A Basic Introduction to Surgery Theory

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1. The s-cobordism theorem and Whitehead torsion

Theorem 1.1 (s-cobordism theorem) *Let M_0 be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M_0)$. Then*

1. *Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion*

$$\tau(W, M_0) \in \mathrm{Wh}(\pi)$$

vanishes;

2. *The function assigning to an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of h-cobordism over M_0 to the Whitehead group $\mathrm{Wh}(\pi)$.*

Definition 1.2 *An* n-dimensional cobordism $(W; M_0, f_0, M_1, f_1)$ *consists of a compact oriented* n*-dimensional manifold* W*, closed* $(n-1)$*-dimensional manifolds* M_0 *and* M_1*, a disjoint decomposition* $\partial W = \partial_0 W \coprod \partial_1 W$ *of the boundary* ∂W *of* W *and orientation preserving diffeomorphisms* $f_0 : M_0 \to \partial W_0$ *and* $f_1 : M_1^- \to \partial W_1$*.*

We call a cobordism $(W; M_0, f_0, M_1, f_1)$ *an* h-cobordism *if the inclusions* $\partial_i W \to W$ *for* $i = 0, 1$ *are homotopy equivalences.*

Theorem 1.3 (Poincaré conjecture) *The Poincaré Conjecture is true for a closed* n*-dimensional manifold* M *with* $\dim(M) \geq 5$*, namely, if* M *is homotopy equivalent to* S^n*, then* M *is homeomorphic to* S^n*.*

Remark 1.4 The Poincaré Conjecture is not true if one replaces homeomorphic by diffeomorphic.

Remark 1.5 The s-Cobordism Theorem 1.1 is one step in a program to decide whether two closed manifolds M and N are diffeomorphic. This is in general a very hard question. The idea is to construct an h-cobordism $(W; M, f, N, g)$ with vanishing Whitehead torsion and to apply the s-cobordism theorem. So the *surgery program* is:

1. Construct a simple homotopy equivalence $f : M \to N$;

2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id}) : (W; M, N) \to (N \times [0, 1], N \times \{0\}, N \times \{1\})$;

3. Modify W and F relative boundary by so called surgery such that F becomes a homotopy equivalence and thus W becomes an h-cobordism. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial.

In the sequel let W be an n-dimensional manifold for $n \geq 6$ whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$.

Definition 1.6 *The* n-dimensional handle of index q *or briefly* q*-handle is* $D^q \times D^{n-q}$*. Its* core *is* $D^q \times \{0\}$*. The* boundary of the core *is* $S^{q-1} \times \{0\}$*. Its* cocore *is* $\{0\} \times D^{n-q}$ *and its* transverse sphere *is* $\{0\} \times S^{n-q-1}$*.*

Notation 1.7 *If* $\phi^q : S^{q-1} \times D^{n-q-1} \to \partial_1 W$ *is an embedding, then we say that the manifold* $W + (\phi^q)$ *defined by* $W \cup_{\phi^q} D^q \times D^{n-q}$ *is obtained from* W *by attaching a handle of index* q *by* ϕ^q*. Notice that* $\partial_0 W$ *is unchanged. Put*

$$\begin{aligned} \partial_0(W + (\phi^q)) &:= \partial_0 W; \\ \partial_1(W + (\phi^q)) &:= \partial(W + (\phi^q)) - \partial_0 W. \end{aligned}$$

Lemma 1.8 *Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then W possesses a handlebody decomposition relative $\partial_0 W$, i.e. W is up to diffeomorphism relative $\partial_0 W = \partial_0 W \times \{0\}$ of the form*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \ldots + \sum_{i=1}^{p_n} (\phi_i^n),$$

Lemma 1.9 (Cancellation lemma) *Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be an embedding. Let $\psi^{q+1} : S^q \times D^{n-1-q} \to \partial_1(W + (\phi^q))$ be an embedding. Suppose that $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of the handle (ϕ^q) and meets the transverse sphere in exactly one point. Then there is a diffeomorphism relative $\partial_0 W$ from W to $W + (\phi^q) + (\psi^{q+1})$.*

Definition 1.10 *Let* $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ *be the based free* $\mathbb{Z}\pi$*-chain complex whose* q*-th chain group is* $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ *and whose* q*-th differential is given by the composition*

$$H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \xrightarrow{\partial_p} H_q(\widetilde{W_{q-1}})$$

$$\xrightarrow{i_q} H_{q-1}(\widetilde{W_{q-1}}, \widetilde{W_{q-2}}),$$

where ∂_q *is the boundary operator of the long homology sequence associated to the pair* $(\widetilde{W_p}, \widetilde{W_{p-1}})$ *and* i_q *is induced by the inclusion.*

Lemma 1.11 *There is a* CW*-complex* X *such that there is a bijection between the* q*-handles of* W *and the* q*-cells of* X *and a homotopy equivalence* $f : W \to X$ *which respects the filtrations. The cellular* $\mathbb{Z}\pi$*-chain complex* $C_*(\widetilde{X})$ *is based isomorphic to the* $\mathbb{Z}\pi$*-chain complex* $C_*(\widetilde{W})$*.*

Remark 1.12 Notice that one can never get rid of one handle alone, there must always be involved at least two handles simultaneously.

Lemma 1.13 *The following statements are equivalent*

1. *The inclusion* $\partial_0 W \to W$ *is* 1*-connected;*

2. *We can find a diffeomorphism relativ* $\partial_0 W$

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \ldots + \sum_{i=1}^{p_n} (\overline{\phi}_i^n).$$

Lemma 1.14 (Homology lemma) *Suppose $n \geq 6$. Fix $2 \leq q \leq n-3$ and $i_0 \in \{1, 2, \ldots p_q\}$. Let $S^q \to \partial_1 W_q$ be an embedding. Then the following statements are equivalent*

1. *f is isotopic to an embedding $g : S^q \to \partial_1 W_q$ such that g meets the transverse sphere of $(\phi^q_{i_0})$ transversally and in exactly one point and is disjoint from transverse spheres of the handles (ϕ^q_i) for $i \neq i_0$;*

2. *Let $\widetilde{f} : S^q \to \widetilde{W_q}$ be a lift of f under $p|_{\widetilde{W_q}} : \widetilde{W_q} \to W_q$. Let $[\widetilde{f}]$ be the image of the class represented by $\widetilde{f}$ under the obvious composition*

$$\pi_q(\widetilde{W_q}) \to \pi_q(\widetilde{W_q}, \widetilde{W_{q-1}})$$

$$\to H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) = C_q(\widetilde{W}).$$

Then there is $\gamma \in \pi$ with

$$[\widetilde{f}] = \pm\gamma \cdot [\phi^q_{i_0}].$$

Remark 1.15 Notice that in the proof of the implication (2) $\Rightarrow$ (1) of the Homology Lemma 1.14 the Whitney trick comes in and that the Whitney trick forces us to assume $n = \dim(M_0) \geq 5$ in the s-cobordism Theorem 1.1. For $n = 4$ the s-cobordism theorem is false by results of Donaldson in the smooth category and is true for so called good fundamental groups in the topological category by results of Freedman. Counterexamples in dimension $n = 3$ have been constructed by Cappell and Shaneson.

Lemma 1.16 (Normal form lemma) *Let $(W; \partial_0 W, \partial_1 W)$ be an n-dimensional oriented compact h-cobordism for $n \geq 6$. Let q be an integer with $2 \leq q \leq n-3$. Then there is a handlebody decomposition which has only handles of index q and $(q+1)$, i.e. there is a diffeomorphism relative $\partial_0 W$*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^r) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Define the *Whitehead group* $\mathrm{Wh}(\pi)$ as the abelian group of equivalence classes of invertible matrices A of arbitrary size with entries in $\mathbb{Z}\pi$. We call A and B equivalent, if we can pass from A to B by a sequence of the following operations:

1. B is obtained from A by adding the k-th row multiplied with x from the left to the l-th row for $x \in \mathbb{Z}\pi$ and $k \neq l$;

2. B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

3. The inverse to operation (2)

4. B is obtained from A by multiplying the i-th row from the left with an element $\pm\gamma$ for $\gamma \in \pi$;

5. B is obtained from A by interchanging two rows or two columns.

Lemma 1.17 *1. Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact oriented h-cobordism for $n \geq 6$ and A be the matrix defined above. If $[A] = 0$ in* $\mathrm{Wh}(\pi)$*, then the h-cobordism W is trivial relative $\partial_0 W$;*

2. Consider an element $u \in \mathrm{Wh}(\pi)$, a closed oriented manifold M of dimension $n - 1 \geq 5$ with fundamental group π and an integer q with $2 \leq q \leq n-3$. Then we can find an h-cobordism of the shape

$$W = M \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^r) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

such that $[A] = u$.

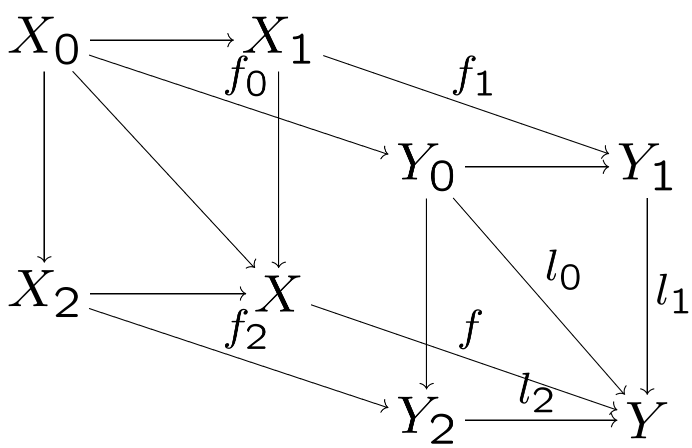
Lemma 1.17 (2) implies the s-Cobordism Theorem Theorem 1.1. Beforehand we have to define the *Whitehead torsion*

$$\tau(f) \in \mathrm{Wh}(\pi_1(Y)$$

of a homotopy equivalence $f : X \to Y$ of finite CW-complexes and to establish its main properties listed below.

Theorem 1.18 *1. Sum formula*

Consider the commutative diagram of finite CW-complexes



such that the back square and the front square are cellular pushouts and f_0, f_1 and f_2 are homotopy equivalences. Then f is a homotopy equivalence and

$$\tau(f) = (l_1)_*\tau(f_1)+(l_2)_*\tau(f_2)-(l_0)_*\tau(f_0);$$

2. *Homotopy invariance*

Let $f \simeq g : X \to Y$ be homotopic. Then $f_* = g_* : \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y))$. If additionally f and g are homotopy equivalences, then

$$\tau(g) = \tau(f);$$

3. *Composition formula*

   Let $f : X \to Y$ *and* $g : Y \to Z$ *be homotopy equivalences of finite* CW*-complexes. Then*

$$\tau(g \circ f) = g_*\tau(f) + \tau(g);$$

4. *Product formula*

   Let $f : X' \to X$ *and* $g : Y' \to Y$ *be homotopy equivalences of connected finite* CW*-complexes. Then*

$$\tau(f \times g) = \chi(X) \cdot j_*\tau(g) + \chi(Y) \cdot i_*\tau(f);$$

5. *Topological invariance*

   Let $f : X \to Y$ *be a homeomorphism of finite* CW*-complexes. Then*

$$\tau(f) = 0.$$

We briefly give the definition of Whitehead torsion. Let $C_*(\widetilde{f}) : C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ be the $\mathbb{Z}\pi$-chain homotopy equivalence induced by the lift $\widetilde{f}$ of f to the universal covering for $\pi = \pi_1(X) = \pi_1(Y)$. Let cone_* be its mapping cone. This is a contractible based free $\mathbb{Z}\pi$-chain complex. Let γ_* be a chain contraction. Then

$$(c + \gamma)_{\mathrm{odd}} : \mathrm{cone}_{\mathrm{odd}} \xrightarrow{\cong} \mathrm{cone}_{\mathrm{ev}}$$

is bijective. Its matrix A is an invertible matrix over $\mathbb{Z}\pi$. Define

$$\tau(f) \; := \; [A] \quad \mathrm{Wh}(\pi). \tag{1.19}$$

Given an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0, we define its Whitehead torsion $\tau(W, M_0)$ by the Whitehead torsion of the inclusion $\partial_0 W \to W$. Notice that we get CW-structures on $\partial_0 W$ and W from any smooth triangulation and the choice of triangulation does not affect the Whitehad torsion. This is the invariant appearing in the s-Cobordism Theorem 1.1 and in Lemma 1.17.

Definition 1.20 *A homotopy equivalence $f : X \to Y$ of finite CW-complexes is called* simple *if $\tau(f) = 0$.*

We have the inclusion of spaces $S^{n-2} \subset S_+^{n-1} \subset S^{n-1} \subset D^n$, where $S_+^{n-1} \subset S^{n-1}$ is the upper hemisphere. The pair (D^n, S_+^{n-1}) carries an obvious relative CW-structure. Namely, attach a $(n-1)$-cell to S_+^{n-1} by the attaching map $\mathrm{id} : S^{n-2} \to S^{n-2}$ to obtain S^{n-1}. Then we attach to S^{n-1} an n-cell by the attaching map $\mathrm{id} : S^{n-1} \to S^{n-1}$ to obtain D^n. Let X be a CW-complex. Let $q : S_+^{n-1} \to X$ be a map satisfying $q(S^{n-2}) \subset X_{n-2}$ and $q(S_+^{n-1}) \subset X_{n-1}$. Let Y be the space $D^n \cup_q X$, i.e. the push out

$$\begin{array}{ccc} S_+^{n-1} & \xrightarrow{q} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ D^n & \xrightarrow[g]{} & Y \end{array}$$

where i is the inclusion. Then Y inherits a CW-structure by putting $Y_k = j(X_k)$ for $k \leq n-2$, $Y_{n-1} = j(X_{n-1}) \cup g(S^{n-1})$ and $Y_k = j(X_k) \cup g(D^n)$ for $k \geq n$.

