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On the cut-and-paste property of higher signatures of a closed oriented manifold

Eric Leichtnam[a], Wolfgang Lück[b,*], Matthias Kreck[c]

[a]Institut de Chevaleret, Plateau E, 7[eme] étage (Algèbres d' opérateurs), 175 Rue de Chevaleret, 75013 Paris, France
[b]Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
[c]Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany

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Abstract

We extend the notion of the symmetric signature $\sigma(\bar{M},r)\in L^n(R)$ for a compact n-dimensional manifold M without boundary, a reference map $r: M \to BG$ and a homomorphism of rings with involutions $\beta: \mathbb{Z}G \to R$ to the case with boundary ∂M, where $(\bar{M},\overline{\partial M}) \to (M,\partial M)$ is the G-covering associated to r. We need the assumption that $C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R$ is R-chain homotopy equivalent to a R-chain complex D_* with trivial mth differential for $n=2m$ resp. $n=2m+1$. We prove a glueing formula, homotopy invariance and additivity for this new notion. Let Z be a closed oriented manifold with reference map $Z \to BG$. Let $F \subset Z$ be a cutting codimension one submanifold $F \subset Z$ and let $\bar{F} \to F$ be the associated G-covering. Denote by $\alpha_m(\bar{F})$ the mth Novikov–Shubin invariant and by $b_m^{(2)}(\bar{F})$ the mth L^2-Betti number. If for the discrete group G the Baum–Connes assembly map is rationally injective, then we use $\sigma(\bar{M},r)$ to prove the additivity (or cut and paste property) of the higher signatures of Z, if we have $\alpha_m(\bar{F}) = \infty^+$ in the case $n=2m$ and, in the case $n=2m+1$, if we have $\alpha_m(\bar{F}) = \infty^+$ and $b_m^{(2)}(\bar{F}) = 0$. This additivity result had been proved (by a different method) in (On the Homotopy Invariance of Higher Signatures for Mainfolds with Boundary, preprint, 1999, Corollary 0.4) when G is Gromov hyperbolic or virtually nilpotent. We give new examples, where these conditions are not satisfied and additivity fails.

We explain at the end of the introduction why our paper is greatly motivated by and partially extends some of the work of Leichtnam et al. (On the Homotopy Invariance of Higher Signatures for Mainfolds with Boundary, preprint, 1999), Lott (Math. Ann., 1999) and Weinberger (Contemporary Mathematics, 1999, p. 231). © 2002 Elsevier Science Ltd. All rights reserved.

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* Corresponding author.

E-mail addresses: leicht@math.jussieu.fr (E. Leichtnam), lueck@math.uni-muenster.de (W. Lück), kreck@mathi.uni-heidelberg.de (M. Kreck).

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0. Introduction

Let M be an oriented compact n-dimensional manifold possibly with boundary. Let G be a (discrete) group and $r: M \to BG$ be a (continuous) reference map to its classifying space. Fix an (associative) ring R (with unit and) with involution and a homomorphism $\beta: \mathbb{Z}G \to R$ of rings with involution. Let $\overline{\partial M} \to \partial M$ and $\bar{M} \to M$ be the G-coverings associated to the maps $r|_{\partial M}: \partial M \to BG$ and $r: M \to BG$. Following [10, Section 4.7] and [9, Assumption 1 and Lemma 2.3], we make an assumption about $(\partial M, r|_{\partial M})$.

Assumption 0.1. *Let m be the integer for which either $n = 2m$ or $n = 2m + 1$. Let $C_*(\overline{\partial M})$ be the cellular $\mathbb{Z}G$-chain complex. Then we assume that the R-chain complex $C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R$ is R-chain homotopy equivalent to a R-chain complex D_* whose mth differential $d_m: D_m \to D_{m-1}$ vanishes.*

We will discuss this assumption later (see Lemma 3.1).

We first consider the easier and more satisfactory case $n = 2m$. Under Assumption 0.1 we will assign (in Section 2) for $n = 2m$ to (M, r) the element

$$\sigma(\bar{M}, r) \in L^{2m}(R), \tag{0.2}$$

which we will call the symmetric signature, in the symmetric L-group $L^{2m}(R)$. (Here and in the sequel we are considering the projective version and omit in the standard notation $L_p^{2m}(R)$ the index p.)

This element $\sigma(\bar{M}, r)$ agrees with the symmetric signature in the sense of [18, Proposition 2.1], [19, p. 26], provided that ∂M is empty. If ∂M is non-empty and $D_m = 0$ then $\sigma(\bar{M}, r)$ was previously considered in [23] and [11, Appendix A].

The main properties of this invariant will be that it occurs in a glueing formula, is a homotopy invariant and is related to higher signatures as explained in Theorems 0.3, 0.5 and Corollary 0.7. Given an oriented manifold M, we will denote by M^- the same manifold with the reversed orientation.

Theorem 0.3. (a) Glueing formula: *Let M and N be two oriented compact 2m-dimensional manifolds with boundary and let $\phi: \partial M \to \partial N$ be an orientation preserving diffeomorphism. Let $r: M \cup_\phi N^- \to BG$ be a reference map. Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption* 0.1. *Then*

$$\sigma(\overline{M \cup_\phi N^-}, r) = \sigma(\bar{M}, r|_M) - \sigma(\bar{N}, r|_N).$$

(b) Additivity: *Let M and N be two oriented compact 2m-dimensional manifolds with boundary and let $\phi, \psi: \partial M \to \partial N$ be orientation preserving diffeomorphisms. Let $r: M \cup_\phi N^- \to BG$ and $s: M \cup_\psi N^- \to BG$ be reference maps such that $r|_M \simeq s|_M$ and $r|_N \simeq s|_N$ holds, where $\simeq$ means homotopic. Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption* 0.1. *Then $\sigma(\overline{M \cup_\phi N^-}, r) = \sigma(\overline{M \cup_\psi N^-}, s)$.*

(c) Homotopy invariance: *Let M_0 and M_1 be two oriented compact 2m-dimensional manifolds possibly with boundaries together with reference maps $r_i: M_i \to BG$ for $i = 0,1$. Let $(f, \partial f): (M_0, \partial M_0) \to (M_1, \partial M_1)$ be an orientation preserving homotopy equivalence of pairs with $r_1 \circ f \simeq r_0$. Suppose that $(\partial M_0, r_0|_{\partial M_0})$ satisfies Assumption* 0.1. *Then*

$$\sigma(\overline{M_0}, r_0) = \sigma(\overline{M_1}, r_1).$$

Next, we consider the case $n = 2m + 1$. Then we need besides Assumption 0.1 the following additional input. Assumption 0.1 implies that $H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R)$ is a finitely generated projective R-module and that we get from Poincaré duality the structure of a (non-degenerate) $(-1)^m$-symmetric form μ on it. Following [9, Section 3], we will assume that we have specified a stable Lagrangian $L \subset H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R)$. The existence of a stable Lagrangian follows automatically if 2 is a unit in R (see Lemma 2.4). Under Assumption 0.1 and after the choice of a stable Lagrangian L we can assign for $n = 2m + 1$ to (M, r, L) an element, which we will call the symmetric signature, in the symmetric L-group $L^{2m+1}(R)$ (see Section 2)

$$\sigma(\bar{M}, r, L) \in L^{2m+1}(R). \tag{0.4}$$

It agrees with the symmetric signature in the sense of [18, Proposition 2.1], [19, p. 26], provided that ∂M is empty.

Theorem 0.5. (a) Glueing formula: *Let M and N be oriented compact $(2m + 1)$-dimensional manifolds with boundary and let $\phi : \partial M \to \partial N$ be an orientation preserving diffeomorphism. Let $r : M \cup_\phi N^- \to BG$ be a reference map. Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption 0.1. Suppose that we have fixed two stable Lagrangians $K \subset H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R)$ and $L \subset H_m(C_*(\overline{\partial N}) \otimes_{\mathbb{Z}G} R)$ such that the isomorphism $H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R) \xrightarrow{\cong} H_m(C_*(\overline{\partial N}) \otimes_{\mathbb{Z}G} R)$ of $(-1)^m$-symmetric forms induced by ϕ sends K to L stably. Then*

$$\sigma(\overline{M \cup_\phi N^-}, r) = \sigma(\bar{M}, r|_M, K) - \sigma(\bar{N}, r|_N, L).$$

(b) Additivity: *Let M and N be oriented compact $(2m + 1)$-dimensional manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be two orientation preserving diffeomorphisms. Let $r : M \cup_\phi N^- \to BG$ and $s : M \cup_\psi N^- \to BG$ be reference maps together with homotopies h_M: $r|_M \simeq s|_M$ and h_M: $r|_N \simeq s|_N$. Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption 0.1. Fix a stable Lagrangian $K \subset H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R)$. The restriction of the homotopies h_M and h_N to ∂M and ∂N induce a homotopy k: $r|_{\partial M} \circ \psi^{-1} \circ \phi \simeq r|_{\partial M}$. We get from $\psi^{-1} \circ \phi$ and k an automorphism of the $(-1)^m$-symmetric form $(H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R), \mu)$. Let $L \subset H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R)$ be the stable Lagrangian which is the image of K under this automorphism. Thus we get a formation $(H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R), \mu, K, L)$ which defines an element in $[H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R), \mu, K, L] \in L^{2m+1}(R)$ by suspension. Then*

$$\sigma(\overline{M \cup_\phi N^-}, r) - \sigma(\overline{M \cup_\psi N^-}, s) = [H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R), \mu, K, L].$$

(c) Homotopy invariance: *Let M_0 and M_1 be oriented compact $(2m + 1)$-dimensional manifolds possibly with boundaries together with reference maps $r_i : M_i \to BG$ for $i = 0,1$. Let $(f, \partial f) : (M_0, \partial M_0) \to (M_1, \partial M_1)$ be an orientation preserving homotopy equivalence of pairs together with a homotopy h: $r_1 \circ f \simeq r_0$. Suppose that $(\partial M_0, r_0|_{\partial M_0})$ satisfies Assumption 0.1. Suppose that we have fixed stable Lagrangians $L_0 \subset H_m(C_*(\overline{\partial M_0}) \otimes_{\mathbb{Z}G} R)$ and $L_1 \subset H_m(C_*(\overline{\partial M_1}) \otimes_{\mathbb{Z}G} R)$. Let L_0' be the image of L_0 under the isomorphism of $(-1)^m$-symmetric forms $(H_m(C_*(\overline{\partial M_0}) \otimes_{\mathbb{Z}G} R), \mu_0) \xrightarrow{\cong} (H_m(C_*(\overline{\partial M_1}) \otimes_{\mathbb{Z}G} R), \mu_1)$ induced by ∂f and the restriction of the homotopy h to ∂M_0. We get a stable formation $(H_m(C_*(\overline{\partial M_1}) \otimes_{\mathbb{Z}G} R), \mu_1, L_0', L_1)$ and thus by suspension an element*

$$[H_m(C_*(\overline{\partial M_1}) \otimes_{\mathbb{Z}G} R), \mu_1, L_0', L_1] \in L^{2m+1}(R).$$

