J. reine angew. Math. **498** (1998), 1–33

Journal für die reine und angewandte Mathematik
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Berlin · New York 1998

Torsion and fibrations

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Abstract. We study the behaviour of analytic torsion under smooth fibrations. Namely, let $F \to E \xrightarrow{f} B$ be a smooth fiber bundle of connected closed oriented smooth manifolds and let V be a flat vector bundle over E. Assume that E and B come with Riemannian metrics. Suppose that $\dim(E)$ is odd and V is unimodular and comes with an arbitrary Riemannian metric or that $\dim(E)$ is even and V comes with a unimodular (not necessarily flat) Riemannian metric. Let $\varrho_{\mathrm{an}}(E;V)$ be the analytic torsion of E with coefficients in V, let $\varrho_{\mathrm{an}}(F_b;V)$ be the analytic torsion of the fiber over b with coefficients in V restricted to F_b and let Pf_B be the Pfaffian $\dim(B)$-form. Let $H^q_{\mathrm{dR}}(F;V)$ be the flat vector bundle over B whose fiber over $b \in B$ is $H^q_{\mathrm{dR}}(F_b;V)$ with the Riemannian metric which comes from the Hodge-deRham decomposition and the Hilbert space structure on the space of harmonic forms induced by the Riemannian metrics. Let $\varrho_{\mathrm{an}}\big(B;H^q_{\mathrm{dR}}(F;V)\big)$ be the analytic torsion of B with coefficients in this bundle. The Leray-Serre spectral sequence for deRham cohomology determines a certain correction term $\varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)$. We prove

$$\varrho_{\mathrm{an}}(E;V) = \int_B \varrho_{\mathrm{an}}(F_b;V)\cdot \mathrm{Pf}_B + \sum_q (-1)^q \cdot \varrho_{\mathrm{an}}\big(B;H^q_{\mathrm{dR}}(F;V)\big) + \varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)\,.$$

This formula simplifies in special cases such as bundles with S^n as fiber or base, in which case the correction term $\varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)$ reduces to the torsion of the associated Gysin or Wang sequence, resp.

0. Introduction

Let M be a connected closed smooth manifold with Riemannian metric and V be a flat vector bundle with a not necessarily flat Riemannian metric. The definition of analytic torsion due to Ray and Singer [20] for an orthogonal representation V, or, equivalently, for a flat Riemannian metric on V, still makes sense in the setting above ([1], page 35 and [18], page 730). Namely, let $\zeta_p(s)$ be the zeta-function of the Laplace operator

$$\Delta^p : \Omega^p(E;V) \to \Omega^p(E;V)$$

which is for $\mathrm{Re}(s) \gg 0$ the holomorphic function $\sum_{\lambda > 0} \lambda^{-s}$ where λ runs over the positive eigenvalues of Δ^p. It has a meromorphic extension to the complex plane which is analytic in zero. Define

$$(0.1) \qquad \varrho_{\mathrm{an}}(E;V) := \frac{1}{2} \cdot \sum_{q \geqq 0} (-1)^q \cdot q \cdot \zeta_q'(0) \quad \in \mathbb{R}\,.$$

We want to study it for smooth fiber bundles. The main result of this paper is

Theorem 0.2. *Let $F \to E \xrightarrow{f} B$ be a smooth fiber bundle of connected closed oriented smooth manifolds and let V be a flat vector bundle over E. Assume that E and B come with Riemannian metrics. Suppose that $\dim(E)$ is odd and V is unimodular and comes with an arbitrary Riemannian metric or that $\dim(E)$ is even and V comes with a unimodular Riemannian metric or that $\dim(E)$ is even and V comes with a unimodular Riemannian metric. Then*

$$\varrho_{\mathrm{an}}(E;V) = \int_B \varrho_{\mathrm{an}}(F_b;V) \cdot \mathrm{Pf}_B + \sum_q (-1)^q \cdot \varrho_{\mathrm{an}}\big(B; H^q_{\mathrm{dR}}(F;V)\big) + \varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)\,. \quad \square$$

Here are some explanations of the assumptions and the formula in Theorem 0.2.

For a path w in E the fiber transport gives a linear isomorphism $V_w : V_{w(0)} \to V_{w(1)}$ which depends only on the homotopy class relative endpoints of w since V is flat. We call V *unimodular* if for one (and hence all) $e \in E$ and all loops w with base point e we get $|\det(V_w : V_e \to V_e)| = 1$. We call a Riemannian metric on V *unimodular* if for any path w in E we get $\det(V_w^* \circ V_w : V_{w(0)} \to V_{w(0)}) = 1$ where V_w^* is the adjoint of V_w with respect to the Hilbert space structure on the fibers of V given by the Riemannian metric. This is a weaker condition than being a flat Riemannian metric what would mean that V_w is always an isometry. Notice that V is unimodular if and only if it carries a unimodular Riemannian metric.

Of course $\varrho_{\mathrm{an}}(E;V)$ is just the analytic torsion with respect to the given Riemannian metrics on E and V. These induce also a metric on $H^p_{\mathrm{dR}}(E,V)$. Each fiber $F_b = p^{-1}(b)$ inherits a Riemannian metric from E by restriction. We denote the restriction of V to F_b again by V. Hence $\varrho_{\mathrm{an}}(F_b;V)$ is defined and is a smooth function in $b \in B$.

Let Pf_B be the Pfaffian $\dim(B)$-form on the oriented Riemannian manifold B. It is a representative of the Euler class of B in Chern-Weil theory and satisfies by the Gauss-Bonnet theorem

$$\int_B \mathrm{Pf}_B = \chi(B)$$

where $\chi(B)$ is the Euler characteristic. If $\dim(B)$ is odd, then Pf_B is defined to be zero.

Let $\mathscr{H}^q(F_b;V)$ be the space of harmonic q-forms, i.e. the kernel of the Laplace operator $\Delta^q : \Omega^q(F_b;V) \to \Omega^q(F_b;V)$. It inherits an inner product from the Riemannian metrics on F_b and V. The *harmonic Hilbert structure* on the deRham cohomology $H^q_{\mathrm{dR}}(F_b;V)$ is the Hilbert space structure for which the canonical Hodge isomorphism

$$\mathscr{H}^q(F_b;V) \to H^q_{\mathrm{dR}}(F_b;V)$$

is isometric. Thus we get a Riemannian metric on the canonical flat vector bundle $H^q_{\mathrm{dR}}(F;V)$ whose fiber over $b \in B$ is $H^q_{\mathrm{dR}}(F_b;V)$. Hence the analytic torsion $\varrho_{\mathrm{an}}(B;H^q_{\mathrm{dR}}(F;V))$ is defined, and $H^p_{\mathrm{dR}}(B,H^q_{\mathrm{dR}}(F_b;V))$ inherits a natural Hilbert space structure.

There is the following natural descending filtration of the deRham complex $\Omega^*(E;V)$. Define $F_p\Omega^n(E;V)$ to be those n-forms with coefficients in V which can be written as finite sums of n-forms on E with coefficients in V of the shape $\omega \wedge f^*\eta$ for $\omega \in \Omega^{n-k}(E;V)$ and $\eta \in \Omega^k(B)$ for some $k \geqq p$. This filtration is compatible with the differential since $d(\omega \wedge f^*\eta) = d(\omega) \wedge f^*\eta \pm \omega \wedge f^*d(\eta)$. The associated spectral cohomology sequence is the Leray-Serre spectral sequence for deRham cohomology, which we recall in Section 4.

Part of the Leray-Serre spectral sequence for deRham cohomology is the filtration of the cohomology $H^n(E;V)$

$$\{0\} = F^{n+1,-1} \subset \cdots \subset F^{p+1,n-p-1} \subset F^{p,n-p} \subset \cdots \subset F^{0,n} = H^n_{\mathrm{dR}}(E;V),$$

the natural identification of the E_2-term

$$V_2^{p,q}: H^p_{\mathrm{dR}}(B;H^q_{\mathrm{dR}}(F;V)) \xrightarrow{\cong} E_2^{p,q}, \tag{0.3}$$

the identification of the cohomology of the r-th term of the spectral sequence with the $(r+1)$-th term and the identification of the E_∞-term with the filtration quotients

$$\begin{aligned} \phi_r^{p,q}: H^0(E_r^{p+r*,q-(r-1)*}) &\xrightarrow{\cong} E_{r+1}^{p,q}, \\ \psi^{p,q}: F^{p,q}/F^{p+1,q-1} &\xrightarrow{\cong} E_\infty^{p,q}. \end{aligned}$$

For r sufficiently large, the differentials in $E_r^{*,*}$ are trivial.

Next we explain the term $\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)$ appearing in Theorem 0.2.

For a linear isomorphism $f: V \to W$ of finite-dimensional real Hilbert spaces, set

$$[\![f]\!] := \frac{1}{2}\ln\left(|\det(f^*f)|\right) \in \mathbb{R}. \tag{0.4}$$

Let $C = C^*$ be an acyclic finite Hilbert cochain complex. Define

$$\varrho(C) := [\![(c^* + \gamma^*): C^{\mathrm{ev}} \to C^{\mathrm{odd}}]\!] \in \mathbb{R} \tag{0.5}$$

where c^* is the differential and γ^* a chain contraction. If $f: C \to D$ is a chain homotopy equivalence of finite Hilbert cochain complexes, $\mathrm{cone}(f)$ is the cochain complex with n-th differential

$$\begin{pmatrix} c^n & 0 \\ f^n & -d^{n-1} \end{pmatrix}: C^n \oplus D^{n-1} \to C^{n+1} \oplus D^n.$$

It is acyclic and we define

$$t(f) := \varrho(\mathrm{cone}(f)). \tag{0.6}$$

Let C be a finite Hilbert cochain complex such that $H(C^*)$ carries a Hilbert structure. There is up to homotopy precisely one chain map $i\colon H(C) \to C$ whith $H(i) = \mathrm{id}$, where we consider $H(C)$ as a cochain complex with the trivial differential. Define

$$\varrho(C) := -t(i) \in \mathbb{R}. \tag{0.7}$$

In the Leray-Serre spectral sequence, equip $E_2^{p,q}$ with the Hilbert space structure which makes $V_2^{p,q}$ to an isometry. Equip inductively $E_r^{p+r*,q-(r-1)*}$, $H(E_r^{p+r*,q-(r-1)*})$ ($r \geqq 2$) and $E_\infty^{p,q}$ with the Hilbert sub- and quotient structures. In particular, $\phi_r^{p,q}$ become isometries. Equip $F^{p,q} \subset H_{\mathrm{dR}}^{p+q}(E,V)$ and $F^{p,q}/F^{p+1,q-1}$ with the Hilbert sub- and quotient structures. Now define

$$\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f) := \sum_{r \geqq 2} \sum_{p=0}^{r-1} \sum_q (-1)^{p+q} \cdot \varrho(E_r^{p+r*,q-(r-1)*}) - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!].$$

This number depends only on f and the Riemannian metrics on E, B and V.

In general the correction term $\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)$ is very involved and is as complicated as the Leray-Serre spectral cohomology sequence is. However, there are cases where $\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)$ and the whole formula in Theorem 0.2 are easy to understand. Namely, we will prove under the assumptions of Theorem 0.2 the following three corollaries. The first one generalizes a result of Fried [7] for orthogonal V.

Corollary 0.8. *Suppose that $H_{\mathrm{dR}}^q(F;V)$ vanishes for all q. Then $\chi(B) \cdot \varrho_{\mathrm{an}}(F_b;V)$ is independent of b and*

$$\varrho_{\mathrm{an}}(E;V) = \chi(B) \cdot \varrho_{\mathrm{an}}(F_b;V). \quad \square$$

Corollary 0.9. *Suppose that F is S^n and $V = f^*W$ for a flat vector bundle with Riemannian metric over B. Let G^* be the acyclic cochain complex of finite-dimensional Hilbert spaces given by the Gysin sequence*

$$\cdots \xrightarrow{\int} H_{\mathrm{dR}}^p(B;W) \xrightarrow{\wedge e(f)} H_{\mathrm{dR}}^{p+n+1}(B;W) \xrightarrow{f^*} H_{\mathrm{dR}}^{p+n+1}(E;V)$$
$$\xrightarrow{\int} H_{\mathrm{dR}}^{p+1}(B;W) \xrightarrow{\wedge e(f)} \cdots$$

where $\wedge e(f)$ is the product with the Euler class $e(f) \in H_{\mathrm{dR}}^{n+1}(B)$ of the sphere bundle, $\int$ is integration over the fiber and $G^1 = H_{\mathrm{dR}}^0(E;V)$. Then the torsion $\varrho(G^*) \in \mathbb{R}$ is defined and we get:

$$\varrho_{\mathrm{an}}(E;V) = \chi(S^n) \cdot \varrho_{\mathrm{an}}(B;W) + \varrho(G^*). \quad \square$$

The condition $V = f^*W$ is no loss of generality, provided that $n \geqq 2$ or that f induces an isomorphism $\pi_1(E) \to \pi_1(B)$.

