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Isomorphism Conjectures in K- and L-Theory

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Preface

This manuscript is not finished. Most of it is in nearly final form, whereas Chapters 8 and 25 are still under construction. Comments are very welcome.

The Isomorphism Conjectures due to Baum and Connes and to Farrell and Jones aim at the topological K-theory of reduced group C^*-algebras and the algebraic K-and L-theory of group rings. These theories are of major interest for many reasons. For instance, the algebraic L-groups are the recipients for various surgery obstructions and hence highly relevant for the classification of manifolds. Other important obstructions such as Wall's finiteness obstruction and Whitehead torsion take values in algebraic K-groups. The topological K-groups of C^*-algebras play a central role in index theory and the classification of C^*-algebras.

In general these K- and L-groups are very hard to analyze for group rings or group C^*-algebras. The Isomorphism Conjectures identify them with equivariant homology groups of classifying spaces for families of subgroups. As an illustration, let us consider the special case that G is a torsionfree group and R is a regular ring (with involution). Then the Isomorphism Conjectures predict that the so-called assembly maps

$$\begin{aligned} H_n(BG;\mathbf{K}(R)) &\xrightarrow{\cong} K_n(RG);\\ H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) &\xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG);\\ K_n(BG) &\xrightarrow{\cong} K_n(C_r^*(G)), \end{aligned}$$

are isomorphisms for all $n \in \mathbb{Z}$. The target is the algebraic K-theory of the group ring RG, the algebraic L-theory of RG with decoration $\langle -\infty\rangle$, or the topological K-theory of the reduced group C^*-algebra $C_r^*(G)$. The source is the evaluation of a specific homology theory on the classifying space BG, where $H_n(\{\bullet\};\mathbf{K}(R)) \cong K_n(R)$, $H_n(\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}(R)) \cong L_n^{\langle -\infty\rangle}(R)$, and $K_n(\{\bullet\}) \cong K_n(\mathbb{C})$ hold for all $n \in \mathbb{Z}$.

Since the sources of these assembly maps are much more accessible than the targets, the Isomorphism Conjectures are key ingredients for explicit computations of the K-and L-groups of group rings and reduced group C^*-algebras. These often are motivated by and have applications to concrete problems that arise, for instance, in the classification of manifolds or C^*-algebras.

The Baum-Conjecture and the Farrell-Jones Conjecture imply many other prominent conjectures. In a lot of cases these conjectures were not known to be true for certain groups until the Baum-Connes Conjecture or the Farrell-Jones Conjecture was proved for them. Examples for such prominent conjectures are the Borel Conjecture about the topological rigidity of aspherical closed manifolds, the (stable) Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature on closed Spin-manifolds, the Kaplansky' Idempotent Conjecture and the Kadison Conjecture on the non-existence of non-trivial idempotents in the group ring or the reduced group C^*-algebra of torsionfree groups, the Novikov Conjecture about the homotopy invariance of higher signatures, and the conjectures about the vanishing of the reduced projective class group of $\mathbb{Z}G$ and the Whitehead group of G for a torsionfree group G.

The Baum-Connes Conjecture and the Farrell-Jones Conjecture are still open in general (at the time of writing). However, tremendous progress has been made on the class of groups for which they are known to be true. The techniques of the sophisticated proofs stem from algebra, dynamical systems, geometry, group theory, operator theory, and topology. The extreme broad scope of the Baum-Connes Conjecture and the Farrell-Jones Conjecture is both the main challenge and main motivation for writing this book. We hope that, after having read parts of this monograph, the reader will share the enthusiasm of the author for the Isomorphism Conjectures.

The monograph is a guide for and gives a panorama of Isomorphism Conjectures and related topics. It presents or at least indicates the (at the time of writing) most advanced results and developments. It can be used by various groups of readers, such as experts on the Baum-Connes Conjecture or the Farrell-Jones Conjecture, experienced mathematicians, who may not be experts on these conjectures but want to learn or just apply them, and also, of course, advanced undergraduate and graduate students. References for further reading and information have been inserted.

We will give more information about the organization of the book and a user's guide in Section 1.10.

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Chapter 1
Introduction

The Isomorphism Conjectures due to Paul Baum and Alain Connes and to Tom Farrell and Lowell Jones are important conjectures, which have many interesting applications and consequences. However, they are not easy to formulate and it is a priori not clear why the actual versions are the most promising ones. The current versions are the final upshot of a longer process, which has led to them step by step. They have been influenced and steered by various new results that have been proved during the last decades and given new insight into the objects, problems, and constructions at which these conjectures aim.

In this introduction we want to motivate these conjectures by explaining how one can be led to them by general considerations and certain facts. We present brief surveys about applications of these conjectures, their status, and the methods of proof. We give information about contents of this monograph including a user's guide.

1.1 The Statement of the Baum-Connes Conjecture and of the Farrell-Jones Conjecture

Next we record the statements of the Baum-Connes Conjecture and Farrell-Jones Conjecture. Explanations and motivations will follow. The versions stated below will be generalized later.

Conjecture 1.1 (Baum-Connes Conjecture). Let G be a group. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called *assembly map*,

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Conjecture 1.2. (Farrell-Jones Conjecture for $K_*(RG)$). Let G be a group. Let R be an associative ring with unit. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called *assembly map*,

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Conjecture 1.3. (Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$). Let G be a group. Let R be an associative ring with unit and involution. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called *assembly map*,

$$H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

The general pattern is that the target of the assembly map is what we want to understand or to compute, namely K-and L-theory of group rings and group C^*-algebras, and that the source is a homological expression, which is much more accessible than the source and depends only the values of the K- or L-groups under considerations on finite subgroups or on virtually cyclic subgroups of G. The spaces $\underline{E}G$ and $\underline{\underline{E}}G$ are classifying spaces for the family of finite subgroups and the family of virtually cyclic subgroups.

1.2 Motivation for the Baum-Connes Conjecture

We will start with the easiest and most convenient to state and motivate Isomorphism Conjecture, the Baum-Connes Conjecture for the topological K-theory of reduced group C^*-algebras. Then we will pass to the Farrell-Jones Conjecture for the algebraic K- and L-theory of group rings, which is more complicate due to the appearance of Nil-terms.

1.2.1 Topological *K*-Theory of Reduced Group *C**-Algebras

The target of the Baum-Connes Conjecture is the topological K-theory of the *reduced C^*-algebra* $C_r^*(G)$ of a group G. We will consider discrete groups G only. One defines the *topological K-groups* $K_n(A)$ for any Banach algebra A to be the abelian group $K_n(A) = \pi_{n-1}(\mathrm{GL}(A))$ for $n \geq 1$. The famous *Bott Periodicity Theorem* gives a natural isomorphism $K_n(A) \xrightarrow{\cong} K_{n+2}(A)$ for $n \geq 1$. Finally one defines $K_n(A)$ for all $n \in \mathbb{Z}$ so that the Bott isomorphism theorem is true for all $n \in \mathbb{Z}$. It turns out that $K_0(A)$ is the same as the *projective class group* of the ring A, which is the Grothendieck group of the abelian monoid of isomorphism classes of finitely generated projective A-modules with the direct sum as addition. The topological K-theory of $\mathbb{C} = C_r^*(\{1\})$ is trivial in odd dimensions and isomorphic to $\mathbb{Z}$ in even dimensions. More generally, for a finite group G the topological K-theory of $C_r^*(G)$ is the complex representation ring $R_{\mathbb{C}}(G)$ in even dimensions and is trivial in odd dimensions.

Let P be an appropriate elliptic differential operator (or more generally an elliptic complex) on a closed n-dimensional Riemannian manifold M, for instance the Dirac operator or the signature operator. Then one can consider its *index* in $K_n(\mathbb{C})$ which $\dim_{\mathbb{C}}(\ker(P)) - \dim_{\mathbb{C}}(\mathrm{coker}(P)) \in \mathbb{Z}$ for even n and is zero for odd n. If M comes with an isometric G-action of a finite group G and P is compatible with the G-action, then $\ker(P)$ and $\mathrm{coker}(P)$ are complex finite dimensional G-representations and one obtains an element in $K_n(C_r^*(G)) = R_{\mathbb{C}}(G)$ by $[\ker(P)] - [\operatorname{coker}(P)]$ for even n. Suppose that G is an arbitrary discrete group and that M is a (not necessarily compact) n-dimensional smooth manifold without boundary with a proper cocompact G-action, a G-invariant Riemannian metric, and an appropriate elliptic differential operator P compatible with the G-action. An example is the universal covering $M = \widetilde{N}$ of an n-dimensional closed Riemannian manifold N with $G = \pi_1(N)$ and the lift $\widetilde{P}$ to $\widetilde{N}$ of an appropriate elliptic differential operator P on N. Then one can define an *equivariant index* of P which takes values in $K_n(C_r^*(G))$. Therefore the interest in $K_*(C_r^*(G))$ comes from the fact that it is the natural recipient for indices of certain equivariant differential operators. All this will be explained in Chapter 10.

1.2.2 Homological Aspects

A first basic problem is to compute $K_*(C_r^*(G))$ or to identify it with more familiar terms. The key idea comes from the observation that $K_*(C_r^*(G))$ has some homological properties. More precisely, if G is the amalgamated free product $G = G_1 *_{G_0} G_2$ for subgroups $G_i \subseteq G$, then there is a long exact sequence

(1.4)
$$\begin{aligned}
\cdots &\xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) \xrightarrow{K_n(C_r^*(i_1)) \oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \\
&\xrightarrow{K_n(C_r^*(j_1)) - K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \\
&\xrightarrow{K_{n-1}(C_r^*(i_1)) \oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1)) \\
&\xrightarrow{K_{n-1}(C_r^*(j_1)) - K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots
\end{aligned}$$

where i_1,i_2, j_1, and j_2 are the obvious inclusions, see [796, Theorem 18 on page 632]. If $\phi\colon G \to G$ is a group automorphism and $G \rtimes_\phi \mathbb{Z}$ is the associated semidirect product, then there is a long exact sequence

(1.5)
$$\begin{aligned}
\cdots &\xrightarrow{\partial_{n+1}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(\phi)) - \mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_\phi \mathbb{Z})) \\
&\xrightarrow{\partial_n} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(\phi)) - \mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots
\end{aligned}$$

where k is the obvious inclusion, see [795, Theorem 3.1 on page 151] or more generally [796, Theorem 18 on page 632].

