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Wolfgang Lück

Isomorphism Conjectures in *K*- and *L*-Theory

With contributions by Christoph Winges

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Preface

The Isomorphism Conjectures due to Baum and Connes and to Farrell and Jones aim at the topological K-theory of reduced group C^*-algebras and the algebraic K-and L-theory of group rings. These theories are of major interest for many reasons. For instance, the algebraic L-groups are the recipients for various surgery obstructions and hence highly relevant for the classification of manifolds. Other important obstructions such as Wall's finiteness obstruction and Whitehead torsion take values in algebraic K-groups. The topological K-groups of C^*-algebras play a central role in index theory and the classification of C^*-algebras.

In general these K- and L-groups are very hard to analyze for group rings or group C^*-algebras. The Isomorphism Conjectures identify them with equivariant homology groups of classifying spaces for families of subgroups. As an illustration, let us consider the special case that G is a torsionfree group and R is a regular ring (with involution). Then the Isomorphism Conjectures predict that the so-called assembly maps

$$\begin{aligned} H_n(BG;\mathbf{K}(R)) &\xrightarrow{\cong} K_n(RG); \\ H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) &\xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG); \\ K_n(BG) &\xrightarrow{\cong} K_n(C_r^*(G)), \end{aligned}$$

are isomorphisms for all $n \in \mathbb{Z}$. The target is the algebraic K-theory of the group ring RG, the algebraic L-theory of RG with decoration $\langle -\infty\rangle$, or the topological K-theory of the reduced group C^*-algebra $C_r^*(G)$. The source is the evaluation of a specific homology theory on the classifying space BG, where $H_n(\{\bullet\};\mathbf{K}(R)) \cong K_n(R)$, $H_n(\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}(R)) \cong L_n^{\langle -\infty\rangle}(R)$, and $K_n(\{\bullet\}) \cong K_n(\mathbb{C})$ hold for all $n \in \mathbb{Z}$.

Since the sources of these assembly maps are much more accessible than the targets, the Isomorphism Conjectures are key ingredients for explicit computations of the K-and L-groups of group rings and reduced group C^*-algebras. These often are motivated by and have applications to concrete problems that arise, for instance, in the classification of manifolds or C^*-algebras.

The Baum-Connes Conjecture and the Farrell-Jones Conjecture imply many other well-known conjectures. In a lot of cases these conjectures were not known to be true for certain groups until the Baum-Connes Conjecture or the Farrell-Jones Conjecture was proved for them. Examples of such prominent conjectures are the Borel Conjecture about the topological rigidity of aspherical closed manifolds, the (stable) Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature on closed Spin-manifolds, Kaplansky's Idempotent Conjecture and the Kadison Conjecture on the non-existence of non-trivial idempotents in the group ring or the reduced group C^*-algebra of torsionfree groups, the Novikov Conjecture about the homotopy invariance of higher signatures, and the conjectures about the vanishing of the reduced projective class group of $\mathbb{Z}G$ and the Whitehead group of G for a torsionfree group G.

The Baum-Connes Conjecture and the Farrell-Jones Conjecture are still open in general at the time of writing. However, tremendous progress has been made on the class of groups for which they are known to be true. The techniques of the sophisticated proofs stem from algebra, dynamical systems, geometry, group theory, operator theory, and topology. The extreme broad scope of the Baum-Connes Conjecture and the Farrell-Jones Conjecture is both the main challenge and the main motivation for writing this book. We hope that, after having read parts of this monograph, the reader will share the enthusiasm of the author for the Isomorphism Conjectures.

The monograph is a guide to and gives a panorama of Isomorphism Conjectures and related topics. It presents or at least indicates the most advanced results and developments at the time of writing. It can be used by various groups of readers, such as experts on the Baum-Connes Conjecture or the Farrell-Jones Conjecture, experienced mathematicians, who may not be experts on these conjectures but want to learn or just apply them, and also, of course, advanced undergraduate and graduate students. References for further reading and information have been inserted.

We will give more information about the organization of the book and a user's guide in Section 1.11.

Bonn, May 2025 *Wolfgang Lück*

Contents

Part I: Introduction to K- and L-theory

Part II: The Isomorphism Conjectures

Part III: Methods of Proofs

Chapter 1
Introduction

The Isomorphism Conjectures due to Paul Baum and Alain Connes and to Tom Farrell and Lowell Jones are important conjectures, which have many interesting applications and consequences. However, they are not easy to formulate and it is a priori not clear why the actual versions are the most promising ones. The current versions are the final upshot of a longer process, which has led to them step by step. They have been influenced and steered by various new results that have been proved during the last decades and given new insight into the objects, problems, and constructions at which these conjectures aim.

In this introduction we want to motivate these conjectures by explaining how one can be led to them by general considerations and certain facts. We present brief surveys about applications of these conjectures, their status, and the methods of proof. We give information about the contents of this monograph including a user's guide.

1.1 Why Should we Care about Isomorphism Conjectures in K- and L-Theory?

In this section we give some background and motivation for the reader who has no previous knowledge about the Baum-Connes Conjecture and the Farrell-Jones Conjecture. An expert may skip this section.

The Baum-Connes Conjecture aims at the topological K-theory of the reduced group C^*-algebra of a group, whereas the Farrell-Jones Conjecture is devoted to the algebraic K- and L-theory of the group ring of a group. K- and L-theory are rather sophisticated theories. Group rings are very difficult rings, for instance, they are in general not commutative, are not Noetherian or regular, and may have zero-divisors. So studying the algebraic K-theory and L-theory of group rings is hard and seems at first glance to be a very special problem. So why should one care?

The answer to this question is that information about the K- or L-theory of group rings or the topological K-theory of group C^*-algebras have many applications to algebra, geometry, group theory, topology, and operator algebras and that meanwhile these conjectures are known for a large class of groups.

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1.1.1 Projective Class Group

Let us illustrate this by considering the most prominent and easy to define K-group, *the projective class group* $K_0(S)$ *of a ring* S. It is the abelian group which we obtain from the Grothendieck construction applied to the abelian semigroup of isomorphism classes of finitely generated projective S-modules under direct sum. Equivalently, it can be described as the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective S-modules P and for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective S-modules we require the relation $[P_1] = [P_0] + [P_2]$. The *reduced projective class group* $\widetilde{K}_0(S)$ *of a ring* S is obtained from $K_0(S)$ by dividing out the subgroup generated by all finitely generated free S-modules. Any finitely generated projective S-module P defines an element $[P]$ in $K_0(S)$ and hence also a class $[P]$ in $\widetilde{K}_0(S)$. The decisive property of $\widetilde{K}_0(S)$ is that $[P] = 0$ holds in $\widetilde{K}_0(S)$ if and only if P is *stably free*, i.e., there are natural numbers m and n satisfying $P \oplus S^m \cong_S S^n$. So roughly speaking, $[P] \in \widetilde{K}_0(S)$ measures the deviation of a finitely generated projective S-module P from being stably free.

Why are we especially interested in the case $S = RG$, where R is a ring, G is a group, and RG is the *group ring*? (The precise definition of RG can be found in Subsection 2.8.) One reason is that a representation of G with coefficients in R is the same as an RG-module. Another reason is that for a connected manifold or CW-complex its universal covering comes with an action of the fundamental group π and the cellular $\mathbb{Z}$-chain complex of the universal covering is actually a free $\mathbb{Z}\pi$-chain complex. The latter observation opens the door to connections of algebraic K-theory to topological problems, as described next.

A CW-complex X is called *finitely dominated* if there is a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to id_X. Often one can construct a finitely dominated CW-complex with interesting properties but one needs to know whether it is homotopy equivalent to a finite CW-complex. This problem is decided by the *finiteness obstruction* of Wall. A finitely dominated connected CW-complex X with fundamental group π determines an element $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}\pi)$, which vanishes if and only if X is homotopy equivalent to a finite CW-complex, see Theorem 2.39. So it is interesting to know whether $\widetilde{K}_0(\mathbb{Z}\pi)$ vanishes because then $\widetilde{o}(X)$ is automatically trivial. One can actually show for a finitely presented group G that $\widetilde{K}_0(\mathbb{Z}G)$ vanishes if and only if every finitely dominated connected CW-complex with fundamental group isomorphic to G is homotopy equivalent to a finite CW-complex. So we have an algebraic assertion and a topological assertion for a group G which turn out to be equivalent.

The question whether a finitely dominated CW-complex is homotopy equivalent to a finite CW-complex appears naturally in the construction of closed manifolds with certain properties, since a closed manifold is homotopy equivalent to a finite CW-complex, and one may be able to construct a finitely dominated CW-complex as a first approximation up to homotopy. This is explained in Section 2.5. The *Spherical Space Form Problem* 9.205 is a prominent example. It aims at the classification of closed manifolds whose universal coverings are diffeomorphic or homeomorphic to the standard sphere.

