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EQUIVARIANT LEFSCHETZ CLASSES

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0. Introduction

This paper studies equivariant fixed point theory of G-complexes with cellular methods. We introduce the *universal Lefschetz ring* $UL(G)$ and the *Lefschetz ring* $L(G)$ of a compact Lie group G. They are both quotients of the set of G-endomorphisms of finite G-complexes by an equivalence relation based on Lefschetz numbers of the induced maps on X^H/WH_0 and X^H. The ring $L(G)$ bears a similar relation to $UL(G)$ as the Burnside ring $A(G)$ to the universal additive invariant $U(G)$. There is a commutative square of ring homomorphisms

$$\begin{tikzcd} UL(G) \arrow[r] & L(G) \\ U(G) \arrow[u] \arrow[r] & A(G) \arrow[u] \end{tikzcd}$$

where the horizontal arrows are quotient maps and the vertical arrows are inclusions sending a finite G-complex X to $id\colon X \to X$.

The groups $UL(G)$ and $U(G)$ are in fact defined for arbitrary topological groups G by certain universal properties. This universal property is mainly used for constructing homomorphisms with $UL(G)$ as source.

Let $Con\ G$ be the set of conjugacy classes of subgroups $\{(H) \mid H \leq G\}$ and $con\ G$ be the set of conjugacy classes of elements $\{\langle g\rangle \mid g \in G\}$. Denote by $Cl(ZG)$ the free abelian group generated by $con\, G$ or, equivalently, the group of class functions $G \to Z$ with finite support. Under a mild technical condition which is satisfied for compact or discrete G we can define the *universal Lefschetz class*

$$UL(f) \in \bigoplus_{(H) \in Con\, G} Cl(Z\pi_0 WH)$$

of a G-endomorphism f of a finite G-complex by applying the Hattori-Stallings trace to the induced chain map on the cellular $Z\pi_0 WH$-chain complex

$C^c(X^H, X^{>H})$ for $(H)\in Con\ G$. We can use $UL(f)$ to define the *universal Euler class* $U\chi(X)\in \bigoplus_{(H)} Z$. Denote by $r(w)$: $G/H\to G/H$ for $w\in WH$ the G-map $gH\to gwH$. Let $l(g)$ for $g\in G$ always denote left multiplication with g.

Theorem A. *Suppose for G that any G-map $G/H\to G/H$ is a G-homeomorphism (This holds for compact G). Then $UL(f)$ and $U\chi(X)$ induce isomorphisms*

$$UL(G)\cong \bigoplus_{(H)} Cl(Z\pi_0 WH) \quad \textit{and} \quad U(G)\cong \bigoplus_{(H)} Z$$

where the sums run over $Con\ G$. A Z-base for $UL(G)$ is $\{[r(w)]\,|\,\langle w\rangle\in con\ \pi_0 WH, (H)\in Con\ G\}$ and $\{[G/H]\,|\,(H)\in Con\ G\}$ is a Z-base for $U(G)$.

The groups $UL(G)$ and $U(G)$ give rise to a general method of constructing homotopy invariants of G-maps f: $X\to X$ (resp. spaces X): assign to the basis elements $[r(w)]$ (resp. $[G/H]$) arbitrary values in an abelian group. This framework covers Brown's equivariant Euler characteristics of discrete group actions [3, Ch. IX. 7] and tom Dieck's Burnside ring of a compact Lie group [6, Ch. IV]. Indeed, the first one is obtained by mapping $[G/H]$ to the Euler characteristic of H in the sense of group cohomology, whereas the second one results by considering the Euler characteristics of the spaces G/H and their fixed point sets.

For the rest of the introduction, let G be a compact Lie group. If f: $X\to X$ is a self-map of a finite G-complex X and $H\leq G$ then the Lefschetz numbers

$$\bar{L}^H(f)(w) = L(X^H/WH_0,\ l(w^{-1})\circ f^H/WH_0),\ w\in\pi_0 WH \tag{0.1}$$

define a class function $\bar{L}^H(f)$ on $\pi_0 WH=WH/WH_0$. The universal Lefschetz ring $UL(G)$ is obtained by identifying f_1 and f_2 when $\bar{L}^H(f_1)=\bar{L}^H(f_2)$ for each $H\leq G$. The homomorphisms $\bar{L}^H$ define an injective group homomorphism

$$\bar{L}\colon UL(G)\to \Pi Cl(Z\pi_0 WH)$$

which is a ring homomorphism only for finite groups. To get an invariant more accessible to computations, consider the class functions $L^H(f)$: $\pi_0 WH\to Z$ defined by

$$L^H(f)(w) = L(X^H,\ l(w^{-1})\circ f^H), \qquad w\in WH \tag{0.2}$$

If f_1 and f_2 are identified when $L^H(f_1)=L^H(f_2)$ for each $H\leq G$ the result is the Lefschetz ring $L(G)$. It admits a ring embedding L: $L(G)\to \prod_{(H)} Cl(Z\pi_0 WH)$. For finite groups G the rings $UL(G)$ and $L(G)$ coincide. In general, the class of $r(w)$: $G/H\to G/H$ in $UL(G)$ maps to zero in $L(G)$ if w has infinite centralizer in WH, and the remaining generators form a basis of $L(G)$:

$$L(G)\cong \oplus Cl_\phi(Z\pi_0 WH)$$

where $Cl_\phi(Z\pi_0 WH)$ is the free abelian group on those conjugacy classes $\langle w \rangle$ of $\pi_0 WH$ for which $C_{WH}(w)$ is finite. The ring $UL(G)$ is of theoretical interest whereas $L(G)$ is adequate for explicit computations.

The quotient set of G-maps under the coarse relation based on the Lefschetz numbers $L(X^H, f^H)$ turns out to be the Burnside ring $A(G)$. The fact that $A(G)$ is also the quotient of the same relation on spaces has important consequences and has no counterpart for $UL(G)$ and $U(G)$.

So far the spaces have been finite G-complexes. Next we apply homological algebra in the category of modules over the orbit category to weaken the finiteness assumption. The idea is to approximate the cellular chain complexes of all fixed point sets simultaneously by finite projective complexes, replacing thus the arguments based on induction over orbit types. This approach was used in the context of finiteness obstructions of finite groups by tom Dieck [5].

We are ready to state the main results. Let X be a Z-homology finite G-complex, i.e. $H_*(X^H; Z)$ is finitely generated for each $H \leq G$. Then we prove that $H_*(X^H/WH_0, Z)$ is also finitely generated for each $H \leq G$, so that the class functions $\bar{L}^H(f)$ and $L^H(f)$ from (0.1) and (0.2) are defined for all G-maps $f: X \to X$. Recall that $\bar{L}^H$ and L^H define homomorphisms $UL(G) \to Cl(Z\pi_0 WH)$.

Theorem B. *Let G be a compact Lie group and let X be a finite-dimensional Z-homology finite G-complex of finite orbit type. Then every G-map $f: X \to X$ has an equivariant Lefschetz class $[f]$ in $UL(G)$ such that $\bar{L}^H(f) = \bar{L}^H([f])$ and $L^H(f) = L^H([f])$ for each subgroup $H \leq G$.*

Thus the relations between $\bar{L}^H(f)$ or $L^H(f)$ for various H are the same as those that occur for the maps $r(w): G/K \to G/K$. In particular

Corollary C. *With the assumptions of theorem B the Lefschetz numbers $L(f^H)$ satisfy the Burnside ring congruences: let $H \leq L$ be closed subgroups of G.*

i) *If L/H is finite then*

$$L(f^H) \equiv -\sum \phi(|K/H|) L(f^K) \bmod |L/H|$$

summed over those $K \leq L$ which correspond to non-trivial cyclic subgroups of L/H.

ii) *If L/H is a torus then $L(f^H) = L(f^L)$.*

Corollary D. (Lefschetz fixed point formula). *With the assumptions of theorem B, $L(g) = \chi(X^g)$ for each element $g \in G$.*

Verdier [18] and Brown [4] have proved versions of Corollary D for finite groups.

Section 1 contains the definition of $UL(G)$ for topological groups G and the proof of Theorem A. In section 2 we relate it to ordinary Lefschetz numbers, when G is a compact Lie group, and prove the main results in the special case of finite G-complexes. Section 3 deals with homological algebra over the orbit category. In section 4 we give axioms for Lefschetz invariants of chain mappings. In section 5 we apply the algebra to G-complexes and prove Theorem B and Corollaries C and D in slightly more general form allowing arbitrary coefficients. Section 6 is devoted to homotopy representations which formed the authors' original motivation for constructing Lefschetz classes in $A(G)$ for finite G-complexes in [11] and [12].

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1. The universal Lefschetz group of a topological group

Let G be a topological group. Topological groups as well as G-spaces are supposed to be Hausdorff. For a general discussion of G-complexes we refer to [6, II. 1+2]. We call a G-complex pointed if we have chosen a base point, i.e. a G-fixed point in the zero-skeleton X_0. A G-map is pointed if it preserves the base point. Given a G-map $f\colon X\to Y$, let X_+ and Y_+ be the pointed G-spaces $X_+=X\coprod\{G/G\}$ and $Y_+=Y\coprod\{G/G\}$ with base point G/G and $f_+\colon X_+\to Y_+$ be the pointed G-map $f\coprod \mathrm{id}$. Denote by $[X, Y]^G$ (resp. $[X, Y]^G_+$) the set of (pointed) G-homotopy classes of (pointed) G-maps from X to Y. If S^n has trivial G-action and $\nabla\colon S^n\to S^n\vee S^n$ denotes the pinch map then $[f]+[g]=[f\vee g\circ\nabla]$ defines a group structure on $[S^n\wedge X, S^n\wedge X]^G_+$ for $n\geq 1$ which is abelian when $n\geq 2$.

A *Lefschetz invariant* for G consists of an abelian group A and a function assigning to a pointed G-endomorphism $f\colon X\to X$ of a pointed finite G-complex an element $L(f)\in A$ satisfying

i) *Homotopy invariance*

If $f, g\colon X\to X$ are pointed G-homotopic then $L(f)=L(g)$.

ii) *Commutativity*

If $f\colon X\to Y$ and $g\colon Y\to X$ are pointed G-maps then $L(g\circ f)=L(f\circ g)$.

iii) *Additivity*

Consider the commutative diagram of pointed finite G-complexes with i the inclusion of such G-complexes

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g"] & X|A \arrow[d, "h"] \\ A \arrow[r, "i"] & X \arrow[r] & X|A \end{tikzcd}$$

Then $L(f)-L(g)+L(h)=0$.

iv) *Linearity*
For f, g: $S^1 \wedge X \to S^1 \wedge X$ we have $L(f+g) = L(f) + L(g)$. (This makes sense because of i).

Remark 1.1. Notice that homotopy invariance and commutativity imply that $L(f) = L(g)$ if there is a G-homotopy commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

with a pointed G-homotopy equivalence h.

Example 1.2. Consider the function assigning to a pointed G-endomorphism f: $X \to X$ of a pointed finite G-complex the ordinary reduced Lefschetz number of f/G: $X/G \to X/G$ in Z. This is a Lefschetz invariant for G.

In the sequel we need the following condition (0) on G which is satisfied for all abelian or compact or discrete groups G

(0) The Weyl group $WH = NH/H = \{g \in G \mid g^{-1}Hg = H\}/H$ is open in $G/H^H = \{g \in G \mid g^{-1}Hg \subset H\}/H$ for any $H \leq G$.

