COMMUTING HOMOTOPY LIMITS AND SMASH PRODUCTS

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Abstract. In general the processes of taking a homotopy inverse limit of a diagram of spectra and smashing spectra with a fixed space do not commute. In this paper we investigate under what additional assumptions these two processes do commute. In fact we deal with an equivariant generalization that involves spectra and smash products over the orbit category of a discrete group. Such a situation naturally occurs if one studies the equivariant homology theory associated to topological cyclic homology. The main theorem of this paper will play a role in the generalization of the results obtained by Bökstedt, Hsiang and Madsen about the algebraic K-theory Novikov Conjecture to the assembly map for the family of virtually cyclic subgroups.

to Hyman Bass on his seventieth birthday

1. Introduction

Let $\mathcal{I}$ be a small category and let $\mathbf{E} : \mathcal{I} \to \Omega\text{-}\mathsf{SPECTRA}$ be a contravariant functor, then for any space X there is a natural map

$$X_+ \wedge \operatorname{holim}_{\mathcal{I}} \mathbf{E} \to \operatorname{holim}_{\mathcal{I}} X_+ \wedge \mathbf{E}.$$

In general this map is not a weak equivalence. A special case of our main result says that the map is an equivalence if we assume that:

(i) The spectra $\mathbf{E}(c)$ for objects c in $\mathcal{I}$ are uniformly bounded below.
(ii) The homology of X is degreewise finitely generated as an abelian group.
(iii) The category $\mathcal{I}$ admits a finite dimensional contravariant classifying space (compare Definition 3.4).

In fact we will generalize the question in several directions. On the one hand we will work throughout in an equivariant setting. On the other hand we will investigate general mapping space constructions, of which homotopy limits are just special cases. Moreover we will also try to obtain weaker assumptions that still suffice to conclude that the map induces an isomorphism on rationalized homotopy groups. And finally we will discuss a chain complex analogue of our result.

In order to state our main result we introduce some more notation. Let G be a discrete group. Let $\mathsf{Or}(G)$ be the *orbit category*, whose objects are the homogeneous left G-spaces G/H and whose morphisms are G-maps. Every left G-space X, in particular every G-CW-complex (compare Section 8), gives rise to a contravariant $\mathsf{Or}(G)$-space, i.e. a contravariant functor from the orbit category to spaces, by sending G/H to $X^H = \operatorname{map}_G(G/H, X)$. If we furthermore have a covariant functor $\mathbf{E}$ from $\mathsf{Or}(G)$ to $\mathsf{SPECTRA}$ (from now on called an $\mathsf{Or}(G)$-spectrum) then we can form the smash product over the orbit category

$$X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}$$

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to obtain an ordinary spectrum (compare Section 3). This is an important construction since sending X to $\pi_*(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E})$ yields an equivariant homology theory.

Our basic conventions concerning the categories $\mathsf{SPECTRA}$ and Ω-$\mathsf{SPECTRA}$ can be found in Section 2. In particular we will denote by

$$\mathbf{R} : \mathsf{SPECTRA} \to \Omega\text{-}\mathsf{SPECTRA}$$

a *fibrant replacement* functor that replaces a spectrum by a weakly equivalent Ω-spectrum, and $\mathbf{r}\colon \mathbf{E} \to \mathbf{R}(\mathbf{E})$ will denote the natural equivalence from a spectrum to its replacement, compare (2.2).

The *homotopy limit* of a contravariant functor $\mathbf{E} : \mathcal{I} \to \mathsf{SPECTRA}$ is defined as

$$\operatorname{holim}_{\mathcal{I}} \mathbf{E} = \operatorname{map}_{\mathcal{I}}(E\mathcal{I}_+, \mathbf{R} \circ \mathbf{E}),$$

see Definition 3.6. (If $\mathbf{E}$ takes already values in Ω-$\mathsf{SPECTRA}$, then one can work with $\mathbf{E}$ instead of $\mathbf{R} \circ \mathbf{E}$, but in general the replacement is crucial.) The $\mathcal{I}$-mapping spectrum construction is explained in Section 3 and $E\mathcal{I}$ is the contravariant classifying space of the category $\mathcal{I}$ (see Definition 3.4). In particular $E\mathcal{I}$ is a contravariant $\mathcal{I}$-CW-complex, i.e. a contravariant functor from $\mathcal{I}$ to spaces which is built up out of free cells (see Definition 3.3). Even if one is eventually only interested in homotopy limits it is important for the proof of our main result to deal with other $\mathcal{I}$-CW-complexes in place of $E\mathcal{I}$, because the proof will proceed via induction over the skeleta.

We will now state the question we want to investigate in full generality. Let $\mathcal{I}$ be a small category and Y a contravariant $\mathcal{I}$-CW-complex. Furthermore let

$$\mathbf{E}\colon \mathcal{I}^{\mathrm{op}} \times \mathsf{Or}(G) \to \Omega\text{-}\mathsf{SPECTRA}$$

be a functor and X a G-CW-complex. There are maps of spectra $\mathbf{t}'$ and $\mathbf{t}''$, which are natural in X, Y and $\mathbf{E}$. They are given as follows:

$$\mathbf{t}'\colon X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E}) \to \operatorname{map}_{\mathcal{I}}(Y_+, X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E})$$

sends $x \wedge \phi$ to the map from Y to $X_+ \wedge \mathbf{E}$ which is given by $y \mapsto x \wedge \phi(y)$ and

$$\mathbf{t}''\colon \operatorname{map}_{\mathcal{I}}(Y_+, X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}) \to \operatorname{map}_{\mathcal{I}}\left(Y_+, \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\right)\right)$$

is induced by the map $\mathbf{r}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\right)$, the weak equivalence to the fibrant replacement.

Question 1.1. *Under what assumptions is the map*

$$\mathbf{t}\colon X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E}) \to \operatorname{map}_{\mathcal{I}}\left(Y_+, \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\right)\right),$$

defined as the composition $\mathbf{t}'' \circ \mathbf{t}'$*, a weak equivalence?*

We still need to explain a mild condition that we will always impose on our functor $\mathbf{E}$. Let $\mathsf{GROUPOIDS}$ denote the category of small groupoids and functors between them. For a G-set S let $\mathcal{G}^G(S)$ denote the *transport groupoid*, whose set of objects is S and for which the set of morphisms from s_1 to s_2 consists of $\{g \in G | gs_1 = s_2\}$. Composition of morphisms is induced from the group structure on G. We obtain a covariant functor $\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}$ by assigning $\mathcal{G}^G(G/H)$ to G/H. We call a functor $\mathbf{F}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ a *homotopy functor* if it sends an equivalence of groupoids to a weak equivalence of spectra. It seems that all interesting examples of $\mathsf{Or}(G)$-spectra can be obtained by composing a suitable homotopy functor $\mathbf{F} : \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ with the functor $\mathcal{G}^G$. For a subgroup $H \subset G$ we denote by $Z_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ its *centralizer.*

Theorem 1.2. *Consider the following data:*

- *A functor* $\mathbf{E}\colon \mathcal{I}^{\mathrm{op}} \times \mathsf{Or}(G) \to \Omega\text{-}\mathsf{SPECTRA}$ *which factorizes as*
$$\mathbf{E} = \mathbf{F} \circ (\mathrm{id} \times \mathcal{G}^G),$$
where $\mathbf{F}$ *is such that for each object* c *in* $\mathcal{I}$ *the functor* $\mathbf{F}(c,-)$ *is a homotopy functor.*
- *A contravariant* $\mathcal{I}$*-CW complex* Y.
- *A* G*-CW complex* X.

Suppose that there are numbers d, n *and* $N \in \mathbb{Z}$ *with* $d \geq 0$ *such that the following conditions are satisfied:*

(A) Y *is* d*-dimensional.*

(B) *The spectra* $\mathbf{E}(c, G/H)$ *are uniformly* $(N-1)$*-connected, i.e. for all objects* c *in* $\mathcal{I}$ *and all orbits* G/H *we have* $\pi_q(\mathbf{E}(c, G/H)) = 0$ *for* $q < N$.

(C) *The* G*-CW complex* X *has only finite isotropy groups and only finitely many orbit types.*

(D) *For each finite subgroup* $H \subset G$ *and every* $p \leq n + d - N$ *the homology group*
$$H_p(Z_G H \backslash X^H; \mathbb{Z})$$
is a finitely generated $\mathbb{Z}$*-module.*

Then for each $p \leq n$ *the map*
$$\pi_p(\mathbf{t})\colon \pi_p\left(X_+ \wedge_{\mathsf{Or}(G)} \mathrm{map}_{\mathcal{I}}(Y_+, \mathbf{E})\right) \to \pi_p\left(\mathrm{map}_{\mathcal{I}}\left(Y_+, \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}\right)\right)\right)$$
is an isomorphism.

This theorem will be proven in Section 5. In Example 1.9 we discuss that none of the assumptions can be dropped. A chain complex version of Theorem 1.2 will be discussed in Section 6.

If one is only interested in the question whether the map $\mathbf{t}$ induces an isomorphism on rational homotopy groups one can obtain results where assumption (D) is replaced by a weaker assumption. A very naive guess here would be to require the rationalized homology groups to be finitely generated as $\mathbb{Q}$-vector spaces. But this is too weak, compare Remark 5.6. In order to formulate the weakest possible assumption and strongest conclusion we introduce the notions of almost finitely generated modules and almost isomorphisms, compare Section 4.

A $\mathbb{Z}$-module M is *almost trivial* if there is an element $r \in \mathbb{Z}, r \neq 0$ such that $rm = 0$ holds for all $m \in M$. A $\mathbb{Z}$-module M is *almost finitely generated* if $M/\operatorname{tors}(M)$ is a finitely generated $\mathbb{Z}$-module and $\operatorname{tors}(M)$ is almost trivial. A $\mathbb{Z}$-homomorphism is an *almost isomorphism* if its kernel and cokernel are almost trivial. An almost isomorphism becomes an isomorphism after rationalization.

Addendum 1.3. *If we weaken assumption (D) in Theorem 1.2 by requiring the homology groups to be almost finitely generated* $\mathbb{Z}$*-modules, then we obtain the conclusion that the map* $\pi_p(\mathbf{t})$ *is an almost isomorphism and in particular a rational isomorphism for all* $p \leq n$.

As already explained we obtain results about homotopy limits by specializing to classifying spaces. For the convenience of the reader we formulate the corresponding result explicitly.

Corollary 1.4. *If* $\mathcal{I}$ *admits a finite dimensional model for its contravariant classifying space* $E\mathcal{I}$*, then there is a natural equivalence*
$$X_+ \wedge_{\mathsf{Or}(G)} \mathrm{holim}_{\mathcal{I}}\, \mathbf{E} \to \mathrm{holim}_{\mathcal{I}}\, X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}$$
for a functor $\mathbf{E} : \mathcal{I}^{\mathrm{op}} \times \Omega\text{-}\mathsf{SPECTRA}$ *as in Theorem 1.2 and a* G*-CW complex* X *provided that:*

(B) *The spectra* $\mathbf{E}(c, G/H)$ *are uniformly bounded below.*

(C) *The G-CW complex X has only finite isotropy groups and only finitely many orbit types.*

(D) *For every finite subgroup $H \subset G$ and every n the homology group*

$$H_n(Z_G H \backslash X^H; \mathbb{Z})$$

is a finitely generated $\mathbb{Z}$-module.

Of course there is again a version with an "almost finitely generated"-assumption and correspondingly an "almost isomorphism"-conclusion for the map induced on homotopy groups. Also there is the analogous refinement specifying the range of dimensions where we have to impose the finiteness conditions in order to obtain the conclusion for a given dimension.

In the case where G is the trivial group assumption (C) is redundant, assumption (D) simply says that the ordinary homology groups of X are degreewise finitely generated $\mathbb{Z}$-modules and the conclusion is about the map

$$X_+ \wedge \operatorname{holim}_{\mathcal{I}} \mathbf{E} \to \operatorname{holim}_{\mathcal{I}} X_+ \wedge \mathbf{E}.$$

Example 1.5. An important example is the case where the category is $\mathcal{N}$, the category whose objects are the non-negative integers and for which $\operatorname{mor}_{\mathcal{N}}(i,j)$ is empty if $i > j$, and consists of one element if $i \leq j$. In that case a contravariant functor $\mathbf{E} : \mathcal{N} \to \mathsf{SPECTRA}$ is simply an inverse system of spectra

$$\mathbf{E}_0 \leftarrow \mathbf{E}_1 \leftarrow \mathbf{E}_2 \leftarrow \dots .$$

There exists a 1-dimensional model for the contravariant classifying space $E\mathcal{N}$, compare Section 7.

