Induction Theorems and Isomorphism Conjectures for K- and L-Theory

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Abstract

The Farrell-Jones and the Baum-Connes Conjecture say that one can compute the algebraic K- and L-theory of the group ring and the topological K-theory of the reduced group C^*-algebra of a group G in terms of these functors for the virtually cyclic subgroups or the finite subgroups of G. By induction theory we want to reduce these families of subgroups to a smaller family, for instance to the family of subgroups which are either finite hyperelementary or extensions of finite hyperelementary groups with $\mathbb{Z}$ as kernel or to the family of finite cyclic subgroups. Roughly speaking, we extend the induction theorems of Dress for finite groups to infinite groups.

Key words: K- and L-groups of group rings and group C^*-algebras, induction theorems.

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0. Introduction

The goal of this paper is to reduce the family of virtually cyclic subgroups to a smaller subfamily in the statements of the (Fibered) Farrell-Jones Conjecture for algebraic K- and L-theory of group rings. The strategy is to extend the classical induction results for finite groups to infinite groups.

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Let $\mathcal{FIN}$ and $\mathcal{VCY}$ respectively be the class of finite groups and virtually cyclic groups respectively. Let $\mathcal{F} \subseteq \mathcal{FIN}$ be a subclass of the class of finite groups which is closed under isomorphism of groups and taking subgroups. Define

$$\mathcal{F}' \quad \subseteq \quad \mathcal{VCY}$$

to be the class of groups V for which either

(a) there exists an extension $1 \to \mathbb{Z} \to V \to F \to 1$ for a group $F \in \mathcal{F}$ or

(b) $V \in \mathcal{F}$ holds.

With this notion we get $\mathcal{VCY} = \mathcal{FIN}'$.

Let p be a prime. A finite group G is called *p-elementary* if it is isomorphic to $C \times P$ for a cyclic group C and a p-group P such that the order $|C|$ is prime to p. A finite group G is called *p-hyperelementary* if it can be written as an extension $1 \to C \to G \to P \to 1$ for a cyclic group C and a p-group P such that the order $|C|$ is prime to p. A finite group G is called *elementary* or *hyperelementary* respectively if it is *p-elementary* or *p-hyperelementary* respectively for some prime p. Let $\mathcal{FCY}$ be the class of finite cyclic groups. Let $\mathcal{E}_p$ and $\mathcal{H}_p$ respectively be the class of groups which are p-elementary groups and p-hyperelementary respectively for a prime p. Let $\mathcal{E}$ and $\mathcal{H}$ respectively be the class of groups which are elementary and hyperelementary respectively.

A family of subgroups of G is a set of subgroups which is closed under conjugation and taking subgroups. For a class $\mathcal{F}$ of groups which is closed under isomorphism of groups and taking subgroups we denote by $\mathcal{F}(G)$ the family of subgroups of G whose members are in $\mathcal{F}$. Thus $\mathcal{FIN}(G)$ is the family of finite subgroups of G. If G is clear from the context we will often abuse notation and write simply $\mathcal{F}$ for $\mathcal{F}(G)$.

The Isomorphism Conjecture of Farrell and Jones appeared in [10, 1.6]. For a survey of this conjecture see [22].

Theorem 0.1 (Induction theorem for algebraic K-theory). *Let G be a group and let N be an integer. Then the following hold.*

(a) The group G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}'$.

(b) Let p be a prime. Then G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$ if and only if G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}_p'$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$.

(c) *Suppose that R is regular and $\mathbb{Q} \subseteq R$. Then the group G satisfies the Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}$.*

   If we assume that R is regular and $\mathbb{C} \subseteq R$, then we can replace $\mathcal{H}$ by $\mathcal{E}$.

(d) *Suppose that R is regular and $\mathbb{Q} \subseteq R$. Let p be a prime. Then G satisfies the Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$ if and only if G satisfies the Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{H}_p$ after applying $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$.*

   If we assume that R is regular and $\mathbb{C} \subseteq R$, then we can replace $\mathcal{H}_p$ by $\mathcal{E}_p$.

Note that in (c) and (d) above there are no claims about the Fibered Isomorphism Conjecture. The problem is that we do not know whether the relevant Nil-groups for amalgamated products and HNN-extensions with rational coefficients vanish for all groups. If the group $\mathbb{Z}$ satisfies the Fibered Isomorphism Conjecture for algebraic K-theory with coefficients in $\mathbb{Q}$ for the trivial family, then the Nil groups $NK_*(\mathbb{Q}\Gamma)$ vanishes for all groups Γ. It seems to be not known whether these Nil groups vanish. (Of course for the group $\mathbb{Z}$ the families $\mathcal{H}$, $\mathcal{H}_p$ and $\mathcal{FIN}$ are all trivial.)

Remark 0.2 (Relative assembly maps). We have stated the main theorems in terms of the (Fibered) Isomorphism Conjectures. The main new result we prove is that the relevant relative assembly maps are bijective. This statement is true in general and is independent of the question whether the Isomorphism Conjecture is true or not but implies our main results.

For instance, the assembly map for the family $\mathcal{H}'$ factorizes as

$$\mathcal{H}_n(E_{\mathcal{H}'}G) \to \mathcal{H}_n(E_{\mathcal{FCY}}G) \to \mathcal{H}_n^G(\{\mathrm{pt}\,.\}) = K_n(RG)$$

where $\mathcal{H}_*^G$ is a certain G-homology theory related to algebraic K-theory of group rings with coefficients in a ring R, the first map is a relative assembly map which we will prove is bijective, and the second map is the assembly map for $\mathcal{VCY}$. The Isomorphism Conjecture for algebraic K-theory for $\mathcal{H}'$ or $\mathcal{VCY}$ respectively says that the assembly map for $\mathcal{H}'$ or $\mathcal{VCY}$ respectively is bijective for $n \in \mathbb{Z}$. The same remark applies to the L-theory version below.

Theorem 0.3 (Induction theorem for algebraic L-theory). *Let G be a group. Then the following hold.*

(a) *The group G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic L-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the (Fibered) Isomorphism Conjecture for algebraic L-theory with coefficients in R for the family $\left(\mathcal{H}_2 \cup \bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p\right)'$.*

(b) *The group G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic L-theory with coefficients in R for the family $\mathcal{VCY}$ after applying $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} -$ if and only if G satisfies the (Fibered) Isomorphism Conjecture for algebraic L-theory with coefficients in R for the family $\bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p$ after applying $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} -$.*

Remark 0.4 (Rationalized versions). We omit the discussion of the rationalized versions, i.e. the versions obtained after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$. In this case one gets more precise information as discussed in detail in [22, Section 8]. The results presented there are based on [1], [13], [17] and [19].

The next result is due to Mislin and Matthey [23] for the complex case. We will give a proof for both the complex and the real case in our framework. It is not clear to us whether it is possible to extend the methods of [23] to the real case.

Theorem 0.5 (Induction theorem for topological K-theory). *Let G be a group.*

Then the relative assembly map

$$K_n^G(E_{\mathcal{FCY}}G) \to K_n^G(E_{\mathcal{FIN}}G)$$

is bijective for all $n \in \mathbb{Z}$.

In particular G satisfies the Baum-Connes Conjecture if and only if G satisfies the Baum-Connes Conjecture for the family $\mathcal{FCY}$.

The corresponding statements are also true if one replaces complex equivariant K-homology K_*^G by real equivariant K-homology KO_*^G and uses the real reduced group C^*-algebra in the Baum-Connes Conjecture.

The rational version of the Induction theorem for topological real K-theory has been used by Stolz [29, p.695]. Using Artin Induction [28, Theorem 26 on page 97] the methods we use to proof the Induction Theorem for algebraic K-theory can also be used to give a simpler proof of the rational version of Theorem 0.5.

The paper is organized as follows

1. Transitivity Principles

In this section we fix an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules for a commutative associative ring Λ with unit in the sense of [19, Section 1]. This essentially means that we get for each group G a G-homology theory $\mathcal{H}^G_*$ which assigns to a (not necessarily proper or cocompact) pair of G-CW-complexes (X, A) a $\mathbb{Z}$-graded Λ-module $\mathcal{H}^G_n(X, A)$, such that there exists natural long exact sequences of pairs and G-homotopy invariance, excision, and the disjoint union axiom are satisfied. Moreover, an induction structure is required which in particular implies for a subgroup $H \subseteq G$ and a H-CW-pair (X, A) that there is a natural isomorphism $\mathcal{H}^H_n(X, A) \xrightarrow{\cong} \mathcal{H}^G_n(G \times_H (X, A))$.

Recall that a family $\mathcal{F}$ of subgroups of G is a set of subgroups which is closed under conjugation and taking subgroups. Examples are the families $\mathcal{FIN}(G)$ of finite subgroups and $\mathcal{VCY}(G)$ of virtually cyclic subgroups. Given a group homomorphism $\phi\colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family $\phi^*\mathcal{F}$ of subgroups of K by

$$\phi^*\mathcal{F} \ := \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}.$$

If ϕ is an inclusion of subgroups, we also write

$$\phi^*\mathcal{F} = K \cap \mathcal{F} \ = \ \{H \subseteq K \mid H \in \mathcal{F}\} \ = \ \{L \cap K \mid L \in \mathcal{F}\}.$$

If $\psi\colon H \to K$ is another group homomorphism, then

$$\psi^*(\phi^*\mathcal{F}) \quad = \quad (\phi \circ \psi)^*\mathcal{F}. \tag{1.1}$$

Associated to a family $\mathcal{F}$ there is a G-CW-complex $E_{\mathcal{F}}G$ (unique up to G-homotopy equivalence) with the property that the fixpoint sets $(E_{\mathcal{F}}G)^H$ are contractible for $H \in \mathcal{F}$ and empty for $H \notin \mathcal{F}$. It is called the *classifying space of the family $\mathcal{F}$*. For more information about these spaces we refer for instance to [20], [33, pp.46].

Definition 1.2 ((Fibered) Isomorphism Conjecture). *Fix an equivariant homology theory $\mathcal{H}^?_*$ with values in Λ-modules for a commutative associative ring Λ. A group G together with a family of subgroups $\mathcal{F}$ satisfies the* Isomorphism Conjecture (in the range $\leq N$) *if the projection* $\mathrm{pr}\colon E_{\mathcal{F}}G \to \{\mathrm{pt}.\}$ *induces an isomorphism*

$$\mathcal{H}^G_n(\mathrm{pr})\colon \mathcal{H}^G_n(E_{\mathcal{F}}G) \xrightarrow{\cong} \mathcal{H}^G_n(\{\mathrm{pt}.\})$$

for $n \in \mathbb{Z}$ (with $n \leq N$).

