“The Universal Functorial Lefschetz Invariant”

by

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Abstract

We introduce the universal functorial Lefschetz invariant for endomorphisms of finite CW-complexes in terms of Grothendieck groups of endomorphisms of finitely generated free modules. It encompasses invariants like Lefschetz number, its generalization to the Lefschetz invariant, Nielsen number and L^2-torsion of mapping tori. We examine its behaviour under fibrations.

Key words: Universal functorial Lefschetz invariants, Grothendieck group of endomorphisms of modules, transfer maps

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Introduction

Given an endomorphism $f : X \longrightarrow X$ of finite CW-complex X (with $\pi_0(f) = \mathrm{id}$), we introduce an abelian group $U(f)$ and an invariant $u(f)$ in 2.2 based on an algebraic invariant for chain complexes which we will define in Definition 1.2. The algebraic version of $U(f)$ for a ring R with an endomorphism $\phi : R \longrightarrow R$ is the Grothendieck group of ϕ-linear endomorphisms of finitely generated free R-modules. The pair (U, u) is a *functorial Lefschetz invariant* on the category $\mathrm{End}(\mathcal{C})$ of endomorphisms $f : X \longrightarrow X$ of finite CW-complexes (with $\pi_0(f) = \mathrm{id}$) in the sense of Definition 2.3, i.e. U is a functor from $\mathrm{End}(\mathcal{C})$ into the category of abelian groups and for any object $f : X \longrightarrow X$ there is an invariant $u(f) \in U(f)$ such that (U, u) satisfies a push out formula, for a morphism $h : (X, f) \longrightarrow (Y, g)$ in $\mathrm{End}(\mathcal{C})$ $U(h) : U(f) \longrightarrow U(g)$ depends only on the homotopy class of h, $u(\mathrm{id} : \emptyset \longrightarrow \emptyset) = 0$ and $U(h)$ maps $u(f)$ to $u(g)$ and is bijective if h is a homotopy equivalence. We call a functorial Lefschetz invariant (A, a) *universal* if for any other functorial Lefschetz invariant (B, b) there is precisely one natural transformation $\xi : A \longrightarrow B$ which satisfies $\xi(f)(a(f)) = b(f)$ for all objects $f : X \longrightarrow X$ in $\mathrm{End}(\mathcal{C})$ (see Definition 2.4). One of the main results of the paper is proven in Section 4

Theorem 2.5. *The pair (U, u) is the universal functorial Lefschetz invariant for endomorphisms of finite CW-complexes.* □

The universal functorial Lefschetz invariant is unique and carries maximal information compared with any other functorial Lefschetz invariant. In particular any result about the universal functorial Lefschetz invariant carries over to any other functorial Lefschetz invariant. We will give in Section 3 examples of functorial Lefschetz invariants such as the (classical) Lefschetz number, its generalization to the Lefschetz invariant and the Nielsen number which have been extensively studied in the literature, and of a new one which is essentially given by the L^2-torsion of the mapping torus. The last one can be used to compute the volume of a hyperbolic closed 3-manifold given by a mapping torus of a pseudo-Anosov selfhomeomorphism f of a closed hyperbolic 2-dimensional manifold and hence the volume can be derived from $u(f)$. We do not know how to get the volume from the other functorial Lefschetz invariants mentioned above.

In Section 5 we investigate the behaviour of $u(f)$ under fibrations by assigning to a fibration a transfer map 5.6 which computes the invariants on the total space level by the one on the basis level (Theorem 5.8). We investigate this transfer map algebraically in Section 6. We obtain a down-up-formula (see Lemma 6.5), give explicit calculations for S^n as fiber (see Example 6.9) and prove

Theorem 6.7. *Let $f : X \longrightarrow X$ be a S^1-endomorphism of a finite S^1-CW-complex X. Denote by $i : (X^{S^1}, f^{S^1}) \longrightarrow (X, f)$ the morphism in $End(\mathcal{C})$ induced by the inclusion of the fixed point set. Then we have:*

$$U(i)(u(X^{S^1}, f^{S^1})) = u(X, f) \qquad \in U(X, f).$$

In particular $u(X, f)$ vanishes if the S^1-action has no fixed points. □

In Section 7 we construct a diagram

7.1
$$\begin{array}{ccc} U(R,\phi) & \xrightarrow{\tau} & U(\widehat{R}[t,t^{-1}]_{\widehat{\phi}}, \mathrm{id}) \\ \eta\downarrow & & \eta\downarrow \\ \prod_{m\geq 1}\Lambda(R,\phi^m) & \xrightarrow{\tau=\prod_{m\geq 1}\tau_m} & \prod_{m\geq 1}\Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}}, \mathrm{id}) \end{array}$$

The horizontal maps are given by the mapping torus construction whereas the vertical maps are given by the Lefschetz invariants of the various iterates of an endomorphism. In the case where R is commutative and $\phi = \mathrm{id}$ this corresponds to the passage to the Lefschetz Zeta-function. We discuss the question which of the maps are injective.

The paper is organized as follows:

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1 The universal Lefschetz invariant for chain complexes

In this section we introduce the universal Lefschetz invariant for finite free chain complexes. This is the algebraic version of the universal Lefschetz invariant for spaces which we will introduce in Definition 2.4. Modules are always left-modules unless explicitely stated differently.

Recall that an R-chain complex is *finitely generated* resp. *free* if each of its chain modules have this property. It is called *finite-dimensional* if C_n is zero for $n < 0$ and $n > N$ for some N. We call it *finite* if it is both finitely generated and finite-dimensional. Given an R-module F, let $\mathrm{el}(F, n)$ be the associated *n-dimensional elementary chain complex* which is concentrated in dimension n and $n - 1$ and has as n-th differential $\mathrm{id} : F \longrightarrow F$.

Definition 1.1. Let R be a an associative ring with unit and $\phi : R \longrightarrow R$ be a ring homomorphism respecting the unit. An *additive invariant* for the category of ϕ-endomorphisms of finite free R-chain complexes is a pair (A, a) which consists of an abelian group A and a function which assigns to each R-chain map $f : C \longrightarrow \phi^*C$ for C a finite free R-chain complex an element

$$a(f) \in A$$

where ϕ^*C_* is the R-chain complex obtained from C_* by restriction with ϕ. such that the following holds:

1. Additivity

   For a commutative diagram of finite free R-chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i} & D & \xrightarrow{p} & E & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & \phi^*C & \xrightarrow{\phi^*i} & \phi^*D & \xrightarrow{\phi^*p} & \phi^*E & \longrightarrow & 0 \end{array}$$

   we have

$$a(f) - a(g) + a(h) = 0;$$

2. Homotopy invariance

   Let $f, g : C \longrightarrow \phi^*C$ be R-chain maps of finite free R-chain complexes. If f and g are R-chain homotopic, then

$$a(f) = a(g);$$

3. Elementary chain complexes

   We have for any finitely generated free R-module F and $n \geq 1$

$$a(0 : \mathrm{el}(F, n) \longrightarrow \phi^* \mathrm{el}(F, n)) \;=\; 0.$$

We call an additive invariant (U, u) *universal* if for any additive invariant (A, a) there is precisely one homomorphism $\xi : U \longrightarrow A$ of abelian groups satisfying $\xi(u(f)) = a(f)$ for all $f : C \longrightarrow \phi^* C$. $\square$

In particular an R-map $f : F \longrightarrow \phi^* Q$ is map of abelian groups such that $f(rx) = \phi(r) f(x)$ holds for all $r \in R$ and $x \in F$.

Definition 1.2. Let R be a ring and $\phi : R \longrightarrow R$ be a ring homomorphism. Let $U(R, \phi)$ be the abelian group defined by generators and relations as follows. Generators $[A]$ are given by (n, n)-matrices A with entries in R for $n \geq 1$. If A is in block form for square matrices B and D

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

then $[A] = [B] + [D]$. If A is a (n, n)-matrix, U is an invertible (n, n)-matrix and $\phi(U)$ denotes the matrix obtained from U by applying ϕ to each entry, then $[\phi(U) A U^{-1}] = [A]$.

Given a R-map $f : F \longrightarrow \phi^* F$ for a finitely generated free R-module F, define

$$[f] \in U(R, \phi)$$

by $[A]$, where A is the matrix descibing $\phi^* u^{-1} \circ f \circ u : R^n \longrightarrow \phi^* R^n$ for any R-isomorphism $u : R^n \longrightarrow F$. Given an R-chain map $f : C \longrightarrow \phi^* C$ for C a finite free R-chain complex C, we define:

$$u(f) \;:=\; \sum_{n \geq 0} (-1)^n \cdot [f_n : C_n \longrightarrow \phi^* C_n] \qquad \in U(R, \phi). \quad \square$$

We have defined $U(R, \phi)$ in terms of matrices since the square matrices form a set, whereas the R-maps $F \longrightarrow \phi^* F$ for finitely generated free R-modules do not form a set. Notice that for each commutative diagram of R-maps of finitely generated free R-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{i} & F_2 & \xrightarrow{p} & F_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & \phi^* F_1 & \xrightarrow{\phi^* i} & \phi^* F_2 & \xrightarrow{\phi^* p} & \phi^* F_3 & \longrightarrow & 0 \end{array}$$

we get the relation

$$[f_1] - [f_2] + [f_3] = 0.$$

Given rings with endomorphisms (R, ϕ) and (S, ψ) and a ring homomorphism $h : R \longrightarrow S$ with $\psi \circ h = h \circ \phi$, induction with h induces a homomorphism

$$h_* : U(R, \phi) \longrightarrow U(S, \psi) \qquad [g : F \longrightarrow \phi^* F] \mapsto [h_* g : h_* F \longrightarrow \psi^* h_* F], \tag{1.3}$$

where $h_* g$ sends $s \otimes f \in S \otimes_h F$ to $\psi(s) \otimes g(f) \in \psi^*(S \otimes_h F)$.

Theorem 1.4. *$(U(R, \phi), u)$ is the universal additive invariant for ϕ-endomorphisms of finite free R-chain complexes.*

Proof. We first show that $(U(R,\phi),u)$ is an additive invariant. Additivity follows directly from the definitions of u and $U(R,\phi)$. Obviously $u(0:\mathrm{el}(F,n) \longrightarrow \mathrm{el}(F,n)) = 0$. It remains to check homotopy invariance. Let $h : C \longrightarrow \phi^*C$ be an R-chain homotopy from f to g. Denote by ΣC the suspension of C and by $\mathrm{cone}(C)$ the mapping cone of C. Then one obtains an R-chain map $k : \mathrm{cone}(C) \longrightarrow \mathrm{cone}(C)$ by putting

$$k_n = \begin{pmatrix} f_{n-1} & 0 \\ h_{n-1} & g_n \end{pmatrix} : C_{n-1} \oplus C_n \longrightarrow C_{n-1} \oplus C_n$$

such that there is a commutative diagram of R-chain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i} & \mathrm{cone}(C) & \xrightarrow{p} & \Sigma C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle \Sigma g}\downarrow & & \\ 0 & \longrightarrow & \phi^*C & \xrightarrow{\phi^* i} & \phi^*\mathrm{cone}(C) & \xrightarrow{\phi^* p} & \phi^*\Sigma C & \longrightarrow & 0 \end{array}$$

We conclude from additivity:

$$u(f) - u(g) = u(f) + u(\Sigma g) = u(k).$$

Notice that $\mathrm{cone}(C)$ is a contractible R-chain complex. Hence it suffices to show

$$u(f : C \longrightarrow \phi^*C) = 0,$$

provided C is contractible. We do this by induction over the dimension of C. If $d \leq 2$, then the claim follows from the definitions of u and $U(R,\phi)$. The induction step from $d \geq 2$ to $d+1$ is done as follows. Let D be the R-subchain complex of C given by $D_{d+1} = C_{d+1}$, $D_d = \ker(c_d)$ and $D_p = 0$ for $p \neq d, d+1$. Let E be the cokernel of the inclusion $D \xrightarrow{i} C$. Since C is a finite free contractible R-chain complex, we can assume without loss of generality that D and E are finite free contractible R-chain complexes, otherwise add to C the elementary chain complex $\mathrm{el}(C_{d+1}, d)$. The R-chain map f induces R-chain maps $g : D \longrightarrow \phi^*D$ and $h : E \longrightarrow \phi^*E$. Now one gets from additivity:

$$u(g) - u(f) + u(h) = 0.$$

We get from the induction beginning $u(g) = u(h) = 0$. We conclude $u(f) = 0$. This finishes the proof that $(U(R,\phi),u)$ is an additive invariant.

It remains to check the universal property. Let (A,a) be an additive invariant for ϕ-endomorphisms of finite free chain complexes. There is precisely one homomorphism

$$\xi : U(R,\phi) \longrightarrow A$$

which sends a generator represented by an R-map $f : F \longrightarrow \phi^*F$ to $a(f)$, where we interpret f as an R-chain map of finite 0-dimensional R-chain complexes. It remains to show for an R-chain map $f : C \longrightarrow \phi^*C$ for C a finite free R-chain complex

$$a(f) = \sum_{n \geq 0} (-1)^n \cdot \xi(u(f_n)).$$

We do this by induction over the dimension d of C. The induction beginning $d = 0$ follows from the definition of ξ. Let D be the R-chain complex which is concentrated in dimension d and satisfies $D_d = C_d$. Denote by $C|_{d-1}$ the $d-1$-dimensional R-chain complex obtained by truncating C. Let $g : C|_{d-1} \longrightarrow \phi^* C|_{d-1}$ and $h : D \longrightarrow \phi^* D$ be the R-chain maps induced by f. From additivity and the obvious exact sequence of R-chain complexes $0 \to C|_{d-1} \to C \to D \to 0$ we conclude

$$a(f) = a(g) + a(h).$$

Because of the induction hypothesis applied to $C|_{d-1}$ and $\Sigma^{-1} D$ it suffices to show

$$a(h) = -a(\Sigma^{-1} h).$$

There is an obvious exact sequence $0 \to \Sigma^{-1} D \to \operatorname{el}(C_d, d) \to D \to 0$ and an R-chain map $\overline{h} : \operatorname{el}(C_d, d) \longrightarrow \phi^* \operatorname{el}(C_d, d)$ compatible with the exact sequence above. We conclude

$$a(\Sigma^{-1} h) + a(h) = a(\overline{h}) = a(0 : \operatorname{el}(C_d, d) \longrightarrow \phi^* \operatorname{el}(C_d, d)) = 0.$$

This finishes the proof of Theorem 1.4. □

There is a canonical homomorphism

$$s : \mathbb{Z} \longrightarrow U(R, \phi) \qquad m - n \mapsto [0 : R^m \longrightarrow \phi^* R^m] - [0 : R^n \longrightarrow \phi^* R^n].$$

Suppose for simplicity that R has the property that $R^n \cong R^m$ implies $n = m$. This condition is satisfied in our main example, namely in the case where R is the integral group ring of a group. Then each finitely generated R-module F has a well-defined dimension $\dim_R(F) \in \mathbb{Z}$ and we obtain a homomorphism

$$\dim : U(R, \phi) \longrightarrow \mathbb{Z} \qquad [F, f] \mapsto \dim_R(F). \tag{1.5}$$

satisfying $\dim \circ s = \mathrm{id}$. Recall for a finite free R-chain complex C that its *Euler characteristic* is defined by

$$\chi(C) := \sum_{n \geq 0} (-1)^n \cdot \dim_R(C) \qquad \in \mathbb{Z}.$$

Now we can show that a crucial property of Lefschetz type or trace type invariants, commutativity, follows from additivity and homotopy invariance.

Lemma 1.6. *Let $v : C \longrightarrow D$ and $f : D \longrightarrow \phi^* C$ be R-chain maps of finite free R-chain complexes. Then we get*

$$u(f \circ v) + s(\chi(D)) = u(\phi^* v \circ f) + s(\chi(C)).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} D \oplus C & \xrightarrow{\begin{pmatrix} 0 & 0 \\ f & f \circ v \end{pmatrix}} & \phi^* D \oplus \phi^* C \\ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \Big\downarrow & & \Big\downarrow \begin{pmatrix} 1 & \phi^* v \\ 0 & 1 \end{pmatrix} \\ D \oplus C & \xrightarrow[\begin{pmatrix} \phi^* v \circ f & 0 \\ f & 0 \end{pmatrix}]{} & \phi^* D \oplus \phi^* C \end{array}$$

Since the vertical arrows are isomorphisms, we get from additivity

$$u\begin{pmatrix} 0 & 0 \\ f & f\circ v \end{pmatrix} = u\begin{pmatrix} \phi^*v\circ f & 0 \\ f & 0 \end{pmatrix}.$$

We derive from additivity

$$\begin{aligned} u(f\circ v) + u(0: D \longrightarrow \phi^*D) &= u\begin{pmatrix} 0 & 0 \\ f & f\circ v \end{pmatrix}; \\ u(\phi^*v\circ f) + u(0: C \longrightarrow \phi^*C) &= u\begin{pmatrix} \phi^*v\circ f & 0 \\ f & 0 \end{pmatrix}. \end{aligned}$$

This finishes the proof of Lemma 1.6. □

Example 1.7. Suppose that R is commutative and that $\phi = \mathrm{id}$. Then $U(R,\mathrm{id})$ is computed in [1, page 377] and [16, Corollary 3, page 442]. Namely, it is given by

$$U(R,\mathrm{id}) \longrightarrow \mathbb{Z} \times \left\{ \frac{1+a_1t+\ldots a_nt^n}{1+b_1t+\ldots b_mt^m} \,\middle|\, a_i,b_i\in R \right\} \qquad [F,f] \mapsto (\dim_R(F), \det(1-tf)),$$

where $\det(1-tf)$ is the characteristic polynomial in the variable t of f. □

Let $c\colon \mathbb{C} \longrightarrow \mathbb{C}$ be complex conjugation. As an illustration we want to investigate $U(\mathbb{C},c)$. For a complex number $z\in\mathbb{C}$ define $\mathbb{C}$-linear maps

$$\begin{aligned} R_z : \mathbb{C} &\longrightarrow c^*\mathbb{C} & u &\mapsto zc(u); \\ S_z : \mathbb{C}\oplus\mathbb{C} &\longrightarrow c^*(\mathbb{C}\oplus\mathbb{C}) & (a,b) &\mapsto (zc(b), c(a)). \end{aligned}$$

Recall that we have introduced the map dim in 1.5. Define a map

$$\eta : U(\mathbb{C},c) \longrightarrow \prod_{n\geq 1}\mathbb{C} \qquad [f] \mapsto \left(\mathrm{tr}_{\mathbb{C}}(f^{2n})\right)_{n\geq 1}. \tag{1.8}$$

Theorem 1.9. *1. $U(\mathbb{C},c)$ is the free abelian group with basis*

$$B := \{[R_r] \mid r\in\mathbb{R}, r\geq 0\} \coprod \{[S_s] \mid s\in\mathbb{R}, s<0\} \coprod \{[S_z] \mid z\in\mathbb{C}, \Im(z)>0\};$$

2. The map $\dim\times\eta : U(\mathbb{C},c) \longrightarrow \mathbb{Z}\times\prod_{n\geq 1}\mathbb{C}$ is injective;

Proof. We first show that the set

$$\{[R_z] \mid z\in\mathbb{C}\} \quad \coprod \quad \{[S_z] \mid z\in\mathbb{C}\} \tag{1.10}$$

generates $U(\mathbb{C},c)$. We show by induction over the dimension of the complex vector space V that the class $[f]\in U(\mathbb{C},c)$ of a $\mathbb{C}$-linear map $f : V \longrightarrow c^*V$ is a linear combination of elements of this set. Notice that $f^2 : V \longrightarrow V$ is a $\mathbb{C}$-linear endomorphism and hence has a non-trivial eigenvector $v\in V$ with eigenvalue μ. Consider the subspace U of V spanned by v and $f(v)$. Obviously f induces $\mathbb{C}$-linear maps $f_0 : U \longrightarrow c^*U$ and $f_1 : V/U \longrightarrow c^*V/U$ such that we get in $U(\mathbb{C}, c)$

$$[f] = [f_0] + [f_1].$$

If U is different from V, the induction step follows from the induction hypothesis. Hence it remains to treat the case $U = V$. Suppose that v and $f(v)$ are linearly dependent. Then there is $z \in \mathbb{C}$ with $f(v) = zv$ and hence $[f] = [R_z]$. Hence it remains to treat the case where $\{v, f(v)\}$ is a basis for V. If we conjugate f with the $\mathbb{C}$-isomorphism $\mathbb{C}^2 \longrightarrow V$ which maps $(1, 0)$ to v and $(0, 1)$ to $f(v)$, then we obtain S_μ and hence $[f] = [S_\mu]$.