Notice that WH is always closed in G/H^H as $H \leq G$ is closed, Hence G/H^H is the topological sum $WH \coprod G/H^{>H}$. Therefore the G-complex structure on X induces a relative WH-complex structure on $(X^H, X^{>H})$. If R is a commutative ring and H_* denotes singular homology with R-coefficients define the cellular chain complex $C^c(X^H, X^{>H})$ with R-coefficients by

$$\cdots \xrightarrow{\Delta_{n+1}} H_n((X^H, X^{>H})_n, (X^H, X^{>H})_{n-1}) \xrightarrow{\Delta_n}$$
$$H_{n-1}((X^H, X^{>H}))_{n-1}, (X^H, X^{>H})_{n-2}) \xrightarrow{\Delta_{n-1}} \cdots$$

where Δ_n is the boundary operator of the corresponding triple. By naturality and homotopy invariance $C^c(X^{H>}, X^H)$ is a $R\pi_0 WH$-chain complex. It is finite free becasue of $H_n((X^H, X^{>H})_n, (X^H, X^{>H})_{n-1}) \cong \oplus H_n(WH \times (D^n, S^{n-1})) \cong \oplus H_0(WH)$ where the sum runs over the n-dimensional WH-cells in $X^H \backslash X^{>H}$. A cellular G-map f: $X \to X$ induces $C^c(f^H, f^{>H})$: $C^c(X^H, X^{>H}) \to C^c(X^H, X^{>H})$. We make the convention that for a pointed G-complex X with base point x $X^{>G}$ is $\{x\}$. Then we have $C^c(X^H, X^{>H}) = C^c(X^H_+, X^{>H}_+)$ for a G-complex X and

iv) *Linearity*
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Let K be a group. Denote by $Cl(RK)$ the free R-module generated by the conjugacy classes $\langle k\rangle$ of elements $k\in K$. There is a canonical epimorphism of R-modules

$$(1.3)\qquad T\colon RK\to Cl(RK)$$

with the group ring RK as source. It is bijective if and only if K is abelian. Consider an endomorphism $f\colon P\to P$ of a finitely generated projective RK-module. In [8], [17] and [1] there is defined a trace $\mathrm{Tr}_{RK}(f)\in Cl(RK)$ as follows. Choose a finitely generated RK-module Q and an isomorphism $h\colon RK^n\to P\oplus Q$ from the based free RK-module of rank n. The endomorphism $h^{-1}\circ(f\oplus 0)\circ h$ of RK^n is given by a (n, n)-matrix $A=(a_{i,j})$. Then the *Hattori-Stallings trace* of f is

$$(1.4)\qquad \mathrm{Tr}_{RK}(f)=\sum_{i=1}^{n}T(a_{ii})\in Cl(RK)\,.$$

Consider a finitely generated projective RK-chain complex C and a RK-chain map $f\colon C\to C$. Define

$$(1.5)\qquad L_{RK}(f)\in Cl(RK)\,.$$

by $\sum_{i=0}^{\infty}(-1)^i\mathrm{Tr}_{RK}(f_i)$. This is a Lefschetz invariant in the sense of section 4, i.e. homotopy invariance, additivity, linearity and commutativity are satisfied.

Consider a (pointed) G-endomorphism $f\colon X\to X$ of a (pointed) finite G-complex. Let $UL^H(f)\in Cl(Z\pi_0WH)$ be $L_{Z\pi_0WH}(C^c(X^H, X^{>H}), C^c(g))$ for any cellular G-map g with $f\simeq g$. Define the *universal Lefschetz class*

$$(1.6)\qquad UL(f)\in\bigoplus_{(H)}Cl(Z\pi_0WH)$$

by the collection $\{UL^H(f)\}$ where (H) runs over the set $Con\ G$ of conjugacy classes (H) of subgroups $H\leq G$. This makes sense as $UL^H(f)$ depends only on (H) and is different from zero only for $H\in\mathrm{Iso}\ X$. One easily checks using the remarks above that the universal Lefschetz class defines a Lefschetz invariant for G. In particular we get from remark 1.1 that $UL(f)$ does not depend on the G-complex structure on X. We will see that UL is the most general Lefschetz invariant.

We call a Lefschetz invariant (A, L) *universal* if for any Lefschetz invariant (A', L') there is exactly one homomorphism $\phi\colon A\to A'$ such that $\phi(L(f))=L'(f)$ holds for any pointed G-endomorphism f of a pointed finite G-complex. Up to unique isomorphism there is only one universal Lefschetz invariant. One can construct a model by introducing on the abelian group generated by the isomorphism classes of pointed G-endomorphisms of pointed finite G-complexes the necessary relations corresponding to the axioms.

DEFINITION 1.7. Let $UL(G)$ together with the function $f \to [f] \in UL(G)$ be the universal Lefschetz invariant of the topological group G. We call $UL(G)$ the *universal Lefschetz group* of G.

Notice that we obtain by the universal Lefschetz class $UL(f)$ and the universal property $UL(G)$ a homomorphism

$$UL: UL(G) \to \bigoplus_{(H)} Cl(Z\pi_0 WH) \tag{1.8}$$

uniquely determined by $UL([f]) = UL(f)$. We can define a homomorphism

$$\psi: \bigoplus_{(H)} Cl(Z\pi_0 WH) \to UL(G) \tag{1.9}$$

by sending the base element $\langle w \rangle \in Cl(Z\pi_0 WH)$ represented by $w \in \pi_0 WH$ to $[r(w)_+]$ where $r(w): G/H \to G/H$ sends gH to gwH. This is independent of the choice of w by homotopy invariance and commutativity since any path from w to w' in WH induces a G-homotopy between $r(w)$ and $r(w')$ and $r(w) \circ r(v) = r(vw)$ holds. One checks directly that $UL \circ \psi$ is the identity. Now assume that G satisfies the condition

(EI) Any G-endomorphism of a homogeneous G-space is a G-homeomorphism.

This is equivalent to $G/H^H = WH$ so that (EI) implies (0). If G is compact (EI) is satisfied. Now we can prove Theorem A of the introduction.

Theorem 1.10. *If G satisfies* (EI) *then UL and ψ are inverse isomorphisms.*

Proof. It remains to prove that ψ is surjective. We write $(H) \leq (K)$ if $H \leq K$ holds for appropriate representatives H and K. This is equivalent to the existence of a G-map $G/H \to G/K$. We get from condition (EI) that $(H) \leq (K)$ and $(K) \leq (H)$ implies $(H) = (K)$. Consider a pointed G-map $f: X \to X$ of a pointed finite G-complex. Numerate $\{(H) \mid H \in \mathrm{Iso}\, X\} = \{(H_1), (H_2), \cdots, (H_r)\}$ such that $(H_i) \leq (H_j)$ implies $i \geq j$. Then $X(k) = \bigcup_{i=1}^{k} X^{(H_i)}$ is a G-subcomplex of X and f induces by restriction $f(k): X(k) \to X(k)$. We obtain from additivity

$$[f] = \sum_{k=1}^{\infty} [f(k)/f(k-1)]\,.$$

Similarly we obtain from the skeletal filtration

$$[f] = \sum_{n=0}^{\infty} [f_n/f_{n-1}]\,.$$

Hence it suffices to show $[f] \in \mathrm{image}\,\psi$ under the assumption $X = \bigvee_{i=1}^{r} G/H_+ \wedge S^n$ for $n \geq 0$. By additivity and homotopy invariance $[S^1 \wedge f] = -[f]$ holds as we have the cofibration $X \to I \wedge X \to S^1 \wedge X$ and $I \wedge X$ is contractible. Therefore we can also suppose $n \geq 2$. If $M_r(Z\pi_0 WH)$ is the ring of (r, r)-matrices over $Z\pi_0 WH$ we next construct an isomorphism of abelian groups

$$F\colon M_r(Z\pi_0 WH) \to [X, X]^G_+ .$$

Let X be a space. One shows inductively for $n \geq 2$ that $S^n \wedge X_+$ is $(n-1)$-connected and the Hurewicz homomorphism $\pi_n(S^n \wedge X_+) \to \tilde{H}_n(S^n \wedge X_+)$ is bijective. In the start $n=2$ use the theorem of Seifert-van Kampen. Hence we obtain an isomorphism of abelian groups for $n \geq 2$

$$H_0(X) = \tilde{H}_0(X_+) \xrightarrow{\cong} \tilde{H}_n(X_+) \to [S^n, S^n \wedge X_+]_+ .$$

We define F as the compostion

$$\begin{aligned} M_r(Z\pi_0 WH) &= \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} H_0(WH) = \bigoplus_{i=1}^{r} \tilde{H}_0(\bigvee_{j=1}^{r} G/H^H_+) \\ &= \bigoplus_{i=1}^{r} [S^n, \bigvee_{j=1}^{r} G/H^H_+ \wedge S^n]_+ = \bigoplus_{i=1}^{r} [G/H_+ \wedge S^n, \bigvee_{j=1}^{r} G/H_+ \wedge S^n]^G_+ \\ &= [\bigvee_{i=1}^{r} G/H_+ \wedge S^n, \bigvee_{j=1}^{r} G/H_+ \wedge S^n]^G_+ = [X, X]^G_+ . \end{aligned}$$

Let $A \in M_r(Z\pi_0 WH)$ be given. Let $\delta_{i,j}$ be the Kronecker symbol: $\delta_{i,j}=0$ for $i \neq j$ and $\delta_{i,i}=1$, The matrix $E(i, j)=(\delta_{i,i'} \cdot \delta_{j,j'})_{i',j'}$ has always 0 as entry except at (i, j) where it is 1. We get

$$F(A) = \sum_{i,j} F(a_{i,j} \cdot E(i, j)) .$$

Applying additivity to

$$\begin{tikzcd} \bigvee\limits_{\substack{k=1 \\ k \neq i}}^{r} G/H_+ \wedge S^n \arrow[r] \arrow[d, "F(0)"] & \bigvee\limits_{k=1}^{r} G/H_+ \wedge S^n \arrow[r] \arrow[d, "F(a_{i,j}E(i,j))"] & G/H_+ \wedge S^n \arrow[d, "\delta_{i,j}F(a_{i,j})"] \\ \bigvee\limits_{\substack{k=1 \\ k \neq i}}^{r} G/H_+ \wedge S^n \arrow[r] & \bigvee\limits_{k=1}^{r} G/H_+ \wedge S^n \arrow[r] & G/H_+ \wedge S^n \end{tikzcd}$$

and linearity to $O+O=O$ yields

$$[F(a_{i,j}E(i, j))] = \delta_{i,j}[F(a_{i,j})] .$$

Hence it remains to show for $a \in Z\pi_0 WH$ that $[F(a)] \in \operatorname{im} \psi$ holds for $F(a)$: $G/H_+ \wedge S^n \to G/H_+ \wedge S^n$. Since we can write $a = \sum a_w \cdot w$ we can assume $a=w$. But $[F(w)]$ is $(-1)^n[r(w)_+]$ and $[r(w)_+] \in \operatorname{im} \psi$ is obvious. □

Let $U(G)$ be the universal additive invariant for pointed finite G-complexes (see [6, IV.1.]). It is universal with respect to homotopy invariance and additivity. By the universal property we obtain unique homomorphisms

$$I\colon U(G) \to UL(G),\ [X] \to [\mathrm{id}\colon X \to X]. \tag{1.11}$$

Let $\beta(X, H, n)$ be the number of cells of type $G/H\times D^n$ in (X, x). Define $U\chi^H(X)\in Z$ by $\sum_{n=0}^{\infty}(-1)^n\beta(X, H, n)$. Suppose that G satisfies (0). Since $\mathrm{Tr}_{Z\pi_0(WH)}(Z\pi_0 WH, \mathrm{id})\in Cl(Z\pi_0 WH)$ is the base element given by the unit $e\in WH$ we have $UL^H(\mathrm{id}\colon X\to X)=U\chi^H(X)\cdot[e]$. Hence we get a well-defined homomorphism

$$U\chi\colon U(G) \to \bigoplus_{(H)} Z,\ [X] \to (U\chi^H(X))_{(H)} \tag{1.12}$$

such that the map $i\colon \bigoplus_{(H)} Z \to Cl(Z\pi_0 WH)$ sending $(n_H\in Z)_{(H)}$ to $(n_H\cdot[e]\in Cl(Z\pi_0 WH))_{(H)}$ makes the following diagram commute

$$\begin{tikzcd} UL(G) \arrow[r, "UL"] & \bigoplus_{(H)} Cl(Z\pi_0 WH) \\ U(G) \arrow[u, "I"] \arrow[r, "U\chi"] & \bigoplus_{(H)} Z \arrow[u, "i"'] \end{tikzcd} \tag{1.13}$$

If G satisfies (EI) the map $U\chi$ is an isomorphism. We call $U\chi(X)$ the *universal Euler characteristic*. The possibility of defining equivariant Euler characteristics for general groups was suggested to us by Sören Illman.

If one drops in the definition of the Lefschetz invariants the linearity axiom one is led to larger universal groups (see Dold [7], Okonek [15]).

2. Lefschetz invariants for compact Lie groups

In this section G is always a compact Lie group. We continue the study of $UL(G)$ and $U(G)$ and the universal Lefschetz class UL. Next we show how to compute UL by ordinary Lefschetz numbers. This is based on the following observation for a finite G-complex X.

There is a relative $\pi_0 WH=WH/WH_0$-complex structure on $(X^H, X^{>H})/WH_0$. Let $C^c((X^H, X^{>H})/WH_0)$ be its cellular $Z\pi_0\ WH$-chain complex.