If R is a field and the group G is torsionfree, then the *Idempotent Conjecture* of Kaplansky predicts that the group ring RG has only trivial idempotent, namely, 0 and 1. Roughly speaking, non-trivial idempotents in a ring can be used to decompose the ring into smaller pieces; think for instance of the theorems of Wedderburn and Maschke which imply that for a finite group G and a field F of characteristic zero the group ring FG is a product of matrix algebras over skew-fields. The Idempotent Conjecture shows that this does not apply to torsionfree groups. On the other hand, the non-existence of non-trivial idempotents gives hope that one can embed the group ring FG of a torsionfree group and a field F into a skew-field as conjectured by Malcev, which opens the door to many applications in group theory and topology, see Remark 2.85. This ring theoretic conjecture due to Kaplansky is related to the projective class group, since it is known to be true if $\widetilde{K}_0(RG)$ vanishes. There are many instances of groups where no algebraic proof is known for the Idempotent Conjecture, but one can show with geometric, homotopy theoretic, and K-theoretic methods that $\widetilde{K}_0(RG)$ vanishes.

A special version of the Farrell-Jones Conjecture, see Conjecture 2.60, predicts that $\widetilde{K}_0(RG)$ vanishes if G is torsionfree and R is $\mathbb{Z}$ or a field.

All of this is explained in detail in Chapter 2.

1.1.2 The Whitehead Group

Here is another example of a nice connection between algebraic K-theory and topology. One can define $K_1(S)$ of a ring S as the abelianization of the general linear group $\mathrm{GL}(S)$ or, equivalently, as the abelian group generated by conjugacy classes of automorphisms of finitely generated projective S-modules with relations concerning exact sequences and composites of such automorphisms. Given a group G, the *Whitehead group* $\mathrm{Wh}(G)$ is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup generated by trivial units. For more details we refer to Definition 3.1, Theorem 3.12, and Definition 3.23. This is related to topology as follows.

Given a closed manifold M, an *h-cobordism W over M* is a compact manifold W such that its boundary ∂W can be written as a disjoint union $\partial W = \partial_0 W \amalg \partial_1 W$, there is a preferred identification of M with $\partial_0 W$, and the inclusions $\partial_k W \to W$ are homotopy equivalences for $k = 0, 1$. The set of isomorphism classes relative M of h-cobordisms over M can be identified with $\mathrm{Wh}(\pi)$ if M is connected, has dimension ≥ 5, and π denotes its fundamental group, see Theorem 3.47. This is remarkable since the set of isomorphism classes of h-cobordism relative M over M a priori depends on M, whereas $\mathrm{Wh}(\pi)$ depends only on the fundamental group. In the classification of closed manifolds it is often a key step to decide whether an *h-cobordism W over M* is trivial, i.e., isomorphic relative M to $M \times [0,1]$, since this has the consequence that M and $\partial_1 W$ are isomorphic. It is not hard to show that $\mathrm{Wh}(\{1\})$ is trivial which, together with the results above, implies the Poincaré

Conjecture in dimensions ≥ 5, see Theorem 3.51. One can show for a finitely presented group G and any natural number $n \geq 5$ that $\mathrm{Wh}(G)$ is trivial if and only if for every connected n-dimensional closed manifold M with fundamental group isomorphic to G every h-cobordism over M is trivial. So we have again an algebraic assertion and a topological assertion for a group G which turn out to be equivalent.

A special version of the Farrell-Jones Conjecture, see Conjecture 3.110, predicts that $\mathrm{Wh}(G)$ vanishes if G is torsionfree. All of this is explained in detail in Chapter 3.

1.1.3 The Borel Conjecture and the Novikov Conjecture

One of the author's favorite conjectures is the *Borel Conjecture*. It predicts that an aspherical closed manifold is topologically rigid. *Aspherical* means that the universal covering is contractible and *topologically rigid* means that every homotopy equivalence from a closed manifold to M is homotopic to a homeomorphism. In particular it implies that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic. One may view the Borel Conjecture as the topological counterpart of Mostow rigidity, see Remark 9.169.

If G denotes the fundamental group of an aspherical closed manifold of dimension ≥ 5, then the Borel Conjecture for M holds if G satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture for $\mathbb{Z}G$, see Theorem 9.171. Moreover, all proofs of the Borel Conjecture in dimensions ≥ 4 are based on the Farrell-Jones Conjecture. So we see again that the Farrell-Jones Conjecture has interesting applications to topology.

L-theory, which one may think of as the algebraic K-theory of quadratic forms over finitely generated projective modules, is an important ingredient in the so-called Surgery Program 3.53, whose highlight is the Surgery Exact Sequence, see Theorem 9.127. It aims at the classification of closed manifolds, see Remark 3.53, and was initiated by the classification of exotic spheres, see Remark 3.52.

All this is explained in Chapter 9. In particular, we refer to Sections 9.12, 9.14, and 9.15.

Note that both the Baum-Connes Conjecture and the Farrell-Jones Conjecture imply the prominent *Novikov Conjecture* about the homotopy invariance of higher signatures, see Remark 9.143 and Theorem 14.29. The Novikov Conjecture and its link to both the Baum-Connes Conjecture and the Farrell-Jones Conjecture triggered a lot of interesting interactions and transfer of methods and techniques between topology and operator theory.

1.1.4 Further Applications

There are many more striking applications of the Farrell-Jones Conjecture and the Baum-Connes Conjecture to algebra, geometric group theory, geometry, topology, and operator algebras, which are listed in Sections 13.12 and 14.8. We hope that, by browsing through these sections, the reader will be convinced of the great interest of these conjectures.

1.1.5 Status of the Full Farrell-Jones Conjecture and the Baum-Connes Conjecture with Coefficients

The Full Farrell-Jones Conjecture 13.30 implies all the variants of the Farrell-Jones Conjecture scattered in this monograph, see Theorem 13.65. A list of all the versions of the Farrell-Jones Conjecture can be found in Subsection 13.11.1.

The Baum Connes Conjecture with coefficients 14.11 is the most general variant in the Baum-Connes setting.

The class of groups for which the Full Farrell-Jones Conjecture 13.30 is known to be true is discussed in Theorem 16.1, whereas the class of groups for which the Baum Connes Conjecture with coefficients 14.11 is known to be true is discussed in Theorem 16.7. The question whether the Full Farrell-Jones Conjecture 13.30 might be true for all groups and how one might find counterexamples is treated in Section 16.10. This should convince the reader that in many interesting cases one knows that these conjectures are known to be true. Roughly speaking, in "daily life" one can expect that the Farrell-Jones Conjecture is known to be true and one can just apply it.

If one wants to figure out quickly whether a specific class of groups satisfies one of these conjectures, one should take a look at Section 16.8. Open cases are discussed in Section 16.9.

At the time of writing, no counterexamples to the Full Farrell-Jones Conjecture 13.30 are known. This is also true for the Baum-Connes Conjecture 14.11 (without coefficients). Counterexamples to the Baum Connes Conjecture with coefficients 14.11 are discussed in Remark 14.12.

1.1.6 Proofs

The proofs of the Farrell-Jones Conjecture or the Baum-Connes Conjecture are sophisticated and require a lot of different techniques. The proof of inheritance properties, such as the passage to subgroups, are usually based on homotopy theoretic methods. The proofs for specific classes of groups, such as hyperbolic groups, are based on transfer methods in the Farrell-Jones setting and on KK-theory in the Baum-Connes setting and for both conjectures require additional geometric input, which is the interesting and surprising part. For instance flow spaces play a prominent role in the proof of the Farrell-Jones Conjecture for hyperbolic groups or finite-dimensional CAT(0)-groups. It is intriguing and astonishing that the proofs of the Idempotent Conjecture of Kaplansky, which is a purely ring theoretic statement, are based for many groups on the proof of the Farrell-Jones Conjecture and thus use geometric input such as flows and compactifications of certain spaces on which the group in question acts. Often purely algebraic methods are not sufficient to prove the Idempotent Conjecture.

The reader who wants to get a first impression about the proofs should consult Chapter 19.

1.2 The Statement of the Baum-Connes Conjecture and of the Farrell-Jones Conjecture

Next we record the statements of the Baum-Connes Conjecture and Farrell-Jones Conjecture. Explanations and motivations will follow. The versions stated below will be generalized later.

Conjecture 1.1 (Baum-Connes Conjecture). Let G be a group. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called an *assembly map*,

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Conjecture 1.2. (Farrell-Jones Conjecture for $K_*(RG)$). Let G be a group. Let R be an associative ring with unit. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called an *assembly map*,

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Conjecture 1.3. (Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$). Let G be a group. Let R be an associative ring with unit and involution. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called an *assembly map*,

$$H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

The general pattern is that the target of the assembly map is what we want to understand or to compute, namely, the K- and L-theory of group rings and group C^*-algebras, and that the source is a homological expression, which is much more accessible than the source and depends only on the values of the K- or L-groups under considerations for finite subgroups or for virtually cyclic subgroups of G. The spaces $\underline{E}G$ and $\underline{\underline{E}}G$ are classifying spaces for the family of finite subgroups and the family of virtually cyclic subgroups, which are inserted in specific G-homology theories.

