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Computations of K- and L-Theory of Cocompact Planar Groups

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Abstract. The verification of the isomorphism conjectures of Baum and Connes and Farrell and Jones for certain classes of groups is used to compute the algebraic K- and L-theory and the topological K-theory of cocompact planar groups (= cocompact N.E.C-groups) and of groups G appearing in an extension $1 \to \mathbb{Z}^n \to G \to \pi \to 1$ where π is a finite group and the conjugation π-action on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$. These computations apply, for instance, to two-dimensional crystallographic groups and cocompact Fuchsian groups.

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0. Introduction

The goal of this paper is to compute the algebraic K-groups $K_p(\mathbb{Z}G)$ for $p \leqslant 1$, the algebraic L-groups $L_p(\mathbb{Z}G)$ for $p \in \mathbb{Z}$ (mostly after inverting 2) of the integral group ring $\mathbb{Z}G$ and the topological K-groups $K_p(C_r^*(G))$ for $p \in \mathbb{Z}$ of the reduced C^*-algebra $C_r^*(G)$ for certain infinite (discrete) groups G. Namely, we assume that G is either a cocompact planar group or that there is an exact sequence $1 \to \mathbb{Z}^n \to G \to \pi \to 1$, where π is a finite group and the conjugation action of π on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$. A cocompact planar group is a discontinuous group of isometries of S^2, $\mathbb{R}^2$ or $\mathbb{H}^2$ with compact quotient. More information about these groups and the result of the explicit computations will be given in Section 4 (see Theorems 4.4 and 4.9).

For our techniques to work it is crucial to have very good information on the structure of the finite subgroups, as well as their normalizers, and the infinite virtually cyclic subgroups of cocompact planar groups. More explicitly, we use that all maximal finite subgroups are either cyclic or dihedral, and that a common subgroup of any two different maximal finite subgroups has at most two elements. Furthermore, the normalizer $N_G M$ of a maximal finite subgroup M satisfies $N_G M = M$, except if M is generated by a single reflection, in which case $N_G M \cong \mathbb{Z} \times \mathbb{Z}/2$. These exceptions are responsible for the summands indexed by B and B'' in Theorem 4.4, whereas the other maximal finite subgroups correspond to the summands indexed by A. Finally, we have a complete list of the infinite virtually cyclic subgroups of a cocompact planar group; they are given as subgroups of very simple amalgams. This will allow us to reduce the computations in algebraic K- and L-theory from the family of virtually cyclic subgroups to the family of finite subgroups. All the facts about cocompact planar groups mentioned above will be recollected in Theorem 4.3 and Lemma 4.5 and 4.6.

Examples of cocompact planar groups are cocompact Fuchsian groups. Next we give the result in this comparatively easy case as an illustration.

THEOREM 0.1. *Let F be a cocompact Fuchsian group with presentation*

$$\begin{aligned} F &= \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_t \mid c_1^{\gamma_1} = \cdots \\ &= c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1}[a_1, b_1] \cdots [a_g, b_g] = 1 \rangle \end{aligned}$$

for integers $g, t \geqslant 0$ and $\gamma_i > 1$. Then

(a) *the inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism*

$$\bigoplus_{i=1}^{t} \mathrm{Wh}_q(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_q(F)$$

for $q \leqslant 1$. If the isomorphism conjecture for algebraic K-theory (see 1.9 and Theorem 1.10) holds for F also in dimensions $q \geqslant 2$, then this is an isomorphism for all $q \in \mathbb{Z}$. (Information about $\mathrm{Wh}_q(\mathbb{Z}/\gamma_i)$ is given in Theorem 3.2 (d);

(b) *there are isomorphisms*

$$L_q(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1 + \sum_{i=1}^{t} \left[\frac{\gamma_i}{2}\right]\right) \cdot \mathbb{Z}[1/2] & q \equiv 0(4), \\ (2g) \cdot \mathbb{Z}[1/2] & q \equiv 1(4), \\ \left(1 + \sum_{i=1}^{t} \left[\frac{\gamma_i - 1}{2}\right]\right) \cdot \mathbb{Z}[1/2] & q \equiv 2(4), \\ 0 & q \equiv 3(4), \end{cases}$$

where $[r]$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r;

(c) *there are isomorphisms*

$$K_q(C_r^*(F)) \cong \begin{cases} \left(2 + \sum_{i=1}^{t} (\gamma_i - 1)\right) \cdot \mathbb{Z} & q = 0, \\ (2g) \cdot \mathbb{Z} & q = 1. \end{cases}$$

We will give more information about the algebraic L-theory of cocompact Fuchsian groups without inverting 2 in Remark 4.10. The algebraic K-theory in dimensions $\leqslant 1$ of cocompact Fuchsian groups has been computed in [7].

Other examples of cocompact planar groups are two-dimensional crystallographic groups. Their K- and L-theory is explicitly computed in Section 5. For a two-dimensional crystallographic group G the algebraic K-theory of $\mathbb{Z}G$ in dimensions $\leqslant 1$ has already been determined in [32], and the topological K-theory of $C_r^*(G)$ in [51].

In Section 6 we will prove a result which is similar to that of Theorem 4.4 but applies to certain virtually Abelian groups whose classifying space for the family of finite subgroups $\underline{E}G = E(G, \mathcal{FIN})$ is of higher dimension. (For an infinite cocompact planar group G, a model for $\underline{E}G$ is $\mathbb{R}^2$ or $\mathbb{H}^2$ with the obvious G-action, and for the groups G appearing below, $\mathbb{R}^n$ with a certain G-action is a model for $\underline{E}G$.)

THEOREM 0.2. *Let $1 \to \mathbb{Z}^n \to G \to \pi \to 1$ be a group extension for a finite group π such that the conjugation action of π on $\mathbb{Z}^n$ is free, i.e. the only element in π with a fixed point in $\mathbb{Z}^n$ different from zero is the identity element in π. Let $\{M_\alpha \mid \alpha \in A\}$ be a complete system of representatives of conjugacy classes of maximal finite subgroups of G. Then*

(a) *the natural map induced by the inclusions of subgroups*

$$\bigoplus_{\alpha \in A} \mathrm{Wh}_q(M_\alpha) \to \mathrm{Wh}_q(G)$$

is an isomorphism for $q \leqslant 1$, and $K_q(\mathbb{Z}G)$ is trivial for $q \leqslant -2$.
If the isomorphism conjecture in algebraic K-theory (see 1.9 and Theorem 1.10) holds also for $q \geqslant 2$, then the map above is an isomorphism for all $q \in \mathbb{Z}$;

(b) *there are short exact sequences*

$$\begin{aligned} 0 \to &\bigoplus_{\alpha \in A} \widetilde{L}_q(\mathbb{Z}M_\alpha)[1/2] \to L_q(\mathbb{Z}G)[1/2] \\ &\to H_q(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z}))[1/2] \to 0, \end{aligned}$$

where $\mathbf{L}(\mathbb{Z})$ is the L-theory spectrum associated to the ring $\mathbb{Z}$, $H_*(-; \mathbf{L}(\mathbb{Z}))$ is the associated homology theory and the first map is induced by the various inclusions $M_\alpha \to G$.
If we invert $2|\pi|$, this sequence splits and we obtain isomorphisms

$$\left(\bigoplus_{\alpha \in A} \widetilde{L}_q(\mathbb{Z}M_\alpha)\left[\frac{1}{2|\pi|}\right]\right) \oplus H_q(\pi\backslash T^n; \mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right] \xrightarrow{\cong} L_q(\mathbb{Z}G)\left[\frac{1}{2|\pi|}\right],$$

where the π-action on T^n is induced by the conjugation action of π on $\mathbb{Z}^n$;

(c) *there are short exact sequences*

$$0 \to \bigoplus_{\alpha \in A} \widetilde{K}_q(C_r^*(M_\alpha)) \to K_q(C_r^*(G)) \to K_q(G\backslash \underline{E}G) \to 0,$$

where $K_q(G\backslash \underline{E}G)$ is the topological complex K-homology of $G\backslash \underline{E}G$ and the first map is induced by the various inclusions $M_\alpha \to G$.
If we invert $|\pi|$, this sequence splits and we obtain isomorphisms

$$\left(\bigoplus_{\alpha \in A} \widetilde{K}_q(C_r^*(M_\alpha))\left[\frac{1}{|\pi|}\right]\right) \oplus K_q(\pi\backslash T^n)\left[\frac{1}{|\pi|}\right] \xrightarrow{\cong} K_q(C_r^*(G))\left[\frac{1}{|\pi|}\right].$$

We will present more detailed information on the L-theory (without inverting 2) of groups as in Theorem 0.2 in Remark 6.4. Furthermore, we can generalize the methods of Theorem 0.1 and 0.2 to yield similar results for groups which are given as extensions of the form $1 \to \mathbb{Z}^n \to G \to \pi \to 1$, where $\pi \cong D_{2m}$ is a dihedral group of order $2m$ such that the subgroup $\mathbb{Z}/m$ acts freely on $\mathbb{Z}^n$. See Remark 6.5 for more information on this.

Our computations use the isomorphism conjectures in K-and L-theory due to Farrell and Jones and to Baum and Connes which are known to be true for the groups we consider here. We exploit the unified treatment of these conjectures of [15]. Thus the computation is reduced to the investigation of the homology of certain spaces over the orbit category with coefficients in K- and L-spectra over the orbit category which will be carried out by homological methods, mainly Mayer–Vietoris sequences. There are various spectral sequences to compute these homology groups but they turn out to be too complicated even for the relatively elementary groups we consider here. It seems to be very hard to compute these K- and L-groups integrally (or after inverting 2) for more general groups even if one assumes in the K-theory case that one does know the K-theory of integral group rings of finite subgroups. Rationally these computations can be done via Chern characters and lead to rather general and explicit formulas, since the existence of the Chern characters guarantees that the relevant spectral sequences collapses [27]. The integral computations of K- and L-groups presented here exploit the explicit knowledge and special properties of the virtually cyclic and finite subgroups and their normalizers of the groups under consideration (see Lemma 4.5 and Lemma 6.3).

The paper is organized as follows:

(1) Review of the isomorphism conjectures in K- and L-theory.
(2) Preliminaries about spaces over the orbit category.
(3) Preliminary computations of K- and L-groups of finite groups.
(4) Cocompact planar groups.
(5) Two-dimensional crystallographic groups.
(6) Extensions of finite groups with a free Abelian group references.

1. Review of the Isomorphism Conjectures in *K*- and *L*-Theory

We want to review the isomorphism conjectures in K- and L-theory as far as we will need here. Since we want to do this in the language of spaces and spectra over a category we give some basic facts about these notions. More information can be found for instance in [15, 18].

Given a (discrete) group G, a *family* $\mathcal{F}$ of subgroups is a set of subgroups of G closed under taking subgroups and conjugates. Our main examples for families will be the families $\mathcal{TR}$, $\mathcal{FIN}$, $\mathcal{VC}$ and $\mathcal{ALL}$, respectively, consisting of the trivial subgroup, finite subgroups, virtually cyclic subgroups and all subgroups, respectively. Recall that G is virtually cyclic if G is finite or contains $\mathbb{Z}$ as subgroup of finite index. The *orbit category* $\mathsf{Or}(G, \mathcal{F})$ *of* G with respect to $\mathcal{F}$ has as objects homogeneous spaces G/H with $H \in \mathcal{F}$ and as morphisms G-maps. If $\mathcal{F}$ is the family $\mathcal{ALL}$ of all subgroups, we abbreviate $\mathsf{Or}(G, \mathcal{ALL})$ by $\mathsf{Or}(G)$.

A *contravariant (pointed)* $\mathsf{Or}(G)$*-space* is a contravariant functor from $\mathsf{Or}(G)$ to the category of (pointed) spaces. A morphism between contravariant (pointed) $\mathsf{Or}(G)$-spaces is a natural transformation. A G-space X defines a contravariant $\mathsf{Or}(G)$-space by assigning to G/H its H-fixed point set $X^H = \mathrm{map}_G(G/H, X)$. A *covariant* $\mathsf{Or}(G)$*-spectrum* is a covariant functor from the category $\mathsf{Or}(G)$ into the category $\mathsf{Spectra}$ of spectra. An object $\mathbf{E}$ in $\mathsf{Spectra}$ is a sequence of spaces $(E_n)_{n\in\mathbb{Z}}$ together with structure maps $\sigma_n\colon \Sigma E_n \to E_{n+1}$ for each $n \in \mathbb{Z}$. A map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra is a sequence $(f_n\colon E_n \to F_n)_{n\in\mathbb{Z}}$ of maps satisfying $f_{n+1}\circ\sigma_n^{\mathbf{E}} = \sigma_n^{\mathbf{F}}\circ\Sigma f_n$ for all $n \in \mathbb{Z}$. The *qth homotopy group* $\pi_q(\mathbf{E})$ of a spectrum $\mathbf{E}$ for $q \in \mathbb{Z}$ is the colimit $\mathrm{colim}_{n\to\infty}\,\pi_{q+n}(E_n)$ with respect to the obvious maps $\pi_{q+n}(E_n) \to \pi_{q+n+1}(E_{n+1})$ induced by the structure maps σ_n and the suspension homomorphisms. Next we review our main examples of covariant $\mathsf{Or}(G)$-spectra.

Let $\mathsf{Groupoids}$ be the category of groupoids. Let $\mathsf{Groupoids}^{\mathsf{inj}}$ be the subcategory of $\mathsf{Groupoids}$ which has the same objects as $\mathsf{Groupoids}$ and as morphisms covariant functors $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ which are faithful, i.e., for any two objects x, y in $\mathcal{G}_0$ the induced map $\mathrm{mor}_{\mathcal{G}_0}(x, y) \longrightarrow \mathrm{mor}_{\mathcal{G}_1}(F(x), F(y))$ is injective. A left G-set S defines a groupoid $\mathcal{G}^G(S)$, where $\mathrm{Ob}(\mathcal{G}^G(S)) = S$ and $\mathrm{mor}(s, t) = \{g \in G \mid gs = t\}$ for $s, t \in S$. The composition law is given by group multiplication. Obviously a map of left G-sets defines a functor of the associated groupoids. The category $\mathcal{G}(G/H)$ is equivalent to the groupoid associated with H which has one object and H as set of morphisms, and hence $\mathcal{G}(G/H)$ can serve as a substitute for the subgroup H. Thus we obtain a covariant functor

$$\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{Groupoids}^{\mathsf{inj}}. \tag{1.1}$$

In [15, section 2] covariant functors

$$\mathbf{K}^{\mathrm{alg}}\colon \mathsf{Groupoids} \longrightarrow \mathsf{Spectra},$$

$$\mathbf{L}\colon \mathsf{Groupoids} \longrightarrow \mathsf{Spectra},$$

$$\mathbf{K}^{\mathrm{top}}\colon \mathsf{Groupoids}^{\mathsf{inj}} \longrightarrow \mathsf{Spectra},$$

are constructed using [34] and [40] in the algebraic context. (Unfortunately there is a problem in the actual construction of $\mathbf{K}^{\mathrm{top}}$ concerning the pairing μ in [15, page 217] which will be corrected elsewhere. This does not affect the results of this paper.) We denote their composition with $\mathcal{G}^G$ by the same letters or by the following abbreviations and obtain covariant functors

$$\mathbf{K} = \mathbf{K}_G^{\mathrm{alg}}\colon \mathsf{Or}(G) \longrightarrow \mathsf{Spectra}, \tag{1.2}$$

$$\mathbf{L} = \mathbf{L}_G\colon \mathsf{Or}(G) \longrightarrow \mathsf{Spectra}, \tag{1.3}$$

$$\mathbf{K} = \mathbf{K}_G^{\mathrm{top}}\colon \mathsf{Or}(G) \longrightarrow \mathsf{Spectra}. \tag{1.4}$$

Notice that $\pi_q(\mathbf{L}(G/H)) \cong L_q(\mathbb{Z}H)$, $\pi_q(\mathbf{K}^{\mathrm{alg}}(G/H)) \cong K_q(\mathbb{Z}H)$ and $\pi_q(\mathbf{K}^{\mathrm{top}}(G/H)) \cong K_q(C_r^*(H))$, where $C_r^*(H)$ is the reduced C^*-algebra of the group H and $K_q(C_r^*(H))$ denotes its topological K-theory. Functoriality for a G-map $G/H \to G/K$, $g'H \mapsto g'g^{-1}K$ corresponds under this isomorphism to induction with respect to the injective homomorphism $H \to K$ given by $h \mapsto ghg^{-1}$.

If $*$ denotes the trivial groupoid consisting of one morphism, there is for any groupoid $\mathcal{G}$ the canonical projection $\mathrm{pr}\colon \mathcal{G} \to *$. Denote by $\widetilde{\mathbf{L}}(\mathcal{G})$ the homotopy fiber of the map of spectra $\mathbf{L}(\mathrm{pr})\colon \mathbf{L}(\mathcal{G}) \to \mathbf{L}(*)$. Thus we obtain covariant functors

$$\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}_G^{\mathrm{alg}}\colon \mathsf{Or}(G) \longrightarrow \mathsf{Spectra}, \tag{1.5}$$

$$\widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}_G\colon \mathsf{Or}(G) \longrightarrow \mathsf{Spectra}, \tag{1.6}$$

$$\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}_G^{\mathrm{top}}\colon \mathsf{Or}(G, \mathcal{FIN}) \longrightarrow \mathsf{Spectra}. \tag{1.7}$$

Notice that we have defined $\widetilde{\mathbf{K}}_G^{\mathrm{top}}$ only for $\mathsf{Or}(G, \mathcal{FIN})$. The problem is that $\mathrm{pr}\colon \mathcal{G} \to *$ is not a morphism in $\mathsf{Groupoids}^{\mathsf{inj}}$. However, if we replace the reduced C^*-algebra by the maximal C^*-algebra, then $\mathbf{K}^{\mathrm{top}}$ is indeed a functor on $\mathsf{Groupoids}$, and for amenable groups such as all finite groups and virtually Abelian groups the natural map from the maximal to the reduced C^*-algebra is an isomorphism [33, Theorem 7.3.9 on page 243]. Notice that we only need $\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}_G^{\mathrm{top}}\colon \mathsf{Or}(G, \mathcal{FIN}) \longrightarrow \mathsf{Spectra}$ since it will be only applied in context with $\mathsf{Or}(G)$-spaces X for which $X(G/H)$ is empty for infinite H.