Lemma 2.1. *The canonical projection* pr: $C^c(X^H, X^{>H})\to C^c((X^H, X^{>H})/WH_0)$ *is a base preserving $Z\pi_0 WH$-chain isomorphism.*

Proof. If $\oplus$ runs over the cells of type $G/H\times D^n$ we can write pr_n as the composition of isomorphisms

$$H_n((X^H, X^{>H})_n, (X^H, X^{>H})_{n-1}) = \oplus H_n(WH \times (D^n, S^{n-1}))$$
$$\cong \oplus H_0(WH) \cong \oplus H_0(WH/WH_0) \cong \oplus H_n(WH/WH_0 \times (D^n, S^{n-1}))$$
$$\cong H_n((X^H, X^{>H})_n/WH_0, (X^H, X^{>H})_{n-1}/WH_0) . \qquad \square$$

Denote by $L_Z((Y, B), f)$ the ordinary Lefschetz number of an endomorphism $f: (Y, B) \to (Y, B)$ of a finite relative CW-complex. If Tr_Z denotes the ordinary trace of an endomorphism of a finitely generated abelian group we have for any cellular approximation g of f

$$L_Z(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}_Z(C_n^c(g)) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}_Z(H_n(f)) . \tag{2.2}$$

Consider the (pointed) endomorphism of a (pointed) finite G-complex $f: X \to X$. Given $H \leq G$, let $\langle w \rangle$ be the conjugacy class of $w \in \pi_0 WH$. Write $UL^H(f)$ as $\sum_{\langle w \rangle} UL^H(f)\langle w \rangle \cdot \langle w \rangle$ in $Cl(Z\pi_0 WH)$. Let $l(w^{-1})$ be the map given by left multiplication with w^{-1} and $C_{\pi_0 WH}(w) = \{v \in \pi_0 WH \mid vw = wv\}$ be the centralizer of w in $\pi_0 WH$. Let c be $|C_{\pi_0 WH}(w)|$. Denote by $(\bar{f}^H, \bar{f}^{>H})$ the map $(f^H, f^{>H})/WH_0: (X^H, X^{>H})/WH_0 \to (X^H, X^{>H})/WH_0$.

Theorem 2.3.

$$UL^H(f)\langle w \rangle = \frac{1}{c} \cdot L_Z(X^H, X^{>H})/WH_0, l(w^{-1}) \circ (\bar{f}^H, \bar{f}^{>H})) .$$

Proof. By Lemma 2.1 and the definitions of $L_{Z\pi_0 WH}$ and L_Z it suffices to prove for an endomorphism $\phi: Z\pi_0 WH \to Z\pi_0 WH$

$$\operatorname{Tr}_{Z\pi_0(WH)} \phi = \sum_{\langle w \rangle} \frac{1}{c} \operatorname{Tr}_Z (l(w^{-1}) \circ \phi) \cdot \langle w \rangle .$$

Given $w, w_0 \in Z\pi_0 WH$ let $l(w^{-1}) \circ r(w_0)$ be the endomorphism $Z\pi_0 WH \to Z\pi_0 WH$, $v \to w^{-1} v w_0$. One easily computes

$$\operatorname{Tr}_Z (l(w^{-1}) \circ r(w_0)) = \begin{cases} 0, & \langle w \rangle \neq \langle w_0 \rangle \\ c, & \langle w \rangle = \langle w_0 \rangle \end{cases} . \tag{2.4}$$

This proves the claim. $\square$

The coefficients $L_Z((X^H, X^{>H})/WH_0, l(w^{-1}) \circ (\bar{f}^H, \bar{f}^{>H}))$ are rather difficult to compute in practice since even in the case of a linear representation sphere $X = SV$ of a finite group G the singular set $X^{>H}$ in X^H is a union of subspheres whose intersections form a complicated combinatorial object. We shall therefore study the following absolute numbers

$$\bar{L}^H(f)\langle w \rangle = L_Z(X^H/WH_0, l(w^{-1}) \circ \bar{f}^H) \tag{2.5}$$

(2.6) $$\underline{L}^H(f)=\sum_{\langle w\rangle}\underline{L}^H(f)\langle w\rangle\cdot\langle w\rangle\in Cl(Z\pi_0WH)$$

and their collection $\{\underline{L}^H(f)\mid(H)\in Con\,G\}$ denoted by

(2.7) $$\underline{L}(f)\in\prod_{(H)}Cl(Z\pi_0WH)\,.$$

Since $X\to X^H/WH_0$ is compatible with equivariant homotopy and inclusions of equivariant CW-complexes and $(S^1\wedge X)^H/WH^0=S^1\wedge(X^H/WH_0)$ holds, $\underline{L}$ is a Lefschetz invariant. Hence we get a homomorphism

(2.8) $$\underline{L}\colon UL(G)\to\prod_{(H)}Cl(Z\pi_0WH),\ [f]\to\underline{L}(f)\,.$$

Theorem 2.9. *$\underline{L}$ is injective.*

Proof. By Theorem 1.10 it suffices to show that the composition

$$\bigoplus_{(H)}Cl(Z\pi_0WH)\xrightarrow{\psi}UL(G)\xrightarrow{\underline{L}}\prod_{(H)}Cl(Z\pi_0WH)$$

is injective. Consider a $a=\sum_{(H)}a(H)$ with $a(H)\in Cl(Z\pi_0WH)$ in the kernel of $\underline{L}\circ\psi$. Suppose that a is not zero. Then choose (H) maximal with $a(H)\neq0$. Since $G/K^H\neq\emptyset$ implies $(H)\leq(K)$ $\underline{L}^H\circ\psi(a)$ is $\underline{L}^H\circ\psi(a(H))$. Write $a(H)=\sum_{\langle w\rangle}n_w\cdot\langle w\rangle$. As $G/H^{>H}$ is empty we get from Theorem 2.3 and 2.4

$$\underline{L}^H\circ\psi(a(H))=\sum_{\langle w\rangle}|C_{\pi_0WH}(w)|\cdot n_w\cdot\langle w\rangle\,.$$

This is a contradiction to $|C_{\pi_0WH}(w)|>0$. □

Now we show that the universal property of $UL(G)$ induces the structure of a commutative ring with unit. Let $f\colon X\to X$ be a pointed G-endomorphism of a pointed finite G-complex. If $g\colon Y\to Y$ is another such map over G' $f\wedge g\colon X\wedge Y\to X\wedge Y$ is a pointed $G\times G'$-endomorphism of a pointed finite $G\times G'$-complex and defines $[f\wedge g]\in UL(G\times G')$. One easily checks that $g\to[f\wedge g]\in UL(G\times G')$ is a Lefschetz invariant for G', so that there is a unique homomorphism $\phi(f)\colon UL(G')\to UL(G\times G')$ $[g]\to[f\wedge g]$. Now $f\to\phi(f)\in\operatorname{Hom}(UL(G')\to UL(G\times G'))$ is a Lefschetz invariant for G. The induced homomorphism $UL(G)\to\operatorname{Hom}(UL(G')\to UL(G\times G'))$ can be viewed as a pairing

(2.10) $$P(G,G')\colon UL(G)\otimes UL(G')\to UL(G\otimes G')$$

uniquely determined by the property $P(G,G')\,([f]\otimes[g])=[f\wedge g]$.

Let $i\colon H\to G$ be a subgroup. Consider the pointed endomorphism $f\colon X\to X$ of the pointed finite G-complex X. It follows from the triangulation theorem that there is a pointed finite H-complex and a pointed H-homotopy equivalence $h\colon Y\to\operatorname{res}X$, see [9, Th. A] or [13]. We get $[\operatorname{res}f]\in UL(H)$ by $[h^{-1}\circ f\circ h]$.

This is independent of the choice of h, h^{-1} and Y by homotopy invariance and commutativity. We leave it to the reader to check that $f \to [\mathrm{res}\, f] \in UL(H)$ is a Lefschetz invariant for G. Hence we get a homomorphism

$$(2.11) \qquad i^*: UL(G) \to UL(H)$$

sending $[f]$ to $[\mathrm{res}\, f]$. If $\Delta: G \to G \times G$ is the diagonal map we get from 2.10 and 2.11.

Theorem 2.12. *The composition $\Delta^* \circ P(G, G)$: $UL(G) \otimes UL(G) \to UL(G)$ induces the structure of an associative commutative ring with unit $[\mathrm{id}_+: G/G_+ \to G/G_+]$ on $UL(G)$.*

One should compare this with [14, section 6]. Because of Theorem 2.9 and 2.12 we can also define $UL(G)$ as the set of equivalence classes $[f]$ of pointed G-endomorphisms of pointed finite G-complexes under the equivalence relation $f \sim g \Leftrightarrow \bar{L}(f) = \bar{L}(g)$. The ring structure is induced from $\vee$ and $\wedge$. Given $f: X \to X$, an inverse of $[f]$ under addition is given by $[f \wedge \mathrm{id}_Y]$ for any finite CW-complex Y with trivial G-action and ordinary Euler characteristi $\chi(Y) = -1$.

The evaluation of the product in $UL(G)$ is in practice very difficult when $\dim G > 0$ so we study a weaker equivalence relation. Call two pointed G endomorphisms $f: X \to X$ and $g: Y \to Y$ of pointed finite G-complexes equivalent if we have for any $H \subset G$ and $w \in \pi_0 WH$ that $L_Z(l(w^{-1}) \circ f^H) = L_Z(l(w^{-1}) \circ g^H)$ holds. Let $L(G)$ be the set of equivalence classes. It becomes a commutative ring with unit $[\mathrm{id}_+: G/G_+ \to G/G_+]$ by $\vee$ and $\wedge$. We call $L(G)$ the *Lefschetz ring* of G. Let $L^H(f) \in Cl(Z\pi_0 WH)$ be the element $\sum_{\langle w \rangle} L_Z(l(w^{-1}) \circ f^H) \cdot \langle w \rangle$. The collection $\{L^H(f) \mid (H) \in Con\, G\}$ defines an inductive ring homomorphism

$$(2.13) \qquad L: L(G) \to \prod_{(H)} Cl(Z\pi_0 WH)$$

if the equip $Cl(Z\pi_0 WH)$ with the ring structure induced by $Cl(Z\pi_0 WH) = \prod_{\langle w \rangle} Z$. The advantage of $L(G)$ is that L is a ring homomorphism which is not true for $\bar{L}$ from (2.8) when G is infinite.

One easily checks that the function $f \to L(f) \in \prod_{(H)} Cl(Z\pi_0 WH)$ is a Lefschetz invariant of G. By 2.13 and the universal property we get a ring homomorphism

$$(2.14) \qquad P: UL(G) \to L(G), \quad [f] \to [f] .$$

Theorem 2.15. *Let $con_\Phi(\pi_0 WH)$ be the set of conjugacy classes $\langle wWH_0 \rangle$ of elements $wWH_0 \in WH/WH_0 = \pi_0 WH$ such that $C_{WH}(w)$ is finite. Then a Z-basis for $L(G)$ is given by*

$$B = \{[r(w)] \mid \langle wWH_0 \rangle \in con_\Phi(\pi_0(WH)), (H) \in Con\, G\} .$$

Proof. Consider $w \in WH$ and $v \in WK$. Then $l(v^{-1}) \circ r(w)^K$: $G/H^K \to G/H^K$ is a $C_{WH}(w)$-map under the right $C_{WH}(w)$-action. If $C_{WH}(w)$ is infinite it contains a circle so that $L_Z(l(v^{-1}) \circ r(w)^K)$ vanishes by Lemma 2.16 below. Suppose $K = H$ and $C_{WH}(w)$ to be finite. If $\langle vWH_0 \rangle \neq \langle wWH_0 \rangle$ then $l(v^{-1}) \circ r(w)^H$ has no fixed points so that $L_Z(l(v^{-1}) \circ r(w)^H)$ vanishes by the Lefschetz fixed point theorem. If $\langle vWH_0 \rangle = \langle wWH_0 \rangle$ holds we can suppose $v = w$ by remark 1.1. Then we obtain from the Lefschetz fixed point formula 2.18 below $L_Z(l(w^{-1}) \circ r(w)^H) = \chi(C_{WH}(w)) = |C_{WH}(w)|$.

This shows that the condition "$C_{WH}(w)$ is finite" depends only on $\langle wWH_0 \rangle$. By Theorem 1.10 and the epimorphism 2.14 the set B generates $L(G)$. Suppose that $a = \sum a(H, w) \cdot [r(w)]$ is zero where the sum runs over B. Assume that not all $a(H, w)$ vanish. Choose (H) maximal with $a(H, w) \neq 0$ for some w. Then $L^H(a)\langle w \rangle$ equals $|C_{WH}(w)|$, a contradiction. □

Lemma 2.16. *Let f: $X \to X$ be a G-endomorphism of a finite free G-complex. If G is S^1 we have $L_Z(f) = \chi(X) = 0$. If G is a finite group $L_Z(f) \equiv \chi(X) \equiv 0 \bmod |G|$.*

Proof. Obviously it suffices to show for finite G that $L_Z(f) \equiv 0 \bmod |G|$ is valid since S^1 contains Z/p as a subgroup for all prime numbers p and $L_Z(\mathrm{id}) = \chi(X)$ holds. If f is cellular then $L_{ZG}(C^c(f)) \in Cl\,(ZG)$ is defined. The homomorphism

$$Cl(ZG) \to Z, \ \sum_{\langle g \rangle} a_{\langle g \rangle} \cdot \langle g \rangle \to |G| \cdot a_{\langle e \rangle}$$

maps it to $L_Z(f)$. □

REMARK 2.17. Theorem 2.15 implies that the property "$C_{WH}(w)$ is finite" depends only on the conjugacy class $\langle wWH_0 \rangle$ of $wWH_0 \in WH/WH_0 = \pi_0(WH)$. This can be seen directly as follows. Choose a Cartan subgroup $S \subset WH$ containing w such that wS_0 generates $\pi_0 S = S/S_0$. Then S is finite if and only if $C_{WH}(w)$ is finite and the conjugacy class $\langle wWH_0 \rangle$ determines the conjugacy class of S in WH.