Corollary 0.10. *Suppose that $B = S^n$. Let W^* be the acyclic cochain complex of finite-dimensional Hilbert spaces given by the Wang sequence and the harmonic structures for some $b \in B$:*

$$\cdots \to H^{q-1}_{\mathrm{dR}}(E;V) \to H^{q-1}_{\mathrm{dR}}(F_b;V) \to H^{q-n}_{\mathrm{dR}}(F_b;V) \to H^{q}_{\mathrm{dR}}(E;V) \to \cdots$$

where $W^1 = H^0_{\mathrm{dR}}(E;V)$. Then

$$\varrho_{\mathrm{an}}(E;V) = \chi(S^n) \cdot \varrho_{\mathrm{an}}(F_b;V) + \varrho(W^*) . \quad \square$$

We make some remarks about the proof of Theorem 0.2. It will depend on the following deep results of Bismut-Zhang [1] and Müller [18]. In the sequel M is a connected closed oriented Riemannian manifold and V is a flat vector bundle over M with Riemannian metric. We have introduced the analytic torsion $\varrho_{\mathrm{an}}(M;V)$ above. Its topological counterpart

$$\varrho_{\mathrm{top}}(M;V) \in \mathbb{R} \tag{0.11}$$

is the Milnor torsion of M with respect to some triangulation and the harmonic Hilbert structure on cohomology which we will recall in Definition 3.3.

Theorem 0.12. *If the Riemannian metric on V is unimodular, then*

$$\varrho_{\mathrm{an}}(M;V) = \varrho_{\mathrm{top}}(M;V) . \quad \square$$

Let g_M and $\overline{g_M}$ be two Riemannian metrics on M and g_V and $\overline{g_V}$ be two arbitrary Riemannian metrics on V. Let $\varrho_{\mathrm{an}}(M;V)$ and $\overline{\varrho_{\mathrm{an}}(M;V)}$ be the analytic torsion with respect to (g_M, g_V) and $(\overline{g_M}, \overline{g_V})$. Analogously we denote the Hilbert spaces $H^p_{\mathrm{dR}}(M;V)$ and $\overline{H^p_{\mathrm{dR}}(M;V)}$ equipped with the harmonic Hilbert structures with respect to (g_M, g_V) and $(\overline{g_M}, \overline{g_V})$ and the Hilbert spaces V_x and $\overline{V_x}$ equipped with the Hilbert structure with respect to g_V and $\overline{g_V}$. Denote by Pf_M the Pfaffian with respect to g_M. Let $\widetilde{\mathrm{Pf}}(M, g_M, \overline{g_M})$ be the Chern-Simons $n-1$-form. Its image under the differential is the difference of the two Pfaffians of M with respect to g_M and $\overline{g_M}$. Let $\theta(V, \overline{g_V})$ be the closed 1-form defined in [1], Definition 4.5. It measures the deviation of $\overline{g_V}$ from being unimodular and vanishes if $\overline{g_V}$ is unimodular.

Theorem 0.13. *We get under the conditions and in the notations above:*

1. *If* $\dim(M)$ *is odd, then*

$$\varrho_{\mathrm{an}}(M;V) - \overline{\varrho_{\mathrm{an}}(M;V)} = -\sum_p (-1)^p \cdot \left[\!\left[H^p_{\mathrm{dR}}(M;V) \xrightarrow{\mathrm{id}} \overline{H^p_{\mathrm{dR}}(M;V)} \right]\!\right] .$$

2. *If* $\dim(M)$ *is even, then*

$$\begin{aligned}\varrho_{\mathrm{an}}(M;V) - \overline{\varrho_{\mathrm{an}}(M;V)} = {} & -\sum_p (-1)^p \cdot \left[\!\left[H^p_{\mathrm{dR}}(M;V) \xrightarrow{\mathrm{id}} \overline{H^p_{\mathrm{dR}}(M;V)} \right]\!\right] \\ & + \int_M \left[\!\left[V_x \xrightarrow{\mathrm{id}} \overline{V_x} \right]\!\right] \cdot \mathrm{Pf}_M - \int_M \theta(V, \overline{g_V}) \cdot \widetilde{\mathrm{Pf}}(M, g_M, \overline{g_M}) . \quad \square\end{aligned}$$

Theorem 0.12 and Theorem 0.13 for odd-dimensional M are Theorem 1 and Theorem 2.6 of Müller [18], who generalizes Cheeger's and his proof [3] and [17] of the Ray-Singer Conjecture $\varrho_{\mathrm{an}}(M;V) = \varrho_{\mathrm{top}}(M;V)$ for orthogonal representations V to the unimodular setting. Bismut and Zhang [1], Theorem 0.1 and Theorem 0.2, have generalized Müller's work to all dimensions and to a setting where V is not necessarily unimodular. We say more about their version of Theorem 0.12 in Section 6.

Theorem 0.12 and Theorem 0.13 enable us to show that Theorem 0.2 follows from its topological version. This will be done in Section 6 where we also explain how Corollaries 0.8, 0.9 and 0.10 follow from Theorem 0.2 and its topological version. In the topological case we can treat a more general setting. Namely, we consider a fibration $f\colon E \to B$ such that B is a connected finite CW-complex, the homotopy fiber has the homotopy type of a finite CW-complex and a certain cohomology class $\theta_f \in H^1\big(B;\mathrm{Wh}(E)\big)$ vanishes. Then we get for a local coefficient system V with unimodular Hilbert structure, and fixed Hilbert structures on the relevant singular cohomology groups:

Theorem 5.4.

$$\varrho(E;V) = \chi(B)\cdot\varrho(F;V) + \varrho\Big(B;\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)\Big) + \varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)\,. \quad \square$$

Freed [6] obtained results about torsion and spectral sequences similar to those of Section 4. The latter are used to deduce the topological fibration formula. There is also work of Maumary [13] about Whitehead torsion and spectral sequences.

The topological version includes manifolds with boundary and the question whether Theorem 0.2 generalizes to manifolds with boundary comes down to the question whether Theorem 0.12 and Theorem 0.13 generalize to manifolds with boundary. At least under the condition that V is an orthogonal representation Cheeger and Müller's results for closed manifolds have been extended to manifolds with boundary (see [8], [9] and [11]).

There are also interesting generalizations of the topological versions to other settings. For example one can consider an L^2-version. Or one can substitute V by a local coefficient system of finitely generated projective modules over a ring R. Then the torsion takes value in the algebraic K-theory of R.

Instead of explicitely choosing Hilbert structures to define torsion as a real number, one can consider it as an element in certain determinant spaces. The latter approach has the advantage not to depend on artificial choices. However, it does not allow the generalizations we have mentioned. In particular, one is bound to the finite dimensional setting (i.e. even the singular cochain complex of a finite CW-complex is not allowed). Also, if one carries out explicit computations, in most cases it is preferable to deal with real numbers. Therefore we used the first method. In our context, both languages are completely equivalent and we supply a dictionary to translate between them in Appendix A. One should also mention that there is a third equivalent approach which uses norms constructed on determinant lines (compare [1]).

Finally we mention that in the case where V is orthogonal Dai and Melrose [5] have proven the formula of Theorem 0.2 in the adiabatic limit by completely different methods, namely, by a careful analysis of the heat kernel in the adiabatic limit and the adiabatic version of the Leray-Serre spectral sequence [14]. We remark that in the special case where V is an orthogonal representation $\varrho_{\mathrm{an}}(E;V)$ vanishes if $\dim(E)$ is even but this is not true in general for a non-flat Riemannian metric on V.

The first two authors decidate this paper to their friend and colleage Thomas Thielmann who died in a car accident in November 1994.

1. Simple structures on spaces

In this section we explain additional structures on arbitrary topological spaces and fibrations which allow the definition of torsion invariants on them. This extension from the category of finite CW-complexes serves two purposes: on the one hand there are interesting spaces which are not finite CW-complexes, f.i. classifiying spaces of discrete groups. On the other hand, our approach singles out what exactly is used in the definition of torsion invariants, and this definitely clarifies the exposition.

We will always assume for a pair of spaces (X,A) that the inclusion of A into X is a cofibration. This condition is satisfied if (X,A) is a pair of CW-complexes or if X is a manifold with submanifold A. A map $(F,f):(X,A)\to(Y,B)$ of pairs is a *relative homotopy equivalence* if $F\cup\mathrm{id}:X\cup_f B\to Y$ is a homotopy equivalence. Here $X\cup_f B$ is obtained from $X\coprod B$ by identifying $a\in A$ with $f(a)\in B$. Notice that (F,f) is a homotopy equivalence of pairs if and only if f is a homotopy equivalence and (F,f) is a relative homotopy equivalence. Given a cellular relative homotopy equivalence $(F,f):(X,A)\to(Y,B)$ of pairs of finite CW-complexes, define its Whitehead torsion as the Whitehead torsion of the homotopy equivalence $F\cup\mathrm{id}:X\cup_f B\to Y$ of finite CW-complexes

$$\tau(F,f):=\tau(F\cup\mathrm{id})\quad\in\mathrm{Wh}(Y)\,. \tag{1.1}$$

We refer to [4], § 6, § 21 and § 22, for the definition of the geometric Whitehead group and Whitehead torsion and their identifications with the algebraic Whitehead group $\mathrm{Wh}\big(\pi_1(Y)\big)$ and Whitehead torsion. Notice that $\tau(F,f)$ depends only on the homotopy class of (F,f) and satisfies the composition formula, the formula for pairs and the product formula as stated in 1.6. This follows from the special case $A=\emptyset$ in [4], § 22 and § 23. Given a pair (Y,B), we call two relative homotopy equivalences $(F_i,f_i):(X_i,A_i)\to(Y,B)$ with pairs of finite CW-complexes as source for $i=0$, 1 equivalent if

$$\tau\big((G,g)\circ(F_0,f_0)\big)=\tau\big((G,g)\circ(F_1,f_1)\big)$$

holds for any relative homotopy equivalence $(G,g):(Y,B)\to(Y',B')$ into a pair of finite CW-complexes.

Definition 1.2. A *relative simple structure* on a pair (Y,B) is an equivalence class of relative homotopy equivalence $(F,f):(X,A)\to(Y,B)$ for a pair of finite CW-complexes as source. □

1.3. Let (X,A) be a pair with a relative simple structure and $g:A''\to A$ be a homotopy equivalence with a finite CW-complex as source. Then we can extend g to a representative for the simple structure $(G,g):(X'',A'')\to(X,A)$ which is a homotopy equivalence of pairs as follows.

Choose some representative $(F,f):(X',A')\to(X,A)$ for the given simple structure. Furthermore choose a homotopy inverse $g^{-1}:A\to A''$, a homotopy ϕ from $g^{-1}\circ f$ to a cellular map $A'\to A''$ and a homotopy ψ from $g\circ g^{-1}\circ f$ to f. Let X'' be the finite CW-complex $X'\cup_{\phi_1}A''$. Define $(G,g):(X'',A'')\to(X,A)$ by the following composition of (relative) homotopy equivalences or their homotopy inverses

$$X'\cup_{\phi_1}A''\to X'\times[0,1]\cup_{\phi}A''\leftarrow X'\cup_{g^{-1}\circ f}A''\to X'\cup_{g\circ g^{-1}\circ f}A$$
$$\to X'\times[0,1]\cup_{\psi}A\leftarrow X'\cup_{f}A\to X.$$

The proof that (G,g) represents the given simple structure is done by the results of [4], §5. □

1.4. Given a (relative) simple structure on (X,A) and on A, we construct a preferred simple structure on X as follows. Because of 1.3 there is a homotopy equivalence of pairs $(G,g):(X'',A'')\to(X,A)$ such that it represents the given relative simple structure on (X,A) and g represents the given simple structure on A. The preferred simple structure on X is then represented by G. This is independent of the choice of (G,g) by 1.6. □

Given a relative homotopy equivalence $(F,f):(X,A)\to(Y,B)$ of pairs with relative simple structures, we still can define its (relative) Whitehead torsion

$$\tau(F,f)\in \mathrm{Wh}(Y) \tag{1.5}$$

as follows. Choose representatives $(G,g):(X',A')\to(X,A)$ and $(H,h):(Y',B')\to(Y,B)$ for the relative structures. Because of 1.3 one can arrange that (H,h) is a homotopy equivalence of pairs. Define $\tau(F,f)$ as the image of Whitehead torsion $\tau\big((H,h)^{-1}\circ(F,f)\circ(G,g)\big)$ defined in 1.1 under the map $H_*:\mathrm{Wh}(Y')\to\mathrm{Wh}(Y)$ induced by H.

1.6. We have already mentioned homotopy invariance, the composition formula

$$\tau\big((G,g)\circ(F,f)\big)=\tau(G,g)+G_*\tau(F,f)$$

the formula for pairs

$$\tau(F)=\tau(F,f)+i_*\tau(f)$$

and the product formula

$$\tau\big((F,f)\times \mathrm{id}_Y\big)=\chi(Y)\cdot i_*\tau(F,f)$$

where i denotes the obvious inclusions. One easily checks that they remain true in the more general case that (X,A) is not necessarily a pair of finite CW-complexes, but carries a relative simple structure. □

1.7. Let $f:E\to B$ be a fibration such that B is a finite CW-complex and the fiber has the homotopy type of a finite CW-complex. Suppose that we are given a *cellular base point system* $\{b_c\,|\,c\in I\}$ for B, i.e. a choice of points b_c in the interior c° for each $c\in I$ where

here and elsewhere I_n is the set of n-cells and I is the disjoint union of the I_n-s. Furthermore suppose that we have specified a simple structure on each fiber $F_{b_c} = f^{-1}(b_c)$. We want to define a simple structure on E depending only on these choices as follows. Recall that any homotopy class of paths w from b_0 to b_1 defines a homotopy class of homotopy equivalences $t_w : F_{b_0} \to F_{b_1}$ by the *fiber transport* [21], 15.12.