The homology $H_q^G(X; \mathbf{E})$ of a contravariant $\mathsf{Or}(G)$-space X with coefficients in the covariant $\mathsf{Or}(G)$-spectrum $\mathbf{E}$ is defined for $q \in \mathbb{Z}$ in [15, section 4] using $\mathsf{Or}(G)$-CW-approximations. The above homology groups are functorial in X and $\mathbf{E}$. We get from [15, Lemma 4.4]

LEMMA 1.1. *$H_p^G(X, A; \mathbf{E})$ is an unreduced homology theory on pairs of contravariant $\mathsf{Or}(G)$-spaces which satisfy the WHE-axiom and the disjoint union axiom.*

To be more precise, homology theory means that homotopic maps of pairs of contravariant $\mathsf{Or}(G)$-spaces induce the same maps on the homology groups, that there are long exact sequences of pairs (X, A), and that for any commutative diagram of contravariant $\mathsf{Or}(G)$-spaces

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ x_2 & \xrightarrow[j_2]{} & X \end{array}$$

the map $(j_2, i_1)\colon (X_2, X_0) \longrightarrow (X, X_1)$ induces an isomorphism on homology, provided that the evaluation of the diagram at any object G/H yields a pushout of spaces with a cofibration of spaces as upper horizontal arrow. We will frequently use the associated long exact Mayer–Vietoris sequence

$$\cdots \xrightarrow{\delta} H_p^G(X_0;\mathbf{E}) \to H_p^G(X_1;\mathbf{E}) \oplus H_p^G(X_2;\mathbf{E}) \to H_p^G(X;\mathbf{E}) \\ \xrightarrow{\partial} H_{p-1}^G(X_0;\mathbf{E}) \to \cdots .$$

The disjoint union axiom says that for an arbitrary disjoint union the obvious map from the direct sum of the homology groups of the various summands to the homology of the disjoint union is an isomorphism. The WHE-axiom requires that a weak homotopy equivalence of contravariant $\mathrm{Or}(G)$-spaces induces an isomorphism on homology, where a map $f\colon X \to Y$ of $\mathrm{Or}(G)$-spaces or spectra is called *n-connected* resp. a *weak homotopy equivalence*, if the map $f(G/H)\colon X(G/H) \to Y(G/H)$ is n-connected resp. a weak homotopy equivalence for every object G/H. In order to guarantee the WHE-axiom, $\mathrm{Or}(G)$-CW-approximations are used in the definitions. Notice, however, that we will almost everywhere do calculations with the $\mathrm{Or}(G)$-space X itself and not with its $\mathrm{Or}(G)$-CW-approximations, which will be quite convenient since the $\mathrm{Or}(G)$-spaces X we will deal with will very often be very simple. Namely, we will consider the $\mathrm{Or}(G)$-spaces $\star_{G,\mathcal{F}}$ associated to a family $\mathcal{F}$, which assigns to an object G/H the space $*$ consisting of one point if H belongs to $\mathcal{F}$ and the empty set $\emptyset$ otherwise. For these spaces it will be comparatively easy to check whether the necessary conditions are satisfied for the square above to get a Mayer–Vietoris sequence.

Notice in the sequel that for any covariant $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ there is a canonical isomorphism

$$H_q^G(\star_{G,\mathcal{ALL}};\mathbf{E}) \xrightarrow{\cong} \pi_q(\mathbf{E}(G/G)),$$

which comes from the fact that $\star_{G,\mathcal{ALL}}$ is an $\mathrm{Or}(G)$-CW-complex. The isomorphism conjecture for a group G, a family of subgroups $\mathcal{F}$ and an $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ says that the map induced by the inclusion $\star_{G,\mathcal{F}} \to \star_{G,\mathcal{ALL}}$

$$H_q^G(\star_{G,\mathcal{F}};\mathbf{E}) \to H_q^G(\star_{G,\mathcal{ALL}};\mathbf{E}) = \pi_q(\mathbf{E}(G/G)) \tag{1.8}$$

is an isomorphism for all $q \in \mathbb{Z}$. The philosophy is to compute the groups of interest $\pi_q(\mathbf{E}(G/G))$ by the values of $\mathbf{E}(G/H)$ on the groups in $H \in \mathcal{F}$. The isomorphism conjectures of Farrell and Jones for algebraic K-theory and L-theory are the special cases where $\mathbf{E}$ is given by the $\mathrm{Or}(G)$-spectra $\mathbf{K}$ and $\mathbf{L}$ of (1.2) and (1.3) and $\mathcal{F}$ is the family $\mathcal{VC}$ of virtually cyclic subgroups of G. The Baum–Connes conjecture is the special case where $\mathbf{E}$ is given by the $\mathrm{Or}(G)$-spectra $\mathbf{K}^{\mathrm{top}}$ of (1.4) and $\mathcal{F}$ is the family $\mathcal{FIN}$ of finite subgroups of G. The Farrell and Jones isomorphism conjecture and the Baum–Connes conjecture provide tools to compute $\pi_q(\mathbf{K}^{\mathrm{alg}}(G/G)) = K_q(\mathbb{Z}G)$, $\pi_q(\mathbf{L}(G/G)) = L_q(\mathbb{Z}G)$ and $\pi_q(\mathbf{K}^{\mathrm{top}}(G/G)) = K_q(C_r^*(G))$ in terms of data given by the virtually cyclic subgroups or the finite subgroups of G. If G is torsion free, these conjectures predict that for $q \in \mathbb{Z}$

$$\mathrm{Wh}_q(G) \cong 0,$$
$$L_q(\mathbb{Z}G) \cong H_q(BG; \mathbf{L}(\mathbb{Z})),$$
$$K_q(C_r^*(G)) \cong K_q(BG),$$

where $H_q(BG; \mathbf{L}(\mathbb{Z}))$ is the homology of the classifying space BG with respect to the L-theory spectrum associated to the ring $\mathbb{Z}$, $K_q(BG)$ is the topological complex K-homology of BG and $\mathrm{Wh}_q(G)$ denotes the reduced negative or zeroth K-group $\widetilde{K}_q(\mathbb{Z}G)$ for $q \leqslant 0$, the ordinary Whitehead group $\mathrm{Wh}(G)$ for $q = 1$ and Waldhausen's definition for $q \geqslant 1$ in terms of the fiber of the assembly map $BG_+ \wedge \mathbf{K}(\mathbb{Z}) \to \mathbf{K}(\mathbb{Z}G)$.

We mention that the assembly map appearing in the original conjectures of Farrell and Jones and of Baum and Connes are different from the one presented here. Their identification is discussed in [15, page 239, page 247–248] and based on [11] and [39, Proposition 8.4 on page 421].

THEOREM 1.2. *Let G be a planar group with compact orbit space or a virtually finitely generated Abelian group. Then the assembly maps*

$$H_q^G(\star_{G,\mathcal{VC}}; \mathbf{K}^{\mathrm{alg}}) \to H_q^G(\star_{G,\mathcal{ALL}}; \mathbf{K}^{\mathrm{alg}}) = K_q(\mathbb{Z}G) \quad \text{for } q \leqslant 1,$$
$$H_q^G(\star_{G,\mathcal{FIN}}; \mathbf{L})[1/2] \to H_q^G(\star_{G,\mathcal{ALL}}; \mathbf{L})[1/2] = L_q(\mathbb{Z}G)[1/2] \quad \text{for } q \in \mathbb{Z},$$
$$H_q^G(\star_{G,\mathcal{FIN}}; \mathbf{K}^{\mathrm{top}}) \to H_q^G(\star_{G,\mathcal{ALL}}; \mathbf{K}^{\mathrm{top}}) = K_q(C_r^*(G)) \quad \text{for } q \in \mathbb{Z},$$

are isomorphisms. The first map is surjective for $q = 2$.

Here and in the sequel $A[1/m]$ for an integer $m \geqslant 1$ means $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/m]$ for $\mathbb{Z}[1/m] = \{a \cdot m^b \in \mathbb{Q} \mid a, b \in \mathbb{Z}\}$. Thus $A[1/m]$ is obtained from A by inverting m. The claim for algebraic K- and L-theory is a consequence of the results of [18, Proposition 2.3, Proposition 2.4 and Remark 2.1.3], [50] and Theorem 2.3. Since we will invert 2 in (almost) all our L-theory calculations we do not have to distinguish between the various decorations of L-groups such as L^h, L^s or $L^{-\infty}$ as they differ by 2-torsion because of the Rothenberg sequences. The Baum–Connes-conjecture has been proven for a very large class of groups including virtually finitely generated Abelian groups and cocompact planar groups [21, Theorem 1.12].

2. Preliminaries About Spaces Over the Orbit Category

In this section we prove some facts about spaces over the orbit category and their homology which will be needed later.

Given a homomorphism of groups $i\colon H \to G$, there is an induced functor $I = I(i)\colon \mathrm{Or}(H) \to \mathrm{Or}(G)$ sending H/K to $G \times_i H/K = G/i(K)$. Given a (covariant or contravariant) $\mathrm{Or}(G)$-space Y, we obtain a (covariant or contravariant) $\mathrm{Or}(H)$-space I^*Y called the restriction of Y with I by the composition $Y \circ I$. Given a covariant $\mathrm{Or}(H)$-space X, its induction I_*X is the covariant $\mathrm{Or}(G)$-space defined in [15, Definition 1.8]. Its value at the object G/K of $\mathrm{Or}(G)$ is the quotient space

$$\coprod_{H/L \in \mathrm{Or}(H)} X(H/L) \times \mathrm{map}_G(G/K, G/i(L))/\sim,$$

where $\sim$ is the equivalence relation generated by $(X(\phi)(x), \psi) \sim (x, I(\phi)\circ\psi)$ for a H-map $\phi\colon H/L \to H/L'$, a G-map $\psi\colon G/K \to G(i(L))$ and $x \in X(H/L')$. There is an adjunction homeomorphism for an $\mathrm{Or}(G)$-space X and an $\mathrm{Or}(G)$-space Y (see [15, Lemma 1.9])

$$\hom_{\mathrm{Or}(G)}(I_*X, Y) \xrightarrow{\cong} \hom_{\mathrm{Or}(H)}(X, I^*Y). \tag{2.1}$$

In the sequel we use the identification of the Weyl group $W_GH = N_GH/H$ with the automorphism group $\mathrm{aut}_G(G/H, G/H)$ which sends $gH \in N_GH/H$ to the G-map $R_{g^{-1}}\colon G/H \to G/Hg'H \mapsto g'g^{-1}H$. Notice that $\mathrm{aut}_G(G/H, G/H) = \mathrm{map}_G(G/H, G/H)$ holds for finite H but not in general (see [26, Example 1.32 on page 22]).

LEMMA 2.1. *Let G be a group with subgroups $H, L \subset G$. Let $\{H_\alpha | \alpha \in A\}$ be a complete system of representatives of L-conjugacy classes of subgroups of L which are G-conjugated to H. To every index $\alpha \in A$ choose an isomorphism $\mu_\alpha\colon G/H \to G/H_\alpha$. Let $I = I(L)$ be the functor induced by the inclusion $L \hookrightarrow G$. Then we have for $K \subset L$:*

(a) *The following map is a natural equivalence of functors* $\mathrm{Or}(L) \to \mathrm{Sets}$

$$\begin{aligned} T(L/K)\colon & \coprod_{\alpha \in A} \mathrm{map}_L(L/H_\alpha, L/K) \times \\ & \times_{W_LH_\alpha} W_GH_\alpha \longrightarrow \mathrm{map}_G(G/H, G/K) \\ & [\phi, \psi] \mapsto (\mathrm{Id}_G \times_L \phi)\circ\psi\circ\mu_\alpha, \end{aligned}$$

(b) *There is a homeomorphism, natural in X,*

$$\coprod_{\alpha \in A} X(L/H_\alpha) \times_{W_LH_\alpha} W_GH_\alpha \longrightarrow I_*X(G/H)$$

for every contravariant $\mathrm{Or}(L)$-space X,

(c) *There is a natural isomorphism*

$$H_q^G(I_*X; \mathbf{E}) \xrightarrow{\cong} H_q^L(X; \mathbf{E})$$

for $\mathbf{E}$ one of the spectra associated to G and L in (1.2)–(1.7),

(d) *The map from* $\{g \in L\backslash G/N_G H \mid gHg^{-1} \subset L\}$ *to the set of* L*-conjugacy classes of subgroups of* L *being* G*-conjugated to* H *which sends* LgN_GH *to* $(gHg^{-1})_L$ *is bijective.*

Proof. (a), (b) and (d) are elementary consequences of the definitions. (c) follows from the facts that induction with $I\colon \mathrm{Or}(L) \to \mathrm{Or}(G)$ sends an $\mathrm{Or}(L)$-CW-approximation to an $\mathrm{Or}(G)$-CW-approximation and that the canonical map $\mathbf{E}_L \to I^*\mathbf{E}_G$ is an equivalence of $\mathrm{Or}(L)$-spectra together with the adjunction (2.1) and [15, Lemma 4.6]. □

Sometimes we can use smaller families than $\mathcal{VC}$ such as the family $\mathcal{FIN}$ of finite subgroups as explained in the next result. Notice that for $n = \infty$ and $m = 1$ it is just [18, Theorem A.10].

DEFINITION 2.2. Let G be a group, and let $\mathcal{F}$ be a family of subgroups of G. For a subgroup H of G, we define $H \cap \mathcal{F}$ to be the family of subgroups of H given as $\{H \cap K \mid K \in \mathcal{F}\}$.

THEOREM 2.3. *Let* $\mathcal{F} \subset \mathcal{G}$ *be families of subgroups of the group* G*. Let* $m \geqslant 1$ *and* n *be integers. Suppose for every* $H \in \mathcal{G}$ *that the assembly map*

$$H_q^H(\star_{H,H\cap\mathcal{F}}; I(H)^*\mathbf{E})[1/m] \to H_q^H(\star_{H,\mathcal{ALL}}; I(H)^*\mathbf{E})[1/m]$$

is an isomorphism for $q \leqslant n$*. Then the relative assembly map*

$$H_q^G(\star_{G,\mathcal{F}}; \mathbf{E})[1/m] \to H_q^G(\star_{G,\mathcal{G}}; \mathbf{E})[1/m]$$

is an isomorphism for $q \leqslant n$*.*

Proof. In the sequel we use the identification

$$H_p^G(\star_{G,\mathcal{F}}; \mathbf{E}) = \pi_q(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E})$$

as explained in [15, section 3]. By assumption the map

$$\pi_q\left(\operatorname*{hocolim}_{\mathrm{Or}(H,H\cap\mathcal{F})} I(H)^*\mathbf{E}\right)[1/m] \to \pi_q(\mathbf{E}(G/H))[1/m]$$

is an isomorphism for $q \leqslant n$ and $H \in \mathcal{G}$. A standard spectral sequence argument applied to the Atiyah–Hirzebruch spectral sequence [15, Theorem 4.7] or the Bousfield–Kan spectral sequence [9, XXII 5.7 on page 339] shows that the induced map

$$\pi_q\left(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(H,H\cap\mathcal{F})} I(H)^*\mathbf{E}\right)[1/m] \to \pi_q(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{G})} \mathbf{E})[1/m]$$

is an isomorphism for $q \leqslant n$. There is an equivalence of categories

$$\mathrm{Or}(G,\mathcal{F}) \downarrow (G/H) \xrightarrow{\cong} \mathrm{Or}(H, H\cap\mathcal{F}) \quad (G/F \to G/H) \mapsto H/F,$$

where $\mathrm{Or}(G,\mathcal{F}) \downarrow (G/H)$ denotes the category of objects over G/H. For $\mathbf{E}'(G/F \to G/H) := \mathbf{E}(G/F)$ we get an isomorphism

$$\pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(H,H\cap\mathcal{F})} I(H)^*\mathbf{E}\right) \cong \pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})\downarrow G/H} \mathbf{E}'\right).$$

By [24, Theorem 2.4] the homotopy colimits commute, hence there is an isomorphism for $q \leqslant n$

$$\begin{aligned}
\pi_q\left(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{G})} \mathbf{E}\right)[1/m] &\cong \pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(H,H\cap\mathcal{F})} I(H)^*\mathbf{E}\right)[1/m] \\
&\cong \pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(H,H\cap\mathcal{F})} I(H)^*\mathbf{E}\right)[1/m] \\
&\cong \pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{G})} \operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})\downarrow G/H} \mathbf{E}'\right)[1/m] \\
&\cong \pi_q\left(\operatorname*{hocolim}_{G/H\in \mathrm{Or}(G,\mathcal{F})} \operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{G})\downarrow G/H} \mathbf{E}'\right)[1/m] \\
&\cong \pi_q\left(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}\right)[1/m]. \qquad \square
\end{aligned}$$

Recall that a *classifying space $E(G,\mathcal{F})$ for a family $\mathcal{F}$ of subgroups of G* is a G-CW-complex whose H-fixed point set is contractible if $H \in \mathcal{F}$ and empty otherwise. Such a G-space is unique up to G-homotopy. In particular $E(G,\mathcal{TR})$ is a model for EG. We abbreviate $\underline{E}G = E(G,\mathcal{FIN})$ for the family $\mathcal{FIN}$ of finite subgroups.

For the reader's convenience we briefly sketch a different more geometric proof of Theorem 2.3. Namely, given a model for $E(G,\mathcal{G})$, one can construct a model for $E(G,\mathcal{F})$ by replacing each cell $G/H\times D^n$ in $E(G,\mathcal{G})$ by $G\times_H E(H, H\cap\mathcal{F})\times D^n$ and then use Mayer–Vietoris sequences. Notice that the assumption in Theorem 2.3 implies that for each $H \in \mathcal{G}$ the projection induces an isomorphism

$$\begin{aligned}
&H_q^H(\star_{H,H\cap\mathcal{F}}; I(H)^*\mathbf{E})[1/m] = H_p^G(G\times_H E(H, H\cap\mathcal{F}); \mathbf{E})[1/m] \\
&\xrightarrow{\cong} H_p^G(G/H; \mathbf{E})[1/m] = H_q^H(\star_{H,\mathcal{ALL}}; I(H)^*\mathbf{E})[1/m],
\end{aligned}$$

where we interpret a G-space as a $\mathrm{Or}(G)$-space by assigning to G/L its L-fixed point set.