The pair $(G, \mathcal{F})$ satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) *if for each group homomorphism $\phi\colon K \to G$ the pair $(K, \phi^*\mathcal{F})$ satisfies the Isomorphism Conjecture (in the range $\leq N$).*

Built in into the Fibered Isomorphism Conjecture is the following obvious inheritance property which is not true in general in the non-fibered case.

Lemma 1.3. *Let $\phi\colon K \to G$ be a group homomorphism and let $\mathcal{F}$ a family of subgroups. If $(G, \mathcal{F})$ satisfies the Fibered Isomorphism Conjecture 1.2, then $(K, \phi^*\mathcal{F})$ satisfies the Fibered Isomorphism Conjecture 1.2.*

Proof. If $\psi\colon L \to K$ is a group homomorphism, then $\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}$ from (1.1). ∎

We will use several times the following result proved in [22, Theorem 2.9]

Theorem 1.4 (Transitivity Principle for equivariant homology). *Suppose $\mathcal{F} \subset \mathcal{G}$ are two families of subgroups of the group G. Let N be an integer. If for every $H \in \mathcal{G}$ and every $n \leq N$ the map induced by the projection*

$$\mathcal{H}_n^H(E_{\mathcal{F}\cap H}H) \to \mathcal{H}_n^H(\{\mathrm{pt}.\})$$

is an isomorphism, then for every $n \leq N$ the map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}G \to E_{\mathcal{G}}G$

$$\mathcal{H}_n^G(E_{\mathcal{F}}G) \to \mathcal{H}_n^G(E_{\mathcal{G}}G)$$

is an isomorphism.

This implies the following transitivity principle for the (Fibered) Isomorphism Conjecture. At the level of spectra this transitivity principle is due to Farrell and Jones [10, TheoremA.10].

Theorem 1.5 (Transitivity Principle). *Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the (Fibered) Isomorphism Conjecture for $\mathcal{F} \cap H$ (in the range $\leq N$).*

Then $(G, \mathcal{G})$ satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) if and only if $(G, \mathcal{F})$ satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$).

Proof. We first prove the claim for the Fibered Isomorphism Conjecture. Consider a group homomorphism $\phi\colon K \to G$. Then for every subgroup H of K we conclude

$$(\phi|_H)^*(\mathcal{F} \cap \phi(H)) \;=\; (\phi^*\mathcal{F}) \cap H$$

from (1.1), where $\phi|_H\colon H \to \phi(H)$ is the group homomorphism induced by ϕ. For every element $H \in \phi^*\mathcal{G}$ the map

$$\mathcal{H}_p^H(E_{(\phi|_H)^*(\mathcal{F}\cap\phi(H))}H) = \mathcal{H}_p^H(E_{\phi^*\mathcal{F}\cap H}H) \to \mathcal{H}_p^H(\{\mathrm{pt}.\})$$

is bijective for all $p \in \mathbb{Z}$ (with $p \leq N$) by the assumption that that the element $\phi(H) \in \mathcal{G}$ satisfies the Fibered Isomorphism Conjecture for $\mathcal{F} \cap \phi(H)$ (in the range $\leq N$). Hence by Theorem 1.4 applied to the inclusion $\phi^*\mathcal{F} \subseteq \phi^*\mathcal{G}$ of families of subgroups of K we get an isomorphism

$$\mathcal{H}_p^K(E_{\phi^*\mathcal{F}}K) \xrightarrow{\cong} \mathcal{H}_p^K(E_{\phi^*\mathcal{G}}K)$$

(for $p \leq N$). Therefore the map $\mathcal{H}_p^K(E_{\phi^*\mathcal{F}}K) \to \mathcal{H}_p^K(\{\mathrm{pt}.\})$ is bijective for all $p \in \mathbb{Z}$ (with $p \leq N$) if and only if the map $\mathcal{H}_p^K(E_{\phi^*\mathcal{G}}K) \to \mathcal{H}_p^K(\{\mathrm{pt}.\})$ is bijective for all $p \in \mathbb{Z}$ (with $p \leq N$).

If we specialize this argument to $\phi = \mathrm{id}_G$ then only the Isomorphism Conjecture for $\mathcal{F} \cap H$ (in the range $\leq N$) is needed and this proves the claim for the Isomorphism Conjecture. ∎

As an easy application we see that we can always enlarge the family in the Fibered Isomorphism Conjecture. Note that this argument is not valid for the Isomorphism Conjecture.

Lemma 1.6. *Let G be a group and let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that G satisfies the Fibered Isomorphism Conjecture 1.2 (in the range $n \leq N$) for the family $\mathcal{F}$.*

Then G satisfies the Fibered Isomorphism Conjecture 1.2 (in the range $n \leq N$) for the family $\mathcal{G}$.

Proof. We want to use Theorem 1.5. Therefore we have to show for each subgroup K in $\mathcal{G}$ that it satisfies the Fibered Isomorphism Conjecture for $\mathcal{F} \cap K$. If $i \colon K \to G$ is the inclusion, then $i^*\mathcal{F} = \mathcal{F} \cap K$. Now apply Lemma 1.3. □

We have defined in the introduction for every class $\mathcal{F}$ of finite groups the class $\mathcal{F}'$ consisting of those virtually cyclic groups V for which there exists an extension $1 \to \mathbb{Z} \to V \to F \to 1$ for a group $F \in \mathcal{F}$ or for which $V \in \mathcal{F}$ holds.

Theorem 1.7. *Let $\mathcal{F}$ be a class of finite groups closed under isomorphism and taking subgroups. Suppose that every finite group F satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) with respect to the family $\mathcal{F}(F) = \{H \subseteq F \mid H \in \mathcal{F}\}$. Let G be a group.*

Then G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) with respect to the family $\mathcal{F}'(G)$ if and only if G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) with respect to the family $\mathcal{VCY}(G) = \{H \subseteq G \mid H \in \mathcal{VCY}\}$.

Proof. Consider $V \in \mathcal{VCY}$. Because of Theorem 1.5 we have to show that V satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) for the family $\mathcal{F}'(V) = \mathcal{F}'(G) \cap V$. If V is finite, then the claim follows from the assumptions. It remains to treat the case, where V can be written as an extension $1 \to \mathbb{Z} \to V \xrightarrow{p} F \to 1$ for a finite group F.

Since F satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) for $\mathcal{F}$ by assumption, V satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) for $p^*\mathcal{F}$ by Lemma 1.3. Obviously $p^*\mathcal{F}(F) \subseteq \mathcal{F}'(V)$ but in general these two families of subgroups of V do not agree. Because of Lemma 1.6 V satisfies the Fibered Isomorphism Conjecture (in the range $\leq N$) for $\mathcal{F}'(V)$. □

The Fibered Isomorphism Conjecture is also well behaved with respect to finite intersections of families of subgroups.

Lemma 1.8. *Let G be a groups and $\mathcal{F}$ and $\mathcal{G}$ families of subgroups. Suppose that G satisfies the Fibered Isomorphism Conjecture (in the range $n \leq N$) for both $\mathcal{F}$ and $\mathcal{G}$.*

Then G satisfies the Fibered Isomorphism Conjecture (in the range $n \leq N$) for $\mathcal{F} \cup \mathcal{G}$ and the Fibered Isomorphism Conjecture (in the range $n \leq N-1$) for $\mathcal{F} \cap \mathcal{G}$.

Proof. We conclude from Lemma 1.6 that G satisfies the Fibered Isomorphism Conjecture (in the range $n \leq N$) for $\mathcal{F} \cup \mathcal{G}$. Next we prove the claim for $\mathcal{F} \cap \mathcal{G}$.

Consider a group homomorphism $\phi\colon K \to G$. Choose G-CW-models $E_{\mathcal{F}\cap\mathcal{G}}G$, $E_{\mathcal{F}}G$ and $E_{\mathcal{G}}G$ such that $E_{\mathcal{F}\cap\mathcal{G}}G$ is a G-CW-subcomplex of both $E_{\mathcal{F}}G$ and $E_{\mathcal{G}}G$. This can be arranged by mapping cylinder constructions. Define a G-CW-complex

$$X = E_{\mathcal{F}}G \cup_{E_{\mathcal{F}\cap\mathcal{G}}G} E_{\mathcal{G}}G.$$

For any subgroup $H \subseteq G$ we get

$$X^H = (E_{\mathcal{F}}G)^H \cup_{(E_{\mathcal{F}\cap\mathcal{G}}G)^H} (E_{\mathcal{G}}G)^H.$$

If $(E_{\mathcal{F}}G)^H$ and $(E_{\mathcal{G}}G)^H$ are empty, the same is true for X^H. If $(E_{\mathcal{F}}G)^H$ is empty, then $(E_{\mathcal{G}}G)^H = X^H$. If $(E_{\mathcal{G}}G)^H$ is empty, then $(E_{\mathcal{F}}G)^H = X^H$. If $(E_{\mathcal{F}}G)^H$, $(E_{\mathcal{G}}G)^H$ and $(E_{\mathcal{F}\cap\mathcal{G}}G)^H$ are contractible, the same is true for X^H. Hence X is a model for $E_{\mathcal{F}\cup\mathcal{G}}G$. If we apply restriction with ϕ, we get a decomposition of $E_{\phi^*\mathcal{F}\cup\phi^*\mathcal{G}}K = \phi^* E_{\mathcal{F}\cup\mathcal{G}}G$ as the union of $E_{\phi^*\mathcal{F}}K = \phi^* E_{\mathcal{F}}G$ and $E_{\phi^*\mathcal{G}}K = \phi^* E_{\mathcal{G}}G$ such that the intersection of $E_{\phi^*\mathcal{F}}K$ and $E_{\phi^*\mathcal{G}}K$ is $E_{\phi^*(\mathcal{F}\cap\mathcal{G})}K = \phi^* E_{\mathcal{F}\cap\mathcal{G}}G$. Now the claim follows from the Mayer-Vietoris sequence for $E_{\phi^*\mathcal{F}\cup\phi^*\mathcal{G}}K$ and the Five-Lemma. $\square$

2. General Induction Theorems

Let G be a finite group. Let G-FSETS be the category of finite G-sets. Morphisms are G-maps. Let Λ be an associative commutative ring with unit. Denote by Λ-MOD the abelian category of Λ-modules. A *bi-functor* M from G-FSETS to Λ-MOD consists of a covariant functor

$$M_*\colon G\text{-}\mathsf{FSETS} \to \Lambda\text{-}\mathsf{MOD}$$

and a contravariant functor

$$M^*\colon G\text{-}\mathsf{FSETS} \to \Lambda\text{-}\mathsf{MOD}$$

which agree on objects.