We compare this with group homology in order to explain the analogy with homology. Recall that the *classifying space* BG of a group G is an aspherical CW-complex whose fundamental group is isomorphic to G and that *aspherical* means that all higher homotopy groups are trivial, or, equivalently, that the universal covering is contractible. The classifying space BG is unique up to homotopy. If one has an amalgamated free product $G = G_1 *_{G_0} G_2$, then one can find models for the classifying spaces such that BG_i is a CW-subcomplex of BG and $BG = BG_1 \cup BG_2$ and $BG_0 = BG_1 \cap BG_2$. Thus we obtain a pushout of inclusions of CW-complexes

$$\begin{tikzcd} BG_0 \arrow[r, "Bi_1"] \arrow[d, "Bi_2"'] & BG_1 \arrow[d, "Bj_1"] \\ BG_2 \arrow[r, "Bj_2"'] & BG. \end{tikzcd}$$

It yields a long Mayer-Vietoris sequence for the cellular or singular homology

$$(1.6) \quad \cdots \xrightarrow{\partial_{n+1}} H_n(BG_0) \xrightarrow{H_n(Bi_1)\oplus H_n(Bi_2)} H_n(BG_1) \oplus H_n(BG_2) \xrightarrow{H_n(Bj_1)-H_n(Bj_2)} H_n(BG) \xrightarrow{\partial_n} H_{n-1}(BG_0) \xrightarrow{H_{n-1}(Bi_1)\oplus H_{n-1}(Bi_2)} H_{n-1}(BG_2) \oplus H_{n-1}(BG_1) \xrightarrow{H_{n-1}(Bj_1)-H_{n-1}(Bj_2)} H_{n-1}(BG) \xrightarrow{\partial_{n-1}} \cdots .$$

If $\phi\colon G \to G$ is a group automorphism, then a model for $B(G \rtimes_\phi \mathbb{Z})$ is given by the mapping torus of $B\phi\colon BG \to BG$, which is obtained from the cylinder $BG \times [0,1]$ by identifying the bottom and the top with the map $B\phi$. Associated to a mapping torus, there is the long exact sequence

$$(1.7) \quad \cdots \xrightarrow{\partial_{n+1}} H_n(BG) \xrightarrow{H_n(B\phi)-\mathrm{id}} H_n(BG) \xrightarrow{H_n(Bk)} H_n(B(G \rtimes_\phi \mathbb{Z})) \xrightarrow{\partial_n} H_{n-1}(BG) \xrightarrow{H_{n-1}(B\phi)-\mathrm{id}} H_{n-1}(BG) \xrightarrow{H_n(Bk)} \cdots$$

where k is the obvious inclusion of BG into the mapping torus.

1.2.3 The Baum-Connes Conjecture for Torsionfree Groups

There is an obvious analogy between the sequences (1.4) and (1.6) and the sequences (1.5) and (1.7). On the other hand we get for the trivial group $G = \{1\}$ that $H_n(B\{1\}) = H_n(\{\bullet\})$ is $\mathbb{Z}$ for $n = 0$ and trivial for $n \neq 0$ so that the group homology of BG cannot be the same as the topological K-theory of $C_r^*(\{1\})$. But there is a better candidate, namely take the *topological K-homology* of BG instead of the singular homology. Topological K-homology is a homology theory defined for CW-complexes. At least we mention that for a topologist its definition is a routine, namely, it is the homology theory associated to the K-theory spectrum which defines topological K-theory of

CW-complexes, i.e., the cohomology theory which comes from considering vector bundles over CW-complexes. In contrast to singular homology, the topological K-homology of a point $K_n(\{\bullet\})$ is $\mathbb{Z}$ for even n and is trivial for n odd. So we still get exact sequences (1.6) and (1.7) if we replace H_* by K_* everywhere and we have $K_n(B\{1\}) \cong K_n(C_r^*(\{1\})$ for all $n \in \mathbb{Z}$. This leads to the following conjecture.

Conjecture 1.8 (Baum-Connes Conjecture for torsionfree groups). Let G be a torsionfree group. Then there is for every $n \in \mathbb{Z}$ an isomorphism, called *assembly map*,

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

This is indeed a formulation which will turn out to be equivalent to the Baum-Connes Conjecture 1.1, provided that G is torsionfree. Conjecture 1.8 cannot hold in general as already the example of a finite group G shows. Namely, if G is finite, then the obvious inclusion induces an isomorphism $K_n(B\{1\}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$ for every $n \in \mathbb{Z}$, whereas $K_0(C_r^*(\{1\}) \to K_0(C_r^*(G))$ agrees with the map $R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(G)$ which is rationally bijective if and only if G itself is trivial. Hence Conjecture 1.8 is not true for non-trivial finite groups.

1.2.4 The Baum-Connes Conjecture

What is going wrong? The sequences (1.4) and (1.5) do exist regardless whether the groups are torsionfree or not. More generally, if G acts on a tree, then they can be combined to compute the K-theory $K_*(C_r^*(G))$ of a group G by a certain Mayer-Vietoris sequence from the stabilizers of the vertices and edges, see Pimsner [796, Theorem 18 on page 632]). In the special case where all stabilizers are finite, one sees that $K_*(C_r^*(G))$ is built by the topological K-theory of the finite subgroups of G in a homological fashion. This leads to the idea that $K_*(C_r^*(G))$ can be computed in a homological way, but the building blocks do not only consist of $K_*(C_r^*(\{1\}))$ alone but of $K_*(C_r^*(H))$ for all finite subgroups $H \subseteq G$. This suggest to study *equivariant topological K-theory*. It assigns to every proper G-CW-complex X a sequence of abelian groups $K_n^G(X)$ for $n \in \mathbb{Z}$ such that G-homotopy invariance holds and Mayer-Vietoris sequences exist. A *proper G-CW-complex* is a CW-complex with G-action such that for every $g \in G$ and every open cell e with $e \cap g \cdot e \neq \emptyset$ we have $gx = x$ for all $x \in e$ and all isotropy groups are finite. Two interesting features are that $K_n^G(G/H)$ agrees with $K_n(C_r^*(H))$ for every finite subgroup $H \subseteq G$ and that for a free G-CW-complex X and $n \in \mathbb{Z}$ we have a natural isomorphism $K_n^G(X) \xrightarrow{\cong} K_n(G\backslash X)$. Recall that EG is a free G-CW-complex which is contractible and that $EG \to G\backslash EG = BG$ is the universal covering of BG. We can reformulate Conjecture 1.8 by stating an isomorphism

$$K_n^G(EG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Now suppose that G acts on a tree T with finite stabilizers. Then the computation of Pimsner [796, Theorem 18 on page 632]) mentioned above can be rephrased to the statement that there is an isomorphism

$$K_n^G(T) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular the left hand side is independent of the tree T, on which G acts by finite stabilizers. This can be explained as follows. It is known that for every finite subgroup $H \subseteq G$ the H-fixed point set T is again a non-empty tree and hence contractible. This implies that two trees T_1 and T_2, on which G acts with finite stabilizers, are G-homotopy equivalent and hence have the same equivariant topological K-theory. The same remark applies to $K_n(BG)$ and $K_n^G(EG)$, namely, two models for BG are homotopy equivalent and two models for EG are G-homotopy equivalent and therefore $K_n(BG)$ and $K_n^G(EG)$ are independent of the choice of a model. This leads to the idea to look for an appropriate proper G-CW-complex $\underline{E}G = E_{\mathcal{FIN}}(G)$, which is characterized by a certain universal property and is unique up to G-homotopy, such that for a torsionfree group G we have $EG = \underline{E}G$, for a tree, on which G acts with finite stabilizers, we have $\underline{E}G = T$, and there is an isomorphism

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular for a finite group we would like to have $\underline{E}G = G/G = \{\bullet\}$ and then the desired isomorphism above is true for trivial reasons. Recall that EG is characterized up to G-homotopy by the property that it is a G-CW-complex such that EG^H is empty for $H \neq \{1\}$ and is contractible for $H = \{1\}$. Having the case of a tree, on which G acts with finite stabilizers, in mind, we define the *classifying space for proper G-actions* $\underline{E}G$ to be a G-CW-complex such that $\underline{E}G^H$ is empty for $|H| = \infty$ and is contractible for $|H| < \infty$. Indeed, two models for $\underline{E}G$ are G-homotopy equivalent, a tree, on which G acts with finite stabilizers, is a model for $\underline{E}G$, we have $EG = \underline{E}G$ if and only if G is torsionfree, and $\underline{E}G = G/G = \{\bullet\}$ if and only if G is finite. This leads to the Baum-Connes Conjecture, stated already as Conjecture 1.1. Classifying spaces for families will be treated in detail in Chapter 11.

The Baum-Connes Conjecture 1.1 makes sense for all groups, and no counterexamples are known at the time of writing. The Baum-Connes Conjecture 1.1 reduces in the torsionfree case to Conjecture 1.8 and is consistent with the results by Pimsner [796, Theorem 18 on page 632] for G acting on a tree with finite stabilizers. It is obviously true for finite groups G. Pimsner's result does hold more generally for groups acting on trees with not necessarily finite stabilizers. So one should get the analogous result for the left hand side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. Essentially this boils down to the question whether the analogues of the long exact sequences (1.4) and (1.5) hold for the left side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. This follows for (1.4) from the fact that for $G = G_1 *_{G_0} G_2$ one can find appropriate models for the classifying spaces for proper G-actions such that there is a G-pushout of inclusions of proper G-CW-complexes

$$\begin{tikzcd}
G \times_{G_0} \underline{E}G_0 \arrow[r] \arrow[d] & G \times_{G_1} \underline{E}G_1 \arrow[d] \\
G \times_{G_2} \underline{E}G_2 \arrow[r] & \underline{E}G
\end{tikzcd}$$

and for a subgroup $H \subseteq G$ and a proper H-CW-complex X there is a natural isomorphism

$$K_n^H(X) \xrightarrow{\cong} K_n^G(G \times_H X).$$

Thus the associated long exact Mayer-Vietoris sequence yields the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} K_n^{G_0}(\underline{E}G_0) \to K_n^{G_1}(\underline{E}G_1) \oplus K_n^{G_2}(\underline{E}G_2) \to K_n^G(\underline{E}G) \xrightarrow{\partial_n} \\ K_{n-1}^{G_0}(\underline{E}G_0) \to K_{n-1}^{G_1}(\underline{E}G_1) \oplus K_{n-1}^{G_2}(\underline{E}G_2) \to K_{n-1}^{G_0}(\underline{E}G) \to \cdots \end{aligned}$$

which corresponds to (1.4). For (1.5) one uses the fact that for a group automorphism $\phi\colon G \xrightarrow{\cong} G$ the $G \rtimes_\phi \mathbb{Z}$-$CW$-complex given by the to both sides infinite mapping telescope of the ϕ-equivariant map $\underline{E}\phi\colon \underline{E}G \to \underline{E}G$ is a model for $\underline{E}(G \rtimes_\phi \mathbb{Z})$.