The next result follows from the definitions of $\mathrm{Wh}_q(G)$ in [47, Definition 15.6 on page 228 and Proposition 15.7 on page 229], from the definition of $H_*^G(X, A; \mathbf{E})$ in [15, Section 4] and from [15, Lemma 7.6].

LEMMA 2.4. *For a group G there is an isomorphism*

$$H_q^G(\star_{G,\mathcal{ALL}}, \star_{G,\mathcal{TR}}; \mathbf{K}^{\mathrm{alg}}) = \begin{cases} \mathrm{Wh}_q(G) & q \geqslant 2, \\ \mathrm{Wh}(G) = \mathrm{Wh}_1(G) & q = 1, \\ \widetilde{K}_0(\mathbb{Z}G) = \mathrm{Wh}_0(G) & q = 0, \\ K_q(\mathbb{Z}G) = \mathrm{Wh}_q(G) & q \leqslant -1, \end{cases}$$

which is natural in G.

LEMMA 2.5. *The assembly map*

$$H_q^{\mathrm{Or}(G)}(\star_{G,\mathcal{FIN}};\mathbf{K}) \xrightarrow{\cong} H_q^{\mathrm{Or}(G)}(\star_{G,\mathcal{ALL}};\mathbf{K})$$

is an isomorphism for any $q \in \mathbb{Z}$ if $G = \mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$, and an isomorphism for $q \leqslant 2$ if $G = \mathbb{Z}/2 \times \mathbb{Z}$, $\mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2)$, $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4$, or $\mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$.

Proof. We begin with the case $G = \mathbb{Z}$. A model for $\underline{E}G = E(G,\mathcal{FIN})$ is the universal covering of S^1. Therefore, the source of the assembly map above reduces to $K_q(\mathbb{Z}) \oplus K_{q-1}(\mathbb{Z})$ and the assembly map itself is the restriction of the Bass–Heller–Swan isomorphism (see [4, Chapter XII], [5] and [37, Corollary to Theorem 8 on p. 114])

$$K_q(\mathbb{Z}) \oplus K_{q-1}(\mathbb{Z}) \oplus \mathrm{Nil}_q(\mathbb{Z}) \oplus \mathrm{Nil}_q(\mathbb{Z}) \to K_q(\mathbb{Z}[\mathbb{Z}])$$

to the first two summands. Since the ring $\mathbb{Z}$ is regular and hence all its Nil-groups are trivial, the assembly map is an isomorphism for $G = \mathbb{Z}$.

For $G = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$ there is a model for $\underline{E}G$ with $\mathbb{R}$ as underlying space such that $G\backslash\underline{E}G$ is the unit interval. Then the assembly map in question can be identified with the obvious map $\mathrm{Wh}_q(\mathbb{Z}/2) \oplus \mathrm{Wh}_q(\mathbb{Z}/2) \to \mathrm{Wh}_q(\mathbb{Z}/2 * \mathbb{Z}/2)$ which is bijective by a result of Waldhausen [47, Corollary 11.5 and the following remark].

By Theorem 3.2 (e) the Nil-groups of $\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}^l]$ are trivial for $q \leqslant 2$ and $l \geqslant 0$. So the claim follows for $G = \mathbb{Z} \times \mathbb{Z}/2$ from the Bass–Heller–Swan Theorem [4, Chapter XII], [5] and [37, Corollary to Theorem 8 on p. 114].

In the remaining cases G is of the shape $G = G_1 *_{\mathbb{Z}/2} G_2$ for finite groups G_i such that $\mathrm{Wh}_q(G_i) = 0$ for $q \leqslant 1$ (see Theorem 3.2 (div)). The claim for $q \leqslant -1$ follows from [19, Theorem 2.1]. We obtain from [47, Theorem 1 on page 137] the exact sequence

$$\begin{aligned}\mathrm{Wh}_2(\mathbb{Z}/2) &\to \mathrm{Wh}_2(G_1) \oplus \mathrm{Wh}_2(G_2) \to \mathrm{Wh}_2(G) \to \mathrm{Wh}(\mathbb{Z}/2)\\ &\to \mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G) \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2])\\ &\to \widetilde{K}_0(\mathbb{Z}G_1) \oplus \widetilde{K}_0(\mathbb{Z}G_2) \to \widetilde{K}_0(\mathbb{Z}G) \to 0\end{aligned}$$

where it is not clear a priori why the last map is surjective. This follows from the exact sequence, obtained by applying the same argument to $G \times \mathbb{Z} = (G_1 \times \mathbb{Z}) *_{\mathbb{Z}/2\times\mathbb{Z}} (G_2 \times \mathbb{Z})$,

$$\mathrm{Wh}(G_1 \times \mathbb{Z}) \oplus \mathrm{Wh}(G_2 \times \mathbb{Z}) \to \mathrm{Wh}(G \times \mathbb{Z}) \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}]),$$

the computation $\widetilde{K}_0(\mathbb{Z}/2 \times \mathbb{Z}) = 0$ from Theorem 3.2 (diii) and (e) and the fact that $\widetilde{K}_0(\mathbb{Z}G_1) \oplus \widetilde{K}_0(\mathbb{Z}G_2) \to \widetilde{K}_0(\mathbb{Z}G)$ is a natural direct summand in $\mathrm{Wh}(G_1 \times \mathbb{Z}) \oplus \mathrm{Wh}(G_2 \times \mathbb{Z}) \to \mathrm{Wh}(G \times \mathbb{Z})$ by the Bass–Heller–Swan decomposition. In order to get the claim for $q = 0, 1, 2$, one constructs an exact sequence as above but now with $H_q^G(\star_{G,\mathcal{FIN}};\mathbf{K}^{\mathrm{alg}})$ instead of $\mathrm{Wh}_q(G)$, together with a map between these exact sequences, and uses the five-lemma. The last exact sequence is the

Mayer–Vietoris sequence associated to the G-CW-model for $\underline{E}G$ which has one 1-cell with isotropy group $\mathbb{Z}/2$ and two 0-cells with isotropy groups G_1 and G_2 [43, Theorem 7 on page 32 and Corollary on page 36]. □

LEMMA 2.6. *Let G be a group and let $L \subset G$ be a normal subgroup such that there is an epimorphism $f\colon G \to L$ inducing the identity on L. Denote by $\mathcal{SUB}(L)$ the family of subgroups of G which consists of the subgroups of L. Then there is an isomorphism*

$$H_q^G(\star_{G,\mathcal{SUB}(L)}; \mathbf{L}) \to H_q(B(G/L); \mathbf{L}(\mathbb{Z}L)),$$

and analogously for both versions of K-theory and all the reduced versions of K- and L-theory.

Proof. We only treat the case of the L-theory spectrum $\mathbf{L}$, the others are completely analogous. We get from the definitions and [15, section 7] an isomorphism

$$H_q^G(\star_{G,\mathcal{SUB}(L)}; \mathbf{L}) \cong \pi_q(E(G/L)_+ \wedge_{G/L} \mathbf{L}(G/L)),$$

since EG/L regarded as a G-space via the projection $G \to G/L$ is a model for the classifying space $E(G, \mathcal{SUB}(L))$ of G for the family $\mathcal{SUB}(L)$. From f we obtain a morphism in $\mathsf{Groupoids}^{\mathsf{inj}}$

$$f_*\colon \mathcal{G}^G(G/L) \to \mathcal{G}^L(L/L).$$

Notice that $G/L = \mathrm{aut}_G(G/L, G/L)$ acts on $\mathcal{G}^G(G/L)$ in the obvious way and that this action induces a non-trivial action on $\mathbf{L}(\mathcal{G}^G(G/L))$ although it induces a trivial action on the homotopy groups. If we equip $\mathcal{G}^L(L/L)$ with the trivial G/L-action, the map f_* is L-equivariant. Since f_* is an equivalence of categories, the induced map $\pi_q(\mathbf{L}(\mathcal{G}^G(G/L))) \to \pi_q(\mathbf{L}(\mathcal{G}^L(L/L)))$ is an isomorphism. Hence the induced map

$$\begin{aligned}&\pi_q(E(G/L)_+ \wedge_{G/L} \mathbf{L}(G/L))\\ &\quad\xrightarrow{\cong} \pi_q(E(G/L)_+ \wedge_{G/L} \mathbf{L}(\mathcal{G}^L(L/L))) = H_q(B(G/L); \mathbf{L}(\mathbb{Z}L))\end{aligned}$$

is an isomorphism [15, Lemma 4.6]. □

DEFINITION 2.7. If $\mathbf{E}$ is a spectrum, we denote the generalized homology of a space X which is associated to $\mathbf{E}$ by $H_*(X; \mathbf{E})$. If $\mathbf{E}$ is the topological K-theory spectrum $\mathbf{K} = \mathbf{K}(\mathbb{C})$, then we also write $K_*(X) = H_*(X; \mathbf{K})$.

LEMMA 2.8. *Let G be a discrete group. Let A be a ring with $\mathbb{Z} \subset A \subset \mathbb{Q}$ such that the order of any finite subgroup of G is invertible in A.*

(a) *Let $\mathcal{H}_*$ be any generalized homology theory. Then we obtain a natural isomorphism*

$$\mathcal{H}_*(BG) \otimes_{\mathbb{Z}} A \xrightarrow{\cong} \mathcal{H}_*(G\backslash\underline{E}G) \otimes_{\mathbb{Z}} A.$$

(b) *There is a long exact sequence*

$$\cdots \to H_{q+1}(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \to H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{L}}) \to H_q^G(\star_{G,\mathcal{FIN}};\mathbf{L}) \to H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \to \cdots,$$

where $\mathbf{L}(\mathbb{Z})$ *is the* L*-theory spectrum associated to the ring* $\mathbb{Z}$*. This sequence splits after tensoring with* A*, yielding isomorphisms*

$$H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z}))\otimes_{\mathbb{Z}} A \oplus H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{L}}) \otimes_{\mathbb{Z}} A \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{FIN}};\mathbf{L})\otimes_{\mathbb{Z}} A.$$

(c) *For topological* K*-theory we obtain the long exact sequence*

$$\cdots \to K_{q+1}(G\backslash \underline{E}G) \to H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}}) \to H_q^G(\star_{G,\mathcal{FIN}};\mathbf{K}) \to K_q(G\backslash \underline{E}G) \to \cdots.$$

This sequence splits after tensoring with A *yielding isomorphisms*

$$K_q(G\backslash \underline{E}G)\otimes_{\mathbb{Z}} A \;\oplus\; H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}})\otimes_{\mathbb{Z}} A \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{FIN}};\mathbf{K})\otimes_{\mathbb{Z}} A.$$

Proof. (a) By the Atiyah–Hirzebruch spectral sequence it suffices to check the claimed isomorphism in the special case where $\mathcal{H}_*$ is the cellular homology H_*. The claim follows from the fact that the projection induces a homology equivalence of projective AG-chain complexes $C_*(EG)\otimes_{\mathbb{Z}} A \to C_*(\underline{E}G)\otimes_{\mathbb{Z}} A$ which is then an AG-chain homotopy equivalence and hence induces a chain homotopy equivalence $C_*(EG)\otimes_{\mathbb{Z}G} A \to C_*(\underline{E}G)\otimes_{\mathbb{Z}G} A$.

(b) There are natural maps of $\mathrm{Or}(G)$-spectra $\widetilde{\mathbf{L}} \to \mathbf{L} \to \underline{\mathbf{L}(\mathbb{Z})}$, where $\underline{\mathbf{L}(\mathbb{Z})}$ denotes the constant $\mathrm{Or}(G)$-spectrum with value $\mathbf{L}(\mathbb{Z}) = \mathbf{L}(*)$. Since its evaluation at an object G/H is a fibration of spectra, it induces a long exact sequence

$$\cdots \to H_{q+1}^G(X;\underline{\mathbf{L}(\mathbb{Z})}) \xrightarrow{\delta_{q+1}} H_q^G(X;\widetilde{\mathbf{L}}) \to H_q^G(X;\mathbf{L}) \to H_q^G(X;\underline{\mathbf{L}(\mathbb{Z})}) \to \cdots,$$

for any contravariant $\mathrm{Or}(G)$-space X. Notice that it suffices to check exactness for any $\mathrm{Or}(G)$-CW-complex X and hence for any $\mathrm{Or}(G)$-space of the form $\mathrm{map}_G(G/?, G/K)$ for any fixed object G/K in $\mathrm{Or}(G)$, where the claim reduces to the exactness of the long homotopy sequence associated to a fibration. We get from [15, Lemma 7.6] an identification $H_q^G(\star_{G,\mathcal{FIN}};\underline{\mathbf{L}(\mathbb{Z})})$ with $H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z}))$ and thus the desired long exact sequence by taking $X = \star_{G,\mathcal{FIN}}$.

The composition

$$H_q(BG;\mathbf{L}(\mathbb{Z})) = H_q^G(\star_{G,\mathcal{TR}};\mathbf{L}) \to H_q(\star_{G,\mathcal{FIN}};\mathbf{L}) \to H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z}))$$

becomes an isomorphism after tensoring with A by assertion (a) and thus induces the splitting of the long exact sequence after tensoring with A.

The proof of (c) is analogous to that of assertion (b) taking into account that $K_q(Y) = H_q(Y;\mathbf{K}^{\mathrm{top}}(\mathbb{C}))$ holds by definition. □

3. Preliminary Computations of *K*- and *L*-Groups of Finite Groups

In this section we state some computations about K-and L-groups for finite groups which we will use later in the computations for infinite groups.

DEFINITION 3.1. Let π be a finite group. By $q(\pi)$, $r(\pi)$ and $c(\pi)$, we denote the number of irreducible rational, real respective complex representations of π. Let $r_{\mathbb{C}}(\pi)$ be the number of isomorphisms classes of irreducible real π-representations V which are of complex type, i.e. $\mathrm{aut}_{\mathbb{R}\pi}(V) \cong \mathbb{C}$. Let $RO(\pi)$ resp. $R(\pi)$ be the real resp. complex representation ring. For a positive integer m, we let $\mathbb{Z}/m$ denote the cyclic group of order m, and D_{2m} is the dihedral group of order $2m$.

THEOREM 3.2. *For a finite group π, we have the following:*

(a) *There are isomorphisms $RO(\pi) \cong \mathbb{Z}^{r(\pi)}$ and $R(\pi) \cong \mathbb{Z}^{c(\pi)}$. The number $q(\pi)$ is the number of conjugacy classes of cyclic subgroups in π, the number $c(\pi)$ is the number of conjugacy classes of elements in π and the number $r(\pi)$ is the number of $\mathbb{R}$-conjugacy classes of elements in π, where g_1 and g_2 in π are $\mathbb{R}$-conjugated if g_1 and g_2 or g_1^{-1} and g_2 are conjugated,*

(b) $$K_q(C_r^*(\pi)) \cong \begin{cases} R(\pi) \cong \mathbb{Z}^{c(\pi)} & q = 0, \\ 0 & q = 1. \end{cases}$$

In particular

$$K_0(C_r^*(\mathbb{Z}/m)) \ \cong \ \mathbb{Z}^m$$
$$K_0(C_r^*(D_{2m})) \ \cong \ \begin{cases} \mathbb{Z}^{m/2+3} & m \equiv 0(2), \\ \mathbb{Z}^{(m-1)/2+2} & m \equiv 1(2), \end{cases}$$

(c) $$L_q(\mathbb{Z}\pi)[1/2] \cong L_q(\mathbb{R}\pi)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r(\pi)} & q \equiv 0(4), \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(\pi)} & q \equiv 2(4), \\ 0 & q \equiv 1, 3(4). \end{cases}$$

In particular,

$$L_0(\mathbb{Z}[\mathbb{Z}/m])[1/2] \cong \mathbb{Z}[1/2]^{[(m+2)/2]},$$
$$L_2(\mathbb{Z}[\mathbb{Z}/m])[1/2] \cong \mathbb{Z}[1/2]^{[(m-1)/2]},$$
$$L_0(\mathbb{Z}[D_{2m}])[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{m/2+3} & m \equiv 0(2), \\ \mathbb{Z}[1/2]^{(m-1)/2+2} & m \equiv 1(2), \end{cases}$$
$$L_2(\mathbb{Z}[D_{2m}])[1/2] \cong 0.$$

(d) (i) $K_q(\mathbb{Z}\pi) = 0$ *for* $q \leqslant -2$.

(ii) *The rank of* $\mathrm{Wh}(\pi)$ *as an Abelian group is* $r(\pi) - q(\pi)$. *We have*

$$\mathrm{Wh}(\mathbb{Z}/m) \cong \mathbb{Z}^{[m/2]+1-\delta(m)},$$

where $\delta(m)$ is the number of divisors of m and $[m/2]$ is the largest integer less or equal to $m/2$.