We call the homotopy equivalence j an *elementary expansion* There is a map $r : Y \to X$ with $r \circ j = \mathrm{id}_X$. We call any such map an *elementary collaps*.

Theorem 1.21 *Let $f : X \to Y$ be a map of finite CW-complexes. It is a simple homotopy equivalence if and only if there is a sequence of maps*

$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each f_i is an elementary expansion or elementary collaps and f is homotopic to the composition of the maps f_i.

Finally we give some information about the Whitehead group $\mathrm{Wh}(\pi)$.

- The Whitehead group $\mathrm{Wh}(G)$ is known to be trivial if G is the free abelian group $\mathbb{Z}^n$ of rank n or the free group $*_{i=1}^n \mathbb{Z}$ of rank n;

- The Whitehead group satisfies $\mathrm{Wh}(G * H) = \mathrm{Wh}(G) \oplus \mathrm{Wh}(H)$;

- There is the conjecture that $\mathrm{Wh}(G)$ vanishes for any torsionfree group G. This has been proven by Farrell and Jones for a large class of groups. This class contains any subgroup $G \subset G'$, where G' is a discrete cocompact subgroup of a Lie group with finitely many path components, and any group G which is the fundamental group of a non-positively curved closed Riemannian manifold or of a complete pinched negatively curved Riemannian manifold.

- If G is finite, then Wh(G) is very well understood. Namely, Wh(G) is finitely generated, its rank as abelian group is the number of conjugacy classes of unordered pairs $\{g, g^{-1}\}$ in G minus the number of conjugacy classes of cyclic subgroups, and its torsion subgroup is isomorphic to the kernel $SK_1(G)$ of the change of coefficient homomorphism $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.

  For a finite cyclic group G the Whitehead group Wh(G) is torsionfree. The Whitehead group of the symmetric group S_n is trivial;

- The Whitehead group of $\mathbb{Z}^2 \times \mathbb{Z}/4$ is not finitely generated as abelian group;

- For a ring R the first K-group $K_1(R)$ is defined to be the abelianization of the general linear group

$$GL(R) \; := \; \operatorname{colim}_{n\to\infty} GL(n,R).$$

For $R = \mathbb{Z}G$ the Whitehead group $\mathrm{Wh}(G)$ is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup generated by all $(1,1)$-matrices of the shape $(\pm g)$ for $g \in G$.

Remark 1.22 Given an invertible matrix A over $\mathbb{Z}G$, let A^* be the matrix obtained from A by transposing and applying the involution

$$\mathbb{Z} \to \mathbb{Z}G, \qquad \sum_{g\in G} \lambda_g \cdot g \;\mapsto\; \sum_{g\in G} \lambda_g \cdot g^{-1}.$$

We obtain an involution

$$* : \mathrm{Wh}(G) \to \mathrm{Wh}(G), \; [A] \mapsto [A^*].$$

It corresponds on the level of h-cobordisms to

$$\tau(W,M_0) \; = \; (-1)^{\dim(M_0)} \cdot *(\tau(W,M_1)).$$

2. Poincaré spaces, normal maps and the surgery step

Problem 2.1 *Let X be a topological space. When is X homotopy equivalent to a closed manifold?*

The cap-product yields a $\mathbb{Z}$-homomorphism

$$\begin{aligned} \cap : H_n(X;\mathbb{Z}) &\to [\ C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi} \\ x &\mapsto\ ? \cap x : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}). \end{aligned}$$

Definition 2.2 *A* connected finite n-dimensional Poincaré complex *is a connected finite CW-complex of dimension n together with an element $[X] \in H_n(X;\mathbb{Z})$ called* fundamental class *such that the $\mathbb{Z}\pi$-chain map $? \cap [X] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence. We will call it the* Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence*.*

We call X simple if the Whitehead torsion of the Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence vanishes.

Theorem 2.3 *Let M be a connected oriented closed manifold of dimension n. Then M carries the structure of a simple connected finite n-dimensional Poincaré complex.*

Remark 2.4 The analytic version of Poincaré duality is the fact that the space $\mathcal{H}^p(M)$ of harmonic p-forms on a closed connected oriented Riemannian manifold is canonically isomorphic to $H^p(M;\mathbb{R})$ and the Hodge-star-operator induces an isomorphism

$$* : \mathcal{H}^p(M) \to \mathcal{H}^{\dim(M)-p}(M).$$

From a Morse theoretic point of view Poincaré duality corresponds to the dual handlebody decomposition of a manifold which comes from replacing a Morse function f by $-f$.

This corresponds simplicially to the so called dual cell decomposition associated to a triangulation.

Definition 2.5 *Let X be a finite connected Poincaré complex of dimension $n = 4k$. Define its* intersection pairing *to be the symmetric bilinear non-degenerate pairing*

$$I : H^{2k}(X;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(X;\mathbb{R}) \xrightarrow{\cup} H^n(X;\mathbb{R}) \xrightarrow{\langle -,[X]_{\mathbb{R}}\rangle} \mathbb{R}.$$

Define the signature $\operatorname{sign}(X)$ *to be the signature of the intersection pairing.*

Remark 2.6 The notion of a Poincaré complex can be extended to pairs. One requires the existence of a fundamental class $[X, A] \in H_n(X, A; \mathbb{Z})$ such that the $\mathbb{Z}\pi$-chain maps $? \cap [X, A] : C^{n-*}(\widetilde{X}, \widetilde{A}) \to C_*(\widetilde{X})$ and $? \cap [X, A] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}\pi$-chain equivalences. Also the signature can be defined for Poincaré pairs.

Lemma 2.7 *1. Bordism invariance*

Let (X, A) be a $(4k+1)$-dimensional oriented finite Poincaré pair. Then

$$\sum_{C \in \pi_0(A)} \operatorname{sign}(C) = 0.$$

2. Additivity

Let M and N be compact oriented manifolds and $f : \partial M \to \partial N$ be an orientation reversing diffeomorphism. Then $M \cup_f N$ inherits an orientation from M and N and

$$\operatorname{sign}(M \cup_f N) \;=\; \operatorname{sign}(M) + \operatorname{sign}(N);$$

3. Multiplicativity

Let $p : \overline{M} \to M$ be a finite covering with d sheets of closed oriented manifolds. Then

$$\operatorname{sign}(\overline{M}) \;=\; d \cdot \operatorname{sign}(N).$$

Example 2.8 Wall has constructed a finite connected Poincaré space X together with a finite covering with d sheets $\overline{X} \to X$ such that the signature does not satisfy $\text{sign}(\overline{X}) = d \cdot \text{sign}(X)$ Hence X cannot be homotopy equivalent to a closed manifold by Lemma 2.7.

Next we briefly recall the Pontrjagin-Thom construction. Let $\xi : E \to X$ be a k-dimensional vector bundle over a CW-complex X. Denote by $\Omega_n(X, \xi)$ the set of bordism classes of closed n-dimensional manifolds M together with an embedding $i : M \to \mathbb{R}^{n+k}$ and a bundle map $\overline{f} : \nu(i) \to \xi$ covering a map $f : M \to X$. Let $\text{Th}(\xi)$ be the *Thom space*. Denote the *collapse map* by

$$c : S^{n+k} \ \to \ \text{Th}(\nu(M))$$

Theorem 2.9 (Pontrjagin-Thom construction) *The map*

$$P_n(\xi) : \Omega_n(X, \xi) \xrightarrow{\cong} \pi_{n+k}(\mathsf{Th}(\xi)),$$

which sends the class of $(M, i, f, \overline{f})$ *to the class of the composite*

$$S^{n+k} \xrightarrow{c} \mathsf{Th}(\nu(M)) \xrightarrow{\mathsf{Th}(\overline{f})} \mathsf{Th}(\xi)$$

is bijective. Its inverse is given by making a map $f : S^{n+k} \to \mathsf{Th}(\xi)$ *transversal to the zero section* $X \subset E$ *and taking the restriction to* $f^{-1}(X)$.

Example 2.10 Let $\Omega_n(X)$ be the bordism group of oriented closed manifolds M with reference map $M \to X$. Let $E_k \to BSO(k)$ be the universal bundle and define $\gamma_k : X \times E_k \to X \times BSO(k)$. There is an obvious bundle map $\overline{i_k} : \gamma_k \oplus \mathbb{R} \to \gamma_{k+1}$. We obtain a canonical bijection.

$$\mathsf{colim}_{k\to\infty}\, \Omega_n(\gamma_k) \xrightarrow{\cong} \Omega_n(X).$$

Thus we get an isomorphism of abelian groups natural in X

$$P : \Omega_n(X) \xrightarrow{\cong} \mathsf{colim}_{k\to\infty}\, \pi_{n+k}(\mathsf{Th}(\gamma_k)).$$

Remark 2.11 Notice that this is the beginning of the theory of spectra and stable homotopy theory. A *spectrum* $\mathbf{E}$ consists of a sequence of spaces $(E_k)_{k\in\mathbb{Z}}$ together with so called structure maps $s_k : \Sigma E_k \to E_{k+1}$. The n-th *stable homotopy group* is defined by

$$\pi_n(\mathbf{E}) \;=\; \operatorname{colim}_{k\to\infty} \pi_{n+k}(E_k)$$

with respect to the directed system given by the composites

$$\pi_{n+k}(E_k) \xrightarrow{\sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\pi_{n+k+1}(s_k)} \pi_{n+k+1}(E_{k+1}).$$

Example 2.12 Let Ω_n^{fr} be the bordism ring of stably framed manifolds, i.e. manifolds together with stable trivializations $\nu(M) \xrightarrow{\cong} \underline{\mathbb{R}^{n+k}}$. This is the same as $\operatorname{colim}_{k\to\infty} \Omega_n(\underline{\mathbb{R}^k})$. Thus we get an isomorphism

$$\Omega_n^{\mathsf{fr}} \xrightarrow{\cong} \pi_n^s := \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k).$$

Next we deal with the Spivak spherical fibration which is the analogue of the normal sphere bundle of a closed manifold for a finite Poincaré complex.

A *spherical* $(k-1)$*-fibration* $p : E \to X$ is a fibration, i.e. a map having the homotopy lifting property, whose typical fiber is homotopy equivalent to S^{k-1}. Define its *associated disc fibration* by

$$Dp : DE := \mathsf{cyl}(p) \to X.$$

Define its *Thom space* to be the pointed space

$$\mathsf{Th}(p) \;:=\; \mathsf{cone}(p) = DE/E.$$

We call ξ *orientable* if the fiber transport is trivial. Denote by $\xi * \eta$ the fiberwise join. There are canonical homeomorphisms

$$\begin{aligned} \mathsf{Th}(\xi * \eta) &\cong \mathsf{Th}(\xi) \wedge \mathsf{Th}(\eta); \\ \mathsf{Th}(\xi * \underline{\mathbb{R}^{k-1}}) &\cong \Sigma^{k-1}\,\mathsf{Th}(\xi). \end{aligned}$$

Theorem 2.13 (Thom isomorphism) *Let $p : E \to X$ be an orientable $(k-1)$-spherical fibration. Then there exists a so called* Thom class $U_p \in H^k(DE, E; \mathbb{Z})$ *such that the composite*

$$H^{p+k}(X; \mathbb{Z}) \xrightarrow{H^{p+k}(p)} H^p(DE; \mathbb{Z}) \xrightarrow{? \cup U_p} H^{p+k}(DE, SE; \mathbb{Z})$$

is bijective.

Definition 2.14 *A* Spivak normal fibration *for an n-dimensional connected finite Poincaré complex X is a $(k-1)$-spherical fibration $p = p_X : E \to X$ together with a pointed map $c = c_X : S^{n+k} \to \mathrm{Th}(p)$ such that for some choice of Thom class $U_p \in H^k(DE, E; \mathbb{Z})$ the fundamental class $[X] \in H_n(X; \mathbb{Z})$ and the image $h(c) \in H_{n+k}(\mathrm{Th}(p)) \cong H_{n+k}(DE, E; \mathbb{Z})$ of $[c]$ under the Hurewicz homomorphism $h : \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p), \mathbb{Z})$ are related by the formula*

$$[X] = H_n(p)(U_p \cap h(c)).$$

Remark 2.15 A closed manifold M admits a Spivak normal fibration.