Next we want to show that the set B defined in Theorem 1.9.1 generates $U(\mathbb{C}, c)$. For that purpose it suffices to verify in $U(\mathbb{C}, c)$ the relations

$$[S_{r^2}] = 2[R_r] \qquad \text{for } r \in \mathbb{R}^{>0}; \tag{1.11}$$

$$[R_{|z|}] = [R_z] \qquad \text{for } z \in \mathbb{C}; \tag{1.12}$$

$$[S_z] = [S_{c(z)}] \qquad \text{for } z \in \mathbb{C}. \tag{1.13}$$

We get the following equations of maps from $\mathbb{C} \oplus \mathbb{C}$ to $\mathbb{C} \oplus \mathbb{C}$

$$\begin{aligned} \begin{pmatrix} r & 1 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 0 & c \\ R_{r^2} & 0 \end{pmatrix} &= \begin{pmatrix} R_{r^2} & R_r \\ R_{ir^2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} R_r & 0 \\ R_{ir} & R_r \end{pmatrix} \cdot \begin{pmatrix} r & 1 \\ 0 & i \end{pmatrix}. \end{aligned}$$

Now 1.11 follows from

$$\begin{aligned} [S_{r^2}] &= \left[\begin{pmatrix} R_r & 0 \\ R_{ir} & R_r \end{pmatrix} \right] \\ &= [R_r] + [R_r]. \end{aligned}$$

For $z \in \mathbb{C}$ with $z \neq 0$ choose $\omega \in \mathbb{C}$ satisfying $\omega^{-1} z c(\omega) = |z|$. If we conjugate R_z with $\omega \cdot \mathrm{id}$, we obtain $R_{|z|}$ and hence 1.12 follows.

We obtain 1.13 from

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ c(z) & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & c \\ R_z & 0 \end{pmatrix} &= \begin{pmatrix} R_z & 0 \\ 0 & R_{c(z)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & c \\ R_{c(z)} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ c(z) & 0 \end{pmatrix} \end{aligned}$$

Since the set defined in 1.10 generates $U(\mathbb{C}, c)$, we conclude from 1.11, 1.12 and 1.13 that the set B generates $U(\mathbb{C}, c)$.

Finally we show that the set $(\dim \times \eta)(B)$ is a $\mathbb{Z}$-linear independent subset of $\mathbb{Z} \times \prod_{n \geq 1} \mathbb{C}$. Notice that then Theorem 1.9 follows since B generates $U(\mathbb{C}, c)$. Notice that $[R_0]$ lies in the kernel of η and is mapped to 1 under dim. Hence it suffices to show for sequences $0 < r_1 < r_2 < \ldots r_d$ and $0 > s_1 > s_2 > \ldots > s_e$ of real numbers, and a sequence $z_1, z_2, \ldots, z_f$

of pairwise distinct complex numbers with positive imaginary parts that for any sequences of integers λ_1, ..., λ_d, μ_1, ..., μ_e and ν_1, ..., ν_f which satisfy

$$\sum_{i=1}^{d} \lambda_i \cdot \eta([R_{r_i}]) + \sum_{j=1}^{e} \mu_j \cdot \eta([S_{s_j}]) + \sum_{k=1}^{f} \mu_k \cdot \eta([S_{z_k}]) \quad = 0 \tag{1.14}$$

the equations

$$\lambda_i \quad = 0 \qquad \text{for } i = 1, \ldots d; \tag{1.15}$$

$$\mu_j \quad = 0 \qquad \text{for } j = 1, \ldots e; \tag{1.16}$$

$$\nu_k \quad = 0 \qquad \text{for } k = 1, \ldots f. \tag{1.17}$$

hold. One easily computes

$$\begin{aligned} \eta([R_{r_i}] &= (r_i^{2n})_{n\geq 1}; \\ \eta([S_{s_j}]) &= (2s_j^n)_{n\geq 1}; \\ \eta([S_{z_k}]) &= (z_k^n + c(z_k)^n)_{n\geq 1}. \end{aligned}$$

Hence we get from 1.14

$$\sum_{i=1}^{d} \lambda_i r_i^{2n} + \sum_{j=1}^{e} \mu_j 2s_j^n + \sum_{k=1}^{f} \nu_k (z_k^n + c(z_k)^n) \quad = \quad 0 \qquad \text{for } n \geq 1. \tag{1.18}$$

If we apply the next Lemma 1.19 to the sequences

$$\begin{aligned} w_n &:= r_1^2, \ldots r_d^2, s_1, \ldots s_e, z_1, c(z_1), \ldots z_f, c(z_f); \\ v_n &:= \lambda_1, \ldots \lambda_d, 2\mu_1, \ldots, 2\mu_e, \nu_1, \nu_1, \ldots \nu_f, \nu_f; \end{aligned}$$

then we get 1.15, 1.16 and 1.17. This finishes the proof of Theorem 1.9 except for the proof of Lemma 1.19 we will give next. □

Lemma 1.19. *Let w_1, ..., w_l be a sequence of pairwise distinct non-zero complex numbers and v_1, ..., v_l be a sequence of complex numbers satisfying*

$$\sum_{i=1}^{l} v_i w_i^n = 0 \qquad \text{for } n = 1, 2, \ldots, l.$$

Then $v_i = 0$ for $i = 1, 2 \ldots l$.

Proof. We have to show that the l elements $(w_1^n, w_2^n, \ldots w_l^n) \in \mathbb{C}^l$ for $n = 1, 2 \ldots l$ are $\mathbb{C}$-linearly independent. This follows from the following determinant computation involving Vandermonde's determinant

$$\begin{aligned} \begin{vmatrix} w_1 & w_2 & \ldots & w_l \\ w_1^2 & w_2^2 & \ldots & w_l^2 \\ \vdots & \vdots & \ddots & \vdots \\ w_1^l & w_2^l & \ldots & w_l^l \end{vmatrix} &= w_1 w_2 \ldots w_l \begin{vmatrix} 1 & 1 & \ldots & 1 \\ w_1 & w_2 & \ldots & w_l \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{l-1} & w_2^{l-1} & \ldots & w_l^{l-1} \end{vmatrix} \\ &= w_1 w_2 \ldots w_l \prod_{i>j} (w_i - w_j) \\ &\neq 0. \end{aligned}$$

This finishes the proof of Lemma 1.19 and Theorem 1.9. □

2 The universal Lefschetz invariant for CW-complexes

In this section we want to apply the chain complex invariants of Section 1 to endomorphisms $f : X \longrightarrow X$ of finite CW-complexes satisfying $\pi_0(f) = \mathrm{id}$. Suppose for the moment that X is connected. We make the following choices, namely, of a point $x \in X$ and a path w in X from $y = fx$ to x. Next we want to define an abelian group $U(f, x, w)$ and an invariant

$$u(f, x, w) \quad \in \quad U(f, x, w). \tag{2.1}$$

For this purpose we make additional choices, namely of a model of the universal covering $p : \widetilde{X} \longrightarrow X$ and a point $\widetilde{x} \in \widetilde{X}$ satisfying $p(\widetilde{x}) = x$. Let $\widetilde{y} \in \widetilde{X}$ be the point satisfying $p(\widetilde{y}) = y$ such that w lifts to path in $\widetilde{X}$ from $\widetilde{y}$ to $\widetilde{x}$. There is precisely one lift $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$ satisfying $\widetilde{f}(\widetilde{x}) = \widetilde{y}$. There is a specific left-action of $\pi_1(X, x)$ on $\widetilde{X}$ uniquely determined by the choice of $\widetilde{x}$. Namely, for $\widetilde{z} \in \widetilde{X}$ and $u \in \pi_1(X, x)$ let $u\widetilde{z} \in \widetilde{X}$ be the point such that $p(u\widetilde{z}) = p(\widetilde{z})$ holds and for any paths $\widetilde{a}$ from $\widetilde{x}$ to $\widetilde{z}$ and $\widetilde{b}$ from $\widetilde{x}$ to $u\widetilde{z}$ the loop $p(\widetilde{b}) * p(\widetilde{a}^-)$ represents $u \in \pi_1(X, x)$. Let $c_w : \pi_1(X, y) \longrightarrow \pi_1(X, x)$ be the homomorphism sending u to $w^- * u * w$. Let $\phi = \phi(f, x, w) : \pi_1(X, x) \longrightarrow \pi_1(X, x)$ be the composition $c_w \circ \pi_1(f, x)$. Then $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$ is ϕ-equivariant. The ring homomorphism $\mathbb{Z}\pi_1(X, x) \longrightarrow \mathbb{Z}\pi_1(X, x)$ induced by the group homomorphism ϕ is also denoted by ϕ. We get a $\mathbb{Z}\pi_1(X, x)$-chain map $C(\widetilde{f}) : C(\widetilde{X}) \longrightarrow \phi^* C(\widetilde{X})$. Define (see Definition 1.2)

$$U(f, w, x) := U(\mathbb{Z}\pi_1(X, x), \phi(f, w, x))$$

and

$$u(f, x, w) := u(C(\widetilde{f})) \in U(f, w, x).$$

We have to verify that $u(f, w, x)$ is independent of the choice of $\widetilde{X}$ and $\widetilde{x} \in p^{-1}(x)$. Let $\widetilde{X}'$ and $\widetilde{x}'$ be different choices. The identity map $\mathrm{id} : (X, x) \longrightarrow (X, x)$ lifts uniquely to a $\pi_1(X, x)$-equivariant homeomorphism $\widetilde{\mathrm{id}} : (\widetilde{X}, \widetilde{x}) \longrightarrow (\widetilde{X}', \widetilde{x}')$ such that $\widetilde{f}' \circ \widetilde{\mathrm{id}} = \phi^* \widetilde{\mathrm{id}} \circ \widetilde{f}$ holds. Theorem 1.4 and Lemma 1.6 imply

$$u(C(\widetilde{f})) = u(C(\widetilde{f}')) \in U(f, x, w).$$

Hence $u(f, x, w)$ depends only on (x, w).

Next we examine the dependency on (x, w). Let (x_k, w_k) for $k = 0, 1$ be two such choices. We want to construct a homomorphism

$$\mu = \mu(x_0, w_0, x_1, w_1) : U(f, x_0, w_0) \longrightarrow U(f, x_1, w_1).$$

To do this we choose a path v from x_0 to x_1. Recall that for a path a we get by conjugation a homomorphism $c_a : \pi_1(X, a(0)) \longrightarrow \pi_1(X, a(1))$. One easily checks

$$\phi_1 \circ c_v \quad = \quad c_{w_0^- * f(v) * w_1} \circ \Phi_0,$$

where $\phi_k : \pi_1(X, x_k) \longrightarrow \pi_1(X, x_k)$ is the endomorphism ϕ with respect to the choice (x_k, w_k) for $k = 0, 1$. Given a finitely generated free $\mathbb{Z}\pi_1(X, x_0)$-module F, define an isomorphism of $\mathbb{Z}\pi_1(X, x_1)$-modules

$$\rho(F) : (c_v)_* \phi_0^* F \longrightarrow \phi_1^* (c_v)_* F$$

by

$$\mathbb{Z}\pi_1(X,x_1)\otimes_{c_v}\phi_0^*F \longrightarrow \phi_1^*\left(\mathbb{Z}\pi_1(X,x_1)\otimes_{c_v}F\right) \qquad g\otimes f\mapsto \Phi_1(g)\otimes u^{-1}f,$$

where $u\in\pi_1(X,x_0)$ is given by $w_0 * f(v) * w_1 * v^-$. In order to check that this is well-defined we must show that $gc_v(h)\otimes f$ and $g\otimes\phi_0(h)f$ are mapped to the same elements, i.e. we must show in $\mathbb{Z}\pi_1(X,x_1)\otimes_{c_v}F$

$$\phi_1(gc_v(h))\otimes u^{-1}f \;=\; \phi_1(g)\otimes u^{-1}\phi_0(h)f.$$

We compute

$$\begin{aligned}
\phi_1(gc_v(h))\otimes u^{-1}f &= \phi_1(g)\phi_1(c_v(h))\otimes u^{-1}f\\
&= \phi_1(g)\otimes c_v^{-1}\circ\phi_1\circ c_v(h)u^{-1}f\\
&= \phi_1(g)\otimes c_v^{-1}\circ c_{w_0^-*f(v)*w_1}\circ\phi_0(h)u^{-1}f\\
&= \phi_1(g)\otimes c_{w_0^-*f(v)*w_1*v^-}\circ\phi_0(h)u^{-1}f\\
&= \phi_1(g)\otimes c_u\circ\phi_0(h)u^{-1}f\\
&= \phi_1(g)\otimes u^{-1}\phi_0(h)uu^{-1}f\\
&= \phi_1(g)\otimes u^{-1}\phi_0(h)f.
\end{aligned}$$

We define the desired homomorphism

$$\mu=\mu(x_0,w_0,x_1,w_1):U(f,x_0,w_0)\longrightarrow U(f,x_1,w_1)$$

by

$$\mu\left([f:F\longrightarrow\phi_0^*F]\right) \;=\; [(c_v)_*F\xrightarrow{(c_v)_*f}(c_v)_*\phi_0^*F\xrightarrow{\rho(F)}\phi_1^*(c_v)_*F].$$

One easily checks that this map is independent of the choice of v. Moreover, it sends $u(f,x,w)$ to $u(f,x_1,w_1)$ because of Theorem 1.4 since there is the following commutative diagram

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\widetilde{f_0}} & \widetilde{X}\\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle l_u}\\
\widetilde{X} & \xrightarrow{\widetilde{f_1}} & \widetilde{X}
\end{array}$$

One easily checks

$$\begin{aligned}
\mu(x_0,w_0,x_2,w_2) &= \mu(x_1,w_1,x_2,w_2)\circ\mu(x_0,w_0,x_1,w_1);\\
\mu(x_0,w_0,x_0,w_0) &= \mathrm{id}\,.
\end{aligned}$$

We define $U(f)$ as the abelian group which is the set of equivalence classes of the equivalence relation on $\coprod_{(x,w)}U(f,x,w)$ generated by $u\sim c(x,x',w,w')(u)$ for $u\in U(f,x,w)$. The collection of the $u(f,x,w)$ determines an element

$$u(f) \;\in\; U(f). \qquad (2.2)$$

The obvious map $U(f,w,x) \longrightarrow U(f)$ is an isomorphism and sends $u(f,x,w)$ to $u(f)$ for all (x,w).

Recall that so far we have assumed that X is connected. If X has more than one path component, we will assume that f induces the identity on $\pi_0(X)$ and we define $U(f)$ by the direct sum over the path components C of X of the groups $U(f|_C)$ and $u(f) \in U(f)$ by the collection of the invariants $u(f|_C)$.

Let $\mathcal{C}$ be the category having as objects finite CW-complexes and as morphisms maps between them. The category $\mathrm{End}(\mathcal{C})$ has as objects (X,f) endomorphisms $f : X \longrightarrow X$ in $\mathcal{C}$ such that f induces the identity on $\pi_0(X)$. A morphism $h : (X,f) \longrightarrow (Y,g)$ in $\mathrm{End}(\mathcal{C})$ is a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

Given two such morphisms $h_i : (X,f) \longrightarrow (Y,g)$ for $i = 0,1$, a homotopy from h_0 to h_1 is given by a commutative square in $\mathcal{C}$

$$\begin{array}{ccc} X \times [0,1] & \xrightarrow{f \times \mathrm{id}} & X \times [0,1] \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

such that the restriction of h to $X \times \{i\}$ agrees with h_i for $i = 0,1$. If such a homotopy exists, we call h_0 and h_1 *homotopic*. A *push out* in $\mathrm{End}(\mathcal{C})$ is a commutative square in $\mathrm{End}(\mathcal{C})$

$$\begin{array}{ccc} (X_0,f_0) & \xrightarrow{i_1} & (X_1,f_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ (X_2,f_2) & \xrightarrow{j_2} & (X,f) \end{array}$$

such that the commutative square in $\mathcal{C}$

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

is a push out, f is the push out of f_0, f_1 and f_2, i_2 is an inclusion of CW-complexes, i_1 is cellular and X has the CW-structure induced by the ones on X_i for $i = 0,1,2$.

Definition 2.3. A *functorial Lefschetz invariant on the category of finite CW-complexes* is a pair (Θ, θ) consisting of a functor

$$\Theta : \mathrm{End}(\mathcal{C}) \longrightarrow \mathrm{ABEL}$$

into the category of abelian groups and a function θ which assigns to any object (X, f) in $\mathrm{End}(\mathcal{C})$ an element

$$\theta(X, f) \in \Theta(X, f)$$

such that the following conditions are satisfied:

1. Additivity

   For a push out in $\mathrm{End}(\mathcal{C})$

$$\begin{array}{ccc} (X_0, f_0) & \xrightarrow{i_1} & (X_1, f_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ (X_2, f_2) & \xrightarrow{j_2} & (X, f) \end{array}$$

   we get in $\Theta(X, f)$:

$$\theta(X, f) = \Theta(j_1)(\theta(X_1, f_1)) + \Theta(j_2)(\theta(X_2, f_2)) - \Theta(j_0)(\theta(X_0, f_0)),$$

   where j_0 is $j_1 \circ i_1 = j_2 \circ i_2$;

2. Homotopy invariance

   If $h_i : (X, f) \longrightarrow (Y, g)$ are homotopic morphisms in $\mathrm{End}(\mathcal{C})$ for $i = 0, 1$, then:

$$\Theta(h_0) = \Theta(h_1);$$

3. Invariance under homotopy equivalence

   If $h : (X, f) \longrightarrow (Y, g)$ is a morphism in $\mathrm{End}(\mathcal{C})$ such that $h : X \longrightarrow Y$ is a homotopy equivalence, then

$$\Theta(h) : \Theta(X, f) \longrightarrow \Theta(Y, g)$$

   is bijective and sends $\theta(X, f)$ to $\theta(Y, g)$;

4. Value at the empty set

$$\theta(\mathrm{id} : \emptyset \longrightarrow \emptyset) \;=\; 0 \quad \in \Theta(\emptyset, \mathrm{id}). \quad \square$$

Definition 2.4. We call a functorial Lefschetz invariant (U, u) *universal* if for any functorial Lefschetz invariant (Θ, θ) there is precisely one natural transformation $\tau : U \longrightarrow \Theta$ such that $\tau(f) : U(f) \longrightarrow \Theta(f)$ sends $u(f)$ to $\theta(f)$ for any object $f : X \longrightarrow X$ in $\mathrm{End}(\mathcal{C})$. $\square$

The following theorem is one of the main results of this paper. It explains why the invariant u encompasses a lot of known Lefschetz type invariants and other invariants as we will analyse in Section 3. We will give its proof in Section 4. Analogous results for finiteness obstructions and Whitehead torsion have been proven in [25], [28, Section 6]. Notice that U becomes a functor $U : \mathrm{End}(\mathcal{C}) \longrightarrow \mathrm{ABEL}$ by 1.3.

Theorem 2.5. *The pair (U, u) defined in 2.2 is the universal functorial Lefschetz invariant for endomorphisms of finite CW-complexes in the sense of Definition 2.3.* □

For related universal properties of Lefschetz-type invariants for spaces with group actions we refer for instance to [22], [39].

Remark 2.6. In the definition of $U(X, f)$ we could work with finitely generated projective modules instead of finitely generated free modules. This corresponds in geometry to the passage from finite CW-complexes to finitely dominated CW-complexes. Then the new group would be the direct sum of the version discussed here with

$$\oplus_{C \in \pi_0(X)} \widetilde{K}_0(\mathbb{Z}\pi_1(C))$$

and the new invariant would be the sum with the old one and the collection of Wall's finiteness obstructions $o(C) \in \widetilde{K}_0(\mathbb{Z}\pi_1(C))$ of the components C. □

Remark 2.7. Let $f_k : X \longrightarrow X$ for $k = 0, 1$ be homotopic endomorphisms of a finite CW-complex X with $\pi_0(f) = \mathrm{id}$. Let $h : f_0 \simeq f_1$ be such a homotopy. Let (Θ, θ) be a functorial Lefschetz invariant. We obtain an isomorphism

$$\Theta_h : \ \Theta(X, f_0) \xrightarrow{\Theta(k_0)} \Theta(X \times [0,1], h \times \mathrm{pr}_{[0,1]}) \xrightarrow{\Theta(k_1)^{-1}} \ \Theta(X, f_1), \tag{2.8}$$

where $k_n : X \longrightarrow X \times [0, 1]$ maps x to (x, n) for $n = 0, 1$. Because of invariance under homotopy equivalence, $\Theta(k_n)$ is bijective for $n = 0, 1$ and

$$\Theta_h(\theta(X, f_0)) \ = \ \theta(X, f_1).$$

Notice, however, that Θ_h does depend on h, or more precisely, on the homotopy class of the path $h(x, -)$ as the following concrete calculation for the universal invariant shows.

