relator group for $s \in \{2, 3 \ldots\} \amalg \{\infty\}$ and one non-trivial relation R. Then

$$b_p^{(2)}(G) = \begin{cases} 0 & \text{if } p \neq 1; \\ s-2 & \text{if } p = 1 \text{ and } s < \infty; \\ \infty & \text{if } p = 1 \text{ and } s = \infty. \end{cases}$$

We only treat the case $s < \infty$, the general case is obtained from it by taking the free amalgamated product with a free group. Because the 2-dimensional CW-complex X associated to the given presentation is a model for BG (see [85, chapter III §§9 -11]) and satisfies $\chi(X) = s - 2$, it suffices to prove $b_2^{(2)}(G) = 0$. We sketch the argument of Dicks and Linnell for this claim. Howie [56] has shown that such a group G is locally indicable and hence left-orderable. A result of Linnell [64, Theorem 2] for left-orderable groups says that an element $\alpha \in \mathbb{C}G$ with $\alpha \neq 0$ is a non-zero-divisor in $\mathcal{U}(G)$. This implies that the second differential $c_2^{\mathcal{U}(G)}$ in the chain complex $\mathcal{U}(G) \otimes_{\mathbb{C}G} C_*(EG)$ is injective. Since $\mathcal{U}(G)$ is flat over $\mathcal{N}(G)$, we get from Theorem 4.5

$$\begin{aligned} b_2^{(2)}(G) &= \dim_{\mathcal{N}(G)}\left(H_p^G(EG; \mathcal{N}(G))\right) = \dim_{\mathcal{U}(G)}\left(H_p^G(EG; \mathcal{U}(G))\right) \\ &= \dim_{\mathcal{U}(G)}\left(\ker\left(c_2^{\mathcal{U}(G)}\right)\right) = 0. \end{aligned}$$

Linnell has an extensions of this argument to non-torsion-free one-relator groups G with $s \geq 2$ generators. (The case $s = 1$ is obvious.) Such a group contains a cyclic subgroup $\mathbb{Z}/k$ such that any finite subgroup is subconjugated to $\mathbb{Z}/k$ and then

$$b_p^{(2)}(G) = \begin{cases} 0 & \text{if } p \neq 1; \\ s - 1 - \frac{1}{k} & \text{if } p = 1 \text{ and } s < \infty; \\ \infty & \text{if } p = 1 \text{ and } s = \infty. \end{cases}$$

Example 6.13 (Lattices). Let L be a connected semisimple Lie group with finite center such that its Lie algebra has no compact ideal. Let $G \subseteq L$ be a lattice, i.e. a discrete subgroup of finite covolume. We want to compute its L^2-Betti numbers. There is a subgroup $G_0 \subseteq G$ of finite index which is torsion-free. Since $b_p^{(2)}(G) = [G : G_0] \cdot b_p^{(2)}(G_0)$, it suffices to treat the case $G = G_0$, i.e. $G \subseteq L$ is a torsion-free lattice.

Let $K \subseteq L$ be a maximal compact subgroup. Put $M = G \backslash L/K$. Then the space $L/K = \widetilde{M}$ is a symmetric space of non-compact type. We have already mentioned in Theorem 3.5 that the work of Borel [6] implies for cocompact G that $b_p^{(2)}(G) = b_p^{(2)}(\widetilde{M}) \neq 0$ if and only if $\text{f-rk}(\widetilde{M}) = 0$ and $2p = \dim(M)$. This is actually true without the condition "cocompact", because the condition "finite covolume" is enough.

Next we deal with the general case of a connected Lie group L. Let $\operatorname{Rad}(L)$ be its radical. One can choose a compact normal subgroup $K \subseteq L$ such that $R = \operatorname{Rad}(L) \times K$ is a normal subgroup of L and the quotient $L_1 = L/R$ is a semisimple Lie group such that its Lie algebra has no compact ideal. Then $G_1 = L/L \cap R$ is a lattice in L_1 and $G \cap R$ is a lattice in R. The group $G \cap R$ is a normal amenable subgroup of G. If $G \cap R$ is infinite, we get $b_p^{(2)}(G) = 0$ for all $p \geq 0$ from Theorem 6.4. If $G \cap R$ is finite, we get $b_p^{(2)}(G) = |G \cap R|^{-1} \cdot b_p^{(2)}(G_1)$ for all $p \geq 0$ from Theorem 6.1 (iv). If the center of L_1 is infinite, the center of G_1 must also be infinite and hence $b_p^{(2)}(G_1) = 0$ for all $p \geq 0$ by Theorem 6.4. Suppose that the center of L_1 is finite. Then we know already how to compute the L^2-Betti numbers of G_1 from the explanation above.

Given a lattice G in a connected Lie group, $b_1^{(2)}(G) > 0$ is true if and only if G is commensurable with a torsion-free lattice in $PSL_2(\mathbb{R})$, or, equivalently commensurable with a surface group for genus ≥ 2 or a finitely generated non-abelian free group (see Eckmann [27] or Lott [68, Theorem 2]).

6.4 Deficiency and L^2-Betti Numbers of Groups

Let G be a finitely presented group. Define its *deficiency* $\operatorname{def}(G)$ to be the maximum $g(P) - r(P)$, where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Next we reprove the well-known fact that the maximum appearing in the definition of the deficiency does exist.

Lemma 6.14. *Let G be a group with finite presentation*

$$P = \langle s_1, s_2, \ldots, s_g \mid R_1, R_2, \ldots, R_r \rangle$$

Let $\phi\colon G \to K$ be any group homomorphism. Then

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(K \times_\phi EG; \mathcal{N}(K)) + b_1^{(2)}(K \times_\phi EG; \mathcal{N}(K)) \\ - b_2^{(2)}(K \times_\phi EG; \mathcal{N}(K)).$$

Proof. Given a presentation P with g generators and r relations, let X be the associated finite 2-dimensional CW-complex. It has one 0-cell, g 1-cells, one for each generator, and r 2-cells, one for each relation. There is an obvious isomorphism from $\pi_1(X)$ to G so that we can choose a map $f\colon X \to BG$ which induces an isomorphism on the fundamental groups. It induces a 2-connected K-equivariant map $\overline{f}\colon K \times_\phi \widetilde{X} \to K \times_\phi \widetilde{EG}$. We conclude from Theorem 2.7 (i)a

$$\begin{array}{rcll} b_p^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) & = & b_p^{(2)}(K \times_\phi EG; \mathcal{N}(K)) & \quad \text{for } p = 0, 1; \\ b_2^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) & \geq & b_2^{(2)}(K \times_\phi EG; \mathcal{N}(K)). & \end{array}$$

We conclude from the L^2-Euler-Poincaré formula (see Theorem 2.18 (i))

$$\begin{array}{rcl} g - r & = & 1 - \chi^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) \\ & = & 1 - b_0^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) + b_1^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) \\ & & \qquad -b_2^{(2)}(K \times_\phi \widetilde{X}; \mathcal{N}(K)) \\ & \leq & 1 - b_0^{(2)}(K \times_\phi EG; \mathcal{N}(K)) + b_1^{(2)}(K \times_\phi EG; \mathcal{N}(K)) \\ & & \qquad -b_2^{(2)}(K \times_\phi EG; \mathcal{N}(K)). \quad \square \end{array}$$

Example 6.15 (Deficiency of some groups). Sometimes the deficiency is realized by the "obvious" presentation. For instance the deficiency of a free group $\langle s_1, s_2, \ldots, s_g \mid \emptyset \rangle$ on g letters is indeed g. The cyclic group $\mathbb{Z}/n$ of order n has the presentation $\langle t \mid t^n = 1 \rangle$ and its deficiency is 0. The group $\mathbb{Z}/n \times \mathbb{Z}/n$ has the presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ and its deficiency is -1.

Remark 6.16 (Non-additivity of the deficiency). The deficiency is not additive under free products by the following example which is a special case of a more general example due to Hog, Lustig and Metzler [55, Theorem 3 on page 162]. The group $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \;\mid\; s_0^2 \;=\; t_0^2 \;=\; [s_0, t_0] \;=\; s_1^3 \;=\; t_1^3 \;=\; [s_1, t_1] \;=\; 1 \rangle$$

One may think that its deficiency is -2. However, it turns out that its deficiency is -1, realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \;\mid\; s_0^2 \;=\; 1, [s_0, t_0] \;=\; t_0^2, s_1^3 \;=\; 1, [s_1, t_1] \;=\; t_1^3, t_0^2 \;=\; t_1^3 \rangle.$$

This shows that it is important to get upper bounds on the deficiency of groups. Writing down presentations gives lower bounds, but it is not clear whether a given presentation realizes the deficiency.

Lemma 6.17. *Let G be a finitely presented group and let $\phi\colon G \to K$ be a homomorphism such that $b_1^{(2)}(K \times_\phi EG; \mathcal{N}(K)) = 0$. Then*

(i) $\operatorname{def}(G) \leq 1$*;*

(ii) Let M be a closed oriented 4-manifold with G as fundamental group. Then

$$|\operatorname{sign}(M)| \;\leq\; \chi(M).$$

Proof. (i) This follows directly from Lemma 6.14.
(ii) This is a consequence of the L^2-Signature Theorem due to Atiyah [2]. Details of the proof can be found in [80, Lemma 7.22]. □

Theorem 6.18. *Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ be an exact sequence of infinite groups. Suppose that G is finitely presented and one of the following conditions is satisfied:*

(i) $b_1^{(2)}(H) < \infty$*;*

(ii) The classical first Betti number of H satisfies $b_1(H) < \infty$ and K belongs to $\mathcal{B}_1$.

Then

(i) $\operatorname{def}(G) \leq 1$*;*

(ii) Let M be a closed oriented 4-manifold with G as fundamental group. Then

$$|\operatorname{sign}(M)| \;\leq\; \chi(M).$$

Proof. If condition (i) is satisfied, then $b_p^{(2)}(G) = 0$ for $p = 0, 1$ by Theorem 6.4 (vii), and the claim follows from Lemma 6.17.

Suppose that condition (ii) is satisfied. There is a spectral sequence converging to $H_{p+q}^K(K \times_q EG; \mathcal{N}(K))$ with E^2-term

$$E^2_{p,q} \;=\; \operatorname{Tor}_p^{\mathbb{C}K}(H_q(BH;\mathbb{C}), \mathcal{N}(K))$$

[108, Theorem 5.6.4 on page 143]. Since $H_q(BH;\mathbb{C})$ is $\mathbb{C}$ with the trivial K-action for $q = 0$ and finite dimensional as complex vector space by assumption for $q = 1$, we conclude $\dim_{\mathcal{N}(K)}(E^2_{p,q}) = 0$ for $p + q = 1$ from the assumption $b_1^{(2)}(K) = 0$. This implies $b_1^{(2)}(K \times_q EG; \mathcal{N}(K)) = 0$ and the claim follows from Lemma 6.17. □

Theorem 6.18 generalizes results in [29], [58], where also some additional information is given. Furthermore see [49], [63]. We mention the result of Hitchin [54] that a connected closed oriented smooth 4-manifold which admits an Einstein metric satisfies the stronger inequality $|\operatorname{sign}(M)| \leq \frac{2}{3} \cdot \chi(M)$.

Finally we mention the following result of Lott [68, Theorem 2] (see also [28]) which generalizes a result of Lubotzky [70]. The statement we present here is a slight improvement of Lott's result due to Hillman [53].

Theorem 6.19 (Lattices of positive deficiency). *Let L be a connected Lie group. Let G be a lattice in L. If $\operatorname{def}(G) > 0$, then one of the following assertions holds:*

(i) *G is a lattice in $PSL_2(\mathbb{C})$;*

(ii) $\operatorname{def}(G) = 1$. *Moreover, either G is isomorphic to a torsion-free non-uniform lattice in $\mathbb{R} \times PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$, or G is $\mathbb{Z}$ or $\mathbb{Z}^2$.*

7 G- and K-Theory

In this section we discuss the projective class group $K_0(\mathcal{N}(G))$ of a group von Neumann algebra. We present applications of its computation to $G_0(\mathbb{C}G)$ and the Whitehead group $\operatorname{Wh}(G)$ of a group G.

7.1 The K_0-group of a Group von Neumann Algebra

In this subsection we want to investigate the projective class group of a group von Neumann algebra.

Definition 7.1 (Definition of $K_0(R)$ and $G_0(R)$). *Let R be an (associative) ring (with unit). Define its* projective class group *$K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.*

Define the Grothendieck group of finitely generated modules *$G_0(R)$ analogously but replace finitely generated projective with finitely generated.*

The group K_0 is known for any von Neumann algebra (see for instance [80, Subsection 9.2.1]. For simplicity we only treat the von Neumann algebra $\mathcal{N}(G)$ of a group here.

The next result is taken from [60, Theorem 7.1.12 on page 462, Proposition 7.4.5 on page 483, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525, Theorem 8.4.3 on page 532].

Theorem 7.2 (The universal trace). *There is a map*

$$\operatorname{tr}^u_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{Z}(\mathcal{N}(G))$$

into the center $\mathcal{Z}(\mathcal{N}(G))$ of $\mathcal{N}(G)$ called the center valued trace *or* universal trace *of $\mathcal{N}(G)$, which is uniquely determined by the following two properties:*

(i) *$\operatorname{tr}^u_{\mathcal{N}(G)}$ is a trace with values in the center, i.e. $\operatorname{tr}^u_{\mathcal{N}(G)}$ is $\mathbb{C}$-linear, for $a \in \mathcal{N}(G)$ with $a \geq 0$ we have $\operatorname{tr}^u_{\mathcal{N}(G)}(a) \geq 0$ and $\operatorname{tr}^u_{\mathcal{N}(G)}(ab) = \operatorname{tr}^u_{\mathcal{N}(G)}(ba)$ for all $a, b \in \mathcal{N}(G)$;*

(ii) $\mathrm{tr}^u_{\mathcal{N}(G)}(a) = a$ *for all* $a \in \mathcal{Z}(\mathcal{N}(G))$.

The map $\mathrm{tr}^u_{\mathcal{N}(G)}$ *has the following further properties:*

(iii) $\mathrm{tr}^u_{\mathcal{N}(G)}$ *is faithful, i.e.* $\mathrm{tr}^u_{\mathcal{N}(G)}(a) = 0 \Leftrightarrow a = 0$ *for* $a \in \mathcal{N}(G), a \geq 0$;

(iv) $\mathrm{tr}^u_{\mathcal{N}(G)}$ *is normal, i.e. for a monotone increasing net* $\{a_i \mid i \in I\}$ *of positive elements* a_i *with supremum* a *we have* $\mathrm{tr}^u_{\mathcal{N}(G)}(a) = \sup\{\mathrm{tr}\,(a_i) \mid i \in I\}$, *or, equivalently,* $\mathrm{tr}^u_{\mathcal{N}(G)}$ *is continuous with respect to the ultra-weak topology on* $\mathcal{N}(G)$;

(v) $||\,\mathrm{tr}^u_{\mathcal{N}(G)}(a)|| \ \leq\ ||a||$ *for* $a \in \mathcal{N}(G)$;

(vi) $\mathrm{tr}^u_{\mathcal{N}(G)}(ab) = a\,\mathrm{tr}^u_{\mathcal{N}(G)}(b)$ *for all* $a \in \mathcal{Z}(\mathcal{N}(G))$ *and* $b \in \mathcal{N}(G)$;

(vii) Let p *and* q *be projections in* $\mathcal{N}(G)$. *Then* $p \sim q$, *i.e.* $p = uu^*$ *and* $q = u^*u$ *for some element* $u \in \mathcal{N}(G)$, *if and only if* $\mathrm{tr}^u_{\mathcal{N}(G)}(p) = \mathrm{tr}^u_{\mathcal{N}(G)}(q)$;

(viii) Any linear functional $f\colon \mathcal{N}(G) \to \mathbb{C}$ *which is continuous with respect to the norm topology on* $\mathcal{N}(G)$ *and which is central, i.e.* $f(ab) = f(ba)$ *for all* $a, b \in \mathcal{N}(G)$ *factorizes as*

$$\mathcal{N}(G) \xrightarrow{\mathrm{tr}^u_{\mathcal{N}(G)}} \mathcal{Z}(\mathcal{N}(G)) \xrightarrow{f|_{\mathcal{Z}(\mathcal{N}(G))}} \mathbb{C}.$$

Definition 7.3 (Center valued dimension). *For a finitely generated projective* $\mathcal{N}(G)$*-module* P *define its* center valued von Neumann dimension *by*

$$\dim^u_{\mathcal{N}(G)}(P) := \sum_{i=1}^{n} \mathrm{tr}^u_{\mathcal{N}(G)}(a_{i,i}) \in \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \mathcal{Z}(\mathcal{N}(G)) \mid a = a^*\}$$

for any matrix $A = (a_{i,j})_{i,j} \in M_n(\mathcal{N}(G))$ *with* $A^2 = A$ *such that* $\mathrm{im}(r_A\colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n)$ *induced by right multiplication with* A *is* $\mathcal{N}(G)$*-isomorphic to* P.