Theorem 2.16 (Existence and uniqueness of the Spivak normal fibration) *Let X be a connected finite n-dimensional Poincaré complex. Then for $k > n$ there exists a Spivak normal $(k-1)$-fibration for X. It is unique up to strong fiber homotopy equivalence after stabilization.*

Definition 2.17 *Let X be a connected finite n-dimensional Poincaré complex. A* normal k-invariant (ξ, c) *consists of a k-dimensional vector bundle $\xi : E \to X$ together with an element $c \in \pi_{n+k}(\mathsf{Th}(\xi))$ such that for some choice of Thom class $U_p \in H^k(DE, SE; w^{\mathbb{Z}})$ the equation*

$$[X] = H_n(p)(U_p \cap h(c))$$

holds. The set of normal k-invariants $\mathcal{T}_n(X, k)$ *is the set of equivalence classes of normal k-invariants of X. Define the* set of normal invariants

$$\mathcal{T}_n(X) \;:=\; \operatorname{colim}_{k\to\infty} \mathcal{T}_n(X, k).$$

Let $BO(k)$ be the classifying space for k-dimensional vector bundles and $BG(k)$ be the classifying space for $(k-1)$-spherical fibrations. Let $J(k) : BO(k) \to BG(k)$ be the canonical map. Put

$$\begin{aligned} BO &:= \operatorname{colim}_{k\to\infty} BO(k) \\ BG &:= \operatorname{colim}_{k\to\infty} BG(k) \\ J &:= \operatorname{colim}_{k\to\infty} J(k). \end{aligned}$$

Remark 2.18 A necessary condition for a connected finite n-dimensional Poincaré complex to be homotopy equivalent to a closed manifold is that $\mathcal{T}_n(X) \neq \emptyset$, or equivalently, that the classifying map $s : X \xrightarrow{s_X} BG(k)$ lifts along $J : BO \to BG$. There is a fibration $BO \to BG \to BG/O$. Hence this condition is equivalent to the statement that the composition $X \xrightarrow{s_X} BG \to BG/O$ is homotopic to the constant map. There exists a finite Poincaré complex X which do not satisfy this condition.

Let G/O be the homotopy fiber of $J : BO \to BG$. This is the fiber of the fibration $\widehat{J} : E_J \to BG$ associated to J. Then the following holds

Theorem 2.19 *Let X be a connected finite n-dimensional Poincaré complex. Suppose that $\mathcal{T}_n(X)$ is non-empty. Then there is a canonical group structure on the set $[X, G/O]$ of homotopy classes of maps from X to G/O and a transitive free operation of this group on $\mathcal{T}_n(X)$.*

Notice that Theorem 2.19 yields after a choice of an element in $\mathcal{T}_n(X)$ a bijection of sets $[X, G/O] \xrightarrow{\cong} \mathcal{T}_n(X)$.

Definition 2.20 *Let X be a connected finite n-dimensional Poincare complex together with a k-dimensional vector bundle $\xi : E \to X$. A* normal k-map $(M, i, f, \overline{f})$ *consists of a closed manifold M of dimension n together with an embedding $i : M \to \mathbb{R}^{n+k}$ and a bundle map $(\overline{f}, f) : \nu(M) \to \xi$. A* normal map of degree one *is a normal map such that the degree of $f : M \to X$ is one.*

Definition 2.21 *Denote by* $\mathcal{N}_n(X,k)$ *the set of normal bordism classes of normal* k*-maps to* X*. Define the* set of normal maps to X

$$\mathcal{N}_n(X) \ := \ \mathrm{colim}_{k\to\infty}\mathcal{N}_n(X,k).$$

Theorem 2.22 *The Pontrjagin-Thom construction yields for each a bijection*

$$P(X): \mathcal{N}_n(X) \xrightarrow{\cong} \mathcal{T}_n(X).$$

Remark 2.23 In view of the Pontrjagin Thom construction it is convenient to work with the normal bundle. On the other hand one always needs an embedding and one would prefer an intrinsic definition. This is possible if one defines the normal map in terms of the tangent bundle. Namely one requires bundle data of the form $(\overline{f}, f)$: $TM \oplus \underline{\mathbb{R}^a} \to \xi$. Both approaches are equivalent.

Problem 2.24 *Suppose we have some normal map* $(\overline{f}, f)$ *from a closed manifold* M *to a finite Poincaré complex* X*. Can we change* M *and* f *leaving* X *fixed to get a normal map* $(\overline{g}, g)$ *such that* g *is a homotopy equivalence?*

Remark 2.25 Consider a normal map of degree one $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering $f : M \to Y$. It is a homotopy equivalence if and only if $\pi_k(f) = 0$ for all k. Consider an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ j\downarrow & & \downarrow f \\ D^{k+1} & \xrightarrow[Q]{} & Y \end{array}$$

We can get rid of it by attaching a cell to M according to this diagram. But this destroys the manifold structure on M. Hence we have to find a similar procedure which keeps the manifold structure. This will lead to the surgery step. Here also the bundle data will come in.

Theorem 2.26 (Immersions and bundle monomorphisms) *Let M be a m-dimensional and N be a n-dimensional closed manifold. Suppose that $1 \leq m \leq n$ and that M has a handlebody decomposition consisting of q-handles for $q \leq n-2$. Then taking the differential of an immersion yields a bijection*

$$T : \pi_0(\mathrm{Imm}(M, N)) \xrightarrow{\cong} \operatorname{colim}_{a\to\infty} \pi_0(\mathrm{Mono}(TM \oplus \underline{\mathbb{R}^a}, TN \oplus \underline{\mathbb{R}^a})).$$

Example 2.27 An easy computation shows that $\pi_0(\mathrm{Imm}(S^2, \mathbb{R}^3))$ consist of one element. Hence one turn the sphere inside out by a regular homotopy.

Theorem 2.28 (The surgery step) *Consider a normal map*

$$(\overline{f}, f) : TM \oplus \overline{\mathbb{R}^a} \to \xi$$

and an element $\omega \in \pi_{k+1}(f)$ *for* $k \leq n-2$ *for* $n = \dim(M)$.

1. *We can find a commutative diagram of vector bundles*

$$\begin{array}{ccc} T(S^k \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b}} & \xrightarrow{\overline{q}} & TM \oplus \underline{\mathbb{R}^{a+b}} \\ {\scriptstyle Tj \oplus n \oplus \mathrm{id}_{\underline{\mathbb{R}^{a+b-1}}}} \downarrow & & \downarrow {\scriptstyle \overline{f}} \\ T(D^{k+1} \times D^{n-k}) \oplus \underline{\mathbb{R}^{a+b-1}} & \xrightarrow[\overline{Q}]{} & \xi \oplus \underline{\mathbb{R}^b} \end{array}$$

   covering a commutative diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{q} & M \\ {\scriptstyle j} \downarrow & & \downarrow {\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow[Q]{} & X \end{array}$$

   such that the restriction of the last diagram to $D^{k+1} \times \{0\}$ *represents* ω *and* $q : S^k \times D^{n-k} \to M$ *is an immersion;*

2. *Suppose that the regular homotopy class of the immersion q appearing in* (1) *contains an embedding. Then one can arrange q in assertion* (1) *to be an embedding. If* $2k < n$, *one can always find an embedding in the regular homotopy class of q;*

3. *Suppose that the map q appearing in assertion* (1) *is an embedding.*

   Let W be the manifold obtained from $M \times [0,1]$ *by attaching a handle* $D^{k+1} \times D^{n-k}$ *by* $q : S^k \times D^{n-k} \to M = M \times \{1\}$. *Let* $F : W \to X$ *be the map induced by* $M \times [0,1] \xrightarrow{\mathrm{pr}} M \xrightarrow{f} X$ *and* $Q : D^k \times D^{k+1} \to X$. *After possibly stabilizing* $\overline{f}$ *the bundle maps* $\overline{f}$ *and* $\overline{Q}$ *induce a bundle map* $\overline{F} : TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ *covering* $F : W \to X$. *Thus we get a normal map*

   $$(\overline{F}, F) : TW \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$$

   which extends $(\overline{f} \oplus (f \times \mathrm{id}_{\mathbb{R}^b}), f) : TM \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$;

4. *The normal map* $(\overline{f}', f') : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ *obtained by restricting* $(\overline{F}, F)$ *to* $\partial W - M \times \{0\} =: M'$ *appearing in assertion (3) is a normal map of degree one which is normally bordant to* $(\overline{f}, f)$ *and has as underlying manifold*

$$M' = M - \mathrm{int}(q(S^k \times D^{n-k})) \cup_q D^k \times S^{n-k-1}.$$

We will the result of surgery on $(\overline{f}, f)$ *and* ω.

Theorem 2.29 *Let* X *be a connected finite* n*-dimensional Poincaré complex. Let* $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to \xi$ *be a normal map of degree one covering* $f : M \to X$. *Then we can carry out a finite sequence of surgery steps to obtain a normal map of degree one* $\overline{g} : TN \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ *covering* $g : N \to X$ *such that* $(\overline{f}, f)$ *and* $(\overline{g}, g)$ *are normally bordant and* g *is* k*-connected, where* $n = 2k$ *or* $n = 2k + 1$.

Problem 2.30 (Surgery problem) *Suppose we have some normal map $(\overline{f}, f)$ from a closed manifold M to a finite Poincaré complex X. Can we change M and f leaving X fixed by finitely many surgery steps to get a normal map $(\overline{g}, g)$ from a closed manifold N to X such that g is a homotopy equivalence?*

Remark 2.31 Suppose that X appearing in Problem 2.30 is orientable and of dimension $n = 4k$. Then we see an obstruction to solve the Surgery Problem 2.30, namely $\text{sign}(M) - \text{sign}(X)$ must be zero.

3. The surgery obstruction groups and the surgery exact sequence

We summarize what we have done so far.

- The s-cobordism Theorem;
- The surgery program;
- Whitehead torsion;
- Problem: When is a CW-complex homotopy equivalent to a closed oriented manifold;
- Finite Poincaré complexes;
- Pontrjagin-Thom construction;
- Spivak normal fibration;

- The set $\mathcal{T}_n(X)$ of reductions of the Spivak normal fibration to vector bundles;

- The set $\mathcal{N}_n(X)$ of normal bordism classes of normal maps $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f : M \to X$ of degree one;

- Construction of bijections

$$\mathcal{N}_n(X) \cong \mathcal{T}_n(X) = [X, G/O];$$

- The surgery step and bundle data;

- Making a normal map highly connected by surgery;

- Formulation of the surgery problem;

- The signature is a surgery obstruction.

Theorem 3.1 (Surgery obstruction theorem) *There are L-groups $L_n(\mathbb{Z}\pi)$ which are defined algebraically in terms of forms and formations over $\mathbb{Z}\pi$, and for any normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ there is an element called surgery obstruction*

$$\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi)$$

for $n = \dim(M) \geq 5$ and $\pi = \pi_1(X)$ such that the following holds:

1. *Suppose $n \geq 5$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f) : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ which covers a homotopy equivalence $f' : M' \to X$;*

2. *The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.*

Remark 3.2 We will only give some details in even dimensions $n = 2k$. There the essential problem is to figure out whether an immersion $f : S^k \to M$ is regular homotopic to an embedding. This problem will lead to the notion of quadratic form and the L-group $L_n(\mathbb{Z}\pi)$ and the surgery obstruction in a natural way.

We fix base points $s \in S^k$ and $b \in M$ and assume that M is connected and $k \geq 2$. We will consider pointed immersions (f, w), i.e. an immersion $f : S^k \to M$ together with a path w from b to $f(s)$. Denote by

$$I_k(M)$$

the set of pointed homotopy classes of pointed immersions from S^k to M. It inherits the structure of a $\mathbb{Z}\pi$-module.

Next we want to define the *intersection pairing*

$$\lambda : I_k(M) \times I_k(M) \to \mathbb{Z}\pi. \qquad (3.3)$$

Consider $\alpha_0 = [(f_0, w_0)]$ and $\alpha_1 = [(f_1, w_1)]$ in $I_k(M)$. Choose representatives (f_0, w_0) and (f_1, w_1). We can arrange without changing the pointed regular homotopy class that $D = \text{im}(f_0) \cap \text{im}(f_1)$ is finite, for any $y \in D$ both the preimage $f_0^{-1}(y)$ and the preimage $f_1^{-1}(y)$ consists of precisely one point and for any two points x_0 and x_1 in S^k with $f_0(x_0) = f_1(x_1)$ we have $T_{x_0} f_0(T_{x_0} S^k) + T_{x_1} f_1(T_{x_1} S^k) = T_{f_0(x_0)} M$. Consider $d \in D$. Let x_0 and x_1 in S^k be the points uniquely determined by $f_0(x_0) = f_1(x_1) = d$. Let u_i be a path in S^k from s to x_i. Then we obtain an element $g(d) \in \pi$ by $w_1 * f_1(u_1) * f_0(u_0)^- * w_0^-$. Define $\epsilon(d) = 1$ if the isomorphism of oriented vector spaces

$$T_{x_0} f_0 \oplus T_{x_1} f_1 : T_{x_0} S^k \oplus T_{x_1} S^k \xrightarrow{\cong} T_d M$$

respects the orientations and $\epsilon(d) = -1$ otherwise. Define

$$\lambda(\alpha_0, \alpha_1) \ := \ \sum_{d \in D} \epsilon(d) \cdot g(d).$$

Remark 3.4 One can describe the intersection pairing in terms of algebraic intersection numbers:

$$\lambda(\alpha_0, \alpha_1) \;=\; \sum_{g\in\pi} \lambda_{\mathbb{Z}}(\widetilde{f_0}, l_{g^{-1}} \circ \widetilde{f_1}) \cdot g.$$

Remark 3.5 A necessary condition for an immersion $f : S^k \to M$ to be regularily homotopic to an embedding is

$$\lambda(f, f) \;=\; 0.$$

This condition is only sufficient. In order to get a necessary and sufficient condition we have to deal with selfintersections which will give a refinement of the intersection pairing. Algebraically this corresponds to refine a symmetric form to a quadratic form. In this step the bundle data of a normal map will actually be used.