Let E_n be $f^{-1}(B_n)$. As $B_{n-1} \to B_n$ is a cofibration, the same is true for $E_{n-1} \to E_n$ [22], I. 7.14. Because of the construction 1.4 it suffices to specify a relative simple structure for (E_n, E_{n-1}) for all $n \geqq 0$. This will be done by the next Lemma 1.8 taking into account 1.6 and that the Whitehead torsion of any homotopy equivalence $(D^n, S^{n-1}) \to (D^n, S^{n-1})$ is trivial since D^n is simply-connected and hence $\mathrm{Wh}(D^n)$ is trivial. □

Lemma 1.8. *Suppose we have specified a cellular base point system for B and for each element in I_n an orientation. Then there is a relative homotopy equivalence which is uniquely defined up to homotopy*

$$\coprod_{c \in I_n} F_{b_c} \times (D^n, S^{n-1}) \to (E_n, E_{n-1}).$$

If we change the orientation of the cell c, then the map is changed by the selfhomotopy equivalence $\mathrm{id} \times s : F_{b_c} \times (D^n, S^{n-1}) \to F_{b_c} \times (D^n, S^{n-1})$ *where s is a map of degree -1. If we change the base point b_c of c to b'_c, then the map is changed by the homotopy equivalence* $t \times \mathrm{id} : F_{b_c} \times (D^n, S^{n-1}) \to F_{b'_c} \times (D^n, S^{n-1})$ *where $t : F_{b_c} \to F_{b'_c}$ is given by the fiber transport along any path in c° connecting b_c and b'_c.*

Proof. The map will be constructed as the composition of the following four maps or their homotopy inverses.

There is up to homotopy one orientation preserving homotopy equivalence of pairs $(D^n, S^{n-1}) \to (c^\circ, c^\circ - b_c)$. This gives the first map

$$\coprod_{c \in I_n} F_{b_c} \times (D^n, S^{n-1}) \to \coprod_{c \in I_n} F_{b_c} \times (c^\circ, c^\circ - b_c).$$

Choose a homotopy $\phi_c : c^\circ \times [0,1] \to B$ from the canonical inclusion $c^\circ \to B$ to the constant map with value b_c such that its evaluation at b_c gives a path within c°. By the homotopy lifting property we obtain a strong fiber homotopy equivalence which is unique up to fiber homotopy [21], Proposition 15.11

$$F_{b_c} \times (c^\circ, c^\circ - b_c) \to (E|_{c^\circ}, E|_{c^\circ - b_c}).$$

The second map is the disjoint union of these maps over I_n. The third map is the relative homotopy equivalence given by the inclusion

$$\coprod_{c \in I_n} (E|_{c^\circ}, E|_{c^\circ - b_c}) \to (E_n, E|_{B_n - \{\coprod_{c \in I_n} b_c\}}).$$

The fourth map is the homotopy equivalence of pairs given by the inclusion

$$(E_n, E_{n-1}) \to (E_n, E|_{B_n - \{\coprod_{c \in I_n} b_c\}}).$$

This finishes the proof of Lemma 1.8. □

1.9. Let $\{b_c \mid c \in I\}$ and $\{b'_c \mid c \in I\}$ be two base point systems and suppose F_{b_c} and $F_{b'_c}$ come with simple structures. Let σ and σ' be the simple structures on E given by the construction 1.7 for these two choices. Let $t_c : F_{b_c} \to F_{b'_c}$ be the homotopy equivalence which is given by the fiber transport along any path in c° from b_c to b'_c. Denote by $i(b'_c) : F_{b'_c} \to E$ the inclusion. Then we get from 1.6 and Lemma 1.8

$$\tau\big(\mathrm{id} : (E,\sigma) \to (E,\sigma')\big) = \sum_{n \geqq 0} (-1)^n \cdot \sum_{c \in I_n} i(b'_c)_* \tau(t_c : F_{b_c} \to F_{b'_c}) . \quad \square$$

Next we give a criterion when the choice of base point system does not affect the simple structure on E. Fix a base point $b \in B$. Given an element w in $\pi_1(B,b)$, define

$$\theta_f(w) = i(b)_* \tau(t_w : F_b \to F_b) \in \mathrm{Wh}(E)$$

where t_w is the fiber transport and $i(b) : F_b \to E$ is the inclusion. As $i\big(w(1)\big) \circ t_w \simeq i\big(w(0)\big)$ holds for any path w in B, one easily checks using 1.6 that this defines a homomorphism from $\pi_1(B,b)$ to $\mathrm{Wh}(E)$ and thus a cohomology class which is independent of the choice of $b \in B$

$$\theta_f \in H^1\big(B; \mathrm{Wh}(E)\big) . \tag{1.10}$$

Definition 1.11. A *choice of simple structures on the fibers* is a choice of simple structure on each fiber F_b. It is called *coherent* if for any path w in B we get

$$i\big(w(1)\big)_* \tau(t_w : F_{w(0)} \to F_{w(1)}) = 0 \in \mathrm{Wh}(E) . \quad \square$$

1.12. Notice that a coherent choice of simple structures on the fibers exists if and only if θ_f is trivial. If we fix a coherent choice of simple structures $\{\sigma(F_b) \mid b \in B\}$ on the fibers, the induced simple structure σ on E of 1.7 is independent of the cellular base point system by 1.9. Moreover, if we have a different choice of coherent structures on the fibers, then

$$\tau\big(\mathrm{id} : (E,\sigma) \to (E,\sigma')\big) = \chi(B) \cdot i(b)_*\Big(\tau\big(\mathrm{id} : \big(F_b, \sigma(F_b)\big) \to \big(F_b, \sigma'(F_b)\big)\big)\Big) .$$

In particular we see for a fibration $p : E \to B$ over a finite CW-complex B such that the homotopy fiber has the homotopy type of a finite CW-complex and $\chi(B) = 0$ and $\theta_f = 0$ holds that E has a preferred simple structure. □

Lemma 1.13. *Let $F \to E \xrightarrow{f} B$ be a smooth bundle of compact smooth manifolds. Equip B and each fiber with the simple structure given by a smooth triangulation. This gives a coherent choice of simple structures on the fibers. Then the simple structure on E given by a smooth triangulation agrees with the one given by* 1.7. □

Remark 1.14. The construction 1.7 can be extended to the case where B is not necessarily a CW-complex but carries a simple structure. Namely, choose a representative $g : X \to B$ of the simple structure. The pull back construction yields a fibration $\bar{f} : g^*E \to X$

and a fiber homotopy equivalence $\bar{g}: g^*E \to E$. Equip $\bar{f}$ with the coherent choice of simple structures on the fibers induced by $\bar{g}$ and the given one of f. Construction 1.12 applies to $\bar{f}$ and gives a simple structure on g^*E. Equip E with the simple structure for which $\tau(\bar{g})$ vanishes. It is not hard to check that this is independent of the choice of the representative g. The main step is to show in the case where X and B are finite CW-complexes and g is an elementary expansion that $\tau(\bar{g})$ vanishes with respect to the simple structures on E and g^*E given by construction 1.12. □

2. Milnor torsion for cochain complexes

In this section we give a brief introduction to torsion invariants of cochain complexes as defined in the introduction. We give no proofs but refer to [16] and [12].

Given linear isomorphisms f, g of finite-dimensional real Hilbert spaces, the number $[\![\cdot]\!]$ of 0.4 has the following properties:

$$[\![f]\!] = \ln\big(|\det(A)|\big) \in \mathbb{R} \tag{2.1}$$

where A is the matrix describing f with respect to some choice of orthonormal basis for source and range.

$$[\![f \circ g]\!] = [\![f]\!] + [\![g]\!],$$

$$\left[\!\!\left[\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\right]\!\!\right] = [\![f]\!] + [\![g]\!],$$

$$[\![f^*]\!] = [\![f]\!].$$

Let $C = C^*$ be a finite Hilbert cochain complex, i.e. a cochain complex of finite-dimensional Hilbert spaces such that $C^i = 0$ for $|i| \geqq N$ for some natural number N. If C is acyclic, we defined in (0.5) $\varrho(C) := [\![(c^* + \gamma^*) : C^{\mathrm{ev}} \to C^{\mathrm{odd}}]\!] \in \mathbb{R}$, where c^* is the differential and γ^* a chain contraction. This is independent of the choice of γ^*. For $f: C \to D$ a chain homotopy equivalence of finite Hilbert cochain complexes, $t(f)$ was defined in (0.6). It turns out that $t(f)$ depends only on the homotopy class of f and $t(f \circ g) = t(f) + t(g)$. Notice that with these conventions we get for an isomorphism $f: C \to D$ of finite Hilbert cochain complexes

$$t(f) = \sum_n (-1)^n \cdot [\![f^n]\!].$$

Let C be a finite Hilbert cochain complex such that $H(C^*)$ carries a Hilbert structure, i.e. $H^n(C^*)$ is equipped with a Hilbert space structure for each $n \in \mathbb{Z}$. If $i: H(C) \to C$ is the chain map which induces on cohomology the indentity, we defined $\varrho(C) := -t(i) \in \mathbb{R}$. The minus sign ensures that this definition coincides with the one in 0.5 in the acyclic case. If we fix an orthonormal basis for each C^n and each $H^n(C)$, then the logarithm of the torsion defined by Milnor [16], page 365, is $\varrho(C)$.

Let C be a finite Hilbert cochain complex and equip $\ker(c^p)$ and $\operatorname{im}(c^{p-1})$ with the Hilbert substructures and $H^p(C) = \ker(c^p)/\operatorname{im}(c^{p-1})$ and $C^p/\ker(c^p)$ with the quotient

Hilbert structures. If $\overline{c^p} : C^p/\ker(c^p) \to \operatorname{im}(c^p)$ is the obvious isomorphism induced by c^p and $\overline{\Delta^p} : C^p/\ker(\Delta^p) \to C^p/\ker(\Delta^p)$ is the automorphism induced by the endomorphism $\Delta^p = (c^p)^* \circ c^p + c^{p-1} \circ (c^{p-1})^* : C^p \to C^p$, then we get

$$\varrho(C) = \sum_p (-1)^p \cdot [\![\overline{c^p}]\!] = -\sum_p (-1)^p \cdot p \cdot \frac{1}{2} \cdot \ln\big(\det(\overline{\Delta^p})\big). \tag{2.2}$$

A *simple structure* on a (real) cochain complex C is an equivalence class of chain homotopy equivalences $u : \bar{C} \to C$ with a finite Hilbert cochain complex as source where u and $v : \bar{\bar{C}} \to C$ are equivalent if $t(v^{-1} \circ u)$ vanishes. Let $f : C \to D$ be a chain homotopy equivalence of cochain complexes with simple structure. Define

$$t(f) = t(v^{-1} \circ f \circ u) \in \mathbb{R} \tag{2.3}$$

for any representatives $u : \bar{C} \to C$ and $v : \bar{D} \to D$ of the simple structures. Let C be a cochain complex with simple structure such that $H(C)$ has a Hilbert structure. Define

$$\varrho(C) := \varrho(\bar{C}) \in \mathbb{R} \tag{2.4}$$

for any representative $u : \bar{C} \to C$ where we use the Hilbert structure on $H(\bar{C})$ for which $H(u)$ is an isometry.

Next we collect the basic properties of these invariants. We mention that one firstly verifies them for finite Hilbert cochain complexes and uses this to show that the definitions and results extend to cochain complexes with simple structures.

If $0 \to C \to D \to E \to 0$ is an exact sequence of finite Hilbert cochain complexes, we can view $C^n \to D^n \to E^n$ as an acyclic finite Hilbert cochain complex concentrated in dimension 0, 1 and 2 and define

$$\varrho(C \to D \to E) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot \varrho(C^n \to D^n \to E^n).$$

This extends to an exact sequence $0 \to C \to D \to E \to 0$ of cochain complexes with simple structures as follows. Construct a commutative diagram of cochain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\ & & {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & \\ 0 & \longrightarrow & \bar{C} & \longrightarrow & \bar{D} & \longrightarrow & \bar{E} & \longrightarrow & 0 \end{array}$$

with the property that the rows are exact and the vertical arrows are homotopy equivalences which represent the given simple structures. In particular the lower row is an exact sequence of finite Hilbert cochain complexes and we put

$$\varrho(C \to D \to E) := \varrho(\bar{C} \to \bar{D} \to \bar{E}). \tag{2.5}$$

2.6. In the following list of basic properties all cochain complexes come with simple structures.

1. Homotopy invariance.

$$f \simeq g \Rightarrow t(f) = t(g)\,.$$

2. Composition formula.

$$t(f \circ g) = t(f) + t(g)\,.$$

3. Exactness.

Given a commutative diagram of the shape above with exact rows and homotopy equivalences as vertical arrows, then

$$t(f) - t(g) + t(h) = \varrho(\bar{C} \to \bar{D} \to \bar{E}) - \varrho(C \to D \to E)\,.$$

4. Sum formula.

Let $0 \to C \to D \to E \to 0$ be exact and the cohomolgy of C, D and E come with Hilbert structures. Let LHS be the acyclic finite Hilbert cochain complex given by the long cohomology sequence where $LHS^0 = H^0(C)$. Then:

$$\varrho(D) - \varrho(C) - \varrho(E) = \varrho(LHS) - \varrho(C \to D \to E)\,.$$

5. Transformation formula.

If $f\colon C \to D$ is a homotopy equivalence and the cohomology of C and D come with Hilbert structures, then:

$$\varrho(C) - \varrho(D) = t(f) - \sum_{n \in \mathbb{Z}} (-1)^n \cdot [\![H^n(f)]\!]\,.$$

3. Milnor torsion for spaces and local coefficient systems

The fundamental groupoid $\Pi(X)$ of a space X has as objects points in X and a morphism $w : x \to y$ is a homotopy class relative to end points of paths in X from y to x. For $x \in X$ let $\tilde{X}(x)$ be the set of all morphisms $w : x \to y$ with x as source. The projection $p(x) : \tilde{X}(x) \to X$ sends w to y. Assume that X is locally path-connected and semi-locally simply connected. This condition is always satisfied if X is a CW-complex or manifold. It ensures that there is precisely one topology on $\tilde{X}(x)$ such that $p(x)$ is a model for the universal covering of the path component of X containing x. Thus we get a contravariant functor

$$\tilde{X} : \Pi(X) \to \text{SPACES}\,.$$

Composing it with the covariant functor singular chain complex with real coefficients yields the contravariant functor

$$C_*^{\text{sing}}(\tilde{X}) : \Pi(X) \to \mathbb{R} - \text{CHAIN}\,.$$

A local coefficient system V on X is a contravariant functor $V\colon \Pi(X) \to \mathbb{R} - \text{VECTOR}$ into the category of finite-dimensional real vector spaces. For instance a flat vector bundle over X defines a local coefficient system. Define the *singular cochain complex and the singular cohomology of X with coefficient in V* by

$$\begin{aligned}(3.1)\qquad C^*_{\mathrm{sing}}(X;V) &:= \hom\big(C^{\mathrm{sing}}_*(\tilde{X}),V\big),\\ H^*_{\mathrm{sing}}(X;V) &:= H^*\big(C^*_{\mathrm{sing}}(X;V)\big)\end{aligned}$$

where hom denotes the real vector space of natural transformations.

