THE ORE CONDITION, AFFILIATED OPERATORS, AND THE LAMPLIGHTER GROUP

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Abstract. Let $G = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ be the so called lamplighter group and k a commutative ring. We show that kG does not have a classical ring of quotients (i.e. does not satisfy the Ore condition). This answers a Kourovka notebook problem. Assume that kG is contained in a ring R in which the element $1 - x$ is invertible, with x a generator of $\mathbb{Z} \subset G$. Then R is not flat over kG. If $k = \mathbb{C}$, this applies in particular to the algebra $\mathcal{U}G$ of unbounded operators affiliated to the group von Neumann algebra of G.

We present two proofs of these results. The second one is due to Warren Dicks, who, having seen our argument, found a much simpler and more elementary proof, which at the same time yielded a more general result than we had originally proved. Nevertheless, we present both proofs here, in the hope that the original arguments might be of use in some other context not yet known to us.

1. Notation and Terminology

Let A be a group. Then $A \wr \mathbb{Z}$ indicates the Wreath product with base group $B = \bigoplus_{i=-\infty}^{\infty} A_i$, where $A_i = A$ for all i. Thus $A \wr \mathbb{Z}$ is isomorphic to the split extension $B \rtimes \mathbb{Z}$ where if x is a generator for $\mathbb{Z}$, then $x^i A_0 x^{-i} = A_i$. Also we identify A with A_0. In the case $A = \mathbb{Z}/2\mathbb{Z}$ above, $A \wr \mathbb{Z}$ is often called the lamplighter group.

Let kG denote the group algebra of the group G over the field k, and let $\alpha \in kG$. Write $\alpha = \sum_{g\in G} a_g g$ where $a_g \in k$. Then the support of α is $\{g \in G \mid a_g \neq 0\}$, a finite subset of G.

The augmentation ideal of a group algebra will denoted by the small German letter corresponding to the capital Latin letter used to name the group. Thus if k is a field and G is a group, then $\mathfrak{g}$ is the ideal of kG which has k-basis $\{g-1 \mid g \in G\backslash 1\}$. For the purposes of this paper, it will always be clear over which field we are working when considering augmentation ideals.

1. **Definition.** A ring R satisfies the *Ore condition* if for any $s, t \in R$ with s a non-zerodivisor, there are $x, y \in R$ with x a non-zerodivisor such that $sy = tx$. Formally, this means that $s^{-1}t = yx^{-1}$, and the condition makes sure that a classical ring of quotients, inverting all non-zerodivisors of R, can be constructed.

Then R is an Ore ring means that R satisfies the Ore condition. Equivalently this means that if S is the set of non-zerodivisors of R, then given $r \in R$ and $s \in S$, we can find $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$. In this situation we can form the Ore localization RS^{-1}, which consists of elements rs^{-1} where $r \in R$ and $s \in S$.

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The above definition is really the right Ore condition, though for group rings the right and left Ore conditions are equivalent.

In this note, we study, which group rings satisfy the Ore condition. It is well known that this fails for a non-abelian free group.

On the other hand, abelian groups evidently satisfy the Ore condition. In this note we show that the lamplighter groups (and relatives) do not satisfy it. Note, however, that these groups are 2-step solvable, i.e. close relatives of abelian groups.

Let G be a group, let $\mathcal{N}(G)$ denote the group von Neumann algebra of G, let $\mathcal{U}(G)$ denote the algebra of unbounded linear operators affiliated to $\mathcal{N}(G)$, and let $\mathcal{D}(G)$ denote the division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$. For more information on these notions, see [5, §8 and §9] and [6, §8 and §10]. In particular we have the inclusion of $\mathbb{C}$-algebras

$$\mathbb{C}G \subseteq \mathcal{D}(G) \subseteq \mathcal{U}(G)$$

and it is natural to ask whether $\mathcal{D}(G)$ and $\mathcal{U}(G)$ are flat over $\mathbb{C}G$.

We use the following well-known and easily verified statement without further comment in this paper. If k is a field and $g \in G$ has infinite order, then $1-g$ is a non-zerodivisor in kG, and in the case $k = \mathbb{C}$ we also have that $1-g$ is invertible in $\mathcal{D}(G)$.