Example 1.6. Illuminating is also the extreme case where the only morphisms in $\mathcal{I}$ are the identities. In that case $\mathcal{I}$ can be identified with an index set I and a contravariant functor $\mathcal{I} \to \Omega\text{-}\mathsf{SPECTRA}$ is just a family of Ω-spectra $\{\mathbf{E}_i \mid i \in I\}$ indexed by I. In the case where G is the trivial group the above specializes to results about the map

$$X_+ \wedge \prod_{i \in I} \mathbf{E}_i \ \to \ \prod_{i \in I} \mathbf{R}\,(X_+ \wedge \mathbf{E}_i)$$

and is a variation on [Ada95, Theorem 15.2 on page 326].

Now we would like to briefly explain the concrete situation that motivated the investigations in this paper. In [LRRV03] we study to what extent the methods used by Bökstedt, Hsiang and Madsen in [BHM93] to prove the algebraic K-theory Novikov Conjecture can be applied to prove rational injectivity results for generalized assembly maps in algebraic K-theory. The main tool in both cases is to use the so called cyclotomic trace to topological cyclic homology in order to detect elements in K-theory. The K-theoretic Novikov Conjecture says that the "classical" assembly map corresponding to the family consisting of the trivial subgroup is rationally injective. Using generalized assembly maps we can detect a much bigger portion in K-theory. The method of comparison via the cyclotomic trace thus naturally leads to study generalized assembly maps for topological cyclic homology:

$$EG(\mathcal{F})_+ \wedge_{\mathsf{Or}(G)} TC(\mathbf{A}(?);p) \to TC(\mathbf{A}(G);p).$$

Here $EG(\mathcal{F})$ is the classifying space for a family of subgroups $\mathcal{F}$ (compare Section 8), $\mathbf{A}$ is a fixed coefficient ring(spectrum), and for a fixed prime p the $\mathsf{Or}(G)$-spectrum $TC(\mathbf{A}(?);p)$ is defined as a certain homotopy limit

$$TC(\mathbf{A}(?);p) = \operatorname{holim}_{\mathcal{RF}} THH(\mathbf{A}(?))^{C_{p^n}}.$$

The category $\mathcal{RF}$ is described in Section 7 and we verify in Proposition 7.3 that it has a 2-dimensional model for its contravariant classifying space. Moreover the spectra $THH(\mathbf{A}(G/H))^{C_{p^n}}$ are uniformly bounded below, so that conditions (A) and (B) are satisfied. In order to study the generalized assembly map it is hence particularly important to understand the conditions (C) and (D) in the case where X is the classifying space of a family of subgroups. The following proposition, which is proven in Section 8, says that for certain families the conditions can be formulated entirely in terms of group homology.

Proposition 1.7. *Let $\mathcal{F}$ be a family of finite subgroups of G such that the set of conjugacy classes $(\mathcal{F}) = \{(H) \mid H \in \mathcal{F}\}$ is finite.*

If $H_n(BZ_GH;\mathbb{Z})$ is an almost finitely generated $\mathbb{Z}$-module for all $n \geq 0$ and all $H \in \mathcal{F}$, then the G-CW-complex $EG(\mathcal{F})$ has finite isotropy groups and only finitely many orbit types, and for any finite group $H \subset G$ and any $n \geq 0$ the homology group $H_n(Z_GH\backslash EG(\mathcal{F})^H;\mathbb{Z})$ is an almost finitely generated $\mathbb{Z}$-module.

Remark 1.8. The notions "almost finitely generated $\mathbb{Z}$-module" and "almost isomorphism" allow to prove simultaneously a result about commuting homotopy limits and smash products as in Addendum 1.3 and the above Proposition 1.7.

If one drops "almost", then our main Theorem 1.2 works, but the proof of a corresponding version of Proposition 1.7 breaks down, because then $EG(\mathcal{F})$ may have infinitely many equivariant cells. One could get away with the stronger assumption that $EG(\mathcal{F})$ has a model with finite skeleta. But that is a lot more restrictive and often not true in the applications we have in mind.

On the other hand a version of Proposition 1.7 where the assumption and the conclusion is that the rationalized homology groups are finitely generated as $\mathbb{Q}$-modules does exist. But under such an assumption one cannot prove an analogue of Theorem 1.2, compare Remark 5.6.

We finish this introduction with an example that shows that no proper subset of the conditions appearing in Theorem 1.2 is sufficient.

Example 1.9. Let G be the trivial group and $\mathcal{I}$ be the trivial category. Fix two sequences of integers $0 \leq m_0 \leq m_1 \leq m_2 \leq \ldots$ and $0 \leq n_0 \leq n_1 \leq n_2 \leq \ldots$. Put $X_+ = \bigvee_{i=0}^{\infty} S^{m_i}$ and $Y_+ = (\coprod_{j=0}^{\infty} S^{n_j})_+$. (Notice that here we consider for simplicity pointed spaces instead of spaces with a disjoint base point added). Let $\mathbf{E}$ be an Ω-spectrum. The transformation $\mathbf{t}$ appearing in Question 1.1 can be identified with the map

$$\bigvee_{i=0}^{\infty} S^{m_i} \wedge \prod_{j=0}^{\infty} \operatorname{map}(S^{n_j}, \mathbf{E}) \to \prod_{j=0}^{\infty} \operatorname{map}\left(S^{n_j}, \bigvee_{i=0}^{\infty} S^{m_i} \wedge \mathbf{E}\right) \\ \to \prod_{j=0}^{\infty} \operatorname{map}\left(S^{n_j}; \mathbf{R}\left(\bigvee_{i=0}^{\infty} S^{m_i} \wedge \mathbf{E}\right)\right).$$

Recall that for a spectrum $\mathbf{F}$ we have $\pi_p(S^n \wedge \mathbf{F}) = \pi_{p-n}(\mathbf{F})$ and $\pi_p(\operatorname{map}(S^n, \mathbf{F})) = \pi_{p+n}(\mathbf{F})$, and for a collection $\{X_i \mid i \in I\}$ of pointed CW-complexes the canonical map

$$\bigoplus_{i\in I} \pi_p\left(X_i \wedge \mathbf{F}\right) \xrightarrow{\cong} \pi_p\left(\bigvee_{i\in I} X_i \wedge \mathbf{F}\right)$$

is bijective. Using Lemma 2.3 and Lemma 3.9 one can identify the map above with the canonical map

$$\bigoplus_{i=0}^{\infty}\prod_{j=0}^{\infty} \pi_{n_j-m_i+p}(\mathbf{E}) \to \prod_{j=0}^{\infty}\bigoplus_{i=0}^{\infty} \pi_{n_j-m_i+p}(\mathbf{E}).$$

This map is always injective. But it is surjective if and only if there exists i_0 such that for all $i \geq i_0$ and all j we have $\pi_{n_j - m_i + p}(\mathbf{E}) = 0$. Note that condition (A) corresponds to the existence of a constant D such that $n_j \leq D$ for all j. Condition (B) says that there exists a constant N such that $\pi_q(\mathbf{E}) = 0$ for $q \leq N$. Condition (C) is redundant in the non-equivariant case and condition (D) is satisfied if and only if $\lim_{i\to\infty} m_i = \infty$. It is then easy to find examples showing that we cannot drop any of the assumptions (A), (B) or (D).

2. Basics about spectra

The question we are studying is not even well posed in the stable homotopy category of spectra. We are hence forced to explain what we mean by a spectrum, and in what sense we understand the basic constructions concerning spectra.

Throughout this paper we work in the category of compactly generated spaces (see [Ste67], [Whi78, I.4]). So space means compactly generated space and all constructions like mapping spaces and products are to be understood in this category. We will always assume that the inclusion of the base point into a pointed space is a cofibration and that maps between pointed spaces preserve the base point.

We define the category $\mathsf{SPECTRA}$ as follows. A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n = 0, 1, 2, \dots\}$ is a sequence of pointed spaces $\{E(n) \mid n = 0, 1, 2, \dots\}$ together with pointed maps (called *structure maps*) $\sigma(n) : E(n) \wedge S^1 \to E(n+1)$. A *map* of spectra (sometimes also called function in the literature) $\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps of pointed spaces $f(n) \colon E(n) \to E'(n)$ that are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all n. This should not be confused with the notion of map of spectra in the stable homotopy category (see [Ada95, III.2]). Recall that the homotopy groups of a spectrum are defined as

$$\pi_p(\mathbf{E}) = \mathrm{colim}_{k\to\infty} \pi_{p+k}(E(k)),$$

where the maps in this system are given by the composition

$$\pi_{p+k}(E(k)) \longrightarrow \pi_{p+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{p+k+1}(E(k+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f} \colon \mathbf{E} \to \mathbf{F}$ inducing an isomorphism on all homotopy groups. A spectrum $\mathbf{E}$ is called Ω-*spectrum* if the adjoint of each structure map

$$\overline{\sigma}(n) \colon E(n) \to \Omega E(n+1) = \mathrm{map}(S^1, E(n+1))$$

is a weak homotopy equivalence of spaces. We denote by Ω-$\mathsf{SPECTRA}$ the corresponding full subcategory of $\mathsf{SPECTRA}$.

In order to define the covariant *fibrant replacement* functor

$$\mathbf{R} \colon \mathsf{SPECTRA} \to \Omega\text{-}\mathsf{SPECTRA}$$

we first need to recall the definition and a key property of sequential homotopy colimits. Consider a sequence of pointed spaces and maps $X_0 \to X_1 \to X_2 \to \dots$ This can be thought of equivalently as a covariant functor X from the category $\mathcal{N}$ to pointed spaces (see Example 1.5). Its homotopy colimit is defined as the pointed space

$$\mathrm{hocolim}_{n\to\infty} X_n = \mathrm{hocolim}_{\mathcal{N}} X = E\mathcal{N}_+ \wedge_{\mathcal{N}} X,$$

compare Definition 3.6. Using the model for $E\mathcal{N}$ described in Example 7.1 one gets the well-known reduced mapping telescope construction. Since any map from a compact space to $\mathrm{hocolim}_{n\to\infty} X_n$ factors for some N through the natural map $X_N \to \mathrm{hocolim}_{n\to\infty} X_n$, we have that for any $k \geq 0$

$$\pi_k(\mathrm{hocolim}_{n\to\infty} X_n) \cong \mathrm{colim}_{n\to\infty} \pi_k(X_n). \tag{2.1}$$

Now for any spectrum $\mathbf{E}$ we define its fibrant replacement as the spectrum $\mathbf{RE}$ whose k-th pointed space is $\operatorname{hocolim}_{n\to\infty}\Omega^n E(n+k)$. Here the homotopy colimit is taken with respect to the system of maps

$$\Omega^n\overline{\sigma}(n+k)\colon \Omega^n E(n+k) \to \Omega^n\Omega E(n+k+1) = \Omega^{n+1}E((n+1)+k)$$

where $\overline{\sigma}(k) : E(k) \to \Omega E(k+1)$ is the adjoint of the structure map of $\mathbf{E}$. The adjoints of the structure maps of $\mathbf{RE}$ are given by the composition

$$\begin{aligned}\operatorname{hocolim}_{n\to\infty}\Omega^n E(n+k) &\xrightarrow{\operatorname{hocolim}_{n\to\infty}\Omega^n\overline{\sigma}(k+n)} \operatorname{hocolim}_{n\to\infty}\Omega^n\Omega E(n+k+1)\\ &= \operatorname{hocolim}_{n\to\infty}\Omega\Omega^n E(n+(k+1)) \xrightarrow{\mu} \Omega\operatorname{hocolim}_{n\to\infty}\Omega^n E(n+(k+1)),\end{aligned}$$

where the first map is the obvious weak equivalence induced by shifting the system, and μ is the canonical map, which is also a weak equivalence as one sees applying (2.1) above. Hence $\mathbf{RE}$ is indeed an Ω-spectrum. There is a natural map

$$\mathbf{r}(\mathbf{E})\colon \mathbf{E} \to \mathbf{RE} \tag{2.2}$$

coming from the canonical maps $E(k) \to \operatorname{hocolim}_{n\to\infty}\Omega^n E(n+k)$. Using (2.1) one computes that $\mathbf{r}(\mathbf{E})$ is a weak equivalence.