In general $K_n^G(\underline{E}G)$ is much bigger than $K_n^G(EG) \cong K_n(BG)$ and the canonical map $K_n^G(EG) \to K_n^G(\underline{E}G)$ is rationally injective but not necessarily integrally injective.

1.2.5 Reduced versus Maximal Group C^*-Algebras

All the arguments above do also apply to the *maximal group C^*-algebra* which does even have better functorial properties than the reduced group C^*-algebra. So a priori one may think that one should use the maximal group C^*-algebra instead of the reduced one. However, the version for the maximal group C^*-algebra is not true in general and the version for the reduced group C^*-algebra seems to be the right one. This will be discussed in more detail in subsection 14.5.1.

If one considers instead of the reduced group C^*-algebra the Banach group algebra $l^1(G)$, one obtains the *Bost Conjecture* 14.23.

1.2.6 Applications of the Baum-Connes Conjecture

The assembly map appearing in the Baum-Connes Conjecture 1.1 has an index theoretic interpretation. An element in $K_0^G(\underline{E}G)$ can be represented by a pair (M, P^*) consisting of a cocompact proper smooth n-dimensional G-manifold M with a G-invariant Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M and its image under the assembly map is a certain equivariant index $\operatorname{ind}_{C_r^*(G)}(M, P^*)$ in $K_n(C_r^*(G))$. There are many important consequences of the Baum-Connes Conjecture such as the *Kadison Conjecture*, see Subsection 10.4.2, the *stable Gromov-Lawson-Rosenberg Conjecture*, see Subsection 14.8.2, the *Novikov Conjecture*, see Section 9.14, and the *(Modified) Trace Conjecture*, see Subsections 10.4.1 and 14.8.1.

A summary of all the application of the Baum-Connes Conjecture is given in Section 14.8.

1.3 Motivation for the Farrell-Jones Conjecture for *K*-Theory

Next we want to deal with the algebraic K-groups $K_n(RG)$ of the group ring RG.

1.3.1 Algebraic *K*-Theory of Group Rings

For an associative ring with unit R one defines $K_0(R)$ to be the *projective class group* of R and $K_1(R)$ to be the abelianization of $\mathrm{GL}(R) = \operatorname{colim}_{n\to\infty} \mathrm{GL}_n(R)$. The *higher algebraic K-groups* $K_n(R)$ for $n \geq 1$ are the homotopy group groups of a certain *K-theory space* associated to the category of finitely generated projective R-modules. One can define *negative K-groups* $K_n(R)$ for $n \leq -1$ by a certain contracting procedure applied to $K_0(R)$. Finally there exists a *K-theory spectrum* $\mathbf{K}(R)$ such that $\pi_n(\mathbf{K}(R)) = K_n(R)$ holds for every $n \in \mathbb{Z}$. If $\mathbb{Z} \to R$ is the obvious ring map sending n to $n \cdot 1_R$, then one defines for $n \leq 1$ the *reduced K-groups* to be the cokernel of the induced map $K_n(\mathbb{Z}) \to K_n(R)$. The *Whitehead group* $\mathrm{Wh}(G)$ of a group G is the quotient of $K_1(\mathbb{Z}G)$ by elements given by $(1,1)$-matrices of the shape $(\pm g)$ for $g \in G$.

The *reduced projective class group* $\widetilde{K}_0(\mathbb{Z}G)$ is the recipient for the *finiteness obstruction* of a finitely dominated CW-complex X with fundamental group $G = \pi_1(X)$. *Finitely dominated* means that there is a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. The Whitehead group $\mathrm{Wh}(G)$ is the recipient of the

Whitehead torsion of a homotopy equivalence of finite CW-complexes and of a compact h-cobordism over a closed manifold, where G is the fundamental group. An *h-cobordism W over M* consists of a manifold W whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$ such that both inclusions $\partial_i W \to W$ are homotopy equivalences, together with a diffeomorphism $M \xrightarrow{\cong} \partial_0 W$. The finiteness obstruction and the Whitehead torsion are very important topological obstructions whose vanishing has interesting geometric and topological consequences. The finiteness obstruction vanishes if and only if the finitely dominated CW-complex under consideration is homotopy equivalent to a finite CW-complex. The Whitehead torsion of a compact h-cobordism W over M of dimension ≥ 6 vanishes if and only if W is trivial, i.e., is diffeomorphic to a cylinder $M \times [0,1]$ relative $M = M \times \{0\}$. This explains why topologists are interested in $K_n(\mathbb{Z}G)$ for groups G.

All these definitions and results will be explained in Chapters 2, 3, 4, Chapter 5, and 6.

1.3.2 Appearance of Nil-Terms

The situation for algebraic K-theory of RG is more complicated than the one for the topological K-theory of $C_r^*(G)$. As a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

For algebraic K-theory the analogue is the *Bass-Heller-Swan decomposition*

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R)$$

where certain additional terms, the Nil-*terms* $NK_n(R)$ appear, see Subsection 6.3.4. If one replaces R by RG, one gets

$$K_n(R[G \times \mathbb{Z}]) \cong K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG).$$

Such correction terms in form of Nil-terms appear also, when one wants to get analogues of the sequences (1.4) and (1.5) for algebraic K-theory, see Section 6.9.

1.3.3 The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings and Torsionfree Groups

Let R be a *regular ring*, i.e., it is Noetherian and every R-module possesses a finite dimensional projective resolution. For instance, any principal ideal domain is a regular ring. Then one can prove in many cases for torsionfree groups that the analogues of the sequences (1.4) and (1.5) do hold for algebraic K-theory, see Waldhausen [955] and [958]. The same reasoning as in the Baum-Connes Conjecture for torsionfree groups leads to the following conjecture.

Conjecture 1.9. (Farrell-Jones Conjecture for $K_*(RG)$ for torsionfree groups and regular rings). Let G be a torsionfree group and let R be a regular ring. Then there is for every $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{\cong} K_n(RG).$$

Here $H_*(-; \mathbf{K}(R))$ is the homology theory associated to the K-theory spectrum of R. It is a homology theory with the property that $H_n(\{\bullet\}; \mathbf{K}(R)) = \pi_n(\mathbf{K}(R)) = K_n(R)$ holds for every $n \in \mathbb{Z}$.

1.3.4 The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings

If one drops the condition that G is torsionfree but requires that the order of every finite subgroup of G is invertible in R, then one still can prove in many cases that the analogues of the sequences (1.4) and (1.5) do hold for algebraic K-theory. The same reasoning as in the Baum-Connes Conjecture leads to the following conjecture.

Conjecture 1.10. (Farrell-Jones Conjecture for $K_*(RG)$ for regular rings). Let G be a group. Let R be a regular ring such that $|H|$ is invertible in R for every finite subgroup $H \subseteq G$. Then there is for every $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Here $H_n^G(-; \mathbf{K}_R)$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{K}_R) \cong H_n^H(\{\bullet\}; \mathbf{K}_R) \cong K_n(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$, and the isomorphism above is induced by the G-map $\underline{E}G \to \{\bullet\}$. Conjecture 1.10 reduces to Conjecture 1.9 if G is torsionfree.

1.3.5 The Farrell-Jones Conjecture for $K_*(RG)$

Conjecture 1.9 can be applied in the case $R = \mathbb{Z}$ what is not true for Conjecture 1.10. So what is the right formulation for arbitrary rings R? The idea is that one does not only need to take all finite subgroups into account but also all virtually cyclic subgroups. A group is called *virtually cyclic* if it is finite or contains $\mathbb{Z}$ as subgroup of finite index. Namely, let $\underline{\underline{E}}G = E_{\mathcal{VCY}}(G)$

be the *classifying space for the family of virtually cyclic subgroups*, i.e., a G-CW-complex $\underline{\underline{E}}G$ such that $\underline{\underline{E}}G^H$ is contractible for every virtually cyclic subgroup $H \subseteq G$ and is empty for every subgroup $H \subseteq G$ which is not virtually cyclic. The G-space $\underline{\underline{E}}G$ is unique up to G-homotopy. This considerations lead to the Farrell-Jones Conjecture for $K_*(RG)$ stated already as Conjecture 1.2.

Conjecture 1.2 makes sense for all groups and rings, and no counterexamples are known at the time of writing. We have absorbed all the Nil-phenomena into the source by replacing $\underline{E}G$ by $\underline{\underline{E}}G$. There is a certain prize to pay since often there are nice small geometric models for $\underline{E}G$, whereas the spaces $\underline{\underline{E}}G$ are much harder to analyze and are in general huge. There are up to G-homotopy unique G-maps $EG \to \underline{E}G$ and $\underline{E}G \to \underline{\underline{E}}G$ which yield maps

$$H_n(BG; \mathbf{K}(R)) \cong H_n^G(EG; \mathbf{K}_R) \to H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(\underline{\underline{E}}G; \mathbf{K}_R).$$

We will later see that there is a splitting, see Theorem 13.33,

$$H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \cong H_n^G(\underline{E}G; \mathbf{K}_R) \oplus H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R) \tag{1.11}$$

where $H_n^G(\underline{E}G; \mathbf{K}_R)$ is the comparatively easy homological part and all Nil-type information is contained in $H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R)$. If R is regular and the order of any finite subgroup of G is invertible in R, then $H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{K}_R)$ is trivial and hence the natural map $H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \mathbf{K}_R)$ is bijective. Therefore Conjecture 1.2 reduces to Conjecture 1.9 and Conjecture 1.10 when they apply.

In the Baum-Connes setting the natural map $K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(\underline{\underline{E}}G)$ is always bijective.