If H is the nonabelian free group of rank 2, then we have an exact sequence of $\mathbb{C}H$-modules $0 \to \mathbb{C}H^2 \to \mathbb{C}H \to \mathbb{C} \to 0$. It was shown in [4, Theorem 1.3] (see also [5, Theorem 10.2] and [6, Theorem 10.19 and Lemma 10.39]) that $\mathcal{D}(H)$ is a division ring, so when we apply $\otimes_{\mathbb{C}H}\mathcal{D}(H)$ to this sequence, it becomes $\mathcal{D}(H)^2 \to \mathcal{D}(H) \to 0 \to 0$, since $(1-x)$ is invertible in $D(H)$ for every element $1 \neq x \in H$, but $(1-x)$ acts as the zero operator on $\mathbb{C}$. By counting dimension, we get from this (adding the kernel) a short exact sequence $0 \to \mathcal{D}(H) \to \mathcal{D}(H)^2 \to \mathcal{D}(H) \to 0$. Suppose Q is a ring containing $\mathcal{D}(H)$ which is flat over $\mathbb{C}H$. Then applying $\otimes_{\mathcal{D}(H)}Q$ to the previous sequence, we obtain the exact sequence $0 \to Q \to Q^2 \to Q \to 0$ which contradicts the hypothesis that Q is flat over $\mathbb{C}H$ (in the latter case we would have obtained $0 \to Q^2 \to Q \to 0 \to 0$). In particular if G is a group containing H, then neither $\mathcal{D}(G)$ nor $\mathcal{U}(G)$ is flat over $\mathbb{C}H$. Since $\mathbb{C}G$ is a free $\mathbb{C}H$-module, we conclude that neither $\mathcal{D}(G)$ nor $\mathcal{U}(G)$ is flat over $\mathbb{C}G$.

To sum up the previous paragraph, neither $\mathcal{D}(G)$ nor $\mathcal{U}(G)$ is flat over $\mathbb{C}G$ when G contains a nonabelian free group. On the other hand it was proven in [8, Theorem 9.1] (see also [6, Theorem 10.84]) that if G is an elementary amenable group which has a bound on the orders of its finite subgroups, then $\mathcal{D}(G)$ and $\mathcal{U}(G)$ are flat over $\mathbb{C}G$. Furthermore it follows from [6, Theorems 6.37 and 8.29] that if G is amenable (in particular if G is the lamplighter group), then at least $\mathcal{U}(G)$ is "dimension flat" over $\mathbb{C}G$.

2. Original results and proof

We shall prove

2. Theorem. Let $H \neq 1$ be a finite group and let G be a group containing $H \wr \mathbb{Z}$. Then neither $\mathcal{D}(G)$ nor $\mathcal{U}(G)$ is flat over $\mathbb{C}G$.

Closely related to this question is the problem of when the group algebra kG of the group G over the field k is an Ore ring (in other words does kG have a classical ring of quotients; see Definition 1). Our next result answers a Kourovka Notebook problem [7, 12.47], which was proposed by the first author. The problem there asks if kG has a classical quotient ring in the case $G = \mathbb{Z}_p \wr \mathbb{Z}$ where p is a prime.

3. **Theorem.** Let $H \neq 1$ be a finite group, let k be a field and let G be a group containing $H \wr \mathbb{Z}$. Then kG is not an Ore ring.

4. **Lemma.** Let R be a subring of the ring S and let P be a projective R-module. If $P \otimes_R S$ is finitely generated as an S-module, then P is finitely generated.

Proof. Since P is projective, there are R-modules Q, F with F free such that $P \oplus Q = F$. Let $\mathcal{E}$ be a basis for F. Now $P \otimes_R S \oplus Q \otimes_R S = F \otimes_R S$ and since $P \otimes_R S$ is finitely generated, there exist $e_1, \dots, e_n \in \mathcal{E}$ such that $P \otimes_R S \subseteq e_1 S + \dots + e_n S$. We now see that every element p of P is

(i) An R-linear combination of elements in $\mathcal{E}$ ($\implies p \otimes 1 = \sum_{e \in E} e \otimes r_e$).
(ii) An S-linear combination of $e_1, \dots, e_n$ ($p \otimes 1 = \sum e_i \otimes s_i$).