Suppose that X is a CW-complex. Then $\tilde{X}$ becomes a functor from $\Pi(X)$ into the category of CW-complexes and we can define the cellular versions $C^*_{\mathrm{cell}}(X;V)$ and $H^*_{\mathrm{cell}}(X;V)$ of the definitions above by using the cellular chain complex instead of the singular one. There is a homotopy equivalence

$$(3.2)\qquad C^*_{\mathrm{cell}}(X;V) \to C^*_{\mathrm{sing}}(X;V)$$

which is unique up to homotopy and natural in X and V ([10], page 263). It induces a natural isomorphism $H^*_{\mathrm{cell}}(X;V) \to H^*_{\mathrm{sing}}(X;V)$.

Let X be a space and $V^0, V^1, \ldots, V^r$ local coefficient systems. We say $\sum_{q=0}^{r} (-1)^q \cdot V^q$ is *unimodular* if we have for each automorphism $w : x \to x$ in $\Pi(X)$

$$\prod_{q=0}^{r} |\det(V^q_w : V^q_x \to V^q_x)|^{(-1)^q} = 1 .$$

A *Hilbert structure* on $\sum_{q=0}^{r} (-1)^q \cdot V^q$ is a choice of Hilbert space structure on V^q_x for each $q \in \{1,2,\ldots,r\}$ and $x \in X$. It is called *unimodular* if we have for each morphism $w : y \to x$ in $\Pi(X)$

$$\sum_{q=0}^{r} (-1)^q \cdot [\![V^q_w : V^q_x \to V^q_y]\!] = 0$$

in the notation of 0.4. Notice that $\sum_{q=0}^{r} (-1)^q \cdot V^q$ is unimodular if and only if it admits a unimodular Hilbert structure.

Definition 3.3. Let X be a space and $V^0, V^1, \ldots, V^r$ be local coefficient systems. Assume that X comes with a simple structure, $\sum_{q=0}^{r} (-1)^q \cdot V^q$ with a unimodular Hilbert structure and $H^*_{\mathrm{sing}}(X;V^q)$ with a Hilbert structure. Define the *Milnor torsion*

$$\varrho\left(X; \sum_{q=0}^{r} (-1)^q \cdot V^q\right) := \sum_{q=0}^{r} (-1)^q \cdot \varrho\big(C^*_{\mathrm{sing}}(X;V^q)\big)$$

where $\varrho\big(C^*_{\mathrm{sing}}(X;V^q)\big)$ was defined in (2.4) and we use the simple structure on $C^*_{\mathrm{sing}}(X;V^q)$ defined below. □

Let $f\colon Y \to X$ be a representative for the simple structure on X. Fix a cellular base point system $\{b_c \mid c \in I\}$, i.e. a choice of base points $b_c \in c^\circ$ for each cell c of Y, and an orientation for each cell c. For each cell c there is precisely one lift $\tilde{c} \subset \tilde{Y}(b_c)$ which contains the canonical base point in $\tilde{Y}(b_c)$ given by the constant path at b_c. The orientation of c induces an orientation of $\tilde{c}$. Thus we get an element $[\tilde{c}] \in C^{\mathrm{cell}}_{\dim(c)}(\tilde{Y}(b_c))$. Define an isomorphism

$$(3.4) \qquad C^n_{\mathrm{cell}}(Y, f^* V^q) \to \bigoplus_{c \in I_n} V^q_{f(b_c)}, \quad t \mapsto \big(t(b_c)([\tilde{c}])\big)_{c \in I_n}.$$

Equip $C^n_{\mathrm{cell}}(Y, f^* V^q)$ with the Hilbert structure induced by the isomorphism 3.4 above and the given Hilbert space structure on the various $V^q_{f(b_c)}$-s. The desired simple structure on $C^*_{\mathrm{sing}}(X; V^q)$ is represented by the composition of the chain homotopy equivalences given by (3.2) and f

$$C^*_{\mathrm{cell}}(Y; f^* V^q) \to C^*_{\mathrm{sing}}(Y; f^* V^q) \xrightarrow{(f^*)^{-1}} C^*_{\mathrm{sing}}(X; V^q).$$

The choice of the orientations of the cells does not affect this simple structure. If we change the base point system, the simple structure and hence $\varrho\big(C^*_{\mathrm{sing}}(X; V^q)\big)$ changes. However, $\sum_{q=0}^{r} (-1)^q \cdot \varrho\big(C_{\mathrm{sing}}(X; V)\big)$ does not change because of the formulas 2.6 and the assumption that the Hilbert structure on $\sum_{q=0}^{r} (-1)^q \cdot V^q$ is unimodular.

If $\sum_{q=0}^{r} (-1)^q \cdot V^q$ is unimodular, we can define a homomorphism

$$(3.5) \qquad \Phi\left(\sum_{q=0}^{r} (-1)^q \cdot V^q\right) : \mathrm{Wh}(X) \to \mathbb{R}$$

as follows. An element $[u] \in \mathrm{Wh}(X)$ can be represented by an automorphism $u: M \to M$ of a finitely generated free $\mathbb{Z}\Pi(X)$-module M. Composition with u defines an automorphism $u^q : \hom(M, V^p) \to \hom(M, V^q)$ of a finite-dimensional real vector space. Put

$$\Phi\left(\sum_{q=0}^{r} (-1)^q \cdot V^q\right)([u]) := \sum_{q=0}^{r} (-1)^q \cdot \ln|\det(u^q)|.$$

The unimodularity condition on $\sum_{q=0}^{r} (-1)^q \cdot V^q$ ensures that trivial units are send to 0. The following lemma is a consequence of the definitions and the formulas 2.6.

Lemma 3.6. *Let $f\colon X \to Y$ be a homotopy equivalence of spaces with simple structures. Let $V^0, V^1, \dots, V^r$ be local coefficient systems on Y. Assume that $V := \sum_{q=0}^{r} (-1)^q \cdot V^q$ comes with a unimodular Hilbert structure and $H^*_{\mathrm{sing}}(X; f^* V^q)$* and *$H^*_{\mathrm{sing}}(Y; V^q)$ come with Hilbert structures. Then:*

$$\varrho(Y;V) - \varrho(X;f^*V) = \Phi(V)\big(\tau(f)\big) - \sum_{q=0}^{r} (-1)^q \cdot \sum_{p \geqq 0} (-1)^p \cdot [\![f^p : H^p_{\mathrm{sing}}(Y;V^q) \to H^p_{\mathrm{sing}}(X;f^*V^q)]\!]. \square$$

4. Torsion and spectral sequences

Let C be a cochain complex with finite descending filtration by cochain complexes F_pC

$$C = F_0C \supset F_1C \supset F_2C \supset \cdots \supset F_pC \supset F_{p+1}C \supset \cdots.$$

Finite means that there is a natural number l with $F_lC = \{0\}$. Put $F_pC = C$ for $p \leqq -1$. We recall the construction of the associated spectral cohomology sequence $(E_*^{*,*}, d_*^{*,*})$ converging to $H^*(C)$ since we will need it explicitely (see [2] or [15]). In the sequel ∂ denotes the boundary operator of the long exact cohomology sequence associated to a short exact sequence of cochain complexes. We will abbreviate F_pC by F_p. Define for $r \geqq 1$:

$$\begin{aligned}
Z_r^{p,q} &:= \mathrm{im}\big(H^{p+q}(F_p/F_{p+r}) \to H^{p+q}(F_p/F_{p+1})\big),\\
B_r^{p,q} &:= \mathrm{im}\big(H^{p+q-1}(F_{p-r+1}/F_p) \xrightarrow{\partial} H^{p+q}(F_p/F_{p+1})\big),\\
Z_\infty^{p,q} &:= \mathrm{im}\big(H^{p+q}(F_p) \to H^{p+q}(F_p/F_{p+1})\big),\\
B_\infty^{p,q} &:= \mathrm{im}\big(H^{p+q-1}(C/F_p) \xrightarrow{\partial} H^{p+q}(F_p/F_{p+1})\big),\\
E_r^{p,q} &:= Z_r^{p,q}/B_r^{p,q},\\
E_\infty^{p,q} &:= Z_\infty^{p,q}/B_\infty^{p,q},\\
F^{p,q} &:= \mathrm{im}\big(H^{p+q}(F_p) \to H^{p+q}(C)\big).
\end{aligned}$$

We have the inclusions:

$$\{0\} = B_1^{p,q} \subset \cdots \subset B_r^{p,q} \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset Z_r^{p,q} \subset \cdots \subset Z_1^{p,q} = H^{p+q}(F_p/F_{p+1}).$$

The map $H^{p+q}(F_p/F_{p+r}) \to H^{p+q}(F_p/F_{p+1})$ resp. $H^{p+q}(F_p/F_{p+r}) \to H^{p+q+1}(F_{p+r}/F_{p+r+1})$ is induced by the inclusion resp. is a boundary operator. We get epimorphisms

$$H^{p+q}(F_p/F_{p+r}) \to Z_r^{p,q}/Z_{r+1}^{p,q}$$

and

$$H^{p+q}(F_p/F_{p+r}) \to B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

Now the standard diagram chase shows that these maps have the same kernel. Hence we obtain canonical isomorphisms

$$\gamma_r^{p,q} : Z_r^{p,q}/Z_{r+1}^{p,q} \to B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \quad \text{for } r \geqq 1 .$$

Analogously one gets a natural isomorphism

$$\psi^{p,q} : F^{p,q}/F^{p+1,q-1} \to E_\infty^{p,q} .$$

We define the differential

$$d_r^{p,q} : E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} \to E_r^{p+r,q-r+1} = Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$$

by the composition:

$$Z_r^{p,q}/B_r^{p,q} \to Z_r^{p,q}/Z_{r+1}^{p,q} \xrightarrow{\gamma_r^{p,q}} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \to Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1} .$$

We obtain cochain complexes

$$(E_r^{p+r*,q-(r-1)*}, d_r^{p+r*,q-(r-1)*}), \quad r \geqq 0$$

if we use for $r \geqq 1$ the definition above and define $E_0^{p,*}$ to be the $(-p)$-th suspension $\Sigma^{-p} F_p/F_{q+1}$ of F_p/F_{p+1}. Since $\ker(d_r^{p,q})$ is $Z_{r+1}^{p,q}/B_r^{p,q}$ and $\operatorname{im}(d_r^{p,q})$ is

$$B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} ,$$

we obtain a canonical isomorphism

$$\phi_r^{p,q} : H^0(E_r^{p+r*,q-(r-1)*}) \to E_{r+1}^{p,q}$$

in the case $r \geqq 1$, in the case $r = 0$ we use the identification $H^0(E_0^{p,q+*}) = H^{p+q}(F_p/F_{p+1})$.

4.1. Now suppose we have fixed the following data:

1. Simple structures on F_p/F_{p+1};

2. a Hilbert structure on $H(C)$. $\square$

Equip F_p inductively with the simple structure for which the number defined in (2.5) satisfies

$$\varrho(F_{p+1} \to F_p \to F_p/F_{p+1}) = 0 .$$

In particular we get a preferred simple structure on $C = F_0$ and $\varrho(C)$ is defined. Equip $F^{p,q} \subset H^{p+q}(C)$ with the Hilbert substructure and $F^{p,q}/F^{p+1,q-1}$ with the Hilbert quotient structure. Do iteratively the same for $E_r^{p,q}$ and $H(E_r^{p+r*,q-(r-1)*})$. Observe that $\phi_r^{p,q}$ become isometries. Notice that 4.1.1 implies that $E_r^{p,q}$ is finite-dimensional for $r \geqq 1$.