1.3.6 Applications of the Farrell-Jones Conjecture for $K_*(RG)$

We have $K_n(\mathbb{Z}) = 0$ for $n \leq -1$. Both the map $\mathbb{Z} \xrightarrow{\cong} K_0(\mathbb{Z})$ that sends n to $n \cdot [\mathbb{Z}]$ and the map $\{\pm 1\} \to K_1(\mathbb{Z})$ that sends ± 1 to the class of the $(1,1)$-matrix (± 1) are bijective. Therefore an easy spectral sequence argument shows that Conjecture 1.9 implies

Conjecture 1.12. (Farrell-Jones Conjecture $K_n(\mathbb{Z}G)$ in dimensions $n \leq 1$). Let G be a torsionfree group. Then $\widetilde{K}_n(\mathbb{Z}G) = 0$ for $n \in \mathbb{Z}, n \leq 0$ and $\mathrm{Wh}(G) = 0$.

In particular the finiteness obstruction and the Whitehead torsion are always zero for torsionfree fundamental groups. This implies in particular that every h-cobordism over a simply connected d-dimensional closed manifold for $d \geq 5$ is trivial and thus the Poincaré Conjecture in dimensions ≥ 6 (and with some extra effort also in dimension $d = 5$). This will be explained in

Section 3.5. The Farrell-Jones Conjecture for K-theory, see Conjecture 1.2, implies the *Bass Conjecture*, see Section 2.10. *Kaplansky's Idempotent Conjecture* follows from the Farrell-Jones Conjecture for K-theory for torsionfree groups and regular rings, see Conjecture 1.9, as explained in Section 2.9. Further applications of the Conjecture 1.9, e.g., to pseudoisotopy and to automorphisms of manifolds, will be discussed in Section 9.21.

A summary of all the application of the Farrell-Jones Conjecture is given in Section 13.12.

1.4 Motivation for the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

Next we want to deal with the algebraic L-groups $L_n^{\epsilon}(RG)$ of the group ring RG of a group G with coefficients in an associative ring R with unit and involution.

1.4.1 Algebraic *L*-Theory of Group Rings

Let R be an associative ring with unit. An *involution of rings* $R \to R, r \mapsto \overline{r}$ on R is a map satisfying $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{rs} = \overline{s}\,\overline{r}$, $\overline{0} = 0$, $\overline{1} = 1$, and $\overline{\overline{r}} = r$ for all $r, s \in R$. Given a ring with involution, the group ring RG inherits an involution by $\sum_{g \in G} r_g \cdot g = \sum_{g \in G} \overline{r} \cdot g^{-1}$. If the coefficient ring R is commutative, we usually use the trivial involution $\overline{r} = r$. Given a ring with involution, one can associate to it *quadratic L-groups* $L_n^h(R)$ for $n \in \mathbb{Z}$. The abelian group $L_0^h(R)$ can be identified with the *Witt group* of non-degenerate quadratic forms on finitely generated free R-modules, where every hyperbolic quadratic forms represent the zero element and the addition is given by the orthogonal sum of hyperbolic quadratic forms. The abelian group $L_2^h(R)$ is essentially given by the skew-symmetric versions. One defines $L_1^h(R)$ and $L_3^h(R)$ in terms of automorphism of quadratic forms. The L-groups are four-periodic, i.e., there is a natural isomorphism $L_n^h(R) \xrightarrow{\cong} L_{n+4}^h(R)$ for $n \in \mathbb{Z}$. If one uses finitely generated projective R-modules instead of finitely generated free R-modules, one obtains the *proper quadratic L-groups* $L_n^p(RG)$ for $n \in \mathbb{Z}$. If one uses finitely generated based free RG-modules and takes the Whitehead torsion into account, then one obtains the *simple quadratic L-groups* $L_n^s(RG)$ for $n \in \mathbb{Z}$. For every $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ there are versions $L_n^{\langle j \rangle}(RG)$, where $\langle j \rangle$ is called *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s.

The relevance of the L-groups comes from the fact that they are the recipients for various surgery obstructions. The fundamental *surgery problem* is the following. Consider a map $f \colon M \to X$ from a closed manifold M to a finite Poincaré complex X. We want to know whether we can change it by a process called surgery to a map $g\colon N \to X$ with a closed manifold N as source and the same target such that g is a homotopy equivalence. This may answer the question whether a finite Poincaré complex X is homotopy equivalent to a closed manifold. Note that a space which is homotopy equivalent to a closed manifold must be a finite Poincaré complex, but not every finite Poincaré complex is homotopy equivalent to a closed manifold. If f comes with additional bundle data and has degree 1, we can find g if and only if the so-called *surgery obstruction* of f vanishes which takes values in $L_n^h(\mathbb{Z}G)$ for $n = \dim(X)$ and $G = \pi_1(X)$. If we want g to be a simple homotopy equivalence, the obstruction lives in $L_n^s(\mathbb{Z}G)$. We see that, analogous to the finiteness obstruction in $\widetilde{K}_0(\mathbb{Z}G)$ and the Whitehead torsion in $\mathrm{Wh}(G)$, the algebraic L-groups are the recipients for important obstructions whose vanishing has interesting geometric and topological consequences. Also the question whether two closed manifolds are diffeomorphic or homeomorphic can be decided via surgery theory of which the L-groups are a part.

More explanations about L-groups and surgery theory will be given in Chapter 9.

1.4.2 The Farrell-Jones Conjecture for $L_*(RG)[1/2]$

If we invert 2, i.e., if we consider the localization $L_n^{\langle -j \rangle}(RG)[1/2]$, then there is no difference between the various decorations and the analogues of the sequences (1.4) and (1.5) are true for L-theory, see Cappell [192]. The same reasoning as for the Baum-Connes Conjecture leads to the following conjecture.

Conjecture 1.13. (Farrell-Jones Conjecture for $L_*(RG)[1/2]$). Let G be a group. Let R be an associative ring with unit and involution. Then there is for every $n \in \mathbb{Z}$ and every decoration j an isomorphism

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle j \rangle})[1/2] \xrightarrow{\cong} L_n^{\langle j \rangle}(RG)[1/2].$$

Here $H_n^G(-; \mathbf{L}_R^{\langle j \rangle})$ is an appropriate G-homology theory with the property that $H_n^G(G/H; \mathbf{L}_R^{\langle j \rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle j-\infty \rangle}) \cong L_n^{\langle j \rangle}(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$ and the isomorphism above is induced by the G-map $\underline{E}G \to \{\bullet\}$.

1.4.3 The Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

In general the L-groups $L_n^{\langle j \rangle}(RG)$ do depend on the decoration and often the 2-torsion carries sophisticated information and is hard to handle. Recall that as a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

The L-theory analogues is given by the Shaneson splitting [896]

$$L_n^{\langle j \rangle}(R[\mathbb{Z}]) \cong L_{n-1}^{\langle j-1 \rangle}(R) \oplus L_n^{\langle j \rangle}(R).$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphisms

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j \rangle}(R)) \cong L_{n-1}^{\langle j \rangle}(R) \oplus L_n^{\langle j \rangle}(R).$$

Therefore the decoration $-\infty$ shows the right homological behavior and is the right candidate for the formulation of an isomorphism conjecture.

The analogues of the sequences (1.4) and (1.5) do not hold for $L_*^{\langle j \rangle}(RG)$, certain correction terms, the UNil-*terms* come in, which are independent of the decoration and are always 2-torsion, see Cappell [191], [192]. As in the algebraic K-theory case this leads to the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$, stated already as Conjecture 1.3.

In Conjecture 1.3 the term $H_n^G(-; \mathbf{L}_R^{\langle -\infty \rangle})$ is an appropriate G-homology theory such that $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RH)$ holds for every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$, and the assembly map is induced by the map $\underline{\underline{E}}G \to \{\bullet\}$. Conjecture 1.3 makes sense for all groups and rings with involution, and no counterexamples are known at the time of writing.

After inverting 2 Conjecture 1.3 is equivalent to Conjecture 1.13.

There is an L-theory version of the splitting (1.11)

$$H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle}) \cong H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus H_n^G(\underline{\underline{E}}G, \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}), \tag{1.14}$$

provided that there exists an integer i_0 such that $K_i(RV) = 0$ holds for all virtually cyclic subgroups $V \subseteq G$ and $i \leq i_0$.

1.4.4 Applications of the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

For applications in geometry and topology the simple L-groups $L_n^s(\mathbb{Z}G)$ are the most interesting ones. The difference between the various decorations is measured by the so-called *Rothenberg sequences* and given in terms of the Tate cohomology of $\mathbb{Z}/2$ with coefficients in $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$ and $\mathrm{Wh}(G)$

with respect to the involution coming from the involution on the group ring $\mathbb{Z}G$. Hence the decorations do not matter if $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$ and $\mathrm{Wh}(G)$ vanish. This leads in view of Conjecture 1.12 to the following version of Conjecture 1.3 for torsionfree groups

Conjecture 1.15. (Farrell-Jones Conjecture for $L_*(\mathbb{Z}G)$ for torsion-free groups). Let G be a torsionfree group. Then there is for every $n \in \mathbb{Z}$ and every decoration j an isomorphism

$$H_n(BG; \mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \xrightarrow{\cong} L_n^{\langle j \rangle}(RG).$$

Moreover, the source, target, and the map itself are independent of the decoration j.

Here $H_n(-; \mathbf{L}^{\langle j \rangle}(\mathbb{Z}))$ is the homology theory associated to the L-theory spectrum $\mathbf{L}^{\langle -j \rangle}(\mathbb{Z})$ and satisfies $H_n(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \cong \pi_n\left(\mathbf{L}^{\langle j \rangle}(\mathbb{Z})\right) \cong L_n^{\langle j \rangle}(\mathbb{Z})$.

The L-theoretic assembly map appearing in Conjecture 1.15 has a topological meaning. It appears in the so-called long *exact surgery sequence* which we will discuss in more detail in Section 9.12. Let $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ be the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$. There is a canonical map $\iota\colon H_n(BG; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \to H_n(BG; \mathbf{L}^s(\mathbb{Z}))$. Let N be an aspherical oriented closed manifold with fundamental group G, i.e., an oriented closed manifold homotopy equivalent to BG. Then G is torsionfree, the source of the composite $H_n(BG; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \to L_n^s(RG)$ of the assembly map appearing Conjecture 1.15 with ι consists of bordism classes of normal maps $M \to N$ with N as target, and the composite sends such a normal map to its surgery obstruction. This is analogous to the Baum-Connes setting where the assembly map can be described by assigning to an equivariant index problem its index.