Definition 2.1 (Mackey functor). *A* Mackey functor M for G with values in Λ-modules *is a bifunctor from G-FSETS to Λ-MOD such that*

- *Double Coset formula*

  For any cartesian square of finite G-sets

$$\begin{array}{ccc} S & \xrightarrow{\overline{f_1}} & S_1 \\ {\scriptstyle \overline{f_2}}\downarrow & & \downarrow{\scriptstyle f_1} \\ S_2 & \xrightarrow[f_2]{} & S_0 \end{array}$$

the diagram

$$\begin{array}{ccc} M(S) & \xrightarrow{M_*(\overline{f_1})} & M(S_1) \\ {\scriptstyle M^*(\overline{f_2})}\Big\uparrow & & {\scriptstyle M^*(f_1)}\Big\uparrow \\ M(S_2) & \xrightarrow{M_*(f_2)} & M(S_0) \end{array}$$

commutes.

- *Additivity*

  Consider two finite G-sets S_0 and S_1. Let $i_k \colon S_k \to S_0 \amalg S_1$ for $k = 0, 1$ be the inclusion. Then the map

  $$M^*(i_0) \times M^*(i_1) \colon M(S_0 \amalg S_1) \to M(S_0) \times M(S_1)$$

  is bijective.

One easily checks that the condition Additivity is equivalent to the requirement that

$$M_*(i_0) \oplus M_*(i_1) \colon M(S_0) \oplus M(S_1) \to M(S_0 \amalg S_1)$$

is bijective since the double coset formula implies that $(M^*(i_0) \times M^*(i_1)) \circ (M_*(i_0) \oplus M_*(i_1))$ is the identity.

Let M be a Mackey functor. Let S be a finite non-empty G-set. Then we get another Mackey functor M_S by $(M_S)_*(T) = M_*(S \times T)$ and $(M_S)^*(T) = M^*(S \times T)$. The projection $\mathrm{pr} \colon S \times T \to T$ defines natural transformations of bifunctors from from G-FSETS to Λ-MOD

$$\begin{aligned} \theta_S \colon M_S \to M, \qquad & \theta_S(T) = M_*(\mathrm{pr}); \\ \theta^S \colon M \to M_S, \qquad & \theta^S(T) = M^*(\mathrm{pr}). \end{aligned}$$

We call M *S-projective* if θ_S is split surjective as a natural transformation of bifunctors. We call M *S-injective* if θ^S is split injective as a natural transformation of bifunctors.

Given a Mackey functor M, we can associate to it the Λ-chain complex

$$\ldots \xrightarrow{c_4} M(S^3) \xrightarrow{c_3} M(S^2) \xrightarrow{c_2} M(S^1) \xrightarrow{c_1} M(S^0) \to 0 \to 0 \to \ldots \tag{2.2}$$

and the Λ-cochain complex

$$\ldots \xleftarrow{c^3} M(S^3) \xleftarrow{c^2} M(S^2) \xleftarrow{c^1} M(S^1) \xleftarrow{c^0} M(S^0) \leftarrow 0 \leftarrow 0 \leftarrow \ldots. \tag{2.3}$$

Here $S^n = \prod_{i=1}^n S$ for $n \geq 1$ and $S^0 = G/G$. If $\mathrm{pr}_i^n \colon S^n \to S^{n-1}$ for $i = 1, 2, \ldots n$ is the projection omitting the i-th coordinate, then $c_n = \sum_{i=1}^n (-1)^i \cdot M_*(\mathrm{pr}_i^n)$ and $c^n = \sum_{i=1}^{n+1} (-1)^i \cdot M^*(\mathrm{pr}_i^{n+1})$.

The elementary proof of the following result can be found for instance in [32, Proposition 6.1.3 and Proposition 6.1.6]. The original sources are the papers of Dress [7] and [8].

Lemma 2.4. *Let M be a Mackey functor for the finite group G with values in Λ-modules and let S be a finite G-set. Then*

(a) The following statements are equivalent:

(i) M is S-projective;

(ii) M is S-injective;

(iii) M is a direct summand of M_S as a bifunctor from G-FSETS to Λ-MOD;

(b) M_S is always S-injective and S-projective;

(c) If M is S-projective, then the Λ-chain complex (2.2) *is contractible.*

If M is S-injective, then the Λ-cochain complex (2.3) *is contractible.*

Let Λ be an associative commutative ring with unit. Let Λ-MOD be the abelian category of Λ-modules. For a small category $\mathcal{C}$ let $\Lambda\mathcal{C}$-MOD be the abelian category of contravariant $\Lambda\mathcal{C}$-modules, i.e. objects are contravariant functors $\mathcal{C} \to \Lambda$-MOD and morphisms are natural transformations. In the sequel we use the notation of [18, Sections 9 and 17], where more information about the abelian category $\Lambda\mathcal{C}$-MOD and its homological algebra can be found. If T is a set, then ΛT or $\Lambda(T)$ denote the free Λ-module with the set T as basis. The *orbit category* $\mathsf{Or}(G)$ is the small category whose objects are homogeneous G-spaces G/H and whose morphisms are G-maps.

Lemma 2.5. *Let M be a Mackey functor for the finite group G with values in Λ-modules and let S be a finite non-empty G-set. Suppose that M is S-injective, or, equivalently, that M is S-projective. Let $\mathcal{F}(S)$ be the family of subgroups of $H \subseteq G$ with $S^H \neq \emptyset$. Let $\underline{\Lambda}_{\mathcal{F}(S)}$ be the contravariant $\Lambda\mathsf{Or}(G)$-module which sends G/H to Λ if H belongs to $\mathcal{F}(S)$ and to zero otherwise and which sends a morphism $G/H \to G/K$ to $\mathrm{id}\colon \Lambda \to \Lambda$ if both H and K belong to $\mathcal{F}(S)$ and to the zero map otherwise.*

Then there are natural Λ-isomorphisms

$$\mathrm{Tor}_p^{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, M) \xrightarrow{\cong} \begin{cases} M(G/G) & p = 0; \\ 0 & p \geq 1. \end{cases}$$

$$\mathrm{Ext}^p_{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, M) \xrightarrow{\cong} \begin{cases} M(G/G) & p = 0; \\ 0 & p \geq 1. \end{cases}$$

Proof. Let T be a set, possibly empty. We can assign to it a Λ-chain complex $C_*(T)$ which has as n-th Λ-chain module $\Lambda(T^{n+1})$ and whose n-th differential $c_n\colon \Lambda(T^{n+1}) \to \Lambda(T^n)$ is $\sum_{i=1}^{n+1}(-1)^i \cdot \Lambda(\mathrm{pr}_i^{n+1})$ for $\mathrm{pr}_i^{n+1}\colon T^{n+1} \to T^n$ the projection given by leaving out the i-th coordinate. There are in T natural isomorphisms

$$H_p(C_*(T)) \xrightarrow{\cong} \begin{cases} \Lambda & p = 0 \text{ and } T \neq \emptyset; \\ 0 & p \geq 1 \text{ or } T = \emptyset. \end{cases}$$

Of course the isomorphism $H_0(C_*(T)) \xrightarrow{\cong} \Lambda$ is induced by the augmentation Λ-homomorphism $\Lambda[T] \to \Lambda$ for $T \neq \emptyset$. Fix an element $t_0 \in T$. The maps

$$\Lambda(T^n) \to \Lambda(T^{n+1}), \quad (t_1, t_2, \ldots t_n) \ \mapsto \ -(t_0, t_1, t_2, \ldots t_n)$$

yields the necessary chain homotopies.

Actually, this chain complex belongs to the fat realization of the nerve of the category which has T as set of objects and for which the set of morphisms between two objects consists of exactly one element. Notice that every object t_0 is both initial and terminal.

Next we define a $\Lambda\mathsf{Or}(G)$-chain complex P_* by assigning to G/H the Λ-chain complex $C_*(\mathrm{map}_G(G/H, S))$ for the given finite G-set S. We obtain a natural identification of $\Lambda\mathsf{Or}(G)$-modules

$$H_0(C_*) \ \xrightarrow{\cong} \ \underline{\Lambda}_{\mathcal{F}(S)}.$$

Obviously each $\Lambda\mathsf{Or}(G)$-chain module P_n is finitely generated free and hence in particular projective. Hence P_* is a projective $\Lambda\mathsf{Or}(G)$-resolution of $\underline{\Lambda}_{\mathcal{F}(S)}$. Thus we get by definition

$$\begin{aligned}
\mathrm{Tor}_p^{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, M) &= H_p\left(P_* \otimes_{\Lambda\mathsf{Or}(G)} M_*\right); \\
\mathrm{Ext}^p_{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, M) &= H^p\left(\mathrm{Hom}_{\Lambda\mathsf{Or}(G)}(P_*, M^*)\right).
\end{aligned}$$

There is an obvious identification of Λ-chain complexes $P_* \otimes_{\Lambda\mathsf{Or}(G)} M_*$ with the Λ-chain complex (2.2) if we replace $M(S^0)$ by zero and put $M(S^n)$ as $(n-1)$-th Λ-chain module for $n \geq 1$. Analogously there is an obvious identification of Λ-chain complexes $\mathrm{Hom}_{\Lambda\mathsf{Or}(G)}(P_*, M^*)$ with the Λ-cochain complex (2.3) if we replace $M(S^0)$ by zero and put $M(S^n)$ as $(n-1)$-th Λ-cochain module for $n \geq 1$. Now Lemma 2.5 follows from Lemma 2.4 (c) $\square$

Next we deal with the question whether a given Mackey functor is S-projective or, equivalently, S-injective.

