Algebraic K-theory of von Neumann algebras

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Abstract

To every von Neumann algebra one can associate a (multiplicative) determinant defined on the invertible elements of the algebra with range a subgroup of the abelian group of the invertible elements of the center of the von Neumann algebra. This determinant is a normalization of the usual determinant for finite von Neumann algebras of type I, for the type II_1-case it is the Fuglede-Kadison determinant, and for properly infinite von Neumann algebras the determinant is constant equal to 1. It is proved that every invertible element of determinant 1 is a product of a finite number of commutators. This extends a result of T. Fack and P. de la Harpe for II_1-factors. As a corollary it follows that the determinant induces an injection from the algebraic K_1-group of the von Neumann algebra into the abelian group of the invertible elements of the center. Its image is described. Another group, $K_1^w(A)$, which is generated by elements in matrix algebras over A that induce injective right multiplication maps is also computed. We use the Fuglede-Kadison determinant to detect elements in the Whitehead group $Wh(G)$.

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Introduction

The purpose of this paper is to compute the algebraic K_1-groups $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$ for a von Neumann algebra $\mathcal{A}$. We give their definitions in section 1. One motivation for their study comes from the construction of Reidemeister von Neumann torsion for a compact Riemannian manifold in Lück-Rothenberg [12] which takes values in these K_1-groups for the von Neumann algebra of the fundamental group. Recall that the topological K_1-group of a von Neumann algebra is trivial.

In section 2 we treat von Neumann algebras of type I_f. Since they can be written as product of matrix algebras over abelian von Neumann algebras, the ordinary determinant for commutative rings extends to a normalized determinant $\det_{\text{norm}} : M_k(\mathcal{A}) \longrightarrow Z(\mathcal{A})$ into the center $Z(\mathcal{A})$. We can write $Z(\mathcal{A})$ as the algebra $L^\infty(X;\nu)$ of essentially bounded measurable functions from X to $\mathbb{C} \cup \{\infty\}$ for some compact space X with positive finite measure ν. Let $L^\infty(X;\nu)_{\text{inv}}$ be the abelian group of invertible elements and $\text{Inv}(X;\nu)$ be the (multiplicative) abelian group of measurable functions from X to $\mathbb{C}\cup\{\infty\}$ whose preimage of both 0 and ∞ is a zero-set. We prove in theorem 2.1 that the normalized determinant induces isomorphisms:

$$\det_{\text{norm}} : K_1(\mathcal{A}) \longrightarrow L^\infty(X;\nu)_{\text{inv}}$$
$$\det_{\text{norm}} : K_1^w(\mathcal{A}) \longrightarrow \text{Inv}(X;\nu)$$

Section 3 is devoted to the type II_1-case. The Fuglede-Kadison determinant of an invertible element $A \in M_n(\mathcal{A})$ is defined by

$$\det_{\text{FK}}(A) = \exp\left(\frac{1}{2}\cdot \text{tr}(\log(A^*A))\right) \in Z(\mathcal{A})^+_{\text{inv}}$$

where $Z(\mathcal{A})^+_{\text{inv}}$ denotes the group of positive invertible elements in $Z(\mathcal{A})$. We show in theorem 3.3 that the Fuglede-Kadison determinant induces an isomorphism:

$$\det_{\text{FK}} : K_1(\mathcal{A}) \longrightarrow Z(\mathcal{A})^+_{\text{inv}}$$

and that $K_1^w(\mathcal{A})$ is trivial. The main technical ingredients are proposition 3.1, which is a variation of a technique of Broise [2] and is a kind of "Eilenberg swindle", and proposition 3.9. This result about $K_1(\mathcal{A})$ was proved by Fack and de la Harpe [4] for a II_1-factor. We avoid disintegration theory in our extension of the proof to general von Neumann algebras of type II_1.

We prove in theorem 4.2 in section 4 that $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$ are trivial if $\mathcal{A}$ is properly infinite.

If an invertible matrix represents zero in $K_1(\mathcal{A})$, we obtain a bound which depends only on the type of the von Neumann algebra on the number of commutators needed to write the matrix as a product of commutators.

In section 5 we detect elements in the Whitehead group $Wh(G)$ by the Fuglede-Kadison determinant. Namely, we show in theorem 5.1 for a normal finite subgroup H of a countable discrete group G that the map $Wh(H)^G \longrightarrow Wh(G)$ induced by induction is rationally injective where the G-action on $Wh(H)$ comes from the conjugation action of G on H. This computation is compatible with the much more general isomorphism conjecture on algebraic K-groups by Farrell and Jones [5].

Our interest in $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$ arises from the construction of Reidemeister von Neumann torsion in Lück-Rothenberg [12]. It is a generalization of classical Reidemeister torsion from finite to infinite groups. From the nature of its definition Reidemeister von Neumann torsion does not take values in $K_1(\mathcal{A})$ but in $K_1^w(\mathcal{A})$. Namely, the combinatorial Laplace operator acting on the complement of its kernel is a weak isomorphism, but not necessarily an isomorphism. In particular the relevant K-theory cannot be described by invertible matrices and some of the known techniques involving commutators have to be modified in order to apply them to other K-groups like $K_1^w(\mathcal{A})$.

The analytic counterpart of Reidemeister von Neumann torsion is the analytic L^2-torsion defined by Lott [9] for a closed manifold M. Its definition requires the assumption that the Novikov-Shubin invariants of M are positive. Conjecture 9.1 in Lott-Lück [10] says that this assumption always holds. At the first glance the combinatorial definition does not need the assumption. However, if the von Neumann algebra is of type II_1, then we show that $K_1^w(\mathcal{A})$ is trivial. This indicates that one has to make also in the combinatorial case an assumption on the Novikov-Shubin invariants of the operators coming from the cellular chain complex in order to guarantuee that the K_1-group they take values in are non-trivial. Namely, for weak isomorphisms with positive Novikov-Shubin invariants one can define a generalized Fuglede-Kadison determinant and obtains a non-trivial Reidemeister von Neumann torsion with values in the real numbers, called in this context combinatorial L^2-torsion. This is carried out in Lück [11] and it is conjectured that combinatorial and analytic L^2-torsion agree (see [11, conjecture 3.1] [10, conjecture 9.7]). We mention that the von Neumann algebra of a finitely generated group π is of type I_f if π is virtually a finitely generated free abelian group, (i.e. if π contains a normal finitely generated free abelian group of finite index) and is of type II_f otherwise and that the conjectures mentioned above are known to be true for virtually finitely generated free abelian groups.

We refer to the survey article of Rosenberg [15] for information about connections between topology and algebraic K-theory of operator algebras.

1. Definition of K-groups of a von Neumann algebra

In this section we define the various algebraic K-groups we want to study. Throughout this section let R be an associative ring with unit.

Definition 1.1 *Let $K_1(R)$ and $K_1^w(R)$ be the abelian groups generated by conjugation classes of bijective, respectively, injective R-endomorphisms of finitely generated free R-modules satisfying the following relations*

- $[f]+[h]=[g]$, *if there is an exact sequence of bijective, respectively, injective R-endomorphisms* $0 \longrightarrow (U,f) \xrightarrow{i} (V,g) \xrightarrow{p} (W,h) \longrightarrow 0$;
- $[gf]=[f]+[g]$, *if f and g are bijective resp. injective R-endomorphisms of the same finitely generated free R-module;*
- $[id : V \longrightarrow V]=0$, *if V is a finitely generated free R-module.* ■

The group $K_1(R)$ can be identified with the abelianization $GL(R)_{\text{ab}}$ of the general linear group $GL(R)=\lim_{n\to\infty} GL(n,R)$. The identification is given by interpreting an invertible (n,n)-matrix as an automorphism of R^n and vice versa. The generators of $K_1^w(R)$ can be identified with elements of $M(n,R)$ for which the corresponding endomorphism of R^n given by right multiplication is injective. The description in Definition 1.1 with generators and relations is more natural, but for the computations in this paper the second description will be used.