Let $\alpha \in I_k(M)$ be an element. Let (f, w) be a pointed immersion representing α. We can assume without loss of generality that f is in general position, i.e. there is a finite subset D of $\mathrm{im}(f)$ such that $f^{-1}(y)$ consists of precisely two points for $y \in D$ and of precisely one point for $y \in \mathrm{im}(f) - D$ and for two points x_0 and x_1 in S^k with $x_0 \neq x_1$ and $f(x_0) = f(x_1)$ we have $T_{x_0}f(T_{x_0}S^k) + T_{x_1}f(T_{x_1}S^k) = T_{f_0(x_0)}M$. Now fix for any $d \in D$ an ordering $x_0(d), x_1(d)$ of $f^{-1}(d)$. Analogously to the construction above one defines $\epsilon(x_0(d), x_1(d)) \in \{\pm 1\}$ and $g(x_0(d), x_1(d)) \in \pi$. Define the abelian group

$$Q_{(-1)^k}(\mathbb{Z}\pi) \ := \ \mathbb{Z}\pi / \{u - (-1)^k \cdot \overline{u} \mid u \in \mathbb{Z}\pi\}.$$

Define the *selfintersection element*

$$\mu(\alpha) \ := \ \left[\sum_{d \in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d)) \right] \in Q_{(-1)^k}(\mathbb{Z}\pi).$$

Remark 3.6 The passage from $\mathbb{Z}\pi$ to $Q_{(-1)^k}(\mathbb{Z}\pi)$ ensures that the definition is independent of the choice of the order on $f^{-1}(d)$ for $d \in D$.

Theorem 3.7 *For* $\dim(M) = 2k \geq 6$ *a pointed immersion* (f, w) *of* S^k *in* M *is pointed homotopic to a pointed immersion* (g, v) *for which* $g : S^k \to M$ *is an embedding, if and only* $\mu(f) = 0$.

Fix a normal map of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}}^a \to \xi$ covering $f : M \to X$.

Definition 3.8 *Let* $K_k(\widetilde{M})$ *be the kernel of the* $\mathbb{Z}\pi$*-map* $H_k(\widetilde{f}) : H_k(\widetilde{M}) \to H_k(\widetilde{X})$. *Denote by* $K^k(\widetilde{M})$ *be the cokernel of the* $\mathbb{Z}\pi$*-map* $H^k(\widetilde{f}) : H^k(\widetilde{X}) \to H^k(\widetilde{M})$.

Lemma 3.9 *1. The cap product with* $[M]$ *induces isomorphisms*

$$? \cap [M] : K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

2. Suppose that f *is* k*-connected. Then there is the composition of natural* $\mathbb{Z}\pi$*-isomorphisms*

$$h_k : \pi_{k+1}(f) \xrightarrow{\cong} \pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M});$$

3. Suppose that f *is* k*-connected and* $n = 2k$*. Then there is a natural* $\mathbb{Z}\pi$*-homomophism*

$$t_k : \pi_k(f) \to I_k(M).$$

The Kronecker product induces a pairing

$$\langle\ ,\ \rangle : K^k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Together with the isomorphism

$$? \cap [M] : K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

of Theorem 3.9 (1) it induces the pairing

$$s : K_k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Lemma 3.10 *The following diagram commutes*

$$\begin{array}{ccc} K_k(\widetilde{M}) \times K_k(\widetilde{M}) & \xrightarrow{s} & \mathbb{Z}\pi \\ \downarrow{\scriptstyle \alpha\times\alpha} & & \downarrow{\scriptstyle \mathrm{id}} \\ I_k(M) \times I_k(M) & \xrightarrow[\lambda]{} & \mathbb{Z}\pi \end{array}$$

In the sequel we will sometimes identify P and $(P^*)^*$ by the canonical isomorphism $e(P) : P \xrightarrow{\cong} (P^*)^*$.

Definition 3.11 *An* ϵ-symmetric form (P, ϕ) *over an associative ring* R *with unit and involution is a finitely generated projective* R*-module* P *together with a* R*-map* $\phi : P \to P^*$ *such that the composition* $P = (P^*)^* \xrightarrow{\phi^*} P$ *agrees with* $\epsilon \cdot \phi$*. We call* (P, ϕ) non-degenerate *if* ϕ *is an isomorphism.*

We can rewrite (P, ϕ) as pairing

$$\lambda : P \times P \to \mathbb{Z}\pi, \qquad (p, q) \mapsto \phi(p)(q).$$

Example 3.12 Let P be a finitely generated projective R-module. The *standard hyperbolic* ϵ*-symmetric form* $H^\epsilon(P)$ is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the R-isomorphism

$$\phi : (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P = (P \oplus P^*)^*.$$

If we write it as a pairing we obtain

$$\begin{array}{rcl} (P \oplus P^*) \times (P \oplus P^*) & \to & R \\ ((p, \phi), (p', \phi')) & \mapsto & \phi(p') + \epsilon \cdot \phi'(p). \end{array}$$

Example 3.13 An example of a non-degenerate $(-1)^k$-symmetric form over $\mathbb{Z}\pi$ with the w-twisted involution is $K_k(\widetilde{M})$ with the pairing s above, provided that f is k-connected and $n = 2k$. This uses the fact that $K_k(\widetilde{M})$ is stably finitely generated free and hence in particular finitely generated projective.

For a finitely generated projective R-module P define an involution of R-modules

$$T : \hom_R(P, P^*) \to \hom(P, P^*) \qquad f \mapsto f^*$$

and put

$$\begin{aligned} Q^{\epsilon}(P) &:= \ker\,(1 - \epsilon \cdot T)\,; \\ Q_{\epsilon}(P) &:= \operatorname{coker}\,(1 - \epsilon \cdot T)\,. \end{aligned}$$

Definition 3.14 *A ϵ-quadratic form (P, ψ) is a finitely generated projective R-module P together with an element $\psi \in Q_{\epsilon}(P)$. It is called* non-degenerate *if the associated ϵ-symmetric form $(P, (1 + \epsilon \cdot T)(\psi))$ is non-degenerate, i.e. $(1 + \epsilon \cdot T)(\psi) : P \to P^*$ is bijective.*

An ϵ-quadratic form (P, ϕ) is the same as a triple (P, λ, μ) consisting of pairing

$$\lambda : P \times P \to R$$

satisfying

$$\begin{aligned} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}; \\ \lambda(q, p) &= \epsilon \cdot \overline{\lambda(p, q)}. \end{aligned}$$

and a map

$$\mu : P \to Q_\epsilon(R) = R/\{r - \epsilon \cdot \overline{r} \mid r \in R\}$$

satisfying

$$\begin{aligned} \mu(rp) &= r\mu(p)\overline{r}; \\ \mu(p+q) - \mu(p) - \mu(q) &= \mathrm{pr}(\lambda(p, q)); \\ \lambda(p, p) &= (1 + \epsilon \cdot T)(\mu(p)), \end{aligned}$$

where $\mathrm{pr} : R \to Q_\epsilon(R)$ is the projection and $(1 + \epsilon \cdot T) : Q_\epsilon(R) \to R$ the map sending the class of r to $r + \epsilon \cdot \overline{r}$. Namely, put

$$\begin{aligned} \lambda(p, q) &= ((1 + \epsilon \cdot T)(\psi))(p))(q); \\ \mu(p) &= \psi(p)(p). \end{aligned}$$

Example 3.15 Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$* is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the class in $Q_\epsilon(P \oplus P^*)$ of the R-homomorphism

$$\phi : (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P = (P \oplus P^*)^*.$$

The ϵ-symmetric form associated to $H_\epsilon(P)$ is $H^\epsilon(P)$.

Example 3.16 An example of a non-degenerate $(-1)^k$-quadratic form over $\mathbb{Z}\pi$ with the w-twisted involution is given as follows, provided that f is k-connected and $n = 2k$. Namely, take $K_k(\widetilde{M})$ with the pairing s above and the map

$$t : K_k(\widetilde{M}) \xrightarrow{\alpha} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w).$$

Example 3.17 The effect of doing surgery on $0 \in \pi_{k+1}(f)$ is to replace M by the connected sum $M\sharp(S^k \times S^k)$ and to replace $K_k(\widetilde{M})$ by $K_k(\widetilde{M}) \oplus H_{(-1)^k}(\mathbb{Z}\pi)$.

Remark 3.18 Suppose that $1/2 \in R$. Then the homomorphism

$$(1+\epsilon\cdot T) : Q_\epsilon(P) \xrightarrow{\cong} Q^\epsilon(P) \quad [\psi] \mapsto [\psi+\epsilon\cdot T(\psi)]$$

is bijective. The inverse sends $[u]$ to $[u/2]$. Hence any ϵ-symmetric form carries a unique ϵ-quadratic structure.

Theorem 3.19 *Consider the normal map $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering the k-connected map of degree one $f : M \to N$ of closed connected n-dimensional manifolds for $n = 2k$. Suppose that $k \geq 3$ and that for the non-degenerate $(-1)^k$-quadratic form $(K_k(\widetilde{M}), s, t)$ there are integers $u, v \geq 0$ together with an isomorphism of non-degenerate $(-1)^k$-quadratic forms*

$$(K_k(\widetilde{M}), s, t) \oplus H_{(-1)^k}(\mathbb{Z}\pi^u) \;\cong\; H_{(-1)^k}(\mathbb{Z}\pi^v).$$

Then we can perform a finite number of surgery steps resulting in a normal map of degree one $(\overline{g}, g) : TM' \oplus \underline{\mathbb{R}^{a+b}} \to \xi \oplus \underline{\mathbb{R}^b}$ such that $g : M' \to X$ is a homotopy equivalence.

Proof: Without loss of generality we can choose a $\mathbb{Z}\pi$-basis $\{b_1, b_2, \dots b_v, c_1, c_2, \dots c_v\}$ for $K_k(\widetilde{M})$ such that

$$\begin{array}{lcll} s(b_i, c_i) & = & 1 & i \in \{1, 2, \dots v\}; \\ s(b_i, c_j) & = & 0 & i, j \in \{1, 2, \dots v\},\ i \neq j; \\ s(b_i, b_j) & = & 0 & i, j \in \{1, 2, \dots v\}; \\ s(c_i, c_j) & = & 0 & i, j \in \{1, 2, \dots v\}; \\ t(b_i) & = & 0 & i \in \{1, 2, \dots v\}. \end{array}$$

Notice that f is a homotopy equivalence if and only if the number v is zero. Hence it suffices to explain how we can lower the number v to $(v-1)$ by a surgery step on an element in $\pi_{k+1}(f)$. Of course our candidate is the element ω in $\pi_{k+1}(f)$ which corresponds under the isomorphism $h : \pi_{k+1}(f) \to K_k(\widetilde{M})$ to the element b_v.

Definition 3.20 *Let R be an associative ring with unit and involution. For $n = 2k$ define $L_n(R)$ to be the abelian group of stable isomorphism classes $[(F, \psi)]$ of non-degenerate $(-1)^k$-quadratic forms (F, ψ) whose underlying R-module F is a finitely generated free R-module.*

Definition 3.21 *Consider a normal map of degree one* $(\overline{f}, f) : TM \oplus \mathbb{R}^a \to \xi$ *covering* $f : M \to X$ *for* $n = 2k = \dim(M)$. *Make* f k*-connected by surgery. Define the surgery obstruction*

$$\sigma(\overline{f}, f) \ \in \ L_n(\mathbb{Z}\pi)$$

by the class of the $(-1)^k$*-quadratic non-degenerate form* $(K_k(\widetilde{M}), s, t)$.

Theorem 3.22 *1. The signature defines an isomorphism*

$$\frac{1}{8}\cdot\text{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z},\ [P, \psi] \mapsto \frac{1}{8}\cdot\text{sign}(\mathbb{R}\otimes_{\mathbb{Z}} P, \lambda).$$

The surgery obstruction is given by

$$\sigma(\overline{f}, f) := \frac{1}{8} \cdot (\text{sign}(X) - \text{sign}(M));$$

2. The Arf invariant defines an isomorphism

$$\text{Arf} : L_2(\mathbb{Z}\pi) \xrightarrow{\cong} \mathbb{Z}/2;$$

3. $L_1(\mathbb{Z})$ *and* $L_3(\mathbb{Z})$ *vanish.*

Theorem 3.23 *Let X be a simply connected finite Poincaré complex of dimension n.*

1. *Suppose $n = 4k \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi : E \to X$ such that*

$$\langle \mathcal{L}(\xi)^{-1}, [X] \rangle \;=\; \mathrm{sign}(X);$$

2. *Suppose $n = 4k + 2 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction such that the Arf invariant of the associated surgery problem vanishes;*

3. *Suppose $n = 2k + 1 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction.*

Remark 3.24 One can define the surgery obstruction also for a normal map $\overline{f} : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $(f, \partial f) : (M; \partial M) \to (X, \partial X)$ of degree one provided that ∂f is a homotopy equivalence. Then the obstruction vanishes if and only if one can change f into a homotopy equivalence by surgery on the interior of M. There are also simple versions of the L-groups and the surgery obstruction, where ∂f is required to be a simple homotopy equivalence and the goal is to change f into a simple homotopy equivalence.

Definition 3.25 *Let $(X, \partial X)$ be a compact oriented manifold of dimension n with boundary ∂X. Define the* set of normal maps to $(X, \partial X)$

$$\mathcal{N}_n(X, \partial X)$$

to be the set of normal bordism classes of normal maps of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ with underlying map $(f, \partial f) : (M, \partial M) \to (X, \partial X)$ for which $\partial f : \partial M \to \partial X$ is a diffeomorphism.