Suppose for simplicity in the sequel that X is connected. Let x be a base point and w_k be a path from $f_k(x)$ to x for $k = 0, 1$. Let v be the path $h(x, -)$ from $f_0(x)$ to $f_1(x)$. Write $\pi = \pi_1(X, x)$. Let ϕ_k for $k = 0, 1$ be the endomorphism of π and $\mathbb{Z}\pi$ respectively given by $c_{w_k} \circ \pi_1(f_k, x)$. Notice that $\phi_1 = c_{w_o^- * v * w_1} \circ \phi_0$. Hence we obtain a map

$$\Theta(x, v) : \Theta(\mathbb{Z}\pi, \Phi_0) \ \longrightarrow \ \Theta(\mathbb{Z}\pi, \phi_1), \tag{2.9}$$

which sends the class of $g : F \longrightarrow \phi_0^* F$ to the class of $l_{w_o^- * v * w_1} \circ g : F \longrightarrow \phi_1^* F$, where $l_{w_o^- * v * w_1} : F \longrightarrow c_v^* F$ is the $\mathbb{Z}\pi$-map sending x to $(w_o^- * v * w_1)x$. One easily checks using homotopy invariance

$$\Theta(x, v)(u(f_0, x_0, w_0)) \ = \ u(f_1, x_1, w_1) \tag{2.10}$$

and that Θ_h defined in 2.8 and $\Theta(x, v)$ defined in 2.9 agree under the obvious identifications.

Now suppose further that f_0 and f_1 agree, and for simplicity $f_0(x) = x$. Write $f = f_0 = f_1$. Define *Jiang's subgroup*

$$J(f,x) \quad \subset \quad \pi \tag{2.11}$$

as the subgroup of elements for which there is a homotopy h from f to f such that v is represented by $h(x,-)$. Choose $w_0 = w_1$ to be the trivial path. Then $\phi_0 = \phi_1$ is just $\phi := \pi_1(f,x)$. For $v \in J(f,x)$ the map $\Theta(x,v)$ defined in 2.9 becomes the automorphism

$$U(\mathbb{Z}\pi,\phi) \longrightarrow U(\mathbb{Z}\pi,\phi) \qquad [g : F \longrightarrow \phi^*F] \mapsto [l_v \circ g : F \longrightarrow \phi^*F].$$

Hence $J(f,x)$ acts on $U(\mathbb{Z}\pi)$ and

$$u(f) \quad \in U(\mathbb{Z}\pi,\phi)^{J(f,x)}. \tag{2.12}$$

Jiang's subgroup is studied for instance in [2], [17] and [18] and all the results there about Nielsen numbers can be derived from 2.12 and Theorem 2.5 since they imply in the notation of Example 3.5

$$\lambda(f) \quad \in \Lambda(\mathbb{Z}\pi,\phi)^{J(f,x)},$$

where $u \in J(f,x)$ acts on $[x] \in \Lambda(\mathbb{Z}\pi,\phi)$ by $u[x] = [ux]$. □

3 Examples

In this section we explain that the universal invariant defined in 2.2 encompasses some of the known Lefschetz type invariants and others.

Example 3.1. The *(classical) Lefschetz number* of an endomorphism $f : X \longrightarrow X$ of a finite CW-complex is defined as the integer

$$\lambda_{\text{class}}(f) \;:=\; \sum_{n\geq 0}(-1)^n \cdot \operatorname{tr}(H_n(f;\mathbb{Z})) \qquad \in \mathbb{Z},$$

where $\operatorname{tr}(H_n(f;\mathbb{Z})) \in \mathbb{Z}$ is the trace of the endomorphism $H_n(f;\mathbb{Z})$ of the finitely generated abelian group $H_n(X;\mathbb{Z})$. Recall that the trace of an endomorphism of a finitely generated abelian group A is the trace of the integer square matrix given by the induced endomorphism of the finitely generated free abelian group $A/\operatorname{Tors}(A)$ with respect to some basis of $A/\operatorname{Tors}(A)$. If one takes Θ to be the constant functor with value $\mathbb{Z}$ and $\theta(f) = \lambda_{\text{class}}(f)$, one obtains a functorial Lefschetz invariant for finite CW-complexes.

The unique natural transformation

$$\xi(f) : U(f) \longrightarrow \mathbb{Z}$$

which sends $u(f)$ to $\lambda_{\text{class}}(f)$ (see Definition 2.4 and Theorem 2.5) is given by the homomorphism

$$U(\mathbb{Z}\pi_1(X,x),\phi(f,w,x)) \quad \longrightarrow \quad \mathbb{Z} \tag{3.2}$$

which maps the class of the endomorphism $g : F \longrightarrow \phi^* F$ to the trace of the endomorphism of finitely generated free abelian groups

$$\overline{g} : \mathbb{Z} \otimes_{\mathbb{Z}\pi_1(X,x)} F \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}\pi_1(X,x)} F \qquad n \otimes x \mapsto n \otimes f(x). \quad \square$$

Example 3.3. Let $f : X \longrightarrow X$ be an endomorphism of a finite CW-complex X. Let $\det(I - t \cdot H_n(f;\mathbb{Q}))$ be the characteristic polynomial of the endomorphism $H_n(f;\mathbb{Q})$ of the rational vector space $H_n(X;\mathbb{Q})$. Define the *rational Lefschetz function*

$$r(f) \quad := \quad \prod_{n \geq 0} \det(I - t \cdot H_n(f;\mathbb{Q}))^{(-1)^n}$$

which is a rational function in t with rational coefficients. If we take Θ to be the constant functor with the abelian group $\operatorname{rat}(t)$ of rational functions in t and $\theta(f)$ to be $r(f)$ we obtain a functorial Lefschetz invariant.

Define the *Lefschetz Zeta-function* to be the formal power series in t with rational coefficients

$$\zeta(f) \quad := \quad \sum_{k \geq 1}^{\infty} t^k \frac{\Lambda_{\text{class}}(f^k)}{k}.$$

If we take Θ to be the constant functor with the abelian group of formal power series in t with rational coefficients and $\theta(f)$ to be $\zeta(f)$, we obtain a functorial Lefschetz invariant. These two invariants are related by the following formulas of formal power series (see [38, section 3])

$$\begin{aligned} r(f) &= \exp(-\zeta(f)); \\ \zeta(f) &= -\ln(r(f)). \end{aligned}$$

The natural transformation

$$\xi(f) : U(f) \longrightarrow \operatorname{rat}(t)$$

which sends $u(f)$ to $r(f)$ (see Definition 2.4 and Theorem 2.5) is given by the homomorphism

$$U(\mathbb{Z}\pi_1(X,x), \phi(f,w,x)) \quad \longrightarrow \quad \operatorname{rat}(t) \tag{3.4}$$

which maps the class of the endomorphism $g : F \longrightarrow \phi^* F$ to the characteristic polynomial $\det(I - t\overline{g})$ of the endomorphism of finitely generated free abelian groups

$$\overline{g} : \mathbb{Z} \otimes_{\mathbb{Z}\pi_1(X,x)} F \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}\pi_1(X,x)} F \qquad n \otimes x \mapsto n \otimes f(x).$$

The classical Lefschetz function, variations of it and their relation to Reidemeister torsion have been investigated for instance in [11], [12], [13], [14]. $\square$

Example 3.5. Next we recall the definition of the *(generalized) Lefschetz invariant* (see [41], [42]). Let $f : X \longrightarrow X$ be an endomorphism of a finite CW-complex. Assume for a moment that X is connected. Fix a base point $x \in X$ and a path w from $y = f(x)$ to x. Now make the following additional choices of a model of the universal covering $p : \widetilde{X} \longrightarrow X$ and of a base point $\widetilde{x} \in \widetilde{X}$ with $p(\widetilde{x}) = x$. Let $\widetilde{y} \in \widetilde{X}$ be the point uniquely characterized by the property that w lifts to a path in $\widetilde{X}$ from $\widetilde{y}$ to $\widetilde{x}$. There is precisely one lifting $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$ with $\widetilde{f}(\widetilde{x}) = \widetilde{y}$. Abbreviate $\pi = \pi_1(X, x)$. We obtain a homomorphism of groups $\phi : \pi \longrightarrow \pi$ by the composition of the map $\pi_1(f, x) : \pi_1(X, x) \longrightarrow \pi_1(X, y)$ induced by f and the map $\pi_1(X, y) \longrightarrow \pi_1(X, x)$ given by conjugation with w. In the sequel we let operate π on $\widetilde{X}$ from the left where the operation is defined with respect to the base point $\widetilde{x} \in \widetilde{X}$ (see Section 2). Then the map $\widetilde{f}$ is ϕ-equivariant. It induces a $\mathbb{Z}\pi$-chain map $C(\widetilde{f}) : C(\widetilde{X}) \longrightarrow \phi^*C(\widetilde{X})$ on the cellular $\mathbb{Z}\pi$-chain complex of $\widetilde{X}$.

Let $g : F \longrightarrow \phi^*F$ be a $\mathbb{Z}\pi$-map. Choose a $\mathbb{Z}\pi$-basis $\{b_1, \ldots b_k\}$ for F. Let $A = (A_{i,j})$ be the square (r, r)-matrix over $\mathbb{Z}\pi$ of g with respect to the chosen basis, i.e. $g(b_i) = \sum_j A_{i,j}b_j$. Let $\mathbb{Z}\pi_\phi$ be the abelian group which is the quotient of the abelian group $\mathbb{Z}\pi$ by the abelian subgroup generated by all elements of the form $\phi(v)w - wv$ for all $v, w \in \pi$. We call two elements $w_0, w_1 \in \pi$ *ϕ-conjugated* if there is $u \in \pi$ with $\phi(u)w_0u^{-1} = w_1$. This is an equivalence relation on π and $\mathbb{Z}\pi_\phi$ can be identified with free abelian group generated by the ϕ-conjugacy classes $[w]$ of elements in w in π. For an element $x \in \mathbb{Z}\pi$ let $[x] \in \mathbb{Z}\pi_\phi$ be its image under the obvious projection $\mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi_\phi$. Define

$$\mathrm{tr}_{(\mathbb{Z}\pi,\phi)}(g) := \sum_i [A_{i,i}] \qquad \in \mathbb{Z}\pi_\phi \tag{3.6}$$

and

$$\Lambda(f, x, w) := \mathbb{Z}\pi_\phi; \tag{3.7}$$

$$\lambda(f, x, w) := \sum_{n\geq 0} (-1)^n \cdot \mathrm{tr}_{(\mathbb{Z}\pi,\phi)}(C_n(\widetilde{f})) \qquad \in \Lambda(f, x, w). \tag{3.8}$$

One easily checks the invariant and the group it takes values in are independent of the choice of p, of $\widetilde{x} \in p^{-1}(x)$ and the bases but it depends on the choice of x and the homotopy class relative end points of the path w. Let x' and w' be a second choice. Let v be any path from x to x'. We obtain by conjugation with w a map $\pi_1(X, x) \longrightarrow \pi_1(X, x')$ which induces a map $\mu(x, w, x', w') : \Lambda(f, x, w) \longrightarrow \Lambda(f, x', w')$. This map is indeed independent of the choice of v, sends $\lambda(f, x, w)$ to $\lambda(f, x', w')$ and satisfies $\mu(x', w', x'', w'') \circ \mu(x, w, x', w') = \mu(x, w, x'', w'')$ and $\mu(x, w, x, w) = \mathrm{id}$. Now define $\Lambda(f)$ as the abelian group which is the set of equivalence classes of the equivalence relation on $\coprod_{(x,w)} \Lambda(f, x, w)$ generated by $u \sim \mu(x, w, x', x, w')(u)$ for $u \in \Lambda(f, x, w)$. The collection of the $\lambda(f, x, w)$ determines an element

$$\lambda(f) \in \Lambda(f). \tag{3.9}$$

The obvious map $\Lambda(f, w, x) \longrightarrow \Lambda(f)$ is an isomorphism and sends $\lambda(f, x, w)$ to $\lambda(f)$ for all (x, w).

If X has more than one path component, one defines $\Lambda(f)$ by the direct sum over the path components C of X of the groups $\Lambda(f|_C)$. Recall that we require that f induces the identity on $\pi_0(X)$. Define $\lambda(f) \in \Lambda(f)$ analogously. Then (Λ, λ) is a functorial Lefschetz invariant for finite CW-complexes. Notice that in contrast to the previous Examples 3.1 and 3.3 Λ is not a constant functor.

Consider as example the endomorphism $f_d : S^1 \longrightarrow S^1$ sending z to z^d for $d \in \mathbb{Z}$. Let $\mathbb{Z}/(|d-1|)$ be the cyclic group of order $|d-1|$ if $d \neq 1$ and of infinite order if $d = 1$. Let t

be the image of the generator of $\mathbb{Z}$, written multiplicatively, under the canonical projection onto $\mathbb{Z}/(|d-1|)$. Then there is an obvious isomorphism

$$\Lambda(S^1, f_d) \xrightarrow{\cong} \mathbb{Z}[\mathbb{Z}/(|d-1|)]$$

which sends $\lambda(f_d)$ to $-\sum_{k=1}^{d-1} t^k$ if $d \geq 2$, to $\sum_{k=0}^{|d|} t^{-k}$ if $d \leq 0$ and to 0 if $d = 1$.

We call $\lambda(f) \in \Lambda(f)$ the *(generalized) Lefschetz invariant.* The *Nielsen number* of f is the number of ϕ-conjugacy classes of elements in $\mathbb{Z}\pi_1(X,x)_\phi$ which appear with non-trivial coefficients in $\lambda(f)$. The Nielsen number and the generalized Lefschetz invariant of f vanish if f is homotopic to an endomorphism without fixed points. Any endomorphism homotopic to f has at least $N(f)$-fixed points. Moreover, f is homotopic to an endomorphism with precisely $N(f)$ fixed points and f is homotopic to a map without fixed points if and only if $\lambda(f)$ and $N(f)$ vanish, provided that X is a compact manifold possibly with boundary of dimension different from 2. Next we recall the *Lefschetz fixed point formula.* Suppose that $f : X \longrightarrow X$ is an endomorphism of a connected compact manifold possibly with boundary such that f has only finitely many fixed points z which do not lie on ∂X and satisfy $\det(\mathrm{id} - T_z f) \neq 0$ where $T_z f : T_z X \longrightarrow T_z X$ is the differential of f at z. Then

$$\lambda(f,x,w) \quad = \sum_{z \in Fix(f)} \frac{\det(\mathrm{id} - T_z f)}{|\det(\mathrm{id} - T_z f)|} \cdot [u_z * f(u_z)^{-1} * w], \qquad (3.10)$$

where u_z is any path from x to z. For further information we refer for instance to [2], [7], [11], [12], [18], [19], [20] [21].

The unique natural transformation

$$\xi(f) : U(f) \longrightarrow \Lambda(f)$$

which sends $u(f)$ to $\lambda(f)$ (see Definition 2.4 and Theorem 2.5) is given by the homomorphism

$$U(\mathbb{Z}\pi_1(X,x), \phi(f,w,x)) \quad \longrightarrow \quad \mathbb{Z}\pi_1(X,x)_{\phi(f,w,x))} \qquad [g] \mapsto \mathrm{tr}_{(\mathbb{Z}\pi,\phi)}(g). \qquad (3.11)$$

□

The next example does not seem to be covered by classical Lefschetz-type invariants.

Example 3.12. Let $f : X \longrightarrow X$ be an endomorphism of a finite connected CW-complex. The mapping torus T_f is obtained from the cylinder $X \times I$ by identifying the bottom and top using f. If f and g are homotopic then their mapping tori are simple homotopy equivalent (see [6]). Hence a simple homotopy invariant of T_f is an invariant of the homotopy class of f. For instance one can interpret $r(f)$ introduced in Example 3.3 as the Reidemeister torsion of the canonical infinite cyclic covering of T_f [38, section 3]. One can also apply a more sophisticated invariant to T_f, namely the combinatorial L^2-torsion. It is known that the L^2-Betti numbers of T_f all vanish [30, Theorem 2.1]. We assume in the sequel that T_f is of determinant class in the sense of [4, page 754], we discuss this assumption later. Then the combinatorial L^2-torsion is defined (see for instance [5], [29], [31])

$$\rho^{(2)}(T_f) \in \mathbb{R}.$$

If X is a compact 2-dimensional manifold and f a diffeomorphism, then the L^2-torsion of T_f can be computed in terms of the volumes of the hyperbolic pieces in its decomposition by a minimal family of pairwise non-isotopic incompressible not boundary-parallel embedded 2-tori into Seifert pieces and hyperbolic pieces. This is proven in [35] using [3], [23] and [37]. In particular if $f : F \longrightarrow F$ is a pseudo-Anosov selfhomeomorphism of a closed hyperbolic 2-dimensional manifold, then the mapping torus T_f is a closed hyperbolic 3-manifold and its combinatorial L^2-torsion is $-1/3\pi$ times its volume.

There is a natural homomorphism

$$\rho(f) : U(f) \longrightarrow \mathbb{R} \tag{3.13}$$

which sends an endomorphism $g : F \longrightarrow \phi^* F$ of finitely generated free $\mathbb{Z}\pi_1(X)$-module F to the generalized Fuglede-Kadison-determinant of the endomorphism of finitely generated Hilbert $\mathcal{N}(\pi_1(T_f))$-modules

$$\overline{g} : l^2(\pi_1(T_f)) \otimes_{\mathbb{Z}\pi_1(X)} F \longrightarrow l^2(\pi_1(T_f)) \otimes_{\mathbb{Z}\pi_1(X)} F \qquad u \otimes v \mapsto -ut \otimes g(v) + u \otimes v$$

in the sense of [29, Section 4]. Here $t \in \pi_1(T_f)$ is the element given by the composition of the path $[0,1] \longrightarrow T_f$ $s \mapsto (s,x)$ with some path in X from $f(x)$ to x for some basepoint $x \in X$. Because of the computation of the cellular $\mathbb{Z}\pi_1(T_f)$- chain complex of $\widetilde{T_f}$ in [30, page 207], ρ has the property

$$\rho(f)(u(f)) = \rho^{(2)}(T_f).$$

We see that $u(f)$ determines $\rho^{(2)}(T_f)$. However, the pair $(\mathbb{R}, \rho^{(2)}(T_?))$, which consists of the constant functor with value $\mathbb{R}$ and the function sending f to $\rho^{(2)}(T_f)$ is not quite a functorial Lefschetz invariant because Additivity holds only for those push outs for which for $k = 0,1,2$ and any base point $x_k \in X_k$ the map $\pi_1(j_k, x_k) : \pi_1(X_k, x_k) \longrightarrow \pi_1(X, j_k(x_k))$ is injective [29, Theorem 1.6]. All other axioms are satisfied in full generality.

Next we discuss the assumption of determinant class which is needed to define L^2-torsion or generalized Fuglede-Kadison determinant. Notice that a finite CW-complex X is of determinant class if all its Novikov-Shubin-invariants are positive and that there is the conjecture that the Novikov-Shubin invariant of any finite CW-complex are positive [24, Conjecture 7.2], but it is known only in special cases like T_f in case of an endomorphism of a compact surface. If $\pi_1(X)$ is residually finite or amenable respectively, then $\pi_1(T_f)$ is residually finite or amenable respectively and T_f is of determinant class and 3.13 is well-defined. (see [3, Theorem A in Appendix A] and [8, Theorem 0.2]). □

Remark 3.14. As always in algebraic K-theory it is often useful for computations for group rings to use representations to detect elements. This strategy applies also in our context.