Remark 1.2 The group $K_1(R)$ can be identified with the abelianization $GL(R)_{\text{ab}}$ of the general linear group $GL(R)=\lim_{n\to\infty} GL(n,R)$. The identification is given by interpreting an invertible (n,n)-matrix as an automorphism of R^n and vice versa. A direct description of $K_1^w(R)$ by groups of matrices is not available since injective R-endomorphisms are not necessarily bijective. If R is a finite von Neumann algebra $\mathcal{A}$, generators of $K_1^w(\mathcal{A})$ have the following description in terms of Hilbert $\mathcal{A}$-modules.

Assume that $\mathcal{A}$ is a finite von Neumann algebra with a faithful normal normalized trace tr. Let $L^2(\mathcal{A},tr)$ be the corresponding Hilbert space which is the completion of $\mathcal{A}$ with respect to the inner product $<a,b>=tr(b^*a)$. Then $\oplus_{i=1}^n L^2(\mathcal{A},tr)$ is an $\mathcal{A}-M(n,\mathcal{A})$ bimodule, and every bounded $\mathcal{A}$-endomorphism of $\oplus_{i=1}^n L^2(\mathcal{A},tr)$ is given by right multiplication by an element of $M(n,\mathcal{A})$. An $\mathcal{A}$-endomorphism of $\oplus_{i=1}^n L^2(\mathcal{A},tr)$ is said to be a weak isomorphism if its kernel is zero and its image is dense. Since $\mathcal{A}$ is finite it follows that an endomorphism of $\oplus_{i=1}^n L^2(\mathcal{A},tr)$ is a weak isomorphism if and only if the corresponding element in $M(n,\mathcal{A})$ induces an injective endomorphism on A^n. Hence the generators of $K_1^w(\mathcal{A})$ are weak isomorphisms of $\oplus_{i=1}^n L^2(\mathcal{A},tr)$ for $n\in\mathbb{N}$. It is important for the construction of Reidemeister von Neumann torsion to allow weak isomorphisms and not only isomorphisms (see Lück-Rothenberg [12]). ■

We have the following type decomposition theorem for von Neumann algebras. See [6, 6.5.2].

Theorem 1.3 *Given a von Neumann algebra $\mathcal{A}$, there is a natural unique decomposition:*

$$\mathcal{A}=\mathcal{A}_{I_f}\times\mathcal{A}_{I_\infty}\times\mathcal{A}_{II_1}\times\mathcal{A}_{II_\infty}\times\mathcal{A}_{III}$$

into von Neumann algebras of type I_f, I_∞, II_1, II_∞ and III. In particular there are natural isomorphisms induced by the projections:

$$K_1(\mathcal{A}) = K_1(\mathcal{A}_{I_f}) \times K_1(\mathcal{A}_{I_\infty}) \times K_1(\mathcal{A}_{II_1}) \times K_1(\mathcal{A}_{II_\infty}) \times K_1(\mathcal{A}_{III})$$

and similiarly for $K_1^w(\mathcal{A})$. ■

This theorem reduces the computation of the various K_1-groups of a Neumann algebra to the computation in the case where $\mathcal{A}$ is of type I_f, I_∞, II_1, II_∞ and III. Notice that a von Neumann algebra is properly infinite if and only if its natural decomposition does not contain pieces of type I_f and II_1.

2. The type I_f-case

This section contains the computations of the K_1-groups of a von Neumann algebra of type I_f. We begin by recalling the structure of these von Neumann algebras.

Let $\mathcal{B}_n$ be a von Neumann algebra on a Hilbert space H_n for $n \in \mathbb{N}$. Let $H = \oplus_{i=1}^\infty H_n$ be the Hilbert space direct sum. The *product von Neumann algebra* $\prod_{n=1}^\infty \mathcal{B}_n$ is the von Neumann algebra on H, which elements are sequences $(B_n \in \mathcal{B}_n \mid n \in \mathbb{N})$, such that there exists a number K (depending on the sequence, but not on n), with $\| B_n \| \leq K$ for all $n \in \mathbb{N}$. The embedding of $\prod_{n=1}^\infty \mathcal{B}_n$ in $B(H)$ sends such a sequence to the sum of the operators $B_n : H_n \longrightarrow H_n$.

Every von Neumann algebra $\mathcal{A}$ of type I_f is of the form

$$\prod_{n=1}^\infty \mathcal{A}_n,$$

where $\mathcal{A}_n$ is a von Neumann algebra of type I_n. Furthermore, $\mathcal{A}_n$ is isomorphic to $M_n(Z_n)$, where Z_n is the center of $\mathcal{A}_n$. The center $Z(\mathcal{A})$ of $\mathcal{A}$ is

$$\prod_{n=1}^\infty Z_n.$$

Let $\eta_n : Z_n \to Z_n$ be the map which sends ua into $ua^{1/n}$ when $u, a \in Z_n, u$ is unitary and a is positive. Note that η_n is multiplicative. Let

$$\det : M_k(\mathcal{A}_n) = M_k(M_n(Z_n)) = M_{kn}(Z_n) \to Z_n$$

be the usual determinant, and set

$$\det_{\text{norm}} = \eta_n \circ \det : M_{kn}(\mathcal{A}_n) \to Z_n$$

Then $\det_{\text{norm}}$ is multiplicative and

$$\begin{aligned} \det_{\text{norm}}(U) &= U^n, \quad U \in (Z_n)_{inv}, U^*U = I \\ \det_{\text{norm}}(A) &= A, \quad A \in (Z_n)_{inv}, A^* = A \\ \| \det_{\text{norm}}(A) \| &\leq \| A \|^k, \quad A \in M_k(\mathcal{A}_n). \end{aligned}$$

Define

$$\det_{\text{norm}} : M_k(\mathcal{A}) \to Z(\mathcal{A}), \quad k \in \mathbb{N},$$

by the product of the determinants $\det_{\text{norm}}$ for the $\mathcal{A}_n$-s.

Let $Z(\mathcal{A})_{\text{inv}}$ be the multiplicative group of invertible elements in $Z(\mathcal{A})$. Denote by $Z(\mathcal{A})^w$ the Grothendieck group of the abelian semigroup of elements $a \in Z(\mathcal{A})$, for which multiplication with a induces an injection $m_a : Z(\mathcal{A}) \longrightarrow Z(\mathcal{A}\ )$. If we identify $Z(\mathcal{A})$ with $L^\infty(X;\nu)$, for some measure space $(X;\nu)$, we can identify $Z(\mathcal{A})^w$ with $\text{Inv}(X;\nu)$, i.e. the space of measurable functions from X to $\mathbb{C} \cup \{\infty\}$, for which the preimages of 0 and ∞ are zero sets. In particular, the canonical map

$$Z(\mathcal{A})_{\text{inv}} \longrightarrow Z(\mathcal{A})^w$$

is injective. The next theorem was proved for $\mathcal{A}$ abelian in Lück-Rothenberg [12, section 2].

Theorem 2.1 *The normalized determinant induces isomorphisms for a von Neumann algebra $\mathcal{A}$ of type I_f.*

$$\det_{norm} : K_1(\mathcal{A}) \longrightarrow Z(\mathcal{A})_{inv};$$
$$\det_{norm} : K_1^w(\mathcal{A}) \longrightarrow Z(\mathcal{A})^w. \qquad \blacksquare$$

Notice that the proof of theorem 2.1 is straightforward for K_1 if $\mathcal{A}$ is a finite product $\prod_{i=1}^N \mathcal{A}_n$. In this case the product of the von Neumann algebras is an ordinary product of rings and we get from Morita equivalence and Milnor [13, section 7]

$$K_1(\mathcal{A}) = K_1(\prod_{n=1}^N \mathcal{A}_n) = \prod_{i=1}^N K_1(\mathcal{A}_n) = \prod_{i=1}^N K_1(M_n(Z_n)) = \prod_{i=1}^N K_1(Z_n) = \prod_{i=1}^N (Z_n)_{\text{inv}} = Z(\mathcal{A})_{\text{inv}}.$$

In order to handle the general case and $K_1^w(\mathcal{A})$, we outline a different proof. Namely, theorem 2.1 follows directly from proposition 2.4. The proof of proposition 2.4 uses the following two lemmas whose fairly straightforward proofs are omitted.