Set $E = e_1 R + \dots + e_n R$. Comparing coefficients, the above shows that $P \subseteq E$ and it follows that we have the equation $P \oplus (Q \cap E) = E$. Therefore P is a finitely generated R-module as required. □

5. **Lemma.** Let H be a nontrivial finite group, let k be a field with characteristic which does not divide $|H|$, and let $G = H \wr \mathbb{Z}$. If Q is a ring containing kG such that $k \otimes_{kG} Q = 0$, then $\operatorname{Tor}^{kG}(k, Q) \neq 0$.

Proof. Let d denote the minimum number of elements required to generate G. Then we have exact sequences

$$0 \longrightarrow \mathfrak{g} \longrightarrow kG \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow P \longrightarrow kG^d \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Suppose to the contrary $\operatorname{Tor}^{kG}(k, Q) = 0$. Then the following sequence is exact:

$$0 \longrightarrow \mathfrak{g} \otimes_{kG} Q \longrightarrow kG \otimes_{kG} Q \longrightarrow k \otimes_{kG} Q = 0.$$

Since kG has homological dimension one [1, p. 70 and Proposition 4.12], we also have $0 = \operatorname{Tor}_2^{kG}(k, Q) = \operatorname{Tor}_1^{kG}(\mathfrak{g}, Q)$. Hence we have another exact sequence

$$0 \longrightarrow P \otimes_{kG} Q \longrightarrow kG^d \otimes_{kG} Q \longrightarrow \mathfrak{g} \otimes_{kG} Q \longrightarrow 0. \tag{6}$$

We rewrite this to get the exact sequence

$$0 \longrightarrow P \otimes_{kG} Q \longrightarrow Q^d \longrightarrow Q \longrightarrow 0$$

and we conclude that $P \otimes_{kG} Q$ is a finitely generated Q-module, which is projective since the sequence (6) splits. Now kG has cohomological dimension ≤ 2 [1, p. 70 and Theorem 4.6 and Proposition 4.12], hence P is a projective kG-module. (To see this, let $P' \to P$ be a map from a projective kG-module P' onto P. Because $\operatorname{Ext}^2_{kG}(k, Q) = 0$, this extends to a map $P' \to kG^d \to P$. Since the image of the first arrow is P, this gives a split of the injection $P \to kG^d$, i.e. P is projective.) Therefore P is finitely generated by Lemma 4. But it is well known that $P \cong R/[R, R] \otimes k$ as kG-modules, where $1 \to R \to F \to G \to 1$ is an exact sequence of groups and F is a free group with d generators (compare the proof of [3, (5.3) Theorem]). Consequently, G is almost finitely presented over k as defined in [2] if P is a finitely generated kG-module. But this is a contradiction to [2, Theorem A or Theorem C] (here we use $H \neq 1$), where the structure of almost finitely presented groups such as G is determined. □

7. **Corollary.** *Let H be a nontrivial finite group, let k be a field with characteristic which does not divide $|H|$, and let G be a group containing $H \wr \mathbb{Z}$. Let x be a generator for $\mathbb{Z}$ in G. If Q is a ring containing kG and $1-x$ is invertible in Q, then Q is not flat over kG.*

Proof. Set $L = H \wr \mathbb{Z}$. Since $1-x$ is invertible in Q, we have $\mathbb{C} \otimes_{\mathbb{C}L} Q = 0$, so from Lemma 5 we deduce that $\operatorname{Tor}^{kL}(k, Q) \neq 0$. Furthermore kG is flat over kL, consequently $\operatorname{Tor}^{kG}(k \otimes_{kL} kG, Q) \neq 0$ by [1, p. 2], in particular Q is not flat over kG as required. □