Let $f : X \longrightarrow X$ be an endomorphism of a connected finite CW-complex X. Let A be a commutative ring and V be a right $A\pi$-module such that V as a A-module is finitely generated free. Let $t : V \longrightarrow \phi^* V$ be a $A\pi$-map. Then we obtain a homomorphism

$$R_{V,t} : U(f) \longrightarrow A \tag{3.15}$$

by sending the class of the $\mathbb{Z}\pi$-map $g : F \longrightarrow \phi^* F$ to the trace of the endomorphism of finitely generated free A-modules given by the composition

$$V \otimes_{\mathbb{Z}\pi} F \longrightarrow V \otimes_{\mathbb{Z}\pi} F \qquad v \otimes f \mapsto t(v) \otimes g(f).$$

Computations using representations are given in [21]. We explain its relation to the mapping torus approach in Remark 7.14. $\square$

Other constructions of Lefschetz type invariants taking values in Hochschild-homology and A-theory are given in [15] and [36].

There are higher analogues of the groups $U(R, \phi)$, just apply the standard constructions of Quillen or Waldhausen to the category of ϕ-endomorphisms of finitely generated free R-modules. Analogously one can define an A-theoretic version of the geometric side of endomorphisms of finite CW-complexes and construct a linearization map from the A-theory version to the K-theory version analogously to the linearization map from $A(X)$ to $K(\mathbb{Z}\pi_1(X))$ for a connected finite CW-complex X.

4 Proof of the universal property

This section is devoted to the proof of Theorem 2.5. For this purpose we will need the following notions and constructions.

Let X be a space. A *retractive space over* X is a triple $Y = (Y, i, r)$ which consists of a space Y, a cofibration $i : X \longrightarrow Y$ and a map $r : Y \longrightarrow X$ satisfying $r \circ i = \mathrm{id}$. We often identify X with $i(X)$. Given a retractive space Y over X, define retractive spaces $Y \times_X [0, 1]$ and $C_X Y$ by the push outs

$$\begin{array}{ccc} X \times [0,1] & \xrightarrow{\mathrm{pr}} & X \\ {\scriptstyle i \times \mathrm{id}}\downarrow & & \downarrow \\ Y \times [0,1] & \longrightarrow & Y \times_X [0,1] \end{array}$$

and

$$\begin{array}{ccc} Y \times \{1\} & \xrightarrow{r} & X \\ {\scriptstyle j}\downarrow & & \downarrow \\ Y \times_X [0,1] & \longrightarrow & C_X Y \end{array}$$

where pr resp. j is the canonical projection resp. inclusion and the inclusion of X and the retraction onto X is the obvious one. Define the retractive space $\Sigma_X Y$ by the push out

$$\begin{array}{ccc} Y & \xrightarrow{\widehat{i}} & C_XY \\ {\scriptstyle \widehat{i}}\downarrow & & \downarrow \\ C_XY & \longrightarrow & \Sigma_XY \end{array}$$

where $\widehat{i} : Y \longrightarrow C_XY$ is the inclusion induced by the inclusion $Y \times \{0\} \longrightarrow Y \times [0,1]$. Notice that the composition $\widehat{i} \circ i : X \longrightarrow C_XY$ is a homotopy equivalence relative X with the retraction of C_XY onto X as homotopy inverse relative X. If X consists of one point, then a retractive space over X is just a pointed space and C_XY resp. Σ_XY is the reduced cone resp. suspension of Y.

Given two retractive spaces Y and Z over X and an endomorphism $f : X \longrightarrow X$, define $[(C_XY, Y), (C_XZ, Z)]_f$ to be the set of homotopy classes relative X of maps of pairs $(\widehat{g}, g) : (C_XY, Y) \longrightarrow (C_XZ, Z)$ which induce on X the given endomorphism f. Homotopy class relative X means that the relevant homotopies are stationary on X. Next we want to describe a suspension map

$$\Sigma_X : [(C_XY, Y), (C_XZ, Z)]_f \quad\longrightarrow\quad [(C_X\Sigma_XY, \Sigma_XY), (C_X\Sigma_XZ, \Sigma_XZ)]_f. \tag{4.1}$$

Let $(\widehat{g}, g)$ be a representative of a class in the source. We only explain the definition of a representative $(\widehat{g_\Sigma}, g_\Sigma)$ of the image of the class under this map. Define g_Σ by $\widehat{g} \cup_g \widehat{g}$. Notice that C_X is compatible with push outs so that we can think of $C_X\Sigma_XY$ as the push out of C_X applied to the diagram defining Σ_X, i.e.

$$\begin{array}{ccc} C_XY & \xrightarrow{C_X\widehat{i}} & C_XC_XY \\ {\scriptstyle C_X\widehat{i}}\downarrow & & \downarrow \\ C_XC_XY & \longrightarrow & C_X\Sigma_XY \end{array}$$

In order to define the extension $\widehat{g_\Sigma} : C_X\Sigma_XY \longrightarrow C_X\Sigma_XY$ we will define an endomorphism $\overline{g} : C_XC_XY \longrightarrow C_XC_XY$ extending $\widehat{g}$ and will put $\widehat{g_\Sigma}$ to be $\overline{g} \cup_{\widehat{g}} \overline{g}$. For the definition of $\overline{g}$ it is convenient to rewrite C_XC_XY as follows. Namely, there is a commutative diagram

$$\begin{array}{ccc} Y \times [0,1] \times [0,1] & \xrightarrow{\psi} & Y \times [0,1] \times [0,1] \\ \downarrow & & \downarrow \\ C_XY \times [0,1] \cup_{Y \times [0,1]} Y & \xrightarrow{\overline{\psi}} & C_XC_XY \end{array}$$

where

$$\psi(y, s, t) \;=\; \left(y, \frac{ts}{\max\{t, 1-t\}}, \frac{(1-t)s}{\max\{t, 1-t\}}\right),$$

the vertical arrows are the obvious projections and the space in the left lower corner is the push out of $C_XY \times [0,1] \xleftarrow{j} Y \times [0,1] \xrightarrow{\mathrm{pr}} Y$ for j resp. pr the canonical inclusion resp. projection. One easily checks that $\overline{\psi}$ is a homeomorphism. Conjugating the endomorphism $\widehat{g} \times \mathrm{id} \cup_{g \times \mathrm{id}} g$ with $\overline{\psi}$ yields $\overline{g}$. This finishes the definition 4.1 of the map Σ_X.

Let $f : X \longrightarrow X$ be an endomorphism of a finite CW-complex. Suppose for some time that X is connected. Let $p : \widetilde{X} \longrightarrow X$ be a model of the universal covering, $x \in X$ and $\widetilde{x} \in \widetilde{X}$ base points with $p(\widetilde{x}) = x$, w a path from $f(x)$ to x and $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$ the lift of f for which w lifts to a path from $\widetilde{f}(\widetilde{x})$ to $\widetilde{x}$. Given a retractive space Y, let $\widetilde{Y}$ and $\widetilde{C_X Y}$ be the pull back of $\widetilde{X}$ with the retractions onto X. We call a retractive space Y over X a *d-extension* if Y is obtained from X by attaching finitely many cells in dimension d. If Y is a d-extension of X and $d \geq 2$, then $\pi_1(r)$ is an isomorphism and $\widetilde{Y}$ is the universal covering of Y. Given a map $g : Y \longrightarrow Z$ of d-extensions over X for $d \geq 2$ which induces f on X, define $\widetilde{g} : \widetilde{Y} \longrightarrow \widetilde{Z}$ to be the lift of g uniquely determined by the property that it induces $\widetilde{f}$ on $\widetilde{X}$. Let $\phi : \mathbb{Z}\pi_1(X, x) \longrightarrow \mathbb{Z}\pi_1(X, x)$ be the homomorphism induced as before by the composition $c_w \circ \pi_1(f, x)$.

We have already seen that U is the universal additive invariant for chain complexes. The reason why it turns out to be the universal invariant for spaces is the next lemma which contains the decisive step in the passage from geometry to algebra.

Lemma 4.2. *Let Y and Z be d-extensions of X for $d \geq 2$. Then the map*

$$\eta : [(C_X Y, Y), (C_X Z, Z)]_f \longrightarrow \hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Y}, \widetilde{X}), \phi^* C_d(\widetilde{Z}, \widetilde{X})) \qquad [(\widehat{g}, g)] \mapsto C_d(\widetilde{g}, \widetilde{f})$$

is bijective.

Proof. Choose a push out

$$\begin{array}{ccc} \coprod_{i \in I} S^{d-1} & \xrightarrow{\coprod_{i \in I} q_i} & X \\ k \big\downarrow & & \big\downarrow j \\ \coprod_{i \in I} D^d & \xrightarrow{\coprod_{i \in I} Q_i} & Y \end{array}$$

Define a map $p_i : S^d \longrightarrow Y$ by requiring that p_i on the upper hemisphere S^d_+ is Q_i and on the lower hemisphere S^d_- is $r \circ Q_i$. Then $C_X Y$ is the push out

$$\begin{array}{ccc} \coprod_{i \in I} S^{d} & \xrightarrow{\coprod_{i \in I} p_i} & Y \\ k \big\downarrow & & \big\downarrow \widehat{j} \\ \coprod_{i \in I} D^{d+1} & \xrightarrow{\coprod_{i \in I} P_i} & C_X Y \end{array}$$

Hence a map $(\widehat{g}, g) : (C_X Y, Y) \longrightarrow (C_X Z, Z)$ with $g|_X = f$ is uniquely determined by its compositions $(\widehat{g}, g) \circ (P_i, p_i)$ and any collection of maps $(\widehat{k_i}, k_i) : (D^{d+1}, S^d) \longrightarrow (C_X Z, Z)$ with $k_i|_{S^d_-} = f \circ r \circ Q_i$ determines uniquely such a map $(\widehat{g}, g)$ with $(\widehat{g}, g) \circ (P_i, p_i) = (\widehat{k_i}, k_i)$. Hence the map

$$\mu : [(C_X Y, Y), (C_X Z, Z)]_f \longrightarrow \prod_{i \in I} [(D^{d+1}, S^d), (C_X Z, Z)]_{f \circ r \circ Q_i : S^d_- \longrightarrow X}.$$

sending $[\widehat{g}, g]$ to $([\widehat{g}, g) \circ (P_i, p_i)])_{i \in I}$ is a bijection.

Next we show that the following forgetful map is bijective for $i \in I$:

$$\nu_i : [(D^{d+1}, S^d), (C_X Z, Z)]_{f \circ r \circ Q_i : S^d_- \longrightarrow X} \longrightarrow \pi_{d+1}(C_X Z, Z, f(x_i)),$$

where $s = (0, 0, \ldots, -1) \in S^d$ is a fixed base point and $x_i = r \circ Q_i(s)$. It forgets that the map has to look like $f \circ r \circ Q_i$ on the lower hemisphere and remembers only that the point $s \in S^d$ has to go to $f(x_i)$. It is bijective as the inclusion $\{s\} \longrightarrow S^d_-$ is a homotopy equivalence and the inclusions $\{s\} \longrightarrow S^d_- \longrightarrow S^d \longrightarrow D^{d+1}$ are cofibrations. Choose paths w_i from x_i to x in Y. Composing each ν_i with the isomorphism $\pi_{d+1}(C_X Z, Z, f(x_i)) \longrightarrow \pi_{d+1}(C_X Z, Z, f(x))$ given by the path $f(w_i)$ from $f(x_i)$ to $f(x)$ and taking the product of the resulting isomorphisms yields an isomorphism

$$\nu : \prod_{i \in I} [(D^{d+1}, S^d), (C_X Y, Y)]_{f or \circ Q_i : S^d_- \longrightarrow X} \longrightarrow \prod_{i \in I} \pi_{d+1}(C_X Z, Z, f(x)).$$

Hence the composition

$$\nu \circ \mu : [(C_X Y, Y), (C_X Z, Z)]_f \longrightarrow \prod_{i \in I} \pi_{d+1}(C_X Z, Z, f(x))$$

is bijective.

Each pair $(Q_i, q_i) : (D^{d+1}, S^d) \longrightarrow (C_X Y, Y)$ defines an element in $\pi_{d+1}(C_X Y, Y, x_i)$. Denote its image under the map induced by the path w_i by $b_i \in \pi_{d+1}(C_X Y, Y, x)$. Then $\{b_i \mid i \in I\}$ is $\mathbb{Z}\pi_1(X, x)$-basis for $\pi_{d+1}(C_X Y, Y, x)$. One easily checks that the bijective composition $\nu \circ \mu$ above is given by

$$[(C_X Y, Y), (C_X Z, Z)]_f \longrightarrow \prod_{i \in I} \pi_{d+1}(C_X Z, Z, f(x)) \qquad [(\widehat{g}, g] \mapsto (\pi_{d+1}(\widehat{g}, g, x)(b_i))_{i \in I}.$$

Consider the following isomorphism given by the following composition of isomorphisms or their inverses

$$\sigma_Y : \pi_{d+1}(C_X Y, Y, x) \xrightarrow{\partial} \pi_d(Y, X, x) \xleftarrow{\pi_d(p, \widetilde{x})} \pi_d(\widetilde{Y}, \widetilde{X}, \widetilde{x}) \xrightarrow{h} H_d(\widetilde{Y}, \widetilde{X}) = C_d(\widetilde{Y}, \widetilde{X}).$$

Here ∂ is the boundary operator in the long exact homotopy sequence of the triple $(C_X Y, Y, X)$ which is an isomorphism since the inclusion $X \longrightarrow C_X Y$ is a homotopy equivalence and h is the Hurewicz isomorphism. Analogously define the isomorphism

$$\sigma_Z : \pi_{d+1}(C_X Z, X, f(x)) \longrightarrow C_d(\widetilde{Z}, \widetilde{X}).$$

For a map $(\widehat{g}, g) : (C_X Y, Y) \longrightarrow (C_X Z, Z,)$ with $g|_X = f$ the following diagram commutes

$$\begin{array}{ccc} \pi_{d+1}(C_X Y, Y, x) & \xrightarrow{c_w \circ \pi_{d+1}(\widehat{g}, g, x)} & \pi_{d+1}(C_X Z, Z, f(x)) \\ \sigma_Y \big\downarrow & & \big\downarrow \sigma_Z \\ C_d(\widetilde{Y}, \widetilde{X}) & \xrightarrow{C(\widetilde{g}, \widetilde{f})} & C_d(\widetilde{Z}, \widetilde{X}) \end{array}$$

Now Lemma 4.2 follows. □

Given a functorial Lefschetz invariant (Θ, θ), we want to construct for an endomorphism $f : X \longrightarrow X$ of a connected finite CW-complex and a d-extension Y a map

$$\tau_Y : [(C_X Y, Y), (C_X Y, Y)]_f \longrightarrow \Theta(X, f) \qquad [\widehat{g}, g)] \mapsto \Theta(\widehat{i} \circ i)^{-1} \circ \Theta(\widehat{i})(\theta(Y, g)). \tag{4.3}$$

Recall that $i : X \longrightarrow Y$ and $\widehat{i} : Y \longrightarrow C_X Y$ are the inclusions and $\widehat{i} \circ i$ is a homotopy equivalence so that $\Theta(\widehat{i} \circ i)$ is bijective because of invariance under homotopy equivalence. We have to show that $\Theta(\widehat{i} \circ i)^{-1} \circ \Theta(\widehat{i})(\theta(Y, g))$ depends only on the homotopy class of $(\widehat{g}, g)$ relative X. Let $(\widehat{G}, G)$ be such a homotopy relative X between $(\widehat{g}_0, g_1)$ and $(\widehat{g}_1, g_1)$. For $k = 0, 1$ let l_k be the inclusion $X \longrightarrow X \times [0, 1]$ sending y to (y, k) (and similiar for Y and $C_X Y$). The following diagram in $\operatorname{End}(\mathcal{C})$ commutes:

$$\begin{array}{ccccc}
\Theta(X, f) & \xrightarrow{\Theta(l_0)} & \Theta(X \times [0,1], f \times \mathrm{id}) & \xleftarrow{\Theta(l_1)} & \Theta(X, f) \\
\Theta(i)\downarrow & & \Theta(i \times \mathrm{id})\downarrow & & \Theta(i)\downarrow \\
\Theta(Y, g_0) & \xrightarrow{\Theta(l_0)} & \Theta(Y \times [0,1], G \times \mathrm{pr}_{[0,1]}) & \xleftarrow{\Theta(l_1)} & \Theta(Y, g_1) \\
\Theta(\widehat{i})\downarrow & & \Theta(\widehat{i} \times \mathrm{id})\downarrow & & \Theta(\widehat{i})\downarrow \\
\Theta(C_X Y, \widehat{g}_0) & \xrightarrow{\Theta(l_0)} & \Theta(C_X Y \times [0,1], \widehat{G} \times \mathrm{pr}_{[0,1]}) & \xleftarrow{\Theta(l_1)} & \Theta(C_X Y, \widehat{g}_1)
\end{array}$$

Notice that $\Theta(l_0) = \Theta(l_1) : \Theta(X, f) \longrightarrow \Theta(X \times [0, 1], f \times \mathrm{id})$ holds because of the homotopy invariance of Θ. We have $\Theta(l_k)(\theta(g_k)) = \theta(G \times \mathrm{pr}_{[0,1]})$ for $k = 0, 1$ because of invariance under homotopy equivalence. Now a simple diagram chase shows that $\Theta(\widehat{i} \circ i)^{-1} \circ \Theta(\widehat{i})(\theta(Y, g_k))$ is independent of $k = 0, 1$. Hence the map τ_Y of 4.3 is well-defined.

Next we want to define a map

$$\tau_d : \widetilde{U}(X, f) \longrightarrow \Theta(X, f) \tag{4.4}$$

for an integer $d \geq 2$. We will construct a map

$$\tau_d : \widetilde{U}(\mathbb{Z}\pi_1(X, x), \phi) \longrightarrow \Theta(X, f)$$

for a fixed choice of base point $x \in X$ and path w from $f(x)$ to x and leave it to the reader that the maps for the various choices fit together to give the desired map. Let F be a finitely generated free $\mathbb{Z}\pi$-module and $a : F \longrightarrow \phi^* F$ a $\mathbb{Z}\pi_1(X, x)$-endomorphism. Choose a d-extension Y together with an $\mathbb{Z}\pi_1(X, x)$-isomorphism $b : F \longrightarrow C_d(\widetilde{Y}, \widetilde{X})$. Such Y exists, namely, take the wedge of X with a finite number of copies of S^d with the obvious retraction onto X. Then $b \circ a \circ b^{-1}$ is an element in $\hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Y}, \widetilde{X}), \phi^* C_d(\widetilde{Y}, \widetilde{X}))$. Let $[(\widehat{g}, g)]$ be its preimage under the isomorphism

$$\eta : [(C_X Y, Y), (C_X Y, Y)]_f \longrightarrow \hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Y}, \widetilde{X}), \phi^* C_d(\widetilde{Y}, \widetilde{X}))$$

of Lemma 4.2. Define

$$\tau_d([a]) = (-1)^d \cdot (\tau_Y([\widehat{g}, g]) - \theta(f)) .$$

We must show that this is independent of the choice of Y and b. Suppose Z and c is a second choice yielding an element $[(\widehat{h}, h)]$ in $[(C_X Z, Z), (C_X Z, Z)]_f$. Let $(\widehat{k}, k)$ be the preimage of $c \circ b^{-1}$ under the isomorphism

$$\eta : [(C_X Y, Y), (C_X Z, Z)]_{\mathrm{id}} \longrightarrow \hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Y}, \widetilde{X}), C_d(\widetilde{Z}, \widetilde{X}))$$

of Lemma 4.2 and $(\widehat{k^{-1}}, k^{-1})$ be the preimage of $b \circ c^{-1}$ under the isomorphism

$$\eta : [(C_X Z, Z), (C_X Y, Y)]_{\mathrm{id}} \longrightarrow \hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Z}, \widetilde{X}), C_d(\widetilde{Y}, \widetilde{X}))$$

of Lemma 4.2. We conclude from invariance under homotopy equivalence applied to the morphism $k : (Y, k^{-1} \circ h \circ k) \longrightarrow (Z, k \circ k^{-1} \circ h)$

$$\tau_Y((\widehat{k^{-1}}, k^{-1}) \circ (\widehat{h}, h) \circ (\widehat{k}, k)) = \tau_Z((\widehat{k}, k)) \circ (\widehat{k^{-1}}, k^{-1}) \circ (\widehat{h}, h)).$$

Since $(\widehat{k^{-1}}, k^{-1}) \circ (\widehat{h}, h) \circ (\widehat{k}, k))$ and $(\widehat{g}, g)$ define the same class in $[(C_X Y, Y), (C_X Y, Y)]_f$ because of Lemma 4.2, we get:

$$\tau_Y((\widehat{k^{-1}}, k^{-1}) \circ (\widehat{h}, h) \circ (\widehat{k}, k)) = \tau_Y(\widehat{g}, g).$$

Similarly we get

$$\tau_Z((\widehat{k}, k)) \circ (\widehat{k^{-1}}, k^{-1}) \circ (\widehat{h}, h)) = \tau_Z(\widehat{h}, h).$$

This shows that the definition of $\tau_d(a)$ is independent of the choice of the d-extension Y and the isomorphism b. Hence we have defined τ_d on generators. It remains to check that τ_d is compatible with the relations appearing in the definition of $U(\mathbb{Z}\pi_1(X, x), \phi)$.