1.3 Motivation for and Evolution of the Baum-Connes Conjecture

We will start with the Isomorphism Conjecture that is the easiest and most convenient to state and motivate, the Baum-Connes Conjecture for the topological K-theory of reduced group C^*-algebras. Then we will pass to the Farrell-Jones Conjecture for the algebraic K- and L-theory of group rings, which is more complicated due to the appearance of Nil-terms.

1.3.1 Topological K-Theory of Reduced Group C^*-Algebras

The target of the Baum-Connes Conjecture is the topological K-theory of the *reduced group C^*-algebra* $C_r^*(G)$ of a group G. We will consider discrete groups G only. One defines the *topological K-groups* $K_n(A)$ for any Banach algebra A to be the abelian group $K_n(A) = \pi_{n-1}(\mathrm{GL}(A))$ for $n \geq 1$. The famous *Bott Periodicity Theorem* gives a natural isomorphism $K_n(A) \xrightarrow{\cong} K_{n+2}(A)$ for $n \geq 1$. Finally one defines $K_n(A)$ for all $n \in \mathbb{Z}$ so that the Bott isomorphism theorem is true for all $n \in \mathbb{Z}$. It turns out that $K_0(A)$ is the same as the *projective class group* of the ring A, which is the Grothendieck group of the abelian monoid of isomorphism classes of finitely generated projective A-modules with the direct sum as addition. The topological K-theory of $\mathbb{C} = C_r^*(\{1\})$ is trivial in odd dimensions and isomorphic to $\mathbb{Z}$ in even dimensions. More generally, for a finite group G the topological K-theory of $C_r^*(G)$ is the complex representation ring $R_{\mathbb{C}}(G)$ in even dimensions and is trivial in odd dimensions.

Let P be an appropriate elliptic differential operator (or more generally an elliptic complex) on a closed n-dimensional Riemannian manifold M, for instance the Dirac operator or the signature operator. Then one can consider its *index* in $K_n(\mathbb{C})$, which is $\dim_{\mathbb{C}}(\ker(P)) - \dim_{\mathbb{C}}(\mathrm{coker}(P)) \in \mathbb{Z}$ for even n and is zero for odd n. If M comes with an isometric G-action of a finite group G and P is compatible with the G-action, then $\ker(P)$ and $\mathrm{coker}(P)$ are complex finite-dimensional G-representations and one obtains an element in $K_n(C_r^*(G)) = R_{\mathbb{C}}(G)$ by $[\ker(P)] - [\mathrm{coker}(P)]$ for even n. Suppose that G is an arbitrary discrete group and that M is a (not necessarily compact) n-dimensional smooth manifold without boundary with a proper cocompact G-action, a G-invariant Riemannian metric, and an appropriate elliptic differential operator P compatible with the G-action. An example is the universal covering $M = \widetilde{N}$ of an n-dimensional closed Riemannian manifold N with $G = \pi_1(N)$ and the lift $\widetilde{P}$ to $\widetilde{N}$ of an appropriate elliptic differential operator P on N. Then one can define an *equivariant index* of P which takes values in $K_n(C_r^*(G))$. Therefore the interest in $K_*(C_r^*(G))$ comes from the fact that it is the natural recipient for indices of certain equivariant differential operators. All this will be explained in Chapter 10.

1.3.2 Homological Aspects

A first basic problem is to compute $K_*(C_r^*(G))$ or to identify it with more familiar terms. The key idea comes from the observation that $K_*(C_r^*(G))$ has some homological properties. More precisely, if G is the amalgamated free product $G = G_1 *_{G_0} G_2$ for subgroups $G_i \subseteq G$, then there is a long exact sequence

(1.4)
$$\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) \xrightarrow{K_n(C_r^*(i_1))\oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2))$$
$$\xrightarrow{K_n(C_r^*(j_1))-K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0))$$
$$\xrightarrow{K_{n-1}(C_r^*(i_1))\oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1))$$
$$\xrightarrow{K_{n-1}(C_r^*(j_1))-K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots$$

where i_1,i_2, j_1, and j_2 are the obvious inclusions, see [812, Theorem 18 on page 632]. If $\phi\colon G \to G$ is a group automorphism and $G \rtimes_\phi \mathbb{Z}$ is the associated semidirect product, then there is a long exact sequence

(1.5)
$$\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(\phi))-\mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_\phi \mathbb{Z}))$$
$$\xrightarrow{\partial_n} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(\phi))-\mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots$$

where k is the obvious inclusion, see [811, Theorem 3.1 on page 151] or more generally [812, Theorem 18 on page 632].

We compare this with group homology in order to explain the analogy with homology. Recall that the *classifying space* BG of a group G is an aspherical CW-complex whose fundamental group is isomorphic to G and that *aspherical* means that all higher homotopy groups are trivial, or, equivalently, that the universal covering is contractible. The classifying space BG is unique up to homotopy. If one has an amalgamated free product $G = G_1 *_{G_0} G_2$, then one can find models for the classifying spaces such that BG_i is a CW-subcomplex of BG and $BG = BG_1 \cup BG_2$ and $BG_0 = BG_1 \cap BG_2$. Thus we obtain a pushout of inclusions of CW-complexes

$$\begin{tikzcd}
BG_0 \arrow[r, "Bi_1"] \arrow[d, "Bi_2"'] & BG_1 \arrow[d, "Bj_1"] \\
BG_2 \arrow[r, "Bj_2"'] & BG.
\end{tikzcd}$$

It yields a long Mayer-Vietoris sequence for the cellular or singular homology

(1.6)
$$\cdots \xrightarrow{\partial_{n+1}} H_n(BG_0) \xrightarrow{H_n(Bi_1)\oplus H_n(Bi_2)} H_n(BG_1) \oplus H_n(BG_2)$$
$$\xrightarrow{H_n(Bj_1)-H_n(Bj_2)} H_n(BG) \xrightarrow{\partial_n} H_{n-1}(BG_0)$$
$$\xrightarrow{H_{n-1}(Bi_1)\oplus H_{n-1}(Bi_2)} H_{n-1}(BG_2) \oplus H_{n-1}(BG_1)$$
$$\xrightarrow{H_{n-1}(Bj_1)-H_{n-1}(Bj_2)} H_{n-1}(BG) \xrightarrow{\partial_{n-1}} \cdots .$$

If $\phi\colon G \to G$ is a group automorphism, then a model for $B(G \rtimes_\phi \mathbb{Z})$ is given by the mapping torus of $B\phi\colon BG \to BG$, which is obtained from the cylinder $BG \times [0,1]$ by identifying the bottom and the top with the map $B\phi$. Associated to a mapping torus, there is the long exact sequence

$$(1.7) \quad \cdots \xrightarrow{\partial_{n+1}} H_n(BG) \xrightarrow{H_n(B\phi)-\mathrm{id}} H_n(BG) \xrightarrow{H_n(Bk)} H_n(B(G \rtimes_\phi \mathbb{Z})) \xrightarrow{\partial_n} H_{n-1}(BG) \xrightarrow{H_{n-1}(B\phi)-\mathrm{id}} H_{n-1}(BG) \xrightarrow{H_n(Bk)} \cdots$$

where k is the obvious inclusion of BG into the mapping torus.

1.3.3 The Baum-Connes Conjecture for Torsionfree Groups

There is an obvious analogy between the sequences (1.4) and (1.6) and the sequences (1.5) and (1.7). On the other hand we get for the trivial group $G = \{1\}$ that $H_n(B\{1\}) = H_n(\{\bullet\})$ is $\mathbb{Z}$ for $n = 0$ and trivial for $n \neq 0$ so that the group homology of BG cannot be the same as the topological K-theory of $C_r^*(\{1\})$. But there is a better candidate, namely take the *topological K-homology* of BG instead of the singular homology. Topological K-homology is a homology theory defined for CW-complexes. At least we mention that for a topologist its definition is routine, namely, it is the homology theory associated to the K-theory spectrum which defines the topological K-theory of CW-complexes, i.e., the cohomology theory which comes from considering vector bundles over CW-complexes. In contrast to singular homology, the topological K-homology of a point $K_n(\{\bullet\})$ is $\mathbb{Z}$ for even n and is trivial for n odd. So we still get exact sequences (1.6) and (1.7) if we replace H_* by K_* everywhere and we have $K_n(B\{1\}) \cong K_n(C_r^*(\{1\})$ for all $n \in \mathbb{Z}$. This leads to the following conjecture.

Conjecture 1.8 (Baum-Connes Conjecture for torsionfree groups). Let G be a torsionfree group. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called an *assembly map*,

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

This is indeed a formulation which will turn out to be equivalent to the Baum-Connes Conjecture 1.1, provided that G is torsionfree. Conjecture 1.8 cannot hold in general as the example of a finite group G already shows. Namely, if G is finite, then the obvious inclusion induces an isomorphism $K_n(B\{1\}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$ for every $n \in \mathbb{Z}$, whereas $K_0(C_r^*(\{1\}) \to K_0(C_r^*(G))$ agrees with the map $R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(G)$, which is rationally bijective if and only if G itself is trivial. Hence Conjecture 1.8 is not true for non-trivial finite groups.