The third term in the surgery sequence is given by the so-called *structure set* of N. It is the set of equivalence classes of homotopy equivalences $f_0\colon M_0 \to N$ with a closed topological manifold as source and N as target where $f_0\colon M_0 \to N$ and $f_1\colon M_1 \to N$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic. Conjecture 1.15 implies that this structure set is trivial provided that the dimension of N is greater or equal to five. Hence Conjecture 1.15 implies in dimensions ≥ 5 the following famous conjecture.

Conjecture 1.16 (Borel Conjecture). Let M and N be two aspherical closed topological manifolds whose fundamental groups are isomorphic. Then they are homeomorphic, and every homotopy equivalence from M to N is homotopic to a homeomorphism.

The Borel Conjecture is a topological rigidity theorem for aspherical closed manifolds and analogous to the *Mostow Rigidity Theorem* which says that two hyperbolic closed Riemannian manifolds with isomorphic fundamental groups are isometrically diffeomorphic. The Borel Conjecture is false if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. Its connection to the Borel Conjecture is one of the main features of the Farrell-Jones Conjecture. More details will be given in Subsections 9.15.2 and 9.15.3.

The Farrell-Jones Conjecture for L-theory 1.3 implies the *Novikov Conjecture*, see Section 9.14. It also has applications to the problem whether Poincaré duality groups or torsionfree hyperbolic groups with spheres as boundary are fundamental groups of aspherical closed manifolds, see Sections 9.17 and 9.18. Product decompositions of aspherical closed manifolds are treated in Section 9.20.

A summary of all the application of the Farrell-Jones Conjecture is given in Section 13.12.

1.5 More General Versions of the Farrell-Jones Conjecture

We will also treat versions of the Farrell-Jones Conjecture in equivariant additive categories, or more generally, in equivariant higher categories, see Sections 13.3 and 13.4. There will be versions with finite wreath products, see Section 13.5. The most general versions is the Full Farrell-Jones Conjecture 13.27, see Section 13.6 which implies all other variants of the Farrell-Jones Conjecture, see Section 13.11.

1.6 Status of the Baum-Connes and the Farrell-Jones Conjecture

A detailed report on the groups for which these conjectures have been proved will be given in Chapter 16. For example, the Baum-Connes Conjecture 1.1 is known for a class of groups which includes amenable groups, hyperbolic groups, knot groups, fundamental groups of compact 3-manifolds (possibly with boundary), and one-relator groups, but is open for $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$. The class of groups for which the Farrell-Jones Conjectures 1.2 and 1.3 have been proved contains hyperbolic groups, finite-dimensional CAT(0)-groups, fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary), solvable groups, lattices in almost connected Lie groups, and arithmetic groups, but they are open for amenable groups in general. If one allows coefficients, one can prove for the Baum-Connes Conjecture and the Farrell-Jones Conjecture inheritance properties, e.g., the class of groups for which they are true is closed under taking subgroups, finite direct products, free products, colimits over directed sets whose structure map are injective in the Baum-Connes case and can be arbitrary in the Farrell-Jones case. This will be explained in Sections 13.7 and 14.6.

The Full Farrell-Jones Conjecture 13.27, which implies all other variants of the Farrell-Jones Conjecture, is known to be true for some groups with unusual properties, e.g., groups with expanders, Tarsky monsters, lacunary groups, subgroups of finite products of hyperbolic groups, selfsimilar groups, see Theorem 16.1. At the time of writing we have no specific candidate of a group or of a general property of groups such that the Full Farrell-Jones Conjecture 13.27, or one of its consequences, e.g., the Novikov Conjecture and the Borel Conjecture, might be false. So we have no good starting point for a search for counterexamples, see Section 16.10.

At the time of writing no counterexamples to the Baum-Connes Conjecture is known to the author. There exists a counterexample to the Baum-Connes Conjecture with coefficients, as explained in Section 16.10.

1.7 Structural Aspects

1.7.1 The Meta-Isomorphism Conjecture

The formulations of the Baum-Connes Conjecture 1.1 and of the Farrell-Jones Conjecture 1.2 and 1.3 are very similar in the homological picture. It allows a formulation of the following *Meta-Isomorphism Conjecture*, of which both conjectures are special cases and which has also other very interesting specializations, e.g., for pseudoisotopy, A-theory, topological Hochschild, and topological cyclic homology, see Section 15.2.

Meta-Isomorphism Conjecture 1.17. *Given a group G, a G-homology theory $\mathcal{H}_*^G$, and a family $\mathcal{F}$ of subgroups of G, we say that the* Meta-Isomorphism Conjecture *is satisfied if the G-map $E_{\mathcal{F}}(G) \to \{\bullet\}$ induces for every $n \in \mathbb{Z}$ an isomorphism*

$$A_{\mathcal{F}} \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\{\bullet\}).$$

This general formulation is an excellent framework to construct transformations between the assembly maps appearing in different Isomorphism Conjectures. For instance, the *cyclotomic trace* relates the K-theoretic Farrell-Jones Conjecture with coefficients in $\mathbb{Z}$ to the Isomorphism Conjecture for topological cyclic homology, see Subsection 15.14.3, and via *symmetric signatures* one can link the Farrell-Jones Conjecture for algebraic L-theory with coefficient in $\mathbb{Z}$ to the Baum-Connes Conjecture, see Subsection 15.14.4. Moreover, basic computational tools and techniques for equivariant homology theories apply both to the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3.

1.7.2 Assembly

One important idea is the *assembly principle* which leads to assembly maps in a canonical and universal way by asking for the best approximation of a homotopy invariant functor from G-spaces to spectra by an equivariant homology theory. It is an important ingredient for the identification of the various descriptions of assembly maps appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture. For instance, the assembly map appearing in the Baum-Connes Conjecture 1.1 can be interpreted as assigning to an appropriate equivariant elliptic complex its equivariant index, and the assembly map appearing in the L-theoretic Farrell-Jones Conjecture 1.3 is related to the map appearing the surgery sequence, which assigns to a surgery problem its surgery obstruction. We have already explained above that these identification are the basis for some of applications of the Isomorphisms Conjectures, and we will see that there are also important for proofs. There is a homotopy theoretic approach to the assembly map based on homotopy colimits over the orbit category, which motivates the name assembly. Roughly speaking, the name assembly can be understood as assembling the values of the K-and L-groups of the reduced group C^*-algebra or the group ring of a group G from their values on finite or virtually cyclic subgroups of G. All this will be explained in Chapter 18.

This parallel treatment of the Baum-Connes Conjecture and the Farrell-Jones Conjecture and of other variants is one of the topics of this book. However, the geometric interpretations of the assembly maps in terms of indices, surgery obstructions, or forget control are quite different. Therefore the methods of proof for the Farrell-Jones Conjecture and the Baum-Connes Conjecture use different input. Although there are some similarities in the proofs, its is not clear how to export methods of proof from one conjecture to the other.

1.8 Computational Aspects

In general the target $K_n(C_r^*(G))$ of the assembly map appearing in the Baum-Connes Conjecture 1.1 is very hard to compute, whereas the source $K_n^G(\underline{E}G)$ is much more accessible because one can apply standard techniques from algebraic topology such as spectral sequences and equivariant Chern characters and there are often nice small geometric models for $\underline{E}G$. For the Farrell-Jones Conjectures 1.2 and 1.3, this applies also to the parts $H_n^G(\underline{E}G;\mathbf{K}_R)$ and $H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ respectively appearing in the splittings (1.11) and (1.14). The other parts $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{K}_R)$ or $H_n^G(\underline{\underline{E}}G,\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ are harder to handle, since they involve Nil- or UNil-terms and the G-CW-complex $\underline{\underline{E}}G$ is not proper and in general huge. Most of the known computations of

$K_n(C_r^*(G))$, $K_n(RG)$, and $L_n^{\langle -j \rangle}(RG)$ are based on the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3.

Classifications of manifolds and of C^*-algebras rely on and thus motivate explicit calculations of K- and L-groups. In this context it is often important, not only to determine the K- and L-groups abstractly, but to develop detection techniques so that one can identify or distinguish specific elements associated to the original classification problem or give geometric or index-theoretic interpretations to elements in the K- and L-groups.

A general guide for computations and a list of known cases including applications to classification problems will be given in Chapter 17.

1.9 Are the Baum-Connes Conjecture and the Farrell-Jones Conjecture True in General?

The title of this section is the central and at the time of writing unsolved question. One motivation for writing this monograph is to stimulate some very clever mathematician to work on this problem and finally find an answer. Let us speculate about the possible answer.

We are skeptical about the Baum-Connes Conjecture for two reasons: there are counterexamples for the version with coefficients, and the left side of the Baum-Connes assembly map is functorial under arbitrary group homomorphisms, whereas the right side is not. The Bost Conjecture which predicts an isomorphism

$$K_n^G(\underline{E}G) \to K_n(l^1(G))$$

has a much better chance to be true in general. The possible failure of the Baum-Connes Conjecture may come from the possible failure of the canonical map $K_n(l^1(G)) \to K_n(C_*^r(G))$ to be bijective.

In spite of the Baum-Connes Conjecture, we do not see an obvious flaw with the Bost Conjecture or the Farrell-Jones Conjecture. As explained in Section 1.6 above, we have no starting point for a construction of a counterexample, and all abstract properties we know for the right side do hold for the left side of the assembly map and vice versa. In particular for the Bass Conjecture and for the Novikov Conjecture which follow from the Farrell-Jones Conjecture, the class of groups for which they are known to be true is impressive. There are some conclusions from the Farrell-Jones Conjecture which are not trivial and true for all groups. These are arguments are in favor of a positive answer

The following arguments are in favor of a negative answer. The universe of groups is overwhelming large. We have Gromov's saying on our neck that a statement which is true for all groups is either trivial or false. We have no philosophical reason why the Bost Conjecture or the Farrell-Jones Conjecture should be true. Finding a counterexample will probably require some new ideas, maybe from logic or random groups.

The upshot of this discussion is that the author is sketical about the Baum-Connes Conjecture, but does not dare to make any predictions about the chances for the other conjectures, in particular for the Novikov Conjecture, to be true for all groups.

We will elaborate on this discussion in Section 16.10.