Then

$$\sigma(\bar{M}_0, r_0, L_0) - \sigma(\bar{M}_1, r_1, L_1) = [H_m(C_*(\overline{\partial M_1}) \otimes_{\mathbb{Z}G} R), \mu_1, L_0', L_1].$$

Of particular interest is the case, where R is the real reduced group C^*-algebra $C_r^*(G, \mathbb{R})$ or the complex reduced group C^*-algebra $C_r^*(G)$ and β is the canonical map. Then Assumption 0.1 is equivalent to the assertion that the mth Novikov–Shubin invariant of $\overline{\partial M}$ is ∞^+ in the sense of [12, Definitions 1.8, 2.1 and 3.1] (see Lemma 3.1), and the symmetric L-groups are 2-periodic. Moreover, the invariant $\sigma(\bar{M}, r)$ is linked to higher signatures as follows, provided that ∂M is empty.

Recall that the higher signature $\mathrm{sign}_u(M, r)$ of a closed oriented manifold M with a reference map $r: M \to BG$ for a given class $u \in H^k(BG; \mathbb{Q})$ is the rational number $\langle \mathscr{L}(M) \cup r^* u, [M] \rangle$, where $\mathscr{L}(M) \in \oplus_{i \geqslant 0} H^{4i}(M; \mathbb{Q})$ is the L-class of M, $[M] \in H_{\dim(M)}(M; \mathbb{Q})$ is the homological fundamental class of M and $\langle \, , \rangle$ is the Kronecker pairing. We will consider the following commutative square of $\mathbb{Z}/4$-graded rational vector spaces:

$$\begin{array}{ccccc}
(\Omega_*(BG) \otimes_{\Omega_*(*)} \mathbb{Q})_n & \xrightarrow[\cong]{\bar{D}} & KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\text{induc.}} & K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle A_{\mathbb{R}}} & & \downarrow{\scriptstyle A} \\
L^n(C_r^*(G;\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow[\cong]{\text{sign}} & KO_n(C_r^*(G;\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\text{induc.}} & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

Some explanations are in order. We denote by $\mathbb{Q}$ the $\mathbb{Z}$-graded vector space which is $\mathbb{Q}$ in each dimension divisible by four and zero elsewhere. It can be viewed as a graded module over the $\mathbb{Z}$-graded ring $\Omega_*(*)$ by the signature. Then the $\mathbb{Z}$-graded $\mathbb{Q}$-vector space $\Omega_*(BG) \otimes_{\Omega_*(*)} \mathbb{Q}$ is four-periodic (by crossing with $[\mathbb{CP}^2]$) and hence can be viewed as a $\mathbb{Z}/4$-graded vector space. The map $\bar{D}$ is induced by the $\mathbb{Z}$-graded homomorphism

$$D: \Omega_n(BG) \to KO_n(BG),$$

which sends $[r: M \to BG]$ to the K-homology class of the signature operator of the covering $\bar{M} \to M$ associated to r. The homological Chern character is an isomorphism of $\mathbb{Z}/4$-graded rational vector spaces

$$\mathrm{ch}: KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \oplus_{k \geqslant 0} H_{4k+n}(BG; \mathbb{Q}).$$

By the Atiyah–Hirzebruch index theorem the image $\mathrm{ch} \circ D([M, \mathrm{id}: M \to M])$ of the K-homology class of the signature operator of M in $K_{\dim(M)}(M)$ under the homological Chern character ch is $\mathscr{L}(M) \cap [M]$. This implies for any class $u \in H^k(BG; \mathbb{Q})$

$$\begin{aligned}
\mathrm{sign}_u(M, r) &:= \langle \mathscr{L}(M) \cup r^* u, [M] \rangle \\
&= \langle r^* u, \mathscr{L}(M) \cap [M] \rangle \\
&= \langle u, r_*(\mathscr{L}(M) \cap [M]) \rangle \\
&= \langle u, \mathrm{ch} \circ D([M, r]) \rangle.
\end{aligned} \tag{0.6}$$

Hence, the composition $\mathrm{ch} \circ \bar{D}$: $(\Omega_*(BG) \otimes_{\Omega_*(*)} \mathbb{Q})_n \to \oplus_{k \geqslant 0} H_{4k+n}(BG; \mathbb{Q})$ sends $[r: M \to BG] \otimes 1$ to the image under $H_*(r): H_*(M; \mathbb{Q}) \to H_*(BG; \mathbb{Q})$ of the Poincaré dual $\mathscr{L}(M) \cap [M] \in \oplus_{i \geqslant 0} H_{4i - \dim(M)}(M; \mathbb{Q})$ of the L-class $\mathscr{L}(M)$.

The map $\bar{D}$ is an isomorphism since it is a transformation of homology theories [8, Example 3.4] and induces an isomorphism for the space consisting of one point. The map σ assigns to $[M,r]$ the associated symmetric Poincaré $C_r^*(G;\mathbb{R})$-chain complex $C_*(\bar{M})\otimes_{\mathbb{Z}G}C_r^*(G;\mathbb{R})$. The map $A_{\mathbb{R}}$ resp. A are assembly maps given by taking the index with coefficients in $C_r^*(G;\mathbb{R})$ resp. $C_r^*(G)$. The map sign is in dimension $n=0 \bmod 4$ given by taking the signature of a non-degenerate symmetric bilinear form. Notice that the map sign is bijective by results of Karoubi (see [21, Theorem 1.11]). The maps induc. are given by induction with the inclusion $\mathbb{R}\to\mathbb{C}$ and are injective. Obviously, the right square commutes. In order to show that the diagram commutes it suffices to prove this for the outer square. Here the claim follows from the commutative diagram in [5, p. 81].

The Novikov Conjecture says that $\mathrm{sign}_u(M,r)$ is a homotopy invariant, i.e. if $r:M\to BG$ and $s:N\to BG$ are closed orientable manifolds with reference maps to BG and $f:M\to N$ is a homotopy equivalence with $s\circ f\simeq r$, then $\mathrm{sign}_u(M,r)=\mathrm{sign}_u(N,s)$. Since the homological Chern character is rationally an isomorphism for CW-complexes, one can say by (0.6) that $D(M,r)$ is rationally the same as the collection of all higher signatures. Moreover, the Novikov Conjecture is equivalent to the statement that two elements $[M,r]$ and $[N,s]$ in $\Omega_n(BG)$ represent the same element in $(\Omega_*(BG)\otimes_{\Omega_*(*)}\mathbb{Q})_n$ resp. $KO_n(BG)\otimes_{\mathbb{Z}\mathbb{Q}}$ resp. $K_n(BG)\otimes_{\mathbb{Z}\mathbb{Q}}$, if they are homotopy equivalent.

Notice that the Baum–Connes Conjecture for $C_r^*(G)$ implies that A and hence $A_{\mathbb{R}}$ are rationally injective by the following argument (see [1, Section 7] for details). The map A can be written as the composition of the map $K_n(BG)=K_n^G(EG)\to K_n(\underline{E}G)^G$, which is given by the canonical map from EG to the classifying space $\underline{E}G$ of proper G-actions and is always rationally injective, and the Baum–Connes index map $K_*^G(\underline{E}G)\to K_*(C_r^*(G))$, which is predicted to be bijective by the Baum–Connes Conjecture. Notice that σ is injective if and only if $A_{\mathbb{R}}$ is injective and that the injectivity of A implies the injectivity of $A_{\mathbb{R}}$ and hence of σ. Since for a closed oriented manifold M with reference map $r:M\to BG$ the image of $[r:M\to BG]$ under σ (and $\mathrm{sign}\circ\sigma$) is a homotopy invariant of $r:M\to BG$, the commutativity of the diagram above and the rational injectivity of $A_{\mathbb{R}}$ implies the homotopy invariance of (0.6) and thus the Novikov Conjecture. Moreover, if $A_{\mathbb{R}}$ is rationally injective, $D([M,r])$ contains rationally the same information as $\sigma([M,r])$. We mention that the Baum–Connes Conjecture and thus the rational injectivity of A is known for a large class of groups, namely for all a-T-menable groups [3]. The rational injectivity of A is also known for all Gromov-hyperbolic groups [24].

From Theorems 0.3(b) and 0.5(b), we obtain the following corollary which extends [9, Corollary 0.4] to more general groups G. Notice that the hypothesis of Corollary 0.7 implies those of Theorems 0.3(b) and 0.5(b) because of Lemma 3.1.