Let M, N and L be bi-functors for the finite group G with values in Λ-modules. A pairing

$$M \times N \to L$$

is a family of Λ-bilinear maps

$$\mu(S)\colon M(S) \times N(S) \to L(S), \quad (m, n) \ \mapsto \ \mu(m, n) = m \cdot n$$

indexed by the objects S of G-FSETS such that for every morphism $f\colon S \to T$ in G-FSETS we have

$$\begin{aligned}
L^*(f)(x \cdot y) &= M^*(f)(x) \cdot N^*(f)(y), && x \in M(T), y \in N(T); \\
x \cdot N_*(f)(y) &= L_*(f)(M^*(f)(x) \cdot y), && x \in M(T), y \in N(S); \\
M_*(f)(x) \cdot y &= L_*(f)(x \cdot N^*(f)(y)), && x \in M(S), y \in N(T).
\end{aligned} \tag{2.6}$$

Definition 2.7 (Green functor). *A* Green functor *for the finite group G with values in Λ-modules is a Mackey functor U together with a pairing*

$$\mu\colon U \times U \to U$$

and a choice of elements $1_S \in U(S)$ for each finite G-set S such that for each finite G-set S the pairing $\mu(S)\colon U(S) \times U(S) \to U(S)$ and the element 1_S determine the structure of an associative Λ-algebra with unit on $U(S)$. Moreover, it is required that $U^*(f)(1_T) = 1_S$ for every morphism $f\colon S \to T$ in G-FSETS.

A (left) U-module M is a Mackey functor for the finite group G with values in Λ-modules together with a pairing

$$\nu\colon U \times M \to M$$

such that for every finite G-set S the pairing $\nu(S)\colon U(S) \times M(S) \to M(S)$ defines the structure of a $U(S)$-module on $M(S)$, where 1_S acts as $\mathrm{id}_{M(S)}$.

The proof of the next result can be found for instance in [32, Theorem 6.2.2].

Theorem 2.8 (Criterion for S-projectivity). *Let G be a finite group and let Λ be an associative commutative ring with unit. Let S be a finite G-set. Let U be a Green functor. Then the following statements are equivalent:*

(a) The projection $\mathrm{pr}\colon S \to G/G$ induces an epimorphism $U_*(\mathrm{pr})\colon U(S) \to U(G/G)$;

(b) U is S-injective;

(c) Every U-module is S-injective.

The results of this section have the following application to equivariant homology theories.

Theorem 2.9 (Criterion for induction for G-homology theories). *Let G be a finite group and $\mathcal{F}$ be a family of subgroups of G. Let $\mathcal{H}_*^G$ be a G-homology theory with values in Λ-modules for an associative commutative ring Λ with unit. Suppose that the following conditions are satisfied:*

- *There exists a Green functor U for the finite group G with values in Λ-modules such that the Λ-homomorphism*

$$\bigoplus_{H \in \mathcal{F}} U(\mathrm{pr}_H)\colon \bigoplus_{H \in \mathcal{F}} U(G/H) \to U(G/G)$$

  is surjective, where $\mathrm{pr}_H\colon G/H \to G/G$ is the projection.

- *For every $n \in \mathbb{Z}$ there is a (left) U-module M such that the covariant functor $M_*\colon G$-$\mathsf{FSETS} \to \Lambda$-$\mathsf{MOD}$ is naturally equivalent to the covariant functor*

$$\mathcal{H}_n^G\colon G\text{-}\mathsf{FSETS} \to \Lambda\text{-}\mathsf{MOD}, \quad S \;\mapsto\; \mathcal{H}_n^G(S).$$

Then the projection $\mathrm{pr}\colon E_{\mathcal{F}}G \to G/G$ *induces for all* $n \in \mathbb{Z}$ *a* Λ*-isomorphism*

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}G) \xrightarrow{\cong} \mathcal{H}_n^G(G/G). \tag{2.10}$$

and the canonical map

$$\operatorname{colim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathcal{H}_n^G(G/?) \xrightarrow{\cong} \mathcal{H}_n^G(G/G) \tag{2.11}$$

is bijective, where $\mathsf{Or}_{\mathcal{F}}(G) \subset \mathsf{Or}(G)$ *is the full subcategory of the orbit category whose objects are homogeneous spaces* G/H *with* $H \in \mathcal{F}$.

Proof. Let S be the G-set $\amalg_{H\in\mathcal{F}} G/H$. The first condition together with Lemma 2.4 (a) and Theorem 2.8 implies that M is S-projective. We conclude from the second condition and Lemma 2.5 that there is a canonical Λ-isomorphism

$$\mathrm{Tor}_p^{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, \mathcal{H}_q^G(G/?)) \xrightarrow{\cong} \begin{cases} \mathcal{H}_q^G(G/G) & p = 0; \\ 0 & p \geq 1, \end{cases}$$

for all $q \in \mathbb{Z}$. The cellular $\Lambda\mathsf{Or}(G)$-chain complex of $E_{\mathcal{F}}G$ is a projective $\Lambda\mathsf{Or}(G)$-resolution of $\underline{\Lambda}_{\mathcal{F}(S)}$. Hence $\mathrm{Tor}_p^{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, \mathcal{H}_q^G(G/?)$ agrees with the Bredon homology $H_p^{\Lambda\mathsf{Or}(G)}(E_{\mathcal{F}}G; \mathcal{H}_q^G(G/?))$. But this is exactly the E^2-term in the equivariant Atiyah-Hirzebruch spectral sequence which converges to $\mathcal{H}_{p+q}^G(E_{\mathcal{F}}G)$. This implies that the E^2-term is concentrated on the y-axis and the spectral sequence collapses. Hence the edge homomorphism yields an isomorphism

$$\mathcal{H}_n^G(E_{\mathcal{F}}G) \xrightarrow{\cong} \mathcal{H}_n^G(G/G)$$

for all $n \in \mathbb{Z}$ which can easily be identified with the Λ-map $\mathcal{H}_n^G(\mathrm{pr})$. There is a natural identification

$$\mathrm{Tor}_0^{\Lambda\mathsf{Or}_{\mathcal{F}}(G)}(\underline{\Lambda}_{\mathcal{F}(S)}, \mathcal{H}_q^G(G/?)) \xrightarrow{\cong} \operatorname{colim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathcal{H}_q^G(G/?).$$

This finishes the proof of Theorem 2.9. □

Remark 2.12. The bijectivity of the map (2.11) is a consequence of the exactness of the complex (2.2) at $M(S^0)$ and $M(S^1)$. It is not hard to see that this map factors as

$$\operatorname{colim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathcal{H}_n^G(G/?) \to \mathcal{H}_n^G(E_{\mathcal{F}}G) \to \mathcal{H}_n^G(G/G)$$

and therefore (2.10) is onto. In order to get our applications to assembly maps it is important that (2.10) is in fact an isomorphism. (Surjectivity alone is not helpful, since the Transitivity Principle 1.5 does not apply to surjections.) The proof of the bijectivity of (2.10) is based on the result due to Dress that the complex (2.2) is not only exact at $M(S^0)$ and $M(S^1)$ but is contractible.

The dual version of the map (2.11) leads to the following induction result in [9] that for a group Γ (in their setting a Bieberbach group) which can be written as an extension $1 \to \Gamma_0 \to \Gamma \xrightarrow{p} G \to 1$ for a finite group G the map

$$L_n(\mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \lim L_n(\mathbb{Z}[p^{-1}(C)]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is bijective, where the inverse limit runs over the cyclic subgroups of G. We expect that there is a dual version of (2.10) which leads to kind of dual version of induction theorems for groups mapping surjectively to finite groups.

3. The Swan Group as a Functor on Groupoids

In this section we will construct Green functors and modules over them that will be used in the application of Theorem 2.9 in the proof of Theorem 0.1 in Section 4.

Let $\mathcal{G}$ be a small groupoid. Our main example is the *transport groupoid* $\mathcal{G}^G(S)$ of a G-set S. The set of objects is given by the set S itself. The set of morphisms from s_1 to s_2 consists of those elements $g \in G$ which satisfy $gs_1 = s_2$. Composition comes from the multiplication in G. A G-map $f \colon S \to T$ induces a functor $\mathcal{G}^G(f) \colon \mathcal{G}^G(S) \to \mathcal{G}^G(T)$. Let R be an associative ring with unit. Denote by $R\mathcal{G}$-FGPMOD the category of contravariant functors from $\mathcal{G}$ to finitely generated projective R-modules and by $K_n(R\mathcal{G})$ its K-theory. Let $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGMOD})$ and $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$ respectively be the category of contravariant functors from $\mathcal{G}$ to the category $\mathbb{Z}$-FGMOD of finitely generated $\mathbb{Z}$-modules and to the category $\mathbb{Z}$-FGFMOD of finitely generated free $\mathbb{Z}$-modules respectively. Let $\mathrm{Sw}(\mathcal{G})$ and $\mathrm{Sw}^f(\mathcal{G})$ respectively be the K_0-group of $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGMOD})$ and $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$ respectively. The forgetful map

$$\mathrm{Sw}^f(\mathcal{G}) \xrightarrow{\cong} \mathrm{Sw}(\mathcal{G})$$

is a bijection. This is proved for groups in [25, page 890] and carries easily over to groupoids. We will concentrate our discussion mostly on $\mathrm{Sw}^f(\mathcal{G})$.

Given a contravariant $\mathbb{Z}\mathcal{G}$-module M and a contravariant $R\mathcal{G}$-module N, let $M \otimes_{\mathbb{Z}} N$ be the contravariant $R\mathcal{G}$-module which assigns to an object c the R-module $M(c) \otimes_{\mathbb{Z}} N(c)$. If M belongs to $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$, then the functor $M \otimes_{\mathbb{Z}} -$ is exact. If M and N belong to $\operatorname{func}(\mathcal{C}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$, then $M \otimes_{\mathbb{Z}} N$ belongs to $\operatorname{func}(\mathcal{C}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$. Hence $\otimes_{\mathbb{Z}}$ induce a pairing

$$\mathrm{Sw}^f(\mathcal{G}) \times \mathrm{Sw}^f(\mathcal{G}) \to \mathrm{Sw}^f(\mathcal{G}). \tag{3.1}$$

With this pairing $\mathrm{Sw}^f(\mathcal{G})$ is a commutative associative ring with the class of the constant contravariant $\mathbb{Z}\mathcal{C}$-module with value $\mathbb{Z}$ as unit.

If M belongs to $\operatorname{func}(\mathcal{G}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$, then $M \otimes_{\mathbb{Z}} -$ sends finitely generated projective $R\mathcal{G}$-modules to finitely generated projective $R\mathcal{G}$-modules. This is well-known if the groupoid $\mathcal{C}$ has only one object, i.e. in case of group rings, and hence holds also for a groupoid $\mathcal{G}$. Hence we get for a groupoid a pairing

$$\mathrm{Sw}^f(\mathcal{G}) \times K_n(R\mathcal{G}) \to K_n(R\mathcal{G}) \tag{3.2}$$

which turns $K_n(R\mathcal{G})$ into a $\mathrm{Sw}^f(\mathcal{G})$-module.