1.3.4 The Baum-Connes Conjecture

What is going wrong? The sequences (1.4) and (1.5) exist regardless of whether the groups are torsionfree or not. More generally, if G acts on a tree, then they can be combined to compute the K-theory $K_*(C_r^*(G))$ of a group G by a certain Mayer-Vietoris sequence from the stabilizers of the vertices and edges, see Pimsner [812, Theorem 18 on page 632]). In the special case where all stabilizers are finite, one sees that $K_*(C_r^*(G))$ is built by the topological K-theory of the finite subgroups of G in a homological fashion. This leads to the idea that $K_*(C_r^*(G))$ can be computed in a homological way, but the building blocks do not only consist of $K_*(C_r^*(\{1\}))$ alone but of $K_*(C_r^*(H))$ for all finite subgroups $H \subseteq G$. This suggests to study *equivariant topological K-theory*. It assigns to every proper G-CW-complex X a sequence of abelian groups $K_n^G(X)$ for $n \in \mathbb{Z}$ such that G-homotopy invariance holds and Mayer-Vietoris sequences exist. A *proper G-CW-complex* is a CW-complex with G-action such that for every $g \in G$ and every open cell e with $e \cap g \cdot e \neq \emptyset$ we have $gx = x$ for all $x \in e$ and all isotropy groups are finite. Two interesting features are that $K_n^G(G/H)$ agrees with $K_n(C_r^*(H))$ for every finite subgroup $H \subseteq G$ and that for a free G-CW-complex X and $n \in \mathbb{Z}$ we have a natural isomorphism $K_n^G(X) \xrightarrow{\cong} K_n(G\backslash X)$. Recall that EG is a free G-CW-complex which is contractible and that $EG \to G\backslash EG = BG$ is the universal covering of BG. We can reformulate Conjecture 1.8 by stating an isomorphism

$$K_n^G(EG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Now suppose that G acts on a tree T with finite stabilizers. Then the computation of Pimsner [812, Theorem 18 on page 632]) mentioned above can be rephrased to the statement that there is an isomorphism

$$K_n^G(T) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular the left-hand side is independent of the tree T, on which G acts by finite stabilizers. This can be explained as follows. It is known that for every finite subgroup $H \subseteq G$ the H-fixed point set T is again a non-empty tree and hence contractible. This implies that two trees T_1 and T_2, on which G acts with finite stabilizers, are G-homotopy equivalent and hence have the same equivariant topological K-theory. The same remark applies to $K_n(BG)$ and $K_n^G(EG)$, namely, two models for BG are homotopy equivalent and two models for EG are G-homotopy equivalent and therefore $K_n(BG)$ and $K_n^G(EG)$ are independent of the choice of a model. This leads to the idea to look for an appropriate proper G-CW-complex $\underline{E}G = E_{\mathcal{FIN}}(G)$, which is characterized by a certain universal property and is unique up to G-homotopy, such that for a torsionfree group G we have $EG = \underline{E}G$, for a tree on which G acts with finite stabilizers, we have $\underline{E}G = T$, and there is an isomorphism

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular for a finite group we would like to have $\underline{E}G = G/G = \{\bullet\}$ and then the desired isomorphism above is true for trivial reasons. Recall that EG is characterized up to G-homotopy by the property that it is a G-CW-complex such that EG^H is empty for $H \neq \{1\}$ and is contractible for $H = \{1\}$. Having the case of a tree on which G acts with finite stabilizers in mind, we define the *classifying space for proper G-actions* $\underline{E}G$ to be a G-CW-complex such that $\underline{E}G^H$ is empty for $|H| = \infty$ and is contractible for $|H| < \infty$. Indeed, two models for $\underline{E}G$ are G-homotopy equivalent, a tree on which G acts with finite stabilizers is a model for $\underline{E}G$, we have $EG = \underline{E}G$ if and only if G is torsionfree, and $\underline{E}G = G/G = \{\bullet\}$ if and only if G is finite. This leads to the Baum-Connes Conjecture, stated already as Conjecture 1.1. Classifying spaces for families will be treated in detail in Chapter 11.

The Baum-Connes Conjecture 1.1 makes sense for all groups, and no counterexamples are known at the time of writing. The Baum-Connes Conjecture 1.1 reduces in the torsionfree case to Conjecture 1.8 and is consistent with the result of Pimsner [812, Theorem 18 on page 632] for G acting on a tree with finite stabilizers. It is obviously true for finite groups G. Pimsner's result holds more generally for groups acting on trees with not necessarily finite stabilizers. So one should get the analogous result for the left-hand side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. Essentially this boils down to the question whether the analogs of the long exact sequences (1.4) and (1.5) hold for the left side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. This follows for (1.4) from the fact that for $G = G_1 *_{G_0} G_2$ one can find appropriate models for the classifying spaces for proper G-actions such that there is a G-pushout of inclusions of proper G-CW-complexes

$$\begin{tikzcd}
G \times_{G_0} \underline{E}G_0 \arrow[r] \arrow[d] & G \times_{G_1} \underline{E}G_1 \arrow[d] \\
G \times_{G_2} \underline{E}G_2 \arrow[r] & \underline{E}G
\end{tikzcd}$$

and for a subgroup $H \subseteq G$ and a proper H-CW-complex X there is a natural isomorphism

$$K_n^H(X) \xrightarrow{\cong} K_n^G(G \times_H X).$$

Thus the associated long exact Mayer-Vietoris sequence yields the long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n^{G_0}(\underline{E}G_0) \to K_n^{G_1}(\underline{E}G_1) \oplus K_n^{G_2}(\underline{E}G_2) \to K_n^G(\underline{E}G) \xrightarrow{\partial_n}$$
$$K_{n-1}^{G_0}(\underline{E}G_0) \to K_{n-1}^{G_1}(\underline{E}G_1) \oplus K_{n-1}^{G_2}(\underline{E}G_2) \to K_{n-1}^{G_0}(\underline{E}G) \to \cdots$$

which corresponds to (1.4). For (1.5) one uses the fact that for a group automorphism $\phi\colon G \xrightarrow{\cong} G$ the $G \rtimes_\phi \mathbb{Z}$-$CW$-complex given by the bilaterally infinite mapping telescope of the ϕ-equivariant map $\underline{E}\phi\colon \underline{E}G \to \underline{E}G$ is a model for $\underline{E}(G \rtimes_\phi \mathbb{Z})$.

In general $K_n^G(\underline{E}G)$ is much bigger than $K_n^G(EG) \cong K_n(BG)$ and the canonical map $K_n^G(EG) \to K_n^G(\underline{E}G)$ is rationally injective but not necessarily integrally injective.

1.3.5 Reduced versus Maximal Group C^*-Algebras

All the arguments above also apply to the *maximal group C^*-algebra*, which has even better functorial properties than the reduced group C^*-algebra. So a priori one may think that one should use the maximal group C^*-algebra instead of the reduced one. However, the version for the maximal group C^*-algebra is not true in general and the version for the reduced group C^*-algebra seems to be the right one. This will be discussed in more detail in subsection 14.5.1.

If one considers instead of the reduced group C^*-algebra the Banach group algebra $L^1(G)$, one obtains the *Bost Conjecture* 14.23.

1.3.6 Applications of the Baum-Connes Conjecture

The assembly map appearing in the Baum-Connes Conjecture 1.1 has an index-theoretic interpretation. An element in $K_0^G(\underline{E}G)$ can be represented by a pair (M, P^*) consisting of a cocompact proper smooth n-dimensional G-manifold M with a G-invariant Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M and its image under the assembly map is a certain equivariant index $\mathrm{ind}_{C_r^*(G)}(M, P^*)$ in $K_n(C_r^*(G))$. There are many important consequences of the Baum-Connes Conjecture such as the *Kadison Conjecture*, see Subsection 10.4.2, the *stable Gromov-Lawson-Rosenberg Conjecture*, see Subsection 14.8.4, the *Novikov Conjecture*, see Section 9.14, and the *(Modified) Trace Conjecture*, see Subsections 10.4.1 and 14.8.3.

A summary of the applications of the Baum-Connes Conjecture is given in Section 14.8.

1.4 Motivation for and Evolution of the Farrell-Jones Conjecture for K-Theory

Next we want to deal with the algebraic K-groups $K_n(RG)$ of the group ring RG.

1.4.1 Algebraic K-Theory of Group Rings

For an associative ring with unit R one defines $K_0(R)$ to be the *projective class group* of R and $K_1(R)$ to be the abelianization of $\mathrm{GL}(R) = \mathrm{colim}_{n\to\infty} \mathrm{GL}_n(R)$. The *higher algebraic K-groups* $K_n(R)$ for $n \geq 1$ are the homotopy groups of a certain *K-theory space* associated to the category of finitely generated projective R-modules. One can define *negative K-groups* $K_n(R)$ for $n \leq -1$ by a certain contracting procedure applied to $K_0(R)$. Finally there exists a *K-theory spectrum* $\mathbf{K}(R)$ such that $\pi_n(\mathbf{K}(R)) = K_n(R)$ holds for every $n \in \mathbb{Z}$. If $\mathbb{Z} \to R$ is the obvious ring map sending n to $n \cdot 1_R$, then one defines for $n \leq 1$ the *reduced K-groups* to be the cokernel of the induced map $K_n(\mathbb{Z}) \to K_n(R)$. The *Whitehead group* $\mathrm{Wh}(G)$ of a group G is the quotient of $K_1(\mathbb{Z}G)$ by elements given by $(1,1)$-matrices of the shape $(\pm g)$ for $g \in G$.