Corollary 0.7. *Let M and N be two oriented compact n-dimensional manifolds with boundary and let $\phi,\psi:\partial M\to\partial N$ be orientation preserving diffeomorphisms. Let $r:M\cup_\phi N^-\to BG$ and $s:M\cup_\psi N^-\to BG$ be reference maps such that $r|_M\simeq s|_M$ and $r|_N\simeq s|_N$ holds. Denote by $\overline{\partial M}\to\partial M$ the G-covering associated to $r|_{\partial M}:\partial M\to BG$. If $n=2m$, we assume for the mth Novikov–Shubin invariant $\alpha_m(\overline{\partial M})=\infty^+$ in the sense of* [12]. *If $n=2m+1$, we assume $\alpha_m(\overline{\partial M})=\infty^+$ and for the mth L^2-Betti number $b_m^{(2)}(\overline{\partial M})=0$.* (*We could replace the condition $b_m^{(2)}(\overline{\partial M})=0$ by the weaker but harder to check assumption that the automorphism in Theorem* 0.5(b) *induced by $\psi^{-1}\circ\phi$ and the homotopy k preserve* (*stably*) *a Lagrangian of $H_m(C_*(\overline{\partial M})\otimes_{\mathbb{Z}G}C_r^*(G)$.*) *Suppose furthermore that the map*

$A_{\mathbb{R}}: KO_n(BG) \to K_n(C_r^*(G;\mathbb{R})$ is injective. Then all the higher signatures are additive in the sense that we have for all $u \in H^*(BG,\mathbb{Q})$

$$\mathrm{sign}_u(M \cup_\phi N^-, r) = \mathrm{sign}_u(M \cup_\psi N^-, s). \tag{0.8}$$

We will construct in Example 1.10 many new examples (especially for odd dimensional manifolds) of pairs of cut and paste manifolds $[M \cup_\phi N^-, r]$ and $[M \cup_\psi N^-, s]$ (even with ∂M connected) such that $r|_M \simeq s|_M$ and $r|_N \simeq s|_N$ holds and for which there exist higher signatures which do not satisfy (0.8), i.e. are not additive. There, the assumptions of Corollary 0.7, are not fully satisfied. The fact that, in general, higher signatures of closed manifolds are not cut and paste invariant over BG in the sense of [4], was known before (see, for instance, [11, Section 4.1]). The relationship to symmetric signatures of manifolds-with-boundary, and to the necessity of Assumption 0.1, was pointed out by Weinberger (see [11, Section 4.1]). The problem was raised in [11, Section 4.1] of determining which higher signatures of closed manifolds are cut and paste invariant; we refer to [11, Section 4.1] for further discussion. It is conceivable that our Lemma 2.8 might help to provide, in the future, an answer to this problem.

Finally, we explain why our paper is greatly motivated by and related to the work of Leichtnam et al. [9], Lott [11] and Weinberger [23].

The relevance of a gap condition in the middle degree on the boundary, when considering topological questions concerning manifolds with boundary, comes from Section 4.7 of Lott's paper [10]. This leads to Assumption 1 in the paper of Leichtnam et al. [9] which, from [9, Lemma 2.3], is virtually identical to our Assumption 0.1 and was the motivation for our Assumption 0.1. Our construction of the invariant $\sigma(\bar{M}, r)$ by glueing algebraic Poincaré bordisms is motivated by and extends the one of Weinberger [23] (see also [11, Appendix A]) who uses the more restrictive assumption that $C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R$ is R-chain homotopy equivalent to a R-chain complex D_* with $D_m = 0$. Notice that Weinberger's assumption implies both our Assumption 0.1 and $H_m(C_*(\overline{\partial M}) \otimes_{\mathbb{Z}G} R) = 0$ so that there is only one choice of Lagrangian, namely $L = 0$. The idea of using a Lagrangian subspace, instead of assuming the vanishing of the relevant middle (co-)homology group, is taken from Section 3 of [9].

In the case when $R = C_r^*(G)$ and under Assumption 0.1, an analog of our symmetric signature $\sigma(\bar{M}, r)$ was previously constructed in [9] as a conic index class $\sigma_{\mathrm{conic}} \in K_0(C_r^*(G))$. The homotopy invariance of σ_{conic}, i.e. the analog of our Theorem 0.3(c), was demonstrated in [9, Theorem 6.1]. Furthermore, the analog on the right-hand-side of the equation in our Theorem 0.5(c) previously appeared in [9, Proposition 3.7].

If $\mathscr{B}^\infty$ is a smooth subalgebra of $C_r^*(G)$, the authors of [9] computed the Chern character $\mathrm{ch}(\sigma_{\mathrm{conic}}) \in \bar{H}_*(\mathscr{B}^\infty)$ of their conic index explicitly in terms of the L-form of M and a higher eta-form of ∂M. They identified $\mathrm{ch}(\sigma_{\mathrm{conic}})$ with the $\bar{H}_*(\mathscr{B}^\infty)$-valued higher signature of M introduced in [10]. From this the authors of [9] deduced their main result [9, Theorem 0.1], namely, the homotopy invariance of the $\bar{H}_*(\mathscr{B}^\infty)$-valued higher signature of a manifold with boundary, as opposed to just the homotopy invariance of the "symmetric signature" σ_{conic}. (In fact, this was the motivation for the use of σ_{conic} in [9], instead of $\sigma(\bar{M}, r)$.)

As an immediate consequence of its main result, the paper [9] deduced the additivity of ordinary higher signatures of closed manifolds under Assumption 0.1, i.e. our Corollary 0.7, in the case when G is Gromov-hyperbolic or virtually nilpotent, or more generally when $C_r^*(G)$ admits a smooth subalgebra $\mathscr{B}^\infty$ with the property that all of the group cohomology of G extends to cyclic cocycles on $\mathscr{B}^\infty$ [9, Corollary 0.4]. It is known that the Baum–Connes assembly map is rationally injective for such groups G.

What is new in our paper is the direct and purely algebraic approach to the cut and paste problem of higher signatures of closed manifolds, through the construction of symmetric signatures for manifolds with boundary under Assumption 0.1. This leads to the main goal of the present paper, namely, to the proof of the additivity of higher signatures under Assumption 0.1 in Corollary 0.7, provided that the Baum–Connes assembly map is rationally injective. In this way, our main result, Corollary 0.7, is an extension of [9, Corollary 0.4].

1. Additivity and mapping tori in the bordism group

Throughout this section X is some topological space. Denoted by $\Omega_n(X)$ the bordism group of closed oriented smooth n-dimensional manifolds M together with a reference map $r: M \to X$. Consider quadruples (F, h, r, H) consisting of a closed oriented $(n-1)$-dimensional manifold F together with an orientation preserving self-diffeomorphism $h: F \to F$, a reference map $r: F \to X$ and a homotopy H: $F \times [0,1] \to X$ such that $H(-,0) = r$ and $H(-,1) = r \circ h$. The mapping torus T_h is obtained from the cylinder $F \times [0,1]$ by identifying the bottom and the top by h, i.e. $(h(x), 0) \sim (x,1)$. This is again a closed smooth manifold and inherits a preferred orientation. The map r and the homotopy H yield a reference map $r_{T_h} : T_h \to X$ in the obvious way. Hence, we can associate to such a quadruple an element

$$[F, h, r, H] := [T_h, r_{T_h}] \in \Omega_n(X). \tag{1.1}$$

Given two quadruples $(F, h[0], r[0], H[0])$ and $(F, h[1], r[1], H[1])$ with the same underlying manifold, a homotopy between them is given by a family of such quadruples $(F, h[t], r[t], H[t])$ for $t \in [0,1]$ such that the family $h[t]: F \to F$ is a diffeotopy. One easily checks that for two homotopic quadruples (as above) we have in $\Omega_n(X)$

$$[F, h[0], r[0], H[0]] = [F, h[1], r[1], H[1]]. \tag{1.2}$$

The required cobordism has as underlying manifold $F \times [0,1] \times [0,1]/\sim$, where $\sim$ is the equivalence relation generated by $(x,0,t) \sim (h^{-1}[t](x),1,t)$.

Given two quadruples of the shape (F, h, r, H) and (F, g, r, G), we can compose them to a quadruple $(F, g \circ h, r, H*G)$, where $H*G$ is the obvious homotopy $r \simeq r \circ g \circ h$ obtained from stacking together H and $G \times (h \times \mathrm{id}_{[0,1]})$. One easily checks that in $\Omega_n(X)$

$$[F, g \circ h, r, H*G] = [F, h, r, H] + [F, g, r, G]. \tag{1.3}$$

The desired cobordism has as underlying manifold $F \times [0,1] \times [0,1]/\sim$, where $\sim$ is generated by $(x,0,t) \sim (h^{-1}(x),1,t)$ for $t \in [0,\frac{1}{3}]$ and $(x,1,t) \sim (g^{-1}(x),0,t)$ for $t \in [\frac{1}{3},1]$. We recognize the mapping torus of $g \circ h$ as the part of the boundary which is the image under the canonical projection of the union of $F \times \{0\} \times [\frac{1}{3},\frac{2}{3}]$, $F \times [0,1] \times \{\frac{1}{3}\}$, $F \times \{1\} \times [\frac{1}{3},\frac{2}{3}]$ and $F \times [0,1] \times \{\frac{2}{3}\}$.

Notice that the class of a quadruple in $\Omega_n(X)$ does depend on the choice of the homotopy. Namely, consider two quadruples (F, h, r, H) and (F, h, r, G) which differ only in the choice of the homotopy. Let $u: F \times S^1 \to X$ be the obvious map induced by r and composition of homotopies

$H{*}G^-$: $r \simeq r$. Then we get from (1.2) and (1.3) in $\Omega_n(X)$

$$[F,h,r,H] - [F,h,r,G] = [u: F \times S^1 \to X]. \tag{1.4}$$

The right-hand side of (1.4) is not zero in general. Take, for instance, $F = \mathbb{CP}^2$ and $X = S^1$ and let $u_k: F \times S^1 \to S^1$ be the composition of the projection $F \times S^1 \to S^1$ with a map $S^1 \to S^1$ of degree $k \in \mathbb{Z}$. Then in this situation the right-hand side of (1.4) becomes $[u_k\colon \mathbb{CP}^2 \times S^1 \to S^1]$ for $r: \mathbb{CP}^2 \to S^1$ a constant map. This element $[u_k\colon \mathbb{CP}^2 \times S^1 \to S^1]$ is mapped under the isomorphism $\Omega_5(S^1) \cong \Omega_4(*) \oplus \Omega_5(*) = \mathbb{Z} \oplus \mathbb{Z}/2$ to $(k,0)$.

Let M and N be compact oriented n-dimensional manifolds and let $\phi,\psi: \partial M \to \partial N$ be two orientation preserving diffeomorphisms. By glueing we obtain closed oriented n-dimensional manifolds $M \cup_\phi N^-$ and $M \cup_\psi N^-$. Let $r: M \cup_\phi N^- \to X$ and $s: M \cup_\phi N^- \to X$ be two reference maps such that there exists two homotopies H: $r|_M \simeq s|_M$ and G: $r|_N \simeq s|_N$. By restriction we obtain homotopies $H|_{\partial M}$: $r|_{\partial M} \simeq s|_{\partial M}$ and $G|_{\partial N}$: $r|_{\partial N} \simeq s|_{\partial N}$. Notice that (by construction) $r|_{\partial N} \circ \phi = r|_{\partial M}$ and $s|_{\partial N} \circ \psi = s|_{\partial M}$. Thus, $H|_{\partial M}$ and $G^-|_{\partial N} \circ (\psi \times \mathrm{id})$ can be composed to a homotopy K: $r|_{\partial M} \simeq r|_{\partial M} \circ \phi^{-1} \circ \psi$. Thus we obtain a quadruple $(\partial M, \phi^{-1} \circ \psi, r|_{\partial M}, K)$ in the sense of (1.1).