Lemma 2.2 *Let $(X;\nu)$ be a measure space. Let $T \in M_n(L^\infty(X;\nu))$ be normal, i.e., T and T^* commute. Then there is a unitary $U \in M_n(L^\infty(X;\nu))$ such that U^*TU is diagonal.* $\blacksquare$

Lemma 2.3 *Let $f_i \in L^\infty(X;\nu)$, $1 \le i \le n$, be positive functions. Suppose that their product equals 1. Then there is a measurable map $\sigma : X \longrightarrow \Sigma_n$ into the discrete group of permutations of $\{1, 2, \ldots, n\}$, such that for all $x \in X$ and $1 \le k \le n$,*

$$\min\{f_i(x) \mid 1 \le i \le n\} \le \prod_{i=1}^{k} f_{\sigma(i)(x)} \le \max\{f_i(x) \mid 1 \le i \le n\}. \qquad \blacksquare$$

Proposition 2.4 *Let $T \in M_n(L^\infty(X;\nu))$ be unitary or positive. Assume that the normalized determinant of T equals 1. Then there exist a unitary element $U \in M_n(L^\infty(X;\nu))$ and a unitary, respectively, an invertible, positive element $A \in M_n(L^\infty(X;\nu))$, satisfying:*

$$T = AUA^{-1}U^*;$$
$$\|A\| \le \|T\|;$$
$$\| A^{-1} \| \le \| T^{-1} \| .$$

Proof : By the polar decomposition theorem and lemma 2.2, we may assume that T is either positive or unitary and that T is diagonal. Let t_1, t_2, $\ldots$, t_n be the diagonal entries. Denote by A the diagonal matrix having as (i,i)-th entry the product $\prod_{j=1}^{i} t_j$.

We first treat the case when T is unitary. Let U be the matrix representing the cyclic permutation in Σ_n, sending n to 1 and i to $i+1$ for $1 \le i < n$. Then A and U are unitaries satisfying $T = AUA^{-1}U^{-1}$.

Suppose that T is positive. By lemma 2.3 there is a measurable function $\sigma : X \longrightarrow \Sigma_n$ such that for all $x \in X$ and $1 \le k \le n$, the following inequality

$$\min\{t_i \mid 1 \le i \le n\} \le \prod_{i=1}^{k} t_{\sigma(i)} \le \max\{t_i \mid 1 \le i \le n\}$$

holds. Let P be the permutation matrix associated to σ. Then $P^{-1}TP$ is again a diagonal matrix which is obtained from the diagonal matrix T by permuting the diagonal entries according to σ. Hence we can assume without loss of generality that

$$\min\{t_i \mid 1 \le i \le n\} \le \prod_{i=1}^{k} t_i \le \max\{t_i \mid 1 \le i \le n\}.$$

Since t_i is essentially bounded from above for each $1 \le i \le n$ and $\prod_{i=1}^{n} t_i = 1$ holds by assumption, $1/t_i$ is essentially bounded from above. This shows that T is invertible. Now we can proceed as in the unitary case because the inequality above gives the desired bounds on $\| A \|$ and $\| A^{-1} \|$. $\blacksquare$

Proposition 2.4 shows for $T \in M_k(\mathcal{A})$, satisfying $\det_{\text{norm}}(T) = 1$, that T can be written as a product of two commutators of invertible elements in $M_k(\mathcal{A})$.

3. The type II_1-case

In this section we compute $K_1(\mathcal{A})$ for a von Neumann algebra of type II_1 using the Fuglede-Kadison determinant. This has essentially been proved by Fack and de la Harpe [4] in the case when $\mathcal{A}$ is a II_1-factor. The present proof generalizes Fack and de la Harpe's proof to the non-factor case. For the convenience of the reader, we recall the technique introduced by Broise [2] to express elements as products of commutators.

Proposition 3.1 *Let $\mathcal{A}$ be a von Neumann algebra on H. Let $\{P_n \mid n \in \mathbb{N}\}$ be a sequence of mutually orthogonal projections P_n in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} P_n = 1$. Let $\{T_n \mid n \in \mathbb{N}\}$, $\{A_n \mid n \in \mathbb{N}\}$ and $\{B_n \mid n \in \mathbb{N}\}$ be sequences of elements in $\mathcal{A} \subset B(H)$, satisfying for all $n \in \mathbb{N}$:*

$$T_n = P_n T_n P_n + (1 - P_n);$$
$$A_n = (P_n + P_{n+1}) A_n (P_n + P_{n+1}) + (1 - P_n - P_{n+1});$$
$$B_n = (P_n + P_{n+1}) B_n (P_n + P_{n+1}) + (1 - P_n - P_{n+1});$$
$$B_n A_n T_n = A_n B_n T_{n+1}.$$

Suppose there is a number K such that

$$\| A_n \|, \| B_n \|, \| T_n \| \leq K$$

for all $n \in \mathbb{N}$. Then:

1. *The following products are strongly convergent:*

$$A_{odd} = A_1 A_3 A_5 \ldots \quad ;$$
$$A_{ev} = A_2 A_4 A_6 \ldots \quad ;$$
$$B_{odd} = B_1 B_3 B_5 \ldots \quad ;$$
$$B_{ev} = B_2 B_4 B_6 \ldots \quad ;$$
$$T_{ev} = T_2 T_4 T_6 \ldots \quad ;$$
$$T_{ext} = T_3 T_5 T_7 \ldots \quad .$$

2. *If T_n, A_n and B_n are invertible and*

$$\| A_n^{-1} \|, \| B_n^{-1} \|, \| T_n^{-1} \| \leq K$$

holds for $n \in \mathbb{N}$, then the operators A_{odd}, A_{ev}, B_{odd}, B_{ev}, T_{ev} and T_{ext} are invertible.

3. *If T_n, A_n and B_n induce injections $\mathcal{A} \longrightarrow \mathcal{A}$ for $n \in \mathbb{N}$, then the same is true for the operators A_{odd}, A_{ev}, B_{odd}, B_{ev}, T_{ev} and T_{ext}.*

4. $B_{odd}A_{odd}T_1T_{ext} = A_{odd}B_{odd}T_{ev}$;

$B_{ev}A_{ev}T_{ev} = A_{ev}B_{ev}T_{ext}$.

Proof : 1.) and 2.). Each of the sequences clearly converges on the dense subspace of H spanned by $P_n(H), n \in \mathbb{N}$. The boundedness of the sequences $\|A_n\|, \|B_n\|$ and $\|T_n\|$ now implies that the partial products are also bounded. Hence the six products converge on every vector in H. The inverses of these products are the products of the inverses in reversed order which are strongly convergent if $\|A_n^{-1}\|$, $\|B_n^{-1}\|$ and $\|T_n^{-1}\|$ are bounded.

3.) We give the proof for A_{ev} and the other cases follow in a similar way. Let $x \in \ker A_{\text{ev}}$. Note that A_{2k} and A_{ev} commute with $P_{2n} + P_{2n+1}$ for all $n, k \in \mathbf{N}$. It follows that

$$\begin{aligned} 0 &= (P_{2n} + P_{2n+1})A_{\text{ev}}x = A_{\text{ev}}(P_{2n} + P_{2n+1})x \\ &= A_{2n}(P_{2n} + P_{2n+1})x. \end{aligned}$$

Because A_{2n} is injective, we conclude that $(P_{2n}+P_{2n+1})x = 0$, and since n is arbitrary, $x = 0$ follows.

4.) is easily verified on elements in $P_n(H)$ for all $n \in \mathbb{N}$. As these elements span a dense subspace of H, the claim follows. ■

We will use this proposition to show that the class $[T_1]$ of T_1 in some K_1-group vanishes. This would follow if each of the operators A_{odd}, A_{ev}, B_{odd}, B_{ev}, T_{ev}, T_{ext} and T_1 define a class in the relevant K_1-group since $[AB] = [A] + [B]$ holds. Notice that we avoid to speak of commutators in order to make sure that this lemma also applies in the case where we are dealing with weak isomorphisms. If all maps involved are isomorphisms, the conclusion of the lemma above would be that T_1 is a product of two commutators.