The *reduced projective class group* $\widetilde{K}_0(\mathbb{Z}G)$ is the recipient for the *finiteness obstruction* of a finitely dominated CW-complex X with fundamental group $G = \pi_1(X)$. *Finitely dominated* means that there is a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. The Whitehead group $\mathrm{Wh}(G)$ is the recipient of the *Whitehead torsion* of a homotopy equivalence of finite CW-complexes and of a compact h-cobordism over a closed manifold, where G is the fundamental group. An *h-cobordism W over M* consists of a manifold W whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$ such that both inclusions $\partial_i W \to W$ are homotopy equivalences, together with an isomorphism $M \xrightarrow{\cong} \partial_0 W$. The finiteness obstruction and the Whitehead torsion are very important topological obstructions whose vanishing has interesting geometric and topological consequences. The finiteness obstruction vanishes if and only if the finitely dominated CW-complex under consideration is homotopy equivalent to a finite CW-complex. The Whitehead torsion of a compact h-cobordism W over M of dimension ≥ 6 vanishes if and only if W is trivial, i.e., is isomorphic to a cylinder $M \times [0,1]$ relative $M = M \times \{0\}$. This explains why topologists are interested in $K_n(\mathbb{Z}G)$ for groups G.

All these definitions and results will be explained in Chapters 2, 3, 4, 5, and 6.

1.4.2 Appearance of Nil-Terms

The situation for the algebraic K-theory of RG is more complicated than the one for the topological K-theory of $C_r^*(G)$. As a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

For algebraic K-theory the analog is the *Bass-Heller-Swan decomposition*

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R)$$

where certain additional terms, the Nil-*terms* $NK_n(R)$, appear, see Subsection 6.3.4. If one replaces R by RG, one gets

$$K_n(R[G \times \mathbb{Z}]) \cong K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG).$$

Such correction terms in the form of Nil-terms also appear when one wants to get analogs of the sequences (1.4) and (1.5) for algebraic K-theory, see Section 6.9.

1.4.3 The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings and Torsionfree Groups

Let R be a *regular ring*, i.e., it is Noetherian and every R-module possesses a finite-dimensional projective resolution. For instance, any principal ideal domain is a regular ring. Then one can prove in many cases for torsionfree groups that the analogs of the sequences (1.4) and (1.5) hold for algebraic K-theory, see Waldhausen [974] and [977]. The same reasoning as in the Baum-Connes Conjecture for torsionfree groups leads to the following conjecture.

Conjecture 1.9. (Farrell-Jones Conjecture for $K_*(RG)$ for torsionfree groups and regular rings). Let G be a torsionfree group and let R be a regular ring. Then there is for every $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{\cong} K_n(RG).$$

Here $H_*(-; \mathbf{K}(R))$ is the homology theory associated to the K-theory spectrum of R. It is a homology theory with the property that $H_n(\{\bullet\}; \mathbf{K}(R)) = \pi_n(\mathbf{K}(R)) = K_n(R)$ holds for every $n \in \mathbb{Z}$.

1.4.4 The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings

If one drops the condition that G is torsionfree but requires that the order of every finite subgroup of G is invertible in R, then in many cases one can still prove that the analogs of the sequences (1.4) and (1.5) hold for algebraic K-theory. The same reasoning as in the Baum-Connes Conjecture leads to the following conjecture.

Conjecture 1.10. (Farrell-Jones Conjecture for $K_*(RG)$ for regular rings). Let G be a group. Let R be a regular ring such that $|H|$ is invertible in R for every finite subgroup $H \subseteq G$. Then there is for every $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Here $H_n^G(-;\mathbf{K}_R)$ is an appropriate G-homology theory with the property that $H_n^G(G/H;\mathbf{K}_R) \cong H_n^H(\{\bullet\};\mathbf{K}_R) \cong K_n(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$, and the isomorphism above is induced by the G-map $\underline{E}G \to \{\bullet\}$. Conjecture 1.10 reduces to Conjecture 1.9 if G is torsionfree.

1.4.5 The Farrell-Jones Conjecture for $K_*(RG)$

Conjecture 1.9 can be applied in the case $R = \mathbb{Z}$, which is not true for Conjecture 1.10. So what is the right formulation for arbitrary rings R? The idea is that one not only needs to take all finite subgroups into account but also all virtually cyclic subgroups. A group is called *virtually cyclic* if it is finite or contains $\mathbb{Z}$ as subgroup of finite index. Namely, let $\underline{\underline{E}}G = E_{\mathcal{VCY}}(G)$ be the *classifying space for the family of virtually cyclic subgroups*, i.e., a G-CW-complex $\underline{\underline{E}}G$ such that $\underline{\underline{E}}G^H$ is contractible for every virtually cyclic subgroup $H \subseteq G$ and is empty for every subgroup $H \subseteq G$ which is not virtually cyclic. The G-space $\underline{\underline{E}}G$ is unique up to G-homotopy. These considerations lead to the Farrell-Jones Conjecture for $K_*(RG)$ stated already as Conjecture 1.2.

Conjecture 1.2 makes sense for all groups and rings, and no counterexamples are known at the time of writing. We have absorbed all the Nil-phenomena into the source by replacing $\underline{E}G$ by $\underline{\underline{E}}G$. There is a certain price to pay since often there are nice small geometric models for $\underline{E}G$, whereas the spaces $\underline{\underline{E}}G$ are much harder to analyze and are in general huge. There are up to G-homotopy unique G-maps $EG \to \underline{E}G$ and $\underline{E}G \to \underline{\underline{E}}G$ which yield maps

$$H_n(BG;\mathbf{K}(R)) \cong H_n^G(EG;\mathbf{K}_R) \to H_n^G(\underline{E}G;\mathbf{K}_R) \to H_n^G(\underline{\underline{E}}G;\mathbf{K}_R).$$

We will later see that there is a splitting, see Theorem 13.36,

$$H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) \cong H_n^G(\underline{E}G;\mathbf{K}_R) \oplus H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R) \tag{1.11}$$

where $H_n^G(\underline{E}G;\mathbf{K}_R)$ is the comparatively easy homological part and all Nil-type information is contained in $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$. If R is regular and the order of any finite subgroup of G is invertible in R, then $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$ is trivial and hence the natural map $H_n^G(\underline{E}G;\mathbf{K}_R) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G;\mathbf{K}_R)$ is bijective. Therefore Conjecture 1.2 reduces to Conjecture 1.9 and Conjecture 1.10 when they apply.

In the Baum-Connes setting the natural map $K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(\underline{\underline{E}}G)$ is always bijective.

1.4.6 Applications of the Farrell-Jones Conjecture for $K_*(RG)$

We have $K_n(\mathbb{Z}) = 0$ for $n \le -1$. Both the map $\mathbb{Z} \xrightarrow{\cong} K_0(\mathbb{Z})$ that sends n to $n \cdot [\mathbb{Z}]$ and the map $\{\pm 1\} \to K_1(\mathbb{Z})$ that sends ± 1 to the class of the $(1,1)$-matrix (± 1) are bijective. Therefore an easy spectral sequence argument shows that Conjecture 1.9 implies

Conjecture 1.12. (Farrell-Jones Conjecture $K_n(\mathbb{Z}G)$ in dimensions $n \le 1$). Let G be a torsionfree group. Then $\widetilde{K}_n(\mathbb{Z}G) = 0$ for $n \in \mathbb{Z}, n \le 0$ and $\mathrm{Wh}(G) = 0$.

In particular, the finiteness obstruction and the Whitehead torsion are always zero for torsionfree fundamental groups. This implies that every h-cobordism over a simply connected d-dimensional closed manifold for $d \ge 5$ is trivial and thus the Poincaré Conjecture in dimensions ≥ 6 (and with some extra effort also in dimension $d = 5$). This will be explained in Section 3.5. The Farrell-Jones Conjecture for K-theory, see Conjecture 1.2, implies the *Bass Conjecture*, see Section 2.10. *Kaplansky's Idempotent Conjecture* follows from the Farrell-Jones Conjecture for K-theory for torsionfree groups and regular rings, see Conjecture 1.9, as explained in Section 2.9. Further applications of the Conjecture 1.9, e.g., to pseudoisotopy and to automorphisms of manifolds, will be discussed in Section 9.21.

A summary of the applications of the Farrell-Jones Conjecture is given in Section 13.12.

1.5 Motivation for and Evolution of the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

Next we want to deal with the algebraic L-groups $L_n^{\epsilon}(RG)$ of the group ring RG of a group G with coefficients in an associative ring R with unit and involution.