There is a classification of von Neumann algebras into certain types. We only need to know what the type of a group von Neumann algebra is.

Lemma 7.4. *Let* G *be a discrete group. Let* G_f *be the normal subgroup of* G *consisting of elements* $g \in G$ *whose centralizer has finite index (or, equivalently, whose conjugacy class* (g) *consists of finitely many elements). Then:*

(i) The group von Neumann algebra $\mathcal{N}(G)$ *is of type* I_f *if and only if* G *is virtually abelian;*

(ii) The group von Neumann algebra $\mathcal{N}(G)$ *is of type* II_1 *if and only if the index of* G_f *in* G *is infinite;*

(iii) Suppose that G is finitely generated. Then $\mathcal{N}(G)$ is of type I_f if G is virtually abelian, and of type II_1 if G is not virtually abelian;

(iv) The group von Neumann algebra $\mathcal{N}(G)$ is a factor, *i.e. its center consists of $\{r \cdot 1_{\mathcal{N}(G)} \mid r \in \mathbb{C}\}$, if and only if G_f is the trivial group.*

Proof. (i) This is proved in [61], [103].

(ii) This is proved in [61],[88].

(iii) This follows from assertions (i) and (ii) since for finitely generated G the group G_f has finite index in G if and only if G is virtually abelian.

(iv) This follows from [23, Proposition 5 in III.7.6 on page 319]. □

The next result follows from [60, Theorem 8.4.3 on page 532, Theorem 8.4.4 on page 533].

Theorem 7.5 (K_0 of finite von Neumann algebras). *Let G be a group.*

(i) The following statements are equivalent for two finitely generated projective $\mathcal{N}(G)$-modules P and Q:

(a) P and Q are $\mathcal{N}(G)$-isomorphic;

(b) P and Q are stably $\mathcal{N}(G)$-isomorphic, i.e. $P \oplus V$ and $Q \oplus V$ are $\mathcal{N}(G)$-isomorphic for some finitely generated projective $\mathcal{N}(G)$-module V;

(c) $\dim^u_{\mathcal{N}(G)}(P) = \dim^u_{\mathcal{N}(G)}(Q)$;

(d) $[P] = [Q]$ in $K_0(\mathcal{N}(G))$;

(ii) The center valued dimension induces an injection

$$\dim^u_{\mathcal{N}(G)} \colon K_0(\mathcal{N}(G)) \to \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2} = \{a \in \mathcal{Z}(\mathcal{N}(G)) \mid a = a^*\},$$

where the group structure on $\mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}$ comes from addition. If $\mathcal{N}(G)$ is of type II_1, this map is an isomorphism. □

Remark 7.6 (Group von Neumann algebras and representation theory). Theorem 7.5 shows that the group von Neumann algebra is the right generalization of the complex group ring from finite groups to infinite groups if one is concerned with representation theory of finite groups. Namely, let G be a finite group. Recall that a finite dimensional complex G-representation V is the same as a finitely generated $\mathbb{C}G$-module and that $K_0(\mathbb{C}G)$ is the same as the complex representation ring. Moreover, two finite dimensional G-representations V and W are linearly G-isomorphic if and only if they have the same character. Recall that the character is a class function. One easily checks that the complex vector space of class functions on a finite group G is the same as the center $\mathcal{Z}(\mathbb{C}G)$ and that the character of V contains the same information as $\dim^u_{\mathcal{N}(G)}(V)$.

Remark 7.7 (Factors). Suppose that $\mathcal{N}(G)$ is a factor, i.e. its center consists of $\{r \cdot 1_{\mathcal{N}(G)} \mid r \in \mathbb{C}\}$. By Lemma 7.4 (iv) this is the case if and only if G_f is the trivial group. Then $\dim_{\mathcal{N}(G)} = \dim^u_{\mathcal{N}(G)}$ and two finitely generated projective $\mathcal{N}(G)$-modules P and Q are $\mathcal{N}(G)$-isomorphic if and only if $\dim_{\mathcal{N}(G)}(P) = \dim_{\mathcal{N}(G)}(Q)$ holds. This has the consequence that for a free G-CW-complex X of finite type the p-th L^2-Betti number determines the isomorphism type of $\mathbf{P}H_p^G(X;\mathcal{N}(G))$. In particular we must have $\mathbf{P}H_p^G(X;\mathcal{N}(G)) \cong_{\mathcal{N}(G)} \mathcal{N}(G)^n$ provided that $n = b_p^{(2)}(X;\mathcal{N}(G))$ is an integer. If one prefers to work with reduced L^2-homology, this is equivalent to the statement that $H_p^{(2)}(X;l^2(G))$ is isometrically G-linearly isomorphic to $l^2(G)^n$ provided that $n = b_p^{(2)}(X;\mathcal{N}(G))$ is an integer.

Remark 7.8 (The reduced L^2-cohomology of torsion-free groups). Let G be a torsion-free group. Suppose that it satisfies the Atiyah Conjecture 4.1 for $(G, 1, \mathbb{Q})$. Suppose that there is a model for BG of finite type. Then we get for all p that $\mathbf{P}H_p^G(EG;\mathcal{N}(G)) \cong_{\mathcal{N}(G)} \mathcal{N}(G)^n$, or, equivalently, that $H_p^{(2)}(X;l^2(G))$ is isometrically G-linearly isomorphic to $l^2(G)^n$ if the integer n is given by $n = b_p^{(2)}(X;\mathcal{N}(G))$. This claim is proved in [80, solution to Exercise 10.11 on page 546].

We mention that the inclusion $i\colon \mathcal{N}(G) \to \mathcal{U}(G)$ induces an isomorphism

$$K_0(\mathcal{N}(G)) \xrightarrow{\cong} K_0(\mathcal{U}(G)).$$

The Farrell-Jones Conjecture for $K_0(\mathbb{C}G)$, the Bass Conjecture and the passage in K_0 from $\mathbb{Z}G$ to $\mathbb{C}G$ and to $\mathcal{N}(G)$ is discussed in [80, Section 9.5.2] and [81].

7.2 The K_1-group and the L-groups of a Group von Neumann Algebra

A complete calculation of the K_1-group and of the L-groups of any von Neumann algebra and of the associated algebra of affiliated operators can be found in [80, Section 9.3 and Section 9.4], [83] and [99].

7.3 Applications to G-theory of Group Rings

Theorem 7.9 (Detecting $G_0(\mathbb{C}G)$ by $K_0(\mathcal{N}(G))$ for amenable groups). *If G is amenable, the map*

$$l\colon G_0(\mathbb{C}G) \to K_0(\mathcal{N}(G)), \qquad [M] \mapsto [\mathbf{P}\mathcal{N}(G) \otimes_{\mathbb{C}G} M]$$

is a well-defined homomorphism. If $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ is the forgetful map sending $[P]$ to $[P]$ and $i_*\colon K_0(\mathbb{C}G) \to K_0(\mathcal{N}(G))$ is induced by the inclusion $i\colon \mathbb{C}G \to \mathcal{N}(G)$, then the composition $l \circ f$ agrees with i_*.

Proof. This is essentially a consequence of the dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ (see Theorem 5.1). Details of the proof can be found in [80, Theorem 9.64]. $\square$

Now one can combine Theorem 7.5 and Theorem 7.9 to detect elements in $G_0(\mathbb{C}G)$ for amenable G. In particular one can show

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G_0(\mathbb{C}G)) \quad \geq \quad |\operatorname{con}(G)_{f,cf}|, \tag{7.10}$$

where $\operatorname{con}(G)_{f,cf}$ is the set of conjugacy classes (g) of elements $g \in G$ such that g has finite order and (g) contains only finitely many elements. Notice that $\operatorname{con}(G)_{f,cf}$ contains at least one element, namely the unit element e.

Remark 7.11 (The non-vanishing of $[RG]$ in $G_0(RG)$ for amenable groups). A direct consequence of Theorem 7.9 is that for an amenable group G the class $[\mathbb{C}G]$ in $G_0(\mathbb{C}G)$ generates an infinite cyclic subgroup. Namely, the dimension induces a well-defined homomorphism

$$\dim_{\mathcal{N}(G)} \colon G_0(\mathbb{C}G) \to \mathbb{R}, \quad [M] \;\mapsto\; \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M),$$

which sends $[\mathbb{C}G]$ to 1. This result has been extended by Elek [32] to finitely generated amenable groups and arbitrary fields F, i.e. there is a well-defined homomorphism $G_0(FG) \to \mathbb{R}$, which sends $[FG]$ to 1 and is given by a certain rank function on finitely generated FG-modules.

The class $[RG]$ in $K_0(RG)$ is never zero for a commutative integral domain R with quotient field $R_{(0)}$. The augmentation $RG \to R$ and the map $K_0(R) \to \mathbb{Z}, \quad [P] \mapsto \dim_{R_{(0)}}(R_{(0)} \otimes_R P)$ together induce a homomorphism $K_0(RG) \to \mathbb{Z}$ which sends $[RG]$ to 1. A decisive difference between $K_0(RG)$ and $G_0(RG)$ is that $[RG] = 0$ is possible in $G_0(RG)$ as the following example shows.

Example 7.12 (The vanishing of $[RG]$ in $G_0(RG)$ for groups G containing $\mathbb{Z} * \mathbb{Z}$). We abbreviate $F_2 = \mathbb{Z} * \mathbb{Z}$. Suppose that G contains F_2 as a subgroup. Let R be a ring. Then

$$[RG] \;=\; 0 \quad \in G_0(RG)$$

holds by the following argument. Induction with the inclusion $F_2 \to G$ induces a homomorphism $G_0(RF_2) \to G_0(RG)$ which sends $[RF_2]$ to $[RG]$. Hence it suffices to show $[RF_2] = 0$ in $G_0(RF_2)$. The cellular chain complex of the universal covering of $S^1 \vee S^1$ yields an exact sequence of RF_2-modules $0 \to (RF_2)^2 \to RF_2 \to R \to 0$, where R is equipped with the trivial F_2-action. This implies $[RF_2] = -[R]$ in $G_0(RF_2)$. Hence it suffices to show $[R] = 0$ in $G_0(RF_2)$. Choose an epimorphism $f \colon F_2 \to \mathbb{Z}$. Restriction with f defines a homomorphism $G_0(R\mathbb{Z}) \to G_0(RF_2)$. It sends the class of R viewed as trivial $R\mathbb{Z}$-module to the class of R viewed as trivial RF_2-module. Hence it remains to show $[R] = 0$ in $G_0(R\mathbb{Z})$. This follows from the exact sequence $0 \to R\mathbb{Z} \xrightarrow{s-1} R\mathbb{Z} \to R \to 0$ for s a generator of $\mathbb{Z}$ which comes from the cellular $R\mathbb{Z}$-chain complex of $\widetilde{S^1}$.

Remark 7.11 and Example 7.12 give some evidence for

Conjecture 7.13. (Amenability and the regular representation in G-theory). *Let R be a commutative integral domain. Then a group G is amenable if and only if $[RG] \neq 0$ in $G_0(RG)$.*

Remark 7.14 (The Atiyah Conjecture for amenable groups and $G_0(\mathbb{C}G)$). Assume that G is amenable and that there is an upper bound on the orders of finite subgroups of G. Then the Atiyah Conjecture 4.1 for $(G, d, \mathbb{C})$ is true if and only if the image of the map

$$\dim_{\mathcal{N}(G)} \colon G_0(\mathbb{C}G) \to \mathbb{R}, \qquad [M] \mapsto \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M)$$

is contained in $\{r \in \mathbb{R} \mid d \cdot r \in \mathbb{Z}\}$.

Example 7.15 ($K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ is not necessarily surjective). Let $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$. This abelian group is locally finite. Hence the map

$$\bigoplus_{H \subseteq A, |H| < \infty} K_0(\mathbb{C}H) \to K_0(\mathbb{C}A)$$

is surjective and the image of

$$\dim_{\mathcal{N}(G)} \colon K_0(\mathbb{C}A) \to \mathbb{R}, \quad [P] \mapsto \dim_{\mathcal{N}(A)} (\mathcal{N}(A) \otimes_{\mathbb{C}A} P)$$

is $\mathbb{Z}[1/2]$. On the other hand the argument in [80, Example 10.13] shows that the map

$$\dim_{\mathcal{N}(G)} \colon G_0(\mathbb{C}A) \to \mathbb{R}, \quad [M] \mapsto \dim_{\mathcal{N}(A)} (\mathcal{N}(A) \otimes_{\mathbb{C}A} M)$$

is surjective. In particular the obvious map $K_0(\mathbb{C}A) \to G_0(\mathbb{C}A)$ is not surjective.

7.4 Applications to the Whitehead Group

The *Whitehead group* $\mathrm{Wh}(G)$ of a group G is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup which consists of elements given by units of the shape $\pm g \in \mathbb{Z}G$ for $g \in G$. Let $i \colon H \to G$ be the inclusion of a normal subgroup $H \subseteq G$. It induces a homomorphism $i_0 \colon \mathrm{Wh}(H) \to \mathrm{Wh}(G)$. The conjugation action of G on H and on G induces a G-action on $\mathrm{Wh}(H)$ and on $\mathrm{Wh}(G)$ which turns out to be trivial on $\mathrm{Wh}(G)$. Hence i_0 induces homomorphisms

$$i_1 \colon \mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) \to \mathrm{Wh}(G); \tag{7.16}$$

$$i_2 \colon \mathrm{Wh}(H)^G \to \mathrm{Wh}(G). \tag{7.17}$$

Theorem 7.18 (Detecting elements in $\mathrm{Wh}(G)$). *Let $i \colon H \to G$ be the inclusion of a normal finite subgroup H into an arbitrary group G. Then the maps i_1 and i_2 defined in* (7.16) *and* (7.17) *have finite kernel.*

Proof. See [80, Theorem 9.38]. ∎

We emphasize that Theorem 7.18 above holds for all groups G. It seems to be related to the Farrell-Jones Isomorphism Conjecture.

8 L^2-Betti Numbers and Measurable Group Theory

In this section we want to discuss an interesting relation between L^2-Betti numbers and measurable group theory. We begin with formulating the main result.