Definition 4.2. Define for the finite descending filtration $F_* C$ with respect to the data 4.1 and choices above:

$$\varrho_{\mathrm{fil}}^{\geqq 2}(F_* C) := \sum_{r \geqq 2} \sum_{p=0}^{r-1} \sum_q (-1)^{p+q} \cdot \varrho(E_r^{p+r*, q-(r-1)*}) - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!],$$

$$\begin{aligned} \varrho_{\mathrm{fil}}(F_* C) := \sum_p \varrho(F_p/F_{p+1}) + \sum_{r \geqq 1} \sum_{p=0}^{r-1} \sum_q (-1)^{p+q} \cdot \varrho(E_r^{p+r*, q-(r-1)*}) \\ - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!]; \end{aligned}$$

where ϱ was defined in (2.4) and $[\![\ ]\!]$ in (0.4). □

Remark 4.3. $\varrho_{\mathrm{fil}}(F_* C)$ depends on the choices 4.1.

For the definition of $\varrho_{\mathrm{fil}}^{\geqq 2}(F_* C)$ one can fix Hilbert structures on $E_2^{p,q}$ instead of data 4.1.1 and equip $E_r^{p,q}$ and $H(E_r^{p+r*, q-(r-1)*})$ $(r \geqq 2)$ with the corresponding Hilbert sub- and quotient structures. Then, $\varrho_{\mathrm{fil}}^{\geqq 2}(F_* C)$ depends not on 4.1.1 but on these choices as follows: if $U_2^{p,q} : \bar{E}_2^{p,q} \to E_2^{p,q}$ is the identity on E_2 equipped with two different Hilbert structures, then

$$\overline{\varrho_{\mathrm{fil}}^{\geqq 2}(F_* C)} - \varrho_{\mathrm{fil}}^{\geqq 2}(F_* C) = \sum_{p,q} (-1)^{p+q} [\![U_2^{p,q}]\!]$$

(torsion computed using these two Hilbert structures). This follows from the transformation formula 2.6.

The main result of this section is:

Theorem 4.4. *We get with respect to the data* 4.1 *and the conventions above*

$$\varrho(C) = \varrho_{\mathrm{fil}}(F_* C).$$

It will follow from the next three lemmas.

Lemma 4.5. *It suffices to treat the following special case*: *C itself is a finite Hilbert cochain complex with the simple structure represented by* $\mathrm{id} : C \to C$ *and the Hilbert structures on* F_p *and* F_p/F_{p+1}, *are obtained by the given one on C by taking Hilbert sub- and quotient structures.*

Proof. One easily constructs a finite Hilbert cochain complex D with finite cofiltration $F_* D$ together with a chain homotopy equivalence $f : D \to C$ with the following property: f induces chain homotopy equivalences $F_p f : F_p D \to F_p C$ for all p such that the given simple structure on $F_p C/F_{p+1} C$ is represented by $F_p f/F_{p+1} f : F_p D/F_{p+1} D \to F_p C/F_{p+1} C$. Equip $H(D)$ with the Hilbert structure for which $H(f) : H(D) \to H(C)$ becomes an isometry. We have by assumption and construction for all p that

$$\varrho(F_{p+1} D \to F_p D \to F_p D/F_{p+1} D) = 0 = \varrho(F_{p+1} C \to F_p C \to F_p C/F_{p+1} C).$$

Hence we conclude from 2.6

$$\varrho(C) = \varrho(D).$$

The maps $F_p f$ induce isometries between the $E_r^{p+r*,q-(r-1)*}$ for $r \geqq 1$ and $H(E_r^{p+r*,q-(r-1)*})$ for $r \geqq 0$ associated to $F_* D$ and $F_* C$. Now one easily verifies

$$\varrho_{\mathrm{fil}}(F_* C) = \varrho_{\mathrm{fil}}(F_* D).$$

This finishes the proof of Lemma 4.5. $\square$

From now on we will only consider the special case described in Lemma 4.5. The proof of the next lemma is similiar to the proof in [19], Theorem 2.2.

Lemma 4.6.

$$\varrho(C) = \sum_p \varrho(F_p/F_{p+1}) + \sum_{r \geqq 1} \sum_{p,q} (-1)^{p+q} \cdot [\![\gamma_r^{p,q}]\!] - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!].$$

Proof. We do induction over l with $F_l C = \{0\}$. The begin of induction $l = 0$ is trivial, the induction step from $l-1$ to $l \geqq 1$ done as follows.

Let $\bar{C}$ be $F_1 = F_1 C$ with the cofiltration

$$\bar{F}_p = F_p \bar{C} := F_{p+1} C.$$

We will denote the various data coming from the spectral sequence associated to $\bar{C}$ as the ones for C decorated with an additional bar. Let LHS be the acyclic finite Hilbert cochain complex given by the long cohomology sequence associated to $0 \to \bar{C} \to C \to C/\bar{C} \to 0$. It induces for $n \geqq 0$ an acyclic finite Hilbert cochain complex $D(n)$ concentrated in dimensions 0, 1, 2 and 3 by

$$H^n(C)/F^{1,n-1} \to H^n(C/\bar{C}) \to H^{n+1}(\bar{C}) \to F^{1,n}$$

where we equip $F^{1,n-1} = \operatorname{im}(H^n(\bar{C}) \to H^n(C))$ respectively $H^{n+1}(C)/F^{1,n}$ with the Hilbert substructure respectively quotient structure. Define an acyclic Hilbert subcochain complex $D(n,r)$ for $r \geqq 1$ of $D(n)$ by

$$H^n(C)/F^{1,n-1} \to Z_r^{0,n} \to \bar{F}^{r-1,n+2-r} \to F^{r,n+1-r}.$$

Notice that $D(n,1) = D(n)$. The following diagram of acyclic finite Hilbert cochain complexes concentrated in dimensions 1, 2 and 3 commutes:

$$\begin{array}{ccccc}
Z_r^{0,n}/Z_{r+1}^{0,n} & \longrightarrow & \bar{F}^{r-1,n+2-r}/\bar{F}^{r,n+1-r} & \longrightarrow & F^{r,n+1-r}/F^{r+1,n-r} \\
\gamma_r^{0,n}\big\downarrow & & \bar{\psi}^{r-1,n+2-r}\big\downarrow & & \psi^{r,n+1-r}\big\downarrow \\
B_{r+1}^{r,n+1-r}/B_r^{r,n+1-r} & \longrightarrow & \bar{Z}_\infty^{r-1,n+2-r}/\bar{B}_\infty^{r-1,n+2-r} & \longrightarrow & Z^{r,n+1-r}/B_\infty^{r,n+1-r}
\end{array}$$

where the upper row is the quotient $D(n,r)/D(n,r+1)$ and the lower row is induced by the obvious inclusions and projections if one takes the following identities into account:

$$\begin{aligned}
\bar{B}_{\infty}^{r-1,n+2-r} &= B_r^{r,n+1-r};\\
B_{r+1}^{r,n+1-r} &= B_{\infty}^{r,n+1-r};\\
\bar{Z}_{\infty}^{r-1,n+2-r} &= Z_{\infty}^{r,n+1-r}.
\end{aligned}$$

We conclude from the sum formula and transformation formula 2.6

$$\begin{aligned}
\varrho(C) &= \varrho(\bar{C}) + \varrho(C/\bar{C}) + \varrho(LHS),\\
\varrho(LHS) &= \sum_{n \geqq -1} (-1)^{n+1} \varrho\big(D(n)\big),\\
\varrho\big(D(n)\big) &= \varrho\big(D(n,l)\big) + \sum_{r=1}^{l-1} \varrho\big(D(n,r)/D(n,r+1)\big),\\
\varrho\big(D(n,r)/D(n,r+1)\big) &= -[\![\gamma_r^{0,n}]\!] + [\![\bar{\psi}^{r-1,n+2-r}]\!] - [\![\psi^{r,n+1-r}]\!] \quad \text{for } 1 \leqq r \leqq l.
\end{aligned}$$

Since $D(n,l)$ is concentrated in dimensions 0 and 1 and its zero-th differential is $\psi^{0,n}$, we get

$$\varrho\big(D(n,l)\big) = [\![\psi^{0,n}]\!].$$

We compute using the fact

$$\bar{\gamma}_r^{p,q} = \gamma_r^{p+1,q-1} \quad \text{for } p \geqq 1$$

and the induction hypothesis applied to $\bar{C}$:

$$\begin{aligned}
\varrho(C) &= \varrho(\bar{C}) + \varrho(F_0/F_1)\\
&\quad + \sum_{n \geqq -1} (-1)^{n+1} \cdot \left([\![\psi^{0,n}]\!] + \sum_{r=1}^{l-1} (-[\![\gamma_r^{0,n}]\!] + [\![\bar{\psi}^{r-1,n+2-r}]\!] - [\![\psi^{r,n+1-r}]\!]) \right)\\
&= \sum_{p \geqq 1} \varrho(F_p/F_{p+1}) + \sum_{r \geqq 1} \sum_{p \geqq 1, q} (-1)^{p+q-1} [\![\gamma_r^{p,q-1}]\!] - \sum_{p,q} (-1)^{p+q} \cdot [\![\bar{\psi}^{p,q}]\!]\\
&\quad + \varrho(F_0/F_1)\\
&\quad + \sum_{n \geqq -1} (-1)^{n+1} \cdot \left([\![\psi^{0,n}]\!] + \sum_{r=1}^{l-1} - [\![\gamma_r^{0,n}]\!] + [\![\bar{\psi}^{r-1,n+2-r}]\!] - [\![\psi^{r,n+1-r}]\!] \right)\\
&= \sum_{p} \varrho(F_p/F_{p+1}) + \sum_{r \geqq 1} \sum_{p,q} (-1)^{p+q} \cdot [\![\gamma_r^{p,q}]\!] - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!].
\end{aligned}$$

This finishes the proof of Lemma 4.6. □

Lemma 4.7.

$$\varrho_{\text{fil}}(F_* C) = \sum_{p} \varrho(F_p/F_{p+1}) + \sum_{r \geqq 1} \sum_{p,q} (-1)^{p+q} \cdot [\![\gamma_r^{p,q}]\!] - \sum_{p,q} (-1)^{p+q} \cdot [\![\psi^{p,q}]\!].$$

Proof. We get from 2.2

$$\varrho(E_r^{p+r*,q-(r-1)*}) = \sum_{k} (-1)^k \cdot [\![\gamma_r^{p+rk,q-(r-1)k}]\!].$$

This implies

$$\sum_{r\geqq 1}\sum_{p,q}(-1)^{p+q}\cdot[\![\gamma_r^{p,q}]\!] = \sum_{r\geqq 1}\sum_{p=0}^{r-1}\sum_{q}(-1)^{p+q}\cdot\sum_{k}(-1)^k\cdot[\![\gamma_r^{p+rk,q-(r-1)k}]\!]$$
$$= \sum_{r\geqq 1}\sum_{p=0}^{r-1}\sum_{q}(-1)^{p+q}\cdot\varrho(E_r^{p+r*,q-(r-1)*}).$$

This finishes the proof of Lemma 4 and hence of Theorem 4.4. □

5. The fibration formula for Milnor torsion

For the remainder of this section we suppose that we have a fibration $f\colon E\to B$ so that B is a finite connected CW-complex, the homotopy fiber has the homotopy type of a finite CW-complex, the class $\theta_f\in H^1(B;\mathrm{Wh}(E))$ defined in (1.10) is trivial and E is locally path-connected.

5.1. Moreover, we assume that we are also given the following data:

1. A local coefficient system V on E with a unimodular Hilbert structure;

2. a Hilbert structure on $H^*_{\mathrm{sing}}(E;V)$;

3. a coherent choice of simple structures on the fibers;

4. a unimodular Hilbert structure on $\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)$ for the local coefficient systems $H^q_{\mathrm{sing}}(F;V)$ over B;

5. a Hilbert structure on $H^*_{\mathrm{cell}}(B;H^q_{\mathrm{sing}}(F;V))$. □

We mention that a choice of a unimodular Hilbert structure on $\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)$ is possible because of

Lemma 5.2. *If $f\colon E\to B$ is a fibration as described above, then $\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)$ is unimodular.*

Proof. We get from Lemma 3.6 for a loop w in B with base point b if $t_w\colon F_b\to F_b$ is given by the fiber transport along w:

$$\sum_q(-1)^q\cdot[\![(t_w)^*\colon H^q_{\mathrm{sing}}(F_b;V)\to H^q_{\mathrm{sing}}(F_b;V)]\!] = \varrho(F_b;V)-\varrho(F_b;V)+\Phi(V)\big(\theta_f(w)\big)=0$$

where we think of θ_f as homomorphism $\pi_1(B,b)\to\mathrm{Wh}(E)$ and $\Phi(V)$ was introduced in 3.5. □

Notice that $\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)$ is unimodular but not necessarily each $H^q_{\mathrm{sing}}(F;V)$.

5.3. We will consider the following real numbers:

1. $\varrho(E;V)$.

We get by construction 1.7 a simple structure on E from the data 5.1.3 if we specify a cellular base point system on B. However, by 1.9 the choice of cellular base point system does not matter. Now we use this simple structure and data 5.1.1 and 5.1.2 to define the Milnor torsion $\varrho(E;V)$ according to Definition 3.3. Notice that it depends only on 5.1.1, 5.1.2 and 5.1.3. This is the invariant we want to compute.