1.5.1 Algebraic L-Theory of Group Rings

Let R be an associative ring with unit. An *involution of rings* $R \to R, r \mapsto \overline{r}$ on R is a map satisfying $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{rs} = \overline{s}\,\overline{r}$, $\overline{0} = 0$, $\overline{1} = 1$, and $\overline{\overline{r}} = r$ for all $r, s \in R$. Given a ring with involution, the group ring RG inherits an involution by $\sum_{g \in G} r_g \cdot g = \sum_{g \in G} \overline{r} \cdot g^{-1}$. If the coefficient ring R is commutative, we usually use the trivial involution $\overline{r} = r$. Given a ring with involution, one can associate to it *quadratic L-groups* $L_n^h(R)$ for $n \in \mathbb{Z}$. The abelian group $L_0^h(R)$ can be identified with the *Witt group* of non-degenerate quadratic forms on finitely generated free R-modules, where every hyperbolic quadratic form represents the zero element and the addition is given by the orthogonal sum of hyperbolic quadratic forms. The abelian group $L_2^h(R)$ is essentially given by the skew-symmetric versions. One defines

$L_1^h(R)$ and $L_3^h(R)$ in terms of automorphism of quadratic forms. The L-groups are four-periodic, i.e., there is a natural isomorphism $L_n^h(R) \xrightarrow{\cong} L_{n+4}^h(R)$ for $n \in \mathbb{Z}$. If one uses finitely generated projective R-modules instead of finitely generated free R-modules, one obtains the *proper quadratic L-groups* $L_n^p(R)$ for $n \in \mathbb{Z}$. For every $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 1\}$ there are versions $L_n^{\langle j \rangle}(R)$, where $\langle j \rangle$ is called a *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h. If R is $\mathbb{Z}G$, one uses finitely generated based free $\mathbb{Z}G$-modules and takes the Whitehead torsion into account, then one obtains the *simple quadratic L-groups* $L_n^s(\mathbb{Z}G) = L_n^{\langle 2 \rangle}(\mathbb{Z}G)$ for $n \in \mathbb{Z}$.

The relevance of the L-groups comes from the fact that they are the recipients for various surgery obstructions. The fundamental *surgery problem* is the following. Consider a map $f \colon M \to X$ from a closed manifold M to a finite Poincaré complex X. We want to know whether we can change it by a process called surgery to a map $g \colon N \to X$ with a closed manifold N as source and the same target such that g is a homotopy equivalence. This may answer the question whether a finite Poincaré complex X is homotopy equivalent to a closed manifold. Note that a space which is homotopy equivalent to a closed manifold must be a finite Poincaré complex, but not every finite Poincaré complex is homotopy equivalent to a closed manifold. If f comes with additional bundle data and has degree 1, we can find g if and only if the so-called *surgery obstruction* of f vanishes, which takes values in $L_n^h(\mathbb{Z}G)$ for $n = \dim(X)$ and $G = \pi_1(X)$. If we want g to be a simple homotopy equivalence, the obstruction lives in $L_n^s(\mathbb{Z}G)$. We see that, analogous to the finiteness obstruction in $\widetilde{K}_0(\mathbb{Z}G)$ and the Whitehead torsion in $\mathrm{Wh}(G)$, the algebraic L-groups are the recipients for important obstructions whose vanishing has interesting geometric and topological consequences. Also the question whether two closed manifolds are diffeomorphic or homeomorphic can be decided via surgery theory, of which the L-groups are a part.

More explanations about L-groups and surgery theory will be given in Chapter 9.

1.5.2 The Farrell-Jones Conjecture for $L_*(RG)[1/2]$

If we invert 2, i.e., if we consider the localization $L_n^{\langle -j \rangle}(RG)[1/2]$, then there is no difference between the various decorations and the analogs of the sequences (1.4) and (1.5) are true for L-theory, see Cappell [204]. The same reasoning as for the Baum-Connes Conjecture leads to the following conjecture.

Conjecture 1.13. (Farrell-Jones Conjecture for $L_*(RG)[1/2]$). Let G be a group. Let R be an associative ring with unit and involution. Then there is for every $n \in \mathbb{Z}$ and every decoration j an isomorphism

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle j \rangle})[1/2] \xrightarrow{\cong} L_n^{\langle j \rangle}(RG)[1/2].$$

Here $H_n^G(-;\mathbf{L}_R^{\langle j\rangle})$ is an appropriate G-homology theory with the property that $H_n^G(G/H;\mathbf{L}_R^{\langle j\rangle}) \cong H_n^H(\{\bullet\};\mathbf{L}_R^{\langle j\rangle}) \cong L_n^{\langle j\rangle}(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$ and the isomorphism above is induced by the G-map $\underline{E}G \to \{\bullet\}$.

1.5.3 The Farrell-Jones Conjecture for $L_*^{\langle -\infty\rangle}(RG)$

In general the L-groups $L_n^{\langle j\rangle}(RG)$ depend on the decoration and often the 2-torsion carries sophisticated information and is hard to handle. Recall that as a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

The L-theory analog is given by the Shaneson splitting [913]

$$L_n^{\langle j\rangle}(R[\mathbb{Z}]) \cong L_{n-1}^{\langle j-1\rangle}(R) \oplus L_n^{\langle j\rangle}(R).$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z};\mathbf{L}^{\langle j\rangle}(R)) \cong L_{n-1}^{\langle j\rangle}(R) \oplus L_n^{\langle j\rangle}(R).$$

Therefore the decoration $-\infty$ shows the right homological behavior and is the right candidate for the formulation of an isomorphism conjecture.

The analog of the sequence (1.4) does not hold for $L_*^{\langle j\rangle}(RG)$, certain correction terms, the UNil-*terms* come in, which are independent of the decoration $\langle j\rangle$ and are always (not necessarily finitely generated) 2-primary abelian groups, see Cappell [203], [204]. As in the algebraic K-theory case this leads to the Farrell-Jones Conjecture for $L_*^{\langle -\infty\rangle}(RG)$, stated already as Conjecture 1.3. The analog of the sequence (1.5) holds for $L_*^{\langle -\infty\rangle}(RG)$, see Theorem 13.60.

In Conjecture 1.3 the term $H_n^G(-;\mathbf{L}_R^{\langle -\infty\rangle})$ is an appropriate G-homology theory such that $H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^H(\{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$, and the assembly map is induced by the map $\underline{\underline{E}}G \to \{\bullet\}$. Conjecture 1.3 makes sense for all groups and rings with involution, and no counterexamples are known at the time of writing.

After inverting 2 Conjecture 1.3 is equivalent to Conjecture 1.13.

There is an L-theory version of the splitting (1.11)

$$H_n^G(\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) \oplus H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}), \tag{1.14}$$

provided that there exists an integer i_0 such that $K_i(RV) = 0$ holds for all virtually cyclic subgroups $V \subseteq G$ and $i \leq i_0$.

1.5.4 Applications of the Farrell-Jones Conjecture for $L_*^{\langle -\infty\rangle}(RG)$

For applications in geometry and topology the simple L-groups $L_n^s(\mathbb{Z}G)$ are the most interesting ones. The difference between the various decorations is measured by the so-called *Rothenberg sequences* and given in terms of the Tate cohomology of $\mathbb{Z}/2$ with coefficients in $\widetilde{K}_n(\mathbb{Z}G)$ for $n \le 0$ and $\mathrm{Wh}(G)$ with respect to the involution coming from the standard involution on the group ring $\mathbb{Z}G$ sending $\sum_{g\in G}\lambda_g \cdot g$ to $\sum_{g\in G}\lambda_g \cdot g^{-1}$. Hence the decorations do not matter if $\widetilde{K}_n(\mathbb{Z}G)$ for $n \le 0$ and $\mathrm{Wh}(G)$ vanish. In view of Conjecture 1.12, this leads to the following version of Conjecture 1.3 for torsionfree groups

Conjecture 1.15. (Farrell-Jones Conjecture for $L_*(\mathbb{Z}G)$ for torsionfree groups). Let G be a torsionfree group. Then there is for every $n \in \mathbb{Z}$ and every decoration j an isomorphism

$$H_n(BG;\mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \xrightarrow{\cong} L_n^{\langle j\rangle}(\mathbb{Z}G).$$

Moreover, the source, target, and the map itself are independent of the decoration j.

Here $H_n(-;\mathbf{L}^{\langle j\rangle}(\mathbb{Z}))$ is the homology theory associated to the L-theory spectrum $\mathbf{L}^{\langle -j\rangle}(\mathbb{Z})$ and satisfies $H_n(\{\bullet\};\mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \cong \pi_n\left(\mathbf{L}^{\langle j\rangle}(\mathbb{Z})\right) \cong L_n^{\langle j\rangle}(\mathbb{Z})$.

The L-theoretic assembly map appearing in Conjecture 1.15 has a topological meaning. It appears in the so-called *Surgery Exact Sequence*, which we will discuss in more detail in Section 9.12. Let $\mathbf{L}^s(\mathbb{Z})\langle 1\rangle$ be the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1\rangle$ of $\mathbf{L}^s(\mathbb{Z})$. There is a canonical map $\iota\colon H_n(BG;\mathbf{L}^s(\mathbb{Z})\langle 1\rangle) \to H_n(BG;\mathbf{L}^s(\mathbb{Z}))$. Let N be an aspherical oriented closed manifold with fundamental group G, i.e., an oriented closed manifold homotopy equivalent to BG. Then G is torsionfree, the source of the composite $H_n(BG;\mathbf{L}^s(\mathbb{Z})\langle 1\rangle) \to L_n^s(RG)$ of the assembly map appearing in Conjecture 1.15 with ι consists of bordism classes of normal maps $M \to N$ with N as target, and the composite sends such a normal map to its surgery obstruction. This is analogous to the Baum-Connes setting where the assembly map can be described by assigning to an equivariant index problem its index.