Proposition 2.18. (Lefschetz fixed point formula.) *We have for a finite G-complex X and $g \in G$*

$$L_Z(X, l(g)) = \chi(X^g).$$

Proof. Let C be the closed subgroup of G generated by g. Choose a finite C-complex Y and a C-homotopy equivalence $Y \to X$. Then $L_Z(X, l(g)) = L_Z(Y, l(g))$ and $\chi(X^g) = \chi(Y^g)$, so we may suppose that $X = Y$ and $G = C$ is topologically generated by g. By additivity we can even suppose $Y = C/H$. If $H \neq C$ then $l(g)$: $Y \to Y$ is fixed point free so that $L_Y(l(g)) = 0 = \chi(Y^g)$ holds. If $H = C$ then Y is a point, $l(g) = \mathrm{id}$ and clearly $L_Z(l(g)) = 1 = \chi(Y^g)$. □

Now we look at the ring $L'(G)$ defined analogously to $L(G)$ but using the equivalence relation $f \sim g \Leftrightarrow L(f^H)=L(g^H)$ for all $H \subset G$. The Burnside ring $A(G)$ is the set of equivalence classes $[X]$ of pointed finite G-complexes under the relation $X \sim Y \Leftrightarrow \chi(X^H, x)=\chi(Y^H, y)$ for all $H \subset G$. There is a natural ring homomorphism

$$I': A(G) \to L'(G) \quad [X] \to [\mathrm{id}: X \to X].$$

Let $\mathrm{pr}: L(G) \to L'(G)$ be the obvious epimorphism $[f] \to [f]$. Condider a base element $[r(w): G/H \to G/H]$. If $\langle wWH_0 \rangle \neq \langle eWH_0 \rangle$ then $r(w)$ has no fixed points so that $L(r(w)^K: G/H^K \to G/H^K)$ is zero for all $K \subset H$ by the Lefschetz fixed point theorem. Since $C_{WH}(w)$ is assumed to be finite WH is finite in the case $w=eH$. Hence a set of generators in $L'(G)$ is given by $\{[\mathrm{id}: G/H \to G/H] \mid (H) \in Con\, G,\ WH \text{ finite}\}$. This set is also linearly independent. Suppose that $\sum n(H) \cdot [\mathrm{id}: G/H \to G/H]$ is zero but not all $n(H)$ vanish. Choose (H) maximal with $n(H) \neq 0$. Then the homomorphism $L'(G) \to Z$, $[f] \to L(f^H)$ maps this sum to $n(H) \cdot |WH|$ a contradiction. As $\{[G/H] \mid (H) \in Con\, G,\ WH \text{ finite}\}$ is a Z-base for $A(G)$ we have

Theorem 2.19. *$I': A(G) \to L'(G)$ is a ring isomorphism.*

This implies in particular that the Burnside ring relations of Corollary C in the introduction are valid for the Lefschetz numbers $L(f^H)$ if $f: X \to X$ is a (pointed) G-endomorphism of a (pointed) finite G-complex. This includes the case of a compact smooth G-manifold.

The authors constructed Lefschetz classes $[f]$ in $A(G)$ for finite G-complexes in [11] for finite and in [12] for compact Lie groups. If $J: L(G) \to A(G)$ is $(I')^{-1} \circ \mathrm{pr}$ they are the images of the present classes $[f] \in UL(G)$ under

$$UL(G) \xrightarrow{p} L(G) \xrightarrow{J} A(G).$$

We have already constructed a homomorphism $I: U(G) \to UL(G)$, $[X] \to [\mathrm{id}: X \to X]$ in section 1. The same formula defines $I: A(G) \to L(G)$. Then is a ring homomorphism $P: U(G) \to A(G)$ sending $[X]$ to $[X]$. Hence we obtain a commutative diagram of ring homomorphisms

(2.20)
$$\begin{tikzcd} UL(G) \arrow[r, "P"] & L(G) \\ U(G) \arrow[u, "I"] \arrow[r, "P"] & A(G) \arrow[u, "I"'] \end{tikzcd}$$

The splitting $J: L(G) \to A(G)$ sends explicitly the class of $r(w): G/H \to G/H$

$[G/H]$ when w represents $e \in \pi_0 WH$ and to 0 otherwise. It is a ring homomorphism and satisfies

$$\phi_H(J[f]) = L(f^H), \qquad H \leq G, \tag{2.21}$$

where $\phi_H[X] = \chi(X^H)$. Of course one can define a splitting of abelian groups $UL(G) \to U(G)$ similarly but the analogue of (2.21) is no more valid. For general infinite G it cannot be satisfied by any map $UL(G) \to U(G)$.

Indeed, let $UL'(G)$ denote the set of equivalence classes of endomorphisms of finite G-complexes under the equivalence relation

$$f \sim g \Leftrightarrow L(f^H/WH_0) = L(g^H/WH_0), \qquad H \leq G.$$

The inclusion I': $U(G) \to UL'(G)$ is usually not surjective, so that the projection $UL(G) \to UL'(G)$ cannot factor through it.

EXAMPLE 2.22. Regard $G = O(2)$ as R-automorphism of C. The complex conjugation $c \in O(2)$ has normalizer $N = \langle c, -1 \rangle$ in G. Let $X = G/\langle c \rangle \cong S^1$ and let f: $X \to X$ be multiplication by -1. Then $L(f^H/WH_0) = 1$ for $H = 1$ and 0 for $H \neq 1$. However, if Y is any finite G-complex with $\chi(Y^H/WH_0) = 0$ for $H \neq 1$ then the class of Y in $U(G)$ is a multiple of $[G]$ and $\chi(Y/SO(2))$ is divisible by $\chi(O(2)/SO(2)) = 2$. Thus $[f] \in UL'(O(2))$ does not lie in the image of $U(O(2))$.

3. Homological algebra over the orbit category

The purpose of this section is to reformulate the technique of induction over orbit types in the language of modules over the orbit category. Given a G-space X, each G-map $G/H \to G/K$ gives rise to a map $X^K \to X^H$ so that X can be regarded as a functor from the orbit category consisting of homogenous spaces G/H to the category of spaces. The cellular chain complexes $C_*^c(X^H)$ of a G-complex X, and the singular chain complexes $C_*^s(X^H)$ of a general G-space X, form similar functors from the orbit category to the category of chain complexes. Our aim is to give conditions on X which guarantee that the complexes $C_*(X^H)$ can simultaneously be replaced by finite projective complexes, since these are the ones where Lefschetz numbers can be computed on chain level. Some systematic approach is needed for compact Lie groups G since then the category of modules over the orbit category is not Noetherian. To simplify notation, we shall work with general functor categories. For more details and other applications, see tom Dieck 6, [Ch. I. 11] and Lück [13].

Let Γ be a small category and let R be a commutative ring with unit. An $R\Gamma$-*module* is a contravariant functor M: $\Gamma \to MOD-R$ from Γ to the category of R-modules. A *homomorphism* between $R\Gamma$-modules is a natural transformation. Let MOD-$R\Gamma$ denote the category of $R\Gamma$-modules.

EXAMPLE 3.1. Any group G can be considered as a category with a single object and one morphism for each group element. Contravariant functors $M\colon G \to MOD-R$ are equivalent to right modules over the group ring RG.

The category $MOD-R\Gamma$ inherits a structure of abelian category from $MOD-R$. For example, a sequence of $R\Gamma$-modules is exact if its value at each object of Γ is exact. An $R\Gamma$-module P is *projective*, if it has the following lifting property:

$$\begin{tikzcd} & P \arrow[dl, dashed, "w"'] \arrow[d, "v"] & \\ M \arrow[r, "u"] & N \arrow[r] & 0 \end{tikzcd}$$

if v is a morphism and u is an epimorphism, there exists a morphism w such that $uw=v$. Projective modules are related to free modules in the usual way, once free modules are defined as adjoints to suitable forgetting functors as follows.

A Γ-*set* is a family (B_x) of sets B_x indexed by $Ob(\Gamma)$. A Γ-*map* between two Γ-sets (B_x) and (C_x) is a family $(f_x\colon B_x \to C_x)$ of maps. If $|\Gamma|$ denotes the category having the same objects as Γ and only identities as morphisms, we can interpret Γ-sets as functors $|\Gamma| \to Set$. Each $R\Gamma$-module M has an underlying Γ-set, also denoted by M. It is essential that we forget not only the R-module structure but also the non-identity morphisms in Γ.

An $R\Gamma$-module F is *free with* Γ-set $B \subset F$ as *basis* if each Γ-map $h\colon B \to M$ into an $R\Gamma$-module M has a unique extension to an $R\Gamma$-homomorphism $H\colon F \to M$. A free module F with basis B is unique up to isomorphism by the universal property. It is constructed as follows. Let RS denote the free R-module with basis S. For each object x of Γ the $R\Gamma$-module

$$R\Gamma(x) = R\,\mathrm{Hom}\,(?,\, x)$$

is free with basis $\mathrm{id}_x \in R\,\mathrm{Hom}\,(x, x)$ by the Yoneda Lemma. The free modul over a Γ-set $B=(B_x)$ is now defined as

$$R\Gamma(B) = \bigoplus_{x \in Ob\,\Gamma} \bigoplus_{B_x} R\Gamma(x)\,.$$

It is clear that every $R\Gamma$-module M is a quotient of the free module $R\Gamma(M)$ and that projective $R\Gamma$-modules are precisely the direct summands of free ones.

Let M be an $R\Gamma$-module and let $E \subset M$ be a Γ-subset. The submodul generated by E is the smallest $R\Gamma$-submodule of M containing E, i.e. the imag of the $R\Gamma$-homomorphism $R\Gamma(E) \to M$ extending the inclusion. An $R\Gamma$-modul is *finitely generated* if it is generated by a finite Γ-subset or equivalently, if it i a quotient of some finitely generated free $R\Gamma$-module.

The chain complexes over the abelian category $MOD-R\Gamma$ form an abelian category. We shall always assume that chain complexes C are positive, i.e. $C_n=0$ for $n<0$. We call C *free* (resp. *projective*), if each C_n is free (resp. projective), and *finite-dimensional* if $C_n=0$ for $n\gg 0$. A *finite projective* $R\Gamma$-chain complex is a finite-dimensional projective $R\Gamma$-chain complex C such that each C_n is finitely generated. A chain map $f\colon C\to D$ between $R\Gamma$-chain complexes is a *weak equivalence* if $f_x\colon C(x)\to D(x)$ induces an isomorphism in homology for each object x of Γ. We can now state the

PROBLEM When is a $R\Gamma$-chain complex C weakly equivalent to a finite projective complex P?

Its relevance to topology becomes clear in the following example.

EXAMPLE 3.2. i) Let G be a topological group and let X be a G-space. The *orbit category* $\mathrm{Or}\,G$ has the homogeneous spaces G/H as objects and G-maps as morphisms. The natural bijection $\mathrm{Map}_G(G/H, X)\to X^H$ sending $f\colon G/H\to X$ to $f(eH)\in X^H$ gives rise to a contravariant functor from the orbit category to the category of topological spaces

$$X\colon \mathrm{Or}\,G\to \mathrm{Top}\,,\quad X(G/H)=X^H\,.$$

Explicitly, if $f\colon G/K\to G/H$ is a G-map with $f(eK)=gH$ then $g^{-1}Kg\subset H$ and $X(f)$ is the composite map

$$X(f)\colon X^H\subset X^{g^{-1}Kg}\xrightarrow{l(g)} X^K\,.$$

Let G be a compact Lie group. The discrete orbit category $\mathrm{Or}_d\,G$ is the homotopy category of $\mathrm{Or}\,G$: it has the same objects but homotopy classes of maps as morphisms. Since G/H^K is a disjoint union of finitely many WK-orbits [2, II. 5.7] the space $G/H^K/WK_0$ is discrete. We get an identification

$$[G/K, G/H]^G=\pi_0((G/H)^K)=(G/H)^K/WK_0=\mathrm{Map}_G(G/K, G/H)/WK_0\,.$$

Hence a G-space X gives rise to a contravariant functor

$$\bar{X}\colon \mathrm{Or}_d\,G\to \mathrm{Top}\quad G/H\to X^H/WH_0\,.$$

Composing $\bar{X}$ with the functor singular chain complex with R-coefficients gives an $R\,\mathrm{Or}_d\,G$-chain complex $C^s(X)$, called the *singular $R\,\mathrm{Or}_d\,G$-chain complex* of X.

ii) Let G be a compact Lie group. If X is a G-complex then $\bar{X}$ can be regarded as a functor from $\mathrm{Or}_d\,G$ to CW-complexes. Indeed, $\bar{X}(G/H)=X^H/WH_0$ is a π_0WH-complex, and hence an ordinary CW-complex with skeletons X_n^H/WH_0. The quotient X^H/WH_0 is the largest quotient of X^H with a natural CW-structure. The *cellular $R\,\mathrm{Or}_d\,(G)$-chain complex* $C^c(X)$ of X is the composite of $\bar{X}$ and the functor cellular chain complex with R-coefficients.