Lemma 1.5. *We get in* $\Omega_n(X)$

$$[r: M \cup_\phi N^- \to X] - [s: M \cup_\psi N^- \to X] = [\partial M, \phi^{-1} \circ \psi, r|_{\partial M}, K].$$

Proof. The underlying manifold of the required bordism is obtained by glueing parts of the boundary of $M \times [0,1]$ and of $N^- \times [0,1]$ together as described as follows. Identify $(x,t) \in \partial M \times [0,1]$ with $(\phi(x),t)$ in $\partial N \times [0,1]$ if $0 \leqslant t \leqslant 1/3$, and with $(\psi(x),t)$ in ∂N if $2/3 \leqslant t \leqslant 1$. □

Corollary 1.6. *Suppose in the situation of Lemma* 1.5 *that* $X = BG$ *for a discrete group* G *and that the image* H *of the composition*

$$\pi_1(\partial M) \to \pi_1(M) \xrightarrow{(r|_M)_*} \pi_1(BG) = G$$

satisfies $H_i(BH;\mathbb{Q}) = 0$ *for* $i \geqslant 1$. *Then the higher signatures of* $r: M \cup_\phi N^- \to X$ *and* $s: M \cup_\psi N^- \to X$ *agree.*

Proof. In view of Lemma 1.5, we have to show that the higher signatures of $(\partial M,\phi^{-1} \circ \psi, r|_{\partial M}, K)$ vanish. The homotopy K yields an element $g \in G$ such that the composition of $c(g): G \to G\, g' \mapsto g g' g^{-1}$ with $\pi_1(r|_{\partial M})$ agrees with the composition of $\pi_1(r|_{\partial M})$ with the automorphism $\pi_1(\phi^{-1} \circ \psi)$. Obviously, $c(g)$ induces an automorphism of H. Denote the associated semi-direct product by $H \rtimes \mathbb{Z}$. There is a group homomorphism from $H \rtimes \mathbb{Z}$ to G which sends $h \in H$ to $h \in G$ and the generator of $\mathbb{Z}$ to $g \in G$. Let $p: H \rtimes \mathbb{Z} \to \mathbb{Z}$ be the canonical projection. Then the reference map from the mapping torus $T_{\phi^{-1} \circ \psi}$ to BG factorizes as a map $T_{\phi^{-1} \circ \psi} \xrightarrow{t} B(H \rtimes \mathbb{Z}) \to BG$ and the composition $T_{\phi^{-1} \circ \psi} \xrightarrow{t} B(H \rtimes \mathbb{Z}) \xrightarrow{Bp} B\mathbb{Z} = S^1$ is homotopic to the canonical projection pr: $T_{\phi^{-1} \circ \psi} \to S^1$. Notice that Bp induces an isomorphism $H^*(B\mathbb{Z};\mathbb{Q}) \to H^*(B(H \rtimes \mathbb{Z};\mathbb{Q}))$ since $H^i(BH;\mathbb{Q}) = 0$ for $i \geqslant 1$. Hence, it remains to show that all higher signatures of pr: $T_{\phi^{-1} \circ \psi} \to S^1$ vanish. If $1 \in H^0(S^1;\mathbb{Q}) \cong Q$ and $u \in H^1(S^1;\mathbb{Q}) \cong \mathbb{Q}$ are the obvious generators, it remains to prove that $\mathrm{sign}_1(\mathrm{pr}: T_{\phi^{-1} \circ \psi} \to S^1)$ and $\mathrm{sign}_u(\mathrm{pr}: T_{\phi^{-1} \circ \psi} \to S^1)$ are trivial. (Recall that sign_u has been defined in the introduction.) These numbers are given by ordinary signatures of $T_{\phi^{-1}\circ\psi}$ and ∂M. Since $T_{\phi^{-1}\circ\psi}$ fibers over S^1 and ∂M is nullbordant, these numbers are trivial. $\square$

Example 1.7. Consider the situation of Lemma 1.6 in the special case, where ∂M looks like $\mathbb{RP}^3 \sharp \mathbb{RP}^3 \times N$ for a simply connected oriented closed $(n-3)$-dimensional manifold N for $n \geqslant 6$ such that $H_{m-1}(N;\mathbb{Q})$ or $H_{m-3}(N;\mathbb{C})$ is non-trivial and the map $\pi_1(\partial M) \to \pi_1(M)$ is injective. Recall that m is the integer satisfying $n = 2m$ resp. $n = 2m+1$. The fundamental group of the connected sum $\mathbb{RP}^3 \sharp \mathbb{RP}^3$ is the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$. Notice that there is a two-fold covering $S^1 \times S^2 \to \mathbb{RP}^3 \sharp \mathbb{RP}^3$ and the universal covering $(\mathbb{RP}^3 \sharp \mathbb{RP}^3)^\sim$ of $\mathbb{RP}^3 \sharp \mathbb{RP}^3$ is $\mathbb{R} \times S^2$. Hence, $H_{m-1}((\mathbb{RP}^3 \sharp \mathbb{RP}^3 \times N)^\sim; \mathbb{C}) \cong H_{m-1}(N;\mathbb{C}) \oplus H_{m-3}(N;\mathbb{C})$ is a non-trivial direct sum of finitely many copies of the trivial D_∞-representation $\mathbb{C}$. Since $\alpha_m((\mathbb{RP}^3 \sharp \mathbb{RP}^3 \times N)^\sim) = \alpha_m((S^1 \times S^2 \times N)^\sim)$ [12, Remark 3.9] we conclude that $\alpha_m((\mathbb{RP}^3 \sharp \mathbb{RP}^3 \times N)^\sim)$ is different from ∞^+ (see [13, Example 4.3, Theorem 5.4] or [14, Theorem 8.7(9)]). Hence, Assumption 0.1 is not satisfied because of Lemma 3.1 and we cannot conclude the additivity of the higher signatures from Corollary 0.7. But we can conclude the additivity of the higher signatures from Corollary 1.6 since $H_i(BD_\infty;\mathbb{Q}) \cong H_i(B\mathbb{Z}/2);\mathbb{Q}) \oplus H_i(B\mathbb{Z}/2);\mathbb{Q}) = 0$ holds for $i \geqslant 1$.

More generally one can consider glueing tuples $(M, r_M, N, r_N, \phi, H)$, which consist of two compact oriented n-dimensional manifolds M and N with boundaries with reference maps $r_M : M \to X$ and $r_N : N \to X$, an orientation preserving diffeomorphism $\phi : \partial M \to \partial N$ and a homotopy $H : \partial M \times [0,1] \to X$ between $r_M|_{\partial M}$ and $r_N|_{\partial N} \circ \phi$, i.e. $r_M|_{\partial M} \sim r_N|_{\partial N} \circ \phi$. To such a glueing tuple one can associate an element

$$[M, r_M, N, r_N, \phi, H] := [M \cup_{\partial M \times \{0\}} (\partial M \times [0,1]) \cup_\phi N^-, r] \in \Omega_n(X), \tag{1.8}$$

where r is constructed from r_M, H and r_N in the obvious way. One gets

Lemma 1.9. *Let $(M, r_M, N, r_N, \phi, H)$ and $(N, r_N, P, r_P, \psi, G)$ be two glueing tuples. They can be composed to a glueing tuple $(M, r_M, P, r_P, \psi \circ \phi, K)$, where K is the composition of the homotopies H and $G \circ (\phi \times \mathrm{id})$. Then we get in $\Omega_n(X)$*

$$[M, r_M, P, r_P, \psi \circ \phi, K] = [M, r_M, N, r_N, \phi, H] + [N, r_N, P, r_P, \psi, G].$$

Proof. The required bordism has the following underlying manifold. Take the disjoint union of $M \times [0,1]$, $N \times [0,1]$ and $P \times [0,1]$ and identify $(x,t) \in \partial M \times [0,\frac{1}{3}]$ with $(\phi(x), t) \in \partial N \times [0,\frac{1}{3}]$ and $(y,t) \in \partial N \times [\frac{2}{3},1]$ with $(\psi(y), t) \in \partial P \times [\frac{2}{3},1]$. $\square$

Example 1.10. In odd dimensions additivity of the higher signatures (sometimes also called the cut-and-paste property) fails as badly as possible in the following sense. Let us consider $m \geqslant 2$, a finitely presented group G and *any* element $\omega \in \Omega_{2m+1}(BG)$. Then using the last (surjective) map of Theorem 3.2 of [15] and also the isomorphism given at the bottom of page 57 of [15] (or see the Theorem in the appendix by Matthias Kreck), one can find a quadruple (F, h, r, H) for a $2m$-dimensional closed oriented manifold F with reference map $r : F \to BG$ such that $[F, h, r, H] = \omega$ in $\Omega_{2m+1}(BG)$ and $[F, r] = 0$ in $\Omega_{2m}(BG)$ holds. Fix a nullbordism $R : W \to BG$ for $r : F \to BG$. In the sequel we identity $F = \partial W$. Since F admits a collar neighborhood in W, the inclusion $F \to W$ is a cofibration and thus we can extend the homotopy H: $r \simeq r \circ h$ to a homotopy H': $R \simeq R'$ for some map $R': W \to BG$ such that $R'_{|\partial M} = r \circ h$. Thus we obtain elements $R' \cup_h R: W \cup_h W \to BG$ and $R \cup_{\mathrm{id}} R: W \cup_{\mathrm{id}} W \to BG$ such that $R' \simeq R$. We conclude from Lemma 1.5

$$[R' \cup_h R: W \cup_h W \to BG] - [R \cup_{\mathrm{id}} R: W \cup_{\mathrm{id}} W \to BG] = \omega. \tag{1.11}$$

The theorem of Matthias Kreck which he proves in the appendix shows that for $m \geqslant 2$ one can arrange in the situation above that the reference map $r: F \to BG$ is 2-connected, provided that BG has finite skeleta. (Since we only want to have 2-connected it suffices that BG has finite 2-skeleton.) Consider the special case $m = 2$ and $G = \mathbb{Z}$. Choose in (1.11) a quadruple (F, h, r, H) such that $r: F \to B\mathbb{Z}$ is 2-connected. Then $\bar{F}$ is the universal covering of F. We conclude from [12, Lemma 3.3] that $\alpha_2(\bar{F}) = \alpha_2(\widetilde{S_1}) = \infty^+$. Therefore, Assumption 0.1 is satisfied for $\partial \bar{W} = \bar{F}$ by Lemma 3.1. Notice that there are elements $\omega \in \Omega_5(B\mathbb{Z})$ whose higher signatures do not all vanish, for instance $[\mathbb{C}P^2 \times S^1, r]$ where $r: \mathbb{C}P^2 \times S^1 \to B\mathbb{Z} = S^1$ is the projection onto the second factor. Hence (for such an example), if we set $[M_0, r_0] = [W, R]$ and $[M_1, r_1] = [W, R']$, then formula of (1.11) and Theorem 0.5 show that the right-hand side of the formula of Theorem 0.5(b) is not zero. Thus, Assumption 0.1 is not enough in the case $n = 2m + 1$ (in contrast to the case $n = 2m$) as proven in Theorem 0.3(b) to ensure the additivity of the higher signatures.