The third term in the Surgery Exact Sequence is the so-called *structure set* of N. It is the set of equivalence classes of simple homotopy equivalences $f_0\colon M_0 \to N$ with a closed topological manifold as source and N as target where $f_0\colon M_0 \to N$ and $f_1\colon M_1 \to N$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic. Conjecture 1.15 implies that this structure set is trivial provided that the dimension of N is greater or equal to five. Hence Conjecture 1.15 implies in dimensions ≥ 5 the following famous conjecture if G is isomorphic to the fundamental group.

Conjecture 1.16 (Borel Conjecture). Let M and N be two aspherical closed topological manifolds whose fundamental groups are isomorphic. Then they are homeomorphic, and every homotopy equivalence from M to N is homotopic to a homeomorphism.

The Borel Conjecture is a topological rigidity theorem for aspherical closed manifolds and analogous to the *Mostow Rigidity Theorem*, which says that two hyperbolic closed Riemannian manifolds with isomorphic fundamental groups are isometrically diffeomorphic. The Borel Conjecture is false if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. Its connection to the Borel Conjecture is one of the main features of the Farrell-Jones Conjecture. More details will be given in Subsections 9.15.2 and 9.15.3.

The Farrell-Jones Conjecture for L-theory 1.3 implies the *Novikov Conjecture*, see Section 9.14. It also has applications to the problem whether Poincaré duality groups or torsionfree hyperbolic groups with spheres as boundary are fundamental groups of aspherical closed manifolds, see Sections 9.17 and 9.18. Product decompositions of aspherical closed manifolds are treated in Section 9.20.

A summary of the applications of the Farrell-Jones Conjecture is given in Section 13.12.

1.6 More General Versions of the Farrell-Jones Conjecture

We will also treat versions of the Farrell-Jones Conjecture in equivariant additive categories, or more generally, in equivariant higher categories, see Sections 13.3 and 13.4. There will be versions with finite wreath products, see Section 13.5. The most general version is the Full Farrell-Jones Conjecture 13.30, see Section 13.6, which implies all other variants of the Farrell-Jones Conjecture, see Section 13.11.

1.7 Status of the Baum-Connes and the Farrell-Jones Conjecture

A detailed report on the groups for which these conjectures have been proved will be given in Chapter 16. For example, the Baum-Connes Conjecture 1.1 is known for a class of groups which includes amenable groups, hyperbolic groups, knot groups, fundamental groups of compact 3-manifolds (possibly with boundary), and one-relator groups, but is open for $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, where for a commutative ring R we write $\mathrm{SL}_n(R)$ for the group of invertible (n,n)-matrices with $\det(A) = 1$. The class of groups for which the Farrell-Jones Conjectures 1.2 and 1.3 have been proved contains hyperbolic groups, finite-dimensional $\mathrm{CAT}(0)$-groups, fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary), solvable groups, lattices in almost connected Lie groups, and arithmetic groups, but they are open for amenable groups in general. If one allows coefficients, one can prove inheritance properties for the Baum-Connes Conjecture and the Farrell-Jones Conjecture, e.g., the class of groups for which they are true is closed under taking subgroups, finite direct products, free products, colimits over directed sets whose structure map are injective in the Baum-Connes case and can be arbitrary in the Farrell-Jones case. This will be explained in Sections 13.7 and 14.6.

The Full Farrell-Jones Conjecture 13.30, which implies all other variants of the Farrell-Jones Conjecture, is known to be true for some groups with unusual properties, e.g., groups with expanders, Tarski monsters, lacunary hyperbolic groups, subgroups of finite products of hyperbolic groups, self-similar groups, see Theorem 16.1. At the time of writing we have no specific candidate of a group or of a general property of groups such that the Full Farrell-Jones Conjecture 13.30, or one of its consequences, e.g., the Novikov Conjecture and the Borel Conjecture, might be false. So we have no good starting point for a search for counterexamples, see Section 16.10.

At the time of writing no counterexample to the Baum-Connes Conjecture is known to the author. There exist counterexamples to the Baum-Connes Conjecture with coefficients, as explained in Section 16.10.

1.8 Structural Aspects

1.8.1 The Meta-Isomorphism Conjecture

The formulations of the Baum-Connes Conjecture 1.1 and of the Farrell-Jones Conjecture 1.2 and 1.3 are very similar in the homological picture. It allows a formulation of the following *Meta-Isomorphism Conjecture*, of which both conjectures are special cases and which also has other very interesting specializations, e.g., for pseudoisotopy, A-theory, topological Hochschild homology, and topological cyclic homology, see Section 15.2.

Meta-Isomorphism Conjecture 1.17. *Given a group G, a G-homology theory $\mathcal{H}_*^G$, and a family $\mathcal{F}$ of subgroups of G, we say that the* Meta-Isomorphism Conjecture *is satisfied if the G-map $E_{\mathcal{F}}(G) \to \{\bullet\}$ induces for every $n \in \mathbb{Z}$ an isomorphism*

$$A_{\mathcal{F}} \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\{\bullet\}).$$

This general formulation is an excellent framework to construct transformations between the assembly maps appearing in different Isomorphism Conjectures. For instance, the *cyclotomic trace* relates the K-theoretic Farrell-Jones Conjecture with coefficients in $\mathbb{Z}$ to the Isomorphism Conjecture for topological cyclic homology, see Subsection 15.14.3, and via *symmetric signatures* one can link the Farrell-Jones Conjecture for algebraic L-theory with coefficient in $\mathbb{Z}$ to the Baum-Connes Conjecture, see Subsection 15.14.4. Moreover, basic computational tools and techniques for equivariant homology theories apply both to the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3.

1.8.2 Assembly

One important idea is the *assembly principle*, which leads to assembly maps in a canonical and universal way by asking for the best approximation of a homotopy invariant functor from G-spaces to spectra by an equivariant homology theory. It is an important ingredient for the identification of the various descriptions of assembly maps appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture. For instance, the assembly map appearing in the Baum-Connes Conjecture 1.1 can be interpreted as assigning to an appropriate equivariant elliptic complex its equivariant index, and the assembly map appearing in the L-theoretic Farrell-Jones Conjecture 1.3 is related to the map appearing in the Surgery Exact Sequence, which assigns to a surgery problem its surgery obstruction. We have already explained above that these identifications are the basis for some of applications of the Isomorphism Conjectures, and we will see that they are also important for proofs. There is a homotopy-theoretic approach to the assembly map based on homotopy colimits over the orbit category, which motivates the name assembly. Roughly speaking, the name assembly refers to assembling the values of the K-and L-groups of the reduced group C^*-algebra or the group ring of a group G from their values on finite or virtually cyclic subgroups of G. All this will be explained in Chapter 18.

This parallel treatment of the Baum-Connes Conjecture and the Farrell-Jones Conjecture and of other variants is one of the topics of this book. However, the geometric interpretations of the assembly maps in terms of indices, surgery obstructions, or forget control are quite different. Therefore the methods of proof for the Farrell-Jones Conjecture and the Baum-Connes Conjecture use different input. Although there are some similarities in the proofs, it is not clear how to export methods of proof from one conjecture to the other.

1.9 Computational Aspects

In general the target $K_n(C_r^*(G))$ of the assembly map appearing in the Baum-Connes Conjecture 1.1 is very hard to compute, whereas the source $K_n^G(\underline{E}G)$ is much more accessible because one can apply standard techniques from algebraic topology such as spectral sequences and equivariant Chern characters and there are often nice small geometric models for $\underline{E}G$. For the Farrell-Jones Conjectures 1.2 and 1.3, this applies also to the parts $H_n^G(\underline{E}G;\mathbf{K}_R)$ and $H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ respectively appearing in the splittings (1.11) and (1.14). The other parts $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$ or $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ are harder to handle, since they involve Nil- or UNil-terms and the G-CW-complex $\underline{\underline{E}}G$ is not proper and in general huge. Most of the known computations of $K_n(C_r^*(G))$, $K_n(RG)$, and $L_n^{\langle j\rangle}(RG)$ are based on the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3.

Classifications of manifolds and of C^*-algebras rely on and thus motivate explicit calculations of K- and L-groups. In this context it is often important not only to determine the K- and L-groups abstractly, but to develop detection techniques so that one can identify or distinguish specific elements associated to the original classification problem or give geometric or index-theoretic interpretations to elements in the K- and L-groups.

A general guide for computations and a list of known cases including applications to classification problems will be given in Chapter 17.

1.10 Are the Baum-Connes Conjecture and the Farrell-Jones Conjecture True in General?

The title of this section is the central and at the time of writing unsolved question. One motivation for writing this monograph is to stimulate some very clever mathematician to work on this problem and finally find an answer. Let us speculate about the possible answer.