Given a functor $F \colon \mathcal{C} \to \mathcal{D}$, induction defines an functor

$$RF_* \colon R\mathcal{C}\text{-}\mathsf{MOD} \to R\mathcal{D}\text{-}\mathsf{MOD}$$

and restriction defines a functor

$$RF^*\colon R\mathcal{D}\text{-}\mathsf{MOD} \to R\mathcal{C}\text{-}\mathsf{MOD}.$$

The induction functor is given by $- \otimes_{R\mathcal{C}} R\operatorname{mor}_{\mathcal{D}}(??, F(?))$ and restriction by $- \otimes_{R\mathcal{D}} R\operatorname{mor}_{\mathcal{D}}(F(?), ??)$ for the $R\mathcal{C}$-$R\mathcal{D}$-bimodule $R\operatorname{mor}_{\mathcal{D}}(??, F(?))$ and the $R\mathcal{D}$-$R\mathcal{C}$-bimodule $R\operatorname{mor}_{\mathcal{D}}(F(?), ??)$, where ? runs through the objects in $\mathcal{C}$ and ?? through the objects in $\mathcal{D}$ (see [18, 9.15 on page 166]). We have the obvious equalities $(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_*$ and $(F_2 \circ F_1)^* = (F_1)^* \circ (F_2)^*$ for functors $F_1\colon \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2\colon \mathcal{C}_2 \to \mathcal{C}_3$.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of groupoids. Induction with F sends finitely generated projective $R\mathcal{C}$-modules to finitely generated projective $R\mathcal{D}$-modules and respects direct sums. Hence induction induces for $n \in \mathbb{Z}$ homomorphisms of abelian groups

$$F_*\colon K_n(R\mathcal{C}) \to K_n(R\mathcal{D}).$$

Restriction with F yield exact functors $\operatorname{func}(\mathcal{D}, \mathbb{Z}\text{-}\mathsf{FGFMOD}) \to \operatorname{func}(\mathcal{C}, \mathbb{Z}\text{-}\mathsf{FGFMOD})$ and $\operatorname{func}(\mathcal{D}, \mathbb{Z}\text{-}\mathsf{FGMOD}) \to \operatorname{func}(\mathcal{C}, \mathbb{Z}\text{-}\mathsf{FGMOD})$ and thus ring homomorphisms

$$\begin{array}{rcl} F^*\colon \operatorname{Sw}^f(\mathcal{D}) & \to & \operatorname{Sw}^f(\mathcal{C}); \\ F^*\colon \operatorname{Sw}(\mathcal{D}) & \to & \operatorname{Sw}(\mathcal{C}). \end{array}$$

We call F *admissible* if for each object $c \in \mathcal{C}$ the group homomorphism $\operatorname{aut}_{\mathcal{C}}(c) \to \operatorname{aut}_{\mathcal{D}}(F(c))$ induced by F is injective and its image has finite index and the map $\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ has the property that the preimage of any element in $\pi_0(\mathcal{D})$ is finite. Note that if G is a (not necessary finite) group and $f : S \to T$ is a map of finite G-sets then $\mathcal{G}^G(f)$ is admissible. For admissible F induction and restriction do also induce homomorphisms of abelian groups

$$\begin{array}{rcl} F^*\colon K_n(R\mathcal{D}) & \to & K_n(R\mathcal{C}); \\ F_*\colon \operatorname{Sw}^f(\mathcal{C}) & \to & \operatorname{Sw}^f(\mathcal{D}); \\ F_*\colon \operatorname{Sw}(\mathcal{C}) & \to & \operatorname{Sw}(\mathcal{D}). \end{array}$$

The various claims above are well-known for groups, i.e. groupoids with one object and therefore carry easily over to groupoids.

Let $E, F\colon \mathcal{C} \to \mathcal{D}$ be functors which are naturally equivalent. Then we get the following equalities of homomorphisms:

$$\begin{array}{rcl} E^* = F^*\colon \operatorname{Sw}^f(\mathcal{D}) & \to & \operatorname{Sw}^f(\mathcal{C}); \\ E^* = F^*\colon \operatorname{Sw}(\mathcal{D}) & \to & \operatorname{Sw}(\mathcal{C}); \\ E_* = F_*\colon K_n(R\mathcal{C}) & \to & K_n(R\mathcal{D}) \end{array}$$

and E is admissible if and only if F is and in this case also the following homomorphisms agree

$$\begin{array}{rcl} E_* = F_*\colon \operatorname{Sw}^f(\mathcal{C}) & \to & \operatorname{Sw}^f(\mathcal{D}); \\ E^* = F^*\colon \operatorname{Sw}(\mathcal{C}) & \to & \operatorname{Sw}(\mathcal{D}); \\ E^* = F^*\colon K_n(R\mathcal{D}) & \to & K_n(R\mathcal{C}). \end{array}$$

Let $\mathcal{G}$ be a groupoid with a finite set $\pi_0(\mathcal{G})$ of components. For a component $\mathcal{C} \in \pi_0(\mathcal{G})$ let $i_{\mathcal{C}} : \mathcal{C} \to \mathcal{G}$ be the inclusion. Then we obtain to one another inverse isomorphisms

$$\begin{aligned}
\bigoplus_{\mathcal{C}\in\pi_0(\mathcal{G})} (i_{\mathcal{C}})_* \colon \bigoplus_{\mathcal{C}\in\pi_0(\mathcal{G})} K_n(R\mathcal{C}) &\xrightarrow{\cong} K_n(R\mathcal{G});\\
\prod_{\mathcal{C}\in\pi_0(\mathcal{G})} (i_{\mathcal{C}})^* \colon K_n(R\mathcal{G}) &\xrightarrow{\cong} \prod_{\mathcal{C}\in\pi_0(\mathcal{G})} K_n(R\mathcal{C})
\end{aligned}$$

and to one another inverse isomorphisms

$$\begin{aligned}
\bigoplus_{\mathcal{C}\in\pi_0(\mathcal{G})} (i_{\mathcal{C}})_* \colon \bigoplus_{\mathcal{C}\in\pi_0(\mathcal{G})} \mathrm{Sw}^f(\mathcal{C}) &\xrightarrow{\cong} \mathrm{Sw}^f(\mathcal{G});\\
\prod_{\mathcal{C}\in\pi_0(\mathcal{G})} (i_{\mathcal{C}})^* \colon \mathrm{Sw}^f(\mathcal{G}) &\xrightarrow{\cong} \prod_{\mathcal{C}\in\pi_0(\mathcal{G})} \mathrm{Sw}^f(\mathcal{C})
\end{aligned}$$

and similarly for Sw. If $\mathcal{G}$ is a connected groupoid and x an object in $\mathcal{G}$, then the inclusion $i\colon \mathrm{aut}_{\mathcal{G}}(x) \to \mathcal{G}$ induces to another inverse isomorphisms

$$\begin{aligned}
i_* \colon K_n(R[\mathrm{aut}_{\mathcal{G}}(x)]) &\xrightarrow{\cong} K_n(R\mathcal{G});\\
i^* \colon K_n(R\mathcal{G}) &\xrightarrow{\cong} K_n(R[\mathrm{aut}_{\mathcal{G}}(x)]),
\end{aligned}$$

and analogously for Sw^f and Sw.

Lemma 3.3. *Consider a cartesian square of G-sets*

$$\begin{tikzcd}
S \arrow[r, "\overline{f_1}"] \arrow[d, "\overline{f_2}"'] & S_1 \arrow[d, "f_1"] \\
S_2 \arrow[r, "f_2"'] & S_0
\end{tikzcd}$$

Then the following diagram of functors commutes up to natural equivalence

$$\begin{tikzcd}
R\mathcal{G}^G(S)\text{-}\mathsf{MOD} \arrow[r, "R\mathcal{G}^g(\overline{f_1})_*"] & R\mathcal{G}^G(S_1)\text{-}\mathsf{MOD} \\
R\mathcal{G}^G(S_2)\text{-}\mathsf{MOD} \arrow[u, "R\mathcal{G}^G(\overline{f_2})^*"] \arrow[r, "R\mathcal{G}^g(f_2)_*"] & R\mathcal{G}^G(S_0)\text{-}\mathsf{MOD} \arrow[u, "R\mathcal{G}^G(f_1)^*"']
\end{tikzcd}$$

Proof. The composition

$$R\mathcal{G}^G(S_2)\text{-}\mathsf{MOD} \xrightarrow{R\mathcal{G}^g(f_2)_*} R\mathcal{G}^G(S_0)\text{-}\mathsf{MOD} \xrightarrow{R\mathcal{G}^G(f_1)^*} R\mathcal{G}^G(S_1)\text{-}\mathsf{MOD}$$

is given by

$$- \otimes_{R\mathcal{G}^G(S_2)} R\,\mathrm{mor}_{\mathcal{G}^G(S_0)}(!, \mathcal{G}^G(f_2)(??)) \otimes_{R\mathcal{G}^G(S_0)} R\,\mathrm{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), !),$$

where ?, ?? and ! run through objects in $\mathcal{G}^G(S_1)$, $\mathcal{G}^G(S_2)$ and $\mathcal{G}^G(S_0)$. The composition

$$R\mathcal{G}^G(S_2)\text{-}\mathsf{MOD} \xrightarrow{R\mathcal{G}^g(\overline{f_2})^*} R\mathcal{G}^G(S)\text{-}\mathsf{MOD} \xrightarrow{R\mathcal{G}^G(\overline{f_1})_*} R\mathcal{G}^G(S_1)\text{-}\mathsf{MOD}$$

is given by

$$- \otimes_{R\mathcal{G}^G(S_2)} R\operatorname{mor}_{\mathcal{G}^G(S_2)}(\mathcal{G}^G(\overline{f_2})(!!), ??) \otimes_{R\mathcal{G}^G(S)} R\operatorname{mor}_{\mathcal{G}^G(S_1)}(?, \mathcal{G}^G(\overline{f_1})(!!)),$$

where ?, ?? and !! run through objects in $\mathcal{G}^G(S_1)$, $\mathcal{G}^G(S_2)$ and $\mathcal{G}^G(S)$. Hence it suffices to construct an isomorphism of $R\mathcal{G}^G(S_2)$-$R\mathcal{G}^G(S_1)$-bimodules

$$\begin{aligned} &R\operatorname{mor}_{\mathcal{G}^G(S_0)}(!, \mathcal{G}^G(f_2)(??)) \otimes_{R\mathcal{G}^G(S_0)} R\operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), !) \\ &\qquad \xrightarrow{\cong} R\operatorname{mor}_{\mathcal{G}^G(S_2)}(\mathcal{G}^G(\overline{f_2})(!!), ??) \otimes_{R\mathcal{G}^G(S)} R\operatorname{mor}_{\mathcal{G}^G(S_1)}(?, \mathcal{G}^G(\overline{f_1})(!!)). \end{aligned}$$