(iii) *We have* $\mathrm{Wh}_q(\pi) = 0$ *for* $q \leqslant 1$ *for the following finite groups* $\pi = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_6, D_8. *If* l *is a prime, then* $K_{-1}(\mathbb{Z}[\mathbb{Z}/l]) = K_{-1}(\mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}/l]) = 0$. *We have*

$$\mathrm{Wh}_q(\mathbb{Z}/6) \cong \begin{cases} 0 & q = 0, 1, \\ \mathbb{Z} & q = -1, \end{cases}$$

$$\mathrm{Wh}_q(D_{12}) \cong \begin{cases} 0 & q = 0, 1, \\ \mathbb{Z} & q = -1. \end{cases}$$

(iv) *We have*

$$\begin{aligned} &\mathrm{Wh}_2(\pi) = 0, \ \text{for } \pi = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \\ &|\,\mathrm{Wh}_2(\mathbb{Z}/6)| \leqslant 2, \\ &\mathrm{Wh}_2(D_6) = \mathbb{Z}/2, \\ &\mathrm{Wh}_2((\mathbb{Z}/2)^2) \geqslant (\mathbb{Z}/2)^2. \end{aligned}$$

The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}[\mathbb{Z}/2])$ *is an isomorphism.*

(e) *There are isomorphisms for an integer* $n \geqslant 0$ *and a prime number* l *and* $q \in \mathbb{Z}$

$$\begin{aligned} &L_q(\mathbb{Z}[\mathbb{Z}^n \times \pi])[1/2] \cong \bigoplus_{i=0}^{n} \binom{n}{i} \cdot L_{q-i}(\mathbb{Z}[\pi])[1/2], \\ &K_q(C_r^*(\mathbb{Z}^n \times \pi)) \cong \bigoplus_{i=0}^{n} \binom{n}{i} \cdot K_{q-i}(C_r^*(\pi)), \\ &K_q(\mathbb{Z}[\mathbb{Z}^n \times \mathbb{Z}/l]) \\ &\quad \cong K_q(\mathbb{Z}[\mathbb{Z}/l]) \oplus n \cdot K_{q-1}(\mathbb{Z}[\mathbb{Z}/l]) \oplus \binom{n}{2} \cdot K_{q-2}(\mathbb{Z}[\mathbb{Z}/l]), \quad q \leqslant 2, \\ &\mathrm{Wh}_q(\mathbb{Z}^n \times \mathbb{Z}/l) \\ &\quad \cong \mathrm{Wh}_q(\mathbb{Z}/l) \oplus n \cdot \mathrm{Wh}_{q-1}(\mathbb{Z}/l) \oplus \binom{n}{2} \cdot \mathrm{Wh}_{q-2}(\mathbb{Z}/l) \quad q \leqslant 2, \\ &\mathrm{Nil}_q(\mathbb{Z}[\mathbb{Z}^n \times \mathbb{Z}/l]) = 0, \quad q \leqslant 2. \end{aligned}$$

Proof. (a) is proven in [42, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page 106]. (b) follows from Morita equivalence applied to $\mathbb{C}[\pi] \cong \prod_{i=1}^{c(\pi)} M(n_i, n_i, \mathbb{C})$ and the computation $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$. (c) follows from [40, Proposition 22.34 on page 253]. (d) The computations of $K_q(\mathbb{Z}\pi)$ for $q \leqslant -1$ follow from [13]; that $K_{-1}(\mathbb{Z}\pi) = 0$ for $\pi = \mathbb{Z}/l$ or $\mathbb{Z}/l \times \mathbb{Z}/l$ can also be found in [4, Theorem 10.6, p. 695]. For information about $\mathrm{Wh}(\pi)$ we refer to [31]. The vanishing of $\widetilde{K}_0(\mathbb{Z}\pi)$ is proven for $\pi = D_6$ in [41, Theorem 8.2] and for $\pi = D_8$ in [41, Theorem 6.4]. The cases $\pi = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$, and $(\mathbb{Z}/2)^2$ are in [14, Corollary 5.17]. Finally, $\widetilde{K}_0(\mathbb{Z}D_{12}) = 0$ follows from [14, Theorem 50.29 on page 266] and the fact that $\mathbb{Q}D_{12}$ as a $\mathbb{Q}$-algebra splits into copies of $\mathbb{Q}$ and matrix algebras over $\mathbb{Q}$, so its maximal order has vanishing class group by Morita equivalence.

The claims about $\mathrm{Wh}_2(\mathbb{Z}/n)$ for $n = 2, 3, 4, 6$ and for $\mathrm{Wh}_2((\mathbb{Z}/2)^2)$ are taken from [16, Proposition 5], [17, p.482] and [46, p. 218 and 221]. We get $K_2(\mathbb{Z}D_6) \cong 3{\cdot}\mathbb{Z}/2$ from [46, Theorem 3.1]. The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [17, Theorem on p. 482]. Now construct a commutative diagram

$$\begin{array}{ccc} H_2(B\mathbb{Z}/2;\mathbf{K}(\mathbb{Z})) & \xrightarrow{\cong} & H_2(BD_6;\mathbf{K}(\mathbb{Z})) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ K_2(\mathbb{Z}[\mathbb{Z}/2]) & \longrightarrow & K_2(\mathbb{Z}D_6) \end{array}$$

whose lower horizontal arrow is split injective and whose upper horizontal arrow is an isomorphism by the Atiyah–Hirzebruch spectral sequence. Hence the right vertical arrow is split injective and $\mathrm{Wh}_2(D_6) = \mathbb{Z}/2$.

(e) The claim for L-groups follows from the Shaneson splitting [44, Theorem 5.1] and for topological K-groups for instance from the more general Voiculescu–Pimsner sequence [8, Theorem 10.2.1 on page 83]. The claim for the algebraic K-groups follows for $q \leqslant 0$ from [4, Theorem 10.6 on page 695]. To prove the vanishing of the Nil_1- and Nil_2-terms, consider the following cartesian square of rings:

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}^k] & \xrightarrow{t\mapsto \exp(2\pi i/l)} & \mathbb{Z}[\exp(2\pi i/l)][\mathbb{Z}^k] \\ {\scriptstyle t\mapsto 1}\downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}^k] & \longrightarrow & \mathbb{Z}/l[\mathbb{Z}^k], \end{array}$$

where t is a generator of $\mathbb{Z}/l$. Let $\xi := \exp(2\pi i/l)$. By [30, Theorem 3.3 and 6.2] and the methods of [46, Section 1], this diagram yields a long exact Mayer Vietoris sequence

$$\begin{aligned} K_3(\mathbb{Z}/l[\mathbb{Z}^k]) &\to K_2(\mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}^k]) \to K_2(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_2(\mathbb{Z}[\xi][\mathbb{Z}^k]) \\ &\to K_2(\mathbb{Z}/l[\mathbb{Z}^k]) \to K_1(\mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}^k]) \\ &\to K_1(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\xi][\mathbb{Z}^k]). \end{aligned}$$

The rings $\mathbb{Z}/l$, $\mathbb{Z}[\xi]$ and $\mathbb{Z}$ are regular, so they have trivial Nil-terms in any dimension [37, Corollary to Theorem 8 on p. 122]. Furthermore, since $\mathbb{Z}/l$ is a field, its higher algebraic K-groups are finite [36, Theorem 8 on p. 583], and the K-groups of a Dedekind ring (such as $\mathbb{Z}[\xi]$) are finitely generated [38, Theorem 1 on p. 179]. Therefore, the groups $K_i(\mathbb{Z}/l[\mathbb{Z}^k])$, $K_i(\mathbb{Z}[\mathbb{Z}^k])$ and $K_i(\mathbb{Z}[\xi][\mathbb{Z}^k])$ are finitely generated for $i = 1, 2, 3$, and hence so is $K_i(\mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}^k])$ for $i = 1, 2$.

Using the fact that the Nil-groups are either trivial or infinitely generated [35], we conclude that $\mathrm{Nil}_i(\mathbb{Z}[\mathbb{Z}/l \times \mathbb{Z}^{k-1}]) = 0$ for $i = 1, 2$. □

4. Cocompact Planar Groups

In this section we calculate the K- and L-theory of planar groups, giving explicit formulas in terms of their signature (Theorem 4.4 and Theorem 4.9). We begin with reviewing some facts about planar groups.

DEFINITION 4.1. A *planar group* (sometimes also called NEC group = Non-Euclidean crystallographic group) is a discontinuous group G of isometries of the two-sphere S^2, the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$. It is *cocompact* if the quotient manifold S which is the quotient of S^2, $\mathbb{R}^2$ or $\mathbb{H}^2$ under the G-action is compact.

It is known (see, e.g. [48, Theorem 3 and 4], [52, Theorem 4.5.6 on page 119]) that a cocompact planar group has the following presentation. For fixed integers $r_i \geqslant 1$, $s \geqslant 0$, $t \geqslant 0$, $g \geqslant 0$ and $h \geqslant 0$ such that $h = 0$ or $g = h$, generators are

(1) $x_{i,j}, \quad 1 \leqslant i \leqslant s, \quad 1 \leqslant j \leqslant r_i,$
(2) $e_i, \quad 1 \leqslant i \leqslant s,$
(3) $c_k, \quad 1 \leqslant k \leqslant t,$
(4) $a_p, \quad 1 \leqslant p \leqslant g,$
(5) $b_p, \quad 1 \leqslant p \leqslant h,$

and relations are given by

(6) $x_{i,j}^2 = 1,$
(7) $(x_{i,j}x_{i,j+1})^{n_{i,j}} = 1, \quad 1 \leqslant j < r_i, \quad n_{i,j} \geqslant 2,$
(8) $x_{i,r_i} e_i x_{i,1} e_i^{-1} = 1,$
(9) $c_k^{\gamma_k} = 1, \quad \gamma_k \geqslant 2,$
(10) $e_1^{-1} \cdots e_s^{-1} c_1^{-1} \cdots c_t^{-1} y = 1,$

where $y = a_1^2 \cdots a_g^2$ if $h \neq g$ and $y = [a_1, b_1] \cdots [a_g, b_g]$ if $h = g$. On the other hand, it is also known [52, Theorem 4.7.1 on page 122] that every group with a presentation as above is a cocompact planar group.

Associated to such a presentation is the so-called *signature* which encodes the presentation as follows:

$$(g, \pm, [\gamma_1, \gamma_2, \dots \gamma_t], \{(n_{1,1}, n_{1,2}, \dots, n_{1,r_1-1}), \dots (n_{s,1}, n_{s,2}, \dots n_{s,r_s-1})\}), \tag{4.1}$$

where $g \geqslant 0$ is an integer, $\pm$ a sign, $[\gamma_1, \gamma_2, \dots \gamma_t]$ is an ordered set of integers $\gamma_i \geqslant 0$ and $(n_{i,1}, n_{i,2}, \dots n_{i,r_i-1})$ is an ordered set of integers $n_{i,j} \geqslant 2$ for $i = 1, 2, \dots, s$. It is allowed that $[\gamma_1, \gamma_2, \dots, \gamma_t]$ consists of the empty symbol

[ ], i.e. $t = 0$. Similarly $(n_{i,1}, n_{i,2}, \dots, n_{i,r_i-1})$ may consist of the empty symbol ( ), i.e. $r_i = 1$. It is also possible that $s = 0$, then the fourth entry in the signature consists of the empty set { }. It is clear how a presentation as above defines a signature and vice versa, where of course the sign is positive if S is orientable, or equivalently, $h = g$ in the presentation, and the sign is negative if S is non-orientable, or equivalently, $h \neq g$ in the presentation. In [28, Section 9], we find necessary and sufficient conditions for two such presentations or signatures respectively to describe isomorphic groups and the proof that two planar groups are algebraically isomorphic if and only if they are geometrically isomorphic.

As mentioned above, the quotient manifold S of a cocompact planar group is orientable if and only if $h = g$. The number of boundary components is s, and the genus is g. If G acts on $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$, then this G-space is a model for $\underline{E}G = E(G, \mathcal{FIN})$ and in particular S is a model for $G \backslash \underline{E}G$. This can be found in [52, Section 4.2] or follows from the more general result [3, Corollary 4.14]. A cocompact planar group G is finite if and only if it acts on S^2. One easily computes using the Atiyah-Hirzebruch spectral sequence

$$
\begin{aligned}
&H_0(S;\mathbb{Z}) = \mathbb{Z},\\
&H_1(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g}, & \text{if } S \text{ is orientable and } s = 0,\\ \mathbb{Z}^{2g+s-1}, & \text{if } S \text{ is orientable and } s > 0,\\ \mathbb{Z}/2 \oplus \mathbb{Z}^{g-1}, & \text{if } S \text{ is non-orientable and } s = 0,\\ \mathbb{Z}^{g+s-1}, & \text{if } S \text{ is non-orientable and } s > 0, \end{cases}\\
&H_2(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } S \text{ is orientable and } s = 0,\\ 0, & \text{else}, \end{cases}\\
&K_0(S) = H_0(S,\mathbb{Z}) \oplus H_2(S;\mathbb{Z}),\\
&K_1(S) = H_1(S;\mathbb{Z}),\\
&H_p(S;\mathbf{L}(\mathbb{Z}))[1/2] = H_p(S;\mathbb{Z})[1/2], \quad \text{for } 0 \leqslant p \leqslant 3.
\end{aligned}
$$

As was mentioned in the introduction, the following result yields complete control over the structure of the finite subgroups of a cocompact planar group, thus allowing us to compute its K- and L-theory in terms of the maximal finite subgroups.

THEOREM 4.2. *Let G be an infinite cocompact planar group.*

(a) *The system* $\{\langle c_k \rangle\} \coprod \{\langle x_{i,j}, x_{i,j+1} \rangle\}$ *is a complete system of representatives of the conjugacy classes of maximal finite subgroups with* $(M)_G \neq (\langle x_{i,j} \rangle)_G$. *(Note that if* $r_i = 1$ *in the presentation of G, then* $\langle x_{i,1} \rangle$ *is a maximal finite subgroup isomorphic to* $\mathbb{Z}/2$, *but we want to exclude these from the above system). The group* $\langle c_k \rangle$ *is cyclic of order* γ_k *and the group* $\langle x_{i,j}, x_{i,j+1} \rangle$ *is the dihedral group* $D_{2n_{i,j}}$ *of order* $2n_{i,j}$. *In particular any finite subgroup of G is cyclic or dihedral.*

(b) *Every element of finite order is conjugate to an element* $x_{i,j}$, c_k^q *or* $(x_{i,j}x_{i,j+1})^q$ *for some* $q \in \mathbb{Z}$. *The subgroups* $\langle c_k^q \rangle$ *and* $\langle (x_{i,j}x_{i,j+1})^q \rangle$ *are either trivial or have finite normalizers.*

(c) *Among the non-trivial powers of the* c_k *and the* $x_{i,j}x_{i,j+1}$ *only* $(x_{i,j}x_{i,j+1})^q$ *and* $(x_{i,j}x_{i,j+1})^{-q}$ *are conjugate, and none of these powers is conjugate to one of the* $x_{i,j}$.

(d) *If* $g \in G$ *has infinite order, then* $C_G(g)$ *is isomorphic to one of the following groups:*

$$\mathbb{Z},\ \mathbb{Z}/2 \times \mathbb{Z},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z} \times (\mathbb{Z}/2 * \mathbb{Z}/2),\ \mathbb{Z} \rtimes \mathbb{Z},\ \text{or } \mathbb{Z} *_{2\mathbb{Z}} (2\mathbb{Z} \times \mathbb{Z}/2).$$

If $s = 0$ *in the presentation of* G, *then an element* $g \in G$ *of infinite order has a centralizer isomorphic to* $\mathbb{Z}$, $\mathbb{Z}^2$, *or* $\mathbb{Z} \rtimes \mathbb{Z}$. *If* G *is a cocompact Fuchsian group, i.e.* $s = 0$ *and* F *acts orientation preserving on* $\mathbb{H}^2$, *then an element* $g \in G$ *of infinite order has a centralizer isomorphic to* $\mathbb{Z}$.

(e) *The centralizer of a reflection generator* $x_{i,j}$ *is either* $\mathbb{Z}/2 \times \mathbb{Z}$ *or* $\mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2)$, *where the first factor* $\mathbb{Z}/2$ *is* $\langle x_{i,j} \rangle$.

Proof. (b) follows from [23, Corollary 2 on p. 742] and (c) from [52, Theorem 4.8.1 on page 126]. Assertion (a) is a consequence of (c) and [23, Corollary 2 on p. 742]. We conclude (d) from [22, Corollary 4 on p. 67], [23, Theorem 4 on p. 743] and the fact that $\mathbb{Z}^2$ cannot be a subgroup of a cocompact Fuchsian group. Assertion (e) is proven in [23, Theorem 4 on p. 743]. □

DEFINITION 4.3. Let G be a cocompact planar group with a presentation as above. Let $\{M_\alpha | \alpha \in A\}$ be a full system of representatives of the conjugacy classes of maximal finite subgroups of G with $(M_\alpha)_G \neq (\langle x_{i,j} \rangle)_G$, and let $\{H_\beta \mid \beta \in B\}$ be a full system of representatives of conjugacy classes of subgroups generated by a single reflection generator $x_{i,j}$. For $\alpha \in A$ for which M_α is not cyclic, denote by $C_\alpha \subset M_\alpha$ the cyclic subgroup which is conjugated in G to a cyclic subgroup generated by an element of the form $x_{i,j}x_{i,j+1}$ (see Theorem 4.3 (a) and (c)). For $\alpha \in A$ with cyclic M_α, put $C_\alpha = M_\alpha$. Let $B' = \{\beta \in B \mid N_G H_\beta \cong \mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2)\}$ and $B'' = \{\beta \in B \mid N_G H_\beta \cong \mathbb{Z}/2 \times \mathbb{Z})\}$. Then $B = B' \coprod B''$ by Theorem 4.3. (e).