Definition 3.26 *Let X be a closed oriented manifold of dimension n. We call two orientation preserving simple homotopy equivalences $f_i : M_i \to X$ from closed oriented manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there exists an orientation preserving diffeomorphism $g : M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0. The* simple structure set

$$\mathcal{S}_n^s(X)$$

of X is the set of equivalence classes of orientation preserving simple homotopy equivalences $M \to X$ from closed oriented manifolds of dimension n to X. This set has a preferred base point, namely the class of the identity $\mathrm{id} : X \to X$.

Let

$$\eta : \mathcal{S}_n^s(X) \ \rightarrow \ \mathcal{N}_n^s(X)$$

be the map which sends the class $[f] \in \mathcal{S}_n^s(X)$ represented by a simple homotopy equivalence $f : M \rightarrow X$ to the normal bordism class of the following normal map obtained from f by covering it with bundle data of the form $TM \rightarrow \xi := (f^{-1})^* TM$.

Next we define an action of the abelian group $L_{n+1}^s(\mathbb{Z}\pi, w)$ on the structure set $\mathcal{S}_n^s(X)$

$$\rho : L_{n+1}^s(\mathbb{Z}\pi, w) \times \mathcal{S}_n^s(X) \ \rightarrow \ \mathcal{S}_n^s(X).$$

Fix $x \in L_{n+1}^s(\mathbb{Z}\pi, w)$ and $[f] \in \mathcal{N}_n^s(X)$ represented by a simple homotopy equivalence $f : M \rightarrow X$. We can find a normal map $(\overline{F}, F)$ covering a map of triads $(F; \partial_0 F, \partial_1 F) : (W; \partial_0 W, \partial_1 W) \rightarrow (M \times [0,1], M \times \{0\}, M \times \{1\})$ such that $\partial_0 F$ is a diffeomorphism and $\partial_1 F$ is a simple homotopy equivalence and $\sigma(\overline{F}, F) = u$. Then define $\rho(x, [f])$ by the class $[f \circ \partial_1 F : \partial_1 W \rightarrow X]$.

Theorem 3.27 (The surgery exact sequence) *The so called* surgery sequence

$$\mathcal{N}_{n+1}(X\times[0,1],X\times\{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi,w) \xrightarrow{\partial} \mathcal{S}^s_n(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi,w)$$

is exact for $n \geq 5$ in the following sense. An element $z \in \mathcal{N}_n(X)$ lies in the image of η if and only if $\sigma(z) = 0$. Two elements $y_1, y_2 \in \mathcal{S}^s_n(X)$ have the same image under η if and only if there exists an element $x \in L^s_{n+1}(\mathbb{Z}\pi,w)$ with $\rho(x,y_1) = y_2$. For two elements x_1, x_2 in $L^s_{n+1}(\mathbb{Z}\pi)$ we have $\rho(x_1,[\mathrm{id}: X \to X]) = \rho(x_2,[\mathrm{id}: X \to X])$ if and only if there is $u \in \mathcal{N}_{n+1}(X\times[0,1],X\times\{0,1\})$ with $\sigma(u) = x_1 - x_2$.

Remark 3.28 The surgery sequence of Theorem 3.27 can be extended to infinity to the left.

4. Homotopy spheres

Definition 4.1 *A* homotopy n-sphere Σ *is a closed oriented n-dimensional smooth manifold which is homotopy equivalent S^n.*

Remark 4.2 The Poincaré Conjecture says that any homotopy n-sphere Σ is oriented homeomorphic to S^n and is known to be true for all dimensions except $n = 3$.

Definition 4.3 *Define the n-th* group of homotopy spheres Θ^n *as follows. Elements are oriented h-cobordism classes* $[\Sigma]$ *of oriented homotopy n-spheres Σ. The addition is given by the connected sum. The zero element is represented by S^n. The inverse of* $[\Sigma]$ *is given by* $[\Sigma^-]$, *where* Σ^- *is obtained from Σ by reversing the orientation.*

Remark 4.4 Since in the sequel all spaces are simply connected, we do not have to worry about Whitehead torsion. In dimension $n \geq 5$ the s-cobordim theorem implies that Θ_n is the abelian group of oriented diffeomorphism classes of homotopy n-spheres.

Lemma 4.5 *There is a natural bijection*

$$\alpha : \mathcal{S}_n(S^n) \xrightarrow{\cong} \theta^n \qquad [f : M \to S^n] \mapsto [M].$$

Definition 4.6 *Let $bP^{n+1} \subset \Theta^n$ be the subset of elements $[\Sigma]$ for which Σ is oriented diffeomorphic to the boundary ∂M of a stably parallizable compact manifold M.*

Lemma 4.7 *The subset $bP^{n+1} \subset \Theta^n$ is a subgroup of Θ^n. It is the preimage under the composition*

$$\Theta^n \xrightarrow{\alpha^{-1}} \mathcal{S}_n(S^n) \xrightarrow{\eta} \mathcal{N}_n(S^n)$$

of the base point $[\mathrm{id} : TS^n \to TS^n]$ in $\mathcal{N}_n(S^n)$.

Definition 4.8 *A* stable framing *of a closed oriented manifold M of dimension n is a (strong) bundle isomorphism $\overline{u} : TM \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation. An* almost stable framing *of a closed oriented manifold M of dimension n is a choice of a point $x \in M$ together with a (strong) bundle isomorphism $\overline{u} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{n+a}}$ for some $a \geq 0$ which is compatible with the given orientation on $M - \{x\}$.*

Definition 4.9 *Let Ω_n^{fr} be the abelian group of stably framed bordism classes of stably framed closed oriented manifolds of dimension n.*

Let Ω_n^{alm} be the abelian group of almost stably framed bordism classes of almost stably framed closed oriented manifolds of dimension n. This becomes an abelian group by the connected sum at the preferred base points.

Lemma 4.10 *There are canonical bijections of pointed sets*

$$\beta : \mathcal{N}_n(S^n) \xrightarrow{\cong} \Omega_n^{\mathsf{alm}};$$

$$\gamma : \mathcal{N}_{n+1}(S^n \times [0,1], S^n \times \{0,1\}) \xrightarrow{\cong} \mathcal{N}_{n+1}(S^{n+1}).$$

Theorem 4.11 *The long sequence of abelian groups which extends infinitely to the left*

$$\ldots \to \Omega_{n+1}^{\mathsf{alm}} \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}) \xrightarrow{\partial} \Theta^n \xrightarrow{\eta} \Omega_n^{\mathsf{alm}} \xrightarrow{\sigma} L_n(\mathbb{Z}) \xrightarrow{\partial} \ldots \xrightarrow{\eta} \Omega_5^{\mathsf{alm}} \xrightarrow{\sigma} L_5(\mathbb{Z})$$

is exact.

Proof: One easily checks that the maps are compatible with the abelian groups structures. Now use the identifications above and the general surgery sequence.

Recall that there are isomorphisms

$$\frac{1}{8} \cdot \mathrm{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$$

and

$$\mathrm{Arf} : L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$$

and that $L_{2i+1}(\mathbb{Z}) = 0$ for $i \in \mathbb{Z}$.

Corollary 4.12 *There are for $i \geq 2$ and $j \geq 3$ short exact sequences of abelian groups*

$$0 \to \Theta^{4i} \xrightarrow{\eta} \Omega_{4i}^{\mathrm{alm}} \xrightarrow{\frac{\mathrm{sign}}{8}} \mathbb{Z} \xrightarrow{\partial} bP^{4i} \to 0$$

and

$$0 \to \Theta^{4i-2} \xrightarrow{\eta} \Omega_{4i-2}^{\mathrm{alm}} \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2 \xrightarrow{\partial} bP^{4i-2} \to 0$$

and

$$0 \to bP^{2j} \to \Theta^{2j-1} \xrightarrow{\eta} \Omega_{2j-1}^{\mathrm{alm}} \to 0.$$

We have

$$bP^{2n+1} = 0.$$

There is an obvious forgetful map

$$f : \Omega_n^{\mathsf{fr}} \;\to\; \Omega_n^{\mathsf{alm}}. \qquad (4.13)$$

Define the group homomorphism

$$\partial : \Omega_n^{\mathsf{alm}} \;\to\; \pi_{n-1}(SO) \qquad (4.14)$$

as follows. Given $r \in \Omega_n^{alm}$ choose a representative $(M, x, \overline{u} : TM|_{M-\{x\}} \oplus \underline{\mathbb{R}^a} \to \underline{\mathbb{R}^{n+a}})$. Let $D^n \subset M$ be an embedded disk with origin x. Since D^n is contractible, we obtain a strong bundle isomorphism unique up to isotopy $\overline{v} : TM|_{D^n} \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$. The composition of the inverse of the restriction of $\overline{u}$ to $S^{n-1} = \partial D^n$ and of the restriction of $\overline{v}$ to S^{n-1} is an orientation preserving bundle automorphism of the trivial bundle $\underline{\mathbb{R}^{a+n}}$ over S^{n-1}. This is the same as a map $S^{n-1} \to SO(n+a)$. It composition with the canonical map $SO(n+a) \to SO$ represents an element in $\pi_{n-1}(SO)$ which is defined to be the image of r under $\partial : \Omega_n^{\mathsf{alm}} \to \pi_{n-1}(SO)$.

Let

$$\overline{J} : \pi_n(SO) \ \rightarrow \ \Omega_n^{\mathsf{fr}} \qquad (4.15)$$

be the group homomorphism which assigns to the element $r \in \pi_n(SO)$ represented by a map $\overline{u} : S^n \to SO(n+a)$ the class of S^n with the stable framing $TS^n \oplus \underline{\mathbb{R}^a} \xrightarrow{\cong} \underline{\mathbb{R}^{a+n}}$ coming from r. One easily checks

Lemma 4.16 *The following sequence is a long exact sequence of abelian groups*

$$\ldots \xrightarrow{\partial} \pi_n(SO) \xrightarrow{\overline{J}} \Omega_n^{fr} \xrightarrow{f} \Omega_n^{\mathsf{alm}} \xrightarrow{\partial} \pi_{n-1}(SO) \xrightarrow{\overline{J}} \Omega_{n-1}^{fr} \xrightarrow{f} \ldots .$$

Theorem 4.17 *The Pontrjagin Thom construction yields an isomorphism*

$$\Omega_n^{\mathsf{fr}} \xrightarrow{\cong} \pi_n^s.$$

The *Hopf construction* defines for spaces X, Y and Z a map

$$H : [X \times Y, Z] \rightarrow [X * Y, \Sigma Z] \quad (4.18)$$

as follows. Recall that the join $X * Y$ is defined by $X \times Y \times [0,1]/\sim$ and that the (unreduced) suspension ΣZ is defined by $Z \times [0,1]/\sim$. Given $f : X \times Y \rightarrow Z$, let $H(f) : X * Y \rightarrow \Sigma Z$ be the map induced by $f \times \mathrm{id} : Y \times [0,1] \rightarrow Z \times [0,1]$. Consider the following composition

$$[S^n, SO(k)] \rightarrow [S^n, \mathrm{aut}(S^{k-1})] \rightarrow [S^n \times S^{k-1}, S^{k-1}]$$

$$\xrightarrow{H} [S^n * S^{k-1}, \Sigma S^{k-1}] = [S^{n+k}, S^k].$$

Definition 4.19 *The composition above induces for $n, k \geq 1$ homomorphisms of abelian groups*

$$J_{n,k} : \pi_n(SO(k)) \rightarrow \pi_{n+k}(S^k).$$

Taking the colimit for $k \rightarrow \infty$ induces the so called J-homomorphism

$$J_n : \pi_n(SO) \rightarrow \pi_n^s.$$

Lemma 4.20 *The J-homomorphism is the composite*

$$J : \pi_n(SO) \xrightarrow{\overline{J}} \Omega_n^{\mathsf{fr}}\Omega_n^{\mathsf{fr}} \xrightarrow{\cong} \pi_n^s.$$

It corresponds to the map induced by J : $BO \to BG$ on the homotopy groups $\pi_{n+1}(BO) = \pi_n(SO)$ and $\pi_{n+1}(BG) = \pi_n^s$.

The homotopy groups of O are 8-periodic and given by

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Notice that $\pi_i(SO) = \pi_i(O)$ for $i \geq 1$ and $\pi_0(SO) = 1$. The first stable stems are given by

n	0	1	2	3	4	5	6	7	8
π_n^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2$

The *Bernoulli numbers* B_n for $n \geq 1$ are defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n \geq 1} \frac{(-1)^{n+1} \cdot B_n}{(2n)!} \cdot (z)^{2n}.$$

The first values are given by

n	1	2	3	4	5	6	7	8
B_n	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

The next result is a deep theorem due to Adams.

Theorem 4.21 *1. If* $n \neq 3 \mod 4$, *then the* J*-homomorphism* $J_n : \pi_n(SO) \to \pi_n^s$ *is injective;*

2. The order of the image of the J*-homomorphism*

$$J_{4k-1} : \pi_{4k-1}(SO) \to \pi_{4k-1}^s$$

is denominator$(B_k/4k)$, *where* B_k *is the* k*-th Bernoulli number.*

The boundary operator in the long homotopy sequence yields an isomorphism

$$\delta : \pi_n(BSO) \xrightarrow{\cong} \pi_{n-1}(SO). \quad (4.22)$$

Define a map

$$\gamma : \mathcal{N}_n(S^n) \to \pi_n(BSO) \quad (4.23)$$

by sending the class of the normal map of degree one $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}^a} \to \xi$ covering a map $f : M \to S^n$ to the the class represented by the classifying map $f_\xi : S^n \to BSO(n+k)$ of ξ.