1.10 The Organization of the Book and a User's Guide

We have written the text in a way such that one can read small units, e.g., a single chapter, independently from the rest, concentrate on certain aspects, and extract easily and quickly specific information. We hopefully have found the right mixture between definitions, theorems, examples, and remarks so that reading the book is entertaining and illuminating. We have successfully used parts of this book, sometimes a single chapter, for seminars, reading courses, and advanced lecture courses.

The book consists of three parts and a supplement, which we briefly review next. We will also give some information how to use the book.

1.10.1 Introduction to K- and L-Theory (Part I)

In the first part "Introduction to K- and L-Theory", which encompasses Chapters 2 to 10, we introduce and motivate the relevant theories, namely, algebraic K-theory, algebraic L-theory, and topological K-theory. In these chapters we present some applications and more accessible special versions of the Baum-Connes and the Farrell-Jones Conjecture. They are rather independent of one another and one can start reading each of them without having gone though the others. If a reader may just want to get some information, for instance about Wall's finiteness obstruction, Whitehead torsion, or the projective class group, she or he can directly start reading the relevant chapter, learn the basics about these invariant, and understand the relevant special versions of the Baum-Connes Conjecture or the Farrell-Jones Conjecture without going through the other chapters. Each of these chapters is eligible for a lecture course, seminar, or reading course.

1.10.2 The Isomorphism Conjectures (Part II)

In the second part "The Isomorphism Conjectures", which consists of Chapters 11 to Chapter 18, we introduce the Baum-Connes Conjecture and the Farrell-Jones Conjecture in their most general form, namely, for arbitrary groups and with coefficients. We discuss further applications and in particular how they can be used for computations. We give a report about the status of these conjectures and discuss open problems.

Note that the Farrell-Jones Conjecture comes in different levels. It can be considered for rings (with involution) and hence aims at the algebraic K-theory and L-theory of groups rings. This is the most relevant version for applications, where it often suffices to treat lower and middle K-theory, torsionfree groups, and $\mathbb{Z}$ or a field as coefficients. One may twist the group rings and allow orientation characters. The next level is to pass to equivariant additive categories (with involution) as coefficients, which has the advantage that it automatically leads to useful inheritance properties of the Farrell-Jones Conjecture and does encompass the case of rings as coefficients. For algebraic K-theory one can even allow higher categories as coefficients. This contains the version of additive categories as coefficients and also the versions of the Farrell-Jones Conjecture for Waldhausen's A-theory, for pseudoisotopy, and for Whitehead spaces as special cases. There are also versions "with finite wreath product", where the passage to overgroups of finite index is built in.

So there are many variations of the Farrell-Jones Conjecture, but the Full Farrell-Jones Conjecture 13.27 does imply all of them.

We also state Meta-Conjectures, which reduce to the Baum-Connes Conjecture, the Farrell-Jones Conjecture, or other types of Isomorphism Conjectures if one feeds the right theory into them. There are versions of the Farrell-Jones Conjecture for Waldhausen's A-theory, pseudoisotopy, Whitehead spaces, topological Hochschild homology, topological cyclic homology, and homotopy K-theory.

We also briefly discuss the Farrell-Jones Conjecture for totally disconnected groups and Hecke Algebras, where for the first time a version of the Farrell-Jones Conjecture for a topological groups is considered. The Baum-Connes Conjecture has already been intensively studied for topological groups. However, in this monograph we will confine ourselves to discrete groups.

1.10.3 Methods of Proofs (Part III)

The third part "Methods of Proofs", which ranges from Chapter 19 to Chapter 26, we give a survey on the background, history, philosophy, strategies, and some ingredients of the proofs. We will concentrate on the Farrell-Jones Conjecture in this part III.

The reader, who is interested in proofs, should first go through Chapter 19. There motivations for the proofs of the Farrell-Jones Conjecture and some information about their long history is given without getting lost in technical details. So it will be a soft introduction to the methods of proofs conveying ideas only. Mainly we explain why controlled topology, flows, and transfers come in, which one would not expect at the first glance in view of the homotopy theoretic nature of the Farrell-Jones Conjecture.

In Chapter 20 we isolate some conditions about a group which guarantee that it satisfies the Full Farrell-Jones Conjecture or some special version of it. Note that here K- or L-theory do not yet play any role and one can use the results of this section without any knowledge about them. This is interesting for someone who is already familiar with geometric group theory but has no background in K- or L-theory.

Depending on how ambitious the reader is, she or he should go through the other chapters. We recommend to read Section 24.7, where details of the proof of the Farrell-Jones Conjecture for the surjectivity of the K-theoretic assembly map in dimension 1 is given, which does not use much knowledge about algebraic K-theory but uses all the basic ideas appearing in the proof of the Full Farrell-Conjecture.

The reader, who wants to understand the proof in the most advanced setting, namely the one for higher categories as coefficients, and for the largest class of groups, namely the class of Dress-Farrell-Hsiang-Jones groups, is recommended to read through Chapter 25. For this some background in higher category theory is necessary.

We give a very brief overview for the methods of proof for the Baum-Connes Conjecture in Chapter 26.

1.10.4 Supplement

The book contains a number of exercises. They are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text or improve her or his understanding by trying to solve the exercises. Hints to the solutions of the exercises are given in Chapter 27.

If one wants to find a specific topic, the extensive index of the monograph can be used to find the right spot for a specific topic. The index contains an item “Theorem”, under which all theorems with their names appearing in the book are listed, and analogously there is an item “Conjecture”.

1.10.5 Prerequisites

We require that the reader is familiar with basic notions in topology (CW-complexes, chain complexes, homology, homotopy groups, manifolds, coverings, cofibrations, fibrations, ...), functional analysis (Hilbert spaces, bounded operators, differential operators, ...), algebra (groups, modules, group rings, elementary homological algebra, ...), group theory (presentations, Cayley graphs, hyperbolic groups, ...), and elementary category theory (functors, transformations, additive categories, ...).

1.11 Notations and Conventions

Here is a briefing on our main conventions and notations. Details are of course discussed in the text.

- Ring will mean (not necessarily commutative) associative ring with unit unless explicitly stated otherwise;
- Module means always left module unless explicitly stated otherwise;
- Groups means discrete group unless explicitly stated otherwise;
- We will always work in the category of compactly generated spaces, compare [909] and [987, I.4]. In particular every space is automatically Hausdorff.
- For our conventions concerning spectra see Section 12.4. Spectra are denoted with boldface letters such as $\mathbf{E}$;
- We use the standards symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\widehat{\mathbb{Z}_p}$ for the integers, the rational numbers, the real numbers, the complex numbers, and the p-adic numbers.
- We denote

symbol	name
$\mathbb{Z}/n$	finite cyclic group of order n
S_n	symmetric group of permutations of the set $\{1, 2, \ldots n\}$
A_n	alternating group of even permutations of the set $\{1, 2, \ldots, n\}$
D_∞	infinite dihedral group
D_n	dihedral group of order n

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1.13 Notes

Further information about the Baum-Connes Conjecture and the Farrell-Jones Conjecture can be found in the survey articles [66, 87, 103, 351, 409, 467, 644, 657, 727, 830, 945].

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Chapter 2
The Projective Class Group

2.1 Introduction

This chapter is devoted to the *projective class group* $K_0(R)$ of a ring R.

We give in Section 2.2 three equivalent definitions of $K_0(R)$, namely, by the universal additive invariant for finitely generated projective modules, by the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective modules, and by idempotent matrices, and discuss the significance of $K_0(R)$ for the category of finitely generated projective modules. We explain some calculations for principal ideal domains and Dedekind rings in Section 2.3.

We explain connections to geometry. We prove *Swan's Theorem* 2.27 that relates $K_0(C^0(X))$ for the ring $C^0(X)$ of continuous functions on a compact space X to the Grothendieck group of the abelian monoid of isomorphism classes of vector bundles over X. The relevance of $K_0(\mathbb{Z}G)$ for topologists is illustrated by *Wall's finiteness obstruction*, which also leads to a geometric description of $K_0(\mathbb{Z}G)$ in terms of finitely dominated spaces and is discussed in detail in Section 2.5.

We introduce variants of the *K-theoretic Farrell-Jones Conjecture* for projective class groups in Section 2.8. A prototype asserts that for a torsionfree group G and a regular ring R, e.g., $R = \mathbb{Z}$ or R a field, the change of rings map

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective. It implies the conjecture that for a torsionfree group G the reduced projective class group $\widetilde{K}_0(\mathbb{Z}G)$ vanishes, which is for finitely presented G equivalent to the conjecture that every finitely dominated CW-complex with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW-complex. We also introduce a version where the group is not necessarily torsionfree, but R is a regular ring with $\mathbb{Q} \subseteq R$ or a field of prime characteristic.

In Section 2.9 we consider *Kaplansky' s Idempotent Conjecture*, which asserts for a torsionfree group G and a field F that 0 and 1 are the only idempotents in FG. It is a consequence of the Farrell-Jones Conjecture. We also discuss various *Bass Conjectures*, all of which are implied by the Farrell-Jones Conjecture, in Section 2.10.

Finally, we give a survey of $K_0(\mathbb{Z}G)$ for finite groups G and of $K_0(C_r^*(G))$ in Section 2.12 and of $K_0(\mathcal{N}(G))$ in Section 2.13, where $C_r^*(G)$ is the reduced group C^*-algebra and $\mathcal{N}(G)$ the group von Neumann algebra.

2.2 Definition and Basic Properties of the Projective Class Group

Definition 2.1 (Projective class group $K_0(R)$). Let R be an (associative) ring (with unit). Define its *projective class group* $K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.

Given a ring homomorphism $f\colon R \to S$, we can assign to an R-module M an S-module f_*M by $S \otimes_R M$ where we consider S as a right R-module using f. We say that f_*M is obtained by *induction with* f from M. If M is finitely generated or free or projective, the same is true for f_*M. This construction is natural, compatible with direct sums, and sends an exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules to an exact sequence $0 \to f_*P_0 \to f_*P_1 \to f_*P_2 \to 0$ of finitely generated projective S-modules. Hence we get a homomorphism of abelian groups

$$f_* = K_0(f)\colon K_0(R) \to K_0(S), \ \ [P] \mapsto [f_*P], \tag{2.2}$$

which is also called *change of rings homomorphism*. Thus K_0 becomes a covariant functor from the category of rings to the category of abelian groups.