Define for $\alpha \in A$ a direct summand $S\widetilde{L}_q(\mathbb{Z}M_\alpha)$ of $\widetilde{L}_q(\mathbb{Z}M_\alpha)$ as the kernel of the split-epimorphism

$$\begin{aligned}
&0\colon\ \widetilde{L}_q(\mathbb{Z}M_\alpha) \to 0, \quad M_\alpha \cong \mathbb{Z}/k,\\
&u_*\colon\ \widetilde{L}_q(\mathbb{Z}M_\alpha) \to \widetilde{L}_q(\mathbb{Z}[\mathbb{Z}/2]), \quad M_\alpha \cong D_{2k} k \text{ odd},\\
&u_* \oplus v_*\colon\ \widetilde{L}_q(\mathbb{Z}M_\alpha) \to \widetilde{L}_q(\mathbb{Z}[\mathbb{Z}/2]) \oplus \widetilde{L}_q(\mathbb{Z}[\mathbb{Z}/2]), \quad M_\alpha \cong D_{2k}\ \ k \text{ even},
\end{aligned}$$

where $u\colon M_\alpha \to \mathbb{Z}/2$ is the epimorphism with C_α as kernel and $v\colon M_\alpha \to \mathbb{Z}/2$ is some homomorphism for which $v(C_\alpha) = \mathbb{Z}/2$. Obviously u has a section $s\colon \mathbb{Z}/2 \to M_\alpha$ so that u_* is split surjective. Since there are homomorphisms

$s_1, s_2\colon \mathbb{Z}/2 \to M_\alpha$ such that $u \circ s_1$ and $v \circ s_2$ are the identity and $v \circ s_1$ is trivial, $(s_1)_* + (s_2)_*$ is a section of $u_* \oplus v_*$. Explicitly we get from Theorem 3.2.

$$S\widetilde{L}_q(\mathbb{Z}M_\alpha)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{[k/2]} & q \equiv 0(4) \quad M_\alpha \cong \mathbb{Z}/k, \\ \mathbb{Z}[1/2]^{[k/2]} & q \equiv 0(4) \quad M_\alpha \cong D_{2k}, \quad k \geqslant 2, \\ \mathbb{Z}[1/2]^{[(k-1)/2]} & q \equiv 2(4) \quad M_\alpha \cong \mathbb{Z}/k, \\ 0 & \text{otherwise}. \end{cases}$$

Define the direct summand $S\widetilde{K}_q(C_r^*(M_\alpha))$ of $\widetilde{K}_q(C_r^*(M_\alpha))$ analogously to $S\widetilde{L}_q(\mathbb{Z}[M_\alpha])$. We get explicitly from Theorem 3.2.

$$S\widetilde{K}_q(C_r^*(M_\alpha)) \cong \begin{cases} \mathbb{Z}^{k-1} & q \equiv 0(2) \quad M_\alpha \cong \mathbb{Z}/k, \\ \mathbb{Z}^{[k/2]} & q \equiv 0(2) \quad M_\alpha \cong D_{2k}, \quad k \geqslant 2, \\ 0 & \text{otherwise}. \end{cases}$$

THEOREM 4.4. *Let G be an infinite cocompact planar group with a presentation as above and quotient manifold S, and let M_α and H_β be as in Definition 4.3. We have*

(a) *The map induced by the inclusions*

$$\bigoplus_{\alpha \in A} \mathrm{Wh}_q(M_\alpha) \xrightarrow{\cong} \mathrm{Wh}_q(G)$$

is an isomorphism for $q \leqslant 1$ and surjective for $q = 2$,

(b) *We have split exact sequences*

$$0 \to \left(\bigoplus_{\beta \in B} H_q(BW_G H_\beta; \mathbf{L}(\mathbb{Z})) \oplus \bigoplus_{\alpha \in A} S\widetilde{L}_q(\mathbb{Z}M_\alpha)\right)[1/2]$$
$$\to L_q(\mathbb{Z}G)[1/2] \to H_q(S; \mathbf{L}(\mathbb{Z}))[1/2] \to 0,$$

and

$$H_q(BW_G H_\beta; \mathbf{L}(\mathbb{Z}))[1/2] \cong \begin{cases} \mathbb{Z}[1/2], & q \equiv 0\ (4), \\ \mathbb{Z}[1/2], & \beta \in B'', \quad q \equiv 1\ (4), \\ 0, & \textit{else}, \end{cases}$$

(c) *We have split exact sequences*

$$0 \to \bigoplus_{\beta \in B} K_0(BW_G H) \oplus \bigoplus_{\alpha \in A} S\widetilde{K}_0(C_r^*(M_\alpha))$$
$$\to K_0(C_r^*(G)) \to K_0(S) \to 0,$$
$$0 \to \bigoplus_{\beta \in B''} K_1(B\mathbb{Z}) \to K_1(C_r^*(G)) \to K_1(S) \to 0,$$

and

$$K_0(BW_G H_\beta) \cong \mathbb{Z} \quad \textit{for } \beta \in B.$$

If the isomorphism conjecture in algebraic K-theory (see 1.9 and Theorem 1.10) holds also in dimension 2, the homomorphism appearing in Theorem 4.4 (a) is an isomorphism also for $q = 2$. Furthermore, if $s = 0$ and the isomorphism conjecture is true in any dimension, then it is an isomorphism for any $q \in \mathbb{Z}$. Notice that in the case $s = 0$ we have $B = \emptyset$, so that the families $\mathcal{E}$ and $\mathcal{E}_\alpha$ appearing in the proof of Theorem 4.4 are just the families $\mathcal{TR}$.

The proof of Theorem 4.4 needs some preparations. The following lemma will ensure that the application of Lemma 2.1 yields tractable results.

LEMMA 4.5. *Let G be an infinite cocompact planar group. Define for a maximal finite subgroup M the cyclic subgroup C_M as in Definition 4.3. Then*

(a) *if $H \subset G$ is finite and $M \subset G$ is maximal finite, $(M)_G \neq (\langle x_{i,j}\rangle)_G$, with $H \cap C_M \neq 1$, then $N_G(H \cap C_M)$ and $N_G H$ are finite;*

(b) *let $M \subset G$ be maximal finite, $(M)_G \neq (\langle x_{i,j}\rangle)_G$, and $H \subset M$ be a subgroup with $H \cap C_M \neq 1$. Then*

$$M = N_G M = N_G(H \cap C_M).$$

If M and N are maximal finite subgroups, $(M)_G, (N)_G \neq (\langle x_{i,j}\rangle)_G$, with $C_M \cap C_N \neq 1$, then $M = N$;

(c) *let $M \subset G$ be maximal finite, $(M)_G \neq (\langle x_{i,j}\rangle)_G$. If $K_1, K_2 \subset M$ are subgroups with $K_i \cap C_M \neq 1$ and $(K_1)_G = (K_2)_G$, then $(K_1)_M = (K_2)_M$ and $N_G K_1 = N_M K_1$ and $N_G K_2 = N_M K_2$;*

(d) *let $D_{2m} = \langle s, t \mid s^m = t^2 = 1, tst = s^{-1}\rangle$ be the dihedral group of order $2m$. If m is odd, D_{2m} contains up to conjugacy precisely one subgroup of order 2, namely $\langle t\rangle$ with trivial Weyl group. If m is even, then D_{2m} has up to conjugacy three subgroups of order 2, namely, $\langle s^{m/2}\rangle$ with normalizer D_{2m}, $\langle st\rangle$ with normalizer $\langle s^{m/2}, st\rangle$ and $\langle t\rangle$ with normalizer $\langle s^{m/2}, t\rangle$. The subgroup $\langle s\rangle$ is a characteristic subgroup of D_{2m} if $m \geqslant 3$.*

Proof. (a) By Theorem 4.2 (b), $H \cap C_M$ is conjugated in G to a non-trivial cyclic subgroup generated by some power of an element $x_{i,j}x_{i,j+1}$ or c_k, and $N_G(H \cap C_M)$ is finite. Hence also $N_G H$ is finite since the centralizer $C_G H$ has finite index in $N_G H$ and is contained in $N_G(H \cap C_M)$.

(b) We get $M \subset N_G(H \cap C_M)$ since $H \cap C_M$ is normal in M. Obviously $M \subset N_G M$. As both $N_G(H \cap C_M)$ and $N_G M$ are finite by assertion (a), we get $M = N_G M = N_G(H \cap C_M)$ from the maximality of M. Applying this to $H = C_M \cap C_N$ we conclude $M = N$.

(c) Choose $g \in G$ with $gK_1g^{-1} = K_2$. Then $K_2 \subset gMg^{-1} \cap M$ and hence $gMg^{-1} = M$ by assertion (b). Again from assertion (b) we conclude $g \in M$. If we apply this to the special case $K_1 = K_2$, we conclude $N_G K_1 = N_M K_1$ and $N_G K_2 = N_M K_2$. (d) is a direct calculation. This finishes the proof of Lemma 4.5. □

LEMMA 4.6. *Let G be an infinite cocompact planar group. Let $\Gamma \leqslant G$ be infinite virtually cyclic. Then Γ is isomorphic to one of the following groups, where D denotes the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$:*

$$\mathbb{Z},\ \mathbb{Z}/2 \times \mathbb{Z},\ D,\ \mathbb{Z}/2 \times D,\ \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4 \cong \mathbb{Z} \rtimes \mathbb{Z}/4,\ \mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2.$$

If $s = 0$ in the presentation of G, then G only contains infinite virtually cyclic subgroups isomorphic to $\mathbb{Z}$ or D.

Proof. It is known that for an infinite virtually cyclic group Γ there is an exact sequence $1 \to \pi \to \Gamma \to Q \to 1$ for finite π such that Q is $\mathbb{Z}$ or D [19, Lemma 2.5]. First, suppose that $s = 0$. We have to show that π is trivial. Since D contains a normal subgroup Q' isomorphic to $\mathbb{Z}$, it suffices to treat the case $Q \cong \mathbb{Z}$, otherwise substitute Γ by $p^{-1}(Q')$. The subgroup $p^{-1}(|\pi|! \cdot \mathbb{Z})$ of Γ is isomorphic to $\pi \times \mathbb{Z}$ and hence contained in $C_G(g)$ for some element g of infinite order. Hence π is a subgroup of a torsion free group by Theorem 4.2 (d) and hence trivial.

Now assume that $s \neq 0$. If $Q = \mathbb{Z}$, i.e. $\Gamma = \pi \rtimes \mathbb{Z}$ with π finite, then Γ contains $\pi \times |\pi|! \cdot \mathbb{Z}$, so π is by Theorem 4.2 (d) a subgroup of a group whose finite subgroups all have order $\leqslant 2$ [43, Corollary on page 36]. Hence π is trivial or $\mathbb{Z}/2$. This means that Γ is either $\mathbb{Z}$ or $\mathbb{Z}/2 \times \mathbb{Z}$.

It remains to treat the case $Q = D$, i.e., where Γ surjects onto D with finite kernel π. Then π is either cyclic or dihedral by Theorem 4.2 (a). Suppose $\pi = (\mathbb{Z}/2)^2$. Then D acts on π, and Γ contains an element g of infinite order which centralizes π, i.e. $(\mathbb{Z}/2)^2 \times \langle g \rangle \leqslant C_G(g)$, which is a contradiction to Theorem 4.2 (d). If π is nonAbelian dihedral, then π contains a unique cyclic subgroup of order $\geqslant 3$ by Lemma 4.5 (d) on which D acts. Again, this implies that Γ contains an element of infinite order which is centralized by a cyclic group of order $\geqslant 3$, which contradicts Theorem 4.3 (d). By the same argument, we see that π cannot be cyclic of order $\geqslant 3$. If π is trivial, Γ is D; if π is $\mathbb{Z}/2$, Γ has to be one of the following groups: $\mathbb{Z}/2 \times D$, $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4$, or $\mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$ by [10, Theorem IV.3.12 on page 93]. □

LEMMA 4.7. *Let G be as in Theorem 4.4. There is a bijection of sets*

$$f\colon \coprod_{\alpha \in A, |M_\alpha| \equiv 0(4)} \{(K)_{M_\alpha} \mid K \subset M_\alpha,\ \ K \cap C_\alpha = 1, K \neq 1\} \xrightarrow{\cong} \coprod_{\beta \in B'} \{(U)_{W_G H_\beta} \mid |U| = 2\}.$$

Proof. Given an element $(K)_{M_\alpha}$ in $\{(K)_{M_\alpha} \mid K \subset M_\alpha, K \cap C_\alpha = 1, K \neq 1\}$ for $|M_\alpha| \equiv 0(4)$, we can choose $\beta \in B'$ with $(K)_G = (H_\beta)_G$ and $g \in G$ with $gKg^{-1} = H_\beta$, and M_α is dihedral because of Theorem 4.2 (a) and (c) Since $|M_\alpha| \equiv 0$ (4), we conclude $\beta \in B'$ from Theorem 4.2 (e) and Lemma 4.5 (d). There is a unique element u_α of order 2 in C_α, and u_α is central in M_α. We have $u_\alpha K u_\alpha^{-1} = K$ and hence $g u_\alpha g^{-1} \in N_G H_\beta$. Notice that $g u_\alpha g^{-1} \notin H_\beta$ because of Theorem 4.2

(c). If $\mathrm{pr}\colon N_G H_\beta \to W_G H_\beta$ is the canonical projection, put $U = \langle \mathrm{pr}(g u_\alpha g^{-1}) \rangle$ and define $f((K)_{M_\alpha}) = (U)_{W_G H_\beta}$. This is independent of the choice of g since any other choice is of the shape ug for some $u \in N_G H_\beta$.

Next we define the inverse f^{-1}. Given $\beta \in B'$ and $(U)_{W_G H_\beta}$ with $|U| = 2$, $\mathrm{pr}^{-1}(U))$ contains H_β and has order 4. Choose $\alpha \in A$ and $g \in G$ with $g\mathrm{pr}^{-1}(U)g^{-1} \subset M_\alpha$. Then define $f^{-1}((U)_{W_G H_\beta}) = (g H_\beta g^{-1})_{M_\alpha}$. This is independent of the choice of g, since any other choice is of the form ug for $u \in G$ such that $g\mathrm{pr}^{-1}(U)g^{-1} \subset M_\alpha \cap u^{-1} M_\alpha u$ and hence $u \in M_\alpha$ by Lemma 4.5 (b).

One easily checks that f^{-1} is indeed an inverse of f. □

Now we are ready to prove Theorem 4.4.

Proof. (a) Recall that $\{M_\alpha | \alpha \in A\}$ is a full system of representatives of the conjugacy classes of maximal finite subgroups of G which are not conjugate to some $\langle x_{i,j} \rangle$. Denote by $\mathcal{E}_\alpha$ the family of subgroups $K \leqslant M_\alpha$ with $(K)_G \leqslant (\langle x_{i,j} \rangle)_G$ and by $\mathcal{E}$ the family of finite subgroups of G with $(K)_G \leqslant (\langle x_{i,j} \rangle)_G$. Consider the following diagram of $\mathrm{Or}(G)$-spaces:

$$\begin{array}{ccc} \coprod_{\alpha \in A} (I_\alpha)_*(\star_{M_\alpha, \mathcal{E}_\alpha}) & \longrightarrow & \coprod_{\alpha \in A} (I_\alpha)_*(\star_{M_\alpha, \mathcal{ALL}}) \\ \downarrow & & \downarrow \\ \star_{G,\mathcal{E}} & \longrightarrow & \star_{G,\mathcal{FIN}} \end{array} \tag{4.2}$$

where the horizontal maps are given by inclusions and the vertical maps are the adjoints under the homeomorphism (2.1) of the obvious maps of $\mathrm{Or}(M_\alpha)$-spaces for the obvious functor $I_\alpha\colon \mathrm{Or}(M_\alpha) \to \mathrm{Or}(G)$. Its evaluation at an object G/H gives pushouts of spaces with a cofibration as upper horizontal map since we get for this evaluation from Lemma 2.1 (b) and Lemma 4.5 (c) for appropriate Y

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{Id}} & Y \\ \downarrow & & \downarrow \\ * & \xrightarrow{\mathrm{Id}} & * \end{array} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & * \\ {\scriptstyle\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle\mathrm{Id}} \\ \emptyset & \longrightarrow & *, \end{array} \qquad \begin{array}{ccc} \emptyset & \xrightarrow{\mathrm{Id}} & \emptyset \\ {\scriptstyle\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle\mathrm{Id}} \\ \emptyset & \xrightarrow{\mathrm{Id}} & \emptyset, \end{array}$$

if $H \in \mathcal{E}$, if $H \in \mathcal{FIN}$, $H \notin \mathcal{E}$, else.

Hence we obtain from Lemma 1.1, Lemma 2.1 (c) and (4.2) the isomorphism for $q \in \mathbb{Z}$

$$\bigoplus_{\alpha \in A} H_q^{M_\alpha}\left(\star_{M_\alpha,\mathcal{ALL}}, \star_{M_\alpha,\mathcal{E}_\alpha}; \mathbf{K}\right) \xrightarrow{\cong} H_q^G\left(\star_{G,\mathcal{FIN}}, \star_{G,\mathcal{E}}; \mathbf{K}\right) \tag{4.3}$$

and the long exact Mayer–Vietoris sequence

$$\cdots \to \bigoplus_{\alpha\in A} H_q^{M_\alpha}\left(\star_{M_\alpha,\mathcal{E}_\alpha};\mathbf{K}\right) \to H_q^G(\star_{G,\mathcal{E}};\mathbf{K}) \bigoplus_{\alpha\in A} K_q(C_r^*(M_\alpha)) \to H_q^G(\star_{G,\mathcal{FIN}};\mathbf{K}) \to \cdots. \tag{4.4}$$

If L is $\mathbb{Z}/2$, then the assembly map $H_q^L(\star_{L,\mathcal{TR}};\mathbf{K}) \to H_q^L(\star_{L,\mathcal{ALL}};\mathbf{K})$ is an isomorphism for $q \leqslant 2$ by Theorem 3.2 (div) and Lemma 2.4. Hence the following maps are isomorphisms by Lemma 1.1 and Theorem 2.3 for $q \leqslant 2$

$$H_q^{M_\alpha}\left(\star_{M_\alpha,\mathcal{ALL}},\star_{M_\alpha,\mathcal{TR}};\mathbf{K}\right) \xrightarrow{\cong} H_q^{M_\alpha}\left(\star_{M_\alpha,\mathcal{ALL}},\star_{M_\alpha,\mathcal{E}_\alpha};\mathbf{K}\right), \tag{4.5}$$

$$H_q^G\left(\star_{G,\mathcal{FIN}},\star_{G,\mathcal{TR}};\mathbf{K}\right) \xrightarrow{\cong} H_q^G\left(\star_{G,\mathcal{FIN}},\star_{G,\mathcal{E}};\mathbf{K}\right). \tag{4.6}$$

Combining (4.3), (4.4) and (4.5) yields an isomorphism for $q \leqslant 2$

$$\bigoplus_{\alpha\in A} H_q^{M_\alpha}\left(\star_{M_\alpha,\mathcal{ALL}},\star_{M_\alpha,\mathcal{TR}};\mathbf{K}\right) \xrightarrow{\cong} H_q^G\left(\star_{G,\mathcal{FIN}},\star_{G,\mathcal{TR}};\mathbf{K}\right). \tag{4.7}$$

By Lemma 2.5 and Lemma 4.6 the assumptions in Theorem 2.3 are satisfied. Thus Theorem 2.3 and Lemma 1.1 imply that the inclusion $\star_{G,\mathcal{FIN}},\star_{G,\mathcal{TR}} \to \star_{G,\mathcal{VC}},\star_{G,\mathcal{TR}}$ induces an isomorphism

$$H_q^G\left(\star_{G,\mathcal{FIN}},\star_{G,\mathcal{TR}};\mathbf{K}\right) \xrightarrow{\cong} H_q^G\left(\star_{G,\mathcal{VC}},\star_{G,\mathcal{TR}};\mathbf{K}\right). \tag{4.8}$$