Definition 3.2 *Let $\mathcal{A}$ be a von Neumann algebra of type II_1 and let tr denote the center valued trace on $\mathcal{A}$. Extend tr the standard way to $M_n(\mathcal{A})$ so that tr takes values in $Z(\mathcal{A})$ for all $n \in \mathbb{N}$. The Fuglede-Kadison determinant is in [6] defined to be*

$$\det{}_{FK}(A) = \exp(\tfrac{1}{2} tr(\log(A^*A))) \in Z(\mathcal{A})^+_{inv},$$

for $A \in GL(n, \mathcal{A})$. ■

It is proved in [6] (see also [8, I.6.10]) that

$$\det{}_{\text{FK}}(AB) = \det{}_{\text{FK}}(A)\det{}_{\text{FK}}(B), \quad A, B \in GL(n, \mathcal{A})$$
$$\det{}_{\text{FK}}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det{}_{\text{FK}}(A)\det{}_{\text{FK}}(B), \quad A \in GL(n, \mathcal{A}), B \in GL(m, \mathcal{A})$$
$$\det{}_{\text{FK}}(A) = A, \quad A \in Z(\mathcal{A})^+_{\text{inv}}.$$

In particular, the determinant of every element in the derived group $[GL(\mathcal{A}), GL(\mathcal{A})]$ is 1, and so we get a homomorphism

$$\det{}_{\text{FK}} : K_1(\mathcal{A}) \to Z(\mathcal{A})^+_{\text{inv}},$$

which is surjective by the last identity above.The following theorem is the main result of this section.

Theorem 3.3 *Let $\mathcal{A}$ be a von Neumann algebra of type II_1.*

1. *The Fuglede-Kadison determinant defines an isomorphism:*

$$\det_{FK} : K_1(\mathcal{A}) \longrightarrow Z(\mathcal{A})^+_{inv};$$

2. *The weak K-group $K_1^w(\mathcal{A})$ vanishes.* ■

The rest of the section is devoted to proving that $\det_{\mathrm{FK}}$ is injective and that $K_1^w(\mathcal{A})$ is trivial. We prove the former by showing that if $\det_{\mathrm{FK}}(A) = 1$, then A is a product of at most nine commutators. Notice that we cannot define the homomorphism above for $K_1^w(\mathcal{A})$, since for a weak isomorphism A the logarithm $\log(A^*A)$ is not necessarily a bounded operator.

Recall that our Hilbert spaces are assumed to be separable. In particular all von Neumann algebras we consider are countably decomposable. Hence we get from Kadison-Ringrose [8], exercise 6.9.20 on page 447, and 6.9.27 on page 448:

Lemma 3.4 *Let $\mathcal{B}$ be a von Neumann algebra with no central portion of type I_n with n odd. Let $\mathcal{A} \subset \mathcal{B}$ be a maximal abelian subalgebra. Then $\mathcal{A}$ contains a projection P satisfying $P \sim 1 - P$.* ■

Lemma 3.5 *Let $\mathcal{A}$ be a von Neumann algebra either of type II_1 or properly infinite.*

1. *If $T \in \mathcal{A}$ is normal, then there is a projection $P \in \mathcal{A}$ commuting with T and elements T_1 and T_2 in $\mathcal{A}$ satisfying:*

$$P \sim 1 - P;$$
$$T = T_1T_2;$$
$$T_1 = PT_1P + (1 - P);$$
$$T_2 = (1 - P)T_2(1 - P) + P.$$

   If T is positive, unitary, or induces an injection $r_T : \mathcal{A} \longrightarrow \mathcal{A}$, or invertible, then the same is true for T_1 and T_2.

2. *Let T in $\mathcal{A}$ be positive and invertible such that $\det_{FK}(T) = 1$. Then there are positive invertible operators T_1 and T_2, invertible operators A and B and a projection $P \in \mathcal{A}$ commuting with T satisfying:*

$P \sim 1 - P$;

$T = T_1 T_2 ABA^{-1}B^{-1}$;

$T_1 = PT_1P + (1-P)$;

$T_2 = (1-P)T_2(1-P) + P$;

$\det_{FK}(T_1) = \det_{FK}(T_2) = 1$.

Proof : By lemma 3.4 there is a projection P commuting with T satisfying $P \sim 1 - P$. Set

$$\widetilde{T}_1 = PTP + (1-P);$$
$$\widetilde{T}_2 = (1-P)T(1-P) + P.$$

Then $T_1 = \widetilde{T}_1$ and $T_2 = \widetilde{T}_2$ will satisfy 1.). In 2.) let $C \in Z(\mathcal{A})$ be $\det_{\mathrm{FK}}(\widetilde{T}_1)$ and choose a partial isometry V satisfying $P = V^*V$ and $1 - P = VV^*$. Now define:

$$T_1 = PC^{-2}TP + (1-P);$$
$$T_2 = (1-P)C^2T(1-P) + P;$$
$$A = PC + (1-P)C^{-1};$$
$$B = V + V^*. \qquad \blacksquare$$

Lemma 3.6 *Let $\mathcal{A}$ be of type II_1 and P be a projection satisfying $P \sim 1 - P$. Then there is a sequence of projections $\{P_n \mid n \in \mathbb{N}\}$ satisfying*

1. *The projections P_n are mutually orthogonal;*
2. $P_1 = P$;
3. $\mathrm{tr}(P_n) = 2^{-n}$;
4. $\sum_{n=1}^{\infty} P_n = 1$.

Proof : Use [8], Theorems 8.4.3 and 8.4.4 to find projections P_n, $n \geq 2$, in $\mathcal{A}$ such that 1.) and 3.) are satisfied. Since tr is normal,

$$\mathrm{tr}(\sum_{n=1}^{\infty} P_n) = \sum_{n=1}^{\infty} \mathrm{tr}(P_n) = 1,$$

and so 4.) holds. $\blacksquare$

The following lemma is a consequence of theorem 8.4.4 in [8] on page 533.

Lemma 3.7 *Let $\mathcal{A}$ be a von Neumann algebra of type II_1. Let P_0 and P_1 be projections in $\mathcal{A}$ and $C \in Z(\mathcal{A})$ be such that $P_0 \leq P_1$ and $tr(P_0) \leq C \leq tr(P_1)$. Then there is a projection $P \in \mathcal{A}$ such that $P_0 \leq P \leq P_1$ and $tr(P) = C$.* ■

Lemma 3.8 *Let $\mathcal{A}$ be of type II_1 and $T \in \mathcal{A}$ be selfadjoint. Then there is a selfadjoint element $C \in Z(\mathcal{A})$ and a projection $F \in \mathcal{A}$ satisfying:*

1. $FT \leq FC$;
2. $(1-F)T \geq (1-F)C$;
3. $F \sim (1-F)$;
4. *F commutes with T.*

Proof : Let E_λ for $\lambda \in \mathbb{R}$ be the spectral projection of T in $\mathcal{A}$ for the interval $]-\infty,\lambda]$. Denote by P_λ the spectral projection of $\mathrm{tr}(E_\lambda)$ in $Z(\mathcal{A})$ corresponding to the interval $[1/2,\infty[$. Then $\{P_\lambda \mid \lambda \in \mathbb{R}\}$ is a spectral family in $Z(\mathcal{A})$ and we can define a selfadjoint element C in $Z(\mathcal{A})$ by

$$C = \int_{\mathbb{R}} \lambda dP_\lambda.$$

Since we have $1/2 \cdot P_\mu \leq \mathrm{tr}(E_\mu)P_\mu$ and $1/2 \cdot (1-P_\lambda) \geq \mathrm{tr}(E_\lambda)(1-P_\lambda)$, we get for $\lambda \leq \mu$:

$$\mathrm{tr}(E_\lambda)(P_\mu - P_\lambda) \ \leq \frac{1}{2} \cdot (P_\mu - P_\lambda) \ \leq \mathrm{tr}(E_\mu)(P_\mu - P_\lambda).$$