For this purpose it suffices to construct an isomorphism of $\mathcal{G}^G(S_2)$-$\mathcal{G}^G(S_1)$-bisets

$$\begin{aligned} &\operatorname{mor}_{\mathcal{G}^G(S_0)}(!, \mathcal{G}^G(f_2)(??)) \otimes_{\mathcal{G}^G(S_0)} \operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), !) \\ &\qquad \xrightarrow{\cong} \operatorname{mor}_{\mathcal{G}^G(S_2)}(\mathcal{G}^G(\overline{f_2})(!!), ??) \otimes_{\mathcal{G}^G(S)} \operatorname{mor}_{\mathcal{G}^G(S_1)}(?, \mathcal{G}^G(\overline{f_1})(!!)) \end{aligned} \tag{3.4}$$

where $\otimes_{\mathcal{G}^G(S)}$ is now to be understood with respect to the category of sets. There is an obvious bijection of $\mathcal{G}^G(S_2)$-$\mathcal{G}^G(S_1)$-bisets

$$\begin{aligned} &\operatorname{mor}_{\mathcal{G}^G(S_0)}(!, \mathcal{G}^G(f_2)(??)) \otimes_{\mathcal{G}^G(S_0)} \operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), !) \\ &\qquad\qquad \xrightarrow{\cong} \operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), \mathcal{G}^G(f_2)(??)) \end{aligned}$$

which sends $u \otimes v$ to $u \circ v$. Its inverse sends $w\colon \mathcal{G}^G(f_1)(?) \to \mathcal{G}^G(\overline{f_2})(??)$ to $\operatorname{id}_{\mathcal{G}^G(\overline{f_2})(??)} \otimes w$. There is a map of $\mathcal{G}^G(S_2)$-$\mathcal{G}^G(S_1)$-bisets

$$\begin{aligned} &\operatorname{mor}_{\mathcal{G}^G(S_2)}(\mathcal{G}^G(\overline{f_2})(!!), ??) \otimes_{\mathcal{G}^G(S)} \operatorname{mor}_{\mathcal{G}^G(S_1)}(?, \mathcal{G}^G(\overline{f_1})(!!)) \\ &\qquad\qquad \xrightarrow{\cong} \operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), \mathcal{G}^G(f_2)(??)) \end{aligned}$$

which sends $u \otimes v$ to $\mathcal{G}^G(f_2)(u) \circ \mathcal{G}^G(f_1)(v)$. This makes sense, since $\mathcal{G}^G(f_1) \circ \mathcal{G}^G(\overline{f_1})$ and $\mathcal{G}^G(f_2) \circ \mathcal{G}^G(\overline{f_2})$ coincide. In order to show that this map is a bijection of bisets, we need the assumption that the square of G-sets appearing in Lemma 3.3 is cartesian. Namely, we construct the inverse

$$\begin{aligned} &\operatorname{mor}_{\mathcal{G}^G(S_0)}(\mathcal{G}^G(f_1)(?), \mathcal{G}^G(f_2)(??)) \\ &\qquad\qquad \xrightarrow{\cong} \operatorname{mor}_{\mathcal{G}^G(S_2)}(\mathcal{G}^G(\overline{f_2})(!!), ??) \otimes_{\mathcal{G}^G(S)} \operatorname{mor}_{\mathcal{G}^G(S_1)}(?, \mathcal{G}^G(\overline{f_1})(!!)) \end{aligned}$$

as follows. Consider a morphism $w\colon \mathcal{G}^G(f_1)(?) \to \mathcal{G}^G(\overline{f_2})(??)$. It is given by an element $g \in G$ satisfying $g \cdot f_1(?) = f_2(??)$. The element $(g\cdot ?, ??) \in S_1 \times S_2$ defines a unique element $!! \in S$ since $f_1(g\cdot ?) = f_2(??)$. We have $\overline{f_1}(!!) = g\cdot ?$ and $\overline{f_2}(!!) = ??$. The element g defines a morphism $u\colon ? \to \mathcal{G}^G(\overline{f_1})(!!)$. Now define the image of w by $\operatorname{id}_{??} \otimes u$. One easily checks that these two maps of bisets are inverse to one another. The desired map of bisets (3.4) is given by the two bijections of bisets above. This finishes the proof of Lemma 3.3. $\square$

We can now define the Green functors that we will need. Let $\phi : K \to G$ be a group homomorphism whose target is a finite group G. Then $S \mapsto \mathrm{Sw}^f(\mathcal{G}^G(S))$ and $S \mapsto \mathrm{Sw}^f(\mathcal{G}^K(\phi^*S))$ are Green functors. (Here ϕ^*S denotes the finite K-set obtained by restricting the finite G-set with ϕ.) The covariant functorial structure comes from induction and the contravariant functorial structure from restriction with $\mathcal{G}^G(f)$ and $\mathcal{G}^K(\phi^*f)$. The double coset formula for $\mathrm{Sw}^f(\mathcal{G}^G(-))$ follows directly from Lemma 3.3 and for $\mathrm{Sw}^f(\mathcal{G}^K(\phi^*-)$ from Lemma 3.3 using the fact that ϕ^* sends a cartesian square of finite G-sets to a cartesian square of finite K-sets. The required pairings come from (3.1). We leave the verification of the conditions (2.6) to the reader, they follow from certain natural equivalences of functors on the level of $R\mathcal{G}$-modules.

The group homomorphism ϕ induces a functor $\mathcal{G}^\phi(S)\colon \mathcal{G}^K(\phi^*S) \to \mathcal{G}^G(S)$ for every finite F-set which is natural in S. Restriction with it induces a morphisms of Green functors for G

$$\phi^*\colon\ \mathrm{Sw}^f(\mathcal{G}^G(-)) \to \mathrm{Sw}^f(\mathcal{G}^K(\phi^*-).$$

Analogously one constructs a left $\mathrm{Sw}^f(\mathcal{G}^K(\phi^*-)$-module $K_n(\mathcal{G}^K(\phi^*-)$. The Mackey structure on $K_n(\mathcal{G}^K(\phi^*-)$ comes from induction and restriction and the desired pairing from (3.2). Using $\phi^*\colon\ \mathrm{Sw}^f(\mathcal{G}^G(-)) \to \mathrm{Sw}^f(\mathcal{G}^K(\phi^*-)$ we obtain a left $\mathrm{Sw}^f(\mathcal{G}^G(-))$-module structure on $K_n(\mathcal{G}^K(\phi^*-)$.

If $\mathbb{C} \subseteq R$, we can replace $\mathrm{Sw}^f(\mathcal{G})$ by the version $\mathrm{Sw}(\mathbb{C}\mathcal{G})$, where one uses modules over $\mathbb{C}$ instead of $\mathbb{Z}$. If H is a finite group, then $\mathrm{Sw}(\mathbb{C}H)$ agrees with the complex representation ring of H.

4. Proof of the Main Result for Algebraic K-Theory

Let R be an associative ring with unit. Let $\mathcal{H}^?_*(-;\mathbf{K}_R)$ be the equivariant homology theory associated to the covariant functor $\mathbf{K}_R$ from the category $\mathsf{GROUPOIDS}$ of small groupoids to the category of Ω-$\mathsf{SPECTRA}$ which sends a groupoid $\mathcal{G}$ to the (non-connective) algebraic K-theory spectrum of the category of finitely generated projective contravariant $R\mathcal{G}$-modules. Notice that

$$K_n(R\mathcal{G}) = \pi_n(\mathbf{K}_R(\mathcal{G}))$$

and

$$\mathcal{H}^G_n(G/H;\mathbf{K}_R) \;=\; \mathcal{H}^H_n(\{\mathrm{pt}.\};\mathbf{K}_R) \;=\; K_n(RH)$$

holds for all $n \in \mathbb{Z}$. (See [22, Chapter 6] for more details). Recall that the Farrell-Jones Conjecture for algebraic K-theory and a group G says that the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}G \to G/G$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}^G_n(\mathrm{pr};\mathbf{K}_R)\colon \mathcal{H}^G_n(E_{\mathcal{VCY}}G;\mathbf{K}_R) \;\xrightarrow{\cong}\; \mathcal{H}^G_n(G/G;\mathbf{K}_R) = K_n(RG).$$

Let Λ be a commutative associative ring with unit such that Λ is flat as a $\mathbb{Z}$-module, or, equivalently, Λ is torsionfree as an abelian group. Let $\phi\colon K \to G$ be a group homomorphism. Then $X \mapsto \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_*^K(\phi^* X; \mathbf{K}_R)$ defines a G-homology theory, where for a G-CW-complex X we denote by $\phi^* X$ the K-CW-complex obtained from X by restriction with ϕ.

Lemma 4.1. *Let $\mathcal{F}$ be a class of finite groups closed under isomorphism and taking subgroups. Let $\phi\colon K \to G$ be a group homomorphism with a finite group G as target.*

Then the G-homology theory $\Lambda \otimes_{\mathbb{Z}} \mathcal{H}_*^K(\phi^* -; \mathbf{K}_R)$ satisfies the assumptions appearing in Theorem 2.9 for the family $\mathcal{F}(G) = \{H \subseteq G \mid H \in \mathcal{F}\}$ in the following cases:

(a) $\mathcal{F}$ is the class $\mathcal{H}$ of hyperelementary groups and $\Lambda = \mathbb{Z}$;

(b) $\mathcal{F}$ is the class $\mathcal{E}$ of elementary groups and $\Lambda = \mathbb{Z}$, provided $\mathbb{C} \subseteq R$;

(c) For a given prime p the family $\mathcal{F}$ is the class $\mathcal{H}_p$ of p-hyperelementary groups and $\Lambda = \mathbb{Z}_{(p)}$;

(d) For a given prime p the family $\mathcal{F}$ is the class $\mathcal{E}_p$ of p-elementary groups and $\Lambda = \mathbb{Z}_{(p)}$, provided $\mathbb{C} \subseteq R$;

(e) $\mathcal{F}$ is the class $\mathcal{FCY}$ of finite cyclic groups and $\Lambda = \mathbb{Q}$.