Definition 8.1 (Measure equivalence). *Two countable groups G and H are called* measure equivalent *if there exist commuting measure-preserving free actions of G and H on some standard Borel space (Ω, μ) with non-zero Borel measure μ such that the actions of both G and H admit measure fundamental domains X and Y of finite measure.*

The triple (Ω, X, Y) is called a measure coupling *of G and H. The index of (Ω, X, Y) is the quotient $\frac{\mu(X)}{\mu(Y)}$.*

Here are some explanations. A *Polish space* is a separable topological space which is metrizable by a complete metric. A *measurable space* $\Omega = (\Omega, \mathcal{A})$ is a set Ω together with a σ-algebra $\mathcal{A}$. It is called a *standard Borel space* if it is isomorphic to a Polish space with its Borel σ-algebra. (The Polish space is not part of the structure, only its existence is required.) More information about this notion of measure equivalence can be found for instance in [36], [37] and [46, 0.5E].

The following result is due to Gaboriau [38, Theorem 6.3]. We will discuss its applications and sketch the proof based on homological algebra and the dimension function due to R. Sauer [100].

Theorem 8.2 (Measure equivalence and L^2-Betti numbers). *Let G and H be two countable groups which are measure equivalent. If $C > 0$ is the index of a measure coupling, then we get for all $p \geq 0$*

$$b_p^{(2)}(G) \;=\; C \cdot b_p^{(2)}(H).$$

The general strategy of the proof of Theorem 8.2 is as follows. In the first step one introduces the notion of a standard action $G \curvearrowright X$ and of a weak orbit equivalence of standard actions of index C and shows that two groups G and H are measure equivalent of index C if and only if there exist standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ which are weakly orbit equivalent with index C. In the second step one assigns to a standard action $G \curvearrowright X$ L^2-Betti numbers $b_p^{(2)}(G \curvearrowright X)$, which involve only data that is invariant under orbit equivalence. Hence $b_p^{(2)}(G \curvearrowright X)$ itself depends only on the orbit equivalence class of $G \curvearrowright X$. In order to deal with weak orbit equivalences, one has to investigate the behaviour of the L^2-Betti numbers of $b_p^{(2)}(G \curvearrowright X)$ under restriction. Finally one proves that the L^2-Betti numbers of a standard action $G \curvearrowright X$ agree with the L^2-Betti numbers of G itself.

A version of Theorem 8.2 for the L^2-torsion is presented in Conjecture 11.30.

8.1 Measure Equivalence and Quasi-Isometry

Remark 8.3 (Measure equivalence is the measure theoretic version of quasi-isometry). The notion of measure equivalence can be viewed as the measure theoretic analogue of the metric notion of quasi-isometric groups. Namely, two finitely generated groups G_0 and G_1 are *quasi-isometric* if and only if there exist commuting proper (continuous) actions of G_0 and G_1 on some locally compact space such that each action has a cocompact fundamental domain [46, 0.2 C_2' on page 6].

Example 8.4 (Infinite amenable groups). Every countable infinite amenable group is measure equivalent to $\mathbb{Z}$ (see [94]). Since obviously all the L^2-Betti numbers of $\mathbb{Z}$ vanish, Theorem 8.2 implies the result of Cheeger and Gromov that all the L^2-Betti numbers of an infinite amenable group vanish.

Remark 8.5 (L^2-Betti numbers and quasi-isometry). If the finitely generated groups G_0 and G_1 are quasi-isometric and there exist finite models for BG_0 and BG_1, then $b_p^{(2)}(G_0) = 0 \Leftrightarrow b_p^{(2)}(G_1) = 0$ holds (see [46, page 224], [95]). But in general it is not true that there is a constant $C > 0$ such that $b_p^{(2)}(G_0) = C \cdot b_p^{(2)}(G_1)$ holds for all $p \geq 0$ (cf. [39, page 7], [46, page 233], [109]).

Remark 8.6 (Measure equivalence versus quasi-isometry). If F_g denotes the free group on g generators, then define $G_n := (F_3 \times F_3) * F_n$ for $n \geq 2$. The groups G_m and G_n are quasi-isometric for $m, n \geq 2$ (see [21, page 105 in IV-B.46], [109, Theorem 1.5]) and have finite models for their classifying spaces. One easily checks using Theorem 6.1 that $b_1^{(2)}(G_n) = n$ and $b_2^{(2)}(G_n) = 4$.

Theorem 8.2 due to Gaboriau implies that G_m and G_n are measure equivalent if and only if $m = n$ holds. Hence there are finitely presented groups which are quasi-isometric but not measure equivalent.

The converse is also true. The groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are infinite amenable and hence measure equivalent. But they are not quasi-isometric for different m and n since n is the growth rate of $\mathbb{Z}^n$ and the growth rate is a quasi-isometry invariant.

Notice that Theorem 8.2 implies that the sign of the Euler characteristic of a group G is an invariant under measure equivalence, which is not true for quasi-isometry by the example of the groups G_n above.

Let the two groups G and H act on the same metric space X properly and cocompactly by isometries. If X is second countable and proper, then G and H are measure equivalent. [100, Theorem 2.36]. If X is a geodesic and proper, then G and H are quasi-isometric.

Remark 8.7 (Kazhdan's property (T)). Kazhdan's property (T) is an invariant under measure equivalence [36, Theorem 8.2]. There exist quasi-isometric finitely generated groups G_0 and G_1 such that G_0 has Kazhdan's property (T) and G_1 not (see [39, page 7]). Hence G_0 and G_1 are quasi-isometric but not measure equivalent.

The rest of this section is devoted to an outline of the proof of Theorem 8.2 due to R. Sauer [100] which is simpler and more algebraic than the original one of Gaboriau [38] and may have the potential to apply also to L^2-torsion.

8.2 Discrete Measured Groupoids

A *groupoid* is a small category in which all morphisms are isomorphisms. We will identify a groupoid $\underline{G}$ with its set of morphisms. Then the set of objects $\underline{G}^0$ can be considered as a subset of $\underline{G}$ via the identity morphisms. There are four canonical maps,

$$\begin{array}{llcll} \text{source map} & s\colon \underline{G} & \to & \underline{G}^0, & (f\colon x\to y) \mapsto x; \\ \text{target map} & t\colon \underline{G} & \to & \underline{G}^0, & (f\colon x\to y) \mapsto y; \\ \text{inverse map} & i\colon \underline{G} & \to & \underline{G}, & f \mapsto f^{-1}; \\ \text{composition} & \circ\colon \underline{G}^2 & \to & \underline{G}, & (f,g)\mapsto f\circ g, \end{array}$$

where $\underline{G}^2$ is $\{(f,g)\in \underline{G}\times\underline{G} \mid s(f)=t(g)\}$. We will often abbreviate $f\circ g$ by fg.

A *discrete measurable groupoid* is a groupoid $\underline{G}$ equipped with the structure of a standard Borel space such that the inverse map and the composition are measurable maps and $s^{-1}(x)$ is countable for all objects $x\in\underline{G}^0$. Then $\underline{G}^0\subseteq\underline{G}$ is a Borel subset, the source and the target maps are measurable and $t^{-1}(x)$ is countable for all objects $x\in\underline{G}^0$.

Let μ be a probability measure on $\underline{G}^0$. Then for each measurable subset $A\subseteq\underline{G}$ the function

$$\underline{G}^0\to\mathbb{C},\quad x\mapsto |s^{-1}(x)\cap A|$$

is measurable and we obtain a σ-finite measure μ_s on $\underline{G}$ by

$$\mu_s(A) \;:=\; \int_{\underline{G}^0} |s^{-1}(x)\cap A|\; d\mu(x).$$

It is called the left counting measure of μ. The right counting measure μ_t is defined analogously replacing the source map s by the target map t. We call μ invariant if $\mu_s=\mu_t$, or, equivalently, if $i_*\mu_s=\mu_s$. A discrete measurable groupoid $\underline{G}$ together with an invariant measure μ on $\underline{G}^0$ is called a *discrete measured groupoid*. Given a Borel subset $A\subseteq\underline{G}^0$ with $\mu(A)>0$, there is the *restricted discrete measured groupoid* $\underline{G}|_A=s^{-1}(A)\cap t^{-1}(A)$, which is equipped with the *normalized measure* $\frac{1}{\mu(A)}\cdot\mu|_A$.

An *isomorphism of discrete measured groupoids* $f\colon\underline{G}\to\underline{H}$ is an isomorphisms of groupoids which preserves the measures. Given measurable subsets $A\subseteq\underline{G}^0$ and $B\subseteq\underline{H}^0$ such that $t(s^{-1}(A))$ and $t(s^{-1}(B))$ have full measure in $\underline{G}^0$ and $\underline{H}^0$ respectively, we call an isomorphism of discrete measured groupoids $f\colon\underline{G}_A\to\underline{H}_B$ *a weak isomorphism of discrete measured groupoids.*

Example 8.8 (Orbit equivalence relation). Consider the countable group G with an action $G \curvearrowright X$ on a standard Borel space X with probability measure μ by μ-preserving isomorphisms. The *orbit equivalence relation*

$$\mathcal{R}(G \curvearrowright X) \ := \ \{(x, gx) \mid x \in X, g \in G\} \ \subseteq X \times X$$

becomes a discrete measured groupoid by the obvious groupoid structure and measure.

An action $G \curvearrowright X$ of a countable group G is called *standard* if X is a standard Borel measure space with a probability measure μ, the action is by μ-preserving Borel isomorphisms and the action is *essentially free*, i.e. the stabilizer of almost every $x \in X$ is trivial. Every countable group G admits a standard action, which is given by the shift action on $\prod_{g \in G}[0,1]$. Notice that this G-action is not free but essentially free.

Two standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are *weakly orbit equivalent* if there are Borel subsets $A \subseteq X$ and $B \subseteq Y$, which meet almost every orbit and have positive measure in X and Y respectively, and a Borel isomorphism $f\colon A \to B$, which preserves the normalized measures on A and B and satisfies

$$f(G \cdot x \cap A) \ = \ H \cdot f(x) \cap B$$

for almost all $x \in A$. If A has full measure in X and B has full measure in Y, then the two standard actions are called *orbit equivalent*. The map f is called a *weak orbit equivalence* or *orbit equivalence* respectively. The *index of a weak orbit equivalence* of f is the quotient $\frac{\mu(A)}{\mu(B)}$. The next result is due to Furman [37, Theorem 3.3].

Theorem 8.9 (Measure equivalence and weak orbit equivalence). *Two countable groups are measure equivalent with respect to a measure coupling of index $C > 0$ if and only if there exist standard actions of G and H which are weakly orbit equivalent with index C.*

8.3 Groupoid Rings

Let $\underline{G}$ be a discrete measured groupoid with invariant measure μ on $\underline{G}^0$. For a function $\phi\colon \underline{G} \to \mathbb{C}$ and $x \in \underline{G}^0$ put

$$\begin{aligned}
S(\phi)(x) &:= |\{g \in \underline{G} \mid \phi(g) \neq 0, s(g) = x\} & \in \{0,1,2\ldots\} \amalg \{\infty\}; \\
T(\phi)(x) &:= |\{g \in \underline{G} \mid \phi(g) \neq 0, t(g) = x\} & \in \{0,1,2\ldots\} \amalg \{\infty\}.
\end{aligned}$$

Let $\mu_{\underline{G}} = \mu_s = \mu_t$ be the measure on $\underline{G}$ induced by μ. Let $L^\infty(\underline{G}) = L^\infty(\underline{G}; \mu_{\underline{G}})$ be the $\mathbb{C}$-algebra of equivalence classes of essentially bounded measurable functions $\underline{G} \to \mathbb{C}$. Define $L^\infty(\underline{G}^0) = L^\infty(\underline{G}^0; \mu)$ analogously. Define the *groupoid ring* of $\underline{G}$ as the subset

$$\mathbb{C}\underline{G} \ := \ \{\phi \in L^\infty(\underline{G}) \mid S(\phi) \text{ and } T(\phi) \text{ are essentially bounded on } \underline{G}\}. \quad (8.10)$$

The addition comes from the pointwise addition in $L^\infty(\underline{G})$. Multiplication comes from the convolution product

$$(\phi \cdot \psi)(g) \;=\; \sum_{\substack{g_1, g_2 \in \underline{G} \\ g_2 \circ g_1 = g}} \phi(g_1) \cdot \psi(g_2).$$

An involution of rings on $\mathbb{C}\underline{G}$ is defined by $(\phi^*)(g) := \overline{\phi(i(g))}$. Define the augmentation homomorphism $\epsilon\colon \mathbb{C}\underline{G} \to L^\infty(\underline{G}^0)$ by sending ϕ to $\epsilon(\phi)\colon \underline{G}^0 \to \mathbb{C}, \quad x \mapsto \sum_{g \in s^{-1}(x)} \phi(g)$. Notice that ϵ is in general not a ring homomorphism, it is only compatible with the additive structure. It becomes a homomorphism of $\mathbb{C}\underline{G}$-modules if we equip $L^\infty(\underline{G}^0)$ with the following $\mathbb{C}\underline{G}$-module structure

$$\phi \cdot f := \epsilon(\phi \cdot j(f)) \quad \text{for } \phi \in \mathbb{C}\underline{G}, f \in L^\infty(\underline{G}^0),$$

where $j\colon L^\infty(\underline{G}^0) \to \mathbb{C}\underline{G}$ is the inclusion of rings, which is given by extending a function on $\underline{G}^0$ to $\underline{G}$ by putting it to be zero outside $\underline{G}^0$.

Given a group G and a ring R together with a homomorphism $c\colon G \to \operatorname{aut}(R)$, define the *crossed product ring* $R *_c G$ as the free R-module with G as R-basis and the multiplication given by

$$\left(\sum_{g \in G} r_g \cdot g\right) \cdot \left(\sum_{g \in G} s_g \cdot g\right) \;=\; \sum_{g \in G} \left(\sum_{\substack{g_1, g_2 \in G, \\ g = g_1 g_2}} r_{g_1} \cdot c(g_1)(s_{g_2})\right) \cdot g.$$

Given a standard action $G \curvearrowright X$, let $L^\infty(X) * G$ be the crossed product ring $L^\infty(X) *_c G$ with respect to the group homomorphism $c\colon G \to \operatorname{aut}(L^\infty(X))$ sending g to the automorphism given by composition with $l_{g^{-1}}\colon X \to X, \quad x \mapsto g^{-1}x$. We obtain an injective ring homomorphism

$$k\colon L^\infty(X) * G \to \mathbb{C}\mathcal{R}(G \curvearrowright X)$$

which sends $\sum_{g \in G} f_g \cdot g$ to the function $(gx, x) \;\mapsto\; f_g(gx)$. In the sequel we will regard $L^\infty(X) * G$ as a subring of $\mathbb{C}\mathcal{R}(G \curvearrowright X)$ using k.