Let G_λ be the spectral projection for $T-C$ in $\mathcal{A}$ corresponding to the interval $]-\infty,\lambda]$. Denote by $G_{<0}$ the spectral decomposition for $T-C$ in $\mathcal{A}$ corresponding to the interval $]-\infty,0[$. Notice for the sequel that P_λ and C are central and E_λ commutes with G_μ for all $\lambda,\mu \in \mathbb{R}$. Since we have $\gamma G_\gamma \geq G_\gamma(T-C)$ and $\gamma(1-G_\gamma) \leq (1-G_\gamma)(T-C)$, we obtain for $\lambda,\mu,\gamma \in \mathbb{R}$ with $\lambda < \mu$:

$$\gamma(1-E_\lambda)(P_\mu - P_\lambda)G_\gamma \geq (1-E_\lambda)(P_\mu - P_\lambda)G_\gamma(T-C)$$
$$\gamma E_\mu(P_\mu - P_\lambda)(1-G_\gamma) \leq E_\mu(P_\mu - P_\lambda)(1-G_\gamma)(T-C)$$

As $(1-E_\lambda)T \geq \lambda(1-E_\lambda)$, $E_\mu T \leq \mu E_\mu$, $P_\mu C \leq \mu P_\mu$ and $(1-P_\lambda)C \geq \lambda(1-P_\lambda)$ hold, we get for $\lambda,\mu,\gamma \in \mathbb{R}$ with $\lambda < \mu$:

$$(1-E_\lambda)(P_\mu - P_\lambda)G_\gamma(T-C) \geq (\lambda-\mu)(1-E_\lambda)(P_\mu - P_\lambda)G_\gamma;$$
$$E_\mu(P_\mu - P_\lambda)(1-G_\gamma)(T-C) \leq (\mu-\lambda)E_\mu(P_\mu - P_\lambda)(1-G_\gamma).$$

This implies for $\lambda,\mu,\gamma \in \mathbb{R}$ with $\lambda < \mu$:

$$\gamma(1-E_\lambda)(P_\mu - P_\lambda)G_\gamma \geq (\lambda-\mu)(1-E_\lambda)(P_\mu - P_\lambda)G_\gamma;$$
$$\gamma E_\mu(P_\mu - P_\lambda)(1-G_\gamma) \leq (\mu-\lambda)E_\mu(P_\mu - P_\lambda)(1-G_\gamma).$$

We conclude:

$$(P_\mu - P_\lambda)G_\gamma \leq E_\lambda(P_\mu - P_\lambda)G_\gamma \qquad , \text{if } \gamma < \lambda - \mu \leq 0;$$
$$(P_\mu - P_\lambda)(1 - G_\gamma) \leq (1 - E_\mu)(P_\mu - P_\lambda)(1 - G_\gamma) \qquad , \text{if } \gamma > \mu - \lambda \geq 0.$$

Hence we obtain:

$$(P_\mu - P_\lambda)G_\gamma \leq E_\lambda(P_\mu - P_\lambda) \qquad , \text{if } \gamma < \lambda - \mu \leq 0;$$
$$(P_\mu - P_\lambda)(1 - G_\gamma) \leq (1 - E_\mu)(P_\mu - P_\lambda) \qquad , \text{if } \gamma > \mu - \lambda \geq 0.$$

Combining these inequalities with the first inequality appearing in this proof yields:

$$\mathrm{tr}(G_\gamma)(P_\mu - P_\lambda) \leq \tfrac{1}{2} \cdot (P_\mu - P_\lambda) \qquad , \text{if } \gamma < \lambda - \mu \leq 0;$$
$$\mathrm{tr}(1 - G_\gamma)(P_\mu - P_\lambda) \leq \tfrac{1}{2} \cdot (P_\mu - P_\lambda) \qquad , \text{if } \gamma > \mu - \lambda \geq 0.$$

This implies:

$$\mathrm{tr}(G_\gamma) \leq \tfrac{1}{2} \cdot 1 \qquad \text{for } \gamma < 0;$$
$$\mathrm{tr}(G_\gamma) \geq \tfrac{1}{2} \cdot 1 \qquad \text{for } \gamma > 0.$$

Hence we get:

$$\mathrm{tr}(G_{<0}) \leq \frac{1}{2} \cdot 1 \leq \mathrm{tr}(G_0).$$

By lemma 3.7 there is a projection $F \in \mathcal{A}$ satisfying:

$$\mathrm{tr}(F) = \frac{1}{2} \cdot 1 \text{ and } G_{<0} \leq F \leq G_0.$$

By construction we have

$$F \sim 1 - F;$$
$$F(T - C) \leq 0;$$
$$(1 - F)(T - C) \geq 0.$$

As $F = G_{<0} + (F - G_{<0})$, $(T - C)(F - G_{<0}) = (F - G_{<0})(T - C) = 0$ and $G_{<0}$ commutes with $T - C$, F commutes with $T - C$ and hence with T. ■

Proposition 3.9 *Let $\mathcal{A}$ be a von Neumann algebra of type II_1 and let $T \in \mathcal{A}$. Suppose P is a projection such that $P \sim 1 - P$ and $T = PTP + (1 - P)$.*

1. *Assume that T is positive, invertible and $\det_{FK}(T) = 1$. Then there are sequences of invertible operators T_n, A_n and B_n for $n \in \mathbb{N}$ such that the assumptions of proposition 3.1 are satisfied. In particular, T is the product of two commutators of invertible elements in $\mathcal{A}$ and its class $[T]$ in $K_1(\mathcal{A})$ is trivial.*

2. *Assume that T is unitary. Then there are sequences of unitary operators T_n, A_n and B_n for $n \in \mathbb{N}$ such that the assumptions of proposition 3.1 are satisfied. In particular, T is the product of two commutators in unitary elements in $\mathcal{A}$ and its class $[T]$ in $K_1(\mathcal{A})$ is trivial.*

3. *Assume that T is positive, $0 \leq T \leq 1$ and T induces an injection $\mathcal{A} \longrightarrow \mathcal{A}$. Then there are sequences of operators T_n, A_n and B_n for $n \in \mathbb{N}$, inducing injections $\mathcal{A} \longrightarrow \mathcal{A}$ such that the assumptions of proposition 3.1 are satisfied. In particular, the class $[T]$ in $K_1^w(\mathcal{A})$ is trivial.*

Proof : We first prove assertion 1.) Choose a sequence of projections P_n as in lemma 3.6. Choose $\alpha \geq 1$ such that $\alpha^{-1} \leq \| T^{-1} \|^{-1} \leq \| T \| \leq \alpha$ holds. Now we will define sequences of invertible operators

$$T_n, A_n, B_n \in \mathcal{A};$$

$$K_n, L_n \in Z(P_n \mathcal{A} P_n)$$

with the following properties:

(a) $T_n = P_n T_n P_n + (1 - P_n)$;

(b) $A_n = (P_{n+1} + P_n) A_n (P_{n+1} + P_n) + (1 - P_n - P_{n+1})$;
$B_n = (P_{n+1} + P_n) B_n (P_{n+1} + P_n) + (1 - P_n - P_{n+1})$;

(c) $B_n A_n T_n = A_n B_n T_{n+1}$;

(d) $\| T_n \|, \| T_n^{-1} \| \leq \alpha^2$;
$\| A_n \|, \| A_n^{-1} \| \leq \alpha^2$;
$\| B_n \|, \| B_n^{-1} \| \leq \alpha$

(e) $L_n = \alpha^2 \cdot K_n$;

(f) $K_n \leq P_n T_n P_n \leq L_n$;

(g) $\det_{FK}(T_n) = 1$.

Notice that if we have completed this construction, then assertion 1.) will follow from proposition 3.1.

We construct the operators T_n, A_{n-1}, B_{n-1}, K_n and L_n inductively. Put:

$$T_1 = T;$$

$$K_1 = \alpha^{-1} \cdot P_1;$$

$$L_1 = \alpha \cdot P_1.$$

Next we prove the induction step from n to $n+1$. Apply lemma 3.8 to $P_n T_n P_n \in P_n \mathcal{A} P_n$ to obtain a projection $F_n \in P_n\mathcal{A}P_n$ and an invertible positive element $C_n \in Z(P_n\mathcal{A}P_n)$ satisfying:

$F_nT_n \le F_nC_n$;

$(P_n - F_n)T_n \ge (P_n - F_n)C_n$.