Remark 2.3 (The universal property of the projective class group). One should view $K_0(R)$ together with the assignment sending a finitely generated projective R-module P to its class $[P]$ in $K_0(R)$ as the *universal additive invariant* or the *universal dimension function* for finitely generated projective R-modules. Namely, suppose that we are given an abelian group and an assignment d that associates to a finitely generated projective R-module an element $d(P) \in A$ such that $d(P_0) + d(P_2) = d(P_1)$ holds for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules. Then there is precisely one homomorphism of abelian groups $\phi\colon K_0(R) \to A$ such that $\phi([P]) = d(P)$ holds for every finitely generated projective R-module P. The analogous statement holds for $G_0(R)$ if we consider finitely generated R-modules instead of finitely generated projective R-modules.

A ring is an *integral domain* if every zero-divisor is trivial, i.e., if $r, s \in R$ satisfy $rs = 0$, then $r = 0$ or $s = 0$. A *principal ideal domain* is a commutative integral domain for which every ideal is a *principal ideal*, i.e., of the form $(r) = \{r'r \mid r' \in R\}$ for some $r \in R$.

Example 2.4 ($K_0(R)$ and $G_0(R)$ of a principal ideal domain). Let R be a principal ideal domain. Then we get isomorphisms of abelian groups

$$\mathbb{Z} \xrightarrow{\cong} K_0(R), \quad n \mapsto [R^n];$$
$$K_0(R) \xrightarrow{\cong} G_0(R), \quad [P] \mapsto [P].$$

This follows from the structure theorem of finitely generated R-modules over principal ideal domains. It implies for any finitely generated R-module M that it can be written as a direct sum $R^n \oplus T$ for some torsion R-module T for which there exists an exact sequence of R-modules of the shape $0 \to R^s \to R^s \to T \to 0$. Moreover, M is projective if and only if T is trivial and $R^m = R^n \iff m = n$.

Definition 2.5 (Reduced projective class group $K_0(R)$). Define the *reduced projective class group* $\widetilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the abelian subgroup $\{[R^m]-[R^n] \mid n, m \in \mathbb{Z}, m, n \geq 0\}$, which is the same as the abelian subgroup generated by the class $[R]$.

We conclude from Example 2.4 that the reduced projective class group $\widetilde{K}_0(R)$ is isomorphic to the cokernel of the homomorphism

$$f_*\colon K_0(\mathbb{Z}) \to K_0(R)$$

where f is the unique ring homomorphism $\mathbb{Z} \to R$, $n \mapsto n \cdot 1_R$.

Remark 2.6 (The projective class group as a Grothendieck group). Let $\operatorname{Proj}(R)$ be the abelian semigroup of isomorphisms classes of finitely generated projective R-modules with the addition coming from the direct sum. Let $K_0'(R)$ be the associated abelian group given by the Grothendieck construction applied to $\operatorname{Proj}(R)$. There is a natural homomorphism

$$\phi\colon K_0'(R) \xrightarrow{\cong} K_0(R)$$

sending the class of a finitely generated projective R-module P in $K_0'(R)$ to its class in $K_0(R)$. This is a well-defined isomorphism of abelian groups.

The analogous definition of $G_0'(R)$ and the construction of a homomorphism $G_0'(R) \to G_0(R)$ makes sense, but the latter map is *not* bijective in general. It works for $K_0(R)$ because every exact sequence of projective R-modules $0 \to P_0 \to P_1 \to P_2 \to 0$ splits and thus yields an isomorphism $P_1 \cong P_0 \oplus P_2$. In general K-theory deals with exact sequences, not with direct sums. Therefore Definition 2.1 of $K_0(R)$ reflects better the underlying idea of K-theory than its definition in terms of the Grothendieck construction.

Exercise 2.7. Prove that the homomorphism $\phi\colon K_0'(R) \to K_0(R)$ appearing in Remark 2.6 is a well-defined isomorphism of abelian groups.

Remark 2.8 (What does the reduced projective class group measure?). Let P be a finitely generated projective R-module. Then we conclude from Remark 2.6 that its class $[P] \in \widetilde{K}_0(R)$ is trivial if and only if P is *stably finitely generated free*, i.e., $P \oplus R^r \cong R^s$ for appropriate integers $r, s \geq 0$. So the reduced projective class group $\widetilde{K}_0(R)$ measures the deviation of a finitely generated projective R-module to be stably finitely generated free. Note that stably finitely generated free does in general not imply finitely generated free as Examples 2.9 and 2.28 will show.

Example 2.9 (Dunwoody's example). An interesting $\mathbb{Z}G$-module P that is stably finitely generated free but not finitely generated free is constructed by Dunwoody [303] for G the torsionfree one-relator group $\langle a, b \mid a^2 = b^3 \rangle$, which is the fundamental group of the trefoil knot. Note that $\widetilde{K}_0(\mathbb{Z}G)$ is known to be trivial, in other words, every finitely generated projective RG-module is stably finitely generated free. It is also worth while mentioning that $\mathbb{Z}G$ contains no idempotent besides 0 and 1. Hence any direct summand in $\mathbb{Z}G$ is free.

More examples of this kind are given in Berridge-Dunwoody [127].

One basic feature of algebraic K-theory is *Morita equivalence.*

Theorem 2.10 (Morita equivalence for $K_0(R)$). *For every ring R and integer $n \geq 1$, there is a natural isomorphism*

$$\mu \colon K_0(R) \xrightarrow{\cong} K_0(\mathrm{M}_n(R)).$$

Proof. We can consider R^n as $\mathrm{M}_n(R)$-R-bimodule, denoted by ${}_{\mathrm{M}_n(R)}R^n{}_R$. Then μ sends $[P]$ to $[{}_{\mathrm{M}_n(R)}R^n{}_R \otimes_R P]$. We can also consider R^n as an R-$\mathrm{M}_n(R)$-bimodule denoted by ${}_R R^n{}_{\mathrm{M}_n(R)}$. Define $\nu \colon K_0(\mathrm{M}_n(R)) \to K_0(R)$ by sending $[Q]$ to $[{}_R R^n{}_{\mathrm{M}_n(R)} \otimes_{\mathrm{M}_n(R)} Q]$. Then μ and ν are inverse to one another. □

Exercise 2.11. Check that μ and ν are inverse to one another.

We omit the easy proof of the next lemma.

Lemma 2.12. *Let R_0 and R_1 be rings. Denote by $\mathrm{pr}_i \colon R_0 \times R_1 \to R_i$ for $i = 0, 1$ the projection. Then we obtain an isomorphism*

$$(\mathrm{pr}_0)_* \times (\mathrm{pr}_1)_* \colon K_0(R_0 \times R_1) \xrightarrow{\cong} K_0(R_0) \times K_0(R_1).$$

Example 2.13 (Rings with non-trivial $\widetilde{K}_0(R)$). We conclude from Example 2.4 and Lemma 2.12 that for a principal ideal domain R we have

$$K_0(R \times R) \cong \mathbb{Z} \oplus \mathbb{Z};$$
$$\widetilde{K}_0(R \times R) \cong \mathbb{Z}.$$

The $R \times R$-module $R \times \{0\}$ is finitely generated projective but not stably finitely generated free. It is a generator of the infinite cyclic group $\widetilde{K}_0(R \times R)$.

Notation 2.14 ($\mathrm{M}(R)$, $\mathrm{GL}(R)$ and $\mathrm{Idem}(R)$). Let $\mathrm{M}_{m,n}(R)$ be the set of (m, n)-matrices over R. For $A \in \mathrm{M}_{m,n}(R)$, let $r_A \colon R^m \to R^n,\ x \to xA$ be the R-homomorphism of (left) R-modules given by right multiplication with A. Let $\mathrm{M}_n(R)$ be the ring of (n,n)-matrices over R. Denote by $\mathrm{GL}_n(R)$ the group of invertible (n,n)-matrices over R. Let $\mathrm{Idem}_n(R)$ be the subset of $\mathrm{M}_n(R)$ of idempotent matrices A, i.e., (n,n)-matrices satisfying $A^2 = A$. There are embeddings $i_{t,n}\colon \mathrm{M}_n(R) \to \mathrm{M}_{n+1}(R)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix}$ for $t = 0,1$ and $n \geq 1$. The embedding $i_{1,n}$ induces an embedding $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ of groups. Let $\mathrm{GL}(R)$ be the union of the $\mathrm{GL}_n(R)$-s, which is a group again. Denote by $\mathrm{M}(R)$ the union of the $\mathrm{M}_n(R)$-s with respect to the embeddings i_0. This is a ring without unit. Let $\mathrm{Idem}(R)$ be the set of idempotent elements in $M(R)$. This is the same as the union of the $\mathrm{Idem}_n(R)$-s with respect to the embeddings $\mathrm{Idem}_n(R) \to \mathrm{Idem}_{n+1}(R)$ coming from the embeddings $i_{0,n}\colon \mathrm{M}_n(R) \to \mathrm{M}_{n+1}(R)$.

Remark 2.15 (The projective class groups in terms of idempotent matrices). The projective class groups $K_0(R)$ can also be defined in terms of idempotent matrices. Namely, the conjugation action of $\mathrm{GL}_n(R)$ on $\mathrm{M}_n(R)$ induces an action of $\mathrm{GL}(R)$ on $\mathrm{M}(R)$ which leaves $\mathrm{Idem}(R)$ fixed. One obtains a bijection of sets

$$\phi\colon \mathrm{GL}(R)\backslash \mathrm{Idem}(R) \to \mathrm{Proj}(R), \quad [A] \mapsto \mathrm{im}\left(r_A\colon R^n \to R^n\right).$$

This becomes a bijection of abelian semigroups if we equip the source with the addition coming from $(A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and the target with the one coming from the direct sum. So we can identify $K_0(R)$ with the Grothendieck group associated to the abelian semigroup $\mathrm{GL}(R)\backslash \mathrm{Idem}(R)$ by Remark 2.6.

Exercise 2.16. Show that the map ϕ appearing in Remark 2.15 is a well-defined isomorphism of abelian semigroups.

Example 2.17 (A ring R with trivial $K_0(R)$). Let F be a field and let V be an F-vector space with an infinite countable basis. Consider the ring $R = \mathrm{end}_F(V)$. Next we prove that $K_0(R)$ is trivial.