If J_n is the set of n-dimensional equivariant cells of X, choose a characteristic map $\phi_j\colon G/H_j\times(D^n, S^{n-1})\to(X_n, X_{n-1})$ for each $j\in J_n$. By restriction to $eH_j\times D^n$ and composition with the projection it induces

$$\psi_j\colon (D^n, S^{n-1})\to(X_n^H/WH_0, X_{n-1}^H/WH_0)\,.$$

The image under ψ_j of a generator $w\in H_n(D^n, S^{n-1}; R)$ is an element

$$b_j\in C_n^c(X)(G/H)=H_n(X_n^H/WH_0, X_{n-1}^H/WH_0; R)\,.$$

The set $B_n=\{b_j\mid j\in J_n\}$ forms a basis for the R Or_d G module $C_n^c(X)$, since

$$C_n^c(X)(G/H)=H_n(X_n^H/WK_0, X_{n-1}^H/WK_0)\cong\bigoplus_{j\in J_n}H_n((G/H_j)^H/WH_0\times(D^n, S^{n-1}))$$

and $(G/H_j)^H/WH_0$ is precisely the Hom-set $[G/H, G/H_j]^G$.

The cellular chain complex $C^c(X)$ is thus a free R Or_d G-chain complex, which is finite-dimensional (resp. finite) if X is finite-dimensional (resp. finite).

REMARK 3.3. The Bredon-Illman equivariant cohomology of a G-complex X with coefficients in an R Or_d G-module M is obtained from the cellular chain complex $C^c(X)$ by setting

$$H_G^*(X, M)=H^*(\mathrm{Hom}(C^c(X), M))\,,$$

where Hom means homomorphism of R Or_d G-modules.

For arbitrary G-spaces X we can difine an equivariant cohomology theory by

$$H_G^*(X; M)=H^*(\mathrm{Hom}(C^s(X), M))\,.$$

If G is finite, it agrees with Illman's equivariant singular cohomology. We conjecture that this holds for compact Lie groups, too. Equivariant homology theories $H_*^G(X; M)$ are constructed similarly using tensor product of R $\mathrm{Or}_d(G)$ modules, see [6, Ch. II. 9].

To compare the cellular and singular chain complexes of a G-complex we first recall the following standard lemma (see eg. [10, Ch. I]).

Lemma 3.4. *Let P be a projective $R\Gamma$-chain complex. Any weak equivalence $f\colon C\to D$ of $R\Gamma$-chain complexes induces an isomorphism*

$$f_*\colon [P, C]\to[P, D]\,.$$

There is a functor D from CW-complexes to chain complexes over Z and natural transformations $i\colon D\to C^s$ and $j\colon D\to C^c$ such that $i(X)$ and $j(X)$ are homology equivalences for any CW-complex X, see Wall [19, Lemma 1]. If is a G-complex, the composite functor $D \circ \bar{X}$ gives rise to a third $Z\ \mathrm{Or}_d\ G$-chain complex $D(X)$ together with natural weak equivalences $i(X)\colon D(X) \to C^s(X)$, $j(X)\colon D(X) \to C^c(X)$. Since $C^c(X)$ is free (Example 3.2ii), applying Lemma 3.4 twice yields

Proposition 3.5. *Let G be a compact Lie group and let X be a G-complex. There is a weak equivalence of $Z\ \mathrm{Or}_d\ G$-chain complexes $C^c(X) \to C^s(X)$ which is natural in X up to chain homotopy.*

(In the derived category of $MOD-R\Gamma$ where weak equivalences are formally inverted, i and j define directly a natural isomorphism $C^c(X) \to C^s(X)$, cf. [10, Ch. IX].)

The orbit categories $\mathrm{Or}\, G$ and $\mathrm{Or}_d\, G$ of a compact Lie group G have the property that each endomorphism is an isomorphism. A small category Γ having this property is called an *EI-category*. If Γ is an EI-category, we can define a partial order on the set $\mathrm{Is}(\Gamma)$ of isomorphism classes $\bar{x}$ of objects $x \in \mathrm{Ob}(\Gamma)$ by setting $\bar{x} \leq \bar{y} \Leftrightarrow \mathrm{Hom}(x, y) \neq \emptyset$. For the orbit categories this means that $\overline{G/H} \leq \overline{G/K}$ if and only if H is subconjugate to K.

Let Γ be an EI-categroy. In the sequel $R[x]$ stands for the group ring $R[\mathrm{Aut}(x)]$ and $MOD-R[x]$ is the category of right $R[x]$-modules. For each object x of Γ we introduce a *splitting functor*

$$S_x\colon MOD-R\Gamma \to MOD-R[x] \tag{3.6}$$

and an *extension functor*

$$E_x\colon MOD-R[x] \to MOD-R\Gamma \tag{3.7}$$

as follows. Given an $R\Gamma$-module M, let $M_s(x)$ be the R-submodule of $M(x)$ generated by the images of $M(f)\colon M(y) \to M(x)$ where $f\colon x \to y$ runs through the morphisms with $\bar{x} \neq \bar{y}$. Then $M_s(x)$ is an $R[x]$-submodule of $M(x)$, and we set $S_xM = M(x)/M_s(x)$. If N is a right $R[x]$-module, we define an $R\Gamma$-module $E_xN = N \underset{R[x]}{\otimes} R\,\mathrm{Hom}(?, x)$.

The functors S_x and E_x are right exact and additive. They are easily seen to preserve the properties freeness and finite generation. In particular, the image of a finitely generated projective module under S_x and E_x is again a finitely generated projective module.

EXAMPLE 3.8. Let G be a compact Lie group. The automorphism group of G/H in $\mathrm{Or}_d\, G$ is $(\pi_0 WH)^{op}$ and so right $R[G/H]$-modules correspond to left $R\pi_0 WH$-modules. Let X be a G-complex with cellular $R\ \mathrm{Or}_d\ G$ chain complex $C^c(X)$. Then $S_{G/H}C^c(X)$ is isomorphic to $C^c(X^H/WH_0, X^{>H}/WH_0)$ as a complex of left $R\pi_0 WH$-modules. If X is a single cell G/K, this equals $R\pi_0 WH$ if $G/K \cong G/H$ and 0 otherwise. On the other hand, $E_{G/H}$ maps $R\pi_0 WH$ to $C^c(G/H)$.

Let $F \subset \mathrm{Is}(\Gamma)$ be a subset. An $R\Gamma$-module M is of *type* F if it is generated by a Γ-subset E such that $E_x = \emptyset$ when $\bar{x} \notin F$. An $R\Gamma$-chain complex C_* is of type F if each C_n has type F. A module or chain complex has *finite type*, if it is of type F for some finite F. This should not be confused with finite generation. For example, $\mathrm{Is}(\mathrm{Or}_d G)$ can be identified with the set of conjugacy classes (H) of closed subgroups of G. If X is a G-complex then $C^c(X)$ is of type $F = \{(H) \mid H \in \mathrm{Iso}\, X\}$, and it finite type if and only if X has finite orbit type.

Let $F \subset \mathrm{Is}(\Gamma)$ be a finite family. Choose a maximal element $\bar{x} \in F$, i.e. $\bar{y} \in F$, $\bar{x} \leq \bar{y}$ implies $\bar{x} = \bar{y}$. Let M be an $R\Gamma$-module of type F. Then $M(y) = 0$ for $\bar{x} < \bar{y}$, so that $M_s(x) = 0$ and $S_x M = M(x)$. We define a natural transformation $I_x \colon E_x \circ S_x \to \mathrm{Id}$ of functors on the category of modules of type F by the formula

$$I_x M \colon M(x) \underset{R[x]}{\otimes} R\,\mathrm{Hom}(?, x) \to M(?),\ m \otimes f \to M(f)(m)\,.$$

The cokernel of $I_x M$ is denoted by $\mathrm{Cok}_x M$. It is an $R\Gamma$-module of type $F \setminus \{\bar{x}\}$. Let $\mathrm{Pr}_x M \colon M \to \mathrm{Cok}_x M$ be the projection. The result is an exact sequence

$$E_x S_x M \xrightarrow{I_x M} M \xrightarrow{\mathrm{Pr}_x M} \mathrm{Cok}_x M \to 0$$

which is natural in M.

So far M has been an arbitrary module of type F. For projective modules we can say more.

Theorem 3.9. *Let Γ be an EI-category. If P be a projective $R\Gamma$-module of finite type F and $\bar{x} \in F$ is maximal, then $E_x S_x P$ is projective of type $\{\bar{x}\}$ and $\mathrm{Cok}_x P$ is projective of type $F \setminus \{\bar{x}\}$. The sequence*

$$0 \to E_x S_x P \xrightarrow{I_x P} P \xrightarrow{\mathrm{Pr}_x P} \mathrm{Cok}_x P \to 0$$

is exact and splits.

Proof. Since the sequence above is compatible with direct sums, we may assume that P is free, and has the typical form $R = R\Gamma[y]$. If $\bar{x} = \bar{y}$, then $I_x(P)$ is an isomorphism and $\mathrm{Cok}_x P = 0$. Otherwise $E_x S_x P = 0$ and $\mathrm{Cok}_x P = P$.

Theorem 3.9 implies by induction a splitting $P \cong \bigoplus_{x \in F} E_x S_x P$, see [6, Th. 11.18 p. 83]. However, this splitting is not natural. Since we study Lefschetz invariants of mappings $f \colon P \to P$, we shall have to use the naturality properties of I_x and Pr_x. If X is a G-complex with finite orbit type and $x = G/H$ is an orbit of maximal type, the sequences of 3.9 for $P = C_n^c(X)$ take the familiar form

$$0 \to C_n^c(X^{(H)}) \to C_n^c(X) \to C_n^c(X, X^{(H)}) \to 0\,.$$

We are now ready to handle the problem of approximating complexes by finite projective ones. The induction step will use

Lemma 3.10. *Let Γ be a small category. Assume that two members of the exact sequence*

$$0 \to C_1 \to C \to C_2 \to 0$$

of $R\Gamma$-chain complexes are weakly equivalent to finite projective complexes. Then so is the third one, and there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d, "h_1"] & P \arrow[r] \arrow[d, "h"] & P_2 \arrow[r] \arrow[d, "h_2"] & 0 \\ 0 \arrow[r] & C_1 \arrow[r] & C \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$

with exact rows where P_1, P and P_2 are finite projective complexes and h_1, h and h_2 are weak equivalences.

Proof. Standard manipulation with the cone and cylinder functors, see [10, Prop. I 6.10] for a proof of the dual statement about injective complexes.

Theorem 3.11. *Let Γ be an EI-category with finite* Hom*-sets. Let R be a Noetherian ring. If C is a finite-dimensional projective $R\Gamma$-chain complex of finite type F and $H_*(C(x))$ is finitely generated over R for $\bar{x}\in F$, then C is homotopy equivalent to a finite projective $R\Gamma$-chain complex.*

Proof. We use induction over the cardinality of F. If $|F|=0$ then $C=0$ and the claim holds trivially. In the induction step, let $\bar{x}\in F$ be maximal. Then $S_xC=C(x)$ is a finite-dimensional projective chain complex over the Noetherian ring $R[x]$. Since its homology is finitely generated, $S_x C$ is homotopy equivalent to a finite projective $R[x]$-chain complex P [3, Lemma IX 5.4]. Then E_xS_xC is homotopy equivalent to the finite projective $R\Gamma$-chain complex E_xP. Theorem 3.9 gives an exact sequence

$$0 \to E_xS_xC \to C \to \mathrm{Cok}_x\, C \to 0\,.$$

If $y\in F$ then $H_*(C(y))$ is finitely generated by assumption and $H_*(E_xS_xC(y)) \cong H_*(E_xP(y))$ is finitely generated since $\mathrm{Hom}\,(y, x)$ is finite, Since R is Noetherian, $H_*(\mathrm{Cok}_x P(y))$ is a finitely generated R-module. As $\mathrm{Cok}_x\, C$ has type $F\backslash\{x\}$, it is homotopy equivalent to a finite projective complex by the induction assumption. Lemma 3.10 implies that C is weakly equivalent to a finite projective $R\Gamma$-chain complex. But a weak equivalence between projective complexes is a homotopy equivalence by Lemma 3.4.