Many constructions for spaces can be carried over to spectra by applying them levelwise, i.e. to each of the individual spaces $E(n)$. Usually there is then an obvious way to define either the structure maps or their adjoints. For example the *product* $\prod_{i\in I}\mathbf{E}_i$ of a collection of spectra $\{\mathbf{E}_i \mid i \in I\}$ is the spectrum whose n-th space is $\prod_{i\in I} E_i(n)$. For the structure maps observe that Ω commutes with products. For us the main reason to consider Ω-spectra is the following lemma, which will play an important role below and does not hold for arbitrary spectra.

Lemma 2.3. *Let $\{\mathbf{E}_i \mid i \in I\}$ be a family of Ω-spectra, where I is an arbitrary index set. Then the canonical map induced by the various projections* $\mathrm{pr}_i\colon \prod_{i\in I}\mathbf{E}_i \to \mathbf{E}_i$

$$\pi_n\left(\prod_{i\in I}\mathbf{E}_i\right) \xrightarrow{\cong} \prod_{i\in I}\pi_n(\mathbf{E}_i)$$

is bijective for all $n \in \mathbb{Z}$.

Proof. Notice that for an Ω-spectrum $\mathbf{F}$ the canonical map

$$\pi_n(F(0)) \to \pi_n(\mathbf{F}) = \operatorname{colim}_{k\to\infty}\pi_n(\Omega^k F(n+k))$$

is bijective and that $\prod_{i\in I}\mathbf{E}_i$ is an Ω-spectrum. Hence the claim follows from the corresponding statement for spaces. □

The following example shows that Lemma 2.3 does not hold without the assumption that each $\mathbf{E}_i$ is an Ω-spectrum.

Example 2.4. Let $\mathbf{E}$ be an Ω-spectrum such that $A = \pi_0(\mathbf{E}) \cong \pi_k(E(k))$ is a nontrivial abelian group. Denote by $\mathbf{E}_{(i,\infty)}$ the spectrum obtained from $\mathbf{E}$ by replacing the spaces $E(0)$, $E(1)$, ..., $E(i)$ by the one-point space $\{\mathrm{pt}\}$. We have

$$\pi_k(E_{(i,\infty)}(k)) = \begin{cases} 0 & \text{if } k \leq i \\ A & \text{if } k > i \end{cases}$$

and the maps $\pi_k(E_{(i,\infty)}(k)) \to \pi_{k+1}(E_{(i,\infty)}(k+1))$ are isomorphisms for $k > i$. We see that for all i we have $\pi_0(\mathbf{E}_{(i,\infty)}) = A$ and

$$\pi_0(\prod_{i\in\mathbb{N}}\mathbf{E}_{(i,\infty)}) = \operatorname{colim}_k \pi_k(\prod_{i\in\mathbb{N}} E_{(i,\infty)}(k)) = \operatorname{colim}_k \bigoplus_{i=1}^{k} A = \bigoplus_{i=1}^{\infty} A.$$

The natural map

$$\pi_0(\prod_{i\in\mathbb{N}} \mathbf{E}_{(i,\infty)}) \to \prod_{i\in\mathbb{N}} \pi_0(\mathbf{E}_{(i,\infty)})$$

can be identified with the natural inclusion $\bigoplus_{i\in\mathbb{N}} A \to \prod_{i\in\mathbb{N}} A$ and is not an isomorphism.

Now we fix some notations and conventions about homotopy pullbacks and homotopy pushouts of spectra. Consider a commutative square $D_{\mathbf{E}}$ of spectra

$$\begin{array}{ccc} \mathbf{E}_0 & \xrightarrow{\mathbf{f}_1} & \mathbf{E}_1 \\ {\scriptstyle \mathbf{f}_2}\downarrow & & \downarrow{\scriptstyle \mathbf{g}_1} \\ \mathbf{E}_2 & \xrightarrow[\mathbf{g}_2]{} & \mathbf{E}_{12}. \end{array}$$

We denote by $\operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1)$ the levelwise homotopy pullback spectrum and by $\operatorname{hopo}(\mathbf{E}_2 \leftarrow \mathbf{E}_0 \to \mathbf{E}_1)$ the levelwise homotopy pushout spectrum, i.e. the k-th spaces are given by the homotopy pullback respectively the homotopy pushout of the corresponding diagrams of pointed spaces. For the structure maps use the fact that homotopy pullbacks commute with Ω and homotopy pushouts commute with $S^1 \wedge -$ up to natural homeomorphisms. There are canonical maps of spectra

$$\mathbf{a}\colon \mathbf{E}_0 \to \operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1), \qquad \mathbf{b}\colon \operatorname{hopo}(\mathbf{E}_2 \leftarrow \mathbf{E}_0 \to \mathbf{E}_1) \to \mathbf{E}_{12}.$$

We call $D_{\mathbf{E}}$ *homotopy cartesian* if the map $\mathbf{a}$ is a weak equivalence of spectra; dually we call $D_{\mathbf{E}}$ *homotopy cocartesian* if $\mathbf{b}$ is a weak equivalence. We use the analogous terminology for spaces.

Remark 2.5. A homotopy pullback is a special case of a homotopy limit (the indexing category is $2 \leftarrow 12 \to 1$ for contravariant functors). In Definition 3.6 below, we discuss that in order to define a homotopy limit of a diagram of spectra, one should first replace all spectra in sight with Ω-spectra by applying the fibrant replacement functor $\mathbf{R}$, and then take the levelwise homotopy limit. At first glance this seems to be inconsistent with the definition of homotopy pullback given above. But Lemma 2.6 below implies in particular that the canonical map $\mathbf{r}$ induces a weak equivalence of spectra

$$\operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1) \to \operatorname{hopb}(\mathbf{R}\mathbf{E}_2 \to \mathbf{R}\mathbf{E}_{12} \leftarrow \mathbf{R}\mathbf{E}_1).$$

Finally we recall the well-known fact that a commutative square of spectra is homotopy cartesian if and only if it is homotopy cocartesian. We would like to stress that it is important for later proofs to have such a statement for $\operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1)$ as opposed to $\operatorname{hopb}(\mathbf{R}\mathbf{E}_2 \to \mathbf{R}\mathbf{E}_{12} \leftarrow \mathbf{R}\mathbf{E}_1)$. We have not been able to find an explicit reference (in particular none dealing with this extra subtlety) in the literature, therefore we include a proof.

Lemma 2.6. *A commutative square $D_{\mathbf{E}}$ of spectra is homotopy cocartesian if and only if it is homotopy cartesian. In this case there is a natural long exact Mayer-Vietoris sequence*

$$\ldots \xrightarrow{\partial_{n+1}} \pi_n(\mathbf{E}_0) \xrightarrow{\pi_n(\mathbf{f}_1)\oplus\pi_n(\mathbf{f}_2)} \pi_n(\mathbf{E}_1) \oplus \pi_n(\mathbf{E}_2) \xrightarrow{\pi_n(\mathbf{g}_1)-\pi_n(\mathbf{g}_2)} \pi_n(\mathbf{E}_{12}) \xrightarrow{\partial_n} \pi_{n-1}(\mathbf{E}_0) \xrightarrow{\pi_{n-1}(\mathbf{f}_1)\oplus\pi_{n-1}(\mathbf{f}_2)} \ldots$$

Example 2.7. A special case of this Lemma yields the following well-known fact: For any pair of spectra $\mathbf{E}_1$ and $\mathbf{E}_2$ the natural map $\mathbf{E}_1 \vee \mathbf{E}_2 \to \mathbf{E}_1 \times \mathbf{E}_2$ is a weak equivalence.

Proof of Lemma 2.6. The key step in the proof is to show the following

Claim: If a commutative square of spectra is levelwise a homotopy cocartesian square of spaces, then it is homotopy cartesian.

We first explain why this claim implies the lemma. Any levelwise homotopy cartesian square of spectra yields a natural long exact Mayer-Vietoris sequence. Using the natural map $\mathbf{a}$ we get a natural Mayer-Vietoris sequence for every homotopy cartesian square. For every commutative square $D_{\mathbf{E}}$ we consider the levelwise homotopy cocartesian square D_{po}

$$\begin{array}{ccc} \mathbf{E}_0 & \xrightarrow{\mathbf{f}_1} & \mathbf{E}_1 \\ {\scriptstyle \mathbf{f}_2}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{g}}_1} \\ \mathbf{E}_2 & \xrightarrow[\overline{\mathbf{g}}_2]{} & \mathrm{hopo}(\mathbf{E}_2 \leftarrow \mathbf{E}_0 \rightarrow \mathbf{E}_1). \end{array}$$

The claim implies that D_{po} is homotopy cartesian, and therefore there is a long exact Mayer-Vietoris sequence associated to D_{po}.

Now assume that $D_{\mathbf{E}}$ is homotopy cartesian. In this case both D_{po} and $D_{\mathbf{E}}$ induce long exact Mayer-Vietoris sequences, and the map $\mathbf{b}$ induces a natural transformation between them. The Five-Lemma then implies that $\mathbf{b}$ is a weak equivalence of spectra, i.e. $D_{\mathbf{E}}$ is homotopy cocartesian. Conversely, assume that $D_{\mathbf{E}}$ is homotopy cocartesian. Using the natural weak equivalence $\mathbf{b}$ we get a long exact Mayer-Vietoris sequence for $D_{\mathbf{E}}$ from the one for D_{po}. Then a Five-Lemma argument shows that the map $\mathbf{a}$ is a weak equivalence, i.e. $D_{\mathbf{E}}$ is homotopy cartesian.

It remains to prove the claim. This is based on the following well-known stable range statement about homotopy cocartesian squares of spaces (a version of Blakers-Massey homotopy excision theorem, see for example [Goo92, Section 2, in particular Theorem 2.3]). Assume that the commutative square of spaces

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_2} \\ X_2 & \xrightarrow[g_1]{} & X_{12} \end{array}$$

is homotopy cocartesian, and that f_1 is k_1-connected and f_2 is k_2-connected. Then the natural map : $a\colon X_0 \to \mathrm{hopb}(X_2 \to X_{12} \leftarrow X_1)$ is at least (k_1+k_2-1)-connected.

Now suppose we are given a levelwise homotopy cocartesian square $D_{\mathbf{E}}$ of spectra. Since homotopy pushouts commute with suspensions we have for every $n, k \in \mathbb{N}$ a homotopy cocartesian square of spaces

$$\begin{array}{ccc} \Sigma^k E_0(n) & \xrightarrow{\Sigma^k f_1(n)} & \Sigma^k E_1(n) \\ {\scriptstyle \Sigma^k f_2(n)}\downarrow & & \downarrow{\scriptstyle \Sigma^k g_2(n)} \\ \Sigma^k E_2(n) & \xrightarrow[\Sigma^k g_1(n)]{} & \Sigma^k E_{12}(n). \end{array}$$

Now consider the following diagram

$$\begin{array}{ccc} \Sigma^k E_0(n) & \xrightarrow{\Sigma^k a(n)} & \Sigma^k\, \mathrm{hopb}(E_2(n) \to E_{12}(n) \leftarrow E_1(n)) \\ \| & & \downarrow{\scriptstyle \eta} \\ \Sigma^k E_0(n) & \xrightarrow{a_{k,n}} & \mathrm{hopb}(\Sigma^k E_2(n) \to \Sigma^k E_{12}(n) \leftarrow \Sigma^k E_1(n)) \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \mu} \\ E_0(n+k) & \xrightarrow{a(n+k)} & \mathrm{hopb}(E_2(n+k) \to E_{12}(n+k) \leftarrow E_1(n+k)) \end{array}$$

where σ comes from the structure maps of $\mathbf{E}_0$, η is the natural map, and μ the map induced by the structure maps of $\mathbf{E}_2$, $\mathbf{E}_{12}$ and $\mathbf{E}_1$. One checks that the diagram commutes and that the composition $\mu \circ \eta$ is the structure map of the spectrum $\operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1)$ (recall that for the homotopy pullback spectrum one actually defines the adjoints of the structure maps, using the fact that homotopy pullbacks and loops commute up to natural homeomorphism). From the Blakers-Massey connectivity statement recalled above we get that the natural map $a_{k,n}$ in the middle row of the diagram is at least $(2k-3)$-connected. Hence we can apply Lemma 2.8 below to conclude that the natural map of spectra $\mathbf{a}\colon \mathbf{E}_0 \to \operatorname{hopb}(\mathbf{E}_2 \to \mathbf{E}_{12} \leftarrow \mathbf{E}_1)$ is a weak equivalence, i.e. $D_{\mathbf{E}}$ is homotopy cartesian. □

Lemma 2.8. *Let $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ be a map of spectra and $l \in \mathbb{Z}$ be an integer. Suppose that for all $n \in \mathbb{N}$ there exist a $k = k(l,n) \in \mathbb{N}$ and a commutative diagram*

$$\begin{array}{ccc} \Sigma^k E(n) & \xrightarrow{\Sigma^k f(n)} & \Sigma^k F(n) \\ \downarrow & & \downarrow \\ E_{k,n} & \xrightarrow{f_{k,n}} & F_{k,n} \\ \downarrow & & \downarrow \\ E(k+n) & \xrightarrow{f(k+n)} & F(k+n) \end{array}$$

where $f_{k,n}$ is $(l+k+n)$-connected, and the outer vertical maps are given by the structure maps of the spectra. Then $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ is l-connected.