Counterexamples to additivity in odd dimensions yield also counterexamples in even dimensions by crossing with S^1. In the situation of (1.11) with $\omega \in \Omega_{2m+1}(BG)$, we get in $\Omega_{2m+2}(B(G \times \mathbb{Z})) = \Omega_{2m+2}(BG \times S^1)$

$$\begin{aligned} &[R' \times \mathrm{id}_{S^1} \cup_{h \times \mathrm{id}_{S^1}} R \times \mathrm{id}_{S^1} : W \times S^1 \cup_{h \times \mathrm{id}_{S^1}} W \times S^1 \to BG \times S^1] \\ &\quad - [R \times \mathrm{id}_{S^1} \cup_{\mathrm{id}_{\partial W \times S^1}} R \times \mathrm{id}_{S^1} : W \times S^1 \cup_{\mathrm{id}_{\partial W \times S^1}} W \times S^1 \to BG \times S^1] \\ &\quad = \omega \times [\mathrm{id}_{S^1}] \end{aligned}$$

and, since the L-class of $\omega \times S^1$ may be identified to the one of ω, for any $u \in H^*(BG; \mathbb{Q})$ we have $\mathrm{sign}_{u \times [S^1]}(\omega \times [\mathrm{id}_{S^1}]) = \mathrm{sign}_u(\omega)$ where $[S^1] \in H^1(S^1; \mathbb{Q})$ is the fundamental class. Hence, if $\omega \in \Omega_{2m+1}(BG)$ admits at least a higher signature which is not zero, then $W \times S^1 \cup_{h \times \mathrm{id}} W \times S^1$ admit a higher signature which is not cut and paste invariant.

2. Computations in symmetric L-groups

In this section, we carry out some algebraic computations and constructions of classes in symmetric L-groups which correspond on the geometric side to defining higher signatures of manifolds with boundaries (under Assumption 0.1) and to glueing processes along boundaries.

We briefly recall some basic facts about (symmetric) Poincaré chain complexes and the (symmetric) L-groups defined in terms of bordism classes of such chain complexes. For details we refer the reader to [17] and to the Section 1 of [19].

Let R be a ring with involution $R \to R: r \mapsto \bar{r}$. Two important examples are the group ring $\mathbb{Z}G$ with the involution given by $\bar{g} = g^{-1}$ and the reduced C^*-algebra $C_r^*(G)$ of a group G. Given a left R-module V, let the dual V^* be the (left) R-module $\hom_R(V, R)$ with the R-multiplication given by $(rf)(x) = f(x)\bar{r}$. Given a chain complex $C_* = (C_*, c_*)$ of (left) R-modules, define C^{n-*} to be the

R-chain complex whose ith chain module is $(C_{n-i})^*$ and whose ith differential is $c^*_{n-i+1}: C^*_{n-i} \to C^*_{n-i+1}$. We call C_* finitely generated projective if C_i is finitely generated projective for all $i \in \mathbb{Z}$ and vanishes for $i \leqslant 0$. An n-dimensional (finitely generated projective symmetric) Poincaré R-chain complex (C_*, ϕ) consists of an n-dimensional finitely generated projective R-chain complex C_* together with a R-chain homotopy equivalence $\phi^0_*: C^{n-*} \to C_*$ which the part for $s = 0$ of a representative $\{\phi^s \mid s \geqslant 0\}$ of an element in ϕ in the hypercohomology group $Q^n(C_*) = H^n(\mathbb{Z}/2; \hom(C^*, C_*))$. The element ϕ^1 is a chain homotopy $(\phi^0)^{n-*} \simeq \phi^0_*$, where $(\phi^0)^{n-*}$ is obtained from ϕ^0 in the obvious way using the canonical identification $P \to (P^*)^*$ for a finitely generated projective R-module P. The elements ϕ^{s+1} are higher homotopies for $\phi^s_* \simeq (\phi^s)^{n-*}$.

Consider a connected finite CW-complex X with universal covering $\tilde{X}$ and fundamental group π. It is an n-dimensional Poincaré complex if the (up to $\mathbb{Z}\pi$-chain homotopy well-defined) $\mathbb{Z}\pi$-chain map $-\cap[X]: C^{n-*}(\tilde{X}) \to C_*(\tilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence. Then for any normal covering $\bar{X} \to X$ with group of deck transformations G, the fundamental class $[X]$ determines an element in $\phi \in Q^n(C_*(\bar{X}))$, for which ϕ^0_* is the $\mathbb{Z}G$-chain map induced by $-\cap[X]$ and $(C_*(\bar{X}), \phi)$ is an n-dimensional Poincaré $\mathbb{Z}G$-chain complex [18, Proposition 2.1, p. 208].

The (symmetric) L-group $L^n(R)$ is defined by the algebraic bordism group of n-dimensional finitely generated projective Poincaré R-chain complexes. The algebraic bordism relation mimics the geometric bordism relation. The general philosophy, which we will frequently use without writing down the details, is that any geometric construction for geometric Poincaré pairs, such as glueing along a common boundary with a homotopy equivalence, or taking mapping tori or writing down certain bordisms, can be transferred to the category of algebraic Poincaré chain complexes.

However, there is one important difference between the geometric bordism group $\Omega_n(X)$ and the L-group $L^n(R)$ concerning homotopy invariance. Let G be a group and let M, N be two closed oriented n-dimensional manifolds with reference maps $r: M \to BG$ and $s: N \to BG$. Suppose that $f: M \to N$ is a homotopy equivalence such that $s \circ f \simeq r$. Then this does not imply that the bordism classes $[M, r]$ and $[N, s]$ agree. But the Poincaré $\mathbb{Z}G$-chain complexes $C_*(\bar{M})$ and $C_*(\bar{N})$ are $\mathbb{Z}G$-chain homotopy equivalent, and this does imply that their classes in $L^n(\mathbb{Z}G)$ agree [17, Proposition 3.2, p. 136].

The following lemma explains the role of Assumption 0.1. Its elementary proof is left to the reader.

Lemma 2.1. *Let C_* be a projective R-chain complex. Then the following assertions are equivalent.*
(a) *C_* is R-chain homotopy equivalent to a R-chain complex D_* with trivial mth differential*;
(b) $\operatorname{im}(c_m)$ *is a direct summand in C_{m-1} where $c_m: C_{m-1} \to C_m$ is the differential*;
(c) *There is a finitely generated projective R-subchain complex $D_* \subset C_*$ with $D_m = \ker(c_m)$, $D_{m-1} \oplus \operatorname{im}(c_{m-1}) = C_{m-1}$ and $D_i = C_i$ for $i \neq m$, $m-1$ such that the mth differential of D_* is zero and the inclusion $D_* \to C_*$ is a R-chain homotopy equivalence.*

Fix a non-negative integer n. Let m be the integer for which either $n = 2m$ or $n = 2m + 1$. Next, we give an algebraic construction which allows to assign to a (finitely generated projective symmetric) Poincaré pair $(i_*: C_* \to \bar{C}_*, (\delta\phi, \phi))$ of n-dimensional R-chain complexes an element in $L^n(R)$, provided that C_* is chain homotopy equivalent to a R-chain complex with trivial mth differential. In geometry this would correspond to assign to an inclusion $i: \partial M \to M$ of a manifold M with boundary ∂M together with a reference map $r: M \to X$ an element in $\Omega_n(X)$, where C_* resp.

$\bar{C}_*$ resp. i_* plays the role of $C_*(\partial M)$, $C_*(M)$ and $C_*(i)$. The idea would be to glue some preferred nullbordism to the boundary. This can be carried out in the more flexible algebraic setting under rather weak assumptions.

We begin with the case $n = 2m$. Recall that we assume that C_* is chain homotopy equivalent to an R-chain complex D_* such that $d_m : D_m \to D_{m-1}$ is trivial. Notice that we can arrange that D_* is $(2m-1)$-dimensional finitely generated projective by Lemma 2.1. Fix such a chain homotopy equivalence $u_* : C_* \to D_*$. Equip D_* with the Poincaré structure ψ induced by ϕ on C_* and u_*. Define $\bar{D}_*$ as the quotient chain complex of D_* for which $\bar{D}_i = D_i$ if $0 \leqslant i \leqslant m-1$ and $\bar{D}_i = 0$ otherwise. Let $j_* : D_* \to \bar{D}_*$ be the canonical projection. Notice that it is indeed a chain map since d_m vanishes. There is a canonical extension of the Poincaré structure ψ on D_* to a Poincaré structure $(\delta\psi, \psi)$ on the pair $j_* : D_* \to \bar{D}_*$, namely, take $\delta\psi$ to be zero. Now we can glue the Poincaré pairs $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$ and $(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi))$ along the R-chain homotopy equivalence $u_* : C_* \to D_*$ [17, Section 3], [19, 1.7]. We obtain a $2m$-dimensional Poincaré R-chain complex which presents a class in $L^{2m}(R)$. Since chain homotopy equivalent Poincaré R-chain complexes define the same element in the (symmetric) L-groups, this class is independent of the choice of $u_* : C_* \to D_*$. We denote it by

$$\sigma(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi)) \in L^{2m}(R). \tag{2.2}$$

Notice that a chain homotopy equivalence $u_* : D_* \to E_*$ of $(2m-1)$-dimensional chain complexes with trivial mth differential induces a chain equivalence $\bar{u}_* : \bar{D}_* \to \bar{E}_*$ such that u_* and $\bar{u}_*$ are compatible with the maps $D_* \to \bar{D}_*$ and $E_* \to \bar{E}_*$ constructed above. Since chain homotopy equivalent Poincaré R-chain complexes define the same element in the (symmetric) L-groups, the class defined in (2.2) is independent of the choice of $u_* : C_* \to D_*$.