Let $F \subset \mathrm{Is}(\Gamma)$ be a subset. An $R\Gamma$-module M is of *type* F if it is generated by a Γ-subset E such that $E_x = \emptyset$ when $\bar{x} \notin F$. An $R\Gamma$-chain complex C_* is of type F if each C_n has type F. A module or chain complex has *finite type,* if it is of type F for some finite F. This should not be confused with finite generation. For example, $\mathrm{Is}(\mathrm{Or}_d G)$ can be identified with the set of conjugacy classes (H) of closed subgroups of G. If X is a G-complex then $C^c(X)$ is of type $F = \{(H) \mid H \in \mathrm{Iso}\, X\}$, and it finite type if and only if X has finite orbit type.

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Theorem 3.9. *Let Γ be an EI-category. If P be a projective $R\Gamma$-module of finite type F and $\bar{x} \in F$ is maximal, then $E_x S_x P$ is projective of type $\{\bar{x}\}$ and* $\mathrm{Cok}_x P$ *is projective of type $F \setminus \{\bar{x}\}$. The sequence*

$$0 \to E_x S_x P \xrightarrow{I_x P} P \xrightarrow{\mathrm{Pr}_x P} \mathrm{Cok}_x P \to 0$$

is exact and splits.

Proof. Since the sequence above is compatible with direct sums, we may assume that P is free, and has the typical form $R = R\Gamma[y]$. If $\bar{x} = \bar{y}$, then $I_x(P)$ is an isomorphism and $\mathrm{Cok}_x P = 0$. Otherwise $E_x S_x P = 0$ and $\mathrm{Cok}_x P = P$.

Theorem 3.9 implies by induction a splitting $P \cong \bigoplus_{x \in F} E_x S_x P$, see [6, Th. 11.18 p. 83]. However, this splitting is not natural. Since we study Lefschetz invariants of mappings $f\colon P \to P$, we shall have to use the naturality properties of I_x and Pr_x. If X is a G-complex with finite orbit type and $x = G/H$ is an orbit of maximal type, the sequences of 3.9 for $P = C_n^c(X)$ take the familiar form

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We are now ready to handle the problem of approximating complexes by finite projective ones. The induction step will use

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with exact rows where P_1, P and P_2 are finite projective complexes and h_1, h and h_2 are weak equivalences.

Proof. Standard manipulation with the cone and cylinder functors, see [10, Prop. I 6.10] for a proof of the dual statement about injective complexes.

Theorem 3.11. *Let Γ be an EI-category with finite* Hom*-sets. Let R be a Noetherian ring. If C is a finite-dimensional projective $R\Gamma$-chain complex of finite type F and $H_*(C(x))$ is finitely generated over R for $\bar{x}\in F$, then C is homotopy equivalent to a finite projective $R\Gamma$-chain complex.*

Proof. We use induction over the cardinality of F. If $|F|=0$ then $C=0$ and the claim holds trivially. In the induction step, let $\bar{x}\in F$ be maximal. Then $S_xC=C(x)$ is a finite-dimensional projective chain complex over the Noetherian ring $R[x]$. Since its homology is finitely generated, S_xC is homotopy equivalent to a finite projective $R[x]$-chain complex P [3, Lemma IX 5.4]. Then E_xS_xC is homotopy equivalent to the finite projective $R\Gamma$-chain complex E_xP. Theorem 3.9 gives an exact sequence

$$0 \to E_xS_xC \to C \to \mathrm{Cok}_x\, C \to 0\,.$$

If $y\in F$ then $H_*(C(y))$ is finitely generated by assumption and $H_*(E_xS_xC(y)) \cong H_*(E_xP(y))$ is finitely generated since $\mathrm{Hom}(y, x)$ is finite. Since R is Noetherian, $H_*(\mathrm{Cok}_x P(y))$ is a finitely generated R-module. As $\mathrm{Cok}_x C$ has type $F\backslash\{x\}$, it is homotopy equivalent to a finite projective complex by the induction assumption. Lemma 3.10 implies that C is weakly equivalent to a finite projective $R\Gamma$-chain complex. But a weak equivalence between projective complexes is a homotopy equivalence by Lemma 3.4.

Corollary 3.12. *Let G be a compact Lie group and let X be a finite-dimensional G-complex of finite orbit type. If R is a Noetherian ring and $H_*(X^H/WH_0; R)$ is a finitely generated R-module for each $H \in \mathrm{Iso}(X)$, then the cellular $R\,\mathrm{Or}_d\,G$-chain complex $C^c(X)$ is homotopy equivalent to a finite projective $R\,\mathrm{Or}_d\,G$-complex.*

Proof. The Hom-sets $[G/K, G/H]_G = \pi_0((G/H)^K)$ are finite and $C^c(X)$ has finite type $F = \{(H) \mid H \in \mathrm{Iso}(X)\}$. $\square$

4. Trace and Lefschetz invariants

We give axioms for trace invariants of $R\Gamma$-module homomorphisms and Lefschetz invariants of $R\Gamma$-chain mappings. We show how the trace invariants for finitely generated projective modules determine the Lefschetz invariants for finite projective complexes and how they extend uniquely to complexes having a finite projective approximation. The universal trace and Lefschetz invariants are computed for EI-categories. If G is a compact Lie group, the universal invariant group relevant to cellular chain complexes turns out to be the universal Lefschetz group $UL(G)$.

Let Γ be a small category and let R be a commutative ring. We fix first notation for some subcategories of the category $MOD-R\Gamma$ of $R\Gamma$-modules and the category $C-R\Gamma$ of $R\Gamma$-chain complexes. Considering a module as a chain complex concentrated in dimension zero, we have following inclusions of full subcategories

$$\begin{array}{ccccc} P-R\Gamma & \subset & HP-R\Gamma & \subset & MOD-R\Gamma \\ \cap & & \cap & & \cap \\ C(P)-R\Gamma & \subset & HC(P)-R\Gamma & \subset & C-R\Gamma\,. \end{array}$$

Here $P-R\Gamma$ consists of finitely generated projective modules and $HP-R\Gamma$ of modules having a finite projective resolution. Similarly, $C(P)-R\Gamma$ stands for finite projective complexes and $HC(P)-R\Gamma$ for complexes C having a finite projective approximation, i.e. a weak equivalence $P \to C$ with finite projective P.

DEFINITION 4.1. A *trace invariant* (A, T) for $P-R\Gamma$ is an abelian group A together with a function T assigning an element $T(f) \in A$ to each endomorphism $f\colon M \to M$ in $P-R\Gamma$ such that axioms a, b and c hold.

(a) *Additivity.* If the following diagram commutes and has exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

then $T(f)=T(f_1)+T(f_2)$.

(b) *Linearity.* If $f, g\in \mathrm{End}(M)$, then $T(f+g)=T(f)+T(g)$.

(c) *Commutativity.* If $f\colon M\to N$, $g\colon N\to M$, then $T(fg)=T(gf)$.

A trace invariant (A, T) is *universal*, if for any trace invariant (A', T') there is a unique homomorphism $\phi\colon A\to A'$ such that $T'(f)=\phi(T(f))$.

A universal trace invariant is unique and can be constructed by taking as generators all isomorphism classes of endomorphisms and introducing the relations that follow from a, b and c.

EXAMPLE 4.2. Let Γ be a group G considered as a category as in example 3.1. The universal trace invariant for finitely generated projective RG-modules was determined by Hattori [8] and Stallings [17]. Namely, it is the Hattori-Stallings trace $\mathrm{Tr}_{RG}(f)\in Cl(RG)$ of 1.4.

The axioms of trace invariants make sense also for chain complexes and chain mappings. Each trace invariant (A, T) for $P-R\Gamma$ defines an invariant (A, L_T) for chain maps in $C(P)-R\Gamma$ by

$$L_T(f_*)=\sum_{n\geq 0}(-1)^n T(f_n)\,. \tag{4.3}$$

It is clear that L_T satisfies again the axioms, and that is extends T from the subcategory $P-R\Gamma$. We would like to show that L_T is the unique extension. For this we have to require homotopy invariance.

DEFINITION 4.4. A *Lefschetz invariant* (B, L) for $C(P)-R\Gamma$ is an abelian group B together with a function L assigning an element $L(f)\in B$ to each chain map $f\colon C\to C$ in $C(P)-R\Gamma$ such that L satisfies axioms a, b, c and

(d) *Homotopy invariance.* If $f\simeq g$ then $L(f)=L(g)$.

Proposition 4.5. *If (A, T) is a trace invariant for $P-R\Gamma$, then (A, L_T) is its unique extension to a Lefschetz invariant for $C(P)-R\Gamma$.*

Proof. Homotopy invariance for L_T follows easily from the linearity and commutativity of T. Hence L_T is a Lefschetz invariant extending T. If L is another extension, additivity implies by induction that $L(f_*)=\sum_{n\geq 0}L(f_n)$, where $f_n\colon C_n\to C_n$ is considered as a chain map concentrated in dimension n. Hence we have to prove $L(f_n)=(-1)^n T(f_n)$. If M is any module, the mapping cone $\mathrm{Cone}(M)$ of $\mathrm{id}\colon M\to M$ is contractible and the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & \mathrm{Cone}(M) \arrow[r] \arrow[d, "f"] & M[1] \arrow[r] \arrow[d, "f[1]"] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathrm{Cone}(M) \arrow[r] & M[1] \arrow[r] & 0
\end{tikzcd}$$

implies that $L(f[1])=-L(f)$ by additivity and homotopy invariance. Hence $L(f[n])=(-1)^n L(f)=(-1)^n T(f)$. □

In particular, the universal Lefschetz invariant for $C(P)-R\Gamma$ has the same value group as the universal trace invariant for $P-R\Gamma$.

Next we want to extend a Lefschetz invariant (B, L) to complexes having a finite projective approximation. Let C be an $R\Gamma$-chain complex, P a finite projective $R\Gamma$-chain complex and $h\colon P\to C$ a weak equivalence. If $f\colon C\to C$ is a chain map, it follows from Lemma 3.4 that the diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, dashed, "g"'] & C \arrow[d, "f"] \\ P \arrow[r, "h"] & C \end{tikzcd}$$

can be filled to a homotopy commutative square by a lift g which is unique up to homotopy. Thus $\bar{L}_h(f)=L(g)$ depends only on f and h. For other choice $h'\colon P'\to C$ of a projective approximation and lift $g'\colon P'\to P'$, we may construct similarly a weak equivalence $k\colon P\to P'$ such that $g'k\simeq kg$. Then k is a homotopy equivalence with inverse k^{-1}

$$\bar{L}_{h'}(f) = L(g') = L(kgk^{-1}) = L(g) = \bar{L}_h(f)$$

by the homotopy invariance and commutativity of L. Hence $\bar{L}(f)=L(g)$ depends only on f.

Proposition 4.6. *If (B, L) is a Lefschetz invariant for $C(P)-R\Gamma$, then $(B, \bar{L})$ is its unique extension to $HC(P)-R\Gamma$ which satisfies axioms a to d and*

(e) *Homology invariance. Given a homotopy commutative square with a weak equivalence h*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "g"'] & C \arrow[d, "f"] \\ D \arrow[r, "h"] & C \end{tikzcd}$$

then $L(f)=L(g)$.

The pair (B, L) is called a Lefschetz invariant for $HC(P)-R\Gamma$ if it satisfies axioms a to e.

Proof. Axiom (e) is designed to guarantee the uniqueness of the extension. We have to show that $\bar{L}$ satisfies all axioms. Only the verification of additivity is non-trivial. It follows from the next lemma.

Lemma 4.7. *Let Γ be a small category. Let*

$$0 \to C_1 \to C \to C_2 \to 0$$

be a short exact sequence of complexes in $HC(P)-R\Gamma$. Let $f\colon C \to C$ be a chain map which preserves C_1 and denote by $f_i\colon C_i \to C_i$ the induced maps. For any finite projective approximation

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "h_1"'] & P \arrow[r] \arrow[d, "h"'] & P_2 \arrow[r] \arrow[d, "h_2"'] & 0 \\
0 \arrow[r] & C_1 \arrow[r] & C \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

as in Lemma 3.10 *there exists a lift $g\colon P \to P$ of f such that g preserves P_1 and that the induced maps g_i; $P_i \to P_i$ are lifts of f_i, $i=1, 2$.*

Proof. A chain map $g_1: P_1 \to P_1$ together with a chain homotopy $\phi_1: P_1 \to C_1$ between $f_1 \circ h_1$ and $h_1 \circ g_1$ is equivalent to a commutative square of chain complexes

$$\begin{tikzcd}
P_1 \arrow[r, hook] \arrow[d, "f_1 \circ h_1"'] & \text{Cone}\,(P_1) \arrow[d, dashed, "(g_1,\, \phi_1)"] \\
C_1 \arrow[r, hook] & \text{Cone}\,(h_1)
\end{tikzcd}$$

The canonical inclusion $P_1 \hookrightarrow \text{Cone}\,(P_1)$ has a projective cokernel and is therefore a cofibration. As Cone (h_1) is acyclic, an extension (g_1, ϕ_1) exists.