2. $\varrho(F;V)$.

Choose $b \in B$. Then we get from Definition 3.3 applied to data 5.1.1, 5.1.3 and 5.1.4 the Milnor torsion $\varrho(F_b;V)$. Here and elsewhere we supress in the notation that we view V as a local coefficient system over F_b by the inclusion of F_b into E. We get from Lemma 3.6 that $\varrho(F_b\colon V)$ is independent of b since B is path connected. We abbreviate

$$\varrho(F;V) = \varrho(F_b;V)\,.$$

This number depends only on 5.1.1, 5.1.3 and 5.1.4.

3. $\varrho\big(B;\sum_q (-1)^q \cdot H^q_{\mathrm{sing}}(F;V)\big)$.

This is the Milnor torsion and depends on the data 5.1.4 and 5.1.5.

4. $\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)$.

We have the Leray-Serre spectral sequence for singular cohomology associated to the fibration $f\colon E \to B$. Namely, the skeletal filtration of B induces a filtration $E_p = f^{-1}(B_p)$. It yields a cofiltration on $C^*_{\mathrm{sing}}(E;V)$ by putting

$$F_p C^*_{\mathrm{sing}}(E;V) = C^*_{\mathrm{sing}}(E, E_{p-1};V)\,.$$

Recall that $C^*_{\mathrm{sing}}(E, E_{p-1};V)$ is the kernel of the canonical map $C^*_{\mathrm{sing}}(E;V) \to C^*_{\mathrm{sing}}(E_{p-1};V)$ induced from the inclusion . We will later recall in (5.5) the isomorphism which identifies the E_2-term of the associated spectral sequence

$$U_2^{p,q}\colon H^p_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big) \to E_2^{p,q}\,.$$

Equip $E_2^{p,q}$ with the Hilbert structure so that $U_2^{p,q}$ becomes an isometry. Using Definition 4.2 put

$$\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f) := \varrho^{\geqq 2}_{\mathrm{fil}}\big(F_* C^*_{\mathrm{sing}}(X;V)\big) + \sum_{p,q} (-1)^{p+q} [\![U_2^{p,q}]\!]\,.$$

Using Remark 4.3, $\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)$ depends only on the data 5.1.2 and 5.1.5.

The main result of this section is

Theorem 5.4. *If F is a fibration as described above with data* 5.1 *then*

$$\varrho(E;V)=\chi(B)\cdot\varrho(F;V)+\varrho\big(B;\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)\big)+\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)\,.$$

Proof. Fix a cellular base point system $\{b_c\,|\,c\in I\}$ for B and an orientation for each cell $c\in I$. The homotopy equivalence of Lemma 1.8 together with the suspension cochain homotopy equivalence and the obvious identification of F_p/F_{p+1} with $C^*_{\mathrm{sing}}(E_p,E_{p-1};V)$ yields a cochain homotopy equivalence unique up to homotopy

$$U_0^{p,*}\colon\bigoplus_{c\in I_p}\Sigma^p C^*_{\mathrm{sing}}(F_{b_c};V)\to F_p/F_{p+1}\,.$$

We have defined an isomorphism in (3.4)

$$\mu^{p,q}\colon C^p_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)\to\bigoplus_{c\in I_p}H^q_{\mathrm{sing}}(F_{b_c};V)\,.$$

Recall that we equip $C^p_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)$ with the Hilbert structure for which $\mu^{p,q}$ becomes an isometry. Define an isomorphism by the composition

$$\begin{aligned}U_1^{p,q}\colon C^p_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)&\xrightarrow{\mu^{p,q}}\bigoplus_{c\in I_p}H^q_{\mathrm{sing}}(F_{b_c};V)=\bigoplus_{c\in I_p}H^{p+q}\big(\Sigma^pC^*_{\mathrm{sing}}(F_{b_c};V)\big)\\&\xrightarrow{H^{p+q}(U_0^{p,*})}H^{p+q}(F_p/F_{p+1})=H^0(E_0^{p,q+*})\xrightarrow{\phi_0^{p,q}}E_1^{p,q}\,.\end{aligned}$$

This isomorphism is independent of the choice of base point system and orientation of the cells and is compatible with the differentials on $C^*_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)$ and $E_1^{*,q}$. Hence we get a canonical isomorphism

$$(5.5)\quad U_2^{p,q}\colon H^p_{\mathrm{cell}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)\xrightarrow{H^p(U_1^{*,q})}H^p(E_1^{*,q})=H^0(E_1^{p+*,q})\xrightarrow{\phi_1^{p,q}}E_2^{p,q}\,.$$

The simple structure on F_{b_c} induces one on $C^*_{\mathrm{sing}}(F_{b_c};V)$. Put on F_p/F_{p+1} the simple structure for which the torsion $t\big(U_0^{p,*}\colon\bigoplus_{c\in I_p}\Sigma^pC^*_{\mathrm{sing}}(F_{b_c};V)\to F_p/F_{p+1}\big)$ defined in 2.3 vanishes. Equip F_p inductively with the simple structure for which $\varrho(F_{p+1}\to F_p\to F_p/F_{p+1})$ defined in (2.5) vanishes. Then this simple structure on $F_0=C^*_{\mathrm{sing}}(E;V)$ agrees with the one we get from the simple structure on E which we have defined in 1.7 by an inductive procedure over the E_n-s and Lemma 1.8. We conclude from Theorem 4.4:

$$\varrho(E;V)=\varrho_{\mathrm{fil}}\big(F_*C^*_{\mathrm{sing}}(E;V)\big)\,.$$

Because $\phi_r^{p,q}$ are isometries, we get from the transformation formula 2.6:

$$\begin{aligned}\sum_p\varrho(F_p/F_{p+1})&=\sum_p\sum_{c\in I_c}\varrho\big(\Sigma^pC^*_{\mathrm{sing}}(F_{b_c};V)\big)+\sum_p\sum_q(-1)^q\cdot\big[\!\big[H^q(U_0^{p,*})\big]\!\big]\\&=\sum_p(-1)^p\cdot\sum_{c\in I_p}\varrho(F;V)+\sum_{p,q}(-1)^{p+q}\cdot\big[\!\big[H^{p+q}(U_0^{p,*})\big]\!\big]\\&=\chi(B)\cdot\varrho(F;V)+\sum_{p,q}(-1)^{p+q}\cdot\big[\!\big[U_1^{p,q}\big]\!\big],\end{aligned}$$

$$\begin{aligned}\sum_q (-1)^q \cdot \varrho(E_1^{*,q}) &= \sum_q (-1)^q \cdot \varrho\big(C^*_{\mathrm{cell}}\big(B; H^q_{\mathrm{sing}}(F;V)\big)\big) - \sum_q (-1)^q \cdot \sum_p (-1)^p \cdot \llbracket U_1^{p,q} \rrbracket \\ &\quad + \sum_q (-1)^q \cdot \sum_p (-1)^p \cdot \llbracket H^p(U_1^{*,q}) \rrbracket \\ &= \varrho\Big(B; \sum_q (-1)^q \cdot H^q(F;V)\Big) - \sum_{p,q} (-1)^{p+q} \cdot \llbracket U_1^{p,q} \rrbracket \\ &\quad + \sum_{p,q} (-1)^{p+q} \cdot \llbracket U_2^{p,q} \rrbracket .\end{aligned}$$

We get from 5.3.4 and Definition 4.2:

$$\begin{aligned}\varrho_{\mathrm{fil}}\big(F_* C^*_{\mathrm{sing}}(E;V)\big) &= \sum_p \varrho(F_p/F_{p+1}) + \sum_q (-1)^q \cdot \varrho(E_1^{*,q}) \\ &\quad + \varrho^{\mathrm{LS}}_{\mathrm{sing}}(f) - \sum_{p,q} (-1)^{p+q} \cdot \llbracket U_2^{p,q} \rrbracket .\end{aligned}$$

Now Theorem 5.4 follows. □

Remark 5.6. One can extend Theorem 5.4 to the case where B is not necessarily a finite CW-complex but carries a simple structure and one requires in data 5.1.5 a Hilbert structure on $H^*_{\mathrm{sing}}\big(B; H^q(F;V)\big)$ instead of $H^*_{\mathrm{cell}}\big(B; H^q(F;V)\big)$. Then we still get a simple structure on E by Remark 1.14. It remains to define $\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)$. Choose an arbitrary homotopy equivalence $h: Y \to B$ for a finite CW-complex Y. Let $\bar{f}: h^*E \to Y$ be the pull back fibration and $\bar{h}: h^*E \to E$ be the canonical fiber homotopy equivalence. Equip $H^*_{\mathrm{sing}}(h^*E; \bar{h}^*V)$ and $H^*_{\mathrm{cell}}\big(Y; h^*H^q_{\mathrm{sing}}(F;V)\big)$ with the Hilbert structures for which the following isomorphisms become isometries (see (3.2))

$$H^*_{\mathrm{sing}}(h^*E; \bar{h}^*V) \xrightarrow{(\bar{h}^*)^{-1}} H^*_{\mathrm{sing}}(E;V)$$

and

$$H^*_{\mathrm{cell}}\big(Y; h^*H^q_{\mathrm{sing}}(F;V)\big) \to H^*_{\mathrm{sing}}\big(Y; h^*H^q_{\mathrm{sing}}(F;V)\big) \xrightarrow{(h^*)^{-1}} H^*_{\mathrm{sing}}\big(B; H^q_{\mathrm{sing}}(F;V)\big) .$$

Then $\varrho^{\mathrm{LS}}_{\mathrm{sing}}(\bar{f})$ is defined and we put

$$\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f) = \varrho^{\mathrm{LS}}_{\mathrm{sing}}(\bar{f}) .$$

This is independent of the choice of h and Theorem 5.4 remains true.

6. The fiber bundle formula for analytic torsion

In this section we give the proof of Theorem 0.2 and Corollaries 0.8, 0.9 and 0.10.

We begin with the proof of Theorem 0.2. In the first step we reduce the claim to the case where the Riemannian metric on V is unimodular. By assumption we only have to treat the case where $\dim(E)$ is odd. We will vary the Riemannian metric on V, but fix the

Riemannian metrics on E and B. Denote by V the flat bundle V with the given Riemannian metric and by $\hat{V}$ the flat vector bundle V with some unimodular Riemannian metric. Let $\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)$ and $\widehat{\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)}$ be the two values of the correction term for the choice of these two Riemannian metrics on V. Notice that either Pf_B is zero or $\dim(F_b)$ is odd. We get from Theorem 0.13 and Definition A.7

$$\varrho_{\mathrm{an}}(E;V) - \varrho_{\mathrm{an}}(E;\hat{V}) = -\sum_p (-1)^p \cdot \left[\!\left[H_{\mathrm{dR}}^p(E;V) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^p(E;\hat{V}) \right]\!\right],$$

$$\begin{aligned}
&\int_B \varrho_{\mathrm{an}}(F_b;V)\cdot \mathrm{Pf}_B - \int_B \varrho_{\mathrm{an}}(F_b;\hat{V})\cdot \mathrm{Pf}_B \\
&\qquad = -\int_B \Big(\sum_q (-1)^q \cdot \left[\!\left[H_{\mathrm{dR}}^q(F_b;V) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^q(F_b;\hat{V}) \right]\!\right] \Big) \cdot \mathrm{Pf}_B,
\end{aligned}$$

$$\begin{aligned}
&\varrho_{\mathrm{an}}\big(B; H_{\mathrm{dR}}^q(F;V)\big) - \varrho_{\mathrm{an}}\big(B; H_{\mathrm{dR}}^q(F;\hat{V})\big) \\
&\qquad = -\sum_p (-1)^p \cdot \left[\!\left[H_{\mathrm{dR}}^p\big(B; H_{\mathrm{dR}}^q(F;V)\big) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^p\big(B; H_{\mathrm{dR}}^q(F;\hat{V})\big) \right]\!\right] \\
&\qquad\quad + \int_B \left[\!\left[H_{\mathrm{dR}}^q(F_b;V) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^q(F_b;\hat{V}) \right]\!\right] \cdot \mathrm{Pf}_B,
\end{aligned}$$

$$\begin{aligned}
\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f) - \widehat{\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)} &= \sum_{p,q} (-1)^{p+q} \cdot \left[\!\left[H_{\mathrm{dR}}^p\big(B; H_{\mathrm{dR}}^q(F;V)\big) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^p\big(B; H_{\mathrm{dR}}^q(F;\hat{V})\big) \right]\!\right] \\
&\quad - \sum_p (-1)^p \cdot \left[\!\left[H_{\mathrm{dR}}^p(E;V) \xrightarrow{\mathrm{id}} H_{\mathrm{dR}}^p(E;\hat{V}) \right]\!\right].
\end{aligned}$$

We conclude from the equations above:

$$\begin{aligned}
&\varrho_{\mathrm{an}}(E;V) - \Big(\int_B \varrho_{\mathrm{an}}(F_b;V)\cdot \mathrm{Pf}_B + \sum_q (-1)^q \cdot \varrho_{\mathrm{an}}\big(B; H_{\mathrm{dR}}^q(F;V)\big) + \varrho_{\mathrm{dR}}^{\mathrm{LS}}(f) \Big) \\
&= \varrho_{\mathrm{an}}(E;\hat{V}) - \Big(\int_B \varrho_{\mathrm{an}}(F_b;\hat{V})\cdot \mathrm{Pf}_B + \sum_q (-1)^q \cdot \varrho_{\mathrm{an}}\big(B; H_{\mathrm{dR}}^q(F;\hat{V})\big) + \widehat{\varrho_{\mathrm{dR}}^{\mathrm{LS}}(f)} \Big).
\end{aligned}$$

Hence we can assume in the sequel without loss of generality that the Riemannian metric on V is unimodular.