Now the assertion (a) follows from the Isomorphism Conjectures (see Theorem 1.2), Lemma 2.4 and the isomorphisms (4.7) and (4.8). (b) is proven analogously to assertion (c), only easier because we invert 2. (c) We get from Theorem 4.2 (e)

$$W_G H_\beta \cong \begin{cases} \mathbb{Z} \rtimes \mathbb{Z}/2, & \text{if } \beta \in B', \\ \mathbb{Z}, & \text{if } \beta \in B'', \end{cases} \tag{4.9}$$

where $\mathbb{Z}/2$ acts on $\mathbb{Z}$ by $-\mathrm{Id}$. The induced action of $\mathbb{Z}/2$ on $K_q(S^1)$ is by Id if $q = 0$ and by $-\mathrm{Id}$ if $q = 1$. Hence we obtain the following isomorphisms from the Atiyah–Hirzebruch spectral sequence, applied to the obvious fibration $S^1 \to BW_GH_\beta \to B\mathbb{Z}/2$:

$$K_0(BW_GH_\beta) \cong K_0(S^1) \cong \mathbb{Z}, \tag{4.10}$$

$$K_1(BW_GH_\beta)[1/2] \cong K_0(S^1)[1/2] \cong \mathbb{Z}[1/2]. \tag{4.11}$$

We obtain a long exact sequence

$$\cdots \to K_{q+1}(S) \to H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}}) \to K_q(C_r^*(G)) \to K_q(S) \to \cdots \tag{4.12}$$

and its rational splitting into short split-exact sequences from Theorem 1.2 and Lemma 2.8 (c). Define a diagram of $\mathrm{Or}(G)$-spaces

$$\begin{array}{ccc} \coprod_{\beta\in B} I(H_\beta)_* \star_{N_G H_\beta, \mathcal{TR}} & \longrightarrow & \coprod_{\beta\in B} I(H_\beta)_* \star_{N_G H_\beta, \mathcal{SUB}(H_\beta)} \\ \downarrow & & \downarrow \\ \star_{G,\mathcal{TR}} & \longrightarrow & \star_{G,\mathcal{E}} \end{array}$$

where $I(H_\beta)\colon \mathrm{Or}(N_G H_\beta) \to \mathrm{Or}(G)$ is the functor induced by the inclusion, $\mathcal{SUB}(H_\beta)$ is the family of subgroups of $N_G H_\beta$ consisting of the subgroups of H_β, the horizontal maps are induced by inclusions and the vertical maps by the obvious maps of $\mathrm{Or}(N_G H_\beta)$-spaces and the adjunction (2.1). One easily checks using Lemma 2.1 (b) that evaluation of this diagram at any object in $\mathrm{Or}(G)$ is a push-out of spaces with a cofibration as upper horizontal map. Since $\widetilde{\mathbf{K}}(G/1)$ has trivial homotopy groups, we get from this diagram, Lemma 1.1 and Lemma 2.1 (c) an isomorphism

$$\bigoplus_{\beta\in B} H_q^{N_G H_\beta}(\star_{N_G H_\beta, \mathcal{SUB}(H_\beta)}; \widetilde{\mathbf{K}}) \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{E}}; \widetilde{\mathbf{K}}).$$

By Theorem 4.2 (e), H_β is a direct factor of $N_G H_\beta$. Hence we get from Lemma 2.6

$$H_q^{N_G H_\beta}(\star_{N_G H_\beta, \mathcal{SUB}(H_\beta)}; \widetilde{\mathbf{K}}) \xrightarrow{\cong} H_q(BW_G H_\beta; \widetilde{\mathbf{K}}(C_r^*(\mathbb{Z}/2))). \tag{4.13}$$

The ring homomorphism $\mathbb{C}[\mathbb{Z}/2] \to \mathbb{C}$ sending $a + bt$ to $a - b$ induces a map of spectra $\widetilde{\mathbf{K}}(C_r^*(\mathbb{Z}/2)) \to \mathbf{K}(\mathbb{C})$ which induces an isomorphism on homotopy groups. Hence it yields an isomorphism [15, Lemma 4.6]

$$H_q(BW_G H_\beta; \widetilde{\mathbf{K}}(C_r^*(\mathbb{Z}/2))) \xrightarrow{\cong} K_q(BW_G H_\beta). \tag{4.14}$$

Thus we obtain isomorphisms

$$\bigoplus_{\beta\in B} K_q(BW_G H_\beta) \cong H_q^G(\star_{G,\mathcal{E}}; \widetilde{\mathbf{K}}), \tag{4.15}$$

$$\bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} K_q(BW_{M_\alpha} K) \cong H_q^{M_\alpha}(\star_{M_\alpha, \mathcal{E}_\alpha}; \widetilde{\mathbf{K}}), \tag{4.16}$$

where the construction of the second isomorphism is analogous to the one for G, just replace G by M_α and $\mathcal{E}$ by $\mathcal{E}_\alpha$. The long exact Mayer–Vietoris sequence (4.4) becomes under the isomorphisms (4.15) and (4.16)

$$\begin{aligned}\cdots \to \bigoplus_{\alpha\in A}\ \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} K_q(BW_{M_\alpha}K)\\ \to \bigoplus_{\beta\in B} K_q(BW_G H_\beta)\oplus\bigoplus_{\alpha\in A}\widetilde{K}_q(C_r^*(M_\alpha))\\ \to H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}})\to\bigoplus_{\alpha\in A}\ \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}} K_{q-1}(BW_{M_\alpha K})\to\cdots.\end{aligned}\tag{4.17}$$

The composition of the homomorphism induced by the inclusions

$$\bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} \widetilde{K}_q(C_r^*(K))\to\widetilde{K}_q(C_r^*(M_\alpha))$$

with the homomorphism appearing in the definition of $S\widetilde{K}_q(\mathbb{Z}M_\alpha)$ is an isomorphism since it can be identified with $\mathrm{Id}\colon\ 0\to 0$, $\mathrm{Id}\colon\ \widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])\to\widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])$ or $\begin{pmatrix}1&0\\1&1\end{pmatrix}\colon\ \widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])\oplus\widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])\to\widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])\oplus\widetilde{K}_q(\mathbb{Z}[\mathbb{Z}/2])$, respectively. We obtain a commutative diagram with split exact columns and an isomorphism as lower horizontal map:

$$\begin{array}{ccc}0 & & 0\\ \downarrow & & \downarrow\\ \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}}\widetilde{K}_q(BW_{M_\alpha}K) & \longrightarrow & S\widetilde{K}_q(C_r^*(M_\alpha))\\ \downarrow & & \downarrow\\ \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}}K_q(BW_{M_\alpha}K) & \longrightarrow & \widetilde{K}_q(C_r^*(M_\alpha))\\ \downarrow & & \downarrow\\ \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}}K_q(*) & \xrightarrow{\cong} & \widetilde{K}_q(C_r^*(M_\alpha))/S\widetilde{K}_q(C_r^*(M_\alpha))\\ \downarrow & & \downarrow\\ 0 & & 0.\end{array}$$

Hence the sequnce (4.17) reduces to the long exact sequence

$$\begin{aligned}\cdots \to & \bigoplus_{\alpha\in A} \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} \widetilde{K}_q(BW_{M_\alpha}K) \\ \xrightarrow{\epsilon_q} & \bigoplus_{\beta\in B} K_q(BW_G H_\beta) \oplus \bigoplus_{\alpha} S\widetilde{K}_q(C_r^*(M_\alpha)) \\ \to & H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}}) \to \bigoplus_{\alpha\in A} \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} \widetilde{K}_{q-1}(BW_{M_\alpha}K) \xrightarrow{\epsilon_{q-1}} \cdots .\end{aligned} \tag{4.18}$$

Since $W_{M_\alpha}K$ is trivial or $\mathbb{Z}/2$ for $K \in \mathcal{E}\setminus\{1\}$ by Theorem 4.2 (e), $\widetilde{K}_q(BW_{M_\alpha}K)$ $[1/2]$ is trivial for all $q \in \mathbb{Z}$ by the Atiyah–Hirzebruch spectral sequence. From (4.12) and (4.18), we therefore get the long exact sequence

$$\begin{aligned}K_{q+1}(S)[1/2] \xrightarrow{\delta_{q+1}} & \bigoplus_{\beta\in B} K_q(BW_G H_\beta)[1/2] \oplus \bigoplus_{\alpha\in A} S\widetilde{K}_q(C_r^*(M_\alpha))[1/2] \\ & \longrightarrow K_q(C_r^*(G))[1/2] \to K_q(S)[1/2] \xrightarrow{\delta_q} \cdots ,\end{aligned} \tag{4.19}$$

which splits into short split-exact sequences rationally. Since the boundary map δ_{q+1} in (4.19) is rationally zero and its target does not contain torsion (see (4.10) and (4.11)), δ_{q+1} itself is zero. Hence the long exact sequence (4.19) reduces to short exact sequences

$$\begin{aligned}0 \to & \bigoplus_{\beta\in B} K_q(BW_G H_\beta)[1/2] \oplus \bigoplus_{\alpha\in A} S\widetilde{K}_q(C_r^*(M_\alpha)[1/2] \\ \to & K_q(C_r^*(G))[1/2] \to K_q(S)[1/2] \to 0.\end{aligned}$$

In the case of L-theory, we are done at this point because $H_q(S;\mathbf{L}(\mathbb{Z}))[1/2]$ is free. It remains to show the claim for topological K-theory without inverting 2.

To do this, we further investigate the homomorphism ϵ_q appearing in the long exact sequence (4.18). For $\beta \in B''$, $K_q(BW_G H) = K_q(S^1)$ is torsion free, and so is $S\widetilde{K}_q(C_r^*(M_\alpha))$. Since $\widetilde{K}_q(BW_{M_\alpha}H)$ is 2-torsion, it suffices to determine the kernel and cokernel of the part denoted in the same way:

$$\epsilon_q\colon \bigoplus_{\alpha\in A} \bigoplus_{(K)_{M_\alpha},\ K\in\mathcal{E}\setminus\{1\}} \widetilde{K}_q(BW_{M_\alpha}K) \to \bigoplus_{\beta\in B'} K_q(BW_G H_\beta). \tag{4.20}$$

Fix $\alpha \in A$, $(K)_{M_\alpha}$, $K \in \mathcal{E}\setminus\{1\}$ and $\beta \in B'$. The part of ϵ_q going from the summand in the source belonging to $(K)_{M_\alpha}$ to the summand in the target belonging to β is denoted by

$$\epsilon_q(\alpha,(K),\beta)\colon \widetilde{K}_q(BW_{M_\alpha}K) \to K_q(BW_G H_\beta). \tag{4.21}$$

If $(H_\beta)_G \neq (K)_G$, then $\epsilon_q(\alpha, (K), \beta)$ is trivial. If $H_\beta = K$, then $\epsilon_q(\alpha, (K), \beta)$ is induced by the obvious inclusion $W_{M_\alpha} K \to W_G K$.

Using the bijection f in Lemma 4.7 and the facts that for $K \subset M_\alpha$, $K \neq 1$, $|M_\alpha| \equiv 0\ (4)$ the condition $K \cap C_\alpha = 1$ is equivalent to $(K)_G = (x_{i,j})_G$ for appropriate i, j and that $W_{M_\alpha} K = 1$ and hence $\widetilde{K}_q(BW_{M_\alpha} K) = 0$ if $|M_\alpha| \equiv 2(4)$, one easily constructs a commutative diagram with isomorphisms as vertical maps

$$\begin{array}{ccc}
\bigoplus_{\alpha \in A} \bigoplus_{(K)_{M_\alpha},\ K \in \mathcal{E}\setminus\{1\}} \widetilde{K}_q(BW_{M_\alpha} K) & \xrightarrow{\epsilon_q} & \bigoplus_{\beta \in B'} K_q(BW_G H_\beta) \\
\Big\downarrow \cong & & \Big\downarrow \mathrm{Id} \\
\bigoplus_{\beta \in B'} \bigoplus_{(U)_{W_G H_\beta}, |U|=2} \widetilde{K}_q(BU) & \xrightarrow{\bigoplus_{\beta \in B'} \mu(H_\beta)_q} & \bigoplus_{\beta \in B'} K_q(BW_G H_\beta)
\end{array}$$

where $\mu(H_\beta)_q\colon \bigoplus_{(U)_{W_G H_\beta}, |U|=2} \widetilde{K}_q(BU) \to K_q(BW_G H_\beta)$ is induced by the various inclusions $U \to W_G H_\beta$. Since we get $\widetilde{K}_0(B\mathbb{Z}/2) = 0$ and $\widetilde{K}_1(B\mathbb{Z}/2) = \mathbb{Z}[1/2]/\mathbb{Z} = \mathbb{Z}/2^\infty$ from the Atiyah–Hirzebruch spectral sequence and [25, Proposition 2.11], the long exact sequence (4.25) becomes under this identification

$$\begin{aligned}
0 \to & \bigoplus_{\beta \in B} K_0(BW_G H_\beta) \oplus \bigoplus_{\alpha \in A} S\widetilde{K}_0(C_r^*(M_\alpha)) \to H_0^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}) \\
\to & \bigoplus_{\beta \in B'} \bigoplus_{(U)_{W_G H_\beta}, |U|=2} \widetilde{K}_1(BU) \xrightarrow{\bigoplus_{\beta \in B'} \mu(H_\beta)_1 \oplus 0} \bigoplus_{\beta \in B'} K_1(BW_G H_\beta) \oplus \times \\
\times & \bigoplus_{\beta \in B''} K_1(BW_G H_\beta) \to H_1^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}) \to 0.
\end{aligned} \tag{4.22}$$

Now the map $\widetilde{K}_1(B\mathbb{Z}/2)^2 \to K_1(B(\mathbb{Z}/2 * \mathbb{Z}/2)$ induced by the inclusion of two nonconjugated $\mathbb{Z}/2$ is an isomorphism by the Mayer–Vietoris sequence, so the exact sequence (4.22) reduces to the isomorphisms

$$\bigoplus_{\beta \in B} K_0(BW_G H_\beta) \oplus \bigoplus_{\alpha \in A} S\widetilde{K}_0(C_r^*(M_\alpha)) \xrightarrow{\cong} H_0^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}),$$

$$\bigoplus_{\beta \in B''} K_1(S^1) \xrightarrow{\cong} H_1^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}).$$

This shows that the Abelian groups $H_q^G(\star_{G,\mathcal{FIN}};\widetilde{\mathbf{K}})$ are free, so from the rationally split exact sequence (4.18), we get exact sequences

$$0 \to \bigoplus_{\beta\in B''} K_1(S^1) \to K_1(C_r^*(G)) \to K_1(S) \to 0,$$
$$0 \to \bigoplus_{\beta\in B} K_0(BW_G H_\beta) \oplus \bigoplus_{\alpha\in A} S\widetilde{K}_0(C_r^*(M_\alpha)) \to K_0(C_r^*(G)) \to K_0(S) \to 0.$$

It remains to show that the exact sequences above split. The group $K_0(S)$ is always free Abelian. If $s = 0$ and $h \neq g$, then the quotient space is a closed non-orientable surface, so $K_1(S)$ is not free but contains a direct summand $\mathbb{Z}/2$. On the other hand, in this case B and hence B'' is empty, so $K_1(C_r^*(G)) \cong K_1(S)$. If $h = g$ or $s > 0$, then $K_1(S))$ is free, so the claim follows. □

Next we state the K-groups and L-theory explicitly in terms of the signature.

DEFINITION 4.8. Let G be an infinite cocompact planar group with signature

$$(g, \pm, [\gamma_1, \gamma_2, \ldots \gamma_t], \{(n_{1,1}, n_{1,2}, \ldots n_{1,r_1-1}), \ldots (n_{s,1}, n_{s,2}, \ldots n_{s,r_s-1})\}).$$

Define b' to be the number of elements $n_{i,j}$ which are even and b'' to be the number of integers $i \in \{1, 2, \ldots s\}$ for which $r_i = 1$ or each $n_{i,j}$ for $j = 1, 2, \ldots, r_i - 1$ is odd. (With this definition, $b' = b'' = 0$ if $s = 0$).

THEOREM 4.9. *Let G, b' and b'' be given as in Definition 4.8. Recall that for a real number r we denote by $[r]$ the largest integer which is less than or equal to r. Then*

$$L_0(\mathbb{Z}G)[1/2] = \left(b' + b'' + 1 + \sum_{k=1}^{t}[\gamma_k/2] + \sum_{i=1}^{s}\sum_{j=1}^{r_i-1}[n_{i,j}/2]\right) \cdot \mathbb{Z}[1/2],$$
$$L_1(\mathbb{Z}G)[1/2] = \begin{cases} (b'' + 2g) \cdot \mathbb{Z}[1/2], & \textit{for } +, s = 0,\\ (b'' + 2g + s - 1) \cdot \mathbb{Z}[1/2], & \textit{for } +, s > 0,\\ (b'' + g + s - 1) \cdot \mathbb{Z}[1/2], & \textit{for } -, \end{cases}$$
$$L_2(\mathbb{Z}G)[1/2] = \begin{cases} \left(1 + \sum_{k=1}^{t}[(\gamma_k - 1)/2]\right) \cdot \mathbb{Z}[1/2], & \textit{for } +, s = 0,\\ \left(\sum_{k=1}^{t}[(\gamma_k - 1)/2]\right) \cdot \mathbb{Z}[1/2], & \textit{otherwise}, \end{cases}$$
$$L_3(\mathbb{Z}G)[1/2] = 0,$$

$$K_0(C_r^*(G)) = \begin{cases} \left(b' + b'' + 2 + \sum_{k=1}^{t}(\gamma_k - 1) + \sum_{i=1}^{s}\sum_{j=1}^{r_s-1}[n_{i,j}/2]\right) \cdot \mathbb{Z}, & \textit{for } +, s = 0,\\ \left(b' + b'' + 1 + \sum_{k=1}^{t}(\gamma_k - 1) + \sum_{i=1}^{s}\sum_{j=1}^{r_s-1}[n_{i,j}/2]\right) \cdot \mathbb{Z}, & \textit{otherwise}, \end{cases}$$

$$K_1(C_r^*(G)) = \begin{cases} (b'' + 2g) \cdot \mathbb{Z} & \textit{for } +, \quad s = 0, \\ (b'' + 2g + s - 1) \cdot \mathbb{Z} & \textit{for} +, \quad s > 0, \\ \mathbb{Z}/2 \oplus (b'' + g - 1) \cdot \mathbb{Z} & \textit{for } -, \quad s = 0, \\ (b'' + g + s - 1) \cdot \mathbb{Z} & \textit{for } -, \quad s > 0. \end{cases}$$

Proof. This follows from Theorem 4.4 and the computation of $K_q(S)$ and $H_p(S; \mathbf{L}(\mathbb{Z}))[1/2]$ stated above as soon as we know that b' and b'', respectively, is the cardinality of B' and B'', respectively. (The sets B' and B'' have been introduced in Definition 4.3). This follows for b' from Theorem 4.2 (e) and Lemma 4.7 since the sets $\{(K)_{M_\alpha} \mid K \cap C_\alpha = 1\}$ for $\alpha \in A$ with $|M_\alpha| \equiv 0$ (4) and the sets $\{(U)_{W_G H_\beta} \mid |U| = 2\}$ for $\beta \in B'$ appearing in Lemma 4.7 all have cardinality 2.