We omit the elementary proof, which just uses the definition of homotopy groups of spectra, basic properties of sequential colimits, and diagram chases.

3. Basics about spaces, spectra and modules over a category

In this section we recall some basic notions about spaces, spectra and modules over a small category $\mathcal{I}$. More information for spaces and spectra can be found in [DL98] and for modules in [Lüc89, II.9].

A *covariant (contravariant) pointed $\mathcal{I}$-space* X is a covariant (contravariant) functor from $\mathcal{I}$ to the category of pointed (compactly generated) spaces. A map between $\mathcal{I}$-spaces is a natural transformation of such functors. For pointed $\mathcal{I}$-spaces X and Y of the same variance let $\operatorname{map}_{\mathcal{I}}(X,Y)$ be the space of pointed maps from X to Y with the subspace topology coming from the obvious inclusion into $\prod_{c\in \operatorname{obj}(\mathcal{I})} \operatorname{map}(X(c), Y(c))$. Let X be a contravariant and Y a covariant $\mathcal{I}$-space. Define their *smash product* to be the space

$$X \wedge_{\mathcal{I}} Y \; = \bigvee_{c\in \operatorname{obj}(\mathcal{I})} X(c) \wedge Y(c) / \sim, \tag{3.1}$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi\colon c \to d$ in $\mathcal{I}$ and all points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If $\mathcal{I}$ is the trivial category with only one object and only one morphism, then X and Y are just pointed spaces and $X \wedge_{\mathcal{I}} Y$ is the ordinary smash product $X \wedge Y$. For an ordinary pointed space Z let $\operatorname{map}(Y,Z)$ denote the obvious contravariant pointed $\mathcal{I}$-space whose value at an object c is the mapping space $\operatorname{map}(Y(c), Z)$ of base point preserving maps. One easily checks the following lemma.

Lemma 3.2. *Let X be a contravariant pointed $\mathcal{I}$-space, Y be a covariant pointed $\mathcal{I}$-space and Z be a pointed space. Then there is a homeomorphism*

$$T\colon \operatorname{map}(X \wedge_{\mathcal{I}} Y, Z) \xrightarrow{\cong} \operatorname{map}_{\mathcal{I}}(X, \operatorname{map}(Y,Z))$$

which is natural in X, Y and Z.

A *covariant (contravariant) $\mathcal{I}$-spectrum* is a covariant (contravariant) functor $\mathbf{E}\colon \mathcal{I} \to \mathsf{SPECTRA}$. A *covariant (contravariant) $\mathcal{I}$-Ω-spectrum* is a covariant (contravariant) functor $\mathbf{E}\colon \mathcal{I} \to \Omega\text{-}\mathsf{SPECTRA}$. Given a contravariant pointed $\mathcal{I}$-space X, a covariant $\mathcal{I}$-spectrum $\mathbf{E}$ and a contravariant $\mathcal{I}$-spectrum $\mathbf{F}$ we obtain spectra

$$X \wedge_{\mathcal{I}} \mathbf{E} \quad \text{and} \quad \mathrm{map}_{\mathcal{I}}(X, \mathbf{F})$$

in the obvious way by applying the constructions discussed above levelwise. Notice that for an $\mathcal{I}$-Ω-spectrum $\mathbf{E}$ the spectrum $X \wedge_{\mathcal{I}} \mathbf{E}$ is not necessarily an Ω-spectrum.

Definition 3.3. A *contravariant $\mathcal{I}$-CW-complex* X is a contravariant unpointed $\mathcal{I}$-space $X\colon \mathcal{I} \to \mathsf{SPACES}$ together with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots \subset X = \bigcup_{n \geq 0} X_n$$

such that $X = \mathrm{colim}_{n\to\infty} X_n$ and for any $n \geq 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by attaching free contravariant $\mathcal{I}$-n-cells, i.e. there exists a pushout of $\mathcal{I}$-spaces of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} \mathrm{mor}_{\mathcal{I}}(-, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} \mathrm{mor}_{\mathcal{I}}(-, c_i) \times D^{n} & \longrightarrow & X_n, \end{array}$$

where the vertical maps are inclusions, I_n is an index set, and the c_i are objects of $\mathcal{I}$. (In [DL98, Definition 3.2 on page 221] this is called a free $\mathcal{I}$-CW-complex, but we will drop the word free here.)

Definition 3.4. A contravariant $\mathcal{I}$-CW-complex $E\mathcal{I}$ whose evaluation at each object of $\mathcal{I}$ is a contractible space is called a *contravariant classifying space* for $\mathcal{I}$.

Remark 3.5. Any two models for $E\mathcal{I}$ are $\mathcal{I}$-homotopy equivalent. A bar-construction model for $E\mathcal{I}$, which is functorial in $\mathcal{I}$, is given in [DL98, page 230].

Definition 3.6. If X is a covariant pointed $\mathcal{I}$-space and Y a contravariant pointed $\mathcal{I}$-space the *homotopy colimit* respectively the *homotopy limit* are defined as the pointed spaces given by

$$\mathrm{hocolim}_{\mathcal{I}} X = E\mathcal{I}_+ \wedge_{\mathcal{I}} X \quad \text{respectively} \quad \mathrm{holim}_{\mathcal{I}} Y = \mathrm{map}_{\mathcal{I}}(E\mathcal{I}_+, Y),$$

compare [DL98, Definition 3.13 on page 225]. For a covariant $\mathcal{I}$-spectrum $\mathbf{E}$ and a contravariant $\mathcal{I}$-spectrum $\mathbf{F}$ we define similarly

$$\mathrm{hocolim}_{\mathcal{I}} \mathbf{E} = E\mathcal{I}_+ \wedge_{\mathcal{I}} \mathbf{E} \quad \text{and} \quad \mathrm{holim}_{\mathcal{I}} \mathbf{F} = \mathrm{map}_{\mathcal{I}}(E\mathcal{I}_+, \mathbf{R} \circ \mathbf{F}).$$

But note the replacement functor that appears in the homotopy limit and the fact that even if $\mathbf{E}$ is an $\mathcal{I}$-Ω-spectrum the homotopy colimit need not be an Ω-spectrum.

Remark 3.7. The homotopy type of the homotopy colimit or homotopy limit does not depend on the choice of the model for $E\mathcal{I}$. If the bar-construction model for $E\mathcal{I}$ is used, then our definition agrees with the obvious modification of the one given in [BK72, Chapters XI and XII] for diagrams of simplicial sets. Some properties of homotopy colimits and homotopy limits become obvious with our definition. For instance Ω commutes with homotopy limits of spaces because of the adjunctions

$$\mathrm{map}(S^1, \mathrm{map}_{\mathcal{I}}(E\mathcal{I}_+, X)) \cong \mathrm{map}_{\mathcal{I}}(S^1 \wedge_{\mathcal{I}} E\mathcal{I}_+, X) \cong \mathrm{map}_{\mathcal{I}}(E\mathcal{I}_+, \mathrm{map}(S^1, X))$$

(see [DL98, Lemma 1.6 on page 206]). This fact is used to define the structure maps of the homotopy limit of spectra.

We collect some results that will be needed below (compare [DL98, Theorem 3.11 on page 224 and Lemma 4.6 on page 234]).

Lemma 3.8. *Let Y be a contravariant $\mathcal{I}$-CW-complex.*

(i) *Let $f\colon Z_1 \to Z_2$ be a map of covariant or contravariant respectively $\mathcal{I}$-spaces such that $f(c)$ is a weak equivalence for all objects c in $\mathcal{I}$. Then the induced maps*
$$Y \wedge_{\mathcal{I}} Z_1 \to Y \wedge_{\mathcal{I}} Z_2$$
and
$$\operatorname{map}_{\mathcal{I}}(Y, Z_1) \to \operatorname{map}_{\mathcal{I}}(Y, Z_2)$$
respectively are weak equivalences.

(ii) *Let $f\colon \mathbf{E} \to \mathbf{F}$ be a map of covariant $\mathcal{I}$-spectra such that $f(c)$ is a weak equivalence for all objects c in $\mathcal{I}$. Then the induced map*
$$Y \wedge_{\mathcal{I}} \mathbf{E} \to Y \wedge_{\mathcal{I}} \mathbf{F}$$
is a weak equivalence.

(iii) *Let $f\colon \mathbf{E} \to \mathbf{F}$ be a map of contravariant $\mathcal{I}$-Ω-spectra such that $f(c)$ is a weak equivalence for all objects c in $\mathcal{I}$. Then the induced map*
$$\operatorname{map}_{\mathcal{I}}(Y, \mathbf{E}) \to \operatorname{map}_{\mathcal{I}}(Y, \mathbf{F})$$
is a weak equivalence.

Lemma 3.9. *Let $\mathbf{E}\colon \mathcal{I} \to \Omega$-SPECTRA be a contravariant functor and let Y be a contravariant $\mathcal{I}$-space. Then $\operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E})$ is an Ω-spectrum.*

Proof. The adjoint of the k-th structure map of $\operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E})$ is the composition
$$\operatorname{map}_{\mathcal{I}}(Y_+, E(k)) \xrightarrow{\operatorname{map}_{\mathcal{I}}(\mathrm{id}, \overline{\sigma}(k))} \operatorname{map}_{\mathcal{I}}(Y_+, \Omega E(k+1)) \xrightarrow{\cong} \Omega \operatorname{map}_{\mathcal{I}}(Y_+, E(k+1)),$$
where the first map is a weak homotopy equivalence by Lemma 3.8 (i) and the second one is the canonical homeomorphism (compare Remark 3.7). □

The constructions discussed above for spaces and spectra all have analogues in the world of modules and chain complexes. We will usually refer to this analogy as the "linear case". We collect some basics. Fix a commutative associative ring R with unit. A *covariant (contravariant) $R\mathcal{I}$-module* is a covariant (contravariant) functor from $\mathcal{I}$ to the category of R-modules. Morphisms are natural transformations.

Given a contravariant $R\mathcal{I}$-module M and a covariant $R\mathcal{I}$-module N, their *tensor product* is the R-module
$$M \otimes_{R\mathcal{I}} N \;=\; \bigoplus_{c \in \operatorname{obj}(\mathcal{I})} M(c) \otimes_R N(c)/\sim,$$
where $\sim$ is defined similar as in (3.1). Given two $R\mathcal{I}$-modules M and N of the same variance, define $\hom_{R\mathcal{I}}(M, N)$ to be the R-module of natural transformations from M to N. The obvious analogue of Lemma 3.2 is true for $\otimes_{R\mathcal{I}}$ and $\hom_{R\mathcal{I}}$.