We have to show for a block endomorphism

$$a \;=\; \begin{pmatrix} a_0 & a_2 \\ 0 & a_1 \end{pmatrix} : F_0 \oplus F_1 \longrightarrow \phi^* F_0 \oplus \phi^* F_1$$

that $\tau_d(a) = \tau_d(a_0) + \tau_d(a_1)$ holds. Choose d-extensions Y_k and $\mathbb{Z}\pi_1(X, x)$-isomorphisms $b_k : F_k \longrightarrow C_d(\widetilde{Y_k}, \widetilde{X})$ for $k = 0, 1$. Put $Y = Y_0 \cup_X Y_1$. Notice that $C_X Y = C_X Y_0 \cup_X C_X Y_1$, $i = i_0 \cup_X i_1$ and $\widehat{i} = \widehat{i_0} \cup_X \widehat{i_1}$ holds where $i_k : X \longrightarrow Y_k$, $i : X \longrightarrow Y$, $\widehat{i_k} : Y_k \longrightarrow C_X Y_k$ and $\widehat{i} : Y \longrightarrow C_X Y$ are the canonical inclusions. Then the direct sum of b_0 and b_1 yields a $\mathbb{Z}\pi_1(X, x)$-isomorphism $b : F_0 \oplus F_1 \longrightarrow C_d(\widetilde{Y}, \widetilde{X})$. Let $(\widehat{g}, g) : (C_X Y, Y) \longrightarrow (C_X Y, Y)$ be a map such that it induces a map $(\widehat{g_0}, g_0) : (C_X Y_0, Y_0) \longrightarrow (C_X Y_0, Y_0)$ and under the identifications b_0 and b above the induced chain endomorphisms agree with a_0 and a. Denote by $j : (C_X Y_1, Y_1) \longrightarrow (C_X Y_0 \cup_X C_X Y_1, C_X Y_0 \cup_X Y_1)$ the canonical inclusion and denote by $p : (C_X Y_0 \cup_X C_X Y_1, C_X Y_0 \cup_X Y_1) \longrightarrow (C_X Y_1, Y_1)$ the canonical projection. Because of the identification $C_X Y_0 \cup_X Y_1 = C_X Y_0 \cup_{Y_0} (Y_0 \cup_X Y_1)$ we can view $(\widehat{g}, \widehat{g_0} \cup_{g_0} g)$ as an endomorphism of $(C_X Y_0 \cup_X C_X Y_1, C_X Y_0 \cup_X Y_1)$. Then the endomorphism $p \circ (\widehat{g}, \widehat{g_0} \cup_{g_0} g) \circ j$ induces under the identification b_1 the map a_1. We conclude from the definition of τ_d

$$\begin{aligned} \tau_d(a_0) &= \Theta(\widehat{i_0} \circ i_0)^{-1} \circ \Theta(\widehat{i_0})(\theta(g_0)) - \theta(f) \\ \tau_d(a) &= \Theta(\widehat{i} \circ i)^{-1} \circ \Theta(\widehat{i})(\theta(g)) - \theta(f) \\ \tau_d(a_1) &= \Theta(\widehat{i_1} \circ i_1)^{-1} \circ \Theta(\widehat{i_1})(\theta(p \circ (\widehat{g_0} \cup_{g_0} g) \circ j)) - \theta(f). \end{aligned}$$

Applying additivity to the push out diagram

$$\begin{array}{ccc}
(Y_0, g_0) & \xrightarrow{\mathrm{id}_{Y_0} \cup_X i_1} & (Y_0 \cup_X Y_1, g) \\
\widehat{i_0} \downarrow & & \downarrow \widehat{i_0} \cup_X \mathrm{id}_{Y_1} \\
(C_X Y_0, \widehat{g_0}) & \xrightarrow{\mathrm{id}_{C_X Y_0} \cup_X i_1} & (C_X Y_0 \cup_{Y_0} (Y_0 \cup_X Y_1), \widehat{g_0} \cup_{g_0} g)
\end{array}$$

and the identification $C_X Y_0 \cup_X Y_1 = C_X Y_0 \cup_{Y_0} (Y_0 \cup_X Y_1)$ yields

$$\theta(\widehat{g_0} \cup_{g_0} g) \;=\; \Theta(\widehat{i_0} \cup_X \mathrm{id}_{Y_1})(\theta(g)) + \Theta(\mathrm{id}_{C_X Y_0} \cup_X i_1)(\theta(\widehat{g_0})) - \Theta((\mathrm{id}_{C_X Y_0} \cup_X i_1 \circ \widehat{i_0})(\theta(g_0)).$$

Applying $\Theta(\widehat{i \circ i})^{-1} \circ \Theta(\mathrm{id}_{C_X Y_0} \cup_X \widehat{i_1})$ to this formula and invariance under homotopy equivalence applied to $\widehat{i_0} : (X, f) \longrightarrow (C_X Y_0, \widehat{g_0})$ yields

$$\begin{aligned}
\Theta(\widehat{i \circ i})^{-1} \circ \Theta(\mathrm{id}_{C_X Y_0} \cup_X \widehat{i_1})(\theta(\widehat{g_0} \cup_{g_0} g)) \;&=\; \Theta(\widehat{i \circ i})^{-1} \circ \Theta(\widehat{i})(\theta(g)) \\
&+\; \theta(f) - \Theta(\widehat{i_0} \circ i_0)^{-1} \circ \Theta(i_0))(\theta(g_0)).
\end{aligned}$$

Hence it remains to show

$$\Theta(\widehat{i \circ i}))^{-1} \circ \Theta(\mathrm{id}_{C_X Y_0} \cup_X \widehat{i_1})(\theta(\widehat{g_0} \cup_{g_0} g)) \;=\; \Theta(\widehat{i_1} \circ i_1)^{-1} \circ \Theta(\widehat{i_1})(\theta(p \circ (\widehat{g_0} \cup_{g_0} g) \circ j)).$$

Applying to $j : (Y_1, p \circ \widehat{g_0} \cup_{g_0} g \circ j) \longrightarrow (C_X Y_0 \cup_X Y_1, j \circ p \circ \widehat{g_0} \cup_{g_0} g)$ invariance under homotopy equivalence yields

$$\Theta(j)(\theta(p \circ \widehat{g_0} \cup_{g_0} g \circ j) = \theta(j \circ p \circ \widehat{g_0} \cup_{g_0} g).$$

Hence it suffices to show

$$\Theta(\widehat{i \circ i}))^{-1} \circ \Theta(\mathrm{id}_{C_X Y_0} \cup_X \widehat{i_1})(\theta(\widehat{g_0} \cup_{g_0} g)) \;=\; \Theta(\widehat{i \circ i}))^{-1} \circ \Theta(\mathrm{id}_{C_X Y_0} \cup_X \widehat{i_1})(\theta(j \circ p \circ \widehat{g_0} \cup_{g_0} g)).$$

This follows analogous to the proof that τ_Y is well-defined using the fact that $p \circ j$ is the identity and $j \circ p$ is homotopic relative X to the identity and hence $(j \circ p \circ \widehat{g}, j \circ p \circ \widehat{g_0} \cup_{g_0} g)$ and $(\widehat{g}, \widehat{g_0} \cup_{g_0} g)$ are homotopic relative X. This finishes the proof that τ_d in 4.4 of well-defined.

Next we show that τ_d is independent of $d \geq 2$. If Y is a d-extension of X, then $\Sigma_X Y$ is a $d+1$-extension and there is a bijective correspondence between the d cells in $Y - X$ and the $d+1$-cells in $\Sigma_X Y - X$. In particular, the suspension of $C(\widetilde{Y}, \widetilde{X})$ is $C(\widetilde{\Sigma_X Y}, \widetilde{X})$. One easily checks that the following diagram commutes

$$\begin{array}{ccc}
[(C_X Y, Y), (C_X Y, Y)]_f & \xrightarrow{\Sigma_X} & [(C_X \Sigma_X Y, \Sigma_X Y), (C_X \Sigma_X Y, \Sigma_X Y]_f \\
\eta \downarrow & & \downarrow \eta \\
\hom_{\mathbb{Z}\pi_1(X,x)}(C_d(\widetilde{Y}, \widetilde{X}), \phi^* C_d(\widetilde{Z}, \widetilde{X})) & \xrightarrow{\Sigma} & (C_{d+1}(\widetilde{\Sigma_X Y}, \widetilde{X}), \phi^* C_{d+1}(\widetilde{\Sigma_X Y}, \widetilde{X}))
\end{array}$$

Hence it suffices to show

$$\theta(\Sigma_X g) - \theta(j \circ i)(\theta(f) \;=\; -\Theta(j)(\theta(g)) + \theta(j \circ i)(\theta(f) \tag{4.5}$$

for a map $(\widehat{g}, g) : (C_X Y, Y) \longrightarrow (C_X Y, Y)$ with $g|_X = f$ where $j : Y \longrightarrow \Sigma_X Y$ is the canonical inclusion. Since $\Sigma_X g$ is defined as $\widehat{g} \cup_g \widehat{g}$ and $\widehat{i} \circ i$ is a homotopy equivalence, this follows from additivity and homotopy invariance under homotopy equivalences. This finishes the proof of $\tau_{d+1} = \tau_d$ for $d \geq 2$. In the sequel we write

$$\tau \quad := \quad = \tau_d \qquad \text{for } d \geq 2. \tag{4.6}$$

We have defined a map

$$\tau(X, f) : U(X, f) \longrightarrow \Theta(X, f)$$

for an endomorphism $f : X \longrightarrow X$ of a finite connected CW-complex X. If X is not connected, we always assume $\pi_0(f) = \mathrm{id}$ and we define $\tau(X, f)$ by requiring that the following diagram commutes whose horizontal arrows are induced by the various inclusions of the components of X and whose upper horizontal arrow is an isomorphism

$$\begin{array}{ccc}
\oplus_{C \in \pi_0(X)} U(C, f|_C) & \xrightarrow{\cong} & U(X, f) \\
{\scriptstyle \oplus_{C \in \pi_0(X)} \tau(C, f|_C)} \downarrow & & \downarrow {\scriptstyle \tau(X,f)} \\
\oplus_{C \in \pi_0(X)} \Theta(C, f|_C) & \longrightarrow & \Theta(X, f)
\end{array}$$

Next we show that τ defines a natural transformation, i.e. we must show for any morphism $h : (X_0, f_0) \longrightarrow (X_1, f_1)$ in $\mathrm{End}(\mathcal{C})$ that $\tau(X_1, f_1) \circ U(h) = \Theta(h) \circ \tau(X_0, f_0)$. It suffices to prove this in the special case where X_0 and X_1 are connected. Let Y_0 be a d-extension of X for some $d \geq 2$ and $(\widehat{g_0}, g_0) : (C_{X_0} Y_0, Y_0) \longrightarrow (C_{X_0} Y_0, Y_0)$ be an endomorphism with $g|_X = f_0$. Let Y_1 be the retractive space over X given by the push out of $Y_0 \xleftarrow{i_0} X_0 \xrightarrow{h} X_1$ and let $(\widehat{g_1}, g_1) : (C_{X_1} Y_1, Y_1) \longrightarrow (C_{X_1} Y_1, Y_1)$ be given by the push out property and g_0, f_0 and f_1. Then the induction of the $\mathbb{Z}\pi_1(X_0, x_0)$-chain map $C(\widetilde{g_0}, \widetilde{f_0}) : C(\widetilde{Y_0}, \widetilde{X_0}) \longrightarrow \phi_0^* C(\widetilde{Y_0}, \widetilde{X_0})$ with the map induced by $\pi_1(h, x) : \pi_1(x_0, x_0) \longrightarrow \pi_1(X_1, h(x_0))$ is the $\mathbb{Z}\pi_1(X_1, h(x_0))$-chain map $C(\widetilde{g_1}, \widetilde{f_1}) : C(\widetilde{Y_1}, \widetilde{X_1}) \longrightarrow \phi_1^* C(\widetilde{Y_1}, \widetilde{X_1})$ where we have fixed a base point $x_0 \in X_0$ and a path w from $f_0(x_0)$ to x_0 and use for X_1 the base point $h(x_0)$ and the path $h(w)$. Hence it suffices to show

$$\Theta(h) \circ \Theta(\widehat{i_0} \circ i_0)^{-1} \circ \Theta(i_0)(\theta(g_0)) - \Theta(h)(\Theta(f_0) = \Theta(\widehat{i_1} \circ i_1)^{-1} \circ \Theta(i_1)(\theta(g_1)) - \theta(f_1).$$

This follows from additivity applied to $g_1 = g_0 \cup_{f_0} f_1$. This finishes the definition of the natural transformation

$$\tau : U \quad \longrightarrow \quad \Theta \tag{4.7}$$

between functors from $\mathrm{End}(\mathcal{C})$ to ABEL.

Next we have to show that $\tau(X, f)$ maps $u(X, f)$ to $\theta(X, f)$ and that the natural transformation τ is uniquely determined by this property. Let Y_n be the push out

$$\begin{array}{ccc}
X_{n-1} & \xrightarrow{j_{n-1}} & X \\
{\scriptstyle k_{n-1}} \downarrow & & \downarrow {\scriptstyle i_n} \\
X_n & \xrightarrow{l_n} & Y_n
\end{array}$$

where all arrows are canonical inclusions. There is a canonical retraction $r_n : Y_n \longrightarrow X$ induced by the inclusions of X_{n-1}, X_n and X in X. If $f_k : X_k \longrightarrow X_k$ is the restriction of f to X_k, then the push out above yields an endomorphism $g_n : Y_n \longrightarrow Y_n$ defined by $g_n = f_n \cup_{f_{n-1}} f$. We obtain in $\Theta(Y_n)$ from additivity

$$\theta(g_n) = \Theta(i_n)(\theta(f)) + \Theta(l_n)(\theta(f_n)) - \Theta(l_n \circ k_{n-1})(\theta(f_{n-1})).$$

Applying $\Theta(r_n)$ yields

$$\Theta(r_n)(\theta(g_n)) = \theta(f) + \Theta(j_n)(\theta(f_n)) - \Theta(j_{n-1})(\theta(f_{n-1})).$$

We conclude in $\Theta(X, f)$:

$$\theta(f) \quad = \quad \sum_{n=0}^{\dim(Y)} \left(\Theta(r_n)(\theta(g_n)) - \theta(f)\right). \tag{4.8}$$

In particular we get for (U, u) in $U(X, f)$

$$u(f) \quad = \quad \sum_{n=0}^{\dim(Y)} \left(U(r_n)(u(g_n)) - u(f)\right). \tag{4.9}$$

If $n \geq 2$, then Y_n is a n-extension of X for $n \geq 2$ and because of Theorem 1.4

$$U(r_n)(u(g_n)) - u(f) = u(C_n(\widetilde{g_n}, \widetilde{f})).$$

Since $r_n \circ g_n = f \circ r_n$ holds, there is a canonical extension $\widehat{g_n}$ of g_n. Since $\widehat{i} \circ i \circ r_n$ and $\widehat{i}$ are homotopic morphisms from (Y, g_n) to $(C_X, \widehat{g_n})$ in $\mathrm{End}(\mathcal{C})$ we conclude from homotopy invariance $\Theta(r_n) = \Theta(\widehat{i} \circ i)^{-1} \circ \Theta(i)$. Hence the definition of $\tau(X, f)$ implies for $n \geq 2$

$$\tau(X, f)\left(U(r_n)(u(g_n)) - u(f)\right) \quad = \quad \Theta(r_n)(\theta(g_n)) - \theta(f). \tag{4.10}$$

Next we show that equation 4.10 holds also for $n = 0, 1$. The following two equations are direct consequences of 4.5 (which is true for all $n \geq 0$)

$$\begin{aligned} \Theta(r_n)(\theta(g_n)) - \theta(f) &= -\Theta(\Sigma_X r_n)(\theta(\Sigma_X g_n)) + \theta(f); \\ \tau(U(r_n)(u(g_n)) - u(f)) &= -\tau(U(\Sigma_X r_n)(u(\Sigma_X g_n)) + u(f). \end{aligned}$$

Notice that $\Sigma_X Y$ is a $n+1$-extension if Y is an n-extension and $\Sigma_X r_n$ plays the role of r_{n+1} for $\Sigma_X Y$. Hence 4.10 holds for n if it is true for $n+1$. Therefore 4.10 is true for all $n \geq 0$. Now 4.8, 4.9 and 4.10 imply

$$\tau(X, f)(u(X, f)) \quad = \quad \theta(X, f). \tag{4.11}$$

Notice that τ is uniquely determined by property 4.11 since any element in $U(X, f)$ can be realized for an d-extension Y for $d \geq 2$ and endomorphism $(\widehat{g}, g) : (C_X Y, Y) \longrightarrow (C_X Y, Y)$ by

$$U(\widehat{i} \circ i)^{-1} \circ U(\widehat{i})(u(g)) - u(f).$$

This finishes the proof of Theorem 2.5. □

Remark 4.12. One may think that one could also use the following easier construction instead of the map τ_Y. Namely, given an endomorphism $f : X \longrightarrow X$ and a retractive space Y over X one may consider endomorphisms $g : Y \longrightarrow Y$ satisfying $r \circ g = f \circ r$. If $[Y,Y]_f^r$ is the set of homotopy classes of such maps where the homotopy h also satisfies $r \circ h_t = f \circ r$ for all $t \in [0,1]$, one can define a map

$$\Theta'_{Y,f} : [Y,Y]_f^r \longrightarrow \Theta(X,f)$$

by sending $[g]$ to $\Theta(r)(\theta(g))$. There is an obvious map

$$e : [Y,Y]_f^r \longrightarrow [(C_X Y, Y), C_X Y)]_f$$

satisfying $\tau_Y \circ e = \tau'_Y$ since any map $g : Y \longrightarrow Y$ with $r \circ g = f \circ r$ has a canonical extension $\widehat{g} : C_X Y \longrightarrow C_X Y$ and $\Theta(\widehat{i \circ i})^{-1} \circ \Theta(i) = \Theta(r)$ holds. The problem is, however, that e is not bijective and hence the passage from geometry to algebra in Lemma 4.2 does not work for $[Y,Y]_f^r$. That e is not bijective, can be easily seen from the example $Y = X \vee S^2$. Then a map $g : Y \longrightarrow Y$ satisfying $r \circ g = f \circ r$ is the same as a pointed map $S^2 \longrightarrow S^2$ and $[Y,Y]_f^r$ is isomorphic to $\mathbb{Z}$ whereas $[(C_X Y, Y), C_X Y)]_f$ is in general larger, namely $\hom_{\mathbb{Z}\pi}(\mathbb{Z}\pi, \phi^*\mathbb{Z}\pi)$. □

5 The construction of the transfer map

Suppose we are given the following geometric data:

Data 5.1. Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of spaces of the homotopy type of connected finite CW-complexes and let

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & B \end{array}$$

be a commutative square. □

In this section we want to assign to these data a (natural) homomorphism

$$\operatorname{trf}_{\overline{f},f} : U(f) \quad \longrightarrow \quad U(\overline{f}). \tag{5.2}$$

such that $\operatorname{trf}_{\overline{f},f}(u(f)) = u(\overline{f})$ holds and examine its properties in Section 6. Since our invariant $u(f)$ determines other invariants as explained in Section 3 these results for u give also information for the other invariants. Transfer questions for fixed point theory have also been investigated for instance in [9], [10], and [18, Chapter IV].