Proof of Theorem 2. Set $L = H \wr \mathbb{Z}$ and let x be a generator for $\mathbb{Z}$ in L. Then $1-x$ is a non-zerodivisor in $\mathcal{N}(G)$ and therefore is invertible in $\mathcal{U}(G)$, and hence also invertible in $\mathcal{D}(G)$. The result now follows from Corollary 7 □

8. **Lemma.** Let p be a prime, let k be a field of characteristic p, let A be a group of order p, let $G = A \wr \mathbb{Z}$ with base group B, let a be a generator for A, and let $x \in G$ be a generator for $\mathbb{Z}$. Then there does not exist $\alpha, \sigma \in kG$ with $\sigma \notin \mathfrak{b}kG$ such that $(1-a)\sigma = (1-x)\alpha$.

Proof. Suppose there does exist α and σ as above. Observe that $\alpha \in \mathfrak{b}kG$, since $\mathfrak{b}kG$ is the kernel of the map $kG \to k\mathbb{Z}$ induced from the obvious projection $G = A \wr \mathbb{Z} \to \mathbb{Z}$ mapping x to x, and since $1-x$ is not a zerodivisor in $k\mathbb{Z}$.

Thus we may write $\sigma = \tau + \sum_i s_i x^i$ where $s_i \in k$, $\tau \in \mathfrak{b}kG$ and not all the s_i are zero, and $\alpha = \sum_i x^i \alpha_i$ where $\alpha_i \in \mathfrak{b}$. Then the equation $(1-a)\sigma = (1-x)\alpha$ taken mod $\mathfrak{b}^2 kG$ yields

$$\sum_i (1-a) s_i x^i = \sum_i (1-x) x^i \alpha_i.$$

Set $b_i = 1 - x^{-i} a x^i$. By equating the coefficients of x^i, we obtain $s_i b_i = \alpha_i - \alpha_{i-1}$ mod $\mathfrak{b}^2 kG$ for all i. Since $\alpha_i \neq 0$ for only finitely many i, we deduce that $\sum s_i b_i = 0$ mod $\mathfrak{b}^2$. Also $\mathfrak{b}/\mathfrak{b}^2 \cong B \otimes k$ as k-vector spaces via the map induced by $b - 1 \mapsto b \otimes 1$ and the elements $x^{-i} a x^i \otimes 1$ are linearly independent in $B \otimes k$, consequently the b_i are linearly independent over k mod $\mathfrak{b}^2$. We now have a contradiction and the result follows. □

9. **Lemma.** Let k be a field, let H be a locally finite subgroup of the group G, and let $\alpha_1, \dots, \alpha_n \in \mathfrak{h}kG$. Then there exists $\beta \in kH \setminus 0$ such that $\beta\alpha_i = 0$ for all i.

Proof. Let T be a right transversal for H in G, so G is the disjoint union of $\{Ht \mid t \in T\}$. For each i, we may write $\alpha_i = \sum_{t \in T} \beta_{it} t$ where $\beta_{it} \in \mathfrak{h}$. Let B be the subgroup generated by the supports of the β_{it}. Then B is a finitely generated subgroup of H and thus B is a finite p-group. Also $\beta_{it} \in \mathfrak{b}$ for all i, t. Set $\beta = \sum_{b \in B} b$. Then $\beta\mathfrak{b} = 0$ and the result follows. □

10. **Lemma.** Let p be a prime, let k be a field of characteristic p, let A be a group of order p, and let G be a group containing $A \wr \mathbb{Z}$. Then kG does not satisfy the Ore condition.

Proof. Let $H = A \wr \mathbb{Z}$, which we may regard as a subgroup of G, with $x \in H \subset G$ a generator of $\mathbb{Z}$. Let B be the base group of H and let T be a right transversal for H in G, so G is the disjoint union of $\{Ht \mid t \in T\}$. Note that $1-x$ is a non-zerodivisor in kG. Let a be a generator for A. Suppose $(1-a)\sigma = (1-x)\alpha$ where $\alpha, \sigma \in kG$ and