Notice that $x_{i,j}$ is conjugated to $x_{i,j+1}$ if $n_{i,j}$ is odd by Lemma 4.5 (d), $x_{i,1}$ is always conjugated to x_{i,r_i} and $N_G\langle x_{i,j}\rangle$ cannot be isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}$ for even $n_{i,j}$ by Lemma 4.5 (d). This shows for any $\beta \in B''$ that there is an index $i \in \{1, 2, \ldots s\}$ such that $\langle x_{i,j}\rangle$ is conjugated to H_β for each $j \in \{1, 2, \ldots, r_i\}$. Hence it remains to show that for two elements $x_{i,j}$ and $x_{i',j'}$ which are conjugated, $i = i'$ holds. This follows from the geometric descriptions in terms of fundamental polygons (see e.g. [28, 48, 52]). Namely, a conjugating element $g \in G$ maps the fixed point set of the reflections $x_{i,j}$ and $x_{i',j'}$ to one another. Hence the images of their fixed point set under the quotient map onto S agree. This shows that $x_{i,j}$ and $x_{i',j'}$ belong to the same boundary component of S and thus $i = i'$. □

Remark 4.10. A cocompact Fuchsian group is a planar groups acting isometrically on the hyperbolic plane with $s = 0$, $h = g$, i.e. the quotient space is a closed orientable surface of genus g. Hence Theorem 4.4 implies Theorem 0.1 mentioned in the introduction.

We want to briefly sketch the computation of $L_n^\epsilon(\mathbb{Z}G)$ without inverting 2 for a Fuchsian group F as in Theorem 0.1. For this purpose we will need for L^ϵ for $\epsilon = -\infty, p, h$ or s that the isomorphism conjecture is true for G with respect to the family of virtually cyclic subgroups. Notice that Farrell and Jones [18] formulate their Isomorphism Conjecture only for $L^{-\infty}$ and that they have shown that it cannot be true simultaneously for $\epsilon = h$ and $\epsilon = s$ in the case $G = \mathbb{Z}^2 \times \mathbb{Z}/5$ [20]. However, using the various Rothenberg sequences together with the explicit computations of the lower and middle K-theory and the five lemma one can show in this particular case that the isomorphism conjecture with respect to $\mathcal{VC}$ and without inverting 2 is true for all of the decorated L-groups $L^{-\infty}$, L^p, L^h, L^s if it holds for one of them. Since it is known for $L^{-\infty}$ for F [18, Theorem 2.1 and Remark 2.1.3], the computations below are true without any assumptions.

Let V be any subgroup of F isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Its commutator $[V, V]$ is an infinite cyclic subgroup which is characteristic in V. The centralizer $C_G[V, V]$ is again an infinite cyclic subgroup by Theorem 4.2 (d). This implies that $V_{\max} := N_G[V, V]$ contains V and is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Since $[V, V]$ is a characteristic subgroup of V, we conclude $N_G V \subset V_{\max}$. Let $V \subset W \subset G$ be subgroups such that both V and W are isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Then $[V, V] \subset [W, W] \subset C_G[W, W] \subset C_G[V, V]$. Since $C_G[V, V]$ is Abelian, we get $C_G[V, V] \subset C_G[W, W]$. This implies $C_G[V, V] = C_G[W, W]$ and hence $V_{\max} = W_{\max}$.

Let $\{V_\delta \mid \delta \in D\}$ be a full system of representatives of the conjugacy classes of subgroups $V \subset G$ which are maximal among the subgroups of G isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Hence each subgroup $V \subset G$ with $V \cong \mathbb{Z}/2 * \mathbb{Z}/2$ is subconjugated to precisely one V_δ, namely the one with $(V_\delta) = (V_{\max})$, and we have $N_G V = N_{V_{\max}} V$.

Using these facts one easily verifies that the following diagram is a pushout of contravariant $\mathrm{Or}(G)$-spaces with a cofibration as upper horizontal map:

$$\begin{array}{ccc} \coprod_{\delta\in D} (I_\delta)_*(\star_{V_\delta,\mathcal{FIN}}) & \longrightarrow & \coprod_{\delta\in D} (I_\delta)_*(\star_{V_\delta,\mathcal{ALL}}) \\ \downarrow & & \downarrow \\ \star_{G,\mathcal{FIN}} & \longrightarrow & \star_{G,\mathcal{VC}}. \end{array}$$

Hence this diagram yields a long exact sequence

$$\cdots \to \bigoplus_{\delta\in D} UNil_{q+1} \to H_q^F(\star_{F,\mathcal{FIN}}; \mathbf{L}^\epsilon) \to L_q^\epsilon(\mathbb{Z}F) \to \bigoplus_{\delta\in D} UNil_q \to \cdots,$$

where $UNil_q \cong H_q^G(\star_{V_\delta,\mathcal{ALL}}, \star_{V_\delta,\mathcal{FIN}}; \mathbf{L})$ is the $UNil$-term appearing in the splitting

$$\widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/2]) \oplus \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/2]) \oplus UNil_q \to \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/2 * \mathbb{Z}/2])$$

due to [12, Theorem 10] and

$$\begin{aligned} \cdots \;\to\; & H_{q+1}(S; \mathbf{L}(\mathbb{Z})) \to \bigoplus_{i=1}^{t} \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/\gamma_i]) \to H_q^F(\star_{F,\mathcal{FIN}}; \mathbf{L}^\epsilon) \\ \to\; & H_q(S; \mathbf{L}(\mathbb{Z})) \to \cdots, \end{aligned}$$

where the last sequence splits after inverting the least common multiple m of the γ_i's. The first exact sequence splits, too; a splitting is given by the obvious map

$$\bigoplus_{\delta\in D} UNil_q \to \bigoplus_{\delta\in D} L_q^\epsilon(\mathbb{Z}V_\delta) \to L_q^\epsilon(\mathbb{Z}F).$$

Hence we obtain an exact sequence which splits after inverting m

$$\begin{aligned} \cdots \;\to\; & H_{q+1}(S; \mathbf{L}(\mathbb{Z})) \to \bigoplus_{i=1}^{t} \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/\gamma_i]) \oplus \bigoplus_{\delta\in D} UNil_q \\ \to\; & L_q^\epsilon(\mathbb{Z}F) \to H_q(S; \mathbf{L}(\mathbb{Z})) \to \cdots. \end{aligned}$$

From now on suppose that each γ_i is odd. Then the number m above is odd. Since $H_q(S;\mathbf{L}(\mathbb{Z}))$ and $\bigoplus_{i=1}^t \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/\gamma_i])$ (see Theorem 3.2 (c)) contain no odd torsion, we obtain isomorphisms for $q \in \mathbb{Z}$

$$\bigoplus_{i=1}^{t} \widetilde{L}_q^\epsilon(\mathbb{Z}[\mathbb{Z}/\gamma_i]) \oplus H_q(S;\mathbf{L}(\mathbb{Z})) \cong L_q^\epsilon(\mathbb{Z}F).$$

Explicitly, we get from the computations of $\widetilde{L}_q^\epsilon(\mathbb{Z}/\gamma_i)$ in [1, Theorem 1, 3 and 5] and [2, Corollary 4.3 on page 58] for $\epsilon = p$ and s

$$L_q^\epsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \dfrac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 0(4), \\ (2g)\cdot\mathbb{Z} & q \equiv 1(4), \\ \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \dfrac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 2\ (4), \\ (2g)\cdot\mathbb{Z}/2 & q \equiv 3(4). \end{cases}$$

For $\epsilon = h$, there is no general formula known for the 2-torsion contained in $\widetilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$, m odd, since it is given by the term $\widehat{H}^2(\mathbb{Z}/2;\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [1, Theorem 2].

5. Two-Dimensional Crystallographic Groups

In this section we give a complete description of the algebraic K- and L-groups of the integral group ring and the topological K-theory of the reduced C^*-algebra of all two-dimensional crystallographic groups. The algebraic K-theory in dimension $\leqslant 1$ has been determined in [32], and the C^*-K-theory has been computed in [51]. Note the difference between the group $K_0(C_r^*(Cmm))$ given here and in [51, page 102]. We believe that this difference comes from a fixed point which has been overlooked in [51]. If we use the methods of [51], we get $\mathbb{Z}^6$, as in the following table. As far as we know, this is the first computation of the L-groups of two-dimensional crystallographic groups.

A two-dimensional crystallographic group is the same as a cocompact planar group G acting on $\mathbb{R}^2$. The signatures of crystallographic groups have been listed in [28, page 1204]. Hence the results for the L-theory and the topological K-theory below follow directly from Theorem 4.9. In the computation of the L-theory of the groups $P3$ and Pg we do not have to invert 2 because these two groups only contain infinite virtually cyclic groups which are isomorphic to $\mathbb{Z}$. Hence the isomorphism conjecture for the family of finite subgroups without inverting 2 is true in this case, and a careful analysis as in the proof of Theorem 4.5 (c) shows our results. In the case of $P3$, note that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3]) = 0$ by Theorem 3.2 (diii), so there is no 2-torsion coming from the maximal finite subgroups for any decoration ϵ.

The computations of $\mathrm{Wh}_q(\mathbb{Z}G)$ for $q \leqslant 1$ follow directly from Theorem 3.2 (d), Theorem 4.2 (a) and Theorem 4.4 (a). For $q = 2$ we assume that the the isomorphism conjecture in algebraic K-theory (see (1.8) and Theorem 1.2) holds also in Dimension 2. In some cases we can drop this assumption for the computation of $\mathrm{Wh}_2(\mathbb{Z}G)$. Since the assembly map is surjective in dimension 2, we can at least conclude $\mathrm{Wh}_2(G) = 0$ if the second Whitehead groups of all finite subgroups vanish. Furthermore, in some cases (like $P6$ or $P31m$), there is only one finite subgroup with (possibly) non-trivial Wh_2 which splits off from G, allowing us to compute $\mathrm{Wh}_2(G)$.

Our notation for the two-dimensional crystallographic groups follows that of [28].

Group	Signature	$\mathrm{Wh}_q \neq 0, q \leqslant 2$	$L_q(\mathbb{Z}G)$	$K_q(C_r^*(G))$
$P1$	(1, +, [ ], { })		$L_0 = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_0 = \mathbb{Z}^2$
			$L_1 = \mathbb{Z}^2$	$K_1 = \mathbb{Z}^2$
			$L_2 = \mathbb{Z} \oplus \mathbb{Z}/2$	
			$L_3 = (\mathbb{Z}/2)^2$	
$P2$	(0, +, [2, 2, 2, 2], { })		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$	$K_0 = \mathbb{Z}^6$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P3$	(0, +, [3, 3, 3], { })		$L_0 = \mathbb{Z}^4 \oplus \mathbb{Z}/2$	$K_0 = \mathbb{Z}^8$
			$L_1 = 0$	$K_1 = 0$
			$L_2 = \mathbb{Z}^4 \oplus \mathbb{Z}/2$	
			$L_3 = 0$	
$P4$	(0, +, [2, 4, 4], { })		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$	$K_0 = \mathbb{Z}^9$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P6$	(0, +, [2, 3, 6], { })	$K_{-1} = \mathbb{Z}$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$	$K_0 = \mathbb{Z}^{10}$
		$\mathrm{Wh}_2 = \mathrm{Wh}_2(\mathbb{Z}/6)$	$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$	
			$L_3\left[\frac{1}{2}\right] = 0$	

Group	Signature	$\mathrm{Wh}_q \neq 0, q \leqslant 2$	$L_q(\mathbb{Z}G)$	$K_q(C_r^*(G))$
Cm	$(1, -, [\,], \{(\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^2$	$K_0 = \mathbb{Z}^2$
			$L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^2$	$K_1 = \mathbb{Z}^2$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
Pm	$(0, +, [\,], \{(\,), (\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$	$K_0 = \mathbb{Z}^3$
			$L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$	$K_1 = \mathbb{Z}^3$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
Pg	$(2, -, [\,], \{\,\})$		$L_0 = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_0 = \mathbb{Z}$
			$L_1 = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_1 = \mathbb{Z} \oplus \mathbb{Z}/2$
			$L_2 = \mathbb{Z}/2$	
			$L_3 = (\mathbb{Z}/2)^2$	
Cmm	$(0, +, [2], \{(2, 2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)^2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^6$	$K_0 = \mathbb{Z}^6$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
Pmm	$(0, +, [\,], \{(2, 2, 2, 2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)^4$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^9$	$K_0 = \mathbb{Z}^9$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
Pmg	$(0, +, [2, 2], \{(\,)\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$	$K_0 = \mathbb{Z}^4$
			$L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$	$K_1 = \mathbb{Z}$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
Pgg	$(1, -, [2, 2], \{\,\})$		$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^3$	$K_0 = \mathbb{Z}^3$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = \mathbb{Z}/2$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	

Group	Signature	$\mathrm{Wh}_q \neq 0, q \leqslant 2$	$L_q(\mathbb{Z}G)$	$K_q(C_r^*(G))$
$P3m1$	$(0, +, [3], \{(3)\})$	$\mathrm{Wh}_2 = \mathbb{Z}/2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^4$	$K_0 = \mathbb{Z}^5$
			$L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$	$K_1 = \mathbb{Z}$
			$L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P31m$	$(0, +, [\ ], \{(3, 3, 3)\})$	$\mathrm{Wh}_2 = (\mathbb{Z}/2)^3$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$	$K_0 = \mathbb{Z}^5$
			$L_1\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]$	$K_1 = \mathbb{Z}$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P6m$	$(0, +, [\ ], \{(2, 3, 6)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4) \oplus \mathbb{Z}/2$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^8$	$K_0 = \mathbb{Z}^8$
		$\oplus\, \mathrm{Wh}_2(D_{12})$	$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
		$K_{-1} = \mathbb{Z}$	$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P4m$	$(1, +, [\ ], \{(2, 4, 4)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^9$	$K_0 = \mathbb{Z}^9$
		$\oplus\, \mathrm{Wh}_2(D_8)^2$	$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = 0$	
			$L_3\left[\frac{1}{2}\right] = 0$	
$P4g$	$(0, +, [4], \{(2)\})$	$\mathrm{Wh}_2 = \mathrm{Wh}_2(D_4)$	$L_0\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^5$	$K_0 = \mathbb{Z}^6$
			$L_1\left[\frac{1}{2}\right] = 0$	$K_1 = 0$
			$L_2\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{2}\right]^1$	
			$L_3\left[\frac{1}{2}\right] = 0$	

6. Extensions of Finite Groups with a Free Abelian Group

In this section we give the proof of Theorem 0.2. Some preparations are needed.

LEMMA 6.1. *Let G be as in Theorem 0.2. Then*

(a) *every finite subgroup of G is isomorphic to a subgroup of π. For a non-trivial finite subgroup $H \subset G$ its normalizer $N_G H := \{g \in G \mid gHg^{-1} = H\}$ is finite;*

(b) *every infinite virtually cyclic subgroup of G is either cyclic and contained in $\mathbb{Z}^n$ or isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$.*

Proof. (a) Let $H \subset G$ be finite. Then $p|_H$ has trivial kernel, since it is a finite subgroup of $\mathbb{Z}^n$. Hence H embeds into π.

Consider $g \in N_G H$. The map c_g on H given by conjugation with g is an automorphism of the finite group H and therefore of finite order itself, so there is a natural number k with $(c_g)^k = c_{g^k} = \mathrm{Id}$. Thus we have $g^k h g^{-k} = h$ for all $h \in H$. Hence we get $g^{km} \in \mathbb{Z}^n$ and $hg^{km}h^{-1} = g^{km}$ for all $h \in H$ and $m = |\pi|$. Since H is non-trivial and π acts freely on $\mathbb{Z}^n$, we get $g^{km} = 1$. Thus, every element of $N_G H$ has finite order and cannot lie in the kernel of p. Hence $N_G H$ is isomorphic to a subgroup of π, which is finite.

(b) Let $C \subset G$ be infinite virtually cyclic. There is an extension $1 \to \mathbb{Z} \to C \xrightarrow{q} F \to 1$ with a finite group F. Restricting p to C, we get an extension $1 \to Z \to C \to p(C) \to 1$ where Z is a nontrivial subgroup of $\mathbb{Z}^n$. Its image under q is a subgroup of the finite group F and hence $Z \cap \ker(q)$ has finite index in Z. Therefore, Z must be infinite cyclic. Denote a generator of Z by z. As $\mathrm{aut}(\mathbb{Z}) \cong \mathbb{Z}/2$ and the action of $p(C)$ on Z is free, we must have an injection $p(C) \to \mathbb{Z}/2$. If $p(C) = 1$, $C \cong Z$. If $p(C) = \mathbb{Z}/2$, choose $t \in C$ whose image generates $p(C)$. Then we must have $t^2 = z^k$ for some $k \in \mathbb{Z}$ since $p(t)^2 = 1$. Thus,

$$z^{2k} = t^4 = tz^k t = tt(t^{-1}z^k t) = z^k z^{-k} = 1 \ \Rightarrow k = 0,$$

and C is a nontrivial semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 * \mathbb{Z}/2$. □

LEMMA 6.2. *Let π be a finite group acting linearly on $\mathbb{R}^n$ such that the action is free outside the origin. If $|\pi|$ is odd, there exists to every prime divisor p of $|\pi|$ a unique subgroup of π of order p. If $|\pi|$ is even, there is a unique (and therefore central) element of order 2 in π.*

Proof. If $|\pi|$ is even, π contains exactly one non-trivial element of order two [29, Remark on page 624]. Hence it remains to treat the case where $|\pi|$ is odd.