Next we briefly explain how one can associate to the groupoid ring $\mathbb{C}\underline{G}$ of a discrete measured groupoid $\underline{G}$ a von Neumann algebra $\mathcal{N}(\underline{G})$, which is finite, or, equivalently, which possesses a faithful finite normal trace. One can define on $\mathbb{C}\underline{G}$ an inner product

$$\langle \phi, \psi \rangle \;=\; \int_{\underline{G}} \phi(g) \cdot \overline{\psi(g)}\; d\mu_{\underline{G}}.$$

Then $\mathbb{C}\underline{G}$ as a $\mathbb{C}$-algebra with involution and the scalar product above satisfies the axioms of a *Hilbert algebra* A, i.e. we have $\langle y, x \rangle = \langle x^*, y^* \rangle$ for $x, y \in A$, $\langle xy, z \rangle = \langle y, x^* z \rangle$ for $x, y, z \in A$ and the map $A \to A,\ y \mapsto yx$ is continuous for all $x \in A$. Let H_A be the Hilbert space completion of A with respect to the given inner product. Define the von Neumann algebra $\mathcal{N}(A)$ associated to

A by the $\mathbb{C}$-algebra with involution $\mathcal{B}(H_A)^A$ which consists of all bounded left A-invariant operators $H_A \to H_A$. The standard trace is given by

$$\operatorname{tr}_{\mathcal{N}(A)} \colon \mathcal{N}(A) \to \mathbb{C}, \quad f \ \mapsto \ \langle f(1_A), 1_A \rangle .$$

We do get a dimension function as in Theorem 1.11 for $\mathcal{N}(A)$.

Our main example will be $\mathcal{N}(G \curvearrowright X) := \mathcal{N}(\mathbb{C}\mathcal{R}(G \curvearrowright X))$ for a standard action $G \curvearrowright X$ of G.

If G is a countable group and $\underline{G} = G$ is the associated discrete measured groupoid with one object, then $\mathbb{C}\underline{G} = \mathbb{C}G$, $l^2(G) = H_{\mathbb{C}\underline{G}}$ and the definition of $\mathcal{N}(\underline{G})$ and $\operatorname{tr}_{\mathcal{N}(\underline{G})}$ above agrees with the previous Definition 1.1 of $\mathcal{N}(G)$ and $\operatorname{tr}_{\mathcal{N}(G)}$.

Remark 8.11 (Summary and Relevance of the algebraic structures associated to a standard action). Let $G \curvearrowright X$ be a standard action. We have the following commutative diagram of inclusions of rings

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}G \arrow[r, "="] \arrow[d] & \mathbb{C}G \arrow[r] \arrow[d] & \mathcal{N}(G) \arrow[d] \\
L^\infty(X) \arrow[r] & L^\infty(X) * G \arrow[r] & \mathbb{C}\mathcal{R}(G \curvearrowright X) \arrow[r] & \mathcal{N}(G \curvearrowright X)
\end{tikzcd}$$

There is a $\mathbb{C}\underline{G}$-module structure on $L^\infty(\mathcal{R}(G \curvearrowright X)^0) = L^\infty(X)$. Its restriction to $L^\infty(X) * G \subseteq \mathbb{C}\mathcal{R}(G \curvearrowright X)$ is the obvious $L^\infty(X) * G$-module structure on $L^\infty(X)$.

The following observation will be crucial. Given two standard actions $G \curvearrowright X$ and $G \curvearrowright Y$, an orbit equivalence f from $G \curvearrowright X$ to $H \curvearrowright Y$ induces isomorphisms of rings, all denoted by f_*, such that the following diagram with inclusions as horizontal maps commutes

$$\begin{tikzcd}
L^\infty(X) \arrow[r] \arrow[d, "f_*"', "\cong"] & \mathbb{C}\mathcal{R}(G \curvearrowright X) \arrow[r] \arrow[d, "f_*"', "\cong"] & \mathcal{N}(G \curvearrowright X) \arrow[d, "f_*"', "\cong"] \\
L^\infty(Y) \arrow[r] & \mathbb{C}\mathcal{R}(H \curvearrowright Y) \arrow[r] & \mathcal{N}(G \curvearrowright Y)
\end{tikzcd}$$

It is *not* true that f induces a ring map $L^\infty(X) * G \to L^\infty(Y) * H$, since we only require that f maps orbits to orbits but nothing is demanded about equivariance of f with respect to some homomorphism of groups from $G \to H$. The crossed product ring $L^\infty(X) * G$ contains too much information about the group G itself. Hence we shall only involve $L^\infty(X)$, $\mathbb{C}\mathcal{R}(G \curvearrowright X)$, and $\mathcal{N}(G \curvearrowright X)$ in any algebraic construction which is designed to be invariant under orbit equivalence.

8.4 L^2-Betti Numbers of Standard Actions

Definition 8.12. *Let $\underline{G}$ be a discrete measured groupoid. Define its p-th* L^2-Betti number *by*

$$b_p^{(2)}(\underline{G}) \ = \ \dim_{\mathcal{N}(\underline{G})} \left(\operatorname{Tor}_p^{\mathbb{C}\underline{G}} \left(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0) \right) \right) .$$

Given a standard action $G \curvearrowright X$, define its p-th L^2-Betti number *as the p-th L^2-Betti number of the associated orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$, i.e.*

$$b_p^{(2)}(G \curvearrowright X) \;=\; \dim_{\mathcal{N}(G\curvearrowright X)}\left(\operatorname{Tor}_p^{\mathbb{C}\mathcal{R}(G\curvearrowright X)}\left(\mathcal{N}(G \curvearrowright X), L^\infty(X)\right)\right).$$

Notice that Theorem 8.2 is true if we can prove the following three lemmas.

Lemma 8.13. *If two standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are orbit equivalent, then they have the same L^2-Betti numbers.*

Lemma 8.14. *Let $\underline{G}$ be a discrete measured groupoid. Let $A \subseteq \underline{G}$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in $\underline{G}^0$. Then we get for all $p \geq 0$*

$$b_p^{(2)}(\underline{G}) \;=\; \mu(A) \cdot b_p^{(2)}(\underline{G}|_A).$$

Lemma 8.15. *Let $G \curvearrowright X$ be a standard action. Then we get for all $p \geq 0$*

$$b_p^{(2)}(G \curvearrowright X) \;=\; b_p^{(2)}(G).$$

Lemma 8.13 follows directly from Remark 8.11. The hard part of the proof of Theorem 8.2 is indeed the proof of the remaining two Lemmas 8.14 and 8.15. This is essentially done by developing some homological algebra over finite von Neumann algebras taking the dimension for arbitrary modules into account.

8.5 Invariance of L^2-Betti Numbers under Orbit Equivalence

As an illustration we sketch the proof of Lemma 8.15. It follows from the following chain of equalities which we explain briefly below.

$$\begin{aligned}
b_p^{(2)}(G) &= \dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_p^{\mathbb{C}G}\left(\mathcal{N}(G), \mathbb{C}\right)\right) && (8.16)\\
&= \dim_{\mathcal{N}(G\curvearrowright X)}\left(\mathcal{N}(G \curvearrowright X) \otimes_{\mathcal{N}(G)} \operatorname{Tor}_p^{\mathbb{C}G}\left(\mathcal{N}(G), \mathbb{C}\right)\right) && (8.17)\\
&= \dim_{\mathcal{N}(G\curvearrowright X)}\left(\operatorname{Tor}_p^{\mathbb{C}G}\left(\mathcal{N}(G \curvearrowright X), \mathbb{C}\right)\right) && (8.18)\\
&= \dim_{\mathcal{N}(G\curvearrowright X)}\left(\operatorname{Tor}_p^{L^\infty(X)*G}\left(\mathcal{N}(G \curvearrowright X), L^\infty(X) * G \otimes_{\mathbb{C}G} \mathbb{C}\right)\right) && (8.19)\\
&= \dim_{\mathcal{N}(G\curvearrowright X)}\left(\operatorname{Tor}_p^{L^\infty(X)*G}\left(\mathcal{N}(G \curvearrowright X), L^\infty(X)\right)\right) && (8.20)\\
&= \dim_{\mathcal{N}(G\curvearrowright X)}\left(\operatorname{Tor}_p^{\mathbb{C}\mathcal{R}(G\curvearrowright X)}\left(\mathcal{N}(G \curvearrowright X), L^\infty(X)\right)\right) && (8.21)\\
&= b_p^{(2)}(G \curvearrowright X). && (8.22)
\end{aligned}$$

Equations (8.16) and (8.22) are true by definition. The inclusion of von Neumann algebras $\mathcal{N}(G) \to \mathcal{N}(G \curvearrowright X)$ preserves the traces. This implies that the functor $\mathcal{N}(G \curvearrowright X) \otimes_{\mathcal{N}(G)} -$ from $\mathcal{N}(G)$-modules to $\mathcal{N}(G \curvearrowright X)$-modules is faithfully flat and preserves dimensions. The proof of this fact is completely analogous to the proof of Theorem 1.18. This shows (8.17) and (8.18). For every $\mathbb{C}G$-module M there is a natural $L^\infty(X) * G$-isomorphism

$$L^\infty(X) * G \otimes_{\mathbb{C}G} M \xrightarrow{\cong} L^\infty(X) \otimes_{\mathbb{C}} M.$$

This shows that $L^\infty(X) * G$ is flat as $\mathbb{C}G$-module and that (8.19) and (8.20) are true. The hard part is now to prove (8.21), which is the decisive step, since here one eliminates $L^\infty(X) * G$ from the picture and stays with terms which depend only on the orbit equivalence class of $G \curvearrowright X$. Its proof involves homological algebra and dimension theory. It is not true that the relevant Tor-terms are isomorphic, they only have the same dimension.

This finishes the outline of the proof of Lemma 8.15 and of Theorem 8.2. The complete proof can be found in [100].

9 The Singer Conjecture

In this section we briefly discuss the following conjecture.

Conjecture 9.1 (Singer Conjecture). *If M is an aspherical closed manifold, then*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \qquad \textit{if } 2p \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2p \neq \dim(M); \\ > 0 & \textit{if } 2p = \dim(M). \end{cases}$$

We mention that all the explicit computations presented in Section 3 are compatible with the Singer Conjecture 9.1. A version of the Singer Conjecture for L^2-torsion will be presented in Conjecture 11.28.

9.1 The Singer Conjecture and the Hopf Conjecture

Because of the Euler-Poincaré formula $\chi(M) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{M})$ (see Theorem 2.7 (x)) the Singer Conjecture 9.1 implies the following conjecture in case M is aspherical or has negative sectional curvature.

Conjecture 9.2 (Hopf Conjecture). *If M is an aspherical closed manifold of even dimension, then*

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$*, then*

$$\begin{array}{rclcrcl} (-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \textit{if} & \sec(M) & < & 0; \\ (-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \textit{if} & \sec(M) & \leq & 0; \\ \chi(M) & = & 0 & \textit{if} & \sec(M) & = & 0; \\ \chi(M) & \geq & 0 & \textit{if} & \sec(M) & \geq & 0; \\ \chi(M) & > & 0 & \textit{if} & \sec(M) & > & 0. \end{array}$$

In original versions of the Singer Conjecture 9.1 and the Hopf Conjecture 9.2 the statements for aspherical manifolds did not appear. Every Riemannian manifold with non-positive sectional curvature is aspherical by Hadamard's Theorem.

9.2 Pinching Conditions

The following two results are taken from the paper by Jost and Xin [59, Theorem 2.1 and Theorem 2.3].

Theorem 9.3. *Let M be a closed connected Riemannian manifold of dimension $\dim(M) \geq 3$. Suppose that there are real numbers $a > 0$ and $b > 0$ such that the sectional curvature satisfies $-a^2 \leq \sec(M) \leq 0$ and the Ricci curvature is bounded from above by $-b^2$. If the non-negative integer p satisfies $2p \neq \dim(M)$ and $2pa \leq b$, then*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0.$$

Theorem 9.4. *Let M be a closed connected Riemannian manifold of dimension $\dim(M) \geq 4$. Suppose that there are real numbers $a > 0$ and $b > 0$ such that the sectional curvature satisfies $-a^2 \leq \sec(M) \leq -b^2$. If the non-negative integer p satisfies $2p \neq \dim(M)$ and $(2p-1) \cdot a \leq (\dim(M) - 2) \cdot b$, then*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0.$$

The next result is a consequence of a result of Ballmann and Brüning [3, Theorem B on page 594].

Theorem 9.5. *Let M be a closed connected Riemannian manifold. Suppose that there are real numbers $a > 0$ and $b > 0$ such that the sectional curvature satisfies $-a^2 \leq \sec(M) \leq -b^2$. If the non-negative integer p satisfies $2p < \dim(M) - 1$ and $p \cdot a < (\dim(M) - 1 - p) \cdot b$, then*

$$b_p^{(2)}(\widetilde{M}) \;=\; 0.$$

Theorem 9.4 and Theorem 9.5 are improvements of the older results by Donnelly and Xavier [26].

Remark 9.6 (Right angled Coxeter groups and Coxeter complexes). Next we mention the work of Davis and Okun [20]. A simplicial complex L is called a *flag complex* if each finite non-empty set of vertices which pairwise are connected by edges spans a simplex of L. To such a flag complex they associate a right-angled Coxeter group W_L defined by the following presentation [20, Definition 5.1]. Generators are the vertices v of L. Each generator v satisfies $v^2 = 1$. If two vertices v and w span an edge, there is the relation $(vw)^2 = 1$. Given a finite flag complex L, Davis and Okun associate to it a finite proper W_L-CW-complex Σ_L, which turns out to be a model for the classifying space of the family of finite subgroups $E_{\mathcal{FIN}}(W_L)$ [20, 6.1, 6.1.1 and 6.1.2]. Equipped with a specific metric, Σ_L turns out to be non-positive curved in a combinatorial sense, namely, it is a CAT(0)-space [20, 6.5.3]. If L is a generalized rational homology $(n-1)$-sphere, i.e. a homology $(n-1)$-manifold with the same rational homology as S^{n-1}, then Σ_L is a polyhedral homology n-manifold with rational coefficients [20, 7.4]. So Σ_L is a reminiscence of the universal covering of a closed n-dimensional manifold with non-positive sectional curvature and fundamental group W_L. In view of the Singer Conjecture 9.1 the conjecture makes sense that $b_p^{(2)}(\Sigma_L;\mathcal{N}(W_L)) = 0$ for $2p \neq n$ provided that the underlying topological space of L is S^{n-1} (or, more generally, that it is a homology $(n-1)$-sphere) [20, Conjecture 0.4 and 8.1]. Davis and Okun show that the conjecture is true in dimension $n \leq 4$ and that it is true in dimension $(n+1)$ if it holds in dimension n and n is odd [20, Theorem 9.3.1 and Theorem 10.4.1].

9.3 The Singer Conjecture and Kähler Manifolds

Definition 9.7. *Let (M,g) be a connected Riemannian manifold. A $(p-1)$-form $\eta \in \Omega^{p-1}(M)$ is* bounded *if $||\eta||_\infty := \sup\{||\eta||_x \mid x \in M\} < \infty$ holds, where $||\eta||_x$ is the norm on $\mathrm{Alt}^{p-1}(T_xM)$ induced by g_x. A p-form $\omega \in \Omega^p(M)$ is called* d(bounded) *if $\omega = d(\eta)$ holds for some bounded $(p-1)$-form $\eta \in \Omega^{p-1}(M)$. A p-form $\omega \in \Omega^p(M)$ is called* $\widetilde{d}$(bounded) *if its lift $\widetilde{\omega} \in \Omega^p(\widetilde{M})$ to the universal covering $\widetilde{M}$ is d(bounded).*

The next definition is taken from [45, 0.3 on page 265].

Definition 9.8 (Kähler hyperbolic manifold). *A* Kähler hyperbolic manifold *is a closed connected Kähler manifold (M,h) whose fundamental form ω is $\widetilde{d}$(bounded).*

Example 9.9 (Examples of Kähler hyperbolic manifolds). The following list of examples of Kähler hyperbolic manifolds is taken from [45, Example 0.3]:

(i) M is a closed Kähler manifold which is homotopy equivalent to a Riemannian manifold with negative sectional curvature;

(ii) M is a closed Kähler manifold such that $\pi_1(M)$ is word-hyperbolic in the sense of [44] and $\pi_2(M) = 0$;

(iii) $\widetilde{M}$ is a symmetric Hermitian space of non-compact type;

(iv) M is a complex submanifold of a Kähler hyperbolic manifold;

(v) M is a product of two Kähler hyperbolic manifolds.