σ is a non-zerodivisor in kG. Then we may write $\alpha = \sum_{t\in T} \alpha_t t$ and $\sigma = \sum_{t\in T} \sigma_t t$ with $\alpha_t, \sigma_t \in kH$, and then we have

$$(1-a)\sigma_t = (1-x)\alpha_t$$

for all $t \in T$. If $\sigma_t \in \mathfrak{b}kH$ for all $t \in T$, then by Lemma 9 we see that there exists $\beta \in kB \setminus 0$ such that $\beta\sigma_t = 0$ for all t. This yields $\beta\sigma = 0$ which contradicts the hypothesis that σ is a non-zerodivisor. Therefore we may assume that there exists $s \in T$ such that $\sigma_s \notin \mathfrak{b}kH$. But now the equation

$$(1-a)\sigma_s = (1-x)\alpha_s$$

contradicts Lemma 8, and the result follows. □

Proof of Theorem 3. If the characteristic of k is p and divides $|H|$, then the result follows from Lemma 10. On the other hand if p does not divide $|H|$, we suppose that kG satisfies the Ore condition. Then kG has a classical ring of quotients Q. It is a well known fact that such a classical ring of quotients is always flat over its base (compare [9, p. 57]). In particular, Q is flat over kG. Let x be a generator for $\mathbb{Z}$ in G. Since $1-x$ is a non-zerodivisor in kG, we see that $1-x$ is invertible in Q. We now have a contradiction by Lemma 5 and the result follows. □

3. Warren Dicks' proof

In this section, we give our account of Warren Dicks' proof and generalization of the results presented in Section 2. All credit has to go to him, all mistakes are ours. More precisely, we prove the following theorem (for elements x, y in a group, we use the commutator convention $[x, y] = xyx^{-1}y^{-1}$):

11. Theorem. *Let $2 \le d \in \mathbb{Z}$, let $G = \langle a, x \mid a^d = 1,\ [a, x^l a x^{-l}] = 1;\ l = 1, 2, \ldots\rangle$ be the wreath product $\mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}$, and let k be a nonzero commutative ring with unit. If $u, v \in kG$ are such that $u(a-1) = v(x-1)$, then u is a left zerodivisor in kG. In particular, kG does not satisfy the Ore condition.*

Proof. For the last statement, note that $(x-1)$ is a non-zerodivisor in kG because x has infinite order.

Recall that any presentation $H = \langle S \mid R\rangle$ of a group H gives rise to an exact sequence of left kH-modules

$$\bigoplus_{r\in R} kH \xrightarrow{F} \bigoplus_{s\in S} kH \xrightarrow{\alpha} kH \xrightarrow{\epsilon} k \to 0. \tag{12}$$

Here, ϵ is the augmentation map defined by $\epsilon(h) = 1$ for all $h \in H$, α maps $u\overline{s} \in \bigoplus_{s\in S} kH$ (with $u \in kH$ and $\overline{s}$ the canonical basis element corresponding to the generator $s \in S$) to $u(s-1) \in kH$, and the map F is given by the Fox calculus, i.e. $u\overline{r}$ (where $u \in kH$ and $\overline{r}$ is the canonical basis element corresponding to the relator $r \in R$) is mapped to

$$\sum_{s\in S} u\frac{\partial r}{\partial s}\overline{s}.$$

If $r = s_{i_1}^{\epsilon_1} \ldots s_{i_n}^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$, then the Fox derivative is defined by

$$\frac{\partial r}{\partial s} := \sum_{k=1}^{n} s_{i_1}^{\epsilon_1} \ldots s_{i_{k-1}}^{\epsilon_{k-1}} \frac{\partial s_{i_k}^{\epsilon_k}}{\partial s}.$$

Here $\partial s/\partial s = 1$, $\partial s^{-1}/\partial s = -s^{-1}$ and $\partial t^\epsilon/\partial s = 0$ if $s \neq t \in S$ and $\epsilon = \pm 1$.