Proof. In Section 3 we have constructed the Green functor $\Lambda \otimes \mathrm{Sw}^f(\mathcal{G}^G(-))$ and the $\Lambda \otimes \mathrm{Sw}^f(\mathcal{G}^G(-))$-module $\Lambda \otimes K_n(R\mathcal{G}^K(\phi^* -))$. There is a natural equivalence of covariant functors $G\text{-}\mathsf{FSETS} \to \mathbb{Z} - \mathsf{MOD}$ from $K_n(R\mathcal{G}^K(\phi^* -))$ to $\mathcal{H}_n^G(\phi^* -; \mathbf{K}_R)$. Hence it remains to check the following

(a) The map coming from induction with respect to the various inclusions $H \subseteq G$
$$\bigoplus_{H \in \mathcal{H}} \mathrm{Sw}^f(H) \;\to\; \mathrm{Sw}^f(G)$$
is surjective;

(b) The map coming from induction with respect to the various inclusions $H \subseteq G$
$$\bigoplus_{H \in \mathcal{E}} \mathrm{Sw}(\mathbb{C}H) \;\to\; \mathrm{Sw}(\mathbb{C}G)$$
is surjective;

(c) For a given prime p the map coming from induction with respect to the various inclusions $H \subseteq G$
$$\bigoplus_{H \in \mathcal{H}_p} \mathrm{Sw}^f(H)_{(p)} \;\to\; \mathrm{Sw}^f(G)_{(p)}$$
is surjective;

(d) For a given prime p the map coming from induction with respect to the various inclusions $H \subseteq G$

$$\bigoplus_{H \in \mathcal{E}_p} \operatorname{Sw}(\mathbb{C}H)_{(p)} \;\to\; \operatorname{Sw}(\mathbb{C}G)_{(p)}$$

is surjective;

(e) The map coming from induction with respect to the various inclusions $H \subseteq G$

$$\bigoplus_{H \in \mathcal{FCY}} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Sw}^f(H) \;\to\; \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Sw}^f(G)$$

is surjective.

(a) and (e) are proved by Swan [30, Corollary 4.2] for $\operatorname{Sw}(\mathcal{G}^G(-))$ which is isomorphic to $\operatorname{Sw}^f(\mathcal{G}^G(-))$ by [30, Proposition 1.1]. We conclude from [30, Lemma 4.1] and [31, Section 12] that the torsion elements of $\operatorname{Sw}(G)$ are all nilpotent. Together with (e) and Theorem 2.8 this show that the assumptions of [32, 6.3.3] are satisfied. Thus (c) follows from [32, 6.3.3]. We get (b) and (d) from [28, Theorem 27 and Theorem 28 on page 98], since $\operatorname{Sw}(\mathbb{C}H)$ for a finite group H is the same as the complex representation ring of H. □

Next we give the proof of Theorem 0.1.

Proof. If we combine Theorem 2.9 and Lemma 4.1, then we get for any group homomorphism $\phi\colon K \to G$ whose target G is a finite group that the projection $\mathrm{pr}\colon \phi^* E_{\mathcal{F}(G)}G \to \{\mathrm{pt}.\}$ induces for all $n \in \mathbb{Z}$ bijections

$$\mathrm{id}_\Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n^K(\mathrm{pr}; \mathbf{K}_R) : \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n^K(\phi^* E_{\mathcal{F}(G)}G; \mathbf{K}_R) \;\to\; \Lambda \otimes_{\mathbb{Z}} \mathcal{H}_n^K(\{\mathrm{pt}.\}; \mathbf{K}_R)$$

in the following cases

(a) $\mathcal{F}$ is the class $\mathcal{H}$ of hyperelementary groups and $\Lambda = \mathbb{Z}$;

(b) $\mathcal{F}$ is the class $\mathcal{E}$ of elementary groups and $\Lambda = \mathbb{Z}$, provided that $\mathbb{C} \subseteq R$;

(c) For a given prime p the family $\mathcal{F}$ is the class $\mathcal{H}_p$ of p-hyperelementary groups and $\Lambda = \mathbb{Z}_{(p)}$;

(d) For a given prime p the family $\mathcal{F}$ is the class $\mathcal{E}_p$ of p-elementary groups and $\Lambda = \mathbb{Z}_{(p)}$, provided that $\mathbb{C} \subseteq R$;

(e) $\mathcal{F}$ is the class $\mathcal{FCY}$ of finite cyclic groups and $\Lambda = \mathbb{Q}$.

Since $\phi^* E_{\mathcal{F}(G)}G = E_{\phi^*\mathcal{F}(G)}K$ holds, this shows that for the finite group G and the equivariant homology theory $\Lambda \otimes_{\mathbb{Z}} \mathcal{H}_*^?(-; \mathbf{K}_R)$ the Fibered Isomorphism Conjecture 1.2 for the family $\mathcal{F}(G)$ holds. Now apply Theorem 1.7 and use the fact that the (Fibered) Isomorphism Conjecture in the sense of Definition 1.2 for $\Lambda \otimes_{\mathbb{Z}} \mathcal{H}_*^?(-; \mathbf{K}_R)$ is the same as the original (Fibered) Farrell-Jones Conjecture for algebraic K-theory for group rings with coefficients in R. For the Farrell-Jones Conjecture this is indicated in [6, p.239], a careful proof can be found in [14, Corollary 9.2]. For the Fibered Farrell-Jones Conjecture compare [2, Remark 6.6] and [22, Remark 4.14]. This finishes the proof of assertions (a) and (b) of Theorem 0.1.

For the proof of assertions (c) and (d) use the fact that for regular R with $\mathbb{Q} \subseteq R$ the group G satisfies the (Fibered) Isomorphism Conjecture (in the range $\leq N$) for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if G satisfies the Isomorphism Conjecture for algebraic K-theory with coefficients in R for the family $\mathcal{FIN}$. This is proved in [22, Proposition 2.7]. Now we can reduce from $\mathcal{FIN}$ to $\mathcal{H}$ or to $\mathcal{E}$, provided that $\mathbb{C} \subseteq R$, or after localization at p to $\mathcal{H}_p$ or to $\mathcal{E}_p$, provided that $\mathbb{C} \subseteq R$, by the Transitivity Principle 1.4. $\square$

5. Outline of the Proof of the Main Result for Algebraic L-Theory

We mention that for L-theory we always use the decoration $-\infty$, the Isomorphism Conjecture is not true for the other decorations such as s, h and p (see [12]).

The proof of Theorem 0.3, which is the L-theory version of Theorem 0.1, is analogous. The only difference is that one has to take the involutions into account and replace the functor Sw^f and the results about it due to Swan [30, Corollary 4.2] by its L-theoretic version denoted by $GW(H;R)$ and studied by Dress in [8, Theorem 2].

The reduction from $\mathcal{FIN}$ to $\mathcal{VCY}$ explained in [22, Proposition 2.18] works also in the Fibered case since the relevant UNil-terms for amalgamated products and HNN-extensions vanish for all groups (see [4]).

6. Proof of the Main Result for Topological K-Theory

The proof of Theorem 0.5 will require two ingredients: a Completion Theorem and a Universal Coefficient Theorem for topological K-theory.

In [21, Theorem 6.5] a family version of the Atiyah-Segal Completion Theorem is proved. In the special case, where G is finite, $X = \{\mathrm{pt}.\}$ and the family is the family $\mathcal{FCY}$ of finite cyclic subgroups it yields an isomorphism of pro-groups

$$\{KF_G^*(\mathrm{pr}^{(n)})\}_{n\geq 0}\colon \{KF_G^*(\{\mathrm{pt}.\})\}_{n\geq 0} \xrightarrow{\cong} \{KF_G^*(E_{\mathcal{FCY}}G^{(n)})\}_{n\geq 0}, \quad (6.1)$$

where the first pro-group is given by the constant system with $KF_G^*(\{\mathrm{pt}.\})$ as value, $E_{\mathcal{FCY}}G^{(n)}$ is the n-skeleton of the G-CW-complex $E_{\mathcal{FCY}}G$ and the G-map $\mathrm{pr}^{(n)}\colon E_{\mathcal{FCY}}G^{(n)} \to G/G$ is the projection. This means that for every n

there is $m > n$ and $f_{m,n} : KF^*_G(\{\mathrm{pt}.\}) \to KF^*_G(E_{\mathcal{FCY}}G^{(m)})$ such that

$$\begin{tikzcd}
KF^*_G(E_{\mathcal{FCY}}G^{(n)}) \arrow[d, "\mathrm{pr}^{(n)}"'] & KF^*_G(E_{\mathcal{FCY}}G^{(m)}) \arrow[l] \arrow[d, "\mathrm{pr}^{(n)}"] \\
KF^*_G(\{\mathrm{pt}.\}) & KF^*_G(\{\mathrm{pt}.\}) \arrow[l, "="'] \arrow[ul, "f_{m,n}"]
\end{tikzcd}$$

commutes. The point is that no I-adic completion occurs since the map

$$R_{\mathbb{C}}(G) = KF^0_G(\{\mathrm{pt}.\}) \to \prod_{(C),C \text{ cyclic}} KF^0_H(\{\mathrm{pt}.\}) = R_{\mathbb{C}}(C)$$

is injective and hence I is the zero ideal. Here KF is either complex K-theory ($KF = K$) or real K-theory ($KF = KO$). The complex case occurs already in [15, Theorem 5.1].

Let G be a finite group. In [3] Bökstedt proves Universal Coefficient Theorems that express equivariant K-cohomology in terms of equivariant K-homology. Using S-duality [24, Chapter XVI.7] his results provide also Universal Coefficient Theorems that express equivariant K-homology in terms of equivariant K-cohomology. Let X be a finite G-CW-complex. For complex K-theory Bökstedt's result asserts that there are natural short exact sequences

$$\begin{aligned}
0 \to \mathrm{Ext}^1_{K^*_G(\{\mathrm{pt}.\})}(K^G_*(X), K^G_*(\{\mathrm{pt}.\})) \to K^*_G(X) \\
\to \mathrm{Hom}_{K^*_G(\{\mathrm{pt}.\})}(K^G_*(X), K^G_*(\{\mathrm{pt}.\}) \to 0, \quad (6.2) \\
0 \to \mathrm{Ext}^1_{K^*_G(\{\mathrm{pt}.\})}(K^*_G(X), K^*_G(\{\mathrm{pt}.\})) \to K^G_*(X) \\
\to \mathrm{Hom}_{K^*_G(\{\mathrm{pt}.\})}(K^*_G(X), K^*_G(\{\mathrm{pt}.\}) \to 0. \quad (6.3)
\end{aligned}$$

For KO-theory his results provide spectral sequences

$$E_2^{p,*} = \mathrm{Ext}^p_{KO^*_G(\{\mathrm{pt}.\})}(KO^G_*(X), KO^G_*(\{\mathrm{pt}.\})) \implies KO^*_G(X), \quad (6.4)$$

$$E^2_{p,*} = \mathrm{Ext}^p_{KO^*_G(\{\mathrm{pt}.\})}(KO^*_G(X), KO^*_G(\{\mathrm{pt}.\})) \implies KO^G_*(X) \quad (6.5)$$

such that $E_\infty^{p,*} = 0$ and $E^\infty_{p,*} = 0$ for $p \geq 2$. In [16] the Universal Coefficient Theorem for complex K-theory is generalized to infinite groups and proper G-CW-complexes.