Given a closed smooth manifold M and a flat vector bundle W over M with Riemannian metrics on M and W, the *harmonic Hilbert structure* on the singular cohomology $H_{\mathrm{sing}}^*(M;W)$ is given by the deRham isomorphism

$$H_{\mathrm{dR}}^*(M;W) \to H_{\mathrm{sing}}^*(M;W)$$

and the harmonic Hilbert structure on $H_{\mathrm{dR}}^*(M;W)$ coming from the Hodge decomposition and the Hilbert space structure on the space of harmonic forms as explained in the introduction. If we have choosen a smooth triangulation on M, the harmonic Hilbert structure on $H_{\mathrm{cell}}^*(M;W)$ is induced by the canonical isomorphism

$$H_{\mathrm{cell}}^*(M;W) \to H_{\mathrm{sing}}^*(M;W)$$

induced from (3.2). Denote by $\varrho_{\text{top}}(M;W)$ the Milnor torsion of M defined in 3.3 with respect to the simple structure represented by a smooth triangulation and the harmonic Hilbert structure on $H^*_{\text{sing}}(M;W)$.

Denote by $\overline{H^q_{\text{sing}}(F;V)}$ the flat bundle over B with fibers $H^q_{\text{sing}}(F_b;V)$ equipped with some Riemannian metric such that the induced Hilbert structure on $\sum\limits_q (-1)^q \cdot \overline{H^q_{\text{sing}}(F;V)}$ is unimodular. Recall that this can be done by Lemma 5.2. Equip B and F_b with the CW-structure given by some smooth triangulation. Let $\overline{\varrho(F_b;V)}$ be the Milnor torsion of F_b with respect to $\overline{H^q_{\text{sing}}(F;V)}$ (see Definition 3.3). We have seen in 5.3.2 that it is independent of $b \in B$ and abbreviate it by $\overline{\varrho(F;V)}$. Let $\overline{\varrho^{\text{LS}}_{\text{sing}}(f)}$ be the correction term of 5.3 with respect to the harmonic Hilbert structures on $H^*_{\text{sing}}(E;V)$ and $H^p_{\text{sing}}\big(B;\overline{H^q_{\text{sing}}(F;V)}\big)$. We conclude from Lemma 1.13 and Theorem 5.4:

$$\varrho_{\text{top}}(E;V) = \chi(B) \cdot \overline{\varrho(F;V)} + \varrho\Big(B;\sum_q (-1)^q \cdot \overline{H^q_{\text{sing}}(F;V)}\Big) + \overline{\varrho^{\text{LS}}_{\text{sing}}(f)}\,. \tag{6.1}$$

Next we want to show:

$$\varrho_{\text{an}}(E;V) = \varrho_{\text{top}}(E;V)\,, \tag{6.2}$$

$$\varrho_{\text{an}}(F_b;V) = \varrho_{\text{top}}(F_b;V)\,, \tag{6.3}$$

$$\sum_q (-1)^q \cdot \varrho_{\text{an}}\big(B;\overline{H^q_{\text{sing}}(F;V)}\big) = \varrho\Big(B;\sum_q (-1)^q \cdot \overline{H^q_{\text{sing}}(F;V)}\Big)\,. \tag{6.4}$$

The first two equations follow directly from Theorem 0.12. For the proof of the last equation we must take a closer look at the result of Bismut and Zhang [1], Theorem 0.2. The problem is that the Riemannian metric on $\sum\limits_q (-1)^q \cdot \overline{H^q_{\text{sing}}(F;V)}$ is unimodular but it is not true in general that each of the flat bundles $\overline{H^q_{\text{sing}}(F;V)}$ is unimodular or, equivalently, admits some unimodular Riemannian metric.

Choose a Morse function f on B. Let X be its gradient vector field with respect to some Riemannian metric on B which can be different from the given Riemannian metric such that X satisfies the Smale transversality conditions. We obtain a CW-structure on B, denoted by B', and a cellular base point system $\{b_c \mid c \in I\}$ given by the critical points of f. The simple structure represented by $\text{id}: B' \to B$ is the same as the one represented by any smooth triangulation. Let $C^*_{\text{cell}}\big(B',\{b_c\};\overline{H^q_{\text{sing}}(F;V)}\big)$ be the cellular cochain complex with the Hilbert structure defined in (3.4). We equip $H^p\big(C^*_{\text{cell}}\big(B',\{b_c\};\overline{H^q_{\text{sing}}(F;V)}\big)\big)$ with the harmonic Hilbert structure. We get by inspecting Definition 3.3

$$\varrho\Big(B,\sum_q (-1)^q \cdot \overline{H^q_{\text{sing}}(F;V)}\Big) = \sum_q (-1)^q \cdot \varrho\Big(C^*_{\text{cell}}\big(B',\{b_c\};\overline{H^q_{\text{sing}}(F;V)}\big)\Big)\,.$$

But $C^*_{\text{cell}}\big(B',\{b_c\};\overline{H^q_{\text{sing}}(F;V)}\big)$ is isometrically isomorphic to the Thom-Smale complex. Let vol_{harm} be the volume form associated to the harmonic Hilbert structure. Hence we get from the definitions (cf. [1], Remark 2.3 and Remark 1.10)

$$\varrho_{\mathrm{an}}\big(B;\overline{H^q_{\mathrm{sing}}(F;V)}\big) = -\ln\big(\|\operatorname{vol}_{\mathrm{harm}}\|^{\mathrm{LS}}_{\det H^\bullet_{\mathrm{dR}}(B;\overline{H^q_{\mathrm{sing}}(F;V)})}\big),$$

$$\varrho\big(C^*_{\mathrm{cell}}(B',\{b_c\};\overline{H^q_{\mathrm{sing}}(F;V)}\big) = -\ln\big(\|\operatorname{vol}_{\mathrm{harm}}\|^{\mathscr{M},X}_{\det H^\bullet_{\mathrm{cell}}(B;\overline{H^q_{\mathrm{sing}}(F;V)})}\big),$$

where the terms on the right side are the invariants appearing in [1]. We get from [1], Theorem 0.2

$$\varrho_{\mathrm{an}}\big(B;\overline{H^q_{\mathrm{sing}}(F;V)}\big) - \varrho\big(C^*_{\mathrm{cell}}\big(B',\{b_c\};\overline{H^q_{\mathrm{sing}}(F;V)}\big)\big) = \int_B \theta\big(\overline{H^q_{\mathrm{sing}}(F;V)}\big)\cdot X^*\psi(B)$$

for a certain $\big(\dim(B)-1\big)$-form $\psi(B)$. We conclude

$$\begin{aligned}
&\sum_q (-1)^q\cdot\varrho_{\mathrm{an}}\big(B;\overline{H^q_{\mathrm{sing}}(F;V)}\big) - \varrho\big(B,\sum_q(-1)^q\cdot\overline{H^q_{\mathrm{sing}}(F;V)}\big)\\
&= \sum_q (-1)^q\cdot\varrho_{\mathrm{an}}\big(B;\overline{H^q_{\mathrm{sing}}(F;V)}\big) - \sum_q(-1)^q\cdot\varrho\big(C^*_{\mathrm{cell}}\big(B',\{b_c\};\overline{H^q_{\mathrm{sing}}(F;V)}\big)\big)\\
&= \int_B\sum_q(-1)^q\cdot\theta\big(\overline{H^q_{\mathrm{sing}}(F;V)}\big)\cdot X^*\psi(B)\,.
\end{aligned}$$

Now one easily checks that $\sum_q(-1)^q\cdot\theta\big(\overline{H^q_{\mathrm{sing}}(F;V)}\big)$ vanishes as the Riemannian metric on $\sum_q(-1)^q\cdot\overline{H^q_{\mathrm{sing}}(F;V)}$ is unimodular. This finishes the proof of (6.4).

Denote by $H^q_{\mathrm{dR}}(F;V)$ and $H^q_{\mathrm{sing}}(F;V)$ the flat bundles with the harmonic Riemannian metrics (in contrast to $\overline{H^q_{\mathrm{sing}}(F;V)}$). We get from the transformation formula 2.6 and (6.3)

$$\begin{aligned}
(6.5)\qquad &\chi(B)\cdot\overline{\varrho(F;V)} - \int_B\varrho_{\mathrm{an}}(F_b;V)\cdot\mathrm{Pf}_B\\
&= \int_B\big(\overline{\varrho(F_b;V)}-\varrho_{\mathrm{top}}(F_b;V)\big)\cdot\mathrm{Pf}_B\\
&= -\int_B\sum_q(-1)^q\cdot\big[\!\big[\overline{H^q_{\mathrm{sing}}(F_b;V)}\xrightarrow{\mathrm{id}}H^q_{\mathrm{sing}}(F_b;V)\big]\!\big]\cdot\mathrm{Pf}_B\,.
\end{aligned}$$

We conclude from (6.4) and Theorem 0.13:

$$\begin{aligned}
(6.6)\qquad &\varrho\big(B;\sum_q(-1)^q\cdot\overline{H^q_{\mathrm{sing}}(F;V)}\big) - \sum_q(-1)^q\cdot\varrho_{\mathrm{an}}\big(B;H^q_{\mathrm{dR}}(F;V)\big)\\
&= \sum_q(-1)^q\cdot\big(\varrho_{\mathrm{an}}\big(B;\overline{H^q_{\mathrm{sing}}(F;V)}\big)-\varrho_{\mathrm{an}}\big(B;H^q_{\mathrm{sing}}(F;V)\big)\big)\\
&= \sum_q(-1)^q\cdot\Big(-\sum_p(-1)^p\cdot\big[\!\big[H^p_{\mathrm{dR}}\big(B,\overline{H^q_{\mathrm{sing}}(F;V)}\big)\xrightarrow{\mathrm{id}}H^p_{\mathrm{dR}}\big(B,H^q_{\mathrm{sing}}(F;V)\big)\big]\!\big]\\
&\qquad+\int_B\big[\!\big[\overline{H^q_{\mathrm{sing}}(F;V)}\xrightarrow{\mathrm{id}}H^q_{\mathrm{sing}}(F;V)\big]\!\big]\cdot\mathrm{Pf}_B\Big)\\
&= -\sum_{p,q}(-1)^{p+q}\cdot\big[\!\big[H^p_{\mathrm{dR}}\big(B,\overline{H^q_{\mathrm{sing}}(F;V)}\big)\xrightarrow{\mathrm{id}}H^p_{\mathrm{dR}}\big(B,H^q_{\mathrm{sing}}(F;V)\big)\big]\!\big]\\
&\qquad+\sum_q(-1)^q\cdot\int_B\big[\!\big[\overline{H^q_{\mathrm{sing}}(F_b;V)}\xrightarrow{\mathrm{id}}H^q_{\mathrm{sing}}(F_b;V)\big]\!\big]\cdot\mathrm{Pf}_B\,.
\end{aligned}$$

The deRham isomorphism $H^n_{\mathrm{dR}}(E;V) \to H^n_{\mathrm{sing}}(E;V)$ is compatible with the two filtrations and the identifications of the E_2-term in the Leray-Serre spectral sequence for deRham and singular cohomology (0.3) and (5.5). It yields isomorphisms on $E_r^{p,q}$ for $r \geqq 2$. This implies using Remark 4.3

$$
\begin{aligned}
(6.7) \qquad & \overline{\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)} - \varrho^{\mathrm{LS}}_{\mathrm{dR}}(f) \\
&= \sum_{p,q} (-1)^{p+q} \big[\!\big[H^p_{\mathrm{dR}}\big(B, \overline{H^q_{\mathrm{sing}}(F;V)}\big) \xrightarrow{\mathrm{id}} H^p_{\mathrm{dR}}\big(B, H^q_{\mathrm{sing}}(F;V)\big) \big]\!\big] .
\end{aligned}
$$

Now Theorem 0.2 follows from (6.1), (6.2), (6.5), (6.6) and (6.7). □

Next we prove Corollary 0.8. Recall that $H^q_{\mathrm{dR}}(F;V)$ vanishes by assumption. In particular the E_2-term of the Leray-Serre spectral sequences for cohomology vanishes. Hence Theorem 0.2 implies

$$\varrho_{\mathrm{an}}(E;V) = \int_B \varrho_{\mathrm{an}}(F_b;V) \cdot \mathrm{Pf}_B .$$

If $\dim(B)$ is odd, the right hand side of the equation above and $\chi(B) \cdot \varrho_{\mathrm{an}}\}F_b;V)$ vanish and the claim follows. Suppose that $\dim(B)$ is even. Then $\dim(F_b)$ is even and the Riemannian metric on V is unimodular by assumption or $\dim(F_b)$ is odd. In both cases $\varrho_{\mathrm{an}}(F_b;V)$ is independent of $b \in B$ by Theorem 0.13 and the claim follows. This finishes the proof of Corollary 0.8. □

Next we outline the proof of Corollary 0.9. Analogously to the first step in the proof of Theorem 0.2 we can show that we can assume without loss of generality that the Riemannian metric on W is unimodular. Put $V = f^*W$. There is a canonical isomorphism of local coefficient systems over B

$$H^q_{\mathrm{sing}}(F;\mathbb{Z}) \otimes_{\mathbb{Z}} W \to H^q_{\mathrm{sing}}(F;V) .$$