The above sequence can be considered as the cellular chain complex (with coefficients k) of the universal covering of the standard presentation CW-complex given by $\langle S \mid R\rangle$. Since this space is 2-connected, its first homology vanishes and its zeroth homology is isomorphic to k (by the augmentation), which implies that the sequence is indeed exact. An outline of the proof can be found in [3, II.5 Exercise 3] or in [3, IV.2, Exercises]

Now we specialize to the group G. Let us write $r_0 = a^d$ and $r_l = [a, x^l a x^{-l}]$ for $l \geq 1$. Suppose $u, v \in kG$ with $u(a-1) = v(x-1)$. Then $\alpha(u\overline{a} - v\overline{x}) = 0$. Exactness implies that there exists a positive integer N and $z_l \in kG$ $(0 \leq l \leq N)$ such that $F(\sum_l z_l \overline{r_l}) = u\overline{a} - v\overline{x}$. We want to prove that u is a zerodivisor. Therefore we are concerned only with the $\overline{a}$ component of $F(\sum_{0\leq l\leq N} z_l\overline{r_l})$. This means we first must compute $\partial r/\partial a$ for all the relators in our presentation of G. This is easily done:

$$\frac{\partial a^d}{\partial a} = 1 + a + \cdots + a^{d-1} \tag{13}$$

$$\frac{\partial [a, x^l a x^{-l}]}{\partial a} = \frac{\partial (a x^l a x^{-l} a^{-1} x^l a^{-1} x^{-l})}{\partial a} \tag{14}$$

$$= 1 + a x^l - a x^l a x^{-l} a^{-1} - a x^l a x^{-l} a^{-1} x^l a^{-1}, \tag{15}$$

the latter for $l > 0$. Using the fact that $x^l a x^{-l}$ commutes with a for each l, we can simplify:

$$\frac{\partial [a, x^l a x^{-1}]}{\partial a} = 1 + a x^l - x^l a x^{-l} - x^l = x^l(x^{-l} a x^l - 1) - (x^l a x^{-l} - 1).$$

Since u is the coefficient of $\overline{a}$ in $F(\sum_{0\leq l\leq N} z_l\overline{r_l})$ we see that

$$u = z_0(1 + a + \cdots + a^{d-1}) + \sum_{n=1}^{N} z_n x^n (x^{-n} a x^n - 1) - z_n(x^n a x^{-n} - 1).$$

Now let $C = \langle x^n a x^{-n} \mid 1 \leq |n| \leq N\rangle$, a finite subgroup of the base group $\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}/d\mathbb{Z}$. Then $C\langle a\rangle = C \times \langle a\rangle$. Set $\gamma = (1-a)\sum_{c\in C} c$. Then $\gamma \neq 0$ and $\beta\gamma = 0$. We conclude that u is a left zerodivisor in kG and the result follows. □

16. **Corollary.** *Let $2 \leq d \in \mathbb{Z}$ and let G be a group containing $\mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}$, and let $x \in G$ be a generator for $\mathbb{Z}$. Let k be a nonzero commutative ring with unit and let Q be a ring containing kG such that $1 - x$ becomes invertible in Q. Then Q is not flat over kG.*

Proof. Since kG is free, hence flat, as left $k[\mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}]$-module, by [1, p. 2] we may assume that $G = \mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}$. Now tensor the exact sequence (12) over kG with Q. Then the resulting sequence will also be exact and $\overline{a} - (a-1)(x-1)^{-1}\overline{x}$ will be in the kernel of $\mathrm{id}_Q \otimes \alpha$. Therefore $\overline{a} - (a-1)(x-1)^{-1}\overline{x}$ will be in the image of $\mathrm{id}_Q \otimes F$. However the proof of Theorem 11 shows that if $u\overline{a} - v\overline{x}$ is in the image of $\mathrm{id}_Q \otimes F$, then u is a zerodivisor; the only change is that we want $z_l \in Q$ rather than $z_l \in kG$. Since $u = 1$ in this situation, which is not a zerodivisor, the tensored sequence is not exact. □

In the case k is a subfield of $\mathbb{C}$ and $Q = \mathcal{D}(G)$ or $\mathcal{U}(G)$, Corollary 16 tells us that if G contains $\mathbb{Z}/d\mathbb{Z} \wr \mathbb{Z}$, then $\mathcal{D}(G)$ and $\mathcal{U}(G)$ are not flat over kG.

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