The category $R\mathcal{I}$-MODULES of covariant (contravariant) $R\mathcal{I}$-modules inherits the structure of an abelian category from the category of R-modules. In particular the notion of a projective $R\mathcal{I}$-module makes sense. There are enough injectives and projectives in $R\mathcal{I}$-MODULES (see [Lüc89, 9.16 on page 167], [Wei94, Exercise 2.3.5 on page 42 and Example 2.3.13 on page 43]). Hence standard homological algebra applies to $R\mathcal{I}$-MODULES. For instance, one can define the R-module $\operatorname{Tor}_p^{R\mathcal{I}}(M, N)$ for a contravariant $R\mathcal{I}$-module M and a covariant $R\mathcal{I}$-module N.

An *$R\mathcal{I}$-chain complex* is a chain complex in $R\mathcal{I}$-MODULES, or, equivalently, a covariant (contravariant) functor from $\mathcal{I}$ to the category of R-chain complexes. We denote by $\operatorname{ch}(R\mathcal{I}\text{-MODULES})$ the category of $R\mathcal{I}$-chain complexes.

We now specialize to the case where $\mathcal{I}$ is the *orbit category* $\mathsf{Or}(G)$, whose objects are the homogeneous G-spaces G/H and whose morphisms are G-maps. A (left) G-space X defines a contravariant $\mathsf{Or}(G)$-space

$$X\colon \mathsf{Or}(G) \to \mathsf{SPACES}, \qquad G/H \mapsto \operatorname{map}_G(G/H, X) = X^H.$$

If X is a G-CW-complex, then X considered as a functor is a contravariant $\mathsf{Or}(G)$-CW-complex (see [DL98, Theorem 7.4 on page 250]). Every $\mathsf{Or}(G)$-CW-complex defines a functor from $\mathsf{Or}(G)$ to the category of CW-complexes, which we can compose with the functor cellular chain complex. Thus we associate to an $\mathsf{Or}(G)$-CW-complex X its *cellular* $\mathbb{Z}\mathsf{Or}(G)$*-chain complex* $C_*^{\mathbb{Z}\mathsf{Or}(G)}(X)$. This is always a free $\mathbb{Z}\mathsf{Or}(G)$-chain complex in the sense of [Lüc89, 9.16 on page 167] and in particular projective.

Furthermore we will also need the category $\mathsf{Sub}(G)$, which is a quotient category of $\mathsf{Or}(G)$. Its objects are the subgroups H of G. For two subgroups H and K of G denote by $\operatorname{conhom}_G(H,K)$ the set of group homomorphisms $f\colon H \to K$, for which there exists an element $g \in G$ with $gHg^{-1} \subset K$ such that f is given by conjugation with g, i.e. $f = c(g)\colon H \to K$, $\quad h \mapsto ghg^{-1}$. Notice that $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subset K$ and $g'H(g')^{-1} \subset K$ if and only if $g^{-1}g'$ lies in the centralizer $Z_GH = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G. The group of inner automorphisms of K acts on $\operatorname{conhom}_G(H,K)$ from the left by composition. Define the set of morphisms

$$\operatorname{mor}_{\mathsf{Sub}(G)}(H,K) \;=\; \operatorname{Inn}(K)\backslash \operatorname{conhom}_G(H,K).$$

There is a natural projection

$$\operatorname{pr}\colon \mathsf{Or}(G) \to \mathsf{Sub}(G) \tag{3.10}$$

which sends a homogeneous space G/H to H. Given a G-map $f\colon G/H \to G/K$, we can choose an element $g \in G$ with $gHg^{-1} \subset K$ and $f(g'H) = g'g^{-1}K$. Then $\operatorname{pr}(f)$ is represented by $c(g)\colon H \to K$. Notice that $\operatorname{mor}_{\mathsf{Sub}(G)}(H,K)$ can be identified with the quotient $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)/Z_GH$, where $g \in Z_GH$ acts on $\operatorname{mor}_{\mathsf{Or}(G)}(G/H, G/K)$ by composition with $R_{g^{-1}}\colon G/H \to G/H$, $\quad g'H \mapsto g'g^{-1}H$.

Recall that a functor $\mathbf{F}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ is a homotopy functor if it sends an equivalence of groupoids to a weak equivalence of spectra. This is equivalent to requiring that naturally isomorphic functors between groupoids induce the same map after applying $\pi_*(\mathbf{F}(-))$. Recall also the transport groupoid functor $\mathcal{G}^G$ defined before Theorem 1.2.

Lemma 3.11. *Let* $\mathbf{F}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ *be a homotopy functor, then the functor* $\pi_n(\mathbf{F} \circ \mathcal{G}^G(-))$ *factorizes over the projection* $\operatorname{pr}\colon \mathsf{Or}(G) \to \mathsf{Sub}(G)$.

Proof. It suffices to show that the identity and $\mathcal{G}^G(R_{d^{-1}})\colon \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/H)$ are naturally isomorphic for $d \in Z_GH$. A natural transformation is given at the object gH by $gd^{-1}g^{-1}\colon gH \to gd^{-1}H$. □

4. Almost trivial and almost finitely generated modules

In this section we discuss some elementary properties of the notions "almost finitely generated modules" and "almost isomorphisms" that were used in Addendum 1.3. The main aim is to verify that we have the fundamental properties that are necessary to do spectral sequence comparison arguments.

Let R be an integral domain, i.e. an associative commutative ring with unit such that R has no non-trivial zero-divisor. An R-module M is called *faithful* if its annihilator ideal $\operatorname{Ann}_R(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$ is the trivial ideal $\{0\}$. We call M *almost trivial* or *non-faithful* if M is not faithful, i.e. there exists $r \in R, r \neq 0$ such that $rm = 0$ holds for all $m \in M$. Recall for an R-module M that $\operatorname{tors}(M)$ is the submodule consisting of elements $m \in M$ for which there exists some $r \in R, r \neq 0$ (depending on m) with $rm = 0$. We call M *almost finitely generated* if $\operatorname{tors}(M)$ is almost trivial and $M/\operatorname{tors}(M)$ is a finitely generated R-module. We will often abbreviate $M/\operatorname{tors}(M)$ by M/tors.

Recall that a *Serre subcategory* of an abelian category is an abelian subcategory closed under subobjects, quotients, and extensions (see for example [Wei94, Exercise 10.3.2 on page 384]).

Lemma 4.1. (i) *The full subcategory of R-*MODULES *whose objects are the almost trivial modules is a Serre subcategory.*

(ii) *If R is Noetherian then the full subcategory of R-*MODULES *whose objects are the almost finitely generated modules is a Serre subcategory.*

Proof. (i) Obviously a quotient module and a submodule of an almost trivial module are almost trivial again. Let $M_0 \to M_1 \to M_2$ be an exact sequence of R-modules such M_0 and M_2 are almost trivial. Choose r_i in $\operatorname{Ann}(M_i)$ with $r_i \neq 0$ for $i = 0, 2$. One easily checks that the product $r_0 r_2$ lies in $\operatorname{Ann}(M_1)$ and is different from zero. Hence M_1 is almost trivial.

(ii) Consider an exact sequence of R-modules $0 \to M_0 \to M_1 \to M_2 \to 0$. It suffices to prove that both M_0 and M_2 are almost finitely generated if and only if M_1 is almost finitely generated. Define C to be the cokernel of the induced map $\operatorname{tors}(M_1) \to \operatorname{tors}(M_2)$ and K to be the kernel of the induced map $M_1/\operatorname{tors} \to M_2/\operatorname{tors}$. Then we get exact sequences

$$0 \to \operatorname{tors}(M_0) \to \operatorname{tors}(M_1) \to \operatorname{tors}(M_2) \to C \to 0,$$

$$0 \to M_0/\operatorname{tors} \to K \to C \to 0, \qquad \text{and}$$

$$0 \to K \to M_1/\operatorname{tors} \to M_2/\operatorname{tors} \to 0.$$

Suppose that M_1 is almost finitely generated, i.e. $\operatorname{tors}(M_1)$ is almost trivial and M_1/tors is finitely generated. As R is Noetherian, K is finitely generated. Therefore C is finitely generated and $C = \operatorname{tors}(C)$, hence C is almost trivial. We conclude from assertion (i) that $\operatorname{tors}(M_0)$ and $\operatorname{tors}(M_2)$ are almost trivial. Since R is Noetherian and M_1/tors is finitely generated, both M_0/tors and M_2/tors are finitely generated. This shows that both M_0 and M_2 are almost finitely generated.

Suppose that both M_0 and M_2 are almost finitely generated. We conclude from assertion (i) that C and $\operatorname{tors}(M_1)$ are almost trivial. We obtain an exact sequence

$$\hom_R(K, M_0/\operatorname{tors}) \longrightarrow \hom_R(M_0/\operatorname{tors}, M_0/\operatorname{tors}) \longrightarrow \operatorname{Ext}^1_R(C, M_0/\operatorname{tors}).$$

Since C is almost trivial, we can find $r \in R, r \neq 0$ which annihilates C and hence also $\operatorname{Ext}^1_R(C, M_0/\operatorname{tors})$. Therefore $r \cdot \operatorname{id}\colon M_0/\operatorname{tors} \to M_0/\operatorname{tors}$ is sent to the zero element in $\operatorname{Ext}^1_R(C, M_0/\operatorname{tors})$. Hence we can find a R-map $f\colon K \to M_0/\operatorname{tors}$ whose restriction to M_0/tors is $r \cdot \operatorname{id}\colon M_0/\operatorname{tors} \to M_0/\operatorname{tors}$. Consider $x \in \ker(f)$. Since r annihilates C, the element $r \cdot x$ lies in M_0/tors and belongs to the kernel of $r \cdot \operatorname{id}\colon M_0/\operatorname{tors} \to M_0/\operatorname{tors}$. This implies $r^2 \cdot x = 0$. Since K is a submodule of M_1/tors and hence is torsionfree, we conclude $x = 0$. This shows that $f\colon K \to M_0/\operatorname{tors}$ is an injection. Since M_0/tors is finitely generated and R is Noetherian, K is finitely generated. Since M_2/tors is finitely generated and R is Noetherian, M_1/tors is finitely generated. Hence M_1 is almost finitely generated. □

Definition 4.2. A map $f\colon M \to N$ of R-modules is called an *almost isomorphism* if its kernel and cokernel are almost trivial R-modules.

We conclude from Lemma 4.1 (i) that for two composable R-maps $f\colon L \to M$ and $g\colon M \to N$ all three maps f, g and $g \circ f$ are almost isomorphisms if two of them are. Moreover there is a Five-Lemma for almost isomorphisms. For an almost R-isomorphism f the F-map $F \otimes_R f$ is an F-isomorphism, where F denotes the quotient field of R. An R-module M is almost trivial or almost finitely generated respectively if and only if it is almost isomorphic to the zero module or respectively to a finitely generated R-module.

5. Commuting homotopy limits and smash products

In this section we will prove our main result Theorem 1.2 and the Addendum 1.3. The proof will proceed by induction over the skeleta of Y and hence we start with the following lemma.

Lemma 5.1. *Theorem 1.2 and Addendum 1.3 are true if Y is a zero-dimensional $\mathcal{I}$-CW-complex.*

Proof. For a large part we will treat the proofs of the theorem and of the addendum simultaneously.

By assumption we can find a family of objects $\{c_i \mid i \in I\}$ such that $Y = \coprod_{i \in I} \operatorname{mor}_{\mathcal{I}}(?, c_i)$. Notice that there is a canonical isomorphism

$$\operatorname{map}_{\mathcal{I}}(\operatorname{mor}_{\mathcal{I}}(?, c)_+, \mathbf{F}) \xrightarrow{\cong} \mathbf{F}(c)$$

for any contravariant $\mathcal{I}$-spectrum $\mathbf{F}$ and any object c in $\mathcal{I}$. Hence it suffices to show that the map

$$\mathbf{t}\colon X_+ \wedge_{\mathsf{Or}(G)} \prod_{i \in I} \mathbf{E}(c_i) \to \prod_{i \in I} \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)$$

which is induced by the various projections $\operatorname{pr}_i\colon \prod_{i \in I} \mathbf{E}(c_i) \to \mathbf{E}(c_i)$ followed by the replacement maps $\mathbf{r}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i))\colon X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i) \to \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)$ of (2.2), induces an (almost) isomorphism on homotopy groups π_p for all $p \leq n$.