Remark 5.3. We mention that $(U(f), u(f))$ also makes sense for endomorphisms $f : X \longrightarrow X$ of spaces X of the homotopy type of a finite CW-complex. The algebraic definition of

$u(f : C \longrightarrow C)$ via chain complexes extends to chain complexes of the homotopy type of a finite chain complex using commutativity and homotopy invariance, namely, choose any chain homotopy equivalence $g : C \longrightarrow D$ for a finite chain complex D and define $u(f)$ by $u(g \circ f \circ g^{-1})$ for any chain homotopy inverse g^{-1} of g. □

From the algebraic point of view we will need the following algebraic data to define the transfer.

Data 5.4. 1. A short exact sequence of groups $\{1\} \to \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \to \{1\}$;

2. A commutative diagram of group homomorphisms

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & \Delta & \xrightarrow{i} & \Gamma & \xrightarrow{p} & \pi & \longrightarrow & \{1\} \\ & & \overline{\phi}_f \downarrow & & \overline{\phi} \downarrow & & \phi \downarrow & & \\ \{1\} & \longrightarrow & \Delta & \xrightarrow{i} & \Gamma & \xrightarrow{p} & \pi & \longrightarrow & \{1\} \end{array}$$

3. A finite free $\mathbb{Z}\Delta$-chain complex C together with a Γ-twist L in the sense of [27, Definition 5.1 on page 155]. (A Γ-twist L on C is a collection $\{[L(\gamma)] \mid \gamma \in \Gamma\}$ of $\mathbb{Z}\Delta$-homotopy classes of $\mathbb{Z}\Delta$-chain maps $L(\gamma) : C \longrightarrow c_\gamma^* C$ such that for $\delta \in \Delta$ the class $[L(\delta)]$ is represented by the map $l(\delta)$ given by left multiplication with δ and that the $\mathbb{Z}\Delta$-chain maps $L(\gamma_1) \circ L(\gamma_2)$ and $L(\gamma_1 \cdot \gamma_2)$ from C to $(c_{\gamma_1 \cdot \gamma_2})^* C$ are $\mathbb{Z}\Delta$-chain homotopic where $c_\gamma : \Delta \longrightarrow \Delta$ sends δ to $\gamma \delta \gamma^{-1}$);

4. A $\mathbb{Z}\Delta$-chain homotopy class $[t]$ of $\mathbb{Z}\Delta$-chain maps $t : C \longrightarrow \overline{\phi}_f^* C$ such that for any $\gamma \in \Gamma$ the following diagram commutes up to $\mathbb{Z}\Delta$-chain homotopy:

$$\begin{array}{ccc} C & \xrightarrow{t} & \overline{\phi}_f^* C \\ {\scriptstyle L(\gamma)} \downarrow & & \downarrow {\scriptstyle \overline{\phi}_f^* L(\overline{\phi}(\gamma))} \\ (c_\gamma)^* C & \xrightarrow{(c_\gamma)^* t} & (c_{\overline{\phi}(\gamma)})^* \overline{\phi}^* C = \overline{\phi}_f^* (c_\gamma)^* C \end{array}$$

□

One can think of a Γ-twist as an extension of the Δ-operation to a Γ-operation up to homotopy.

Example 5.5. Next we explain how the geometric data 5.1 yield algebraic data 5.4. Choose a point $e \in E$ and a path w from $\overline{f}(e)$ to e. Put

$$\begin{array}{rcl} \Delta & = & \ker\left(\pi_1(p,e) : \pi_1(E,e) \longrightarrow \pi_1(B,p(e))\right); \\ \Gamma & = & \pi_1(E,e); \\ \pi & = & \pi_1(B,p(e)). \end{array}$$

Then we obtain a commutative square with exact sequence rows

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \Delta & \xrightarrow{i} & \Gamma & \xrightarrow{p} & \pi & \longrightarrow & \{1\} \\
 & & \overline{\phi}_f \downarrow & & \overline{\phi} \downarrow & & \phi \downarrow & & \\
\{1\} & \longrightarrow & \Delta & \xrightarrow{i} & \Gamma & \xrightarrow{p} & \pi & \longrightarrow & \{1\}
\end{array}$$

where i is given by the inclusion, p is $\pi_1(p,e)$ and $\overline{\phi}$ is given by

$$\overline{\phi} : \Gamma = \pi_1(E,e) \xrightarrow{\pi_1(\overline{f},e)} \pi_1(E,f(e)) \xrightarrow{c_w} \Gamma\pi_1(E,e).$$

Any path u from e_0 to e_1 in E defines a pointed homotopy class of pointed maps depending only on the homotopy class of u relative endpoints, the so called *pointed fibre transport* (see [26, section 6])

$$\sigma(u) : (F_{e_1}, e_1) \longrightarrow (F_{e_0}, e_0).$$

We have $\sigma(u * v) = \sigma(u) \circ \sigma(v)$ and σ applied to the trivial path is represented by the identity. In particular we get for each $e \in E$ a homomorphism

$$\sigma_e : \pi_1(E,e) \longrightarrow [(F_e,e),(F_e,e)]^+$$

from the fundamental group into the monoid of pointed homotopy classes of pointed selfmaps of (F_e,e). If $s(u) : (F_e,e) \longrightarrow (F_e,e)$ is a representative of $\sigma(u)$ and $q : (\overline{F_e},\overline{e}) \longrightarrow (F_e,e)$ is the covering of F_e associated to the epimorphism $\pi_1(F_e,e) \longrightarrow \Delta$ induced by the inclusion of F_e into E, then there is a unique lift $\overline{s(u)} : (\overline{F_e},\overline{e}) \longrightarrow (\overline{F_e},\overline{e})$. This map is $c_u : \Delta_e \longrightarrow \Delta_e$-equivariant where as before c_u is conjugation with u. Its c_u-homotopy class depends only on u and not on the choice of $s(u) \in \sigma(u)$. Thus we obtain a $\Gamma = \pi_1(E,e)$-twist on the $\mathbb{Z}\Delta$-chain complex $C(\overline{F_e})$ by

$$L(u) = [C(s(u))].$$

Recall that we have chosen a path w from $\overline{f}(e)$ to e. Let $s(w^-) : (F_e, \overline{f}(e)) \longrightarrow (F_e,e)$ be a representative of $\sigma(w^-)$ where w^- is the inverse of w. Then $s(w) \circ \overline{f}\mid_{F_e} : (F_e,e) \longrightarrow (F_e,e)$ lifts uniquely to a map

$$\overline{s(w) \circ \overline{f}\mid_{F_e}} : (\overline{F_e},\overline{e}) \longrightarrow (\overline{F_e},\overline{e}).$$

This map is $\overline{\phi}$-equivariant and its equivariant homotopy class depends only on the homotopy class relative end points of w. Define a $\mathbb{Z}\Delta$-chain map

$$t = C(\overline{s(w^-) \circ \overline{f}\mid_{F_e}}) : C(\overline{F_e}) \longrightarrow \overline{\phi}^* C(\overline{F_e}).$$

Now we have the data 5.4. □

For the data 5.4 we want to define a map

$$\mathrm{trf} : U(\pi,\phi) \quad \longrightarrow \quad U(\Gamma,\overline{\phi}). \tag{5.6}$$

We will leave it to the reader to check that, given the data 5.1, the collection of the maps 5.6, which are obtained from the data 5.4 for the various choices of $e \in E$ and path w from $\overline{f}(e)$ to e as explained in Example 5.5, fit together to give the desired transfer map 5.2.

Consider a $\mathbb{Z}\pi$-homomorphism $\alpha : \mathbb{Z}\pi \longrightarrow \phi^*\mathbb{Z}\pi$. We can write $\alpha(1) = \sum_{w\in\pi} \lambda_w \cdot w$. Choose for any $w \in \pi$ a lift $\overline{w} \in \Gamma$ and for each $\gamma \in \Gamma$ a representative $L(\gamma)$ of $[L(\gamma)]$. Define a $\mathbb{Z}\Gamma$-chain map

$$X(\alpha) : \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \quad\longrightarrow\quad \overline{\phi}^*\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \qquad \gamma \otimes v \mapsto \sum_{w\in\pi} \lambda_w \cdot \overline{\phi}(\gamma)\overline{w} \otimes L(\overline{w}^{-1}) \circ t(v). \tag{5.7}$$

In order to check that this is a well-defined $\mathbb{Z}\Gamma$-chain map, we have to check that $\gamma \otimes v$ and $\gamma\delta \otimes \delta^{-1}v$ have the same image

$$\overline{\phi}(\gamma\delta)\overline{w}\otimes L(\overline{w}^{-1})t(\delta^{-1}v) = \overline{\phi}(\gamma)\overline{w}\,\overline{w}^{-1}\overline{\phi}(\delta)\overline{w}\otimes\overline{w}^{-1}\overline{\phi}(\delta^{-1})\overline{w}L(\overline{w}^{-1})t(v) = \overline{\phi}(\gamma)\overline{w}\otimes L(\overline{w}^{-1})t(v).$$

One easily checks that the $\mathbb{Z}\Gamma$-chain homotopy class of $X(\alpha)$ is independent of the choices of $\overline{w}$ and $L(\gamma)$. Given a $\mathbb{Z}\pi$-homomorphism

$$\alpha = (\alpha_{i,j})_{i,j} : \oplus_{i=1}^n \mathbb{Z}\pi \longrightarrow \oplus_{j=1}^m \mathbb{Z}\pi$$

we obtain a $\mathbb{Z}\Gamma$-chain map unique up to $\mathbb{Z}\Gamma$-homotopy by

$$X(\alpha) = (X(\alpha_{i,j}))_{i,j} : \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \longrightarrow \oplus_{j=1}^m \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C$$

We want to define the transfer map trf of 5.6 by requiring

$$\operatorname{trf}\left([\alpha : \oplus_{i=1}^n \mathbb{Z}\pi \longrightarrow \oplus_{i=1}^n \phi^*\mathbb{Z}\pi]\right) = u\left(X(\alpha) : \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \longrightarrow \oplus_{i=1}^n \phi^*\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C\right),$$

where u is the invariant of Definition 1.2. Since the $\mathbb{Z}\Gamma$-chain homotopy class of $X(\alpha)$ depends only on α, the expression $u(X(\alpha))$ is well-defined. In order to check that the map trf is well-defined one must verify that the relations in $U(\mathbb{Z}\pi, \phi)$ are respected. Suppose we have a commutative square of $\mathbb{Z}\pi$-maps with isomorphisms as vertical maps:

$$\begin{array}{ccc} \oplus_{i=1}^n \mathbb{Z}\pi & \xrightarrow{\alpha} & \oplus_{i=1}^n \phi^*\mathbb{Z}\pi \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\phi^*\beta} \\ \oplus_{i=1}^n \mathbb{Z}\pi & \xrightarrow{\alpha'} & \oplus_{i=1}^n \phi^*\mathbb{Z}\pi \end{array}$$

Then one can construct a $\mathbb{Z}\Gamma$-chain map

$$X(\beta) : \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \longrightarrow \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C$$

analogously as for α using id instead of t. Then the following square commutes up to $\mathbb{Z}\Gamma$-chain homotopy:

$$\begin{array}{ccc} \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C & \xrightarrow{X(\alpha)} & \oplus_{i=1}^n \overline{\phi}^*\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \\ {\scriptstyle X(\beta)}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\phi}^* X(\beta)} \\ \oplus_{i=1}^n \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C & \xrightarrow{X(\alpha')} & \oplus_{i=1}^n \overline{\phi}^*\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C \end{array}$$

Since $X(\beta^{-1})$ is a $\mathbb{Z}\Gamma$-chain homotopy inverse of $X(\beta)$, we get from homotopy invariance and Lemma 1.6

$$u(X(\alpha)) = u(X(\alpha')).$$

Now consider a block endomorphism

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_0 \\ 0 & \alpha_2 \end{pmatrix} : (\oplus_{i=1}^n \mathbb{Z}\pi) \oplus \left(\oplus_{j=1}^m \mathbb{Z}\pi\right) \longrightarrow (\oplus_{i=1}^n \mathbb{Z}\pi) \oplus \left(\oplus_{j=1}^m \mathbb{Z}\pi\right).$$

Then $X(\alpha)$ also has block form and we get:

$$u(X(\alpha)) = u(X(\alpha_1)) + u(X(\alpha_2)).$$

This finishes the construction of the transfer maps 5.2 and 5.6 and the proof that they are well-defined. The main result of this section is the next theroem. One should compare its proof with the corresponding identifications of algebraic and geometric transfers in algebraic K and L-theory given by finiteness obstructions, Whitehead torsion [26] and surgery obstructions [32].

Theorem 5.8. *Given the geometric data 5.1, we get:*

$$\mathrm{trf}_{f,\overline{f}}(u(f)) = u(\overline{f}).$$

Proof. For each $n \geq 0$ we define (Y_n, g_n) to be the push out in $\mathrm{End}(\mathcal{C})$

$$\begin{array}{ccc} (B_{n-1}, f_{n-1}) & \xrightarrow{j_n} & (B, f) \\ {\scriptstyle k_{n-1}}\downarrow & & \downarrow{\scriptstyle i_n} \\ (B_n, f_n) & \xrightarrow{l_n} & (Y_n, g_n) \end{array}$$

The identity on B induces a retraction $r_n : Y_n \longrightarrow B$. By pulling back $p : (E, \overline{f}) \longrightarrow (B, f)$ with r_n and restricting to B_{n-1}, B_n and B again we obtain a push out

$$\begin{array}{ccc} (E\mid_{B_{n-1}}, \overline{f_{n-1}}) & \xrightarrow{\overline{j_{n-1}}} & (E, \overline{f}) \\ {\scriptstyle \overline{k_{n-1}}}\downarrow & & \downarrow{\scriptstyle \overline{i_n}} \\ (E\mid_{B_n}, \overline{f_n}) & \xrightarrow{\overline{l_n}} & (E_n, \overline{g_n}) \end{array}$$

and a retraction $\overline{r_n} : (E_n, \overline{g_n}) \longrightarrow (E, \overline{f})$ covering $r_n : (Y_n, g_n) \longrightarrow (B, f)$. We obtain from additivity

$$\begin{aligned} u(B, f) &= \sum_{n=0}^{\dim(B)} (U(r_n)(u(Y_n, g_n)) - u(B, f)); \\ u(E, \overline{f}) &= \sum_{n=0}^{\dim(B)} \left(U(\overline{r_n})(u(E_n, \overline{g_n})) - u(E, \overline{f})\right). \end{aligned}$$

Hence it suffices to show for $0 \leq n$

$$\operatorname{trf}_{f,\overline{f}}(U(r_n)(u(Y_n, g_n)) - u(B, f)) \quad = \quad U(\overline{r_n})(u(E_n, \overline{g_n})) - u(E, \overline{f}). \tag{5.9}$$

Next we show analogously as in the corresponding part of the proof of Theorem 2.5 that 5.9 holds for all $n \geq 0$ if we can prove it for $n \geq 2$. Since $r_n : (Y_n, g_n) \longrightarrow (B, f)$ is a morphism in $\operatorname{End}(\mathcal{C})$, we can canonically extend r_n to a retraction $\widehat{r_n} : C_B Y_n \longrightarrow B$ and g_n to $\widehat{g_n} : C_B Y_n \longrightarrow C_B Y_n$ such that $\widehat{r_n} : (C_B Y_n, \widehat{g_n}) \longrightarrow (B, f)$ is also a morphism in $\operatorname{End}(\mathcal{C})$. Define $(Y_{n\Sigma}, g_{n\Sigma})$ to be the push out in $\operatorname{End}(\mathcal{C})$

$$\begin{array}{ccc}
(Y_n, g_n) & \xrightarrow{i} & (C_B Y_n, \widehat{g_n}) \\
{\scriptstyle i}\downarrow & & \downarrow \\
(C_B Y_n, \widehat{g_n}) & \longrightarrow & (Y_{n\Sigma}, g_{n\Sigma})
\end{array}$$

and define the retraction $r_{n\Sigma} : (Y_{n\Sigma}, g_{n\Sigma}) \longrightarrow (B, f)$ by the push out property using $\widehat{r_n}$ and r_n. By pulling back $p : (E, \overline{f}) \longrightarrow (B, f)$ we obtain a push out in $\operatorname{End}(\mathcal{C})$

$$\begin{array}{ccc}
(E_n, \overline{g_n}) & \longrightarrow & (\widehat{E_n}, \overline{\widehat{g_n}}) \\
\downarrow & & \downarrow \\
(\widehat{E_n}, \overline{\widehat{g_n}}) & \longrightarrow & (E_{n\Sigma}, \overline{g_{n\Sigma}})
\end{array}$$

covering the previous push out and a retraction $\overline{r_{n\Sigma}} : (E_{n\Sigma}, \overline{g_{n\Sigma}}) \longrightarrow (E, \overline{f})$ covering $r_{n\Sigma}$. Now we get from additivity and invariance under homotopy equivalence and the fact that the inclusions $B \longrightarrow \widehat{Y}$ and $E \longrightarrow \widehat{E_n}$ are homotopy equivalences

$$\begin{array}{rcll}
U(\overline{r_{n\Sigma}})(u(E_{n\Sigma}, \overline{g_{n\Sigma}})) & = & -U(\overline{r_n})(u(E, \overline{g_n})) & \in U(E, \overline{f}); \\
U(r_{n\Sigma})(u(Y_{n\Sigma}, g_{n\Sigma})) & = & -U(r_n)(u(Y_n, g_n)) & \in U(B, f).
\end{array}$$

Hence

$$\operatorname{trf}_{f,\overline{f}}(U(r_n)(u(Y, g_n))) = u(E, \overline{g_n})$$

is equivalent to

$$\operatorname{trf}_{f,\overline{f}}\left(u(\widehat{Y_{n\Sigma}, g_{n\Sigma}})\right) = u(E_{n\Sigma}, \overline{g_{n\Sigma}}).$$

Notice that $Y_{n\Sigma}$ is a $(n+1)$-extension of B. Thus we have reduced the proof of Theorem 5.8 to the proof of the following statement:

Let $n \geq 2$ and $r : (Y, g) \longrightarrow (B, f)$ be a retraction in $\operatorname{End}(\mathcal{C})$ such that Y is a n-extension of X. Define a retraction $\overline{r} : (r^* E, \overline{g}) \longrightarrow (E, \overline{f})$ covering r by pulling back $p : (E, \overline{f}) \longrightarrow (B, f)$. Then we get in $U(E, f)$

$$\operatorname{trf}_{f,\overline{f}}(U(r)(u(Y, g)) - u(B, f)) \quad = \quad U(\overline{r})(u(r^* E, \overline{g})) - u(E, \overline{f}). \tag{5.10}$$

Fix $e \in E$ and a path w from $\overline{f}(e)$ to e. Choose characteristic maps for the cells in $Y - X$:

$$\begin{array}{ccc} \coprod_{i\in I} S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & B \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^n & \xrightarrow{\coprod_{i\in I} Q_i} & Y \end{array}$$

and paths w_i from $q_i(1)$ to $p(e) \in B$. Using these path each cell defines an element b_i in $\pi_n(Y, b, p(e))$ such that $\{b_i \mid i \in I\}$ is a $\mathbb{Z}\pi_1(B, p(e))$-basis. With respect to this basis we can write $C(\widetilde{g}, \widetilde{f}) : C(\widetilde{Y}, \widetilde{X}) \longrightarrow \phi^* C(\widetilde{Y}, \widetilde{X})$ using the identification of $C(\widetilde{Y}, \widetilde{X})$ with $\pi_n(Y, B, p(e))$ by the Hurewicz isomorphism as a $\mathbb{Z}\pi_1(B, p(e))$-map

$$C(\widetilde{g}, \widetilde{f}) : \oplus_{i\in I}\mathbb{Z}\pi_1(B, p(e)) \longrightarrow \phi^* \oplus_{i\in I} \mathbb{Z}\pi_1(B, p(e)).$$

Notice that $u(C(\widetilde{g}, \widetilde{f})) \in U(\mathbb{Z}\pi_1(B, p(e)), \phi)$ represents $U(r)(u(Y, g)) - u(B, f) \in U(B, f))$. Using the paths w_i above there are up to strong fibre homotopy equivalence unique fibre trivilizations

$$T_i : \pi_1(E, e) \times_{\Delta(e)} \widetilde{F_e} \times (D^n, S^{n-1}) \longrightarrow (\widetilde{Q_i^* E}, \widetilde{q_i^* E}).$$