We can now give the proof of Theorem 0.5.

Proof. Because of the Transitivity Principle 1.5 it suffices to show that

$$KF^G_n(\mathrm{pr})\colon KF^G_n(E_{\mathcal{FCY}}G) \xrightarrow{\cong} KF^G_n(\{\mathrm{pt}.\}) \quad (6.6)$$

is bijective for all finite groups G and $n \in \mathbb{Z}$. There exists a G-CW-model for $E_{\mathcal{FCY}}G$ whose skeleta are all finite G-CW-complexes. This follows for instance from the functorial construction in [6, Section 3 and Lemma 7.6] using the fact that $\mathsf{Or}_{\mathcal{F}}(G)$ is a category with finitely many morphisms.

If we apply (6.3) to $\mathrm{pr}^{(n)} \colon E_{\mathcal{FCY}}G^{(n)} \to G/G$ we obtain a map between two short exact sequences. Since $\mathrm{colim}_{n\to\infty}$ is an exact functor, these sequences stay exact if we apply $\mathrm{colim}_{n\to\infty}$. Because of the isomorphism of pro-groups (6.1), we get isomorphism in the first and third term of (6.3) (see for instance [16]). By the Five-Lemma this implies that

$$\mathrm{colim}_{n\to\infty} K_*^G(E_{\mathcal{FCY}}G^{(n-1)}) \to \mathrm{colim}_{n\to\infty} K_*^G(\{\mathrm{pt}\,.\})$$

is bijective. Since K-homology is compatible with colimits, this map can be identified with the map (6.6) for complex K-theory.

For KO-theory we can proceed similarly. The isomorphism of pro-groups (6.1) yields isomorphisms on the colimit of the E^2-Term of the spectral sequence (6.5). Since $\mathrm{colim}_{n\to\infty}$ is an exact functor this yields also isomorphisms on the colimit of the E^∞ term. Because this E^∞ term has only a finite number of lines a simple diagram chase shows that the induced map

$$\mathrm{colim}_{n\to\infty} KO_*^G(E_{\mathcal{FCY}}G^{(n)}) \to \mathrm{colim}_{n\to\infty} KO_*^G(\{\mathrm{pt}\,.\})$$

on the right hand side of (6.5) is an isomorphisms. Since KO-homology is compatible with colimits, this map can be identified with the map (6.6) for KO-theory. $\square$

7. Versions in Terms of Colimits

The classical induction theorems can be stated in terms of colimits. If we combine Theorem 2.9 and Lemma 4.1 and (its L-theory version) we get for an extension $1 \to K \to G \xrightarrow{p} F \to 1$ for a finite group F isomorphisms

$$\begin{aligned}
\mathrm{colim}_{F/L\in \mathrm{Or}_{\mathcal{H}}(F)} K_n(R[p^{-1}(L)]) &\xrightarrow{\cong} K_n(RG);\\
\mathrm{colim}_{F/L\in \mathrm{Or}_{\mathcal{H}_p}(F)} K_n(R[p^{-1}(L)])_{(p)} &\xrightarrow{\cong} K_n(RG)_{(p)};\\
\mathrm{colim}_{F/L\in \mathrm{Or}_{\mathcal{H}_2\cup\bigcup_{p\text{ prime},p\neq 2}\mathcal{E}_p}(F)} L_n^{\langle-\infty\rangle}(R[p^{-1}(L)]) &\xrightarrow{\cong} L_n^{\langle-\infty\rangle}(RG);\\
\mathrm{colim}_{F/L\in \mathrm{Or}_{\bigcup_{p\text{ prime},p\neq 2}\mathcal{E}_p}(F)} L_n(R[p^{-1}(L)])[1/2] &\xrightarrow{\cong} L_n(RG)[1/2].
\end{aligned}$$

But in general our results cannot be stated in such an elementary way. For instance, Theorem 0.5 says for a finite group G that there is an isomorphism

$$K_0^G(E_{\mathcal{FCY}}G) \xrightarrow{\cong} K_0^G(G/G)$$

In general $K_0^G(G/H)$ is the complex representation ring $R_{\mathbb{C}}(H)$ of the finite group H. It is not true in general that the canonical map

$$\mathrm{colim}_{G/H\in \mathrm{Or}_{\mathcal{FCY}}(G)} R_{\mathbb{C}}(H) \;\to\; R_{\mathbb{C}}(G)$$

is bijective or, equivalently, that the canonical map

$$\mathrm{colim}_{G/H\in \mathrm{Or}_{\mathcal{FCY}}(G)} K_0^G(G/H) \;\to\; K_0^G(G/G)$$

is bijective.

In some special cases one can get formulations of our results in terms of colimits.

Theorem 7.1. *(a) The group G satisfies the Isomorphism Conjecture in the range $n \leq -1$ for algebraic K-theory with coefficients in $R = \mathbb{Z}$ for the family $\mathcal{VCY}$ if and only if*

$$K_n(\mathbb{Z}G) = 0 \text{ for } n \leq -2$$

and the canonical map

$$\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{H}}(G)} K_{-1}(\mathbb{Z}H) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G)$$

is bijective;

(b) Suppose that R is regular and $\mathbb{Q} \subseteq R$. Then the group G satisfies the Isomorphism Conjecture in the range $n \leq 0$ for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ if and only if

$$K_n(RG) = 0 \text{ for } n \leq -1$$

and the canonical map

$$\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{H}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is bijective.

If we assume that R is regular and $\mathbb{C} \subseteq R$, then we can replace $\mathcal{H}$ by $\mathcal{E}$.

Proof. (a) It follows from [11] that for a virtually cyclic group V the assembly map

$$H_n^G(E_{\mathcal{FIN}}V; \mathbf{K}_{\mathbb{Z}}) \to K_n(\mathbb{Z}V)$$

is surjective in the range $n \leq -1$. Moreover, this map is known to be injective in all degrees [1], [27]. Theorem 2.9 and Lemma 4.1 imply that for every finite group F the assembly map

$$H_n^F(E_{\mathcal{H}}F; \mathbf{K}_{\mathbb{Z}}) \xrightarrow{\cong} K_n(\mathbb{Z}F)$$

is bijective. Applying the Transitivity Principle for equivariant homology 1.4 twice we conclude that a group G satisfies the Isomorphism Conjecture for algebraic K-theory with coefficients in $R = \mathbb{Z}$ for the family $\mathcal{VCY}$ in the range $n \leq -1$ if and only if the assembly map

$$H_n^G(E_{\mathcal{H}}G; \mathbf{K}_{\mathbb{Z}}) \xrightarrow{\cong} K_n(\mathbb{Z}G)$$

is bijective for $n \leq -1$. It is proven in [5] that for finite groups F and $n \leq -2$ $K_n(\mathbb{Z}F) = 0$. Now a spectral sequence argument shows that $H_n^G(E_{\mathcal{H}}G; \mathbf{K}_{\mathbb{Z}})$ vanishes for $n \leq -2$ and can be identified with $\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{H}}(F)} K_{-1}(\mathbb{Z}H)$ for

$n = -1$.

(b) By Theorem 0.1 (c) the group G satisfies the Isomorphism Conjecture in the range $n \leq 0$ for algebraic K-theory with coefficients in R for the family $\mathcal{VCY}$ in the range $n \leq 0$ if and only if the canonical map

$$H_n^G(E_{\mathcal{H}}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$$

is bijective for $n \leq 0$. Since RH is regular for a finite group H, $K_q(RH) = 0$ for $q \leq -1$. Now a spectral sequence argument shows that $H_n^G(E_{\mathcal{H}}G; \mathbf{K}_R)$ vanishes for $n \leq -1$ and can be identified with $\operatorname{colim}_{G/H \in \mathrm{Or}_{\mathcal{H}}(G)} K_0(RH)$ for $n = 0$. $\square$

8. On Quinn's Hyperelementary Induction Conjecture

Quinn [26] states the following conjecture

Conjecture 8.1 (Hyperelementary Induction Conjecture). *All groups satisfy induction for the class $\overline{\mathcal{H}}$ of (possibly infinite) hyperelementary groups.*

The phrase that a group G satisfies induction for a family $\mathcal{F}$ of subgroups of G means that the Fibered Farrell-Jones Conjecture is true for the family $\mathcal{F}$ and algebraic K-theory with coefficients in a given associative ring with unit R. The Fibered Isomorphisms Conjecture is due to Farrell-Jones [10] (see also [22, Section 4.2.2]). Quinn calls a group H *p-hyperelementary* if there exists a prime p, a finite p-group P and a (possibly infinite) cyclic group C such that H can be written as an extension $1 \to C \to H \to P \to 1$, and hyperelementary if it is p-hyperelementary for some prime p. The class $\overline{\mathcal{H}}$ appearing in the Hyperelementary Induction Conjecture 8.1 is to be understood with respect to Quinn's definition of hyperelementary group as above. Of course $\overline{\mathcal{H}} \cap \mathcal{FIN}$ is the class $\mathcal{H}$ of finite hyperelementary groups. There is an inclusion $\overline{\mathcal{H}} \subseteq \mathcal{H}'$ but the classes $\overline{\mathcal{H}}$ and $\mathcal{H}'$ are different. One would like to prove that Hyperelementary Induction Conjecture 8.1 follows from the Fibered Isomorphism Conjecture for algebraic K-theory. This is in view of Theorem 0.1 (a) and Theorem 1.5 equivalent to a positive answer to the following question:

Question 8.2. *Is for every group $H \in \mathcal{H}'$ the Hyperelementary Induction Conjecture 8.1 true?*

We have no evidence for a positive answer from general machinery, a proof of a positive answer will need some special information about the Nil-groups $NK_n(RH)$.

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