For all p there is a commutative diagram

$$\begin{array}{ccc} \pi_p\left(X_+ \wedge_{\mathsf{Or}(G)} \prod_{i \in I} \mathbf{E}(c_i)\right) & \xrightarrow{\pi_p(\mathbf{t})} & \pi_p\left(\prod_{i \in I} \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)\right) \\ \downarrow & & \downarrow \cong \\ \prod_{i \in I} \pi_p\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right) & \xrightarrow{\cong} & \prod_{i \in I} \pi_p\left(\mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)\right). \end{array} \tag{5.2}$$

The vertical maps are induced by the obvious projections. The right hand vertical map is an isomorphism by Lemma 2.3. The lower horizontal map is the product of the isomorphisms induced by the fibrant replacement equivalences $\mathbf{r}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i))\colon X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i) \to \mathbf{R}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)$ and hence an isomorphism, compare (2.2). We see that it suffices to show that the left hand vertical map is an (almost) isomorphism for $p \leq n$. This is what we will prove next.

For any covariant $\mathsf{Or}(G)$-spectrum $\mathbf{F}$ and any G-CW-complex Z we get a homological spectral sequence $(E^*_{*,*}, d^*_{*,*})$ which converges to $\pi_{p+q}\left(Z_+ \wedge_{\mathsf{Or}(G)} \mathbf{F}\right)$ and whose E^2-term is given by the Bredon homology

$$E^2_{p,q} \;=\; H_p^{\mathbb{Z}\mathsf{Or}(G)}(Z; \pi_q(\mathbf{F}))$$

This follows from [DL98, Theorem 4.7 on page 234, Theorem 7.4 (3) on page 250]. Recall that the *Bredon homology* of a G-CW-complex Z with coefficients in a covariant $\mathbb{Z}\mathsf{Or}(G)$-module M is defined by

$$H_p^{\mathbb{Z}\mathsf{Or}(G)}(Z; M) = H_p\left(C_*^{\mathbb{Z}\mathsf{Or}(G)}(Z) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M\right).$$

This spectral sequence reduces to the classical Atiyah-Hirzebruch spectral sequence if G is the trivial group. A map of covariant $\mathsf{Or}(G)$-spectra $\mathbf{F} \to \mathbf{F}'$ induces a map of the associated spectral sequences. Let $(E^*_{*,*}, d^*_{*,*})$ be the spectral sequence associated to $\mathbf{F} = \prod_{i \in I} \mathbf{E}(c_i)$ and $(E[i]^*_{*,*}, d[i]^*_{*,*})$ be the spectral sequence associated to $\mathbf{F} = \mathbf{E}(c_i)$. Our assumptions imply that all these spectral sequences live in the first quadrant (possibly shifted down by N). This implies that all convergence questions are trivial, i.e. $E^\infty_{p,q} = E^r_{p,q}$ for $r > p + q - N + 1$ and the filtration of $\pi_{p+q}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i))$, whose quotients are given by the E^∞-term, consists of at most $p + q - N + 1$ stages. Notice that $\prod_{i \in I}$ preserves exact sequences. Hence we can take the product $\prod_{i \in I}(E[i]^*_{*,*}, d[i]^*_{*,*})$ and obtain a spectral sequence which converges to $\prod_{i \in I} \pi_{p+q}\left(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(c_i)\right)$ and whose E^2-term is $\prod_{i \in I} H^{\mathbb{Z}\mathsf{Or}(G)}_p(X; \pi_q(\mathbf{F}))$. The usual spectral sequence comparison argument shows that the left hand vertical map in (5.2) which is induced by the various projections $\mathrm{pr}_i \colon \prod_{i \in I} \mathbf{E}(c_i) \to \mathbf{E}(c_i)$ is an (almost) isomorphism for each $p \leq n$, if the maps induced by the projections $\mathrm{pr}_i \colon \prod_{i \in I} \mathbf{E}(c_i) \to \mathbf{E}(c_i)$

$$H^{\mathbb{Z}\mathsf{Or}(G)}_p\left(X_+; \pi_q\left(\prod_{i \in I} \mathbf{E}(c_i)\right)\right) \to \prod_{i \in I} H^{\mathbb{Z}\mathsf{Or}(G)}_p\left(X_+; \pi_q(\mathbf{E}(c_i))\right)$$

are (almost) isomorphisms for all $p, q \in \mathbb{Z}$ with $p + q \leq n$.

The canonical map

$$\pi_q\left(\prod_{i \in I} \mathbf{E}(c_i)\right) \xrightarrow{\cong} \prod_{i \in I} \pi_q(\mathbf{E}(c_i))$$

is bijective by Lemma 2.3 since we assume that the $\mathbf{E}(c_i)$ are Ω-spectra. The assumptions about $\mathbf{E}$ imply that Lemma 3.11 is applicable and hence the $\mathbb{Z}\mathsf{Or}(G)$-modules $\pi_q(\mathbf{E}(c_i))$ and $\prod_{i \in I} \pi_q(\mathbf{E}(c_i))$ factorize through the canonical projection $\mathrm{pr} \colon \mathsf{Or}(G) \to \mathsf{Sub}(G)$. Let $\mathcal{F}_X = \{H \subset G \mid X^H \neq \emptyset\}$ be the family of isotropy groups of X. Let $\mathsf{Sub}(G; \mathcal{F}_X)$ be the full subcategory $\mathsf{Sub}(G)$ whose objects are in $\mathcal{F}_X$. Let $C^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}_*(X)$ be the contravariant $\mathbb{Z}\mathsf{Sub}(G; \mathcal{F}_X)$-chain complex which is the composition of the "cellular chain complex"-functor with the contravariant functor from $\mathsf{Sub}(G; \mathcal{F}_X)$ to the category of CW-complexes which sends an object H to the CW-complex $Z_G H \backslash X^H$. Given a covariant $\mathbb{Z}\mathsf{Sub}(G; \mathcal{F}_X)$-module N, define

$$H^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}_p(X; N) = H_p\left(C^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}_*(X) \otimes_{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)} N\right).$$

If M is a covariant $\mathbb{Z}\mathsf{Sub}(G)$-module, then we obtain a natural isomorphism

$$H^{\mathbb{Z}\mathsf{Or}(G)}_p(X; \mathrm{pr}^* M) \xrightarrow{\cong} H^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}_p(X; M|_{\mathsf{Sub}(G;\mathcal{F}_X)}),$$

where $\mathrm{pr}^* M$ is the $\mathbb{Z}\mathsf{Or}(G)$-module $M \circ \mathrm{pr}$ and $M|_{\mathsf{Sub}(G;\mathcal{F}_X)}$ is the $\mathbb{Z}\mathsf{Sub}(G; \mathcal{F}_X)$-module given by the restriction of M to $\mathsf{Sub}(G; \mathcal{F}_X)$. This follows from the adjunction of induction and restriction [Lüc89, 9.21 on page 169], since the induction of $C^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}_*(X)$ with respect to the inclusion $\mathsf{Sub}(G; \mathcal{F}_X) \to \mathsf{Sub}(G)$ and the induction of $C^{\mathbb{Z}\mathsf{Or}(G)}_*(X)$ with respect to the projection $\mathrm{pr} \colon \mathsf{Or}(G) \to \mathsf{Sub}(G)$ agree. In the sequel we abbreviate $\mathsf{Sub} = \mathsf{Sub}(G; \mathcal{F}_X)$. Hence it remains to prove that for any family $\{M_i \mid i \in I\}$ of covariant $\mathbb{Z}\mathsf{Sub}$-modules the canonical map induced by the various projections $\prod_{i \in I} M_i \to M_i$

$$H^{\mathbb{Z}\mathsf{Sub}}_p\left(X; \prod_{i \in I} M_i\right) \to \prod_{i \in I} H^{\mathbb{Z}\mathsf{Sub}}_p(X; M_i)$$

is an (almost) isomorphism for all $p \in \mathbb{Z}$ with $p \le n - N$. We will prove that for a contravariant projective $\mathbb{Z}\mathsf{Sub}$-chain complex C_* (such as $C_*^{\mathbb{Z}\mathsf{Sub}(G;\mathcal{F}_X)}(X)$) and a family of covariant $\mathbb{Z}\mathsf{Sub}$-modules M_i the canonical map

$$H_p\left(C_* \otimes_{\mathbb{Z}\mathsf{Sub}} \prod_{i\in I} M_i\right) \to \prod_{i\in I} H_p\left(C_* \otimes_{\mathbb{Z}\mathsf{Sub}} M_i\right) \tag{5.3}$$

induced by the various projections $\prod_{i\in I} M_i \to M_i$ is an (almost) $\mathbb{Z}$-isomorphism for all $p \le n-N$, provided that the $\mathbb{Z}$-module $H_p(C_*(H))$ is (almost) finitely generated for any object H in Sub and $p \le n - N$.

We will first treat the case of the addendum, where we assume that the homology modules are almost finitely generated and want to conclude an almost isomorphism.

There are spectral sequences converging to

$$H_{p+q}\left(C_* \otimes_{\mathbb{Z}\mathsf{Sub}} \prod_{i\in I} M_i\right) \quad \text{respectively} \quad H_{p+q}\left(C_* \otimes_{\mathbb{Z}\mathsf{Sub}} M_i\right)$$

whose $E^2_{p,q}$-term are given by

$$\mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}\left(H_q(C_*), \prod_{i\in I} M_i\right) \quad \text{respectively} \quad \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*), M_i).$$

Taking the product over $i \in I$ of the second spectral sequences yields a spectral sequence which converges to $\prod_{i\in I} H_{p+q}\left(C_* \otimes_{\mathbb{Z}\mathsf{Sub}} M_i\right)$ and whose $E^2_{p,q}$-term is $\prod_{i\in I} \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*), M_i)$. By the standard spectral sequence comparison argument it suffices to prove that the canonical map

$$\mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}\left(H_q(C_*), \prod_{i\in I} M_i\right) \longrightarrow \prod_{i\in I} \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*), M_i) \tag{5.4}$$

is an almost isomorphism for each $p, q \in \mathbb{Z}$ with $p + q \le n - N$. The following diagram commutes

$$\begin{array}{ccc} \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}\left(H_q(C_*), \prod_{i\in I} M_i\right) & \longrightarrow & \prod_{i\in I} \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*), M_i) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}\left(H_q(C_*)/\operatorname{tors}, \prod_{i\in I} M_i\right) & \longrightarrow & \prod_{i\in I} \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*)/\operatorname{tors}, M_i). \end{array}$$

As Sub has only finitely many isomorphism classes of objects and $\operatorname{tors}(H_q(C_*(H)))$ is almost trivial for each object H, there is an integer $k \ne 0$ which annihilates $\operatorname{tors}(H_q(C_*(H)))$ for each object H. Hence $\mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(\operatorname{tors}(H_q(C_*)), N)$ is annihilated by k for each covariant $\mathbb{Z}\mathsf{Sub}$-module N. By the long exact Tor-sequence associated to the exact sequence of contravariant $\mathbb{Z}\mathsf{Sub}$-modules

$$0 \to \operatorname{tors}(H_q(C_*)) \to H_q(C_*) \to H_q(C_*)/\operatorname{tors}(H_q(C_*)) \to 0$$

we conclude that both the kernel and the cokernel of the canonical map

$$\mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*), N) \to \mathrm{Tor}_p^{\mathbb{Z}\mathsf{Sub}}(H_q(C_*)/\operatorname{tors}, N)$$

are annihilated by k for any covariant $\mathbb{Z}\mathsf{Sub}$-module N. This implies that in the commutative square above, the vertical arrows are almost isomorphisms. Hence it suffices to prove that the lower horizontal arrow is an almost $\mathbb{Z}$-isomorphism. By assumption the $\mathbb{Z}$-module $H_q(C_*(H))/\operatorname{tors}$ is finitely generated for all $q \le n - N$.