The following result is due to Gromov [44, Theorem 1.2.B and Theorem 1.4.A on page 274]. A detailed discussion of the proof and the consequences of this theorem can also be found in [80, Chapter 11].

Theorem 9.10. (L^2**-Betti numbers of Kähler hyperbolic manifolds).** *Let M be a Kähler hyperbolic manifold of complex dimension m and real dimension $n = 2m$. Then*

$$\begin{array}{rcll} b_p^{(2)}(\widetilde{M}) & = & 0 & \textit{if } p \neq m; \\ b_m^{(2)}(\widetilde{M}) & > & 0; & \\ (-1)^m \cdot \chi(M) & > & 0. & \end{array}$$

10 The Approximation Conjecture

This section is devoted to the following conjecture.

Conjecture 10.1 (Approximation Conjecture). *A group G satisfies the* Approximation Conjecture *if the following holds:*

Let $\{G_i \mid i \in I\}$ be an inverse system of normal subgroups of G directed by inclusion over the directed set I. Suppose that $\bigcap_{i\in I} G_i = \{1\}$. Let X be a G-CW-complex of finite type. Then $G_i\backslash X$ is a G/G_i-CW-complex of finite type and

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; \lim_{i\in I} b_p^{(2)}(G_i\backslash X;\mathcal{N}(G/G_i)).$$

Remark 10.2 (The Approximation Conjecture for subgroups of finite index). Let us consider the special case where the inverse system $\{G_i \mid i \in I\}$ is given by a nested sequence of normal subgroups of finite index

$$G = G_0 \supset G_1 \supset G_2 \supset G_3 \supset \dots .$$

Notice that then $b_p^{(2)}(G_i\backslash X;\mathcal{N}(G/G_i)) = \frac{b_p(G_i\backslash X)}{[G:G_i]}$, where $b_p(G_i\backslash X)$ is the classical p-th Betti number of the finite CW-complex $G_i\backslash X$. In this special case Conjecture 10.1 was formulated by Gromov [46, pages 20, 231] and proved in [72, Theorem 0.1]. Thus we get an asymptotic relation between the L^2-Betti numbers and Betti numbers, namely

$$b_p^{(2)}(X;\mathcal{N}(G)) \;=\; \lim_{i\to\infty} \frac{b_p(G_i\backslash X)}{[G:G_i]},$$

although the Betti numbers of a connected finite CW-complex Y and the L^2-Betti numbers of its universal covering $\widetilde{Y}$ have nothing in common except the fact that their alternating sum equals $\chi(Y)$ (see Example 6.2).

Interesting variations of this result for not necessarily normal subgroups of finite index and Betti-numbers with coefficients in representations can be found in the paper by Farber [34].

Definition 10.3. *Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following operations:*

(i) Amenable quotient
Let $H \subseteq G$ be a (not necessarily normal) subgroup. Suppose that $H \in \mathcal{G}$ and the quotient G/H is an amenable discrete homogeneous space. (For the precise definition of amenable discrete homogeneous space see for instance [80, Definition 13.8]. If $H \subseteq G$ is normal and G/H is amenable, then G/H is an amenable discrete homogeneous space.)

Then $G \in \mathcal{G}$;

(ii) Colimits
If $G = \operatorname{colim}_{i \in I} G_i$ is the colimit of the directed system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{G}$, then G belongs to $\mathcal{G}$;

(iii) Inverse limits
If $G = \lim_{i \in I} G_i$ is the limit of the inverse system $\{G_i \mid i \in I\}$ of groups indexed by the directed set I and each G_i belongs to $\mathcal{G}$, then G belongs to $\mathcal{G}$;

(iv) Subgroups
If H is isomorphic to a subgroup of the group G with $G \in \mathcal{G}$, then $H \in \mathcal{G}$;

(v) Quotients with finite kernel
Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. If K is finite and G belongs to $\mathcal{G}$, then Q belongs to $\mathcal{G}$.

Next we provide some information about the class $\mathcal{G}$. Notice that in the original definition of $\mathcal{G}$ due to Schick [102, Definition 1.12] the resulting class is slightly smaller: there it is required that the class contains the trivial subgroup and is closed under operations (i), (ii), (iii) and (iv), but not necessarily under operation (v). The proof of the next lemma can be found in [80, Lemma 13.11].

Lemma 10.4. *(i) A group G belongs to $\mathcal{G}$ if and only if every finitely generated subgroup of G belongs to $\mathcal{G}$;*

(ii) The class $\mathcal{G}$ is residually closed, i.e. if there is a nested sequence of normal subgroups $G = G_0 \supset G_1 \supset G_2 \supset \ldots$ such that $\bigcap_{i \geq 0} G_i = \{1\}$ and each quotient G/G_i belongs to $\mathcal{G}$, then G belongs to $\mathcal{G}$;

(iii) Any residually amenable and in particular any residually finite group belongs to $\mathcal{G}$;

(iv) Suppose that G belongs to $\mathcal{G}$ and $f \colon G \to G$ is an endomorphism. Define the "mapping torus group" G_f to be the quotient of $G * \mathbb{Z}$ obtained by introducing the relations $t^{-1}gt = f(g)$ for $g \in G$ and $t \in \mathbb{Z}$ a fixed generator. Then G_f belongs to $\mathcal{G}$;

(v) Let $\{G_j \mid j \in J\}$ be a set of groups with $G_j \in \mathcal{G}$. Then the direct sum $\bigoplus_{j \in J} G_j$ and the direct product $\prod_{j \in J} G_j$ belong to $\mathcal{G}$.

The proof of the next result can be found in [80, Theorem 13.3]. It is a mild generalization of the results of Schick [101] and [102], where the original proof of the Approximation Conjecture for subgroups of finite index was generalized to the much more general setting above and then applied to the Atiyah Conjecture. The connection between the Approximation Conjecture and the Atiyah Conjecture for torsion-free groups comes from the obvious fact that a convergent series of integers has an integer as limit.

Theorem 10.5 (Status of the Approximation Conjecture). *Every group G which belongs to the class $\mathcal{G}$ (see Definition 10.3) satisfies the Approximation Conjecture 10.1.*

11 L^2-Torsion

Recall that L^2-Betti numbers are modelled on Betti numbers. Analogously one can generalize the classical notion of Reidemeister torsion to an L^2-setting, which will lead to the notion of L^2-torsion. The L^2-torsion may be viewed as a secondary L^2-Betti number just as the Reidemeister torsion can be viewed as a secondary Betti number. Namely, the Reidemeister torsion is only defined if all the Betti numbers (with coefficients in a suitable representation) vanish, and similarly the L^2-torsion is defined only if the L^2-Betti numbers vanish. Both invariants give valuable information about the spaces in question.

11.1 The Fuglede-Kadison Determinant

In this subsection we briefly explain the notion of the Fuglede-Kadison determinant. We have extended the notion of the (classical) dimension of a finite dimensional complex vector space to the von Neumann dimension of a finitely generated projective $\mathcal{N}(G)$-module (and later even to arbitrary $\mathcal{N}(G)$-modules). Similarly we want to generalize the classical determinant of an endomorphism of a finite dimensional complex vector space to the Fuglede-Kadison determinant of an $\mathcal{N}(G)$-endomorphism $f\colon P \to P$ of a finitely generated projective $\mathcal{N}(G)$-module P and of an $\mathcal{N}(G)$-map $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ of based finitely generated $\mathcal{N}(G)$-modules. This is necessary since for the definition of Reidemeister torsion one needs determinants and hence for the definition of L^2-torsion one has to develop an appropriate L^2-analogue.

Definition 11.1 (Spectral density function). *Let $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ be an $\mathcal{N}(G)$-homomorphism. Let $\nu(f)\colon l^2(G)^m \to l^2(G)^n$ be the associated bounded G-equivariant operator (see Remark 1.7). Denote by $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator $\nu(f^*f)$. Define the* spectral density function of f *by*

$$F_f\colon \mathbb{R} \to [0,\infty) \quad \lambda \mapsto \dim_{\mathcal{N}(G)}\left(\operatorname{im}(E_{\lambda^2}^{f^*f})\right).$$

The spectral density function is monotone and right-continuous. It takes values in $[0, m]$. Here and in the sequel $|x|$ denotes the norm of an element x of a Hilbert space and $\|T\|$ the operator norm of a bounded operator T. Since $\nu(f)$ and $\nu(f^*f)$ have the same kernel, $\dim_{\mathcal{N}(G)}(\ker(f)) = F_f(0)$.

Example 11.2 (Spectral density function for finite G). Suppose that G is finite. Then $\mathbb{C}G = \mathcal{N}(G) = l^2(G)$ and $\nu(f) = f$. Let $0 \leq \lambda_0 < \ldots < \lambda_r$ be the eigenvalues of f^*f and μ_i be the multiplicity of λ_i, i.e. the dimension of the eigenspace of λ_i. Then the spectral density function is a right continuous step function which is zero for $\lambda < 0$ and has a step of height $\frac{\mu_i}{|G|}$ at each $\sqrt{\lambda_i}$.

Example 11.3 (Spectral density function for $G = \mathbb{Z}^n$). Let $G = \mathbb{Z}^n$. We use the identification $\mathcal{N}(\mathbb{Z}^n) = L^\infty(T^n)$ of Example 1.4. For $f \in L^\infty(T^n)$ the spectral density function F_{M_f} of $M_f \colon L^2(T^n) \to L^2(T^n),\ g \mapsto g \cdot f$ sends λ to the volume of the set $\{z \in T^n \mid |f(z)| \leq \lambda\}$.

Definition 11.4. (Fuglede-Kadison determinant of $\mathcal{N}(G)$-maps $\mathcal{N}(G)^m \to \mathcal{N}(G)^n$). *Let $f \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ be an $\mathcal{N}(G)$-map. Let $F_f(\lambda)$ be the spectral density function of Definition 11.1 which is a monotone non-decreasing right-continuous function. Let dF be the unique measure on the Borel σ-algebra on $\mathbb{R}$ which satisfies $dF(]a, b]) = F(b) - F(a)$ for $a < b$. Then define the* Fuglede-Kadison determinant

$$\det_{\mathcal{N}(G)}(f) \in [0, \infty)$$

by the positive real number

$$\det_{\mathcal{N}(G)}(f) \;=\; \exp\left(\int_{0+}^{\infty} \ln(\lambda)\; dF\right)$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda)\; dF$ converges to a real number and by 0 otherwise.

Notice that in the definition above we do not require $m = n$ or that f is injective or f is surjective.

Example 11.5 (Fuglede-Kadison determinant for finite G). To illustrate this definition, we look at the example where G is finite. We essentially get the classical determinant $\det_{\mathbb{C}}$. Namely, we have computed the spectral density function for finite G in Example 11.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the non-zero eigenvalues of f^*f with multiplicity μ_i. Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f,

$$\det_{\mathcal{N}(G)}(f) \;=\; \exp\left(\sum_{i=1}^{r} \frac{\mu_i}{|G|} \cdot \ln(\sqrt{\lambda_i})\right) \;=\; \prod_{i=1}^{r} \lambda_i^{\frac{\mu_i}{2\cdot|G|}} \;=\; \left(\det_{\mathbb{C}}\left(\overline{f^*f}\right)\right)^{\frac{1}{2\cdot|G|}}.$$

If $f \colon \mathbb{C}G^m \to \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{N}(G)}(f) \;=\; |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}.$$

Example 11.6 (Fuglede-Kadison determinant over $\mathcal{N}(\mathbb{Z}^n)$). Let $G = \mathbb{Z}^n$. We use the identification $\mathcal{N}(\mathbb{Z}^n) = L^\infty(T^n)$ of Example 1.4. For $f \in L^\infty(T^n)$ we conclude from Example 11.3

$$\det_{\mathcal{N}(\mathbb{Z}^n)}\big(M_f\colon L^2(T^n) \to L^2(T^n)\big) = \exp\left(\int_{T^n} \ln(|f(z)|)\cdot \chi_{\{u\in S^1 \mid f(u)\neq 0\}}\, dvol_z\right)$$

using the convention $\exp(-\infty) = 0$.

Here are some basic properties of this notion. A morphism $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ has dense image if the closure $\overline{\operatorname{im}(f)}$ of its image in $\mathcal{N}(G)^n$ in the sense of Definition 1.10 is $\mathcal{N}(G)^n$. The adjoint A^* of a matrix $A = (a_{i,j}) \in M(m,n;\mathcal{N}(G))$ is the matrix in $M(n,m;\mathcal{N}(G))$ given by $(a^*_{j,i})$, where $*\colon \mathcal{N}(G) \to \mathcal{N}(G)$ sends an operator a to its adjoint a^*. The adjoint $f^*\colon \mathcal{N}(G)^n \to \mathcal{N}(G)^m$ of $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ is given by the matrix A^* if f is given by the matrix A. The proof of the next result can be found in [80, Theorem 3.14].

Theorem 11.7 (Fuglede-Kadison determinant).

(i) *Composition*
Let $f\colon \mathcal{N}(G)^l \to \mathcal{N}(G)^m$ and $g\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ be $\mathcal{N}(G)$-homomorphisms such that f has dense image and g is injective. Then

$$\det_{\mathcal{N}(G)}(g \circ f) \;=\; \det_{\mathcal{N}(G)}(g) \cdot \det_{\mathcal{N}(G)}(f);$$

(ii) *Additivity*
Let $f_1\colon \mathcal{N}(G)^{m_1} \to \mathcal{N}(G)^{n_1}$, $f_2\colon \mathcal{N}(G)^{m_2} \to \mathcal{N}(G)^{n_2}$ and $f_3\colon \mathcal{N}(G)^{m_3} \to \mathcal{N}(G)^{n_3}$ be $\mathcal{N}(G)$-homomorphisms such that f_1 has dense image and f_2 is injective. Then

$$\det_{\mathcal{N}(G)}\begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} \;=\; \det_{\mathcal{N}(G)}(f_1)\cdot \det_{\mathcal{N}(G)}(f_2);$$

(iii) *Invariance under adjoint map*

Let $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ be an $\mathcal{N}(G)$-homomorphism. Then

$$\det_{\mathcal{N}(G)}(f) = \det_{\mathcal{N}(G)}(f^*);$$

(iv) *Induction*
Let $i\colon H \to G$ be an injective group homomorphism and let $f\colon \mathcal{N}(H)^m \to \mathcal{N}(H)^n$ be an$\mathcal{N}(H)$-homomorphism. Then

$$\det_{\mathcal{N}(G)}(i_* f) \;=\; \det_{\mathcal{N}(H)}(f).$$

Definition 11.8. (Fuglede-Kadison determinant of $\mathcal{N}(G)$-endomorphisms of finitely generated projective modules). *Let $f: P \to P$ be an endomorphism of a finitely generated projective $\mathcal{N}(G)$-module P. Choose a finitely generated projective $\mathcal{N}(G)$-module Q and an $\mathcal{N}(G)$-isomorphism $u: \mathcal{N}(G)^n \xrightarrow{\cong} P \oplus Q$. Define the* Fuglede-Kadison determinant

$$\det_{\mathcal{N}(G)}(f) \ \in \ [0, \infty)$$

by the Fuglede-Kadison determinant in the sense of Definition 11.4

$$\det_{\mathcal{N}(G)} \left(u^{-1} \circ (f \oplus \mathrm{id}_Q) \circ u\right).$$

This definition is independent of the choices of Q and u by Theorem 11.7. Notice that in Definition 11.8 no $\mathcal{N}(G)$-basis appear but that it works only for endomorphisms, whereas in Definition 11.4 we work with finitely generated free based modules but do not require that the source and target of f are isomorphic. There is an obvious analogue of Theorem 11.7 for the Fuglede-Kadison determinant of endomorphisms of finitely generated projective $\mathcal{N}(G)$-modules.