The proof of the next lemma is straightforward in the sense that one has to figure out the argument for the corresponding geometric statements, which is easy, and then to translate it into the algebraic setting (see also [19, Proposition 1.8.2ii]).

Lemma 2.3. (a) *Let $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$ and $(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi))$ be $2m$-dimensional (finitely generated projective symmetric) Poincaré pairs. Let $u_* : C_* \to D_*$ be a R-chain equivalence such that $Q^{2m-1}(u_*) : Q^{2m-1}(C_*) \to Q^{2m-1}(D_*)$ maps ϕ to ψ. Denote by (E_*, v) the $2m$-dimensional Poincaré chain complex obtained from i_* and j_* by glueing along u_*. Suppose that C_* is R-chain homotopy equivalent to a R-chain complex with trivial mth differential. Then we get in $L^{2m}(R)$*

$$\sigma(E_*, v) = \sigma(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi)) - \sigma(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi)).$$

(b) *Let $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$ and $(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi))$ be two $2m$-dimensional (finitely generated projective symmetric) Poincaré pairs. Let $(\bar{f}_*, f_*) : i_* \to j_*$ be a chain homotopy equivalence of pairs, i.e. R-chain homotopy equivalences $\bar{f}_* : \bar{C}_* \to \bar{D}_*$ and $f_* : C_* \to D_*$ with $\bar{f}_* \circ i_* = j_* \circ f_*$ such that $Q^n(\bar{f}_*, f_*)$ maps $(\delta\phi, \phi)$ to $(\delta\psi, \psi)$. Suppose that C_* is R-chain homotopy equivalent to a R-chain complex with trivial mth differential. Then we get in $L^{2m}(R)$*

$$\sigma(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi)) = \sigma(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi)).$$

Now, the invariant (0.2) is obtained from the invariant (2.2) applied to the Poincaré pair given by the associated chain complexes. Theorem 0.3(a) and (c) follow from Lemma 2.3(a) and (b). Theorem

0.3(b) follows directly from Theorem 0.3(a) and (c) because the right-hand side of the formula appearing in Theorem 0.3(a) does not involve the glueing diffeomorphism. Notice that the geometric version of Lemma 2.3(a) has been considered in Lemma 1.9.

Next, we deal with the case $n = 2m + 1$. Recall that we are considering a $(2m + 1)$-dimensional finitely generated projective Poincaré R-pair $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$ and that we assume that C_* is R-chain homotopy equivalent to a R-chain complex D_* with trivial mth differential. Since $C_* \simeq C^{2m-*} \simeq D^{2m-*}$ holds by Poincaré duality, C_* is also R-chain homotopy equivalent to a R-chain complex, namely, D^{2m-*} whose $(m + 1)$-th differential is trivial. We conclude from Lemma 2.1 that we can fix a R-chain homotopy equivalence $u_* : C_* \to D_*$ to a $2m$-dimensional finitely generated projective R-chain complex D_* such that both d_{m+1} and d_m vanish. This implies also that $H_m(C_*) \cong H_m(D_*) \cong D_m$ is a finitely generated projective R-module and the Poincaré structure on C_* induces the structure of a $(-1)^m$-symmetric (non-degenerate) form μ on $H_m(C_*)$. Recall that a $(-1)^m$-symmetric (non-degenerate) form (P, μ) consists of a finitely generated projective R-module P together with an isomorphism $\mu : P \to P^*$ such that the composition $P \xrightarrow{\cong} (P^*)^* \xrightarrow{\mu^*} P$ of μ^* with the canonical isomorphism $P \to (P^*)^*$ is $(-1)^m \cdot \mu$. The standard $(-1)^m$-symmetric hyperbolic form $H(Q)$ for a finitely generated projective R-module Q is given by

$$\begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix} : H(Q) = Q^* \oplus Q \to (Q^* \oplus Q)^* = Q \oplus Q^*.$$

A Lagrangian for a $(-1)^m$-symmetric form (P, μ) is a direct summand $L \subset P$ with inclusion $j : L \to P$ such that the sequence $0 \to L \xrightarrow{j} P \xrightarrow{j^* \circ \mu} L^* \to 0$ is exact. Any inclusion $j : L \to P$ of a Lagrangian extends to an isomorphism of $(-1)^m$-symmetric forms $H(L) \to (P, \mu)$. A stable Lagrangian for (P, μ) is a Lagrangian in $(P, \mu) \oplus H(Q)$ for some finitely generated projective R-module Q. A formation (P, μ, K, L) consists of a $(-1)^m$-symmetric (non-degenerate) form (P, μ) together with two Lagrangians $K, L \subset P$. A stable formation (P, μ, K, L) on (P, μ) is a formation on $(P, \mu) \oplus H(Q)$ for some finitely generated projective R-module Q. For more informations about these notions we refer to [17, Section 2].

There are natural identifications of $L^0(R, (-1)^m)$ with the Witt groups of equivalence classes of $(-1)^m$-symmetric forms and of $L^1(R, (-1)^m)$ with the Witt group of equivalence classes of $(-1)^m$-symmetric formations [17, Section 5]. There are suspension maps $L^0(R, (-1)^m) \to L^{2m}(R)$ and $L^1(R, (-1)^m) \to L^{2m+1}(R)$. These suspension maps are in contrast to the quadratic L-groups not isomorphism for all rings with involutions, but they are bijective if R contains $\frac{1}{2}$ [17, p. 152]. The class of (C_*, ϕ) vanishes in $L^{2m}(R)$, an algebraic nullbordism is given by $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$. Let $u_* : C_* \to D_*$ be a R-chain homotopy equivalence to a $2m$-dimensional finitely generated projective R-chain complex with trivial mth and $(m - 1)$th differential. Equip D_* with the Poincaré structure ψ induced by the given Poincaré structure ϕ on C_* and u_*. By doing surgery on the projection onto the quotient R-chain complex $D_*|_{m-1}$ whose ith chain module is D_i for $i \leqslant m - 1$ and zero otherwise, in the sense of [17, Section 4], one sees that the class of (C_*, ϕ) in $L^{2m}(R)$ is the image under suspension of the element given by the $(-1)^m$-symmetric form on $H_m(C_*)$. If R contains $\frac{1}{2}$, the suspension map is bijective. Hence, the $(-1)^m$-symmetric (non-degenerate) form on $H_m(C_*)$ represents zero in the Witt group of equivalence classes of $(-1)^m$-symmetric forms. This shows

Lemma 2.4. *Suppose that (M,r) satisfies Assumption* 0.1 *and that $\frac{1}{2}\in R$. Then there exists a stable Lagrangian $L \subset H_m(C_*(\overline{\partial M})\otimes_{\mathbb{Z}G}R)$.*

Now suppose that we have fixed a stable Lagrangian $L \subset H_m(C_*)$. By adding the m-fold suspension of $H(Q)$ for some finitely generated projective R-module Q to C_*, we can arrange that $L \subset H_m(C_*)$ is a (unstable) Lagrangian. Equip D_* with the Poincaré structure ψ induced by ϕ and u_*. Let $K \subset H_m(D_*)$ be the Lagrangian given by L and $H_m(u_*)$. Let $\bar{D}_*$ be the quotient R-chain complex of D_* such that $\bar{D}_i = D_i$ for $i \leqslant m-1$, $\bar{D}_m = K^*$, $\bar{D}_i = 0$ for $i \geqslant m+1$, the ith differential is $d_i : D_i \to D_{i-1}$ for $i \leqslant m-1$ and all other differentials are zero. Let $j_* : D_* \to \bar{D}_*$ be the R-chain map which is the identity in dimensions $i \leqslant m-1$ and given by the obvious composition $D_m = H_m(D_*) \xrightarrow{\cong} H_m(D^{2m-*}) = H_m(D_*)^* \to K^*$. There is a canonical extension of the Poincaré structure ψ on D_* to a structure $(\delta\psi,\psi)$ of a Poincaré pair on $j_* : D_* \to \bar{D}_*$, namely, put $\delta\psi$ to be zero. Now we can glue the pairs $i_* : C_* \to \bar{C}_*$ and $j_* : D_* \to \bar{D}_*$ along u_* to get a $(2m+1)$-dimensional Poincaré R-chain complex. Its class in $L^{2m+1}(R)$ does not depend on the choice of Q, D_* and u_* and is denoted by

$$\sigma(i_* : C_* \to \bar{C}_*, (\delta\phi,\phi), L) \in L^{2m+1}(R). \tag{2.5}$$

Again the proof of the next lemma is straightforward in the sense that one has to figure out the argument for the corresponding geometric statements, which is easy, and then to translate it into the algebraic setting (see also [19, Proposition 1.8.2ii]).