Lemma 4.24 *The following diagram commutes*

$$\begin{array}{ccc} \Omega_n^{alm} & \xrightarrow{\partial} & \pi_{n-1}(SO) \\ \beta^{-1}\big\downarrow & & \big\downarrow\delta^{-1} \\ \mathcal{N}_n(S^n) & \xrightarrow[\gamma]{} & \pi_n(BSO) \end{array}$$

The Hirzebruch signature formula says

$$\text{sign}(M) \;=\; \langle \mathcal{L}(M), [M] \rangle. \quad (4.25)$$

The L-class is a cohomology class which is obtained from inserting the Pontrjagin classes $p_i(TM)$ into a certain polynomial $L(x_1, x_2, \ldots x_k)$. The L-polynomial $L(x_1, x_2, \ldots x_n)$ is the sum of $s_k \cdot x_k$ and terms which do not involve x_k, where s_k is given in terms of the Bernoulli numbers B_k by

$$s_k \;:=\; \frac{2^{2k} \cdot (2^{2k-1} - 1) \cdot B_k}{(2k)!}. \; (4.26)$$

Lemma 4.27 *Let $n = 4k$. Then there is an isomorphism*

$$\phi : \pi_{n-1}(SO) \xrightarrow{\cong} \mathbb{Z}.$$

Define a map

$$p_k : \pi_n(BSO) \to \mathbb{Z}$$

by sending the element $x \in \pi_n(BSO)$ represented by a map $f : S^n \to BSO(m)$ to $\langle p_k(f^*\gamma_m), [S^n]\rangle$ for $\gamma_m \to BSO(m)$ the universal bundle. Let $\delta : \pi_n(BSO) \to \pi_{n-1}(SO)$ be the canonical isomorphism. Put

$$t_k \;:=\; \frac{3-(-1)^k}{2} \cdot (2k-1)! \qquad (4.28)$$

Then

$$t_k \cdot \phi \circ \delta \;=\; p_k.$$

Lemma 4.29 *The following diagram commutes for $n = 4k$*

$$\begin{array}{ccc} \Omega_n^{\mathrm{alm}} & \xrightarrow{\frac{\mathrm{sign}}{8}} & \mathbb{Z} \\ \partial \downarrow & & \uparrow \frac{s_k \cdot t_k}{8}\cdot \mathrm{id} \\ \pi_{n-1}(SO) & \xrightarrow[\cong]{\phi} & \mathbb{Z} \end{array}$$

Proof: Let M be almost stably parallizable. Then for some point $x \in M$ the restriction of the tangent bundle TM to $M - \{x\}$ is stably trivial and hence has trivial Pontrjagin classes. Hence (4.25) implies for a closed oriented almost stably parallizable manifold M of dimension $4k$

$$\text{sign}(M) \; = \; s_k \cdot \langle p_k(TM), [M] \rangle.$$

Now apply Lemma 4.27.

Theorem 4.30 *Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\frac{s_k \cdot t_k}{8} \cdot \left| \text{im} \left(J_{4k-1} \colon \pi_{4k-1}(SO) \to \pi^s_{4k-1} \right) \right| \\ = \; \frac{3 - (-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1} - 1) \\ \cdot \text{numerator}(B_k/(4k)).$$

Proof: $bP^{4k} \; = \; \text{coker} \left(\frac{\text{sign}}{8} \colon \Omega^{\text{alm}}_n \to \mathbb{Z} \right).$

Let

$$\mathsf{Arf} : \pi^s_{4k+2} \;\to\; \mathbb{Z}/2 \qquad (4.31)$$

be the composition of the inverse of the Pontrjagin-Thom isomorphism $\tau : \Omega^{\mathrm{fr}}_n \xrightarrow{\cong} \pi^s_n$, the forgetful homomorphism $f : \Omega^{\mathrm{fr}}_{4k+2} \to \Omega^{\mathrm{alm}}_{4k+2}$ and the map $\mathsf{Arf} : \Omega^{\mathrm{alm}}_{4k+2} \to \mathbb{Z}/2$

Theorem 4.32 *Let $k \geq 3$. Then bP^{4k+2} is a trivial group if the homomorphism* Arf : $\pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (4.31) *is surjective and is $\mathbb{Z}/2$ if the homomorphism* Arf : $\pi^s_{4k+2} \to \mathbb{Z}/2$ *of* (4.31) *is trivial.*

Proof: We conclude from Adam's computations of the J-homomorphism that the forgetful map $f : \Omega^{\mathrm{fr}}_{4k+2} \to \Omega^{\mathrm{alm}}_{4k+2}$ is surjective. Now the claim follows from the exact sequence

$$0 \to \Theta^{4i-2} \xrightarrow{\eta} \Omega^{\mathrm{alm}}_{4i-2} \xrightarrow{\mathsf{Arf}} \mathbb{Z}/2 \xrightarrow{\partial} bP^{4i-2} \to 0.$$

The next result is due to Browder

Theorem 4.33 *The homomorphism* Arf : $\pi_{4k+2}^s \to \mathbb{Z}/2$ *of* (4.31) *is trivial if* $2k+1 \neq 2^l - 1$

The homomorphism Arf : $\pi_{4k+2}^s \to \mathbb{Z}/2$ of (4.31) is also known to be non-trivial for $4k+2 \in \{6, 14, 30, 62\}$ Hence Theorem 4.32 and Theorem 4.33 imply

Corollary 4.34 *The group* bP^{4k+2} *is trivial or isomorphic to* $\mathbb{Z}/2$. *We have*

$$bP^{4k+2} = \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l - 2, k \geq 1; \\ 0 & 4k+2 \in \{6, 14, 30, 62\}. \end{cases}$$

We have already shown

Theorem 4.35 *We have for* $k \geq 3$

$$bP^{2k+1} = 0.$$

Theorem 4.36 *For $n \geq 1$ any homotopy n-sphere Σ is stably parallizable.*

For an almost parallizable manifold M the image of its class $[M] \in \Omega_n^{\text{alm}}$ under the homomorphism $\partial : \Omega_n^{\text{alm}} \to \pi_n(SO(n-1))$ is exactly the obstruction to extend the almost stable framing to a stable framing. Recall that any homotopy n-sphere is almost stably parallizable. The map ∂ is trivial for $n \neq 0 \mod 4$ by Adam's result about the J-homomorphism. If $n = 0 \mod 4$, the claim follows from $\text{sign}(M) = 0$.

Theorem 4.37 1. *If $n = 4k + 2$, then there is an exact sequence*

$$0 \to \Theta^n/bP^{n+1} \to \text{coker}\,(J_n : \pi_n(SO) \to \pi_n^s) \to \mathbb{Z}/2;$$

2. *If $n \neq 2 \mod 4$ or if $n = 4k+2$ with $2k+1 \neq 2^l - 1$, then*

$$\Theta^n/bP^{n+1} \;\cong\; \text{coker}\,(J_n : \pi_n(SO) \to \pi_n^s)\,.$$

Proof: Adam's result about the J-homomorphism implies for $\partial : \Omega_n^{\text{alm}} \to \pi_{n-1}(SO)$

$$\begin{array}{rcll} \ker(\partial) & = & \Omega_n^{\text{alm}} & n \neq 0 \mod 4; \\ \ker(\partial) & = & \ker\left(\frac{\text{sign}}{8} : \Omega_n^{\text{alm}} \to \mathbb{Z}\right) & n = 0 \mod 4; \\ \ker(\partial) & = & \text{coker}\,(J_n : \pi_n(SO) \to \pi_n^s)\,. & \end{array}$$

Now use the exact sequences

$$0 \to \Theta^{4i} \xrightarrow{\eta} \Omega_{4i}^{\text{alm}} \xrightarrow{\frac{\text{sign}}{8}} \mathbb{Z} \xrightarrow{\partial} bP^{4i} \to 0$$

$$0 \to \Theta^{4i-2} \xrightarrow{\eta} \Omega_{4i-2}^{\text{alm}} \xrightarrow{\text{Arf}} \mathbb{Z}/2 \xrightarrow{\partial} bP^{4i-2} \to 0$$

and

$$0 \to bP^{2j} \to \Theta^{2j-1} \xrightarrow{\eta} \Omega_{2j-1}^{\text{alm}} \to 0.$$

Theorem 4.38 Classification of homotopy spheres

1. *Let $k \geq 2$ be an integer. Then bP^{4k} is a finite cyclic group of order*

$$\frac{3-(-1)^k}{2} \cdot 2^{2k-2} \cdot (2^{2k-1}-1) \cdot \text{numerator}(B_k/(4k));$$

2. *Let $k \geq 1$ be an integer. Then bP^{4k+2} is trivial or isomorphic to $\mathbb{Z}/2$. We have*

$$bP^{4k+2} = \begin{cases} \mathbb{Z}/2 & 4k+2 \neq 2^l-2, k \geq 1; \\ 0 & 4k+2 \in \{6, 14, 30, 62\}. \end{cases}$$

3. *If $n = 4k+2$ for $k \geq 2$, then there is an exact sequence*

$$0 \to \Theta^n \to \text{coker}(J_n) \to \mathbb{Z}/2.$$

If $n = 4k$ for $k \geq 2$ or $n = 4k+2$ with $4k+2 \neq 2^l-2$, then

$$\Theta^n \cong \text{coker}(J_n);$$

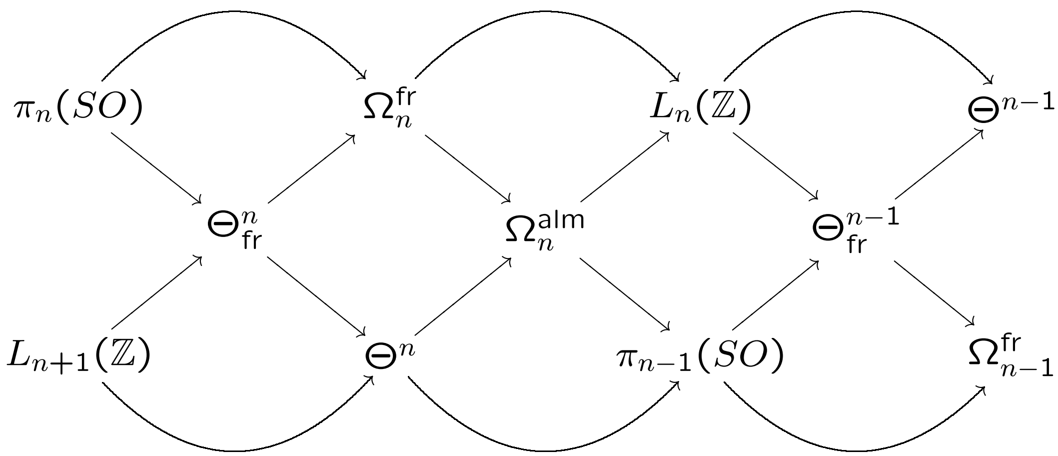
4. Let $n \geq 5$ be odd. Then there is an exact sequence

$$0 \to bP^{n+1} \to \Theta^n \to \text{coker}(J_n) \to 0.$$

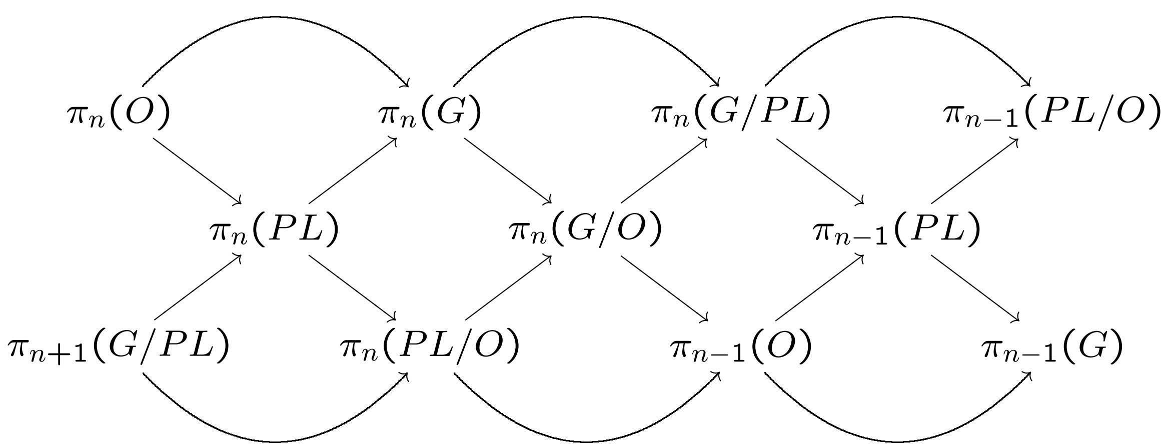
If $n \neq 2^l - 3$, the sequence splits.

n	1	2	3	4	5	6	7	8	9	10
Θ^n	1	1	?	1	1	1	28	2	8	6
bP^{n+1}	1	1	?	1	1	1	28	1	2	1
Θ^n/bP^{n+1}	1	1	1	1	1	1	1	2	4	6

Theorem 4.39 (The Kervaire-Milnor braid) *The following two braids are exact and isomorphic to one another for* $n \geq 5$.



and



Example 4.40 Let $W^{2n-1}(d)$ be the subset of $\mathbb{C}^{n+1}$ consisting of those points $(z_0, z_1, \ldots, z_n)$ which satisfy the equations $z_o^d + z_1^2 + \ldots z_n^2 = 0$ and $||z_0||^2 + ||z_1||^2 + \ldots + ||z_n||^2 = 1$. These are smooth submanifolds and called *Brieskorn varieties*. Suppose that d and n are odd, Then $W^{2n-1}(d)$ is a homotopy $(2n-1)$-sphere. It is diffeomorphic to the standard sphere S^{2n-1} if $d = \pm 1 \mod 8$ and it is an exotic sphere representing the generator of bP^{2n} if $d = \pm 3 \mod 8$.