$F_n \sim P_n - F_n \sim P_{n+1}$;

F_n commutes with T_n;

Find partial isometries V_n, W_n in $P_n\mathcal{A}P_n$ satisfying:

$F_n = V_n^*V_n$;

$P_n - F_n = W_n^*W_n$;

$P_{n+1} = V_nV_n^* = W_nW_n^*$.

We define:

$R_n = V_nT_nV_n^*$;

$S_n = W_nT_nW_n^*$.

Then we get:

$V_nK_nV_n^* \le R_n \le V_nC_nV_n^*$;

$W_nC_nW_n^* \le S_n \le W_nL_nW_n^*$.

Since $V_n^*W_n \in P_n\mathcal{A}P_n$, C_n, K_n, and L_n commute with $V_n^*W_n$ and also with $W_n^*V_n$. This implies:

$V_nC_nV_n^* = W_nC_nW_n^* \in Z(P_{n+1}\mathcal{A}P_{n+1})$;

$V_nK_nV_n^* = W_nK_nW_n^* \in Z(P_{n+1}\mathcal{A}P_{n+1})$;

$V_nL_nV_n^* = W_nL_nW_n^* \in Z(P_{n+1}\mathcal{A}P_{n+1})$.

We conclude:

$$V_nK_nV_n^* \le R_n \le V_nC_nV_n^* = W_nC_nW_n^* \le S_n \le W_nL_nW_n^*.$$

Now we define

$T_{n+1} = R_n^{1/2}S_nR_n^{1/2} + (1 - P_{n+1})$;

$K_{n+1} = V_nK_nC_nV_n^*$;

$L_{n+1} = V_nL_nC_nV_n^*$.

Then we get:

$K_{n+1} \le R_n^{\frac{1}{2}}S_nR_n^{\frac{1}{2}} = P_{n+1}T_{n+1}P_{n+1} \le L_{n+1}$;

$L_{n+1} = \alpha^2 \cdot K_{n+1}.$

The following 3×3 matrix calculation from [4]

$$\begin{pmatrix} 0 & 0 & 1 \\ s^{\frac{1}{2}} r^{\frac{1}{2}} & 0 & 0 \\ 0 & s^{\frac{1}{2}} r^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & s^{-1} & 0 \\ 0 & 0 & 1 \\ s^{-\frac{1}{2}} r^{-\frac{1}{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & s^{-1} & 0 \\ 0 & 0 & 1 \\ s^{-\frac{1}{2}} r^{-\frac{1}{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ s^{\frac{1}{2}} r^{\frac{1}{2}} & 0 & 0 \\ 0 & s^{\frac{1}{2}} r^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^{\frac{1}{2}} s r^{\frac{1}{2}} \end{pmatrix}$$

shows that $B_n A_n T_n = A_n B_n T_{n+1}$ for some $A_n, B_n \in \mathcal{A}$ satisfying (b), and

$$\begin{aligned} \|A_n\| &\leq \max\{\|T_n^{-1}\|, \|T_{n+1}^{-1}\|^{\frac{1}{2}}\}, \quad \|B_n\| \leq \|T_{n+1}\|^{\frac{1}{2}}, \\ \|A_n^{-1}\| &\leq \max\{\|T_n\|, \|T_{n+1}\|^{\frac{1}{2}}\}, \quad \|B_n^{-1}\| \leq \|T_{n+1}^{-1}\|^{\frac{1}{2}}. \end{aligned}$$

In particular we get:

$$\det_{\mathrm{FK}}(T_n) = \det_{\mathrm{FK}}(T_{n+1}) = 1,$$

which, together with $K_{n+1} + (1 - P_{n+1}) \leq T_{n+1} \leq L_{n+1} + (1 - P_{n+1})$, implies

$$K_{n+1} \leq P_{n+1} \leq L_{n+1}.$$

Since $L_{n+1} = \alpha^2 \cdot K_{n+1}$, we get:

$$\alpha^{-2} P_{n+1} \leq K_{n+1} \leq P_{n+1} T_{n+1} P_{n+1} \leq L_{n+1} \leq \alpha^2 P_{n+1}.$$

As $T_{n+1} = P_{n+1} T_{n+1} P_{n+1} + (1 - P_{n+1})$ holds by definition, we conclude:

$\| T_{n+1} \|, \| T_{n+1}^{-1} \| \leq \alpha^2.$

This implies:

$\| A_n \|, \| A_n^{-1} \| \leq \alpha^2, \quad \text{and} \quad \| B_n \|, \| B_n^{-1} \| \leq \alpha.$

This finishes the proof of assertion 1.) of proposition 3.9.

To prove 2.) and 3.) we construct sequences of operators P_n, T_n, A_n and B_n satisfying conditions (a), (b) and (c). The projections P_n are chosen as in the proof of 1.). Assume $F_n \leq P_n$ is a projection which commutes with T_n and satisfies $F_n \sim P_n - F_n$. Consider the following 3×3 matrix identity:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & rs \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & rs \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & rs \end{pmatrix}.$$

It follows that operators A_n, B_n and T_{n+1} can be found so that $B_nA_nT_n = A_nB_nT_{n+1}$, where

$$T_{n+1} = V_nT_nV_n^*W_nT_nW_n^* + (1 - P_{n+1}),$$

$V_n^*V_n = F_n$, $W_n^*W_n = P_n - F_n$ and $V_nV_n^* = W_nW_n^* = P_{n+1}$. Note that A_n, B_n and T_{n+1} are unitaries if T_n is unitary. If T_n induces an injection and $\|T_n\| \leq 1$, then A_n, B_n and T_{n+1} induce injections and $\|A_n\|$, $\|B_n\|$, $\|T_n\| \leq 1$.

The projection F_n can be found using lemma 4.7 if T_n is unitary, and this completes the proof of 2.). In the case 3.) use 4.7 repeatedly to find projections $F(\lambda) \leq P = P_1$ for each dyadic rational $\lambda \in [0,1]$, so $F(\lambda)$ commutes with T, $F(\lambda') \geq F(\lambda)$ if $\lambda' \geq \lambda$, and $trF(\lambda) = \lambda tr\ P$. Set $F_1 = F(1/2)$ and choose V_1 and W_1 so that

$$V_1F(\lambda)V_1^* = W_1(F(\lambda + 1/2) - F(1/2))W_1^*, \quad 0 \leq \lambda \leq 1/2.$$

Then $F_2 = V_1F(1/4)V_1^*$ has the desired properties. This algorithm can be continued to obtain F_n for all n. ■.

Now we can finish the proof of theorem 3.3. It remains to show that the map

$$\det_{\mathrm{FK}} : K_1(\mathcal{A}) \longrightarrow Z(\mathcal{A})^+_{\mathrm{inv}}$$

is injective and $K_1^w(\mathcal{A}) = \{0\}$.

Consider η in $K_1(\mathcal{A})$ satisfying $\det_{FK}(\eta) = 1$. Choose an invertible $S \in M_k(\mathcal{A})$ satisfying $\eta = [S]$. Then S has polar decomposition $S = UT$, with U unitary and T invertible and positive. By Lemma 3.7 there are projections E and F, positive invertible elements T_1 and T_2 with Fuglede-Kadison determinant 1, invertibles A and B, and unitaries U_1 and U_2 satisfying:

$$T = T_1T_2ABA^{-1}B^{-1}, \quad U = U_1U_2, \quad E \sim 1 - E, \quad F \sim 1 - F,$$

$$T_1 = ET_1E + (1 - E), \quad T_2 = (1 - E)T_2(1 - E) + E$$
$$U_1 = FU_2(1 - F), \quad U_2 = (1 - F)U_2(1 - F) + F.$$

Proposition 3.11 now implies that S is a product of nine commutators. In particular $\eta = 0$.