11.2 The Determinant Conjecture

It will be important for applications to geometry to study the Fuglede-Kadison determinant of $\mathcal{N}(G)$-maps $f: \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ which come by induction from $\mathbb{Z}G$-maps or $\mathbb{C}G$-maps. The following example is taken from [80, Example 3.22].

Example 11.9 (Fuglede-Kadison determinant of maps coming from elements in $\mathbb{C}[\mathbb{Z}]$). Consider a non-trivial element $p \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$. We can write

$$p(z) \ = \ C \cdot z^n \cdot \prod_{k=1}^{l} (z - a_k)$$

for non-zero complex numbers $C, a_1, \ldots, a_l$ and non-negative integers n, l. Let $r_p: \mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z})$ be the $\mathcal{N}(\mathbb{Z})$-map given by right multiplication with p. Then

$$\det_{\mathcal{N}(\mathbb{Z})}(r_p) \quad = \quad |C| \cdot \prod_{\substack{1 \le k \le l, \\ |a_k| > 1}} |a_k|.$$

Definition 11.10 (Determinant class). *A group G is of* $\det \geq 1$-class *if for each $A \in M(m, n; \mathbb{Z}G)$ the Fuglede-Kadison determinant (see Definition 11.4) of the morphism $r_A: \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A satisfies*

$$\det_{\mathcal{N}(G)}(r_A) \geq 1.$$

Conjecture 11.11 (Determinant Conjecture). *Every group G is of* $\det \geq 1$*-class.*

The proof of the next result can be found in [80, Theorem 13.3]. It is a mild generalization of the results of Schick [101] and [102].

Theorem 11.12 (Status of the Determinant Conjecture). *Every group G which belongs to the class $\mathcal{G}$ (see Definition 10.3) satisfies the Determinant Conjecture 11.11.*

One easily checks that the Fuglede-Kadison determinant defines a homomorphism of abelian groups

$$\Phi^G : \mathrm{Wh}(G) \quad \to \quad (0,\infty) = \{r \in \mathbb{R} \mid r > 0\} \tag{11.13}$$

with respect to the group structure given by multiplication of positive real numbers on the target. We mention the following conjecture.

Conjecture 11.14 (Triviality of the map induced by the Fuglede-Kadison determinant on $\mathrm{Wh}(G)$). *The map $\Phi^G\colon \mathrm{Wh}(G) \to (0,\infty)$ is trivial.*

Lemma 11.15. *(i) If G satisfies the Determinant Conjecture 11.11, then G satisfies Conjecture 11.14;*

(ii) The Approximation Conjecture 10.1 for G and the inverse system $\{G_i \mid i \in I\}$ is true if each group G_i is of $\det \geq 1$-class.

Proof. See [80, Theorem 13.3 (1) and Lemma 13.6]. ☐

11.3 Definition and Basic Properties of L^2-Torsion

We will consider L^2-torsion only for universal coverings and in the L^2-acyclic case. A more general setting is treated in [80, Section 3.4].

Definition 11.16 ($\det$-L^2-acyclic). *Let X be a finite connected CW-complex with fundamental group $\pi = \pi_1(X)$. Let $C_*^{\mathcal{N}}(\widetilde{X})$ be the $\mathcal{N}(\pi)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X})$ with p-th differential $c_p^{\mathcal{N}} = \mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{Z}} c_p$. We say that X is* $\det$-L^2-acyclic *if for each p we get for the Fuglede-Kadison determinant of $c_p^{\mathcal{N}}$ and for the p-th L^2-Betti number of $\widetilde{X}$*

$$\begin{aligned} \det_{\mathcal{N}(\pi)}\left(c_p^{\mathcal{N}}\right) &> 0; \\ b_p^{(2)}(\widetilde{X}) &= 0. \end{aligned}$$

*If X is $\det$-L^2-acyclic, we define the L^2-*torsion *of $\widetilde{X}$ by*

$$\rho^{(2)}(\widetilde{X}) \;=\; -\sum_{p \geq 0} (-1)^p \cdot \ln\left(\det_{\mathcal{N}(G)}\left(c_p^{\mathcal{N}}\right)\right) \quad \in \mathbb{R}.$$

If X is a finite CW-complex, we call it $\det$-L^2-acyclic if each component C is $\det$-L^2-acyclic. In this case we define

$$\rho^{(2)}(\widetilde{X}) \;:=\; \sum_{C \in \pi_0(X)} \rho^{(2)}(\widetilde{C}).$$

The condition that $\widetilde{X}$ is L^2-acyclic is not needed for the definition of L^2-torsion, but is necessary to ensure the basic and useful properties which we will discuss below.

Remark 11.17 (L^2-torsion in terms of the Laplacian). One can express the L^2-torsion also in terms of the Laplacian which is closer to the notions of analytic torsion and analytic L^2-torsion. After a choice of cellular $\mathbb{Z}\pi$-basis, every $\mathcal{N}(\pi)$-chain module $C_p^{\mathcal{N}}(\widetilde{X})$ looks like $\mathcal{N}(\pi)^{n_p}$ for appropriate non-negative integers n_p. Hence we can assign to $c_p^{\mathcal{N}}: C_p^{\mathcal{N}}(\widetilde{X}) \to C_{p-1}^{\mathcal{N}}(\widetilde{X})$ its adjoint $\left(c_p^{\mathcal{N}}\right)^*: C_{p-1}^{\mathcal{N}}(\widetilde{X}) \to C_p^{\mathcal{N}}(\widetilde{X})$ which is given by the matrix $(a_{j,i}^*)$ if $c_p^{(2)}$ is given by the matrix $(a_{i,j})$. Define the p-th Laplace homomorphism $\Delta_p\colon C_p^{\mathcal{N}}(\widetilde{X}) \to C_p^{\mathcal{N}}(\widetilde{X})$ to be the $\mathcal{N}(\pi)$-homomorphism $\left(c_p^{\mathcal{N}}\right)^* \circ c_p^{\mathcal{N}} + c_{p+1}^{\mathcal{N}} \circ \left(c_{p+1}^{\mathcal{N}}\right)^*$. Then $\widetilde{X}$ is det-L^2-acyclic if and only if Δ_p is injective and has dense image, i.e. the closure of its image in $C_p^{\mathcal{N}}(\widetilde{X})$ is $C_p^{\mathcal{N}}(\widetilde{X})$, and $\det_{\mathcal{N}(\pi)}(\Delta_p) > 0$. In this case we get

$$\rho^{(2)}(\widetilde{X}) \;=\; -\frac{1}{2} \cdot \sum_{p \geq 0} (-1)^p \cdot p \cdot \ln\left(\det_{\mathcal{N}(\pi)}(\Delta_p)\right).$$

This follows from [80, Lemma 3.30].

The next theorem presents the basic properties of $\rho^{(2)}(\widetilde{X})$ and is proved in [80, Theorem 3.96]. Notice the formal analogy between the behaviour of $\rho^{(2)}(\widetilde{X})$ and the classical Euler characteristic $\chi(X)$.

Theorem 11.18. (Cellular L^2-torsion for universal coverings).

(i) *Homotopy invariance*

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Let $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ be its Whitehead torsion (see [17]). Suppose that $\widetilde{X}$ or $\widetilde{Y}$ is det-L^2*-acyclic. Then both $\widetilde{X}$ and $\widetilde{Y}$ are* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X}) \;=\; \Phi^{\pi_1(Y)}(\tau(f)),$$

where $\Phi^{\pi_1(Y)}\colon \mathrm{Wh}(\pi_1(Y)) = \bigoplus_{C \in \pi_0(Y)} \mathrm{Wh}(\pi_1(C)) \to \mathbb{R}$ is the sum of the maps $\Phi^{\pi_1(C)}$ of (11.13)*;*

(ii) *Sum formula*

Consider the pushout of finite CW-complexes such that j_1 is an inclusion of CW-complexes, j_2 is cellular and X inherits its CW-complex structure from X_0, X_1 and X_2

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ X_2 & \xrightarrow[i_2]{} & X \end{array}$$

Assume $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are det-L^2*-acyclic and that for $k = 0, 1, 2$ the map $\pi_1(i_k)\colon \pi_1(X_k) \to \pi_1(X)$ induced by the obvious map $i_k\colon X_k \to X$ is injective for all base points in X_k.*

Then $\widetilde{X}$ is det-L^2*-acyclic and we get*

$$\rho^{(2)}(\widetilde{X}) \;=\; \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0});$$

(iii) *Poincaré duality*

Let M be a closed manifold of even dimension. Equip it with some CW-complex structure. Suppose that $\widetilde{M}$ is det-L^2*-acyclic. Then*

$$\rho^{(2)}(\widetilde{M}) \;=\; 0;$$

(iv) *Product formula*

Let X and Y be finite CW-complexes. Suppose that $\widetilde{X}$ is det-L^2*-acyclic. Then $\widetilde{X \times Y}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{X \times Y}) \;=\; \chi(Y) \cdot \rho^{(2)}(\widetilde{X});$$

(v) *Multiplicativity*

Let $X \to Y$ be a finite covering of finite CW-complexes with d sheets. Then $\widetilde{X}$ is det-L^2*-acyclic if and only if $\widetilde{Y}$ is* det-L^2*-acyclic and in this case*

$$\rho^{(2)}(\widetilde{X}) \;=\; d \cdot \rho^{(2)}(\widetilde{Y}).$$

The next three results are taken from [80, Corollary 3.103, Theorem 3.105 and Theorem 3.111]. There is also a more general version of Theorem 11.19 for fibrations (see [80, Theorem 3.100]).

Theorem 11.19 (L^2-torsion and fiber bundles). *Suppose that $F \to E \xrightarrow{p} B$ is a (locally trivial) fiber bundle of finite CW-complexes with B connected. Suppose that for one (and hence all) $b \in B$ the inclusion of the fiber F_b into E induces an injection on the fundamental groups for all base points in F_b and $\widetilde{F_b}$ is* det-L^2*-acyclic. Then $\widetilde{E}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{E}) \;=\; \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

Theorem 11.20 (L^2-torsion and S^1-actions). *Let X be a connected S^1-CW-complex of finite type. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Then $\widetilde{X}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{X}) \;=\; 0.$$

Theorem 11.21 (L^2-torsion on aspherical closed S^1-manifolds). *Let M be an aspherical closed manifold with non-trivial S^1-action. Then the action has no fixed points and the inclusion of any orbit into M induces an injection on the fundamental groups. Moreover, $\widetilde{M}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{M}) \quad = \quad 0.$$

The assertion for the L^2-torsion in the theorem below is the main result of [106] (see also [107]). Its proof is based on localization techniques.

Theorem 11.22 (L^2-torsion and aspherical CW-complexes). *Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup H and $\pi_1(X)$ is of* det ≥ 1*-class. Then $\widetilde{X}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{X}) \quad = \quad 0.$$

Remark 11.23 (Homotopy invariance of L^2-torsion). Notice that Conjecture 11.14 implies because of Theorem 11.18 (i) the homotopy invariance of the L^2-torsion. i.e. for two homotopy equivalent det-L^2-acyclic finite CW-complexes X and Y we have $\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{Y})$.

11.4 Computations of L^2-Torsion

Remark 11.24 (Analytic L^2-torsion). It is important to know for the following specific calculations that there is an analytic version of L^2-torsion in terms of the heat kernel due to Lott [66] and Mathai [86] and that a deep result of Burghelea, Friedlander, Kappeler and McDonald [9] says that the analytic one agrees with the one presented here.

The following result is due to Hess and Schick [51].

Theorem 11.25 (Analytic L^2-torsion of hyperbolic manifolds).
Let $d = 2n+1$ be an odd integer. To d one can associate an explicit real number $C_d > 0$ with the following property:

For every closed hyperbolic d-dimensional manifold M we have

$$\rho^{(2)}(\widetilde{M}) \;=\; (-1)^n \cdot C_d \cdot \mathrm{vol}(M),$$

where $\mathrm{vol}(M)$ *is the volume of M.*

The existence of a real number C_d with $\rho^{(2)}(\widetilde{M}) \;=\; (-1)^n \cdot C_d \cdot \mathrm{vol}(M)$ follows from the version of the Proportionality Principle for L^2-Betti numbers (see Theorem 3.7) for L^2-torsion (see [80, Theorem 3.183]). The point is that this number C_d is given explicitly. For instance $C_3 = \frac{1}{6\pi}$ and $C_5 = \frac{31}{45\pi^2}$. For each odd d there exists a rational number r_d such that $C_d = \pi^{-n} \cdot r_d$ holds. The proof of this result is based on calculations involving the heat kernel on hyperbolic space.

Remark 11.26 (L^2-torsion of symmetric spaces of non-compact type). More generally, the L^2-torsion $\rho^{(2)}(\widetilde{M})$ for an aspherical closed manifold M whose universal covering $\widetilde{M}$ is a symmetric space is computed by Olbricht [93].

The following result is proved in [84, Theorem 0.6].

Theorem 11.27 (L^2-torsion of 3-manifolds). *Let M be a compact connected orientable prime 3-manifold with infinite fundamental group such that the boundary of M is empty or a disjoint union of incompressible tori. Suppose that M satisfies Thurston's Geometrization Conjecture, i.e. there is a geometric toral splitting along disjoint incompressible 2-sided tori in M whose pieces are Seifert manifolds or hyperbolic manifolds. Let M_1, M_2, ..., M_r be the hyperbolic pieces. They all have finite volume [90, Theorem B on page 52]. Then $\widetilde{M}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{M}) \;=\; -\frac{1}{6\pi} \cdot \sum_{i=1}^{r} \operatorname{vol}(M_i).$$

In particular, $\rho^{(2)}(\widetilde{M})$ is 0 if and only if there are no hyperbolic pieces.

11.5 Some Open Conjectures about L^2-Torsion

All the computations and results above give evidence and are compatible with the following conjectures about L^2-torsion taken from [80, Theorem 11.3].

Conjecture 11.28 (L^2-torsion for aspherical manifolds). *If M is an aspherical closed manifold of odd dimension, then $\widetilde{M}$ is* det-L^2*-acyclic and*

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) \;\geq\; 0.$$

If M is a closed connected Riemannian manifold of odd dimension with negative sectional curvature, then $\widetilde{M}$ is det-L^2*-acyclic and*

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) \;>\; 0.$$

If M is an aspherical closed manifold whose fundamental group contains an amenable infinite normal subgroup, then $\widetilde{M}$ is det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{M}) \;=\; 0.$$

Consider a closed orientable manifold M of dimension n. Let $[M;\mathbb{R}]$ be the image of the fundamental class $[M] \in H_n^{\mathrm{sing}}(M;\mathbb{Z})$ under the change of coefficient map $H_n^{\mathrm{sing}}(M;\mathbb{Z}) \to H_n^{\mathrm{sing}}(M;\mathbb{R})$. Define the L^1-norm on $C_n^{\mathrm{sing}}(M;\mathbb{R})$ by sending $\sum_{i=1}^{s} r_i \cdot [\sigma_i \colon \Delta_n \to M]$ to $\sum_{i=1}^{s} |r_i|$. It induces a seminorm on $H_n(M;\mathbb{R})$. Define the *simplicial volume* $||M|| \in \mathbb{R}$ to be the seminorm of $[M;\mathbb{R}]$. More information about the simplicial volume can be found for instance in [42], [47] and [57], and in [80, Chapter 14], where also the following conjecture is discussed.