Lemma 2.6. (a) *Let $(i_* : C_* \to \bar{C}_*, (\delta\phi,\phi))$ and $(j_* : D_* \to \bar{D}_*, (\delta\psi,\psi))$ be $(2m+1)$-dimensional (finitely generated projective symmetric) Poincaré pairs. Let $u_* : C_* \to D_*$ be a R-chain equivalences such that $Q^{2m}(u_*) : Q^{2m}(C_*) \to Q^{2m}(D_*)$ maps ϕ to ψ. Suppose that C_* is R-chain homotopy equivalent to a R-chain complex with trivial mth differential. Let $K \subset H_m(C_*)$ and $L \subset H_m(D_*)$ be stable Lagrangians such that $H_m(u_*) : H_m(C_*) \to H_m(D_*)$ respects them stably. Let (E_*,ν) be the $2m$-dimensional Poincaré chain complex obtained from i_* and j_* by glueing along u_*. Then we get in $L^{2m}(R)$*

$$\sigma(E_*,\nu) = \sigma(i_* : C_* \to \bar{C}_*, (\delta\phi,\phi), K) - \sigma(j_* : D_* \to \bar{D}_*, (\delta\psi,\psi), L).$$

(b) *Let $(i_* : C_* \to \bar{C}_*, (\delta\phi,\phi))$ and $(j_* : D_* \to \bar{D}_*, (\delta\psi,\psi))$ be two $(2m+1)$-dimensional (finitely generated projective symmetric) Poincaré pairs. Let $(\bar{f}_*, f_*) : i_* \to j_*$ be a chain homotopy equivalence of pairs, i.e. R-chain homotopy equivalences $\bar{f}_* : \bar{C}_* \to \bar{D}_*$ and $f_* : C_* \to D_*$ with $\bar{f}_* \circ i_* = j_* \circ f_*$ such that $Q^{2m+1}(\bar{f}_*, f_*)$ maps $(\delta\phi,\phi)$ to $(\delta\psi,\psi)$. Suppose that C_* is R-chain homotopy equivalent to a R-chain complex with trivial mth differential. Let $K \subset H_m(C_*)$ and $L \subset H_m(D_*)$ be stable Lagrangians. Denote by $K' \subset H_m(D_*)$ the image of K under $H_m(f_*)$. Then we obtain a stable equivalence class of formations $(H_m(D_*),\nu, K', L)$. Let $[H_m(D_*),\nu, K', L] \in L^{2m+1}(R)$ be the image of the element which is represented in the Witt group of equivalence classes of formations under the suspension homomorphism. Then we get in $L^{2m+1}(R)$*

$$\sigma(i_* : C_* \to \bar{C}_*, (\delta\phi,\phi), K) - \sigma(j_* : D_* \to \bar{D}_*, (\delta\psi,\psi), L) = [H_m(D_*),\nu, K', L].$$

Now the invariant (0.4) is obtained from the invariant (2.5) applied to the Poincaré pair given by the associated chain complexes. Theorem 0.5 follows from Lemma 2.6.

The next example shall illustrate that the choice of the homotopies h_M and h_N in Theorem 0.5(b) and of the homotopy h in Theorem 0.5(c) do affect the terms given by the formations. We are grateful to Michel Hilsum who pointed out to us that in an earlier version we did not make this point clear enough.

Example 2.7. Put $R = \mathbb{Z}[\mathbb{Z}]$. Consider (D^1, S^0) with the following two different reference maps $c, e : D^1 = [-1,1] \to B\mathbb{Z} = S^1$, namely, $c(s) = \exp(0)$ and $e(s) = \exp(\pi i(s+1))$. Let $h : D^1 \times [0,1] \to B\mathbb{Z} = S^1$ be the homotopy $e \simeq t$ sending (s,t) to $\exp(\pi i t(s+1))$. Notice that $c^*E\mathbb{Z}|_{S^0}$ and $t^*E\mathbb{Z}|_{S^0}$ agree and that we can choose therefore for both the same Lagrangian $L \subset H_0(\overline{S^0})$. Obviously, Assumption 0.1 is satisfied. We want to show that $\sigma(\overline{D^1}, t, L)$ and $\sigma(\overline{D^1}, e, L)$ are not the same elements in $L^0(\mathbb{Z}[\mathbb{Z}])$. Their difference $\sigma(\overline{D^1}, t, L) - \sigma(\overline{D^1}, e, L)$ is given by the class of the formation $[H_m(C_*(\overline{S^0})), \mu_1, L_0', L_1]$. From Theorem 0.5(c) applied to $(f, \partial f) = id : (D^1, S^0) \to (D^1, S^0)$ and h, we can identify the form $(H_0(S^0), \mu)$ with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}]$$

and choose $L = \{(x,x) | x \in \mathbb{Z}[\mathbb{Z}]\} \subset \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}]$. The homotopy h and the identity on S^0 induce the $\mathbb{Z}$-automorphism of $c^*E\mathbb{Z}|_{S^0} = t^*E\mathbb{Z}|_{S^0} = S^0 \times \mathbb{Z}$ which is the identity on $\{-1\} \times \mathbb{Z}$ and multiplication with t on $\{1\} \times \mathbb{Z}$. The automorphism of the form $(H_0(S^0), \mu)$ induced by $\partial f = \mathrm{id}$ and $h|_{S^0}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} : \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}].$$

Hence, the difference $\sigma(\overline{D^1}, t, L) - \sigma(\overline{D^1}, e, L)$ is represented by the formation

$$\left(\mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}], \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, L', L \right)$$

for $L' = \{(x, tx) | x \in \mathbb{Z}[\mathbb{Z}]\} \subset \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}]$. But the class of this formation under the isomorphism $L^0(\mathbb{Z}[\mathbb{Z}])[1/2] \cong L^0(\mathbb{Z})[1/2] \oplus L^1(\mathbb{Z})[1/2] \cong \mathbb{Z}[1/2] \oplus 0 = \mathbb{Z}[1/2]$ is the generator.

Similarly, one can see from Theorem 0.5(b) that $\sigma(S^1, \mathrm{id})$ and $\sigma(S^1, c)$ are different for the reference maps $\mathrm{id} = S^1 \to S^1 = B\mathbb{Z}$ and the constant map $c : S^1 \to B\mathbb{Z}$ by cutting S^1 open along the embedded $S^0 \subset S^1$. One has to choose homotopies $h_+ : i_+ \simeq c : S^1_+ \to B\mathbb{Z}$ and $h_- : i_- \simeq c : S^1_+ \to B\mathbb{Z}$ for $S^1_\pm$ the upper and lower hemispheres and $i_\pm : S^1_\pm \to B\mathbb{Z} = S^1$ the inclusion. Then the term describing $\sigma(\overline{S^1_+ \cup_{\mathrm{id}} S^1_-}, \mathrm{id}) - \sigma(\overline{S^1_+ \cup_{\mathrm{id}} S^1_-}, c)$ is given again by a formation which does not represent zero in $L^1(\mathbb{Z}[\mathbb{Z}])$. By crossing with $\mathbb{CP}^{2n}$ one gets also examples in dimensions $4n+1$ of this type because crossing with $\mathbb{CP}^{2n}$ induces an isomorphism $L^1(\mathbb{Z}[\mathbb{Z}])[1/2] \to L^{4n+1}(\mathbb{Z}[\mathbb{Z}])[1/2]$.

The next lemma is the algebraic version of Lemma 1.5 (see also [19, Proposition 1.8.2ii]).

Lemma 2.8. *Let n be any positive integer. Let* $(i_* : C_* \to \bar{C}_*, (\delta\phi, \phi))$ *and* $(j_* : D_* \to \bar{D}_*, (\delta\psi, \psi))$ *be two n-dimensional (finitely generated projective symmetric) Poincaré pairs. Let* $u_*, v_* : C_* \to D_*$ *be a R-chain equivalences such that both* $Q^n(u_*)$ *and* $Q^n(v_*)$ *map* $(\delta\phi, \phi)$ *to* $(\delta\psi, \psi)$. *Let* $w_* : C_* \to C_*$ *be a R-chain map with* $u_* \circ w_* \simeq v_*$. *Let* $(E_*(u_*), (\delta v, v)(u_*))$ *and* $(E_*(v_*), (\delta v, v)(v_*))$, *respectively, be the n-dimensional Poincaré chain complexes obtained from* i_* *and* j_* *by glueing along* u_* *and* v_*, *respectively. Let* $(T(w_*), \mu)$ *be the algebraic mapping torus of* w_*. *Its underlying R-chain complex is the mapping cone of* cone(id $- w_*$) (*cf.* [20, p. 264]). *Then we get in* $L^n(R)$

$$\sigma(E_*(u_*), (\delta v, v)(u_*)) - \sigma(E_*(v_*), (\delta v, v)(v_*)) = \sigma(T(w_*), \mu).$$

In general, symmetric signatures and higher signatures are not additive (see Example 1.10). In the situation of Lemma 1.5 the difference of symmetric signatures (and thus of higher signatures) is measured by the symmetric signature of the corresponding mapping torus. If we want to see the difference in $L^n(C_r^*(G))$, we only have to consider the algebraic mapping torus as explained in Lemma 2.8. To detect the image of the class of the mapping torus in $L^n(C_r^*(G))$ under the isomorphism sign: $L^n(C_r^*(G)) \to K_0(C_r^*(G))$ the formula [17, Proposition 4.3] is useful. It reduces the computation of the difference of the element $[r : M \cup_\phi N^- \to X] - [s : M \cup_\psi N^- \to X]$ under the composition $\Omega_n(BG) \xrightarrow{D} K_n(BG) \xrightarrow{A} K_n(C_r^*(G))$ to an expression which only involves the chain complex of $C_*(\overline{\partial M})$ and the map induced by the automorphism $\phi^{-1} \circ \psi$ in a rather close range around the middle dimension.

Remark 2.9. Let Z be a closed oriented n-dimensional manifold with a reference map $r : Z \to BG$. Suppose that we have for the mth Novikov–Shubin invariant $\alpha_m(\bar{Z}) = \infty^+$ in the case $n = 2m - 1$ and, in the case $n = 2m$, we have $\alpha_m(\bar{Z}) = \infty^+$ and for the mth L^2-Betti number $b_m^{(2)}(\bar{Z}) = 0$. Then we conclude from the arguments above and Lemma 3.1 that $\sigma : \Omega_n(BG) \to L^n(C_r^*(G; \mathbb{R}))$ maps $[Z, r]$ to zero. Namely, we have constructed an explicit algebraic nullbordism above. Hence, we conclude that all higher signatures of $[Z, r]$ vanish if the assembly map $A_{\mathbb{R}} : KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to KO_n(C_r^*(G; \mathbb{R}))$ is injective. This follows from the discussion in the Introduction.

3. Novikov–Shubin invariants

Next we reformulate (following [9]) the condition that the middle differential vanishes in terms of spectral invariants.

Let $\mathcal{N}(G)$ be the von Neumann algebra associated to G. Let M be a closed Riemannian manifold with normal covering $\bar{M} \to M$ with deck transformation group G. Let v be the flat $C_r^*(G)$-bundle over M whose total space is $\bar{M} \times_G C_r^*(G)$. Let $H^m(M; v)$ and $\bar{H}^m(M; v)$ resp. be the unreduced and reduced mth cohomology of M, i.e. $\ker(d^m)/\mathrm{im}(d^{m-1})$ and $\ker(d^m)/\overline{\mathrm{im}(d^{m-1})}$ resp. for d the differential in the deRham complex $\Omega^*(M; v)$ of Hilbert $C_r^*(G)$-modules. The next lemma is contained in Lemmas 2.1 and 2.3 of [9].