They induce an explicit isomorphism

$$\rho : \oplus_{i\in I}\mathbb{Z}\pi_1(E, e) \otimes_{\Delta(e)} C(\overline{F_e}) \longrightarrow C(\widetilde{r^* E}, \widetilde{E}).$$

Let

$$X(C(\widetilde{g}, \widetilde{f})) : \oplus_{i\in I}\mathbb{Z}\pi_1(E, e) \otimes_{\Delta(e)} C(\overline{F_e}) \longrightarrow \oplus_{i\in I}\overline{\phi}^*\mathbb{Z}\pi_1(E, e) \otimes_{\Delta(e)} C(\overline{F_e})$$

be the $\mathbb{Z}\pi_1(E, e)$-chain map unique up to $\mathbb{Z}\pi_1(E, e)$-chain homotopy defined in section 7 for the choice of data 5.4 given by $p : E(\overline{f}) \longrightarrow (B, f)$. Then we have by definition of the transfer that $u(X(C(\widetilde{g}, \widetilde{f}))) \in U(\mathbb{Z}\pi_1(E, e), \overline{\phi})$ represents $\operatorname{trf}_{f,\overline{f}} (U(r)(u(Y, g)) - u(B, f))$. Notice that

$$u(C(\widetilde{\overline{g}}, \widetilde{\overline{f}}) : C(\widetilde{r^* E}, \widetilde{E}) \longrightarrow C(\widetilde{r^* E}, \widetilde{E}) \qquad \in U(\mathbb{Z}\pi_1(E, e), \overline{\phi})$$

represents $U(\overline{r}) \left(u(r^* E, \overline{g})) - u(E, \overline{f}) \in U(E, \overline{f})\right)$. Hence we have to show

$$u\left(X(C(\widetilde{g}, \widetilde{f}))\right) = u\left(C(\widetilde{\overline{g}}, \widetilde{\overline{f}})\right) \qquad \in U(\mathbb{Z}\pi_1(E, e), \overline{\phi}).$$

This claim follows if the following diagram of $\mathbb{Z}\pi_1(E, e)$-chain complexes commutes up to $\mathbb{Z}\pi_1(E, e)$-chain homotopy

$$\begin{array}{ccc} \oplus_{i\in I}\mathbb{Z}\pi_1(E, e) \otimes_{\Delta(e)} C(\overline{F_e}) & \xrightarrow{X(C(\widetilde{g}, \widetilde{f}))} & \oplus_{i\in I}\mathbb{Z}\pi_1(E, e) \otimes_{\Delta(e)} C(\overline{F_e}) \\ \rho\downarrow & & \downarrow\rho \\ C(\widetilde{r^* E}, \widetilde{E}) & \xrightarrow{C(\widetilde{\overline{g}}, \widetilde{\overline{f}})} & C(\widetilde{r^* E}, \widetilde{E}) \end{array}$$

The proof is omitted since it is a straightforward modification of the proof of Theorem 2.2 in [26, section 7]. This finishes the proof of Theorem 5.8. □

Remark 5.11. We mention that one can analogously define a transfer map

$$\operatorname{trf}_{\overline{f},f} : \Lambda(f) \longrightarrow \Lambda(\overline{f}) \tag{5.12}$$

satisfying

$$\operatorname{trf}_{f,\overline{f}}(\lambda(f)) = \lambda(\overline{f})$$

for the functorial Lefschetz invariant (Λ, λ) of Example 3.5. It sends the class $[u] \in \Lambda(\mathbb{Z}\pi, \Phi)$ to $\operatorname{tr}_{(\mathbb{Z}\Gamma,\overline{\Phi})}(X(r_u))$, where $\operatorname{tr}_{(\mathbb{Z}\Gamma,\overline{\Phi})}$ has been defined in 3.6, $r_u : \mathbb{Z}\pi \longrightarrow \Phi^*\mathbb{Z}\pi$ maps v to $\Phi(v)u$ and $X(r_u)$ has been defined in 5.7. □

6 Properties of the transfer map

In this section we prove vanishing results and a down-up-formula for the geometric transfer map. Moreover, we consider the special case S^n as fiber and treat (not necessarily free) S^1-actions. There are analogous computations of the transfer maps in algebraic K and L-theory concerning finiteness obstructions, Whitehead torsion [27] and surgery obstructions [33].

Theorem 6.1. *Suppose we are given the data 5.1. Then the transfer map*

$$\operatorname{trf}_{\overline{f},f} : U(B,f) \longrightarrow U(E,\overline{f})$$

and the element

$$u(E,\overline{f}) \in U(E,\overline{f})$$

vanish if one of the following conditions are satisfied:

1. *p is untwisted, i.e. the pointed fibre transport $\sigma_e : \pi_1(E,e) \longrightarrow [(F_{p(e)},e),(F_{p(e)},e)]^+$ (see [26, section 6])) is trivial. The fundamental group $\pi_1(F)$ of F is non-trivial or the Euler characteristic of F satisfies $\chi(F) = 0$. Moreover, the composition of $\overline{f}\,|_{F_b}: F_b \longrightarrow F_{f(b)}$ and $\omega(w) : F_{f(b)} \longrightarrow F_b$ is homotopic to the identity on F_b for some (and hence all) $b \in B$ and path w from b to $f(b)$, where $\omega(w)$ is given by the (free) fibre transport along paths in the base space;*

2. *The map $\pi_1(F,e) \longrightarrow \pi_1(E,e)$ induced by the inclusion is trivial and $\pi_1(B)$ operates trivially on $H_p(F)$ for all $p \geq 0$. The fundamental group $\pi_1(F)$ of F is non-trivial or the Euler characteristic of F satisfies $\chi(F) = 0$. Moreover, the map induced by the composition of $\overline{f}\,|_{F_b}: F_b \longrightarrow F_{f(b)}$ and $\omega(w) : F_{f(b)} \longrightarrow F_b$ on $H_*(F_b)$ is trivial for some (and hence all) $b \in B$ and path w from b to $f(b)$.* □

This theorem is a direct consequence of [27, section 4], Lemma 6.2 and Lemma 6.5 which we will prove below.

Lemma 6.2. *Assume for the data 5.4 that Δ is central in Γ, the $\mathbb{Z}\Delta$-chain map $l(\delta) : C \longrightarrow C$ given by multiplication with δ is $\mathbb{Z}\Delta$-chain homotopic to* id *and the Γ-twist L on C is trivial, i.e., for all $\gamma \in \Gamma$ we have $[L(\gamma)] = [\mathrm{id}]$. Assume furthermore $\overline{\phi}_f = \mathrm{id}$ and $t = \mathrm{id}$. Then* trf *is zero if Δ is non-trivial. If Δ is trivial and we identify Γ and π, the transfer* tr *is given by multiplication with the Euler characteristic $\chi(C) = \sum_{i\geq 0}(-1)^i \cdot \dim_{\mathbb{Z}\Delta}(C_i)$.*

Proof. We get for $\alpha : \oplus_{i=1}^n \mathbb{Z}\pi \longrightarrow \oplus_{j=1}^n \phi^*\mathbb{Z}\pi$

$$\mathrm{trf}(u(\alpha)) = \sum_{i\geq 0} (-1)^i \cdot u\left((X(\alpha)_i : \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C_i \longrightarrow \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C_i\right)$$

One easily checks that for an appropriate $\mathbb{Z}\Gamma$-map $\beta : \oplus_{k=1}^n \mathbb{Z}\Gamma \longrightarrow \oplus_{k=1}^n \overline{\phi}^* \mathbb{Z}\Gamma$ we have for all $i \geq 0$

$$X(\alpha)_i = \oplus_{j=1}^{\dim_{\mathbb{Z}\Delta}(C_i)} \beta.$$

This implies

$$\mathrm{trf}(u(\alpha)) = \chi(C) \cdot u(\beta).$$

If Δ is trivial, β can be chosen to be α if we identify Γ and π. Suppose that Δ is non-trivial. Then the classical Lefschetz number, here with values in the commutative ring $\mathbb{Z}\Delta$, of the $\mathbb{Z}\Delta$-chain map $l(\delta) : C \longrightarrow C$ is $\chi(C) \cdot \delta \in \mathbb{Z}\Delta$ for all $\delta \in \Delta$. Since $l(d)$ and $l(1)$ are $\mathbb{Z}\Delta$-chain homotopic by assumption, we have for all $\delta \in \Delta$:

$$\chi(C) \cdot d = \chi(C) \cdot 1 \in \mathbb{Z}\Delta.$$

This implies $\chi(C) = 0$. This finishes the proof of Lemma 6.2. □

Let $p_* : U(\mathbb{Z}\Gamma, \overline{\phi}) \longrightarrow U(\mathbb{Z}\pi, \phi)$ be the map induced by induction with $p : \mathbb{Z}\Gamma \longrightarrow \mathbb{Z}\pi$. Notice that this map is well-defined since we require $\phi \circ p = p \circ \overline{\phi}$. Let $Sw^f(\pi, \phi)$ respectively $Sw(\pi, \phi)$ be the Grothendieck group of $\mathbb{Z}\pi$-maps $f : M \longrightarrow \Phi^* M$ for $\mathbb{Z}\pi$-modules M which are finitely generated free respectively finitely generated over $\mathbb{Z}$. In other words, its definition is analogous to the definition of $U(\mathbb{Z}\pi, \phi)$ in Definition 1.2 with the exception that all $\mathbb{Z}\pi$-modules are required to be finitely generated free respectively finitely generated over $\mathbb{Z}$ instead of requiring that they are finitely generated free over $\mathbb{Z}\pi$. The tensor product over $\mathbb{Z}$ with the diagonal π-action induces a pairing

$$\otimes_{\mathbb{Z}} : Sw^f(\pi, \phi) \otimes U(\mathbb{Z}\pi, \phi) \quad \longrightarrow \quad U(\mathbb{Z}\pi, \phi). \tag{6.3}$$

Notice that the analogous pairing for $Sw(\pi, \phi)$ is not well-defined because tensoring over $\mathbb{Z}$ with a finitely generated abelian group M is an exact functor if and only if M is free. Given the data 5.4 there is an element in $Sw(\pi, \phi)$

$$h(C, L) \quad := \quad \sum_{i \geq 0} (-1)^i \cdot [H_i(\mathbb{Z} \otimes_{\mathbb{Z}\Delta} t) : H_i(\mathbb{Z} \otimes_{\mathbb{Z}\Delta} C) \longrightarrow \phi^* H_i(\mathbb{Z} \otimes_{\mathbb{Z}\Delta} C)], \tag{6.4}$$

where the action of $w \in \pi$ on $H_i(\mathbb{Z} \otimes_{\Delta} C)$ is induced by

$$\mathbb{Z} \otimes_{\Delta} C \longrightarrow \mathbb{Z} \otimes_{\Delta} C \qquad\qquad n \otimes v \mapsto n \otimes L(\overline{w})(v)$$

for any lift $\overline{w} \in \Gamma$ of w. The next lemma is a down-up formula for the transfer, i.e. it computes the composition of the transfer with the map induced by p.

Lemma 6.5.

1. *The canonical map $I : Sw^f(\pi, \phi) \longrightarrow Sw(\pi, \phi)$ is an isomorphism.*

2. *The composition* $p_* \circ \mathrm{trf} : U(\mathbb{Z}\pi, \phi) \longrightarrow U(\mathbb{Z}\pi, \phi)$ *is given by the pairing 6.3 and the element* $I^{-1}(h(C, L)) \in Sw^f(\pi, \phi)$ *for the element* $h(C, L) \in Sw(\pi, \phi)$ *defined in 6.4.*

3. *Assume that the* π*-action on* $H_*(\mathbb{Z} \otimes_{\mathbb{Z}\Delta} C)$ *is trivial and* $H_i(t)$ *is the identity for all* $i \geq 0$*. Then the composition* $p_* \circ \mathrm{trf} : U(\mathbb{Z}\pi, \phi) \longrightarrow U(\mathbb{Z}\pi, \phi)$ *is multiplication with the Euler characteristic* $\chi(C) = \sum_{i\geq 0}(-1)^i \cdot \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}\Delta} C)$*. If additionally* Δ *is trivial and* $\chi(C) = 0$*, then* trf *vanishes.*

Proof. This proof is a modification of the proof of [27, Corollary 6.4 on page 159] and [40, Lemma 2.2]. □

Corollary 6.6. *Let* G *be a non-trivial connected compact Lie group and let* X *be a free* G*-CW-complex with compact quotient* $G\backslash X$*. Let* $f : X \longrightarrow X$ *be a* G*-map. Then*

$$u(f) = 0. \qquad \in U(f).$$

Proof. This follows from Theorem 6.1.1 since $G \longrightarrow X \longrightarrow G\backslash X$ is an untwisted fibration over a finite CW-complex and $\chi(G) = 0$. □

We have the following result for S^1-actions.

Theorem 6.7. *Let* $f : X \longrightarrow X$ *be a* S^1*-endomorphism of a finite* S^1*-CW-complex* X*. Denote by* $i : (X^{S^1}, f^{S^1}) \longrightarrow (X, f)$ *the morphism in End(*$\mathcal{C}$*) induced by the inclusion of the fixed point set. Then we have:*

$$U(i)(u(X^{S^1}, f^{S^1})) = u(X, f) \qquad \in U(X, f).$$

In particular $u(X, f)$ *vanishes if the* S^1*-action has no fixed points.*

Proof. Since for any subgroup $H \subset S^1$ different from S^1 the quotient S^1/H is isomorphic to S^1 again, one reduces the claim using additivity and induction over the orbit bundles and the skeletons to the following assertion: If Y is the S^1-push out

$$\begin{array}{ccc} \coprod_{i\in I} S^1 \times S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & X \\ \downarrow & & \downarrow i \\ \coprod_{i\in I} S^1 \times D^n & \xrightarrow{\coprod_{i\in I} Q_i} & Y \end{array}$$

for $n \geq 0$ and $(g, f) : (Y, X) \longrightarrow (Y, X)$ is a pair of S^1-maps, then we have

$$U(i)(u(X, f)) = u(Y, g)).$$

Notice that this would follow from Corollary 6.6 if S^1 acts freely on X and we want to reduce the assertion to this case by the following construction. Choose a unitary S^1-representation V such that S^1 acts freely on the unit sphere SV and $\dim(Y) < \dim(SV)$ holds. Then $SV \times X$ with the diagonal S^1-action is free and the projection $\mathrm{pr} : SV \times X \longrightarrow X$ is $n+1$-connected. Let $k : S^1 \times S^{n-1} \longrightarrow SV$ be the composition of the projection onto S^1 and an inclusion of an orbit. Define Z by the S^1-push out:

$$\begin{array}{ccc}
\coprod_{i\in I} S^1 \times S^{n-1} & \xrightarrow{\coprod_{i\in I} k\times q_i} & SV \times X \\
\downarrow & & \downarrow j \\
\coprod_{i\in I} S^1 \times D^n & \xrightarrow{\coprod_{i\in I} P_i} & Z
\end{array}$$

Let $\overline{\mathrm{pr}} : Z \longrightarrow Y$ be the map induced by the commutative diagram

$$\begin{array}{ccccc}
\coprod_{i\in I} S^1 \times D^n & \longleftarrow & \coprod_{i\in I} S^1 \times S^{n-1} & \xrightarrow{\coprod_{i\in I} k\times q_i} & SV \times X \\
\mathrm{id}\downarrow & & \mathrm{id}\downarrow & & \mathrm{pr}\downarrow \\
\coprod_{i\in I} S^1 \times D^n & \longleftarrow & \coprod_{i\in I} S^1 \times S^{n-1} & \xrightarrow{\coprod_{i\in I} q_i} & X
\end{array}$$

Then we obtain a S^1-push out

$$\begin{array}{ccc}
SV \times X & \xrightarrow{\mathrm{pr}} & X \\
j\downarrow & & \downarrow i \\
Z & \xrightarrow{\overline{\mathrm{pr}}} & Y
\end{array}$$

Let $[Z,Z]^{S^1}_{id\times f:SV\times X\longrightarrow SV\times X}$ be the S^1-equivariant homotopy classes of maps $g : Z \longrightarrow Z$ relative $\mathrm{id}\times f : SV \times X \longrightarrow SV \times X$. This means that such a map g extends $\mathrm{id}\times f$ and a S^1-homotopy connecting two such maps is stationary on $SV \times X$. The push out property yields a map

$$l : [Z,Z]^{S^1}_{id\times f:SV\times X\longrightarrow SV\times X} \longrightarrow [Y,Y]^{S^1}_{f:X\longrightarrow X}. \tag{6.8}$$

Next we want to show that this map is bijective. We obtain a bijection

$$[(Z,Z]^{S^1}_{id\times f:SV\times X\longrightarrow SV\times X} \longrightarrow \prod_{i\in I}[D^n,Z]_{(id\times f)\circ(k\times q_i)|_{\{1\}\times S^{n-1}}):S^{n-1}\longrightarrow SV\times X}$$

by sending (g,f) to $\left(g\circ P_i\mid_{\{1\}\times D^n}\right)_{i\in I}$. The composition

$$\overline{\mathrm{pr}}\circ j\circ(\mathrm{id}\times f)\circ(k\times q_i)\mid\{1\}\times S^1 : S^{n-1}\longrightarrow Y$$

is nullhomotopic, a nullhomotopy comes from $Q_i\mid_{\{1\}\times D^n}$. Since $\mathrm{pr} : SV \times X \longrightarrow X$ and hence $\overline{\mathrm{pr}}$ is $(n+1)$-connected we can extend $j\circ(\mathrm{id}\times f)\circ(k\times q_i)\mid_{\{1\}\times S^{n-1}}: S^{n-1}\longrightarrow Z$ to a map $h_i : S^n_- \longrightarrow Z$ for $i \in I$ where S^n_- denotes the lower hemisphere. There is an obvious bijection given by extending with h_i if we think of D^n as the upper hemisphere S^n_+:

$$[D^n,Z]_{(id\times f)\circ(k\times q_i)|_{\{1\}\times S^{n-1}}:S^{n-1}\longrightarrow SV\times X} \longrightarrow [S^n,Z]_{h_i:S^n_-\longrightarrow Z}$$

The forgetful map, which remembers only that the base point $s \in S^{n-1} \subset\in S^n_-$ is mapped to $x_i := j\circ(\mathrm{id}\times f)\circ(k\times q_i)(1,s)$, is a map

$$[S^n,Z]_{h_i:S^n_-\longrightarrow Z} \longrightarrow \pi_n(Z,x_i).$$

Since S^n_- is contractible in S^n, a cofibration argument shows that this map is bijective. Hence we have constructed a bijection

$$[Z,Z]^{S^1}_{id\times f:SV\times X\longrightarrow SV\times X} \longrightarrow \prod_{i\in I}\pi_n(Z,x_i)$$

depending on the choice of the extensions h_i. These extensions h_i induce by composition with $\overline{\mathrm{pr}}$ extensions of $i\circ f\circ q_i\mid_{\{1\}\times S^{n-1}}: S^{n-1}\longrightarrow Y$. With respect to these choices we obtain analogously a bijection $[Y,Y]^{S^1}_{f:X\longrightarrow X}\longrightarrow \pi_n(Y,\mathrm{pr}(x_i))$ such that the following diagram commutes

$$\begin{array}{ccc} [Z,Z]^{S^1}_{id\times f:SV\times X\longrightarrow SV\times X} & \xrightarrow{\cong} & \prod_{i\in I}\pi_n(Z,x_i) \\ l\downarrow & & \downarrow\overline{\mathrm{pr}} \\ [Y,Y]^{S^1}_{f:X\longrightarrow X} & \xrightarrow{\cong} & \prod_{i\in I}\pi_n(Y,\mathrm{pr}(x_i)) \end{array}$$

As SV is $(n+1)$-connected, pr and hence $\overline{\mathrm{pr}}$ are $(n+1)$-connected. Hence the map l in 6.8 is bijective.