Recall that we assume that E and B are oriented. (One can drop this assumption if one allows an additional twist for W.) This implies that $\pi_1(B)$ acts orientation preserving on the homology of the fiber. Hence we can choose an isomorphism of local coefficient systems of $\mathbb{Z}$-modules from $H^q_{\mathrm{sing}}(F;\mathbb{Z})$ to the trivial coefficient system with value $\mathbb{Z}$. Hence we obtain identifications

$$H^q_{\mathrm{sing}}(F;V) = \begin{cases} W, & q = 0, n, \\ 0, & q \neq 0, n . \end{cases}$$

We conclude from Theorem 5.4

$$\varrho_{\mathrm{top}}(E;V) = \chi(B) \cdot \varrho(F_b;V) + \varrho\Big(B; \sum_q (-1)^q \cdot H^q_{\mathrm{sing}}(F;V)\Big) + \varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)$$

where we use on $H^q_{\mathrm{sing}}(F_b;V)$ the unimodular Hilbert structure given by the one on W and the identification above and on $H^q_{\mathrm{sing}}\big(B; \sum_q (-1)^q \cdot H^q_{\mathrm{sing}}(F;V)\big)$ the harmonic one. We get

$$\varrho(F;V) = 0\,,$$
$$\varrho\Big(B;\sum_q(-1)^q\cdot H^q_{\mathrm{sing}}(F;V)\Big) = \chi(S^n)\cdot\varrho_{\mathrm{top}}(B;W)\,,$$
$$\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f) = \varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)\,,$$
$$\varrho_{\mathrm{an}}(E;V) = \varrho_{\mathrm{top}}(E;V)\,,$$
$$\varrho_{\mathrm{an}}(B;W) = \varrho_{\mathrm{top}}(B;W)$$

since there is a canonical isomorphism

$$C^*_{\mathrm{sing}}(F_b;\mathbb{Z})\otimes_{\mathbb{Z}}W\to C^*_{\mathrm{sing}}(F_b;V)$$

and $H^*_{\mathrm{sing}}(F_b;\mathbb{Z})$ is free as $\mathbb{Z}$-module, the deRham isomorphism is compatible with the cofiltrations on the deRham complex and the singular cochain complex and with the identifications of the E_2-terms of the associated Leray-Serre spectral sequences (0.3) and (5.5) and we have Theorem 0.12. Hence it remains to show

$$\varrho^{\mathrm{LS}}_{\mathrm{dR}}(f) = \varrho(G^*)\,.$$

Notice that the E^2-term of the Leray-Serre spectral sequence is trivial except for the 0-th and n-th row. Hence we can splice the spectral sequence together to one exact sequence

$$\cdots\to E_n^{p,n}\xrightarrow{d_{n+1}^{p,n}} E_n^{p+n+1,0}\to H^{p+n+1}(E;V)\to E_{n+1}^{p+1,n}\xrightarrow{d_{n+1}^{p+1,n}}\cdots.$$

If one takes the identification of the E_2-term (0.3) and the obvious identification $E_2^{p,q}=E_n^{p,q}$ into account, one gets an exact sequence whose torsion is precisely $\varrho^{\mathrm{LS}}_{\mathrm{dR}}(f)$. Moreover, it can be identified with the Gysin sequence up to sign. This finishes the proof of Corollary 0.9. The proof of Corollary 0.10 is similar. □

A. Torsion and determinants

We start with a description of the determinant of a finite dimensional vector space. Let V be an n-dimension (real) vector space. Define its *determinant* $\det(V)$ to be the 1-dimensional vector space $\wedge^n V$ given by the n-th exterior power. There are canonical isomorphisms

$$\begin{aligned}
\det(V\oplus W) &\xrightarrow{\ \cong\ } \det(V)\otimes\det(W)\,,\\
\det(V^*) &\xrightarrow{\ \cong\ } \det(V)^*\,,\\
\det(V)^*\otimes\det(W) &\xrightarrow{\ \cong\ } \hom_{\mathbb{R}}\big(\det(V),\det(W)\big)\,,\\
\det(U)^*\otimes\det(V^*)\otimes\det(V)\otimes\det(W) &\xrightarrow{\ \cong\ } \det(U^*)\otimes\det(W)\,,\\
\det(V) &\to \big(\det(V)^*\big)^*.
\end{aligned}$$

Given a homomorphism $f\colon V\to W$, we obtain a well-defined element

$$\det(f) \in \det(V)^* \otimes \det(W).$$

Under the identifications above

$$\det(f \circ g) = \det(f) \otimes \det(g),$$

$$\det\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} = \det(f) \otimes \det(g).$$

If $(V_p)_p$ is a graded vector space graded by nonnegative integers $p \geqq 0$ and almost all V_p are zero, then we define

$$\det\bigl((V_p)_p\bigr) := \bigotimes_{p \geqq 0} \det(V_p)^{(-1)^p}$$

where $\det(V)^1$ is $\det(V)$ and $\det(V)^{-1}$ is defined by $\det(V)^*$. If $(V_{p,q})_{p,q}$ is bigraded by non-negative integers $p, q \geqq 0$ and almost all $V_{p,q}$ are zero, then we define

$$\det\bigl((V_{p,q})_{p,q}\bigr) = \det\bigl((W_n)_n\bigr)$$

where $(W_n)_n$ is the associated graded vector space with $W_n = \bigoplus_{p=0}^{n} V_{p,n-p}$.

Now we will introduce algebraic torsion invariants in the language of determinants. Given an acyclic finite cochain complex, we define

$$\bar{\varrho}(C) \in \det\bigl((C^p)_p\bigr)^{-1} \tag{A.1}$$

by the determinant of the isomorphism $c^* + \gamma^* : C^{\mathrm{ev}} \to C^{\mathrm{odd}}$. For a homotopy equivalence $f : C \to D$ of finite cochain complexes we define

$$\begin{aligned} \bar{\varrho}(f) := \bar{\varrho}(\mathrm{cone}(f)) &\in \det\bigl((C^p)_p\bigr)^{-1} \otimes \det\bigl((D^p)_p\bigr) \\ &\cong \hom_{\mathbb{R}}\bigl(\det(C^p)_p, \det(D^p)_p\bigr). \end{aligned} \tag{A.2}$$

Given a finite cochain complex C, we use the cochain map $i : H(C) \to C$ to define the isomorphism

$$\bar{\varrho}(C) := \bigl(\bar{\varrho}(i)\bigr)^{-1} \quad : \det\bigl((C^p)_p\bigr) \to \det\bigl((H^p(C))_p\bigr). \tag{A.3}$$

The topological torsion $\bar{\varrho}(X)$ of a finite CW-complex X is defined as the torsion of the cellular cochain complex.

The definition of the torsion of a filtration/spectral sequence is a little bit more elaborate.

Let $F_* W$ be a fibration of a finite-dimensional vector space W

$$\{0\} = F_{-1} \subset \cdots \subset F_p \subset F_{p+1} \subset \cdots \subset F_n = W.$$

This determines an isomorphism

$$\bar{\varrho}^{\mathrm{fil}}(F_* W) : \bigotimes_{p \geqq 0} \det(F_p / F_{p-1}) \xrightarrow{\simeq} \det(W) \tag{A.4}$$

which is defined inductively over n. The begin of induction $n=1$ is given by the Milnor torsion of the acyclic 2-dimensional cochain complex $F_0 \to W \to W/F_0$. For the step $n-1$ to n observe that

$$\bar{\varrho}^{\mathrm{fil}}(F_* F_{n-1}) : \bigoplus_{p=0}^{n-1} \det(F_p / F_{p-1}) \xrightarrow{\simeq} \det(F_{n-1})$$

is defined by induction, and we simply compose (after tensoring with $\det(W/F_{n-1})$) with the Milnor torsion of $F_{n-1} \to W \to W/F_{n-1}$.

In particular, suppose $f: E \to B$ is a fibration of closed manifolds as in the introduction. This yields a filtration of $H^n_{\mathrm{dR}}(E;V)$ for $n \geqq 0$. Take the tensor product of the corresponding torsion isomorphisms resp. their inverses depending on the parity of n to obtain an isomorphism

$$\bar{\varrho}^{\mathrm{fil}} : \det\big((F^{p,q}/F^{p+1,q-1})_{p,q}\big) \to \det\big((H^n(E;V))_n\big)\,.$$

In the deRham spectral sequence, we can compute the torsions of the cochain complex $(E_r^{p+r*,q-(r-1)*}, d_r^{p+r*,q-(r-1)*})$. We obtain an isomorphism

$$\bar{\varrho}(E_r^{p+r*,q-(r-1)*}) : \bigotimes_n \det\big(H^n(E_r^{p+r*,q-(r-1)*})^{(-1)^n}\big) \to \bigotimes_n \det(E_r^{p+rn,q-(r-1)n})^{(-1)^n}\,.$$

Taking $\bigotimes_{p=0}^{r-1} \bigotimes_{q \geqq 0} \big(\bar{\varrho}(E_r^{p+r*,q-(r-1)*})\big)^{(-1)^{p+q}}$ yields an isomorphism for $r \geqq 2$

$$\bar{\varrho} : \det\big((H^0(E_r^{p+r*,q-(r-1)*}))_{p,q}\big) \to \det\big((E_{r+1}^{p,q})_{p,q}\big)\,.$$

Together with the natural isomorphisms $V^{p,q}: H^p_{\mathrm{dR}}\big(B, H^q_{\mathrm{dH}}(F,V)\big) \to E_2^{p,q}$ and $\phi_r^{p,q}$ between the homology of E_r and E_{r+1} this defines an isomorphism

$$\bar{\varrho}^{\mathrm{LS}}_{\mathrm{dR}}(f) : \det\big((H^p_{\mathrm{dR}}(B; H^q_{\mathrm{dR}}(F;V))_{p,q})\big) \xrightarrow{\simeq} \det\big((H^n_{\mathrm{dR}}(E;V))_n\big) \tag{A.5}$$

as the composition of isomorphisms (use the fact that E_r and E_∞ are equal for r large enough)

$$\begin{aligned}
&\det\big((H^p_{\mathrm{dR}}(B; H^q_{\mathrm{dR}}(F;V))_{p,q}\big) \xrightarrow{(V^{p,q})_{p,q}} \det\big((E_2^{p,q})_{p,q}\big) \xrightarrow{\bar{\varrho}} \det\big((H^0(E_2^{p+2*,q-*}))_{p,q}\big)\\
&\xrightarrow{(\phi_2^{p,q})_{p,q}} \det\big((E_3^{p,q})_{p,q}\big) \xrightarrow{\bar{\varrho}} \big((H^0(E_3^{p+3*,q-2*}))_{p,q}\big) \xrightarrow{(\phi_3^{p,q})_{p,q}} \det\big((E_4^{p,q})_{p,q}\big) \to \cdots\\
&\to \det\big((E_\infty^{p,q})_{p,q}\big) \xrightarrow{((\psi^{p,q})^{-1})_{p,q}} \det\big((F^{p,q}/F^{p+1,q-1})_{p,q}\big) \xrightarrow{\bar{\varrho}^{\mathrm{fil}}} \det\big((H^n_{\mathrm{dR}}(E;V))_n\big)\,.
\end{aligned}$$

For the spectral sequence associated to an arbitrary filtration $F_* W$ of a finite-dimensional vector space W one can define $\bar{\varrho}_{\mathrm{fil}}(F_* W)$ and $\bar{\varrho}_{\mathrm{fil}}^{\geqq 2}(F_* W)$ in the obvious analogous way.

Now, we have to relate the invariants $\bar{\varrho}$ to the previously defined real numbers ϱ – using given inner products.

If the finite-dimensional vector space V has a Hilbert structure, it induces in a canonical way a Hilbert space structure on $\det(V)$. In particular the norm $\|v\| \in \mathbb{R}^{\geqq 0}$ is defined for any element $v \in V$. If V and W come with Hilbert space structures, for any element

$$u \in \hom_{\mathbb{R}}\big(\det(U), \det(V)\big) = \det(V)^* \otimes \det(W)$$

we get its norm

$$\|u\| \in \mathbb{R}^{\geqq 0}\,. \tag{A.6}$$

It turns out that for a finite Hilbert cochain complex C with given Hilbert structures on $H(C)$

$$\varrho(C) = \ln\big(\|\bar{\varrho}(C)\|\big)\,.$$

In the case of the fibration $f\colon E \to B$ of the introduction, the Riemannian metrics on E, B and V induce harmonic Hilbert structures on $H^n_{\mathrm{dR}}(E; V)$ and $H^p_{\mathrm{dR}}\big(B; H^q_{\mathrm{dR}}(F; V)\big)$ as described there. Then the norm of the element $\bar{\varrho}^{\mathrm{LS}}_{\mathrm{dR}}(f)$ defined in (A.5) is just

$$\varrho^{\mathrm{LS}}_{\mathrm{dR}}(f) = \ln\big(\|\bar{\varrho}^{\mathrm{LS}}_{\mathrm{dR}}(f)\|\big)\,. \tag{A.7}$$

Of course, with similar definitions a similar result can be obtained for $\varrho^{\mathrm{LS}}_{\mathrm{sing}}(f)$. Also, the corresponding equation for ϱ_{fil} of a filtration is true.

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Eingegangen 16. Dezember 1996