Conjecture 11.29 (Simplicial volume and L^2-invariants). *Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then $\widetilde{M}$ is* det-L^2*-acyclic and*

$$\rho^{(2)}(\widetilde{M}) \quad = \quad 0.$$

The simplicial volume is a special invariant concerning bounded cohomology. The point of this conjecture is that it suggests a connection between bounded cohomology and L^2-invariants such as L^2-cohomology and L^2-torsion.

We have already seen that L^2-Betti numbers are up to scaling invariant under measure equivalence. The next conjecture is interesting because it would give a sharper invariant in case all the L^2-Betti numbers vanish, namely the vanishing of the L^2-torsion.

Conjecture 11.30 (Measure equivalence and L^2-torsion). *Let G_i for $i = 0, 1$ be a group such that there is a finite CW-model for BG_i and EG_i is* det-L^2*-acyclic. Suppose that G_0 and G_1 are measure equivalent. Then*

$$\rho^{(2)}(EG_0;\mathcal{N}(G_0)) = 0 \;\Leftrightarrow\; \rho^{(2)}(EG_1;\mathcal{N}(G_1)) = 0.$$

11.6 L^2-Torsion of Group Automorphisms

In this section we explain that for a group automorphism $f\colon G \to G$ the L^2-torsion applied to the $(G \rtimes_f \mathbb{Z})$-CW-complex $E(G \rtimes_f \mathbb{Z})$ gives an interesting new invariant, provided that G is of det ≥ 1-class and satisfies certain finiteness assumptions. It seems to be worthwhile to investigate it further. The following definition and theorem are taken from [80, Definition 7.26 and Theorem 7.27].

Definition 11.31 (L^2-torsion of group automorphisms). *Let $f\colon G \to G$ be a group automorphism. Suppose that there is a finite CW-model for BG and G is of* det ≥ 1*-class. Define the* L^2-torsion *of f by*

$$\rho^{(2)}(f\colon G \to G) \;:= \rho^{(2)}(B\widetilde{(G \rtimes_f}\,\mathbb{Z})) \qquad \in\; \mathbb{R}.$$

Next we present the basic properties of this invariant. Notice that its behaviour is similar to the Euler characteristic $\chi(G) := \chi(BG)$.

Theorem 11.32. *Suppose that all groups appearing below have finite CW-models for their classifying spaces and are of* det ≥ 1*-class.*

(i) Suppose that G is the amalgamated product $G_1 *_{G_0} G_2$ for subgroups $G_i \subseteq G$ and the automorphism $f\colon G \to G$ is the amalgamated product $f_1 *_{f_0} f_2$ for automorphisms $f_i\colon G_i \to G_i$. Then

$$\rho^{(2)}(f) \;=\; \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0);$$

(ii) Let $f\colon G \to H$ and $g\colon H \to G$ be isomorphisms of groups. Then

$$\rho^{(2)}(f \circ g) \;=\; \rho^{(2)}(g \circ f).$$

In particular $\rho^{(2)}(f)$ is invariant under conjugation with automorphisms;

(iii) Suppose that the following diagram of groups

$$\begin{tikzcd} 1 \arrow[r] & G_1 \arrow[r] \arrow[d, "f_1"] & G_2 \arrow[r] \arrow[d, "f_2"] & G_3 \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & G_3 \arrow[r] & 1 \end{tikzcd}$$

commutes, has exact rows and its vertical arrows are automorphisms. Then

$$\rho^{(2)}(f_2) \;=\; \chi(BG_3) \cdot \rho^{(2)}(f_1);$$

(iv) Let $f\colon G \to G$ be an automorphism of a group. Then for all integers $n \geq 1$

$$\rho^{(2)}(f^n) \;=\; n \cdot \rho^{(2)}(f);$$

(v) Suppose that G contains a subgroup G_0 of finite index $[G : G_0]$. Let $f\colon G \to G$ be an automorphism with $f(G_0) = G_0$. Then

$$\rho^{(2)}(f) \;=\; \frac{1}{[G : G_0]} \cdot \rho^{(2)}(f|_{G_0});$$

(vi) We have $\rho^{(2)}(f) = 0$ if G satisfies one of the following conditions:

(a) All the L^2-Betti numbers of G vanish;

(b) G contains an amenable infinite normal subgroup.

Example 11.33 (Automorphisms of surfaces). Using Theorem 11.27 one can compute the L^2-torsion of the automorphism $\pi_1(f)$ for an automorphism $f\colon S \to S$ of a compact connected orientable surface, possibly with boundary. Suppose that f is irreducible. Then the following statements are equivalent: i.) f is pseudo-Anosov, ii.) The mapping torus T_f has a hyperbolic structure and iii.) $\rho^{(2)}(\pi_1(f)) < 0$. Moreover, f is periodic if and only if $\rho^{(2)}(\pi_1(f)) = 0$. (see [80, Subsection 7.4.2]).

The L^2-torsion of a Dehn twist is always zero since the associated mapping torus contains no hyperbolic pieces in his Jaco-Shalen-Johannson-Thurston splitting.

Remark 11.34 (Weaker finiteness conditions). The definition of the L^2-torsion of a group automorphism above still makes sends and has still most of the properties above, if one weakens the condition that there is a finite model for BG to the assumption that there is a finite model for the classifying space of proper G-actions $\underline{E}G = E_{\mathcal{FIN}}(G)$. This is explained in [80, Subsection 7.4.4].

12 Novikov-Shubin Invariants

In this section we briefly discuss Novikov-Shubin invariants. They were originally defined in terms of heat kernels. We will focus on their algebraic definition and aspects.

12.1 Definition of Novikov-Shubin Invariants

Let M be a finitely presented $\mathcal{N}(G)$-module. Choose some exact sequence $\mathcal{N}(G)^m \xrightarrow{f} \mathcal{N}(G)^n \to M \to 0$. Let F_f be the spectral density function of f (see Definition 11.1). Recall that F_f is a monotone increasing right continuous function $[0,\infty) \to [0,\infty)$. Define the *Novikov-Shubin invariant* of M by

$$\alpha(M) \;=\; \liminf_{\lambda \to 0^+} \frac{\ln(F_f(\lambda) - F_f(0))}{\ln(\lambda)} \;\in [0,\infty],$$

provided that $F_f(\lambda) > F_f(0)$ holds for all $\lambda > 0$. Otherwise, one puts formally

$$\alpha(M) \;=\; \infty^+.$$

It measures how fast $F_f(\lambda)$ approaches $F_f(0)$ for $\lambda \to 0^+$. For instance, if $F_f(\lambda) = \lambda^\alpha$ for $\lambda > 0$, then $\alpha(M) = \alpha$. The proof that $\alpha(M)$ is independent of the choice of f is analogous to the proof of [80, Theorem 2.55 (1)].

Definition 12.1 (Novikov-Shubin invariants). *Let X be a G-CW-complex of finite type. Define its p-th* Novikov-Shubin invariant *by*

$$\alpha_p(X;\mathcal{N}(G)) \;=\; \alpha\left(H_{p-1}^{(2)}(X;\mathcal{N}(G))\right) \quad \in [0,\infty] \amalg \{\infty^+\}.$$

If the group G is clear from the context, we abbreviate $\alpha_p(X) \;=\; \alpha_p(X;\mathcal{N}(G))$.

Notice that $H_{p-1}^{(2)}(X;\mathcal{N}(G))$ is finitely presented since $\mathcal{N}(G)$ is semihereditary (see Theorem 1.6) and $C_k^{(2)}(X)$ is a finitely generated free $\mathcal{N}(G)$-module for all $k \in \mathbb{Z}$ because X is by assumption of finite type.

Remark 12.2 (Analytic definition of Novikov-Shubin invariants). Novikov-Shubin invariants were originally analytically defined by Novikov and Shubin (see [91], [92]). For a cocompact smooth G-manifold M without boundary and with G-invariant Riemannian metric one can assign to its p-th Laplace operator Δ_p a density function $F_{\Delta_p}(\lambda) = \operatorname{tr}_{\mathcal{N}(G)}(E_\lambda)$ for $\{E_\lambda \mid \lambda \in [0,\infty)\}$ the spectral family associated to the essentially selfadjoint operator Δ_p. Define $\alpha_p^\Delta(M;\mathcal{N}(G)) \in [0,\infty] \amalg \{\infty^+\}$ by the same expression as appearing in the definition of $\alpha(M)$ above, only replace F_f by F_{Δ_p}. Then $\alpha_p^\Delta(M)$ agrees with $\frac{1}{2} \cdot \min\{\alpha_p(K), \alpha_{p+1}(K)\}$ for any equivariant triangulation K of M. For a proof of this equality see [31] or [80, Section 2.4]. One can define the analytic Novikov-Shubin invariant $\alpha_p^\Delta(M;\mathcal{N}(G))$ also in terms of heat kernels. It measures how fast the function $\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\; dvol_x$ approaches for $t \to \infty$ its limit

$$b_p^{(2)}(M) \quad = \quad \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\; dvol_x.$$

The "thinner" the spectrum of Δ_p is at zero, the larger is $\alpha_p^\Delta(M;\mathcal{N}(G))$.

In view of this original analytic definition the result due to Gromov and Shubin [48] that the Novikov-Shubin invariants are homotopy invariants, is rather surprising.

Remark 12.3 (Analogy to finitely generated $\mathbb{Z}$-modules). Recall Slogan 1.16 that the group von Neumann algebra $\mathcal{N}(G)$ behaves like the ring of integers $\mathbb{Z}$, provided one ignores the properties integral domain and Noetherian. Given a finitely generated abelian group M, the $\mathbb{Z}$-module $M/\operatorname{tors}(M)$ is finitely generated free, there is a $\mathbb{Z}$-isomorphism $M \cong M/\operatorname{tors}(M) \oplus \operatorname{tors}(M)$ and the rank as abelian group of M is $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M)$ and of $\operatorname{tors}(M)$ is 0. In analogy, given a finitely generated $\mathcal{N}(G)$-module M, then $\mathbf{P}M := M/\mathbf{T}M$ is a finitely generated projective$\mathcal{N}(G)$-module, there is an $\mathcal{N}(G)$-isomorphism $M \cong \mathbf{P}M \oplus \operatorname{tors}(M)$ and we get $\dim_{\mathcal{N}(G)}(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M)$ and $\dim_{\mathcal{N}(G)}(\mathbf{T}M) = 0$. Define the so called *capacity* $c(M) \in [0,\infty] \cup \{0^-\}$ of a finitely presented $\mathcal{N}(G)$-module M by

$$c(M) \;=\; \begin{cases} \frac{1}{\alpha(M)} & \text{if } \alpha(M) \in (0,\infty); \\ \infty & \text{if } \alpha(M) = 0; \\ 0 & \text{if } \alpha(M) = \infty; \\ 0^- & \text{if } \alpha(M) = \infty^+. \end{cases}$$

Then the capacity $c(M)$ contains the same information as $\alpha(M)$ and corresponds under the dictionary between $\mathbb{Z}$ and $\mathcal{N}(G)$ to the order of the finite group $\operatorname{tors}(M)$. Notice for a finitely presented $\mathcal{N}(G)$-module M that $M = 0$ is true if and only if both $\dim_{\mathcal{N}(G)}(M) = 0$ and $c(M) = 0^-$ hold. The capacity is at least subadditive, i.e. for an exact sequence $1 \to M_0 \to M_1 \to M_2 \to 0$ of finitely presented $\mathcal{N}(G)$-modules we have $c(M_1) \leq c(M_0) + c(M_2)$ (with the obvious interpretation of $+$ and $\leq$). In particular we get $c(M) \leq c(N)$ for an inclusion of finitely presented $\mathcal{N}(G)$-modules $M \subseteq N$.

Remark 12.4 (Extension to arbitrary $\mathcal{N}(G)$-modules and G-spaces). The algebraic approach presented above has been independently developed in [33] and [74]. The notion of capacity has been extended by Lück-Reich-Schick [82] to so called cofinal-measurable $\mathcal{N}(G)$-modules, i.e. $\mathcal{N}(G)$-modules such that each finitely generated $\mathcal{N}(G)$-submodule is a quotient of a finitely presented $\mathcal{N}(G)$-module with trivial von Neumann dimension. This allows to define Novikov-Shubin invariants for arbitrary G-spaces and also for arbitrary groups G.

12.2 Basic Properties of Novikov-Shubin Invariants

We briefly list some properties of Novikov-Shubin invariants. The proof of the following theorem can be found in [80, Theorem 2.55] and [80, Lemma 13.45].

Theorem 12.5 (Novikov-Shubin invariants).

(i) *Homotopy invariance*

Let $f\colon X \to Y$ be a G-map of free G-CW-complexes of finite type. Suppose that the map $H_p(f;\mathbb{C})\colon H_p(X;\mathbb{C}) \to H_p(Y;\mathbb{C})$ induced on homology with complex coefficients is an isomorphism for $p \leq d-1$. Then we get

$$\alpha_p(X;\mathcal{N}(G)) \;=\; \alpha_p(Y;\mathcal{N}(G)) \qquad \textit{for } p \leq d.$$

In particular we get $\alpha_p(X;\mathcal{N}(G)) = \alpha_p(Y;\mathcal{N}(G))$ for all $p \geq 0$ if f is a weak homotopy equivalence;

(ii) *Poincaré duality*

Let M be a cocompact free proper G-manifold of dimension n which is orientable. Then $\alpha_p(M;\mathcal{N}(G)) = \alpha_{n+1-p}(M,\partial M;\mathcal{N}(G))$ for $p \geq 1$;

(iii) *First Novikov-Shubin invariant*

Let X be a connected free G-CW-complex of finite type. Then G is finitely generated and

(a) *$\alpha_1(X)$ is finite if and only if G is infinite and virtually nilpotent. In this case $\alpha_1(X)$ is the growth rate of G;*

(b) *$\alpha_1(X)$ is ∞^+ if and only if G is finite or non-amenable;*

(c) *$\alpha_1(X)$ is ∞ if and only if G is amenable and not virtually nilpotent;*

(iv) *Restriction to subgroups of finite index*

Let X be a free G-CW-complex of finite type and $H \subseteq G$ a subgroup of finite index. Then $\alpha_p(X;\mathcal{N}(G)) = \alpha_p(\operatorname{res}_G^H X;\mathcal{N}(H))$ holds for $p \geq 0$;

(v) *Extensions with finite kernel*

Let $1 \to H \to G \to Q \to 1$ be an extension of groups such that H is finite. Let X be a free Q-CW-complex of finite type. Then we get $\alpha_p(p^*X;\mathcal{N}(G)) = \alpha_p(X;\mathcal{N}(Q))$ for all $p \geq 1$;

(vi) *Induction*

Let H be a subgroup of G and let X be a free H-CW-complex of finite type. Then $\alpha_p(G \times_H X;\mathcal{N}(G)) = \alpha_p(X;\mathcal{N}(H))$ holds for all $p \geq 1$.

A product formula and a formula for connected sums can also be found in [80, Theorem 2.55]. If X is a finite G-CW-complex such that $b_p^{(2)}(X;\mathcal{N}(G)) = 0$ for $p \geq 0$ and $\alpha_p(X;\mathcal{N}(G)) > 0$ for $p \geq 1$, then X is det-L^2-acyclic [80, Theorem 3.93 (7)].