Since we assume that Sub has only finitely many isomorphism classes of objects and $\operatorname{aut}_{\mathsf{Sub}}(H) = W_G H = N_G H/H \cdot Z_G H$ is a finite group for any object $H \in \mathsf{Sub}$, the category Sub is finite in the sense of [Lüc89, Definition 16.1 on page 325]. (Notice that $\mathsf{Or}(G;\mathcal{F}_X)$ does not have this property and therefore it is crucial to pass to $\mathtt{Sub}$.) Hence the category of $\mathbb{Z}\mathtt{Sub}$-modules is noetherian, in the sense that each submodule of a finitely generated $\mathbb{Z}\mathtt{Sub}$-module is itself finitely generated. Moreover a covariant $\mathbb{Z}\mathtt{Sub}$-module L is finitely generated if and only if $L(H)$ is a finitely generated $\mathbb{Z}$-module for each object $H \in \mathtt{Sub}$ (see [Lüc89, Lemma 16.10 on page 327]). By assumption $H_q(C_*(H))/\operatorname{tors}$ is a finitely generated $\mathbb{Z}$-module for each object H in $\mathtt{Sub}$. This implies that there is a finitely generated free (not necessarily finite-dimensional) $\mathbb{Z}\mathtt{Sub}$-resolution F_* for $H_q(C_*)/\operatorname{tors}$ for $q \leq n - N$ (see [Lüc89, Lemma 17.1 on page 339]). Thus we get for $q \leq n - N$ identifications

$$\operatorname{Tor}_p^{\mathbb{Z}\mathtt{Sub}}(H_q(C_*)/\operatorname{tors}, \prod_{i\in I} M_i) \;=\; H_p(F_* \otimes_{\mathbb{Z}\mathtt{Sub}} \prod_{i\in I} M_i)$$

and

$$\begin{aligned}\prod_{i\in I} \operatorname{Tor}_p^{\mathbb{Z}\mathtt{Sub}}(H_q(C_*)/\operatorname{tors}, M_i) \;&=\; \prod_{i\in I} H_p(F_* \otimes_{\mathbb{Z}\mathtt{Sub}} M_i) \\ &=\; H_p\left(\prod_{i\in I} F_* \otimes_{\mathbb{Z}\mathtt{Sub}} M_i\right),\end{aligned}$$

under which the map

$$\operatorname{Tor}_p^{\mathbb{Z}\mathtt{Sub}}(H_q(C_*), \prod_{i\in I} M_i) \to \prod_{i\in I} \operatorname{Tor}_p^{\mathbb{Z}\mathtt{Sub}}(H_q(C_*), M_i)$$

becomes the map induced on homology by the canonical $\mathbb{Z}$-chain map

$$F_* \otimes_{\mathbb{Z}\mathtt{Sub}} \prod_{i\in I} M_i \to \prod_{i\in I} F_* \otimes_{\mathbb{Z}\mathtt{Sub}} M_i. \tag{5.5}$$

But this chain map is a chain isomorphism since each chain module F_i is a finitely generated free $\mathbb{Z}\mathtt{Sub}$-module. Here one uses the fact that the processes of taking sums over a finite index set and products over an arbitrary index set commute.

The case where we assume the homology groups to be finitely generated $\mathbb{Z}$-modules is similar but easier. Using the fact that the category of $\mathbb{Z}\mathtt{Sub}$-modules is noetherian one can replace the complex C_* which appears in (5.3) by a homotopy equivalent complex F_* of projective $\mathbb{Z}\mathtt{Sub}$-modules which is of finite type. For such a complex a map like the one in (5.5) is an isomorphism and hence so is the map (5.3). □

Remark 5.6. Before we finish the proof of Theorem 1.2 and its addendum we want to discuss where the proof above breaks down if we ask for a rational isomorphisms and weaken assumption (D) to the requirement that $H_n(Z_GH\backslash X^H;\mathbb{Q})$ is a finitely generated $\mathbb{Q}$-module for all n.

We would have to show that the map appearing in (5.4) is a rational isomorphism. The following example shows that this is not possible. We claim that for a fixed prime p the canonical map

$$\operatorname{Tor}_1^{\mathbb{Z}}\left(\bigoplus_{m\geq 2} \mathbb{Z}/p^m, \prod_{n\geq 2} \mathbb{Z}/p^n\right) \;\to\; \prod_{n\geq 2} \operatorname{Tor}_1^{\mathbb{Z}}\left(\bigoplus_{m\geq 2} \mathbb{Z}/p^m, \mathbb{Z}/p^n\right)$$

cannot be a rational isomorphism although $(\bigoplus_{m\geq 2} \mathbb{Z}/p^m) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Namely, the source is $\bigoplus_{m\geq 2} \prod_{n\geq 2} \mathbb{Z}/p^{\min(m,n)}$, which vanishes after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$. The target is $\prod_{n\geq 2} \bigoplus_{m\geq 2} \mathbb{Z}/p^{\min(m,n)}$, which contains $\prod_{n\geq 2} \mathbb{Z}/p^n$ as a submodule and hence does not vanish after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$.

Now we finish the proof of Theorem 1.2 and Addendum 1.3.

Proof of Theorem 1.2 and Addendum 1.3. We use induction over the dimension d of Y. If $d = 0$, the claim has already been proved in Lemma 5.1. The induction step from $d-1$ to $d \geq 1$ is done as follows. We can write Y as a pushout

$$\begin{array}{ccc} Y_0 & \xrightarrow{i_1} & Y_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ Y_2 & \xrightarrow{i_2} & Y \end{array}$$

where Y_1 is a $(d-1)$-dimensional $\mathcal{I}$-CW-complex, $Y_0 = \coprod_{i \in I} \operatorname{mor}_{\mathcal{I}}(?, c_i) \times S^{d-1}$, $Y_2 = \coprod_{i \in I} \operatorname{mor}_{\mathcal{I}}(?, c_i) \times D^d$ and i_2 is the inclusion. In particular Y_1 is a $(d-1)$-dimensional CW-complex and Y_2 is $\mathcal{I}$-homotopy equivalent to the 0-dimensional CW-complex $\coprod_{i \in I} \operatorname{mor}_{\mathcal{I}}(?, c_i)$. Hence the induction hypothesis applies to Y_0, Y_1 and Y_2.

The canonical map from the homotopy pushout of $Y_2 \leftarrow Y_0 \rightarrow Y_1$ to Y is a map of $\mathcal{I}$-CW-complexes whose evaluation at each object c is a homotopy equivalence since $i_2(c)$ is a cofibration. Hence it is a $\mathcal{I}$-homotopy equivalence (see [DL98, Corollary 3.5 on page 222]). Since for any covariant $\mathcal{I}$-space Z the functor $\operatorname{map}_{\mathcal{I}}(-, Z)$ sends a homotopy cocartesian square of spaces to a homotopy cartesian one, the following diagram is levelwise homotopy cartesian for each object $G/H \in \mathsf{Or}(G)$

$$\begin{array}{ccc} \operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E}(G/H)) & \longrightarrow & \operatorname{map}_{\mathcal{I}}((Y_1)_+, \mathbf{E}(G/H)) \\ \downarrow & & \downarrow \\ \operatorname{map}_{\mathcal{I}}((Y_2)_+, \mathbf{E}(G/H)) & \longrightarrow & \operatorname{map}_{\mathcal{I}}((Y_0)_+, \mathbf{E}(G/H)). \end{array}$$

We conclude with Lemma 2.6 that the square is homotopy cocartesian. By the same line of argument the diagram

$$\begin{array}{ccc} \operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{R}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(G/?))) & \longrightarrow & \operatorname{map}_{\mathcal{I}}((Y_1)_+, \mathbf{R}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(G/?))) \\ \downarrow & & \downarrow \\ \operatorname{map}_{\mathcal{I}}((Y_2)_+, \mathbf{R}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(G/?))) & \longrightarrow & \operatorname{map}_{\mathcal{I}}((Y_0)_+, \mathbf{R}(X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}(G/?))) \end{array}$$

is homotopy cocartesian. Since the functor $X_+ \wedge_{\mathsf{Or}(G)} -$ respects homotopy cocartesian squares because of the associativity of $\wedge$, we conclude from Lemma 3.8 (ii) that the following diagram is homotopy cartesian

$$\begin{array}{ccc} X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}(Y_+, \mathbf{E}) & \longrightarrow & X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}((Y_1)_+, \mathbf{E}) \\ \downarrow & & \downarrow \\ X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}((Y_2)_+, \mathbf{E}) & \longrightarrow & X_+ \wedge_{\mathsf{Or}(G)} \operatorname{map}_{\mathcal{I}}((Y_0)_+, \mathbf{E}). \end{array}$$

The various transformations $\mathbf{t}(X, Y_i, \mathbf{E})$ and $\mathbf{t}(X, Y, \mathbf{E})$ yield a map from this homotopy cartesian diagram to the one before. Hence they yield a transformation from the long exact homotopy sequences associated in Lemma 2.6 to these homotopy pullbacks. Now Theorem 1.2 follows from the Five-Lemma for (almost) isomorphisms. □

6. The linear version

In this section we state the linear version of our main Theorem 1.2. This will be needed in [LMR03]. The proof in the linear setting is completely analogous to the one for spaces and spectra presented in Section 5, and therefore we omit it.

Theorem 6.1. *Consider the following data:*

- *A functor* $E_*\colon \mathcal{I}^{\mathrm{op}} \times \mathsf{Or}(G) \to \mathrm{ch}(\mathbb{Z}\text{-}\mathsf{MODULES})$ *which factorizes as*
$$E_* = F_* \circ (\mathrm{id} \times \mathcal{G}^G),$$
where $F_*\colon \mathcal{I}^{\mathrm{op}} \times \mathsf{GROUPOIDS} \to \mathrm{ch}(\mathbb{Z}\text{-}\mathsf{MODULES})$ *has the property that for each object* c *in* $\mathcal{I}$ *the functor* $F_*(c,-)$ *sends equivalences of groupoids to homotopy equivalences.*
- *A contravariant chain complex* D_* *of projective* $\mathbb{Z}\mathcal{I}$*-modules.*
- *A* G*-CW complex* X*.*

Suppose that there are numbers d*,* n *and* $N \in \mathbb{Z}$ *with* $d \geq 0$ *such that the following conditions are satisfied:*

(A) D_* *is* d*-dimensional, i.e.* $D_k(c) = 0$ *for all objects* c *in* $\mathcal{I}$ *and for all* $k \in \mathbb{Z}$ *such that* $k < 0$ *or* $k > d$*.*
(B) *The chain complexes* $E_*(c, G/H)$ *are uniformly homologically* $(N-1)$*-bounded below, i.e. for all objects* c *in* $\mathcal{I}$ *and all orbits* G/H *we have* $H_q(E_*(c, G/H)) = 0$ *for* $q < N$*.*
(C) *The* G*-CW complex* X *has only finite isotropy groups and only finitely many orbit types.*
(D) *For each finite subgroup* $H \subset G$ *and every* $p \leq n + d - N$ *the homology group*
$$H_p(Z_G H \backslash X^H; \mathbb{Z})$$
is an (almost) finitely generated $\mathbb{Z}$*-module.*

Then for each $p \leq n$ *the map*
$$H_p\left(C_*^{\mathbb{Z}\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} \hom_{\mathcal{I}}(D_*, E_*)\right) \xrightarrow{H_p(t_*)} H_p\left(\hom_{\mathcal{I}}\left(D_*, C_*^{\mathbb{Z}\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} E_*\right)\right)$$
is an (almost) isomorphism.

Here of course
$$t_*\colon C_*^{\mathbb{Z}\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} \hom_{\mathcal{I}}(D_*, E_*) \to \hom_{\mathcal{I}}\left(D_*, C_*^{\mathbb{Z}\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} E_*\right)$$
is the chain map that sends $x \otimes \phi$ to the map $D_* \to C_*^{\mathbb{Z}\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} E_*$ given by $y \mapsto x \otimes \phi(y)$. There are also linear versions of Examples 1.6 and 1.9.

We mention that Theorem 6.1 above remains true in a more general setting. Namely we can replace the ring of integers $\mathbb{Z}$ by any commutative Noetherian ring R, and $C_*^{\mathbb{Z}\mathsf{Or}(G)}(X)$ by any contravariant $R\mathsf{Or}(G)$-chain complex satisfying the following conditions: C_* is a complex of projectives concentrated in non-negative degrees; the set $S(C_*) := \{(H) \mid S_{G/H} C_p \neq 0 \text{ for some } p \in \mathbb{Z}\}$ is finite; $(H) \in S(C_*)$ implies that H is finite; and for any finite group $H \subset G$ and any $p \leq n+d-N$ the homology group $H_p((\mathrm{pr}_* C_*)(H))$ is an (almost) finitely generated $\mathbb{Z}$-module.