We conclude that we can find for the given S^1-extension $g:Y\longrightarrow Y$ of $f:X\longrightarrow X$ an S^1-extension $g':Z\longrightarrow Z$ and a homotopy $h:Y\times[0,1]\longrightarrow Y$ relative f from $l(g')$ to g where $l(g'):Y\longrightarrow Y$ is given by the pushout property by $g'\cup_{\mathrm{id}_{SV}\times f} f$. We derive from additivity

$$U(\overline{pr})\left(u(g')-U(j)(u(\mathrm{id}_{SV}\times f))\right)=u(l(g'))-U(i)(u(f)) \qquad \in U(Y,l(g')).$$

Since S^1 acts freely on Z and $SV\times X$, Theorem 6.1 implies $u(g')=0$ and $u(\mathrm{id}_{SV}\times f)=0$. This shows

$$u(l(g'))-U(i)(u(f))=0 \qquad \in U(Y,l(g')).$$

Because of invariance under homotopy and homotopy equivalence we conclude from the commutativity of the following diagram

$$\begin{array}{ccccc} U(X,f) & \xrightarrow{i_0} & U(X\times[0,1],\mathrm{id}\times f) & \xleftarrow{i_1} & U(X,f) \\ {\scriptstyle U(i)}\downarrow & & {\scriptstyle U(i\times\mathrm{id})}\downarrow & & {\scriptstyle U(i)}\downarrow \\ U(Y,l(g')) & \xrightarrow{i_0} & U(Y\times[0,1],h\times\mathrm{pr}_{[0,1]}) & \xleftarrow{i_1} & U(Y,g) \end{array}$$

that

$$U(i_0)\left(u(l(g'))-U(i)(u(f))\right)=u(h\times\mathrm{pr}_{[0,1]})-U(i\times\mathrm{id})(u(\mathrm{id}\times f))=U(i_1)(u(g)-U(i)(u(f)))$$

holds. This implies

$$u(g)-U(i)(u(f)) \;=\; U(i_1)^{-1}U(i_0)\left(u(l(g'))-U(i)(u(f))\right) \;=0 \qquad \in U(Y,g).$$

This finishes the proof of Theorem 6.7. □

Example 6.9. We consider the special case of data 5.1, where F is the sphere S^n for $n \geq 1$. Fix a base point $b \in B$ and a path w from $f(b)$ to b. Put $\pi = \pi_1(B,b)$. Let d be the degree of the endomorphism of S^n_b given by the composition of $\overline{f}\,|_{F_b} : F_b \longrightarrow F_{f(b)}$ and $\omega(w) : F_{f(b)} \longrightarrow F_b$ where $\omega(w)$ is given by the (free) fiber transport. Let $\epsilon : \pi \longrightarrow \{\pm 1\}$ be the homomorphism which sends a path u to the degree of the map $\omega(u) : S^n_b \longrightarrow S^n_b$. Then we get from Lemma 6.5 that $p_* \circ \operatorname{trf} : U(\mathbb{Z}\pi, \phi) \longrightarrow U(\mathbb{Z}\pi, \phi)$ is given by

$$p_* \circ \operatorname{trf}\left([f : M \to \phi^*M]\right) \;=\; [f : M \to \phi^*M] + (-1)^n \cdot [d \cdot f : M^\epsilon \to \phi^*(M^\epsilon)], \tag{6.10}$$

where $f : M \longrightarrow \phi^*M$ is a $\mathbb{Z}\pi$-endomorphism for a finitely generated free $\mathbb{Z}\pi$-module M and M^ϵ is the $\mathbb{Z}\pi$-module which has the same underlying abelian group as M and has the new π-action given by $u \cdot x = \epsilon(u)u \cdot x$ for $x \in M$ and $u \in \pi$.

If ϵ is trivial, $d = 1$ and n is odd, we conclude from 6.10 for $n \geq 3$ and from Lemma 6.1.1 for $n = 1$ that trf is trivial. If ϵ is trivial, $d = 1$ and n is even, we get $p_* \circ \operatorname{trf} = 2 \cdot \operatorname{id}$ from 6.10 and p_* is an isomorphism since $\pi_1(p)$ is bijective.

We have introduced in Section 3 other functorial Lefschetz invariants and we will explain what the results for the universal invariant (U, u) implies for them. This is obvious if ϵ is trivial and $d = 1$ from the computation above. Let us consider in the sequel the case for arbitrary ϵ and d.

We begin with the generalized Lefschetz invariant of Example 3.5 for $n \geq 2$. In the sequel we will identify $\pi_1(E)$ and π by $\pi_1(p)$ and in particular $\Lambda(\mathbb{Z}\pi_1(E), \overline{\Phi})$ with $\Lambda(\mathbb{Z}\pi, \Phi)$. Define a map

$$\operatorname{trf}' : \Lambda(\mathbb{Z}\pi, \Phi) \longrightarrow \Lambda(\mathbb{Z}\pi, \Phi) \qquad \left[\sum_{v \in \pi} \lambda_v \cdot v\right] \mapsto \left[\sum_{v \in \pi} \lambda_v \cdot v\right] - (-1)^n \cdot d \cdot \left[\sum_{v \in \pi} \lambda_v \epsilon(v) \cdot v\right]$$

Then we get under the identifications above

$$\lambda(\overline{f}) \;=\; \operatorname{trf}'(\lambda(f)).$$

Notice that even in the case where n is odd and $d = 1$ it can happen that the (classical Lefschetz number) $\lambda_{\text{class}}(f)$ (see Example 3.1) vanishes, whereas $\lambda_{\text{class}}(\overline{f})$ is not zero. The reason is that for non-trivial ϵ there is no map $\operatorname{trf}'' : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying $e \circ \operatorname{trf}' = \operatorname{trf}'' \circ e$, where $e : \Lambda(\mathbb{Z}\pi, \Phi) \longrightarrow \mathbb{Z}$ maps $\left[\sum_{v \in \pi} \lambda_v \cdot v\right]$ to $\sum_{v \in \pi} \lambda_v$.

Next we consider the invariant L^2-torsion $\rho^{(2)}(T_f) \in \mathbb{R}$ of Example 3.12. If $d = 1$ and $n \geq 1$ we get for all possible ϵ the same result, namely,

$$\rho^{(2)}(T_{\overline{f}}) \;=\; (1 + (-1)^n) \cdot \rho^{(2)}(T_f).$$

If ϵ is trivial, this follows from the computation we have done already above. If ϵ is non-trivial, there is a two-sheeted covering $q : E' \longrightarrow E$ such that ϵ for the S^n- fibration $p \circ q : E' \longrightarrow B$ is trivial. Now the general case follows from the fact that $\rho^{(2)}$ is multiplicative under finite coverings [29, Theorem 1.10]. □

7 The mapping torus approach

In this section we use the construction of the mapping torus to reduce computations to the case where the ring endomorphism ϕ is the identity.

Let R be an associative ring with unit and let $\phi : R \longrightarrow R$ be a ring homomorphism respecting the unit. Denote by $\widehat{R}$ the ring given by the colimit of the direct system of rings indexed by the integers

$$\ldots \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \ldots$$

A model is $\mathbb{Z} \times R/\sim$ where $\sim$ is the equivalence relation for which $r \in n \times R$ and $s \in m \times R$ satisfy $r \sim s$ if and only if $\phi^{k-n}(r) = \phi^{k-m}(s)$ holds for an appropriate integer k with $k \geq m, n$. Let $i : R \longrightarrow \widehat{R}$ be the canonical homomorphism which sends r to the class of $0 \times r$. Notice that its kernel is the union $\cup_{n \geq 1} \ker(\phi^n)$. Let $\widehat{\phi} : \widehat{R} \longrightarrow \widehat{R}$ be the canonical automorphism which sends the class of $n \times r$ to $(n-1) \times r$. We have $\widehat{\phi} \circ i = i \circ \phi$. The ring $\widehat{R}[t, t^{-1}]_{\widehat{\phi}}$ consists of formal finite Laurent series $\sum_{n=k}^{l} t^n a_n$ for integers k and l and $a_n \in \widehat{R}$. Addition is given by

$$\left(\sum_n t^n a_n\right) + \left(\sum_n t^n b_n\right) := \sum_n t^n (a_n + b_n)$$

and multiplication by

$$\left(\sum_n t^n a_n\right) \cdot \left(\sum_n t^n b_n\right) := \sum_n t^n \left(\sum_k \widehat{\phi}^k (a_{n-k}) b_k\right).$$

Notice that multiplication is essentially given by $at = t\widehat{\phi}(a)$ for $a \in \widehat{R}$. Next we want to construct a commutative square

7.1.
$$\begin{array}{ccc} U(R, \phi) & \xrightarrow{\tau} & U(\widehat{R}[t, t^{-1}]_{\widehat{\phi}}, \mathrm{id}) \\ \eta \downarrow & & \eta \downarrow \\ \prod_{m \geq 1} \Lambda(R, \phi^m) & \xrightarrow{\tau = \prod_{m \geq 1} \tau_m} & \prod_{m \geq 1} \Lambda(\widehat{R}[t, t^{-1}]_{\widehat{\phi}}, \mathrm{id}) \end{array}$$

The group $\Lambda(R, \phi)$ is the obvious generalization of the special case of 3.7 where R is $\mathbb{Z}\pi$ and ϕ is induced by a group endomorphism of π, namely

Definition 7.2. Let $H \subset R$ be the abelian subgroup of R generated by all elements of the form $\phi(r)s - sr$. Define the abelian group

$$\Lambda(R, \phi) := R/H. \quad \square$$

Consider an R-map $f : F \longrightarrow \phi^* F$ for a finitely generated free (left) R-module F. Choose a basis $\{b_1, \ldots, b_k\}$ for F. The (k, k)-matrix A with entries in R describing f is determined by the property $f(b_i) = \sum_{j=1}^{k} A_{i,j} b_j$. Define

$$\mathrm{tr}_{(R,\phi)}(f) := \sum_{i=1}^{k} [A_{i,i}] \quad \in \Lambda(R, \phi). \tag{7.3}$$

One easily checks that this definition is independent of the choice of the basis. The m-th component η_m of the map η is given on representatives by

$$\eta_m[f] \quad := \quad [\mathrm{tr}_{(R,\phi^m)}(f^m)]. \tag{7.4}$$

Next we define the two horizontal maps. Let $f : F \longrightarrow \phi^* F$ be an endomorphism of a finitely generated free (left) R-module F. Then we obtain a $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$-endomorphism of a finitely generated free $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$-module

$$r_t \otimes f : \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \otimes_R F \longrightarrow \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \otimes_R F \qquad u \otimes x \mapsto ut \otimes f(x).$$

This is well-defined because of the following calculation

$$ut \otimes f(rx) = ut \otimes \phi(r) f(x) = ut\widehat{\phi}(i(r)) \otimes f(x) = ui(r)t \otimes f(x).$$

Now define the upper horizontal map by

$$\tau\left([f : F \longrightarrow \phi^* F]\right) \quad := \quad \left[r_t \otimes f : \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \otimes_R F \longrightarrow \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \otimes_R F\right]. \tag{7.5}$$

One easily checks that this is compatible with the relations appearing in Definition 1.2. The m-th component τ_m appearing in the lower horizontal map is given on representatives by

$$\tau_m([r]) \quad := \quad [t^m i(r)]. \tag{7.6}$$

One easily checks that the diagram 7.1 commutes.

Remark 7.7. The construction of the ring $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ comes from the mapping torus construction as explained next. Let $f : X \longrightarrow X$ be an endomorphisms of a connected finite CW-complex X. The *mapping torus* T_f is obtained from the cylinder $X \times [0,1]$ by identifying the bottom and the top by f, i.e. by the identification $(y,1) \sim (f(y),0)$ for $y \in X$. If we put $R = \pi_1(X)$ and $\phi : R \to R$ to be the ring homomorphism induced by f, then $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ is just $\mathbb{Z}\pi_1(T_f)$.

Remark 7.8. An interesting problem is to detect the kernels of the maps appearing in the diagrams 7.1. The point for the map $\dim \times \eta$ for dim as defined in 1.5 is that $u(f) \in U(f)$ defined in 2.2 is the invariant we want to know and that its image under η which is the collection of the generalized Lefschetz invariants $\lambda(f^n) \in \Lambda(f^n)$ for $n \geq 1$ of Example 3.9 is easier to compute and the vanishing of $\lambda(f^n)$ has a clear interpretation in terms of fixed points of f^n as explained in Example 3.5. If $\dim \times \eta$ would be injective then one could use the generalized Lefschetz invariants $\lambda(f^n)$ and the Euler characteristic to compute $u(f)$. Notice that $\dim \times \eta$ is injective if R is commutative and $\pi = \mathrm{id}$ by Example 1.7 or if $(R,\phi) = (\mathbb{C}, c)$ by Theorem 1.9 but we have no idea what happens in general.

For the map τ resp. $U(i)$ injectivity would mean that one does not loose information by the mapping torus approach. The advantage of the mapping torus approach is that for the target of the maps τ the endomorphism of the ring is the identity. Of course one has to pay the price that the ring $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ is more complicated than R. At least we can show injectivity of the lower map τ in Lemma 7.9 below. □

Lemma 7.9. *The lower horizontal map* τ *(see 7.6) in diagram 7.1*

$$\tau : \prod_{m\geq 1} \Lambda(R,\phi^m) \longrightarrow \prod_{m\geq 1} \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}},\mathrm{id})$$

is injective if $\Lambda(R,\phi^m)$ *contains no* m*-torsion for* $m \geq 1$*.*

Proof. We have to show for $m \geq 1$ that the map

$$\tau_m : \Lambda(R,\phi^m) \longrightarrow \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}},\mathrm{id}) \qquad [r] \mapsto [t^m i(r)] \tag{7.10}$$

is injective. Define

$$\tau'_m : \Lambda(R,\phi^m) \longrightarrow \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m},\mathrm{id}) \qquad [r] \mapsto [ti(r)]. \tag{7.11}$$

We obtain a ring homomorphism

$$j : \widehat{R}[t,t^{-1}]_{\widehat{\phi}^m} \longrightarrow \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \qquad \sum_n t^n a_n \mapsto \sum_n t^{nm} a_n.$$

It induces a map

$$j_* : \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m},\mathrm{id}) \longrightarrow \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}},\mathrm{id}). \tag{7.12}$$

One easily checks that τ_m is the composition $j_* \circ \tau'_m$. It suffices to show for $m \geq 1$ that τ'_m is injective and the kernel of j_* is m-torsion.

We begin with injectivity of τ'_m. Obviously it suffices to do this for $m = 1$, otherwise substitute ϕ by ϕ^m. Suppose that $[ti(r)] \in \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}},\mathrm{id})$ is trivial. Hence we can rewrite $ti(r)$ in $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ as a finite sum of elements of the form $uv - vu$ for $u,v \in \widehat{R}[t,t^{-1}]_{\widehat{\phi}}$. Inspecting the coefficients of t^1 one sees that $i(r)$ is a finite sum of elements of the form $\widehat{\phi}^a(\widehat{r})\widehat{s} - \widehat{\phi}^{1-a}(\widehat{s})\widehat{r}$ for $\widehat{r},\widehat{s} \in \widehat{R}$ and $a \in \mathbb{Z}$. Since for $\widehat{r},\widehat{s} \in \widehat{R}$ we have

$$\begin{aligned}
&\left(\widehat{\phi}^a(\widehat{r})\widehat{s} - \widehat{\phi}^{1-a}(\widehat{s})\widehat{r}\right) - \left(\widehat{\phi}^{a-1}(\widehat{r})\widehat{\phi}^{-1}(\widehat{s}) - \widehat{\phi}^{1-(a-1)}(\widehat{\phi}^{-1}(\widehat{s}))\widehat{r}\right)\\
&= \widehat{\phi}^a(\widehat{r})\widehat{s} - \widehat{\phi}^{a-1}(\widehat{r})\widehat{\phi}^{-1}(\widehat{s})\\
&= \left(\widehat{\phi}^a(\widehat{r})\widehat{s} - \widehat{s}\widehat{\phi}^{a-1}(\widehat{r})\right) + \left(\widehat{s}\widehat{\phi}^{a-1}(\widehat{r}) - \widehat{\phi}^{a-1}(\widehat{r})\widehat{\phi}^{-1}(\widehat{s})\right),
\end{aligned}$$

we can show by induction over a that the element $i(r)$ is a finite union of elements of the shape $\widehat{\phi}(\widehat{r})\widehat{s} - \widehat{s}\widehat{r}$ for $\widehat{r},\widehat{s} \in \widehat{R}$. For any element $\widehat{r} \in \widehat{R}$ there is $r \in R$ and a non-negative integer k with $\widehat{\phi}^k(\widehat{r}) = i(r)$. Hence there is a non-negative integer l such that $i(\phi^l(r))$ is a finite union of elements of the shape $i\left(\phi(r)s - sr\right)$ for $r,s \in R$. Notice that the kernel of $i : R \longrightarrow \widehat{R}$ is the union $\cup_{n\geq 1}\ker(\phi^n)$ and that $\phi^n(r) - r$ for $n \geq 0$ represents zero in $\Lambda(R,\phi)$ because of

$$\phi^n(r) - r = \phi^n(r)1 - 1\phi^{n-1}(r) + \ldots + \phi(r)1 - 1r.$$

Hence the map τ'_m of 7.11 is injective.

To show that the kernel of the map j_* defined in 7.12 is m-torsion, we define a map

$$\begin{array}{rcl} j^* : \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}}, \mathrm{id}) & \longrightarrow & \Lambda(\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m}, \mathrm{id}) \\ {[u]} & \mapsto & \mathrm{tr}_{\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m},\mathrm{id}} \left(j^* r_u : j^* \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \longrightarrow \widehat{R}[t,t^{-1}]_{\widehat{\phi}} \right), \end{array}$$

where for $u \in \widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ the $\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$-map r_u is given by right multiplication with u, j^* denotes restriction with the ring homomorphism j and $\mathrm{tr}_{\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m},\mathrm{id}}$ has been introduced in 7.3. Since $r_{uv-vu} = r_v \circ r_u - r_u \circ r_v$ holds and $j^*\widehat{R}[t,t^{-1}]_{\widehat{\phi}}$ is the free $\widehat{R}[t,t^{-1}]_{\widehat{\phi^m}}$-module with basis t^0, t^1, ..., t^{m-1}, the map j^* is well-defined. One easily checks that for $u \in \widehat{R}[t,t^{-1}]_{\widehat{\phi}^m}$ $j^*r_{j(u)}$ is conjugated to the direct sum of m-copies of the $\widehat{R}[t,t^{-1}]_{\widehat{\phi^m}}$-map $r_u : \widehat{R}[t,t^{-1}]_{\widehat{\phi^m}} \longrightarrow \widehat{R}[t,t^{-1}]_{\widehat{\phi^m}}$ given by right multiplication with u. Since $\mathrm{tr}_{\widehat{R}[t,t^{-1}]_{\widehat{\phi}^m},\mathrm{id}}(r_u) = u$, we conclude that $j^* \circ j_* = m \cdot \mathrm{id}$. Hence the kernel of j_* is m-torsion. This finishes the proof of Lemma 7.9. □

Remark 7.13. Notice that there is an obvious m-fold covering $p_m : T_{f^m} \longrightarrow T_f$ for an endomorphism $f : X \longrightarrow X$ of a connected finite CW-complex. Let $i_m : X \longrightarrow T_{f^m}$ and $i : X \longrightarrow T_f$ be the canonical inclusions. We obtain morphisms $i_m : (X, f^m) \longrightarrow (T_{f^m}, \overline{f^m})$ and $p_m : (T_{f^m}, \overline{f^m}) \longrightarrow (T_f, \overline{f}^m)$ in $\mathrm{End}(\mathcal{C})$. The composition $p_m \circ i_m$ is just the morphism $i : (X, f) \longrightarrow (T_f, \overline{f}^m)$. In particular we obtain a factorization of $\Lambda(i)$ into $\Lambda(p_m) \circ \Lambda(i_m)$. One easily checks that this corresponds to the factorization of τ_m into $j_* \circ \tau'_m$ appearing in the proof of Lemma 7.9. Moreover, the map j^* is just the transfer map associated to the fibration $p_m : T_{f^m} \longrightarrow T_f$ defined in 5.2.

The assumption in Lemma 7.9 that $\Lambda(R, \phi^m)$ contains no m-torsion is always satisfied in the case where R is an integral group ring $\mathbb{Z}\pi$ and ϕ given by an endomorphism of π because then $\Lambda(\mathbb{Z}\pi, \phi^m)$ is the free abelian group generated by the ϕ^m-conjugacy classes. □

Remark 7.14. We have introduced several homomorphisms with source $U(f)$ which can be used to detect elements in $U(f)$. Notice that the homomorphism 3.2 and the homomorphism 3.11 factorize over η. The map 3.15 factorizes over $\eta \circ \tau = \tau \circ \eta$. However, it seems to be very unlikely that the map ρ of 3.13 factorizes over η or τ. □

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