Assume $S \in \mathcal{A}$ is injective. Then $S = UT$ where U is unitary and T is positive and injective. From above we have that U is a product of four commutators in $\mathcal{A}$, so $[U] = 0$ in $K_1^w(\mathcal{A})$. As in Lemma 3.5, $T = T_1T_2$ where T_1 and T_2 are positive and injective and α_jT_j satisfy the conditions of proposition 3.9 3.) if $\alpha_j \in \mathbf{R}$ is chosen so that $0 < \alpha_j \leq 1$ and $0 \leq \alpha_jT_j \leq 1$. It follows that $[\alpha_jT_j]$ and $[\alpha_j^{-1}]$ vanish in $K_1^w(\mathcal{A})$. Hence $[S] = 0$ in $K_1^w(\mathcal{A})$ which completes the proof. ■

4. The properly infinite case

In this section we show that $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$ are trivial for a properly infinite von Neumann algebra $\mathcal{A}$. This follows for $K_1(\mathcal{A})$ already from de la Harpe-Skandalis [7, Theorem 7.5].

Lemma 4.1 *Let $\mathcal{A}$ be a von Neumann algebra and $T \in \mathcal{A}$ such that T is invertible, respectively right multiplication with T, induces an injection $r_T : \mathcal{A} \longrightarrow \mathcal{A}$. Suppose that there is a countable sequence of mutually orthogonal projections P_1, P_2, ... satisfying:*

$P_n \sim P_{n+1}$;

$\sum_{n=1}^{\infty} P_n = 1$;

$T = P_1 T P_1 + (1 - P_1)$.

Then the class $[T]$ of T in $K_1(\mathcal{A})$, respectively, $K_1^w(\mathcal{A})$ vanishes.

Proof : Choose partial isometries $V_n \in \mathcal{A}$ such that $P_n = V_n^* V_n$ and $P_{n+1} = V_n V_n^*$. Put:

$T_1 = T$;

$T_{n+1} = V_n T_n V_n^* + (1 - P_{n+1})$;

$A_n = T_{n+1}$;

$B_n = V_n + V_n^* + (1 - P_n - P_{n+1})$.

Then B_n is invertible and:

$B_n A_n T_n = A_n B_n T_{n+1}$

$\| A_n \| \| T_n \| = \| T \|$

$\| B_n \| = \| B_n^{-1} \| = 1$

If T is invertible, then A_n is invertible and

$$\| A_n^{-1} \| = \| T^{-1} \|$$

If $r_T : \mathcal{A} \longrightarrow \mathcal{A}$ is injective, then r_{T_n} and r_{A_n} are injective. Now we derive from proposition 3.1 that the class $[T]$ in $K_1(\mathcal{A})$, respectively $K_1^w(\mathcal{A})$, vanishes. This finishes the proof of lemma 4.1. ■

Theorem 4.2 *Let $\mathcal{A}$ be a properly infinite von Neumann algebra. Then $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$ are trivial.*

Proof : Consider η in $K_1(\mathcal{A})$, respectively $K_1^w(\mathcal{A})$. In view of lemma 3.5 we can assume that η is represented by $T \in \mathcal{A}$, such that T is invertible or right multiplication with T induces an injection $r_T : \mathcal{A} \longrightarrow \mathcal{A}$, and that there is a projection P satisfying:

$T = PTP + (1 - P)$;

$P \sim 1 - P$.

Put $P_1 = P$. Since $1 - P$ is properly infinite there is a sequence of projections $P_2, P_3, \cdots$ in $\mathcal{A}$ satisfying

P_2, P_3, …satisfying:

$T = P_1 T P_1 + (1 - P_1)$;

$P_n \sim P_{n+1}$ for $n \geq 1$;

$\sum_{n=1}^{\infty} P_n = 1$.

Now the claim follows from lemma 4.1. ■

Notice that we have shown that any invertible $T \in M_k(\mathcal{A})$ can be written as the product of four commutators of invertible elements in $M_k(\mathcal{A})$, if $\mathcal{A}$ is properly infinite.

5. Detecting elements in $Wh(G)$

Let G be a countable (discrete) group. The Whitehead group $Wh(G)$ is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup of trivial units $\{\pm g \mid g \in G\}$. Denote by $Wh'(G)$ the quotient of $Wh(G)$ by its torsion subgroup. We want to detect elements in $Wh'(G)$ using the Fuglede-Kadison determinant.

Recall that $Z(R)$ denotes the center and R_{inv} denotes multiplicative group of invertible elements if R is a ring. We equip $\mathbb{C}G$ and $\mathbb{Z}G$ with the involution of rings sending $\sum_{g \in G} \lambda_g g$ to $\sum_{g \in G} \overline{\lambda_g} g^{-1}$. It induces involutions on $Z(\mathbb{Z}G)$, $Z(\mathbb{C}G)$, $K_1(\mathbb{C}G)$, $Wh(G)$ and $Wh'(G)$. This involution corresponds to taking adjoints on operator level. Let $Z(\mathbb{Z}G)^{\mathbb{Z}/2}$ be the fixed point set under this involution and $Z(\mathbb{Z}G)_+ \subset Z(\mathbb{Z}G)^{\mathbb{Z}/2}$ be the positive elements, i.e, elements of the shape aa^* for $a \in Z(\mathbb{Z}G)$. Consider a normal subgroup H of G. Then G acts on H by conjugation and this action induces G-actions on $Wh'(H)$. The fixed point set is denoted by $Wh'(H)^G$. The main result of this section is

Theorem 5.1 *For a finite normal subgroup $H \subset G$ the map*

$$i_3 : Wh'(H)^G \longrightarrow Wh'(G)$$

induced by induction is injective. ■

A homomorphism $f : A \longrightarrow B$ of abelian groups is rationally injective, respectively, bijective, i.e., $f \otimes_{\mathbb{Z}} id_{\mathbb{Q}} : A \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow B \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, respectively, bijective if and only if the kernel, respectively, both the kernel and the cokernel are torsion. For the proof of theorem 5.1 we need the following lemma:

Lemma 5.2 *Let A and B be $\mathbb{Z}G$-modules. Let $f : A \longrightarrow B$ be a $\mathbb{Z}G$-homomorphism. If f is rationally injective, respectively, bijective, the same holds for the induced map $f^G : A^G \longrightarrow B^G$*

Proof : For any $\mathbb{Z}G$ -module M there is a natural map

$$T(M) : \hom_{\mathbb{Z}G}(M, A) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \hom_{\mathbb{Q}G}(M \otimes_{\mathbb{Z}} \mathbb{Q}, A \otimes_{\mathbb{Z}} \mathbb{Q})$$

assigning to $f \otimes_{\mathbb{Z}} r$ the $\mathbb{Q}G$-map $M \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $m \otimes_{\mathbb{Z}} s$ to $f(m) \otimes_{\mathbb{Z}} rs$. Obviously $T(\oplus_{i \in I} M_i) = \oplus_{i \in I} T(M_i))$ holds and $T(\mathbb{Z}G)$ is bijective. Hence $T(M)$ is bijective for any projective $\mathbb{Z}G$-module M. Let P_* be a projective $\mathbb{Z}G$-resolution for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. Since $\mathbb{Q}$ is flat as an $\mathbb{Z}$-module, $P_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a projective $\mathbb{Q}G$-resolution for the trivial $\mathbb{Q}G$-module $\mathbb{Q} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. We obtain a chain isomorphism

$$T(P_*) : \hom_{\mathbb{Z}G}(P_*, A) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \hom_{\mathbb{Q}G}(P_* \otimes_{\mathbb{Z}} \mathbb{Q}, A \otimes_{\mathbb{Z}} \mathbb{Q})$$

Since $\mathbb{Q}$ is flat as $\mathbb{Z}$-module, the natural map

$$H^p(\hom_{\mathbb{Z}G}(P_*, A)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^p(\hom_{\mathbb{Z}G}(P_*, A) \otimes_{\mathbb{Z}} \mathbb{Q})$$

is an isomorphism. We obtain a natural isomorphism

$$H^p(\hom_{\mathbb{Z}G}(P_*, A)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^p(\hom_{\mathbb{Q}G}(P_* \otimes_{\mathbb{Z}} \mathbb{Q}, A \otimes_{\mathbb{Z}} \mathbb{Q}))$$