Next we want to extend g_1 to a lift $g\colon P \to P$ of f. We must construct an arrow which makes the following diagram commutative.

$$\begin{tikzcd}
P_1 \arrow[rr] \arrow[dr] \arrow[dd] & & \text{Cone}\,(P_1) \arrow[dr] \arrow[dd] & \\
& P \arrow[rr] \arrow[dd] & & \text{Cone}\,(P) \arrow[dd, dashed] \\
C_1 \arrow[rr] \arrow[dr] & & \text{Cone}\,(h_1) \arrow[dr] & \\
& C \arrow[rr] & & \text{Cone}\,(h)
\end{tikzcd}$$

The sequence $0 \to P_1 \to \text{Cone}\,(P_1) \oplus P \to \text{Cone}\,(P)$ is exact and the image of the last map is a direct summand with projective complement. Since Cone(h) is acyclic, the dotted arrow exists and yields a map $g\colon P\to P$ with a homotopy $\phi\colon f\circ h\simeq h\circ g$ extending g_1 and ϕ_1. The last map $g_2\colon P_2\to P_2$ and the homotopy $\phi_2\colon f_2\circ h_2\simeq h_2\circ g_2$ are induced uniquely. □

The modules having a finite projective resolution can be considered as a subcategory of $HC(P)$--$R\Gamma$. Let $M\in HP$--$R\Gamma$ and let P be a finite projective complex with a weak equivalence $h\colon P\to M$. Each homomorphism $f\colon M\to M$ can be lifted to a chain map $g\colon P\to P$ such that

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "g"'] & M \arrow[d, "f"] \\ P \arrow[r, "h"] & M \end{tikzcd}$$

is a strictly commutative square. If (A, T) is a trace invariant for P—$R\Gamma$ then

$$\bar{T}(f)=\sum_{n\geqq 0}(-1)^n T(g_n)$$

coincides with the Lefschetz invariant $\bar{L}_T(f)$, and is therefore a trace invariant. An easy argument based on additivity shows

Proposition 4.8. *If (A, T) is a trace invariant for P—$R\Gamma$ then $(A, \bar{T})$ is its unique extension to a trace invariant for HP—$R\Gamma$.* □

Propositions 4.5, 4.6 and 4.8 imply

Corollary 4.9. *Let Γ be a small category. A universal trace invariant (A, T) for finitely generated projective $R\Gamma$-modules determines the universal trace invariant $\bar{T}$ for HP—$R\Gamma$ and the universal Lefschetz invariants L_T for $C(P)$—$R\Gamma$ and $\bar{L}_T$ for $HC(P)$—$R\Gamma$. They all have the same value group A.*

From now on, let Γ be an EI-category. We are going to compute the universal invariants. By Corollary 4.9 it suffices to consider finitely generated projective $R\Gamma$-modules P. Let $f\colon P\to P$ be a homomorphism. Using the splitting functor S_x (3.6) we get endomorphisms $S_xf\colon S_xP\to S_xP$ of finitely generated modules over the group ring $R[x]$ for each object $x\in \mathrm{Ob}(\Gamma)$. By example 4.2 their universal trace invariants are the Hattori-Stallings traces $\mathrm{Tr}_{R[x]}$.

Choose for each isomorphism class $\bar{x}\in \mathrm{Is}(\Gamma)$ a representative $x\in\mathrm{Ob}(\Gamma)$ and define a trace invariant (A, T) by

$$A=\bigotimes_{\bar{x}\in \mathrm{Is}(\Gamma)} Cl(R[x]),\quad T(f)=(\mathrm{Tr}_{R[x]}(S_xf))_{\bar{x}\in\mathrm{Is}(\Gamma)}\,. \tag{4.10}$$

For a finitely generated module P we have $S_xP=0$ for almost all $\bar{x}\in \mathrm{Is}(\Gamma)$ so that $T(f)$ lies in the direct sum A.

Theorem 4.11. *Let Γ be an EI-category. Then (A, T) is the universal trace invariant for finitely generated projective $R\Gamma$-modules.*

Proof. Let (A', T') be a universal trace invariant for $P\text{—}R\Gamma$. Since (A, T) is clearly a trace invariant, there is a unique homomorphism $\alpha\colon A' \to A$ satisfying

$$T(f) = \alpha(T'(f)) .$$

For each $x \in \mathrm{Ob}(\Gamma)$ the assignment $f \to T'(E_x f)$ defines a trace invariant for $P - R[x]$ where E_x is the extension functor (3.7). Since $\mathrm{Tr}_{R[x]}$ is universal among such trace invariants there exists a unique homomorphism $\beta_x\colon Cl(R[x]) \to A'$ such that

$$T'(E_x f) = \beta_x(\mathrm{Tr}_{R[x]} f) .$$

A direct sum of the homomorphism β_x over $\bar{x} \in \mathrm{Is}(\Gamma)$ defines a homomorphism $\beta\colon A \to A'$ in the other direction. We claim that α and β are inverse isomorphisms.

Each element $a \in Cl(R[x])$ can be represented as the trace of some endomorphism $f\colon R[x] \to R[x]$. Then

$$\alpha\beta(a) = \alpha\beta_x(\mathrm{Tr}_{R[x]} f) = \alpha(T'E_x f) = T(E_x f) = (\mathrm{Tr}_{R[y]}(S_y E_x f))_{\bar{y}}$$

equals a since $S_x E_x = \mathrm{id}$ and $S_y E_x = 0$ for $\bar{y} \neq \bar{x}$. Hence $\alpha\beta = \mathrm{id}$.

To show that $\beta\alpha = \mathrm{id}$ we have to proceed inductively since $E_x S_x = \mathrm{id}$ only on modules of type $\{\bar{x}\}$. We claim that $T'(f) = \beta(T(f))$ for each endomorphism $f\colon P \to P$ of a finitely generated projective $R\Gamma$-module P. Choose a finite subset $F \subset \mathrm{Is}(\Gamma)$ such that P has type F. We prove the claim by induction on $n = |F|$. The case $n = 0$ is trivial since then $P = 0$. If $n = 1$ and P has type $\{\bar{x}\}$, then $S_y P = 0$ for $\bar{y} \neq \bar{x}$ so that $(T(f))_{\bar{y}} = 0$ when $\bar{y} \neq \bar{x}$. Thus

$$\beta(T(f)) = \beta_x \mathrm{Tr}_{R[x]}(S_x f) = T'(E_x S_x f) = T'(f) .$$

For the induction step choose a maximal $\bar{x} \in F$. Theorem 3.9 gives a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E_x S_x P \arrow[r] \arrow[d, "E_s S_x f"'] & P \arrow[r] \arrow[d, "f"'] & \mathrm{Cok}_x P \arrow[r] \arrow[d, "\mathrm{Cok}_x f"'] & 0 \\
0 \arrow[r] & E_x S_x P \arrow[r] & P \arrow[r] & \mathrm{Cok}_x P \arrow[r] & 0
\end{tikzcd}$$

where $E_x S_x P$ has type $\{\bar{x}\}$, $\mathrm{Cok}_x P$ has type $F\backslash\{\bar{x}\}$, and all modules are projective. By the induction hypothesis the claim $T'(h)=\beta(T(h))$ holds both for $h=E_x S_x f$ and $h=\mathrm{Cok}_x f$. Hence it holds for f by additivity.

EXAMPLE 4.12. Let G be a compact Lie group. The automorphism group of G/H in the discrete orbit category $\mathrm{Or}_d\, G$ is $\pi_0(WH)^{op}$. The universal Lefschetz invariant for $Z\,\mathrm{Or}_d\, G$-chain complexes takes values in the universal Lefschetz group

$$UL(G)=\bigoplus_{(H)} Cl(Z\pi_0 WH)\,.$$

If C is the cellular chain complex of a finite G-complex X and $f_*\colon C\to C$ is induced by a G-map $f\colon X\to X$, then the universal Lefschetz invariant $L(f_*)\in UL(G)$ agrees with the universal Lefschetz class $UL(f)$ of 1.6 (cf. Ex. 3.8).

5. The equivariant Lefschetz class

This section contains a proof of Theorem B and Corollaries C and D.

Let G be a compact Lie group. Homology will mean singular homology with coefficients in a fixed principal ideal domain R. If X is a G-space and $f\colon X\to X$ is a G-map, we define class functions $\bar{L}^H_R(f)$ and $L^H_R(f)$ as in (2.6) and (2.13) using homology with R-coefficients, provided the homology is finitely generated over R. Denote the universal Lefschetz invariant group for $R\,\mathrm{Or}_d\, G$-modules

$$UL(G,R)=UL(G)\underset{Z}{\otimes}R=\bigoplus_{(H)} Cl(R\pi_0 WH)\,. \tag{5.1}$$

The homomorphisms $\bar{L}^H$, $L^H\colon UL(G)\to Cl(Z\pi_0 WH)$ extend linearly to homomorphisms $UL(G,R)\to Cl(R\pi_0 WH)$ which we again denote by $\bar{L}^H$ and L^H. They are characterized by $\bar{L}^H[w]=\bar{L}^H_R(r(w))$ and $L^H[w]=L^H_R(r(w))$ for each generator $[w]$ given by $w\in WH$.

A G-complex Y is called an *R-homology approximation* to the G-space X if there exists a G-map $Y\to X$ which induces an isomorphism

$$H_*(Y^H/WH_0;R)\xrightarrow{\ \sim\ } H_*(X^H/WH_0;R)$$

for each $H\le G$.

Theorem 5.2. *Let X be a G-space which admits a finite-dimensional R-homology approximation of finite orbit type. Assume that $H_*(X^H/WH_0;R)$ is finitely generated over R for each $H\le G$. Then every G-map $f\colon X\to X$ has an equivariant Lefschetz class $[f]_R\in UL(G,R)$ such that $\bar{L}^H_R(f)=\bar{L}^H([f]_R)$ for each $H\le G$.*

Proof. Let $Y\to X$ be an R-homology approximation by a finite-dimensional

G-complex Y of finite orbit type. It induces a weak equivalence $C^s(Y) \to C^s(X)$ between the singular $R\,\mathrm{Or}_d\, G$-chain complexes. For any G-complex Y there is a weak equivalence $C^c(Y) \to C^s(Y)$ between the cellular and singular $R\,\mathrm{Or}_d\, G$-chain complexes by Proposition 3.5. The finiteness assumptions made on X and Y guarantee that $C^c(Y)$ is homotopy equivalent to a finite projective complex P by Corollary 3.12. Composing these equivalences we get a finite projective approximation $h\colon P \to C^s(X)$. Using Proposition 4.6 we can construct a universal Lefschetz invariant $UL(f_*) \in UL(G, R)$ for each chain map $f_*\colon C^s(X) \to C^s(X)$.

A G-map $f\colon X \to X$ induces a chain map $f_*\colon C^s(X) \to C^s(X)$, and we define the equivariant Lefschetz class of f to be $[f]_R = UL(f_*) \in UL(G, R)$. The class functions $\bar{L}^H_R(f)$ can be defined in terms of the $R\,\mathrm{Or}_d\, G$-complex $C = C^s(X)$ by

$$\bar{L}^H_R(f)(w) = L_R(H_*C(G/H),\ l(w^{-1}) \circ f_*(G/H)),\quad w \in \pi_0 WH$$

where L_R is the ordinary Lefschetz number. The right hand side defines a Lefschetz invariant $\bar{L}^H_R(f_*) \in Cl(\pi_0 WH)$ for every chain map $f_*\colon C \to C$ in $HC(P) - R\,\mathrm{Or}_d\, G$. Hence there is a unique homomorphism $\phi^H\colon UL(G, R) \to Cl(\pi_0 WH)$ such that $\bar{L}^H_R(f_*) = \phi^H(UL(f_*))$. We claim that $\phi^H = \bar{L}^H$. It is enough to check this on the generators $[r(w)]$, $w\colon G/H \to G/H$, but then the claim is simply the definition of $\bar{L}^H$.

The theorem holds in particular if X itself is a finite-dimensional G-complex of finite orbit type. If G is finite, we have proved Theorem B since $WH_0 = 1$ and $\bar{L}^H = L^H$ for each $H \le G$. For a general compact Lie group, we have to compare the assumptions on $H_*(X^H/WH_0)$ and $H_*(X^H)$ and to strengthen the conclusion from $\bar{L}^H$ to L^H. We start with a result which might have independent interest.

Proposition 5.3. *Let G be a compact Lie group and let X be a finite-dimensional G-complex of finite orbit type. Let R be a principal ideal domain. If $H_*(X; R)$ is finitely generated over R, then $H_*(X/G; R)$ is finitely generated over R.*

Proof. If G is a finite group or a torus and $R = Z$ or Z_p this is a well-known consequence of P.A. Smith theory and the existence of transfer for finite group actions, see [2, Ch. III]. It is not hard to extend the proof to arbitrary coefficients R. We are going to reduce the theorem to these special cases by using Oliver's transfer for compact Lie groups [16].