By [49, Theorems 5.3.1, 5.3.2 and 5.4.1], π is generated by two elements A, B such that for m and n the order of A and B, respectively, one has $(m, n) = 1$, $BAB^{-1} = A^r$ with $r^n \equiv 1\ (m)$, and if d is the multiplicative order of r in $\mathbb{Z}/m$, then for every prime p dividing d one has $p|n/d$ and $p^2|n$. In particular there is a split extension $1 \to \langle A\rangle \to \pi \to \langle B\rangle \to 1$. Any element $x \in \pi$ can be written uniquely as $A^a B^b$ for $a \in \mathbb{Z}/m$ and $b \in \mathbb{Z}/n$.

Suppose that $x^p = 1$. Since x^p can be written as $x = A^c B^{pb}$ for some integer $c \in \mathbb{Z}/m$, we conclude $pb = 0(n)$. Suppose that p does not divide n. Then this implies $B^b = 1$ and hence $x = A^a$ with $ap = 0(m)$. Hence x lies in the subgroup $\langle A^{m/p}\rangle$. Therefore, $\langle A^{m/p}\rangle$ is the unique subgroup of π of order p if p does not divide n. Suppose that p divides n. Then p does not divide m and $pb = 0(n)$, and hence we can write $b = kn/p$. We have $p|n/d$, and so

$$B^{n/p}AB^{-(n/p)} = A^{r^{(n/p)}} = A^{r^{d(n/pd)}} = A.$$

Therefore, $x^p = A^{ap}B^{pb} = 1$ which implies $x \in \langle B^{n/p}\rangle$. Hence $\langle B^{n/p}\rangle$ is the unique subgroup of π of order p if p divides n. □

LEMMA 6.3. *Let G be as in Theorem 0.2. Then*

(a) *the intersection of any two different maximal finite subgroups of G is trivial;*

(b) *for $H \subset G$ a non-trivial finite subgroup there is precisely one G-conjugacy class of subgroups (M) with $M \subset G$ maximal finite such that $(H) \leqslant (M)$;*

(c) *let $M \subset G$ be a maximal finite subgroup and $K, L \subset M$. Then $N_M K = N_G K$ and $(K)_M = (L)_M \Leftrightarrow (K)_G = (L)_G$.*

Proof. We give only the proof of assertion (a), the other assertions are direct consequences using the conclusion from Lemma 6.1 (a) that $N_G M = M$ holds for a maximal finite subgroup $M \subset G$.

Let $M, N \subset G$ be maximal finite groups with non-trivial intersection $H := M \cap N$. If $|M|$ and $|N|$ are odd, H contains a unique (normal) subgroup U of order p for any p dividing $|H|$ by Lemma 6.2. Since M and N contain exactly one subgroup of order p, they both normalize U, so we must have $N_G U = M = N$, since M and N are maximal and $N_G U$ is finite by Lemma 6.1 (a).

If $|M|$ and $|N|$ are even, M and N contain exactly one element of order 2, say t_M and t_N, respectively. These are central in M resp. N, so they are both contained in the finite group $N_G H$. Since $N_G H$ is finite and acts freely on $\mathbb{Z}^n$ as well by Lemma 6.1 (a), it contains a unique element of order 2, so $t_M = t_N =: t$. Again, we have $N_G\langle t\rangle = M = N$ because of the maximality of M and N and the finiteness of $N_G\langle t\rangle$.

Suppose $|M|$ is even, $|N|$ is odd. Then H contains a subgroup U of prime order with $N_G U = N$. On the other hand, $H \subset M$ is centralized by the unique element of order 2 in M, so $t \in N_G U = N$ which is a contradiction. □

Now we can give the proof of Theorem 0.2.

Proof. (a) Consider the commutative diagram of contravariant $\mathrm{Or}(G)$-spaces

$$\begin{array}{ccc} \coprod_{\alpha\in A} (I_\alpha)_*(\star_{M_\alpha,\mathcal{TR}}) & \longrightarrow & \coprod_{\alpha\in A} (I_\alpha)_*(\star_{M_\alpha,\mathcal{ALL}}) \\ \downarrow & & \downarrow \\ \star_{G,\mathcal{TR}} & \longrightarrow & \star_{G,\mathcal{FIN}} \end{array} \tag{6.1}$$

where the horizontal maps are given by inclusions and the vertical maps are the adjoints under the adjunction homeomorphism (2.1) of the inclusions of $\mathrm{Or}(M_\alpha)$-spaces for the obvious functor $I_\alpha\colon \mathrm{Or}(M_\alpha) \to \mathrm{Or}(G)$. The evaluation of the diagram of $\mathrm{Or}(G)$-spaces above at an object G/H gives pushouts of spaces with a cofibration as upper horizontal map. We get for this evaluation from Lemma 2.2 (b) and Lemma 6.3 (b) and (c)

$$
\begin{array}{ccc}
\coprod_{\alpha\in A} * \times_{M_\alpha} G & \xrightarrow{\mathrm{Id}} & \coprod_{\alpha\in A} * \times_{M_\alpha} G \\
\downarrow & & \downarrow \\
* & \xrightarrow{\mathrm{Id}} & *
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \longrightarrow & * \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\emptyset & \longrightarrow & *
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \xrightarrow{\mathrm{Id}} & \emptyset \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\emptyset & \xrightarrow{\mathrm{Id}} & \emptyset
\end{array}
$$

$$H = 1, \qquad H \in \mathcal{FIN}, H \neq 1, \qquad \text{else.}$$

Lemma 1.1, Theorem 1.2, Lemma 2.1 (c), Theorem 2.3, Lemma 2.5 and Lemma 6.1 (b) show that each map in the following composition is an isomorphism

$$
\begin{aligned}
&\bigoplus_{\alpha\in A} H_q^G(\star_{M_\alpha,\mathcal{ALL}}, \star_{M_\alpha,\mathcal{TR}}; \mathbf{K}) \\
&\quad \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{FIN}}, \star_{G,\mathcal{TR}}; \mathbf{K}) \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{VC}}, \star_{G,\mathcal{TR}}; \mathbf{K}) \\
&\quad \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{ALL}}, \star_{G,\mathcal{TR}}; \mathbf{K}).
\end{aligned}
$$

The assertions now follow from Lemma 2.4 and the fact that $K_q(\mathbb{Z}K) = 0$ for a finite group K and $q \leqslant -2$ [13].

(b) Lemma 1.1 and Lemma 2.1 (c) applied to (6.4) and $\widetilde{\mathbf{L}}$ yield an isomorphism

$$\bigoplus_{\alpha\in A} H_q^{M_\alpha}(\star_{M_\alpha,\mathcal{ALL}}; \widetilde{\mathbf{L}}) \xrightarrow{\cong} H_q^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{L}}).$$

Together with Theorem 1.2 and Lemma 2.8 (b) we obtain an exact sequence which splits into short exact sequences after tensoring with $\mathbb{Z}[1/2|\pi|]$

$$
\begin{aligned}
\cdots \;\to\; & H_{q+1}(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z}))[1/2] \xrightarrow{\delta_{q+1}} \bigoplus_{\alpha\in A} \widetilde{L}_q(\mathbb{Z}M_\alpha)[1/2] \to L_q(\mathbb{Z}G)[1/2] \\
\to\; & H_q(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z}))[1/2] \xrightarrow{\delta_q} \cdots .
\end{aligned}
$$

Since $\delta_q \otimes \mathbb{Q}$ is trivial and its target $\bigoplus_{\alpha\in A} \widetilde{L}_{q-1}(\mathbb{Z}M_\alpha)[1/2]$ is free as a $\mathbb{Z}[1/2]$-module by Theorem 3.2 (c), δ_q is trivial for all $q \in \mathbb{Z}$. Hence assertion (b) follows if we can find an isomorphism

$$H_q(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right] \;\cong\; H_q(\pi\backslash T^n; \mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right].$$

Since $\mathbb{Z}^n$ is torsion free and therefore, acts freely on $\underline{E}G$, $\mathbb{Z}^n\backslash \underline{E}G$ is a model for $B\mathbb{Z}^n$. Choose a homotopy equivalence $f\colon \mathbb{Z}^n\backslash \underline{E}G \to T^n$ which induces on $\pi_1 = H_1$ the obvious map $H_1(\mathbb{Z}^n\backslash E(G,\mathcal{FIN})) \cong \mathbb{Z}^n \cong H_1(\mathbb{R}^n/\mathbb{Z}^n) = H_1(T^n)$. This map f respects the obvious $\pi = \mathbb{Z}^n\backslash G$-operation on $\mathbb{Z}^n\backslash \underline{E}G$ and the π-action on T^n by conjugation up to homotopy since the isomorphism $H_1(f)$ is π-

equivariant. Thus f induces a $\mathbb{Z}\pi$-isomorphism $H_p(f)\colon\ H_p(\mathbb{Z}^n\backslash \underline{E}G) \xrightarrow{\cong} H_p(T^n)$ for $p \geqslant 0$. By the Atiyah–Hirzebruch spectral sequence f induces an isomorphism $H_p(\mathbb{Z}^n\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\left[\frac{1}{2|\pi|}\right] \xrightarrow{\cong} H_p(T^n;\mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\left[\frac{1}{2|\pi|}\right]$. Now the claim follows since the projections induce isomorphisms

$$H_p(\mathbb{Z}^n\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\left[\frac{1}{2|\pi|}\right] \xrightarrow{\cong} H_p(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right],$$

$$H_p(T^n;\mathbf{L}(\mathbb{Z})) \otimes_{\mathbb{Z}\pi} \mathbb{Z}\left[\frac{1}{2|\pi|}\right] \xrightarrow{\cong} H_p(\pi\backslash T^n;\mathbf{L}(\mathbb{Z}))\left[\frac{1}{2|\pi|}\right],$$

(c) is proven analogously to assertion (b). This finishes the proof of Theorem 0.2. □

Remark 6.4. Let G be as in Theorem 0.2. We want briefly sketch the computation of $L_n^\epsilon(\mathbb{Z}G)$ without inverting 2. For this purpose we will need for L^ϵ for $\epsilon = -\infty$, p, h or s that the isomorphism conjecture is true for G with respect to the family of virtually cyclic subgroups. Notice that Farrell and Jones [18] formulate their Isomorphism Conjecture only for $L^{-\infty}$ and that they have shown that it cannot be true simultaneously for $\epsilon = h$ and $\epsilon = s$ in the case $G = \mathbb{Z}^2 \times \mathbb{Z}/5$ [20]. However, using the various Rothenberg sequences together with the explicit computations of the lower and middle K-theory and the five lemma one can show in this particular case that the isomorphism conjecture with respect to $\mathcal{VC}$ and without inverting 2 is true for all of the decorated L-groups $L^{-\infty}$, L^p, L^h, L^s if it holds for one of them. Since it is known for $L^{-\infty}$ for G as in Theorem 0.2 [18, Theorem 2.1 and Remark 2.1.3], the computations below are true without any assumptions. Notice that $\mathbb{Z}^2 \times \mathbb{Z}/5$ does not fall under the groups appearing in Theorem 0.2.

Let $\{V_\delta \mid \delta \in D\}$ be a full system of representatives of the conjugacy classes of subgroups $V \subset G$ which are maximal among the subgroups of G isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Notice for any virtually cyclic subgroup $V \subset G$ with $V \cong \mathbb{Z}/2 * \mathbb{Z}/2$ that $N_G V$ is subconjugated to precisely one V_δ. One can show by the same methods as before that one obtains long exact sequences

$$\cdots \to \bigoplus_{\delta\in D} UNil_{q+1} \to H_q^G(\star_{G,\mathcal{FIN}};\mathbf{L}^\epsilon) \to L_q^\epsilon(\mathbb{Z}G) \to \bigoplus_{\delta\in D} UNil_q \to \cdots,$$

where $UNil_q$ is the $UNil$-term appearing in [12, Theorem 10] and

$$\begin{aligned}\cdots \ \to\ & H_{q+1}(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \to \bigoplus_{\alpha} \widetilde{L}_q^\epsilon(\mathbb{Z}M_\alpha) \to H_q^G(\star_{G,\mathcal{FIN}};\mathbf{L}^\epsilon)\\ \to\ & H_q(G\backslash \underline{E}G;\mathbf{L}(\mathbb{Z})) \to \cdots,\end{aligned}$$

where the last sequence splits after inverting $|\pi|$. The first exact sequence splits, too; a splitting is given by the obvious map

$$\bigoplus_{\delta\in D} UNil_q \to \bigoplus_{\delta\in D} \widetilde{L}_q^{\epsilon}(\mathbb{Z}V_\delta) \to L_q^{\epsilon}(\mathbb{Z}G).$$

Hence we obtain an exact sequence which splits after inverting $|\pi|$

$$\begin{aligned} \cdots \;\to\; & H_{q+1}(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z})) \to \bigoplus_{\alpha} \widetilde{L}_q^{\epsilon}(\mathbb{Z}M_\alpha) \oplus \bigoplus_{\delta\in D} UNil_q \to L_q^{\epsilon}(\mathbb{Z}G) \\ \to\; & H_q(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z})) \to \cdots . \end{aligned}$$

From now on suppose that $|\pi|$ is odd. Since $\bigoplus_\alpha \widetilde{L}_q^{\epsilon}(\mathbb{Z}M_\alpha)$ contains no odd torsion (see Theorem 3.2 (c) and G contains no subgroup of order 2, we obtain a short exact sequence which splits after inverting $|\pi|$

$$0 \to \bigoplus_{\alpha} \widetilde{L}_q^{\epsilon}(\mathbb{Z}M_\alpha) \to L_q^{\epsilon}(\mathbb{Z}G) \to H_q(G\backslash \underline{E}G; \mathbf{L}(\mathbb{Z})) \to 0.$$

Remark 6.5. A variation of the proofs of Theorems 4.4 and 0.2 yield a similar result in the case of an extension $1 \to \mathbb{Z}^n \to G \xrightarrow{p} \pi \to 1$, where π is now a dihedral group $D_{2m} = \langle s, t \mid s^m = t^2 = (st)^2 = 1\rangle$ (note that by Lemma 6.2, π cannot act freely on $\mathbb{Z}^n$ if $m > 1$) such that the cyclic subgroup $\langle s\rangle$ of order m acts freely on $\mathbb{Z}^n$. We only have to replace the system $\{H_\beta\}$ in Theorem 4.4 by a full system of representatives of conjugacy classes of subgroups H of order 2 with $p(H) \cap \langle s\rangle = 1$ and to replace the system $\{M_\alpha\}$ by a full system of conjugacy classes of maximal finite subgroups M of G with $p(M) \cap \langle s\rangle \neq 1$. Put $B' = \{\beta \in B \mid p(N_G H_\beta) \neq p(H_\beta)\}$ and $B'' = B\backslash B'$. Then the claims and proofs for algebraic K- and L-theory are the same as in Theorem 4.4. Let $m(t)$ and $m(\beta)$, respectively be the non-negative integer satisfying $\mathbb{Z}^{m(t)} \cong (\mathbb{Z}^n)^{\langle t\rangle}$ and $\mathbb{Z}^{m(\beta)} \cong (\mathbb{Z}^n)^{p(H_\beta)}$, respectively. For instance for the topological K-theory, we get

(a) Suppose that m is odd. Then $B = B''$ and we obtain short exact sequences

$$\begin{aligned} 0 \;\to\; & |B| \cdot K_q(T^{m(t)}) \oplus \bigoplus_{\alpha} S\widetilde{K}_q(C_r^*(M_\alpha)) \\ \to\; & K_q(C_r^*(G)) \to K_q(G\backslash \underline{E}G) \to 0, \end{aligned}$$

(b) Suppose m is even. Then we obtain the long exact sequence

$$\begin{aligned} \cdots \;\to\; & K_{q+1}(G\backslash \underline{E}G) \to H_q^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}) \to K_q(C_r^*(G)) \\ \to\; & K_q(G\backslash \underline{E}G) \to \cdots , \end{aligned}$$

the short exact sequence

$$0 \to \bigoplus_{\beta \in B} K_0(T^{m(\beta)}) \oplus \bigoplus_{\alpha} S\widetilde{K}_0(C_r^*(M_\alpha)) \to H_0^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}})$$

$$\to \bigoplus_{\beta \in B'} \operatorname{coker}\left(K_0(T^{m(\beta)}) \to K_0(\mathbb{Z}/2\backslash T^{m(\beta)})\right) \to 0$$

and the isomorphism

$$\bigoplus_{\beta \in B''} K_1(T^{m(\beta)}) \xrightarrow{\cong} H_1^G(\star_{G,\mathcal{FIN}}; \widetilde{\mathbf{K}}),$$

where $\mathbb{Z}/2$ acts on $\mathbb{Z}^{m(\beta)}$ by $-\operatorname{Id}$ and thus on $T^{m(\beta)}$.

If we invert 2, these exact sequences reduce to short exact sequences

$$\begin{aligned} 0 \quad \to \quad & \bigoplus_{\beta \in B} K_q(BW_G H_\beta)[1/2] \oplus \bigoplus_{\alpha \in A} S\widetilde{K}_q(C_r^*(M_\alpha))[1/2] \\ \to \quad & K_q(C_r^*(G))[1/2] \to K_q(G\backslash \underline{E}G)[1/2] \to 0. \end{aligned}$$

(c) If we invert m, the short exact sequences above split into short exact sequences and isomorphisms respectively and we obtain isomorphisms

$$\begin{aligned} K_q(D_{2m}\backslash T^n)[1/m] \oplus \bigoplus_{\beta \in B} K_q(BW_G H)[1/m] \times \\ \times \oplus \bigoplus_{\alpha \in A} S\widetilde{K}_q(C_r^*(M_\alpha)[1/m] \xrightarrow{\cong} K_q(C_r^*(G))[1/m]. \end{aligned}$$

For $m(\beta) > 1$, the group $\operatorname{coker}\left(K_0(T^{m(\beta)}) \to K_0(\mathbb{Z}/2\backslash T^{m(\beta)})\right)$ is nontrivial.

For the proof of the above results cf. [45].

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