There are natural identifications

$$H^0(\hom_{\mathbb{Z}G}(P_*, A) = \hom_{\mathbb{Z}G}(\mathbb{Z}, A) = A^G$$

and

$$H^0(\hom_{\mathbb{Q}G}(P_* \otimes_{\mathbb{Z}} \mathbb{Q}, A \otimes_{\mathbb{Z}} \mathbb{Q})) = \hom_{\mathbb{Q}G}(\mathbb{Q}, A \otimes_{\mathbb{Z}} \mathbb{Q}) = (A \otimes_{\mathbb{Z}} \mathbb{Q})^G$$

Hence the natural map

$$A^G \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow (A \otimes_{\mathbb{Z}} \mathbb{Q})^G$$

is an isomorphism. Therefore $f^G \otimes_{\mathbb{Z}} id_{\mathbb{Q}}$ is injective, respectively, bijective, if and only if $(f \otimes_{\mathbb{Z}} id_{\mathbb{Q}})^G$ is. This finishes the proof of lemma 5.2. ■

Now we give the proof of theorem 5.1. Consider the following commutative diagram:

$$
\begin{array}{ccc}
Z(\mathbb{C}H)_{\mathrm{inv}}^{G\times\mathbb{Z}/2} & \xrightarrow{s} & Z(\mathcal{N}(G))_{\mathrm{inv}}^{+} \\
{\scriptstyle k^{G\times\mathbb{Z}/2}}\downarrow & & \uparrow{\scriptstyle \det_{FK}} \\
K_1(\mathbb{C}H)^{G\times\mathbb{Z}/2} & \xrightarrow{i_1} & K_1(\mathbb{C}G) \\
{\scriptstyle j_1^{G\times\mathbb{Z}/2}}\uparrow & & \uparrow{\scriptstyle j_2} \\
K_1(\mathbb{Z}H)^{G\times\mathbb{Z}/2} & \xrightarrow{i_2} & K_1(\mathbb{Z}G) \\
{\scriptstyle p_1^{G\times\mathbb{Z}/2}}\downarrow & & \downarrow{\scriptstyle p_2} \\
Wh'(H)^G & \xrightarrow{i_3} & Wh'(G)
\end{array}
$$

The $G \times \mathbb{Z}/2$-action comes from the G-action and the involution described above which are compatible with one another. Notice that the Fuglede-Kadison determinant $\det_{FK}$ sends the class of an element in $K_1(\mathbb{C}G)$ represented by an element $a \in Z(\mathbb{C}G)_{\mathrm{inv}}$ to $| a |= (aa^*)^{1/2} \in Z(\mathcal{N}(G))_{\mathrm{inv}}^{+}$. Hence the map s is the composition of the injection

$$Z(\mathcal{N}(G))_{\mathrm{inv}}^{+} \longrightarrow Z(\mathcal{N}(G))_{\mathrm{inv}}^{+} \qquad a \mapsto \sqrt{a},$$

the inclusion

$$Z(\mathbb{C}G)_{\mathrm{inv}}^{+} \longrightarrow Z(\mathcal{N}(G))_{\mathrm{inv}}^{+},$$

the inclusion

$$\big(Z(\mathbb{C}H)_{\mathrm{inv}}^{G}\big)^{+} \longrightarrow Z(\mathbb{C}G)_{\mathrm{inv}}^{+}$$

and the rational isomorphism

$$Z(\mathbb{C}H)_{\mathrm{inv}}^{G\times\mathbb{Z}/2} \longrightarrow \big(Z(\mathbb{C}H)_{\mathrm{inv}}^{G}\big)^{+}$$

sending a to $aa^* = a^2$. The map $k : Z(\mathbb{C}H)_{\mathrm{inv}} \longrightarrow K_1(\mathbb{C}H)$ is the canonical map. The maps j_1 and j_2 are change of rings homomorphisms and the maps p_1 and p_2 are the natural projections. Next we show:

Lemma 5.3 *1. The map s is a rationally injective;*

2. *The map k is rationally bijective;*
3. *The map $j_1^{G\times\mathbb{Z}/2}$ is rationally injective;*
4. *The map $p_1^{G\times\mathbb{Z}/2}$ is rationally bijective;*
5. $\det_{FK} \circ j_2$ *maps the kernel of p_2 to the torsion subgroup of $Z(\mathcal{N}(G))_{inv}^{+}$.*

Proof : 1.) follows from the description of s as a composition of maps which are rationally injective.

2.) Since H is finite $\mathbb{C}H$ is a von Neumann algebra of type I_f. Composing k with the isomorphism det_{norm} of theorem 2.1 yields an epimorphism with finite kernel. Now apply lemma 5.2.

3.) Wall [16] has shown for finite H that the kernel $SK_1(\mathbb{Z}H)$ of the change of rings map

$$K_1(\mathbb{Z}H) \longrightarrow K_1(\mathbb{Q}H)$$

is finite and maps under the canonical projection $K_1(\mathbb{Z}H) \longrightarrow Wh(H)$ bijectively onto the torsion subgroup of $Wh(H)$ (see also Oliver [14] page 5 and page 180). The change of rings map

$$K_1(\mathbb{Q}H) \longrightarrow K_1(\mathbb{C}H)$$

is injective (see Oliver [14] page 5 and page 43). Hence

$$j_1 : K_1(\mathbb{Z}H) \longrightarrow K_1(\mathbb{C}H)$$

is rationally injective. Now apply lemma 5.2

4.) The map p_1 is surjective and its kernel is a torsion subgroup, since the subgroup of trivial units $\{\pm h \mid h \in H\}$ in $K_1(\mathbb{Z}H)$ is finite. We derive from lemma 5.2 that $p_1^{G\times\mathbb{Z}/2}$ is rationally bijective. Since the involution on $Wh'(H)$ is trivial (see Wall [16], Oliver [14] page 182) we have $Wh'(H)^G = Wh'(H)^{G\times\mathbb{Z}/2}$ and the claim follows.

5.) Obviously $\det_{FK} \circ j_2$ maps a trivial unit to 1 since a trivial unit represents a unitary operator. The kernel of the projection $p_2' : K_1(\mathbb{Z}G) \longrightarrow Wh(G)$ is the subgroup of trivial units and the kernel of the projection $p_2'' : Wh(G) \longrightarrow Wh'(G)$ is torsion. Since p_2 is the composition $p_2'' \circ p_2'$ the claim follows. ■

Now we can finish the proof of theorem 5.1. Since $Wh'(H)$ is torsion-free, it suffices to show that the map

$$i_3 : Wh'(H)^G \longrightarrow Wh'(G)$$

is rationally injective. This follows from lemma 5.3 and the diagram above.

Corollary 5.4 *Let G be a countable group. If $Wh(G)$ is trivial, then any finite subgroup of the center of G is isomorphic to a product of finitely many copies of $\mathbb{Z}/2$ and $\mathbb{Z}/3$ or of $\mathbb{Z}/2$ and $\mathbb{Z}/4$.*

Proof : Suppose $Wh(G)$ is trivial. Consider a finite subgroup H of the center of G. We derive from theorem 5.1 that $Wh'(H)^G$ is trivial. Since the G-action on $Wh'(H)$ is trivial, $Wh'(H)$ must vanish. The rank of the free abelian group $Wh'(H)$ is the difference $r - q$, where r is the number of $\mathbb{R}$-conjugacy classes and q the number of $\mathbb{Q}$-conjugacy classes in G

(see Oliver [14] page 49). Two elements g and h in the abelian group H are $\mathbb{R}$-conjugated if and only if $g = h$ or $g = h^{-1}$ holds and they are $\mathbb{Q}$-conjugated if they generate the same cyclic subgroup. Hence $Wh'(H)$ is trivial if and only if $r = q$ holds. A finite abelian group H satisfies $r = q$ if and only any cyclic subgroup satisfies this condition. A non-trivial cyclic group $\mathbb{Z}/n$ satisfies $r = q$ if and only if $n = 2, 3, 4, 6$. Hence H is a finite product of copies of $\mathbb{Z}/2$, $\mathbb{Z}/3$ or of $\mathbb{Z}/2$ and $\mathbb{Z}/4$. ■

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