First, since $X/G = (X/G_0)/(G/G_0)$ we may assume that G is connected. Let N be a normalizer of a maximal torus T in G. The transfer

$$\mathrm{tr}_*\colon H_*(X/G) \to H_*(X/N)$$

has the property that its composition with the projection $p_*\colon H_*(X/N) \to H_*(X/G)$ is multiplication by $\chi(G/N)$. Since $\chi(G/N) = 1$, it suffices to prove the claim for the group N. Notice that the N-space X is homotopy-equivalent to a finite-dimensional N-complex Y, and that the orbit structure remains finite [9, Th. A].

The group N is an extension of T by the finite Weyl group W. Hence $X/N=(X/T)/W$ and we are reduced to the classical cases.

Theorem 5.4. *Let G be a compact Lie group and let X be a finite dimensional G-complex of finite orbit type. Then the following conditions are equivalent*

(a) $H_*(X^H; R)$ *is finitely generated over* R *for each* $H \leq G$

(b) $H_*(X^H; R)$ *is finitely generated over* R *for each* $H \in \operatorname{Iso}(X)$

(c) $H_*(X^H/WH_0; R)$ *is finitely generated over* R *for each* $H \leq G$

(d) $H_*(X^H/WH_0; R)$ *is finitely generated over* R *for each* $H \in \operatorname{Iso}(X)$

for any principal ideal domain R.

Proof. The implication (a)$\Rightarrow$(b) is trivial and (b)$\Rightarrow$(d) follows from Proposition 5.3. Similarly (a)$\Rightarrow$(c)$\Rightarrow$(d). Hence it suffices to show that (a) follows from (d). We shall use the Atiyah-Hirzebruch spectral sequence for the equivariant homology theory $X \to H_*(X^H; R)$ with a fixed $H \leq G$. If M_*^H is the coefficient system

$$M_*^H(G/K) = H_*((G/K)^H; R)$$

it takes the form

$$H_p^G(X; M_q^H) \Rightarrow H_{p+q}(X^H; R),$$

where H_*^G is the Bredon-Illman homology $H_*^G(X; M) =: H_*(C^c(X) \otimes_{\mathrm{Or}_d G} M)$. If (d) holds then $C^c(X)$ is homotopy-equivalent as an $R\,\mathrm{Or}_d\,G$-complex to a finite projective complex P by Corollary 3.12. Then the E^2-term $H_*^G(X; M_*) = H_*^G(P \otimes M_*)$ is finitely generated over R since M_* is finitely generated. It follows that the E^∞-term is also finitely generated.

A finite-dimensional G-complex X of finite orbit type which satisfies the conditions of Theorem 5.4 is called *R-homology finite*.

Theorem B. *Let G be a compact Lie group, X be a finite-dimensional G-complex of finite orbit type and R be a principal ideal domain. If X is R-homology finite then each G-map $f\colon X \to X$ has an equivariant Lefschetz class $[f]_R \in UL(G, R)$ such that*

$$\bar{L}_R^H(f) = \bar{L}^H([f]_R), \quad L_R^H(f) = L^H([f]_R)$$

for each $H \leq G$.

Proof. It follows from Theorem 5.4 that $\bar{L}_R^H(f)$ and $L_R^H(f)$ are defined for each $H \leq G$. The existence of $[f]_R$ and the equation for $\bar{L}_R^H$ was shown in

Theorem 5.2. We compute $L_R^H(f)$ from cohomology using the Atiyah-Hirzebruch spectral sequence

$$H_G^p(X; M_H^q) \Rightarrow H^{p+q}(X^H; R), \quad M_H^*(G/K) = H^*(G/K)^H; R).$$

The E_2-term $H_G^p(X; M_H^q) =: H^p(\mathrm{Hom}_{\mathrm{Or}_d G}(C^c(X), M_H^q))$ is finitely generated since $C^c(X)$ is homotopy equivalent to a finite projective complex. By the Hopf trace formula $L_R^H(f)$ can be determined from the E_2-term. Now we define for any $R\,\mathrm{Or}_d\,G$-chain complex C and a chain map $f_*: C \to C$ in $HC(P)$ a Lefschetz invariant

$$L_R^H(f_*) = L_R(H^*(\mathrm{Hom}_{\mathrm{Or}_d G}(C, M_H^*)), f^*).$$

It factors through the universal Lefschetz invariant $[f]_R \in UL(G, R)$. To show that $L_R^H(f_*) = L^H([f]_R)$, it suffices to check the generators $C = C^c(G/K)$, $f = r(w)$, $w \in WK$. But then $\mathrm{Hom}_{\mathrm{Or}_d G}(C, M_H^*) = M_H^*(G/K) = H^*((G/K)^H; R)$ and $L^H([w]) = L_R^H(w)$ by the definition of L^H.

Corollary C. *With the assumptions of Theorem B the Lefschetz numbers $L_R(f^H)$ satisfy the Burnside ring congruences* mod R: *let $H \lhd L$ be closed subgroups of G.*

i) *If L/H is finite, then*

$$L_R(f^H) \equiv -\sum \phi(|K/H|) L_R(f^K) \bmod |L/H| R$$

where ϕ denotes the Euler function and the summation is over non-trivial cyclic subgroups K/H of L/H.

ii) *If L/H is a torus, then $L_R(f^H) = L_R(f^L)$.*

Proof. Since all occuring Lefschetz numbers can be computed from $[f] \in UL(G, R)$, this is a question about the relations between the homomorphisms

$$\phi_H = L^H(e): \ UL(G, R) \to R.$$

They are R-linear extensions of the corresponding integral homomorphisms ϕ_H: $UL(G) \to Z$, which factor through the projection J: $UL(G) \to A(G)$ (see 2.21). The relations between ϕ_H: $A(G) \to Z$ are determined in tom Dieck [6, Ch. IV 5].

Given a compact Lie group G, tom Dieck has shown that there exists a finite upper bound for the numbers $|\pi_0 WH|$ over all closed subgroups H of G [6, Th. IV 6.9]. Hence they have a least common multiple

(5.5)
$$o(G) = \mathrm{l.c.m.}\{|\pi_0 WH| \mid H \leq G\}.$$

If G is finite the number $o(G)$ is simply the order of G.

Corollary D. *With the assumptions of Theorem B*

$$L_R(g) = \chi_R(X^g)$$

for each $g \in G$, *provided char* $R=0$ *and no prime factor of* $o(G)$ *is invertible in* R.

Proof. Let C be the closed subgroup generated by g. Write $[X]_R=[\mathrm{id}_X]_R$ in $UL(G, R)$. We claim that the relation $L^c([X]_R)(g)=L^c([X]_R)(e)$ holds for each X. By Proposition 2.18 it is valid on the image of $I\colon U(G)\to UL(G, R)$ so it is enough to show that $[X]_R$ belongs to this image. The H-component of $[X]_R$ is the rank $r(P)=L(\mathrm{id}_p)\in Cl(R\pi_0 WH)$ of some finite projective $R\pi_0 WH$-complex P. Now $\pi_0 WH$ is a finite group and no prime factor of its order is invertible in R. A theorem of Swan implies that the rank of every finitely generated projective $R\pi_0 WH$-module is an integral multiple of $[e]$, see Bass [1, Th. 4.1]. Hence the H-component of $[X]_R$ has the form $n[e]=I(n[G/H])$ for some integer n and their sum $[X]_R$ lies in $\mathrm{Im}\, I$.

REMARK 5.6. Let p and q be different primes. Conner and Floyd have constructed a simplicial action of the cyclic group Z_{pq} on R^n with $(R^n)^{Z_{pq}}=\emptyset$, see [2, Ch. I 8. B]. The other fixed point sets are Moore spaces

$$(R^n)^{Z_p} = M\left(Z\left[\frac{1}{lq}\right], 1\right), \quad (R^n)^{Z_q} = M\left(Z\left[\frac{1}{kp}\right], 1\right)$$

where k and l are such that $kp+lq=-1$. This shows that the conditions on X and R are necessary in Corollary D.

6. Homotopy representations

We shall study elementary homological properties of homotopy representations of compact Lie groups [6, Ch. II 10]. For these purposes the following weaker notion suffices. Let R be a principal ideal domain.

DEFINITION 6.1. An *R-homology representation* of a compact Lie group G is a finite-dimensional G-complex of finite orbit type such that for each $H\le G$ the fixed point set X^H is an R-homology sphere.

Let $n(H)$ denote the unique integer for which $H_*(X^H; R)\cong H_*(S^{n(H)-1}; R)$. If X^H is empty, then $n(H)=0$. The *dimension function* of X is the integral-valued function $\mathrm{Dim}(X)$ defined on the conjugacy classes of closed subgroups of G by

$$\mathrm{Dim}(X)(H) = n(H)\,. \tag{6.2}$$

Let CX denote the cone on X. For each $H\le G$ we have $H_{n(H)}(CX^H, X^H, R)\cong R$. The action of WH on this homology group defines a homomorphism

(6.3) $$e_{X,H}\colon WH \to \mathrm{Aut}(R) = R^{\times}$$

called the *orientation behaviour* of X at H. By homotopy invariance it actually factors through $\pi_0 WH$. If R is suitably restricted, the dimension function determines the orientation behavior. Recall the number $o(G)$ from (5.5).

Proposition 6.4. *Let X be an R-homology representation of a compact Lie group G. Assume that char$R=0$ and that no prime divisor of $o(G)$ is invertible in R. Then*

$$e_{X,H}(w) = (-1)^{\mathrm{Dim}(X)(H)-\mathrm{Dim}(X)(C)}, \qquad w \in WH,$$

where C is the inverse image in NH of the closed subgroup generated by w in WH.

Proof. We begin with the case $H=1$. The Lefschetz fixed point formula applied to the element $w \in G$ gives

$$1-(-1)^{\mathrm{Dim}(X)(1)} e_{X,1}(w) = L_R(w) = \chi_R(X^c) = 1-(-1)^{\mathrm{Dim}(X)(C)}$$

whence the claim for $H=1$. The general case reduces to this by considering the WH-space X^H, once we show that $o(WH)$ divides $o(G)$.

Let $\bar{K} \le WH$ have normalizer $\bar{L}$ in WH. Denote by K and L their inverse images in NH. Then $L = N_{NH}(K)$. The normalizer NK of K in G may be larger than L, but we claim that they have the same identity component. Indeed, if $g \in (NK)_0$ choose a path p from g to e in $(NK)_0$. Let $c_g\colon K \to K$ be conjugation by g. Then c_g is homotopic to identity via p, and hence it is an inner automorphism of K [6, Ex. I 5.18.1]. In particular, c_g preserves H and $g \in NH$. Since $L = NH \cap NK$ we have $g \in L_0$. The inclusion $L \to NK$ induces thus injective homomorphisms $\pi_0(L) \to \pi_0(NK)$ and $\pi_0(L/K) \to \pi_0(NK/K)$. As $\pi_0(\bar{L}/\bar{K}) \cong \pi_0(L/K)$, $o(WH)$ divides $o(G)$.

Proposition 6.4 holds in particular for homotopy representations, since they are Z-homology representations.

We denote the *Euler characteristic* $[X] \in L(G, R)$ of an R-homology representation X as the class of id_X in the Lefschetz ring. It is determined by the class functions $L^H[X]$, which are now

$$L^H[X](w) = 1-(-1)^{\mathrm{Dim}(H)} e_{X,H}(w), \qquad w \in WH.$$

If R is as in Proposition 6.4, the dimension function determines $[X]$. Let $f\colon X \to X$ be an arbitrary G-map. For its class $[f] \in L(G, R)$ we have

$$L^H[f](w) = 1-(-1)^{\mathrm{Dim}\,X(H)} \deg f^H e_{X,H}(w)$$

making $[f]$ into a mixture of the degree function $(\deg f^H)$ and the orientation behavior $(e_{X,H})$. The product $\{f\} = ([X]-1)([f]-1) \in L(G, R)$ has class functions

$$L^H\{f\}(w) = \deg f^H e_{X,H}(w)^2\,, \qquad w \in WH\,.$$

The projection of $\{f\}$ in $A(G, R)$ has characters $\phi_H\{f\} = \deg f^H$, $H \le G$. This gives

Proposition 6.5. *If $f: X \to X$ is a self-G-map of an R-homology representation X, its degrees* $\deg f^H$ *satisfy the Burnside ring congruences* mod R.

Proposition 6.5 can be used to derive degree relations for G-maps $f: X \to Y$ between different homotopy representations with the same dimension function, as we did for finite groups G in [11, Th. 2]. The key point is the existence of an auxiliary map $h: Y \to X$ which has degrees prime to $o(G)$. For compact Lie groups G the map h is constructed in [6, Th. II 10.20].

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