Theorem 4.41 (Sphere theorem) *Let M be a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \frac{1}{4}$. Then M is homeomorphic to the standard sphere.*

Theorem 4.42 (Differentiable sphere theorem) *There exists a constant δ with $1 > \delta \geq \frac{1}{4}$ with the following property: if M is a complete simply connected Riemannian manifold whose sectional curvature is pinched by $1 \geq \sec(M) > \delta$. then M is diffeomorphic to the standard sphere.*

Remark 4.43 Let Σ be a homotopy n-sphere for $n \geq 5$. Let $D_0^n \to \Sigma$ and $D_1^n \to \Sigma$ be two disjoint embedded discs. Then $W = \Sigma - (\text{int}(D_0^n) \coprod \text{int}(D_1^n))$ is a simply-connected h-cobordism. By the h-cobordism there is a diffeomorphism $(F, \text{id}, f) : \partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$. Hence Σ is oriented diffeomorphic to $D^n \cup_{f : S^{n-1} \to S^{n-1}} (D^n)^-$ for some orientation preserving diffeomorphism $f : S^{n-1} \to S^{n-1}$. If f is isotopic to the identity, Σ is oriented diffeomorphic to S^n. Hence the existence of exotic spheres shows the existence of selfdiffeomorphisms of S^{n-1} which are homotopic but not isotopic to the identity.

5. Assembly maps, Isomorphism Conjectures and the Borel Conjecture

The results of this lecture are partially joint with Jim Davis.

Let $\mathcal{C}$ be a small category.

Example 5.1 Our main example will be the *orbit category* $\text{Or}(G)$ of a group G. It has as objects homogeneous G-spaces G/H. Morphisms are G-maps.

We define the category SPECTRA of spectra as follows. A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence $\{E(n) \mid n \in \mathbb{Z}\}$ of pointed spaces together with pointed (structure) maps $\sigma(n) : E(n) \wedge S^1 \to E(n+1)$. A map of spectra $\mathbf{f} : \mathbf{E} \to \mathbf{E}'$ is a sequence of maps of pointed spaces $f(n) : E(n) \to E'(n)$ compatible with the structure maps.

The homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) \ := \ \text{colim}_{k\to\infty}\,\pi_{i+k}(E(k)).$$

A *weak homotopy equivalence* of spectra is a map $\mathbf{f}:\mathbf{E}\to\mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Definition 5.2 A *covariant $\mathcal{C}$-space* is a covariant functor from $\mathcal{C}$ to the category of topological spaces. Morphisms are natural transformations. Define analogously *covariant pointed space*, *covariant spectrum* and the contravariant notions.

Example 5.3 For a G-space X we get a contravariant $\text{Or}(G)$-space $\text{map}_G(?, X)$ by

$$G/H \mapsto \text{map}_G(G/H, X) = X^H$$

and a covariant $\text{Or}(G)$-space $? \times_G X$ by

$$G/H \mapsto X \times_G G/H = H\backslash X.$$

Remark 5.4 Coproduct, product, pushout, pullback, colimit and limit exist in the category of $\mathcal{C}$-spaces.

Definition 5.5 *Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their* tensor product *to be the space*

$$X \otimes_{\mathcal{C}} Y \;:=\; \coprod_{c \in \mathrm{ob}(\mathcal{C})} X(c) \times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi : c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$.

Definition 5.6 *Given $\mathcal{C}$-spaces X and Y, denote by $\mathrm{hom}_{\mathcal{C}}(X, Y)$ the space of maps of $\mathcal{C}$-spaces from X to Y with the subspace topology coming from the obvious inclusion into $\prod_{c \in \mathrm{ob}(\mathcal{C})} \mathrm{map}(X(c), Y(c))$.*

Lemma 5.7 *Let X be a contravariant $\mathcal{C}$-space, Y be a covariant $\mathcal{C}$-space and Z be a space. Denote by* $\mathrm{map}(Y, Z)$ *the contravariant $\mathcal{C}$-space whose value at an object c is the mapping space* $\mathrm{map}(Y(c), Z)$*. Then $- \otimes_{\mathcal{C}} Y$ and* $\mathrm{map}(Y, -)$ *are adjoint, i.e. there is a homeomorphism natural in X, Y and Z*

$$T = T(X, Y, Z) : \mathrm{map}(X \otimes_{\mathcal{C}} Y, Z) \xrightarrow{\cong} \mathrm{hom}_{\mathcal{C}}(X, \mathrm{map}(Y, Z)).$$

Lemma 5.8 *Let X be a space and let Y and Z be covariant (contravariant) $\mathcal{C}$-spaces. Let $X \times Y$ be the obvious covariant (contravariant) $\mathcal{C}$-space. Then there is an adjunction homeomorphism*

$$T(X, Y, Z) : \mathrm{hom}_{\mathcal{C}}(X \times Y, Z) \xrightarrow{\cong} \mathrm{map}(X, \mathrm{hom}_{\mathcal{C}}(Y, Z)).$$

Remark 5.9 We have introduced the notion of a tensor product and of the mapping space for $\mathcal{C}$-spaces. They can analogously be defined for pointed $\mathcal{C}$-spaces, just replace disjoint unions $\coprod$ and cartesian products $\prod$ by wedges $\vee$ and by smash products $\wedge$. All the adjunction properties carry over.

Consider the set $\text{ob}(\mathcal{C})$ as a small category in the trivial way, i.e. the set of objects is $\text{ob}(\mathcal{C})$ itself and the only morphisms are the identity morphisms. A map of two $\text{ob}(\mathcal{C})$-spaces is a collection of maps $\{f(c) : X(c) \to Y(c) \mid c \in \text{ob}(\mathcal{C})\}$. There is a forgetful functor

$$F : \mathcal{C}\text{-SPACES} \rightarrow \text{ob}(\mathcal{C})\text{-SPACES}.$$

Define a functor

$$B : \text{ob}(\mathcal{C})\text{-SPACES} \rightarrow \mathcal{C}\text{-SPACES}$$

by sending a contravariant $\text{ob}(\mathcal{C})$-space X to $\coprod_{c \in \text{ob}(\mathcal{C})} \text{mor}_{\mathcal{C}}(?, c) \times X(c)$. In the covariant case one uses $\text{mor}_{\mathcal{C}}(c, ?)$.

Lemma 5.10 *The functor B is the left adjoint of F.*

Proof: We have to specify a homeomorphisms

$$T(X,Y) : \text{hom}_{\mathcal{C}}(B(X),Y) \to \text{hom}_{\text{ob}(\mathcal{C})}(X,F(Y))$$

for all $ob(\mathcal{C})$-spaces X and for all $\mathcal{C}$-spaces Y. For

$$f(?) : B(X) = \coprod_{c \in \text{ob}(\mathcal{C})} \text{mor}_{\mathcal{C}}(?,c) \times X(c) \to Y(?)$$

define $T(X,Y)(f)$ by restricting f to $X(?) = \{\text{id}_?\} \times X(?)$. The inverse $T(X,Y)^{-1}$ assigns to a map $g(?) : X(?) \to Y(?)$ of ob($\mathcal{C}$)-spaces the transformation

$$B(X) = \coprod_{c \in \text{ob}(\mathcal{C})} \text{mor}_{\mathcal{C}}(?,c) \times X(c) \to Y(?)$$

given by $B(X)(\phi, x) = Y(\phi) \circ g(c)(x)$.

Definition 5.11 *A* G-CW-complex X *is a* G-*space* X *together with a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

such that $X = \operatorname{colim}_{n\to\infty} X_n$ *and for any* $n \geq 0$ *the* n-skeleton X_n *is obtained from the* $(n-1)$-*skeleton* X_{n-1} *by attaching equivariant cells, i.e. there exists a pushout of* $\mathcal{C}$-*spaces of the form*

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \longrightarrow & X_n \end{array}$$

Definition 5.12 *A* contravariant $\mathcal{C}$-CW-complex X *is a contravariant* $\mathcal{C}$*-space* X *together with a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

such that $X = \operatorname{colim}_{n\to\infty} X_n$ *and for any* $n \geq 0$ *the* n-skeleton X_n *is obtained from the* $(n-1)$*-skeleton* X_{n-1} *by attaching contravariant* $\mathcal{C}$*-*n*-cells, i.e. there exists a pushout of* $\mathcal{C}$*-spaces of the form*

$$\begin{array}{ccc} \coprod_{i\in I_n} \operatorname{mor}_{\mathcal{C}}(?, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} \operatorname{mor}_{\mathcal{C}}(?, c_i) \times D^{n} & \longrightarrow & X_n \end{array}$$

Lemma 5.13 *If* X *is a* G*-*CW*-complex, then* $\operatorname{map}_G(?, X)$ *is a* $\operatorname{Or}(G)$*-*CW*-complex.*

Definition 5.14 *A map* $f : X \to Y$ *of* $\mathcal{C}$*-spaces is* a weak homotopy equivalence *if for all objects* c *the map of spaces* $f(c) : X(c) \to Y(c)$ *is a weak homotopy equivalence.*

Theorem 5.15 *Let $f : Y \to Z$ be a map of $\mathcal{C}$-spaces and X be a $\mathcal{C}$-space. Then f is a weak homotopy equivalence if and only if*

$$f_* : [X, Y]^{\mathcal{C}} \to [X, Z]^{\mathcal{C}}, \qquad [g] \mapsto [g \circ f]$$

is bijective for any $\mathcal{C}$-CW-complex X.

Corollary 5.16 *A weak homotopy equivalence between $\mathcal{C}$-CW-complexes is a homotopy equivalence.*

Definition 5.17 *A $\mathcal{C}$-CW-*approximation *$u : X' \to X$ of a $\mathcal{C}$-space X consists of a $\mathcal{C}$-CW-complex X' together with a weak equivalence u.*

Theorem 5.18 *1. There exists a functorial construction of a $\mathcal{C}$-CW-approximation;*

2. Given a map $f : X \to Y$ of $\mathcal{C}$-spaces and $\mathcal{C}$-CW-approximations $u : X' \to X$ and $v : Y' \to Y$, there exists a map f' making the following diagram commutative up to homotopy

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

The map f' is unique up to homotopy;

Definition 5.19 *Let $\mathbf{E}$ be a covariant $\mathcal{C}$-spectrum. Define for a contravariant $\mathcal{C}$-space X its* homology with coefficients in $\mathbf{E}$ *by*

$$H_p^{\mathcal{C}}(X;\mathbf{E}) \;=\; \pi_p(X'_+ \otimes_{\mathcal{C}} \mathbf{E})$$

for any CW-approximation $u : X' \to X$.

Theorem 5.20 $H_*(-, \mathbf{E})$ *is a generalized homology theory for contravariant* $\mathcal{C}$*-spaces satisfying the disjoint union axiom and the* WHE*-axiom*

Homology theory means that homotopic maps induce the same homomorphism on $H_*(-, \mathbf{E})$, there is a long exact a sequence of a pair and we have a Mayer-Vietoris sequence for any commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

whose evaluation at each object is a pushout of spaces with a cofibration as left vertical arrow. The WHE-axiom means that a weak equivalence of contravariant $\mathcal{C}$-spaces induce isomorphisms on homology. The disjoint union axiom says that there is a natural isomorphism

$$\oplus_{i \in I} H_p^{\mathcal{C}}(X_i; \mathbf{E}) \xrightarrow{\cong} H_p^{\mathcal{C}}(\coprod_{i \in I} X_i; \mathbf{E}).$$

Lemma 5.21 *Let* $\mathbf{f} : \mathbf{E} \to \mathbf{F}$ *be a weak equivalence of covariant* $\mathcal{C}$*-spectra. It induces a natural isomorphism*

$$\mathbf{f}_* : H_*^{\mathcal{C}}(X;\mathbf{E}) \to H_*^{\mathcal{C}}(X;\mathbf{F}).$$

Definition 5.22 *Let* $\mathbf{E}$ *be a covariant* $\mathsf{Or}(G)$*-spectrum. Define for a* G*-space* X

$$H_p^G(X;\mathbf{E}) := H_p^{\mathsf{Or}(G)}(\mathsf{map}_G(?,X);\mathbf{E}).$$

Theorem 5.23 $H_*^G(-,\mathbf{E})$ *is a generalized homology theory for* G*-spaces satisfying the disjoint union axiom and the* WHE*-xiom. We have*

$$H_p^G/H;\mathbf{E}) = \pi_p(\mathbf{E}(G/H)).$$

Theorem 5.24 *The exist covariant* Or(G)-*spectra*

$$\begin{aligned} \mathbf{K} : \mathrm{Or}(G) &\to \Omega\text{-SPECTRA}; \\ \mathbf{L} : \mathrm{Or}(G) &\to \Omega\text{-SPECTRA}; \\ \mathbf{K}^{\mathrm{top}} : \mathrm{Or}(G) &\to \Omega\text{-SPECTRA} \end{aligned}$$

satisfying for all $p \in \mathbb{Z}$

$$\begin{aligned} \pi_p(\mathbf{K}(G/H)) &\cong K_p(\mathbb{Z}H); \\ \pi_p(\mathbf{L}(G/H)) &\cong L_p^{\langle -\infty \rangle}(\mathbb{Z}H); \\ \pi_p(\mathbf{K}^{\mathrm{top}}(G/H)) &\cong K_p^{\mathrm{top}}(C_r^*(H)). \end{aligned}$$

Definition 5.25 *Let* $\mathbf{E}$ *be a covariant* Or(G)-*spectrum and* X *be a* G-*space. Then the associated assembly map is the map induced by the projection* $X \to G/G$

$$\begin{aligned} \mathrm{asmb} : H_p^G(X; \mathbf{E}) \\ \to H_p^G(\{*\}; \mathbf{E}) = \pi_p(\mathbf{E}(G/G)). \end{aligned}$$

Definition 5.26 *Let* G *be a group and* $\mathcal{F}$ *be a family of subgroups, i.e. a set of subgroups closed under conjugation and taking subgroups. A* classifying space $E(G; \mathcal{F})$ of G with respect to $\mathcal{F}$ *is a left* G-CW-*complex such that* $E(G, \mathcal{F})^H$ *is contractible for* $H \in \mathcal{F}$ *and empty otherwise.*

Theorem 5.27 *1. There is a functorial construction of $E(G, \mathcal{F})$;*

2. For any G-CW-complex X whose isotropy groups do belong to $\mathcal{F}$ there is up to G-homotopy precisely one G-map $X \to E(G; \mathcal{F})$. In particular $E(G; \mathcal{F})$ is unique up to G-homotopy;

Remark 5.28 Given a covariant Or(G)-spectrum $\mathbf{E}$ and a family $\mathcal{F}$ of subgroups, we obtain an assembly map

$$\text{asmb} : H_p^G(E(G; \mathcal{F}); \mathbf{E}) \\ \to H_p^G(\{*\}; \mathbf{E}) = \pi_p(\mathbf{E}(G/G)).$$

The Isomorphism Conjecture for $\mathbf{E}$ and $\mathcal{F}$ says that it is an isomorphism.