12.3 Computations of Novikov-Shubin Invariants

Example 12.6 (Novikov-Shubin invariants of $\widetilde{T^n}$). The product formula can be used to show $\alpha_p(\widetilde{T^n}) = n$ if $1 \leq p \leq n$, and $\alpha_p(\widetilde{T^n}) = \infty^+$ otherwise (see [80, Example 2.59].)

Example 12.7 (Novikov-Shubin invariants for finite groups). If G is finite, then $\alpha_p(X;\mathcal{N}(G)) = \infty^+$ for each $p \geq 1$ and G-CW-complex X of finite type. This follows from Example 11.2. This shows that the Novikov-Shubin invariants are interesting only for infinite groups G and have no classical analogue in contrast to L^2-Betti numbers and L^2-torsion.

Example 12.8 (Novikov-Shubin invariants for $G = \mathbb{Z}$). Let X be a free $\mathbb{Z}$-CW-complex of finite type. Since $\mathbb{C}[\mathbb{Z}]$ is a principal ideal domain, we get $\mathbb{C}[\mathbb{Z}]$-isomorphisms

$$H_p(X;\mathbb{C}) \quad \cong \quad \mathbb{C}[\mathbb{Z}]^{n_p} \oplus \left(\bigoplus_{i_p=1}^{s_p} \mathbb{C}[\mathbb{Z}]/((z - a_{p,i_p})^{r_{p,i_p}}) \right)$$

for $a_{p,i_p} \in \mathbb{C}$ and $n_p, s_p, r_{p,i_p} \in \mathbb{Z}$ with $n_p, s_p \geq 0$ and $r_{p,i_p} \geq 1$, where z is a fixed generator of $\mathbb{Z}$. Then we get from [80, Lemma 2.58]

$$b_p^{(2)}(X;\mathcal{N}(\mathbb{Z})) \quad = \quad n_p.$$

If $s_p \geq 1$ and $\{i_p = 1, 2 \ldots, s_p, |a_{p,i_p}| = 1\} \neq \emptyset$, then

$$\alpha_{p+1}(X;\mathcal{N}(\mathbb{Z})) \quad = \quad \min \left\{ \frac{1}{r_{p,i_p}} \mid i_p = 1, 2 \ldots, s_p, |a_{p,i_p}| = 1 \right\},$$

and otherwise

$$\alpha_{p+1}(X;\mathcal{N}(\mathbb{Z})) \quad = \quad \infty^+.$$

Remark 12.9 (Novikov-Shubin invariants and S^1-actions). Under the conditions of Theorem 3.8 and of Theorem 3.9 one can show $\alpha_p(\widetilde{X}) \geq 1$ for all $p \geq 1$ (see [80, Theorem 2.61 and Theorem 2.63]).

Remark 12.10 (Novikov-Shubin invariants of symmetric spaces of non-compact type). The Novikov-Shubin invariants of symmetric spaces of non-compact type with cocompact free G-action are computed by Olbricht [93, Theorem 1.1], the result is also stated in [80, Section 5.3].

Remark 12.11 (Novikov-Shubin invariants of universal coverings of 3-manifolds). Partial results about the computation of the Novikov-Shubin invariants of universal coverings of compact orientable 3-manifolds can be found in [69] and [80, Theorem 4.2].

12.4 Open Conjectures about Novikov-Shubin invariants

The following conjecture is taken from [69, Conjecture 7.1 on page 56].

Conjecture 12.12. (Positivity and rationality of Novikov-Shubin invariants). *Let G be a group. Then for any free G-CW-complex X of finite type its Novikov-Shubin invariants $\alpha_p(X)$ are positive rational numbers unless they are ∞ or ∞^+.*

This conjecture is equivalent to the statement that for every finitely presented $\mathbb{Z}G$-module M the Novikov-Shubin invariant of $\mathcal{N}(G) \otimes_{\mathbb{Z}G} M$ is a positive rational number, ∞ or ∞^+.

Here is some evidence for Conjecture 12.12. Unfortunately, all the evidence comes from computations, no convincing conceptual reason is known. Conjecture 12.12 is true for $G = \mathbb{Z}$ by the explicit computation appearing in Example 12.8. Conjecture 12.12 is true for virtually abelian G by [66, Proposition 39 on page 494]. Conjecture 12.12 is also true for a free group G. Details of the proof appear in the Ph.D. thesis of Roman Sauer [100] following ideas of Voiculescu. The essential ingredients are non-commutative power series and the question whether they are algebraic or rational. All the computations mentioned above are compatible and give evidence for Conjecture 12.12.

Conjecture 12.13 (Zero-in-the-spectrum Conjecture). *Let G be a group such that BG has a closed manifold as model. Then there is $p \geq 0$ with $H_p^G(EG; \mathcal{N}(G)) \neq 0$.*

Remark 12.14 (Original zero-in-the-spectrum Conjecture). The original zero-in-the-spectrum Conjecture, which appears for the first time in Gromov's article [43, page 120], says the following: Let $\widetilde{M}$ be a complete Riemannian manifold. Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (with the Riemannian metric coming from M). Then for some $p \geq 0$ zero is in the spectrum of the minimal closure

$$(\Delta_p)_{\min} \colon \operatorname{dom}((\Delta_p)_{\min}) \subseteq L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p-forms on $\widetilde{M}$.

It follows from [80, Lemma 12.3] that this formulation is equivalent to the homological algebraic formulation appearing in Conjecture 12.13.

Remark 12.15 (Status of the zero-in-the-spectrum Conjecture). The zero-in-the-spectrum Conjecture is true for G if there is a closed manifold model for BG which is Kähler hyperbolic [45], or whose universal covering is hyper-Euclidean [43] or is uniformly contractible with finite asymptotic dimension [110]. The zero-in-the-spectrum Conjecture is true for G if the strong Novikov Conjecture holds for G [67]. More information about zero-in-the-spectrum Conjecture can be found for instance in [67] and [80, Section 12].

Remark 12.16 (Variations of the zero-in-the-spectrum Conjecture). One may ask whether one can weaken the condition in Conjecture 12.13 that BG has a closed manifold model to the condition that there is a finite CW-complex model for BG. This would rule out Poincaré duality from the picture. Or one could only require that BG is of finite type. Without any finiteness conditions on G Conjecture 12.13 is not true in general. For instance $H_p^G(EG; \mathcal{N}(G)) = 0$ holds for all $p \geq 0$ if G is $\prod_{i=1}^{\infty} \mathbb{Z} * \mathbb{Z}$.

The condition aspherical cannot be dropped. Farber and Weinberger [35] proved the existence of a closed non-aspherical manifold M with fundamental group π a product of three copies of $\mathbb{Z}*\mathbb{Z}$ such that $H_p^\pi(\widetilde{M};\mathcal{N}(\pi))$ vanishes for all $p \geq 0$. Later Higson-Roe-Schick [52] proved that one can find for every finitely presented group π, for which $H_p^\pi(E\pi;\mathcal{N}(\pi)) = 0$ holds for $p = 0, 1, 2$, a closed manifold M with π as fundamental group such that $H_p^\pi(\widetilde{M};\mathcal{N}(\pi))$ vanishes for all $p \geq 0$.

Remark 12.17 (Novikov-Shubin invariants and quasi-isometry). Since $\alpha_1(\mathbb{Z}^n) = n$ for $n \geq 1$, the Novikov-Shubin invariants are not invariant under measure equivalence. It is not known whether they are invariant under quasi-isometry. At least it is known that two quasi-isometric amenable groups G_1 and G_2 which possess finite models for BG_1 and BG_2 have the same Novikov-Shubin invariants [100]. Compare also Theorem 8.2, Remark 8.5 and Conjecture 11.30.

13 A Combinatorial Approach to L^2-Invariants

In this section we want to give a more combinatorial approach to the L^2-invariants such as L^2-Betti numbers, Novikov-Shubin invariants and L^2-torsion. The point is that it is in general very hard to compute the spectral density function of an $\mathcal{N}(G)$-map $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$. However in the geometric situation these morphisms are induced by matrices over the integral group ring $\mathbb{Z}G$. We want to exploit this information to get an algorithm which produces a sequence of rational numbers converging to the L^2-Betti number or the L^2-torsion in question.

Let $A \in M(m, n; \mathbb{C}G)$ be an (m, n)-matrix over $\mathbb{C}G$. It induces by right multiplication an $\mathcal{N}(G)$-homomorphism $r_A\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$. We define an involution of rings on $\mathbb{C}G$ by sending $\sum_{g\in G} \lambda_g \cdot g$ to $\sum_{g\in G} \overline{\lambda_g} \cdot g^{-1}$, where $\overline{\lambda_g}$ is the complex conjugate of λ_g. Denote by A^* the (m, n)-matrix obtained from A by transposing and applying the involution above to each entry. Define the $\mathbb{C}G$-*trace* of an element $u = \sum_{g\in G} \lambda_g \cdot g \in \mathbb{C}G$ by the complex number $\operatorname{tr}_{\mathbb{C}G}(u) := \lambda_e$ for e the unit element in G. This extends to a trace of square (n, n)-matrices A over $\mathbb{C}G$ by

$$\operatorname{tr}_{\mathbb{C}G}(A) \quad := \quad \sum_{i=1}^{n} \operatorname{tr}_{\mathbb{C}G}(a_{i,i}) \quad \in \mathbb{C}. \tag{13.1}$$

We get directly from the definitions that the $\mathbb{C}G$-trace $\operatorname{tr}_{\mathbb{C}G}(u)$ for $u \in \mathbb{C}G$ agrees with the von Neumann trace $\operatorname{tr}_{\mathcal{N}(G)}(u)$ introduced in Definition 1.8.

Let $A \in M(m, n; \mathbb{C}G)$ be an (m, n)-matrix over $\mathbb{C}G$. In the sequel let K be any positive real number satisfying $K \geq ||r_A^{(2)}||$, where $||r_A^{(2)}||$ is the operator norm of the bounded G-equivariant operator $r_A^{(2)} : l^2(G)^m \to l^2(G)^n$ induced by right multiplication with A. For $u = \sum_{g\in G} \lambda_g \cdot g \in \mathbb{C}G$ define $||u||_1$ by

$\sum_{g \in G} |\lambda_g|$. Then a possible choice for K is

$$K = \sqrt{(2n-1)m} \cdot \max\{||a_{i,j}||_1 \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Definition 13.2. *The* characteristic sequence *of a matrix $A \in M(m,n;\mathbb{C}G)$ and a non-negative real number K satisfying $K \geq ||r_A^{(2)}||$ is the sequence of real numbers given by*

$$c(A,K)_p := \operatorname{tr}_{\mathbb{C}G}\left(\left(1 - K^{-2} \cdot AA^*\right)^p\right).$$

We have defined $\dim_{\mathcal{N}(G)}(\ker(r_A))$ in Definition 1.12 and $\det_{\mathcal{N}(G)}(r_A)$ in Definition 11.4. The proof of the following result can be found in [73] or [80, Theorem 3.172].

Theorem 13.3. (Combinatorial computation of L^2-invariants). *Let $A \in M(m,n;\mathbb{C}G)$ be an (m,n)-matrix over $\mathbb{C}G$. Let K be a positive real number satisfying $K \geq ||r_A^{(2)}||$. Then:*

(i) Monotony
The characteristic sequence $(c(A,K)_p)_{p\geq 1}$ is a monotone decreasing sequence of non-negative real numbers;

(ii) Dimension of the kernel
We have

$$\dim_{\mathcal{N}(G)}(\ker(r_A)) \;=\; \lim_{p\to\infty} c(A,K)_p;$$

(iii) Novikov-Shubin invariants of the cokernel
Define $\beta(A) \in [0,\infty]$ by

$$\beta(A) := \sup\left\{\beta \in [0,\infty) \;\middle|\; \lim_{p\to\infty} p^\beta \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(G)}(\ker(r_A))\right) = 0\right\}.$$

If $\alpha(\operatorname{coker}(r_A)) < \infty$, then $\alpha(\operatorname{coker}(r_A)) \leq \beta(A)$ and if $\alpha(\operatorname{coker}(r_A)) \in \{\infty, \infty^+\}$, then $\beta(A) = \infty$;

(iv) Fuglede-Kadison determinant
The sum of positive real numbers

$$\sum_{p=1}^{\infty} \frac{1}{p} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(G)}(\ker(r_A))\right)$$

converges if and only if r_A is of determinant class and in this case

$$\begin{aligned}\ln(\det(r_A)) \;=\;& (n - \dim_{\mathcal{N}(G)}(\ker(r_A))) \cdot \ln(K) \\ &- \frac{1}{2} \cdot \sum_{p=1}^{\infty} \frac{1}{p} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(G)}(\ker(r_A))\right);\end{aligned}$$

(v) *Speed of convergence*
Suppose $\alpha(\operatorname{coker}(r_A)) > 0$. *Then* r_A *is of determinant class. Given a real number* α *satisfying* $0 < \alpha < \alpha(\operatorname{coker}(r_A))$, *there is a real number* C *such that we have for all* $L \geq 1$

$$0 \leq c(A,K)_L - \dim_{\mathcal{N}(G)}(\ker(r_A)) \leq \frac{C}{L^\alpha}$$

and

$$\begin{aligned} 0 \quad \leq \quad & -\ln(\det(r_A)) + (n - \dim_{\mathcal{N}(G)}(\ker(r_A))) \cdot \ln(K) \\ & -\frac{1}{2} \cdot \sum_{p=1}^{L} \frac{1}{p} \cdot \big(c(A,K)_p - \dim_{\mathcal{N}(G)}(\ker(r_A))\big) \quad \leq \quad \frac{C}{L^\alpha}. \end{aligned}$$

Remark 13.4 (Vanishing of L^2-Betti numbers and the Atiyah Conjecture). Suppose that the Atiyah Conjecture 4.1 is satisfied for $(G, d, \mathbb{C})$. If we want to show the vanishing of $\dim_{\mathcal{N}(G)}(\ker(r_A))$, it suffices to show that for some $p \geq 0$ we have $c(A,K)_p < \frac{1}{d}$. It is possible that a computer program spits out such a value after a reasonable amount of calculation time.

14 Miscellaneous

The analytic aspects of L^2-invariants are also very interesting. We have already mentioned that L^2-Betti numbers were originally defined by Atiyah [2] in context with his L^2-index theorem. Other L^2-invariants are the L^2-Eta-invariant and the L^2-Rho-invariant (see Cheeger-Gromov [13], [14]). The L^2-Eta-invariant appears in the L^2-index theorem for manifolds with boundary due to Ramachandran [97]. These index theorems have generalizations to a C^*-setting due to Miščenko-Fomenko [89]. There is also an L^2-version of the signature. It plays an important role in the work of Cochran, Orr and Teichner [16] who show that there are non-slice knots in 3-space whose Casson-Gordon invariants are all trivial. Chang and Weinberger [11] show using L^2-invariants that for a closed oriented smooth manifold M of dimension $4k+3$ for $k \geq 1$ whose fundamental group has torsion there are infinitely many smooth manifolds which are homotopy equivalent to M (and even simply and tangentially homotopy equivalent to M) but not homeomorphic to M. The L^2-cohomology has also been investigated for complete non-necessarily compact Riemannian manifolds without a group action. For instance algebraic and arithmetic varieties have been studied. In particular, the Cheeger-Goresky-MacPherson Conjecture [12] and the Zucker Conjecture [111] have created a lot of activity. They link the L^2-cohomology of the regular part with the intersection homology of an algebraic variety.

Finally we mention other survey articles which deal with L^2-invariants: [30], [39], [46, Section 8], [67], [75], [78], [79], [87] and [96].

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Notation

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