We are skeptical about the Baum-Connes Conjecture for two reasons: there are counterexamples for the version with coefficients, and the left side of the Baum-Connes assembly map is functorial under arbitrary group homomorphisms, whereas the right side is not. The Bost Conjecture, which predicts an isomorphism

$$K_n^G(\underline{E}G) \to K_n(L^1(G)),$$

has a much better chance to be true in general. The possible failure of the Baum-Connes Conjecture may come from the possible failure of the canonical map $K_n(L^1(G)) \to K_n(C^r_*(G))$ to be bijective.

In spite of the Baum-Connes Conjecture, we do not see an obvious flaw with the Bost Conjecture or the Farrell-Jones Conjecture. As explained in Section 1.7 above, we have no starting point for the construction of a counterexample, and all abstract properties we know for the right side do hold for the left side of the assembly map and vice versa. In particular for the Bass Conjecture and for the Novikov Conjecture which follow from the Farrell-Jones Conjecture, the class of groups for which they are known to be true is impressive. There are some conclusions from the Farrell-Jones Conjecture which are not trivial and true for all groups. These are arguments in favor of a positive answer

The following arguments are in favor of a negative answer. The universe of groups is overwhelmingly large. We have Gromov's saying on our neck that a statement which holds for all groups is either trivial or false. We have no philosophical reason why the Bost Conjecture or the Farrell-Jones Conjecture should be true in general. Finding a counterexample will probably require some new ideas, maybe from logic or random groups.

The upshot of this discussion is that the author is skeptical about the Baum-Connes Conjecture, but does not dare to make any predictions about the chances for the other conjectures, in particular for the Novikov Conjecture, to be true for all groups.

We will elaborate on this discussion in Section 16.10.

1.11 The Organization of the Book and a User's Guide

We have written the text in a way such that one can read small units, e.g., a single chapter, independently from the rest, concentrate on certain aspects, and extract easily and quickly specific information. Hopefully we have found the right mixture between definitions, theorems, examples, and remarks so that reading the book is entertaining and illuminating. We have successfully used parts of this book, sometimes a single chapter, for seminars, reading courses, and advanced lecture courses.

The book consists of three parts and a supplement, which we briefly review next. We will also give some further information on how to use the book.

Note that not all of the proofs are included in full. At least we convey the basic ideas and include references to sources.

1.11.1 Introduction to K- and L-Theory (Part I)

In the first part "Introduction to K- and L-Theory", which encompasses Chapters 2 to 10, we introduce and motivate the relevant theories, namely, algebraic K-theory, algebraic L-theory, and topological K-theory. In these chapters we present some applications and more accessible special versions of the Baum-Connes and the Farrell-Jones Conjecture. They are rather independent of one another and one can start reading each of them without having gone though the others. If a reader just wants to get some information, for instance about Wall's finiteness obstruction, Whitehead torsion, or the projective class group, she or he can directly start reading the relevant chapter, learn the basics about these invariants, and understand the relevant special versions of the Baum-Connes Conjecture or the Farrell-Jones Conjecture without going through the other chapters. Each of these chapters is eligible for a lecture course, seminar, or reading course.

1.11.2 The Isomorphism Conjectures (Part II)

In the second part "The Isomorphism Conjectures", which consists of Chapters 11 to Chapter 18, we introduce the Baum-Connes Conjecture and the Farrell-Jones Conjecture in their most general form, namely, for arbitrary groups and with coefficients. We discuss further applications and in particular how they can be used for computations. We give a report about the status of these conjectures and discuss open problems.

Note that the Farrell-Jones Conjecture comes in different levels. It can be considered for rings (with involution) as coefficients and hence aims at the algebraic K-theory and L-theory of group rings. This is the most relevant version for applications, where it often suffices to treat lower and middle K-theory, torsionfree groups, and $\mathbb{Z}$ or a field as coefficients. One may twist the group rings and allow orientation characters. The next level is to pass to equivariant additive categories (with involution) as coefficients, which has the advantage that it automatically leads to useful inheritance properties of the Farrell-Jones Conjecture and encompasses the case of rings as coefficients. For algebraic K-theory one can even allow higher categories as coefficients. This contains the version of additive categories as coefficients and also the versions of the Farrell-Jones Conjecture for Waldhausen's A-theory, for pseudoisotopy, and for Whitehead spaces as special cases. There are also versions "with finite wreath product", where the passage to overgroups of finite index is built in.

So there are many variations of the Farrell-Jones Conjecture, but the Full Farrell-Jones Conjecture 13.30 implies all of them.

We also state Meta-Conjectures, which reduce to the Baum-Connes Conjecture, the Farrell-Jones Conjecture, or other types of Isomorphism Conjectures if one feeds the right theory into them. There are versions of the Farrell-Jones Conjecture for Waldhausen's A-theory, pseudoisotopy, Whitehead spaces, topological Hochschild homology, topological cyclic homology, and homotopy K-theory.

We also briefly discuss the Farrell-Jones Conjecture for totally disconnected groups and Hecke Algebras, where for the first time a version of the Farrell-Jones Conjecture for topological groups is considered. The Baum-Connes Conjecture has already been intensively studied for topological groups. However, in this monograph we will confine ourselves to discrete groups.

1.11.3 Methods of Proofs (Part III)

In the third part "Methods of Proofs", which ranges from Chapter 19 to Chapter 25, we give a survey on the background, history, philosophy, strategies, and some ingredients of the proofs. We will concentrate on the Farrell-Jones Conjecture in this part III.

The reader, who is interested in proofs, should first go through Chapter 19. There motivations for the proofs of the Farrell-Jones Conjecture and some information about their long history is given without getting lost in technical details. So it will be a soft introduction to the methods of proofs conveying ideas only. Mainly we explain why controlled topology, flows, and transfers come in, which one would not expect at first glance in view of the homotopy-theoretic nature of the Farrell-Jones Conjecture.

In Chapter 20 we isolate some conditions about a group which guarantee that it satisfies the Full Farrell-Jones Conjecture or some special version of it. Note that here K- or L-theory do not yet play any role and one can use the results of this section without any previous knowledge about them. This will be interesting for someone who is already familiar with geometric group theory but has no background in K- or L-theory.

Depending on how ambitious the reader is, she or he should go through the other chapters. We recommend to read Section 23.7, where details of the proof of the Farrell-Jones Conjecture for the surjectivity of the K-theoretic assembly map in dimension 1 is given, which does not use much knowledge about algebraic K-theory but uses all the basic ideas appearing in the proof of the Full Farrell-Conjecture.

The reader who wants to understand the proof in the most advanced setting, namely the one for higher categories as coefficients, and for the largest class of groups, namely the class of Dress-Farrell-Hsiang-Jones groups, is recommended to read through Chapter 24. For this some background in higher category theory is necessary.

We give a very brief overview of the methods of proof for the Baum-Connes Conjecture in Chapter 25.

1.11.4 Supplement

The book contains a number of exercises. They are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text or improve her or his understanding by trying to solve the exercises. Hints to the solutions of the exercises are given in Chapter 26.

If one wants to find a specific topic, the extensive index of the monograph can be used to find the right spot for a specific topic. The index contains an item "Theorem", under which all theorems with their names appearing in the book are listed, and analogously there is an item "Conjecture".

1.11.5 Prerequisites

We require that the reader is familiar with basic notions in topology (CW-complexes, chain complexes, homology, homotopy groups, manifolds, coverings, cofibrations, fibrations, ...), functional analysis (Hilbert spaces, bounded operators, differential operators, ...), algebra (groups, modules, group rings, elementary homological algebra, ...), group theory (presentations, Cayley graphs, hyperbolic groups, ...), and elementary category theory (functors, transformations, additive categories, ...).

1.12 Notations and Conventions

Here is a briefing on our main conventions and notations. Details are of course discussed in the text.

- Ring will mean (not necessarily commutative) associative ring with unit unless explicitly stated otherwise;
- Module always means left module unless explicitly stated otherwise;
- Group means discrete group unless explicitly stated otherwise;
- We will always work in the category of compactly generated spaces, compare [927] and [1006, I.4]. In particular every space is automatically Hausdorff;
- For our conventions concerning spectra see Section 12.4. Spectra are denoted by boldface letters such as **E**;
- We use the standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\widehat{\mathbb{Z}_p}$, and $\widehat{\mathbb{Q}_p}$ for the integers, the rational numbers, the real numbers, the complex numbers, the p-adic numbers, and the p-adic rationals;
- We use the following symbols to denote various groups:

symbol	name
$\mathbb{Z}/n$	finite cyclic group of order n
S_n	symmetric group of permutations of the set $\{1, 2, \dots n\}$
A_n	alternating group of even permutations of the set $\{1, 2, \dots, n\}$
D_∞	infinite dihedral group
D_{2n}	dihedral group of order $2n$

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1.14 Notes

Further information about the Baum-Connes Conjecture and the Farrell-Jones Conjecture can be found in the survey articles [67, 88, 109, 366, 426, 484, 659, 673, 742, 846, 963].