The point is to find $\mathcal{F}$ as small as possible. If we take $\mathcal{F}$ to be the family of all subgroups, the map above is an isomorphism but this is a trivial and useless fact. The philosophy is to express $\pi_p(\mathbf{E}(G/G))$, which is the group we are interested, in by the groups $\pi_q(\mathbf{E}(G/H))$ for $q \leq p$ and $H \in \mathcal{F}$, which we hopefully understand.

Let $\mathcal{FIN}$ be the family of finite subgroups and $\mathcal{VC}$ be the family of virtually cyclic subgroups.

Conjecture 5.29 (Baum-Connes Conjecture) *Take* $\mathbf{E} = \mathbf{K}^{\text{top}}$ *and* $X = E(G, \mathcal{FIN})$. *Then the assembly map*

$$\text{asmb} : H_p(E(G; \mathcal{FIN}); \mathbf{K}^{\text{top}}) \to K_p^{\text{top}}(C_r^*(G))$$

is an isomorphism.

Conjecture 5.30 (Farrell-Jones Isomorphism Conjecture) *Take* $\mathbf{E} = \mathbf{K}$ *or* $\mathbf{L}$ *and* $X = E(G; \mathcal{VC})$. *Then the assembly maps*

$$\text{asmb} : H_p(E(G; \mathcal{VC}); \mathbf{K} \to K_p(\mathbb{Z}G)$$

and

$$\text{asmb} : H_p(E(G; \mathcal{VC}); \mathbf{L}) \to L_p^{-\infty}(\mathbb{Z}G)$$

are isomorphisms.

Remark 5.31 If one replaces in the Farrell-Jones Isomorphism Conjecture the decoration $\langle -\infty \rangle$ by other decorations such as p, h or s, it becomes false (see Farrell-Jones-L.).

Remark 5.32 The Farrell-Jones Conjecture makes also sense for any coefficient ring R instead of $\mathbb{Z}$. If R is a field F of characteristic zero, one may replace $\mathcal{VC}$ by $\mathcal{FIN}$ in the Farrell-Jones Isomorphism Conjecture for K-theory. In particular it reduces for K_0 to the statement that the canonical map

$$\operatorname{colim}_{H \subset G, |H| < \infty} K_0(FH) \xrightarrow{\cong} K_0(FG)$$

is bijective.

One has to use $\mathcal{VC}$ in general to take Nil-terms into account which appear for instance in the Bass-Heller-Swan decomposition

$$\begin{aligned} K_1(\mathbb{Z}[G \times \mathbb{Z}]) \;\cong\; & K_0(\mathbb{Z}G) \oplus K_1(\mathbb{Z}G) \\ & \oplus \operatorname{Nil}(\mathbb{Z}G) \oplus \operatorname{Nil}(\mathbb{Z}G). \end{aligned}$$

Remark 5.33 Suppose G is torsionfree. Then the Baum-Connes Conjecture reduces to an isomorphism

$$K_p^{\mathrm{top}}(BG) \to K_p^{\mathrm{top}}(C_r^*(G)).$$

The Farrell-Jones Isomorphism Conjecture for $p \leq 1$ is equivalent to the statement that $K_i(\mathbb{Z}G)$ for $i \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Conjecture 5.34 (Borel Conjecture) *Let M and N be closed aspherical manifolds. Then any homotopy equivalence $f : M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic if and only if they have isomorphic fundamental groups.*

Theorem 5.35 *If the Farrell-Jones Isomorphism Conjecture holds for G, then the Borel Conjecture holds for closed aspherical manifolds M and N of dimension ≥ 5 and $\pi_1(M) \cong \pi_1(N) \cong G$.*

Sketch of proof: The Borel Conjecture is equivalent to the claim

$$\mathcal{S}_n^{\mathrm{top}}(M) = \{\mathrm{id} : M \to M\}.$$

We have the surgery exact sequence

$$\ldots \to [\Sigma M, G/TOP] \to L_{p+1}^s(\mathbb{Z}\pi) \to \mathcal{S}_n^{\mathrm{top}}(M) \\ \to [M, G/O] \to L_p(\mathbb{Z}\pi)^s.$$

The K-theory part of the Farrell-Jones Isomorphism Conjecture ensures that we do not have to take care of the decorations for the L-groups. The assembly map in the L-theory part in dimension p and $p+1$ can be identified with the first map and last map appearing in the part of surgery sequence above.

Remark 5.36 The assembly map for a covariant $\mathsf{Or}(G)$-spectrum $\mathbf{E}$ in the special case $X = E(G, \mathcal{F})$ can be identified with the homomorphism induced on homotopy groups by the canonical map

$$\begin{aligned} &\mathrm{hocolim}_{\mathsf{Or}(G,\mathcal{F})} \mathbf{E}|_{\mathsf{Or}(G,\mathcal{F})} \\ &\qquad \to \mathrm{hocolim}_{\mathsf{Or}(G)} \mathbf{E} = \mathbf{E}(G/G). \end{aligned}$$

Remark 5.37 There is an Atiyah-Hirzebruch spectral sequence convering to $H^G_{p+q}(X; \mathbf{E})$ whose E^2-term is given by the Bredon homology

$$E^2_{p,q} \;=\; H_p^{\mathsf{Or}(G)}(X; \pi_q(\mathbf{E}(G/H)).$$

There is another spectral sequence due to Davis-L. which comes from a filtration by chains of subgroups $\{1\} = H_0 \subset H_1 \subset H_2 \subset \ldots H_q \subset G$ with $H_i \neq H_{i+1}$ and H_i a subgroup of an isotropy group of X.

Remark 5.38 The assembly maps in the conjectures above were originally defined differently, for instance in the Baum-Connes Conjecture by an index map. The identifications of the various versions of assembly maps is non-trivial.

A covariant functor

$\mathbf{E} : G\text{–}\mathcal{F}\text{–}CW\text{–}$COMPLEXES $\to$ SPECTRA

is called *(weakly) $\mathcal{F}$-homotopy invariant* if it sends G-homotopy equivalences to (weak) homotopy equivalences of spectra. The functor $\mathbf{E}$ is *(weakly) $\mathcal{F}$-excisive* if it has the following four properties

1. it is (weakly) $\mathcal{F}$-homotopy invariant;

2. $\mathbf{E}(\emptyset)$ is contractible;

3. it respects homotopy pushouts up to (weak) homotopy equivalence;

4. $\mathbf{E}$ respects countable disjoint unions up to (weak) homotopy;

Remark 5.39 *$\mathbf{E}$ is weakly $\mathcal{F}$-excisive if and only if $\pi_q(\mathbf{E}(X))$ defines a homology theory on the category of G-$\mathcal{F}$-CW-complexes satisfying the disjoint union axiom for countable disjoint unions.*

Lemma 5.40 *Let* $\mathbf{T} : \mathbf{E} \to \mathbf{F}$ *be a transformation of (weakly)* $\mathcal{F}$*-excisive functors*

$$\mathbf{E}, \mathbf{F} : G\text{--}\mathcal{F}\text{--}CW\text{--}\mathsf{COMPLEXES} \to \mathsf{SPECTRA}$$

so that $\mathbf{T}(G/H)$ *is a (weak) homotopy equivalence of spectra for all* $H \in \mathcal{F}$*. Then* $\mathbf{T}(X)$ *is a (weak) homotopy equivalence of spectra for all* G*-*$\mathcal{F}$*-*CW*-complexes* X*.*

Theorem 5.41 *Consider a covariant functor*

$$\mathbf{E} : \mathsf{Or}(G; \mathcal{F}) \to \mathsf{SPECTRA}\,.$$

Define

$$E_{\%} : G\text{--}\mathcal{F}\text{--}CW\text{--}\mathsf{COMPLEXES} \to \mathsf{SPECTRA}$$

by sending X *to* $\mathsf{map}_G(?, X) \otimes_{\mathsf{Or}(G;\mathcal{F})} \mathbf{E}$*. Then:*

1. $\mathbf{E}_{\%}$ *is $\mathcal{F}$-excisive;*

2. *For any (weakly) $\mathcal{F}$-homotopy invariant functor*

$$\mathbf{E} : G\text{–}\mathcal{F}\text{–}CW\text{–}\mathsf{COMPLEXES} \to \mathsf{SPECTRA}$$

there is a (weakly) $\mathcal{F}$-excisive functor

$$\mathbf{E}^{\%} : G - \mathcal{F} - CW - \mathsf{COMPLEXES} \to \mathsf{SPECTRA}$$

and natural transformations

$$\begin{aligned} \mathbf{A_E} : \mathbf{E}^{\%} &\to \mathbf{E}; \\ \mathbf{B_E} : \mathbf{E}^{\%} &\to (\mathbf{E}\,|_{\mathsf{Or}(G,\mathcal{F})})_{\%}; \end{aligned}$$

which induce (weak) homotopy equivalences of spectra $\mathbf{A_E}(G/H)$ for all $H \in \mathcal{F}$ and (weak) homotopy equivalences of spectra $\mathbf{B_E}(X)$ for all G-$\mathcal{F}$-CW-complexes X. Given a family $\mathcal{F}' \subset \mathcal{F}$, $\mathbf{E}$ is (weakly) $\mathcal{F}'$-excisive if and only if $\mathbf{A_E}(X)$ is a (weak) homotopy equivalence of spectra for all G-$\mathcal{F}'$-CW-complexes X.

Remark 5.42 The theorem above characterizes the assembly map in the sense that

$$\mathbf{A_E} : \mathbf{E}^{\%} \longrightarrow \mathbf{E}$$

is the universal approximation from the left by a (weakly) $\mathcal{F}$-excisive functor of a (weakly) $\mathcal{F}$-homotopy invariant functor $\mathbf{E}$ from G-$\mathcal{F}$-CW-COMPLEXES to SPECTRA. Namely, let

$$\mathbf{T} : \mathbf{F} \longrightarrow \mathbf{E}$$

be a transformation of functors from G-$\mathcal{F}$-CW-COMPLEXES to SPECTRA such that $\mathbf{F}$ is (weakly) $\mathcal{F}$-excisive and $\mathbf{T}(G/H)$ is a (weak) homotopy equivalence for all $H \in \mathcal{F}$. Then for any G-$\mathcal{F}$-CW-complex X the following diagram commutes

$$\begin{array}{ccc} \mathbf{F}^{\%}(X) & \xrightarrow[\simeq]{\mathbf{A_F}(X)} & \mathbf{F}(X) \\ \mathbf{T}^{\%}(X) \downarrow \simeq & & \downarrow \mathbf{T}(X) \\ \mathbf{E}^{\%}(X) & \xrightarrow{\mathbf{A_E}(X)} & \mathbf{E}(X) \end{array}$$

and $\mathbf{A_F}(X)$ and $\mathbf{T}^{\%}(X)$ are (weak) homotopy equivalences. Hence one may say that $\mathbf{T}(X)$ factorizes over $\mathbf{A_E}(\mathbf{X})$.

Remark 5.43 We can apply the construction above to the the weakly $\mathcal{F}$-homotopy invariant functor

$$\mathbf{E} : G\text{–}\mathcal{F}\text{–}CW\text{–COMPLEXES} \to \text{SPECTRA}$$

which sends X to

$$\mathbf{K}^{\text{top}}(C_r^*(\pi(EG \times_G X))$$
$$\mathbf{K}(\pi(EG \times_G X))$$
$$\mathbf{L}(\pi(EG \times_G X))$$

Then the assembly map appearing in the Isomorphism Conjectures above is given by

$$\pi_p(\mathbf{A_E}(X)) : \pi_p(\mathbf{E}^{\%}(X)) \to \pi_p(\mathbf{E}(X))$$

if one puts $X = E(G, \mathcal{FIN})$ *or* $X = E(G; \mathcal{VC})$*.*