Lemma 3.1. *The following assertions are equivalent for an integer m.*
(a) *The canonical projection* $H^m(\bar{M}; v) \to \bar{H}^m(\bar{M}; v)$ *is bijective.*

(b) *The $C_r^*(G)$-chain complex $C_*(\bar{M})\otimes_{\mathbb{Z}G}C_r^*(G)$ is $C_r^*(G)$-chain homotopy equivalent to a finitely generated projective $C_r^*(G)$-chain complex D_* whose mth differential $d_m : D_m \to D_{m-1}$ is trivial.*
(c) *The $\mathcal{N}(G)$-chain complex $C_*(\bar{M})\otimes_{\mathbb{Z}G}\mathcal{N}(G)$ is $\mathcal{N}(G)$-chain homotopy equivalent to a finitely generated projective $\mathcal{N}(G)$-chain complex D_* whose mth differential $d_m : D_m \to D_{m-1}$ is trivial.*
(d) *The Novikov–Shubin invariant $\alpha_m(\bar{M})$ is ∞^+* (see [12]).
(e) *The Laplacian acting on $L^2(\bar{M}, \Omega^{m-1})/\ker(d^{m-1})$ has a strictly positive spectrum.*

Proof. (a)$\Leftrightarrow$(b) We can interprete the (a priori purely algebraic) $C_r^*(G)$-cochain complex $\hom_{\mathbb{Z}G}(C_*(\bar{M}), C_r^*(G)$ as cochain complexes of Hilbert $C_r^*(G)$-chain complexes with adjointable morphisms as differentials by the identification of each cochain module with the direct sum of finitely many copies of $C_*^r(G)$ using cellular $\mathbb{Z}G$-basis. There is a $C_r^*(G)$-chain homotopy equivalence (by bounded chain maps and homotopies) $\Omega^*(M;v) \to \hom_{\mathbb{Z}G}(C_*(\bar{M}), C_r^*(G))$. Hence, the image of the $(m-1)$th differential in $\Omega^*(M;v)$ is closed if and only if the same is true for the one in $\hom_{\mathbb{Z}G}(C_*(\bar{M}), C_r^*(G))$. The image of a differential in $\hom_{\mathbb{Z}G}(C_*(\bar{M}), C_r^*(G))$ is closed if and only if the image is a direct summand in the purely algebraic sense [22, Corollary 15.3.9]. But this is equivalent to the assertion that $\hom_{\mathbb{Z}G}(C_*(\bar{M}), C_r^*(G))$ is $C_r^*(G)$-chain homotopy equivalent to a finitely generated projective $C_r^*(G)$-cochain complex whose $(m-1)$th codifferential is trivial by Lemma 2.1. This is true if and only if $C_*(\bar{M})\otimes_{\mathbb{Z}G}C_r^*(G)$ is $C_r^*(G)$-chain homotopy equivalent to $C_r^*(G)$-chain complex with trivial mth differential.

(b) $\Rightarrow$ (c) is obvious.

(c)$\Leftrightarrow$(d) follows directly from the interpretation of Novikov–Shubin invariants in terms of the homology of $C_*(\bar{M})\otimes_{\mathbb{Z}G}\mathcal{N}(G)$ [13].

(d)$\Leftrightarrow$(e) follows from the fact that the dilatational equivalence class of the spectral density function of the simplicial mth codifferential and the analytic mth codifferential agree [2].

(e)$\Leftrightarrow$(a). Assertion (a) can be reformulated to the statement that the spectrum of $(d^{m-1})^*d^{m-1}$ for d^{m-1} $(m-1)$th differential in $\Omega^*(M;v)$ has a gap at zero. But this spectrum is the same as the spectrum of $(d^{m-1})^*d^{m-1}$ for d^{m-1} the $(m-1)$th differential in the deRham complex $L^2\Omega^*(\bar{M})$ of Hilbert spaces which has a gap at zero if and only if (e) is true. $\square$

Appendix. Mapping tori of special diffeomorphisms

In this appendix, we consider the image of the bordism group of diffeomorphisms on smooth manifolds over a CW-complex X with finite skeleta under the mapping torus construction. By a diffeomorphism over X we mean a quadruple (M, f, g, h), where M is a closed oriented smooth manifold, g an orientation preserving diffeomorphism on M, $f: M \to X$ a continuous map and h a homotopy between f and $f\circ g$. The role of the homotopy h becomes clear if we consider the mapping torus $M_g := M\times[0,1]/_{(g(x),0)\sim(x,1)}$, which is by projection to the second factor a smooth fibre bundle over S^1. Then h allows an extension of f on the fibre over 0 to a map $\bar{h}([(x,t)]) := h(x,t)$ and any such extension gives a homotopy h with the properties above.

Following [6] we denote the bordism group of these quadruples by $\Delta_n(X)$. Let $\Omega_{n+1}(X)$ be the bordism group of oriented smooth manifolds with reference map to X. The mapping torus construction above gives a homomorphism $\Delta_n(X) \to \Omega_{n+1}(X)$. It was shown in [6] for X simply connected and in [15] for general X that for n even this map is surjective. Recently Wolfgang Lück and Eric Leichtnam asked whether the same statement holds if we only allow quadruples where the map f is 2-connected and (M,f) represents zero in $\Omega_n(X)$. We call such a quadruple a *special diffeomorphism* over X and the subset of $\Delta_n(X)$ represented by special diffeorphisms by $S\Delta_n(X)$ (it is not clear to the author whether this subset forms a subgroup). For X simply connected one can conclude from [6, Section 9], that $S\Delta_{2n}(X) \to \Omega_{2n+1}(X)$ is surjective. In this note we generalize this to arbitrary complexes X.

Theorem A.1. *Let X be a CW-complex with finite skeleta. For $n \geqslant 2$ the mapping torus construction gives a surjection $S\Delta_{2n}(X) \to \Omega_{2n+1}(X)$.*

Proof. Let (N,g) be an element of $\Omega_{2n+1}(X)$. Consider a representative (M,r) of 0 in $\Omega_{2n}(X)$. We use the language and results from [7]. We consider the fibration $p_2 : X \times BSO \to BSO$ and denote it by B. The map $r \times v : M \to B$, where v is the normal Gauss map, is a normal B-structure. By [7, Corollary 1] we can replace $(M, r \times v)$ up to bordism by a n-equivalence $r' \times v' : M' \to X \times BSO$ giving a normal $(n-1)$-structure on M'. In particular $r' : M' \to X$ is 2-connected.

Now we form the disjoint union $(M' \times I) + N$ and consider the map $q : (M' \times I) + N \to X$ given by $r'p_1$ and g, where $p_1 : M' \times I \to M'$ is the projection. We want to replace this manifold by a manifold W diffeomorphic to $M \times I$ which is bordant relative boundary over X to $(M' \times I) + N$. If this is possible we are finished since then we glue the two boundary components of W and the maps together to obtain a mapping torus and a map to X. This is bordant over X to (N,g) since it is bordant to $((M' \times S^1) + N, r'p_1 + g)$ (note that $(M' \times S^1, r'p_1)$ is zero bordant over X).

This idea does not work directly. What we will prove is that there is a bordism W between $M' \# m(S^n \times S^n)$ and $M' \# m(S^n \times S^n)$ for some m equipped with a map to X which on the two boundary components is the composition of the projection from $M \# m(S^n \times S^n)$ to M and r', such that W is diffeomorphic to $(M' \# m(S^n \times S^n)) \times I$. We further achieve that the manifold obtained by glueing the boundary components of W together is over X bordant to (N,g). This is by the considerations above enough to prove the theorem, since our map from $M \# m(S^n \times S^n)$ to $X \times BSO$ is again a n-equivalence.

That this indirect way works follows from [7, Theorem 2], which says that we can replace $(M' \times I) + N$ by a sequence of surgeries over $X \times BSO$ and compatible subtractions of tori by an s-cobordism W between $M' \# m(S^n \times S^n)$ and $M' \# m(S^n \times S^n)$ (the fact that the number of $S^n \times S^n$'s one has to add by Theorem 2 to the boundary components of W is equal follows from the equality of the Euler characteristic of the two boundary components). If $n > 2$ the s-cobordism theorem implies W diffeomorphic to $(M' \# m(S^n \times S^n)) \times I$. If $n = 2$ the same is true by the stable s-cobordism theorem of [16] after further stabilization of W by forming k times a "connected sum" between $(S^2 \times S^2) \times I$ and W along an embeded arc joining the two boundary components of W. To finish the argument one has to note from the definition of compatible subtraction of tori that this process does not affect the bordism class over X for the manifold obtained by glueing the two boundary components together.

To see this we recall the definition of subtraction of tori. Consider two disjoint embeddings of $S^n \times D^{n+1}$ into W such that the map to X is constant on both $S^n \times 0$'s. Join each of these embedded tori by an embedded $I \times D^{2n}$ with the two boundary components and subtract the interior of these embedded submanifolds to obtain W'. This is the subtraction of a pair of tori used in [7, Theorem 2]. The boundary of W consists of two copies of $M\#(S^n \times S^n)$. There is an obvious bordism over X between the manifold obtained from W by identifying the two boundary components and the manifold obtained from W' by identifying the two boundary components. $\square$

Remark A.1. In general, it is difficult to say much about a special diffeomorphism whose mapping torus is bordant to a given pair (N, g). The main difficulty is the determination of the diffeomorphism. One can obtain some information on M'. For example, if $X = S^1$ and $n = 2$ the proof above shows that we can take for M' the following manifold: $S^1 \times S^3 \# \mathbb{CP}^2 \# \mathbb{C}\bar{\mathbb{P}}^2$ and thus the special diffeomorphism lives on $S^1 \times S^3 \# \mathbb{CP}^2 \# \mathbb{C}\bar{\mathbb{P}}^2 \# m(S^2 \times S^2)$ for some unknown integer m. More generally, in dimension 4 for an arbitrary X one can use instead of $S^1 \times S^3$ the boundary of any thickening of the 2-skeleton of X in $\mathbb{R}^5$.

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