Here (H) denotes the conjugacy class of a subgroup $H \subseteq G$ and $S_{G/H}$ is the splitting functor defined in [Lüc89, 9.26 on page 170]; $pr_* C_*$ is the $R\mathsf{Sub}(G)$-chain complex obtained from C_* by induction (see [Lüc89, 9.15 on page 166])) with the projection $\mathrm{pr}\colon \mathsf{Or}(G) \to \mathsf{Sub}(G)$ defined in (3.10).

The situation above is a special case because $C_*^{\mathbb{Z}\mathsf{Or}(G)}(X)$ is a free (in the sense of [Lüc89, 9.16 on page 167 and page 356]) and hence a projective $\mathbb{Z}\mathsf{Or}(G)$-chain complex, the set $S(C_*^{\mathbb{Z}\mathsf{Or}(G)}(X))$ is just the set of conjugacy classes of isotropy groups of X, and $H_p(\mathrm{pr}_* C_*^{\mathbb{Z}\mathsf{Or}(G)}(X)(H)) \cong H_p(Z_G H \backslash X^H; \mathbb{Z})$.

7. Examples of categories with finite-dimensional classifying spaces

In this section we show two examples of categories that have a finite-dimensional contravariant classifying space, in the sense of Definition 3.4. The second example is relevant to the construction of topological cyclic homology and will play a role in the applications discussed at the end of the introduction. In both examples we will denote by $\mathbb{N}$ the set of non-negative integers $\{0, 1, 2, \ldots\}$.

Example 7.1. Let $\mathcal{N}$ be the category associated to the partially ordered set $\mathbb{N}$, i.e. the category whose objects are the non-negative integers $\mathbb{N}$ and for which $\mathrm{mor}_{\mathcal{N}}(i, j)$ is empty if $i > j$, or consists of precisely one element if $i \leq j$. There is a 1-dimensional model for the contravariant classifying space $E\mathcal{N}$. In fact, define $E\mathcal{N}(i)$ to be the interval $[i, \infty)$. The zero-skeleton is given by the intersection of $[i, \infty)$ with $\mathbb{Z}$; the 1-cells are then obvious. (Compare also [DL98, Example 3.9 on page 224].)

Example 7.2. Let $\mathcal{RF}$ be the following category. The set of objects is $\mathbb{N} = \{0, 1, 2, \ldots\}$. The set of morphisms from m to n is $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i + j = n - m\}$. In particular there are no morphisms from m to n if $m > n$. The identity morphism of n is given by $(0, 0)$. The composition of $(i_0, j_0)\colon n_0 \to n_1$ and $(i_1, j_1)\colon n_1 \to n_2$ is $(i_0 + i_1, j_0 + j_1)\colon n_0 \to n_2$.

Notice that a contravariant functor $\mathbf{E}$ from $\mathcal{RF}$ to the category of spectra is the same as a sequence of spectra $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, ... together with a system of maps

$$\cdots \underset{\mathbf{f}_2}{\overset{\mathbf{r}_2}{\rightrightarrows}} \mathbf{E}_2 \underset{\mathbf{f}_1}{\overset{\mathbf{r}_1}{\rightrightarrows}} \mathbf{E}_1 \underset{\mathbf{f}_0}{\overset{\mathbf{r}_0}{\rightrightarrows}} \mathbf{E}_0$$

such that $\mathbf{r}_i \circ \mathbf{f}_{i+1} = \mathbf{f}_i \circ \mathbf{r}_{i+1}$ holds for all $i \in \mathbb{N}$. Namely, put $\mathbf{E}_i = \mathbf{E}(i)$, $\mathbf{r}_i = \mathbf{E}((0,1)\colon i \to (i+1))$ and $\mathbf{f}_i = \mathbf{E}((1,0)\colon i \to (i+1))$. Topological cyclic homology is defined as the homotopy limit of such a system of spectra, where the $\mathbf{E}_n$'s are given by the fixed points of cyclic p-groups acting on topological Hochschild homology, and the structure maps are the so-called Restriction and Frobenius maps (see [BHM93]).

Proposition 7.3. *There is a two-dimensional model for the contravariant classifying space of* $\mathcal{RF}$.

Proof. Define a contravariant $\mathcal{RF}$-space E by sending an object n to the space $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x, y \geq 0\}$ and a morphism $(i, j)\colon m \to n$ to the map $(x, y) \mapsto (x + i, y + j)$. Obviously each space $E(n)$ is contractible. It remains to construct the $\mathcal{RF}$-CW-structure.

Define $E_0(n) = \mathbb{N} \times \mathbb{N}$, $E_1(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x, y \geq 0, x \text{ or } y \in \mathbb{N}\}$ and $E_2(n) = E$. This yields a filtration $E_0 \subset E_1 \subset E_2 = E$ of contravariant $\mathcal{RF}$-spaces. We claim that this is an $\mathcal{RF}$-CW-structure. We have to construct the relevant pushouts of contravariant $\mathcal{RF}$-spaces. Define for each $n \in \mathbb{N}$ embeddings of contravariant $\mathcal{RF}$-spaces

$$\begin{aligned}
Q_0(n)\colon \mathrm{mor}_{\mathcal{RF}}(?, n) &\to E & (i, j) &\mapsto (i, j);\\
Q_1^h(n)\colon \mathrm{mor}_{\mathcal{RF}}(?, n) \times [0,1] &\to E & ((i, j), s) &\mapsto (i + s, j);\\
Q_1^v(n)\colon \mathrm{mor}_{\mathcal{RF}}(?, n) \times [0,1] &\to E & ((i, j), s) &\mapsto (i, j + s);\\
Q_2(n)\colon \mathrm{mor}_{\mathcal{RF}}(?, n) \times [0,1] \times [0,1] &\to E & ((i, j), s, t) &\mapsto (i + s, j + t).
\end{aligned}$$

We claim that $\coprod_{n \in \mathbb{N}} \mathrm{im}(Q_0) = E_0$ and that we get pushouts of contravariant $\mathcal{RF}$-CW-complexes

$$\begin{array}{ccc}
\coprod_{n\in\mathbb{N}}\Big(\mathrm{mor}_{\mathcal{RF}}(?, n) \times \{0,1\} \coprod \mathrm{mor}_{\mathcal{RF}}(?, n) \times \{0,1\}\Big) & \xrightarrow{\coprod_{n\in\mathbb{N}}(Q_1^h| \coprod Q_1^v|)} & E_0\\
\downarrow & & \downarrow\\
\coprod_{n\in\mathbb{N}}\Big(\mathrm{mor}_{\mathcal{RF}}(?, n) \times [0,1] \coprod \mathrm{mor}_{\mathcal{RF}}(?, n) \times [0,1]\Big) & \xrightarrow{\coprod_{n\in\mathbb{N}}(Q_1^h \coprod Q_1^v)} & E_1
\end{array}$$

and

$$\begin{array}{ccc} \coprod_{n\in\mathbb{N}} \operatorname{mor}_{\mathcal{RF}}(?,n) \times \partial\big([0,1]\times[0,1]\big) & \xrightarrow{\coprod_{n\in\mathbb{N}} Q_2|} & E_1 \\ \downarrow & & \downarrow \\ \coprod_{n\in\mathbb{N}} \operatorname{mor}_{\mathcal{RF}}(?,n) \times [0,1]\times[0,1] & \xrightarrow{\coprod_{n\in\mathbb{N}} Q_2} & E \end{array}$$

where $Q_1^h|$, $Q_1^v|$ and $Q_2|$ denote the obvious restrictions. This finishes the proof of Proposition 7.3. □

8. The homology of classifying spaces for families

This section is devoted to the proof of Proposition 1.7. We first recall some definitions.

A *G-CW-complex* is a CW-complex with G-action such that for an open cell e and $g \in G$ with $g \cdot e \cap e \neq \emptyset$ multiplication with g induces the identity on e. For information about G-CW-complexes we refer for instance to [Lüc89, Sections 1 and 2] and [tD87, Chapter II, Section 1]. A G-CW complex is proper if all isotropy groups are finite. Recall that a *family* $\mathcal{F}$ of subgroups of G is a set of subgroups closed under conjugation and taking subgroups. A G-CW-complex $EG(\mathcal{F})$ all whose isotropy groups belong to $\mathcal{F}$ and whose H-fixed point sets are contractible whenever $H \in \mathcal{F}$ is called a *classifying space for the family* $\mathcal{F}$. This notion is due to tom Dieck (see [tD87, Chapter I, Section 6]). Any G-CW-complex whose isotropy groups belong to $\mathcal{F}$ admits up to G-homotopy precisely one G-map to $EG(\mathcal{F})$. In particular any two models for $EG(\mathcal{F})$ are G-homotopy equivalent. A functorial bar-type construction for $EG(\mathcal{F})$ is given in [DL98, Lemma 7.6].

Lemma 8.1. *Let X be a proper G-CW-complex. Suppose that there exists for each $p \geq 0$ a positive integer $d(p)$ such that for any isotropy group H of X multiplication with $d(p)$ annihilates $\widetilde{H}_p(BH;\mathbb{Z})$.*

Then the canonical projection $\operatorname{pr}_X \colon EG \times_G X \to G\backslash X$ induces for all $p \geq 0$ a $\mathbb{Z}$-almost isomorphism

$$H_p(\operatorname{pr}_X;\mathbb{Z}) \colon H_p(EG \times_G X;\mathbb{Z}) \to H_p(G\backslash X;\mathbb{Z}).$$

Proof. Let X_n be the n-skeleton of X. Then both maps $EG \times_G X_n \to EG \times_G X$ and $G\backslash X_n \to G\backslash X$ are n-connected and induce isomorphisms on $H_p(-;\mathbb{Z})$ for $p < n$. Hence it suffices to prove the claim for each n-skeleton X_n. This will be done by induction over n. The induction beginning $n = -1$ is trivial since $X_{-1} = \emptyset$. The induction step from n to $n+1$ is done as follows.

We can write X_{n+1} as a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I} G/H_i \times S^n & \xrightarrow{\coprod_{i\in I} q_i} & X_n \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G/H_i \times D^{n+1} & \xrightarrow{\coprod_{i\in I} Q_i} & X_{n+1}. \end{array}$$

Applying $EG \times_G -$ and $G\backslash -$ one obtains two pushout squares whose left vertical arrows are again cofibrations. The projections pr_Z where Z runs through the four corners of the G-pushout square yield a map between the two new squares. By a Five-Lemma argument, the Künneth formula and the fact that $H_p(-;\mathbb{Z})$ satisfies the disjoint union axiom for arbitrary index sets, the induction step follows from the assumption that the map $H_p(\operatorname{pr}_{G/H};\mathbb{Z}) \colon H_p(EG \times_G G/H;\mathbb{Z}) \to H_p(\{pt\};\mathbb{Z})$ is surjective and its kernel is annihilated by $d(p)$ for each isotropy group H of X. □

Lemma 8.2. *Let $\mathcal{F}$ be a family of finite subgroups of G. Fix an element $H \in \mathcal{F}$. Suppose that there exists for each $p \geq 0$ a positive integer $d(p)$ such that for any $K \in \mathcal{F}$ with $H \subset K$ the annihilator ideal of $\widetilde{H}_p(B(Z_G H \cap K);\mathbb{Z})$ contains $d(p)$. Then $H_p(Z_G H\backslash EG(\mathcal{F})^H;\mathbb{Z})$ is an almost finitely generated $\mathbb{Z}$-module for all $p \geq 1$.*

Proof. Fix $H \in \mathcal{F}$. The projection $EZ_G H \times_{Z_G H} EG(\mathcal{F})^H \to BZ_G H$ is a homotopy equivalence since $EG(\mathcal{F})^H$ is contractible. Now apply Lemma 8.1 to the proper $Z_G H$-CW-complex $EG(\mathcal{F})^H$ and the projection $EZ_G H \times_{Z_G H} EG(\mathcal{F})^H \to Z_G H\backslash EG(\mathcal{F})^H$. □

We are now prepared to prove Proposition 1.7.

Proof of Proposition 1.7. Obviously $EG(\mathcal{F})$ has finite isotropy groups and only finitely many orbit types. There is a number d such that the order of any element $H \in \mathcal{F}$ divides d. Hence $\widetilde{H}_p(BK;\mathbb{Z})$ is annihilated by d for each $K \in \mathcal{F}$ and p [Bro94, Section III.10]. Now apply Lemma 8.2. □

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