By Remark 2.15 it suffices to show for every integer $n \geq 0$ and two idempotent matrices $A, B \in \mathrm{Idem}_n(R)$ that the matrices $A \oplus 0 \oplus 1$ and $B \oplus 1 \oplus 0$ in $\mathrm{M}_{n+2}(R)$ are conjugated by an element in $\mathrm{GL}_{n+2}(R)$. This follows from the observations that both the kernel and the image of the F-linear endomorphisms $r_{A\oplus 0\oplus 1}$ and $r_{B\oplus 0\oplus 1}$ of V^{n+2} have infinite countable dimension, two F-vector spaces of infinite countable dimension are isomorphic, the obvious sequence of F-vector spaces $0 \to \ker(r_{A\oplus 0\oplus 1}) \to V^{n+1} \to \mathrm{im}(r_{A\oplus 0\oplus 1}) \to 0$ is split exact, and the same is true for $r_{B\oplus 1\oplus 0}$.

Lemma 2.18. *Let G be a group. Let R be a commutative integral domain with quotient field F. Then we obtain an isomorphism*

$$K_0(RG) \xrightarrow{\cong} \widetilde{K}_0(RG) \oplus \mathbb{Z}, \quad [P] \;\mapsto\; ([P], \dim_F(F \otimes_{RG} P))$$

where F is considered as an RG-module with respect to the trivial G-action and the inclusion of rings $j\colon R \to F$.

Proof. Since $F \otimes_{RG} P$ is a finite dimensional F-vector space for finitely generated P and $F \otimes_{RG} (P \oplus Q) \cong_G (F \otimes_{RG} P) \oplus (F \otimes_{RG} Q)$, this is a well-defined homomorphism. Bijectivity follows from $\dim_F(F \otimes_{RG} RG^n) = n$. □

2.3 The Projective Class Group of a Dedekind Domain

Let R be a commutative integral domain with quotient field F. A non-zero R-submodule $I \subset F$ is called a *fractional ideal* if for some $r \in R$ we have $rI \subseteq R$. A fractional ideal I is called *principal* if I is of the form $\left\{ \frac{ra}{b} \mid r \in R \right\}$ for some $a, b \in R$ with $a, b \neq 0$.

Definition 2.19 (Dedekind domain). A commutative integral domain R is called a *Dedekind ring* if for any fractional ideal I there exists another fractional ideal J with $IJ = R$.

Note that in Definition 2.19 the fractional ideal J must be given by $\{x \in F \mid x \cdot I \subseteq R\}$.

The fractional ideals in a Dedekind ring form by definition a group under multiplication of ideals with R as unit. The principal fractional ideals form a subgroup. The *class group* $C(R)$ is the quotient of these abelian groups.

A proof of the next theorem can be found for instance in [712, Corollary 11 on page 14] and [844, Theorem 1.4.12 on page 20].

Theorem 2.20 (The reduced projective class group and the class group of Dedekind domains). *Let R be a Dedekind domain. Then every fractional ideal is a finitely generated projective R-module and we obtain an isomorphism of abelian groups*

$$\mathbb{Z} \oplus C(R) \xrightarrow{\cong} K_0(R), \quad (n, [I]) \mapsto n \cdot [R] + [I] - [R].$$

In particular we get an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

A ring is called *hereditary*, if every ideal is projective, or, equivalently, if every submodule of a projective R-module is projective, see [203, Theorem 5.4 in Chapter I.5 on page 14].

Theorem 2.21 (Characterization of Dedekind domains). *The following assertions are equivalent for a commutative integral domain with quotient field F:*

(i) R is a Dedekind domain;

(ii) For every pair of ideals $I \subseteq J$ of R, there exists an ideal $K \subseteq R$ with $I = JK$;
(iii) R is hereditary;
(iv) Every finitely generated torsionfree R-module is projective;
(v) R is Noetherian and integrally closed in its quotient field F and every non-zero prime ideal is maximal.

Proof. This follows from [257, Proposition 4.3 on page 76 and Proposition 4.6 on page 77] and the fact that a finitely generated torsionfree module over an integral domain R can be embedded into R^n for some integer $n \geq 0$. See also [57, Chapter 13]. $\square$

Remark 2.22 (The class group in terms of ideals of R). One calls two ideals I and J in R equivalent if there exists non-zero elements r and s in R with $rI = sJ$. Then $C(R)$ is the same as the equivalence classes of ideals under multiplication of ideals and the class given by the principal ideals as unit. Two ideals I and J of R define the same element in $C(R)$ if and only if they are isomorphic as R-modules, see [844, Proposition 1.4.4 on page 17].

Recall that an *algebraic number field* is a finite algebraic extension of $\mathbb{Q}$ and the *ring of integers* in F is the integral closure of $\mathbb{Z}$ in F.

Theorem 2.23 (The class group of a ring of integers is finite). *Let R be the ring of integers in an algebraic number field. Then R is a Dedekind domain and its class group $C(R)$ and hence its reduced projective class group $\widetilde{K}_0(R)$ are finite.*

Proof. See [844, Theorem 1.4.18 on page 22 and Theorem 1.4.19 on page 23]. $\square$

Remark 2.24 (Class group of $\mathbb{Z}[\exp(2\pi i/p)]$). Let p be a prime number. The ring of integers in the algebraic number field $\mathbb{Q}[\exp(2\pi i/p)]$ is $\mathbb{Z}[\exp(2\pi i/p)]$. Its class group $C(\mathbb{Z}[\exp(2\pi i/p)])$ is finite by Theorem 2.23. However, its structure as a finite abelian group is only known for finitely many small primes, see [712, Remark 3.4 on page 30] or [971, Tables §3 on page 352ff].

Example 2.25 ($\widetilde{K}_0(\mathbb{Z}[\sqrt{-5}])$). The reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\sqrt{-5}])$ of the Dedekind domain $\mathbb{Z}[\sqrt{-5}]$ is cyclic of order two. A generator is given by the maximal ideal $(3, 2+\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$. (For more details see [844, Exercise 1.4.20 on page 25]).

2.4 Swan's Theorem

Let F be the field $\mathbb{R}$ or $\mathbb{C}$. Let X be a compact space. Denote by $C(X, F)$ or briefly by $C(X)$ the ring of continuous functions from X to F. Let ξ and

η be (finite dimensional locally trivial) F-vector bundles over X. Denote by $C(\xi)$ the F-vector space of continuous sections of ξ. This becomes a $C(X)$-module by the pointwise multiplication. If $\underline{F}$ denotes the trivial 1-dimensional vector bundle $X \times F \to X$, then $C(\underline{F})$ and $C(X)$ are isomorphic as $C(X)$-modules. If ξ and η are isomorphic as F-vector bundles, then $C(\xi)$ and $C(\eta)$ are isomorphic as $C(X)$-modules. There is an obvious isomorphism of $C(X)$-modules

$$C(\xi) \oplus C(\eta) \xrightarrow{\cong} C(\xi \oplus \eta). \tag{2.26}$$

Since X is compact, every F-vector bundle has a finite bundle atlas and admits a Riemannian metric. This implies the existence of a F-vector bundle ξ' such that $\xi \oplus \xi'$ is isomorphic as F-vector bundle to a trivial F-vector bundle $\underline{F}^n$. Hence $C(\xi)$ is a finitely generated projective $C(X)$-module. Denote by $\hom(\xi, \eta)$ the $C(X)$-module of morphisms of F-vector bundles from ξ to η, i.e., of continuous maps between the total spaces that commutes with the bundle projections to X and induce linear (not necessarily injective or bijective) maps between the fibers over x for all $x \in X$. This becomes a $C(X)$-module by the pointwise multiplication. Such a morphism $f \colon \xi \to \eta$ induces a $C(X)$-homomorphism $C(f) \colon C(\xi) \to C(\eta)$ by composition. The next result is due to Swan [921].

Theorem 2.27 (Swan's Theorem). *Let X be a compact space and $F = \mathbb{R}, \mathbb{C}$. Then:*

(i) Let ξ and η be F-vector bundles. Then we obtain an isomorphism of $C(X)$-modules

$$\Gamma(\xi, \eta) \colon \hom(\xi, \eta) \to \hom_{C(X)}(C(\xi), C(\eta)), \quad f \mapsto C(f);$$

(ii) We have $\xi \cong \eta \iff C(\xi) \cong_{C(X)} C(\eta)$;

(iii) If P is a finitely generated projective $C(X)$-module, then there exists an F-vector bundle ξ satisfying $C(\xi) \cong_{C(X)} P$.

Proof. (i) Obviously $\Gamma(\xi \oplus \xi', \eta)$ can be identified with $\Gamma(\xi, \eta) \oplus \Gamma(\xi', \eta)$ and $\Gamma(\xi, \eta \oplus \eta')$ can be identified with $\Gamma(\xi, \eta) \oplus \Gamma(\xi, \eta'')$ under the identification (2.26). Since a direct sum of two maps is a bijection if and only if each of the maps is a bijection and for every ξ there is ξ' such that $\xi \oplus \xi'$ is trivial, it suffices to treat the case where $\xi = \underline{F}^m$ and $\eta = \underline{F}^n$ for appropriate integers $m, n \geq 0$. There is an obvious commutative diagram

$$\begin{tikzcd}
\hom(\underline{F}^m, \underline{F}^n) \arrow[r, "\Gamma(\underline{F}^m, \underline{F}^n)"] \arrow[d, "\cong"'] & \hom_{C(X)}(C(\underline{F}^m), C(\underline{F}^n)) \arrow[d, "\cong"] \\
\mathrm{M}_{m,n}(\hom(\underline{F}, \underline{F})) \arrow[r, "\mathrm{M}_{m,n}(\Gamma(\underline{F}, \underline{F}))"'] & \mathrm{M}_{m,n}(C(\underline{F}))
\end{tikzcd}$$

Hence it suffices to treat the claim for $m = n = 1$, which is obvious.

(ii) This follows from assertion (i).

(iii) Given a finitely generated projective $C(X)$-module P, choose a $C(X)$-map $p\colon C(X)^n \to C(X)^n$ satisfying $p^2 = p$ and $\mathrm{im}(p) \cong_{C(X)} P$. Because of assertion (ii) we can choose a morphism of F-vector bundles $q\colon \underline{F}^n \to \underline{F}^n$ with $\Gamma(\underline{F}^n, \underline{F}^n)(q) = p$. We conclude $q^2 = q$ from $p^2 = p$ and the injectivity of $\Gamma(\underline{F}^n, \underline{F}^n)$. Elementary bundle theory shows that the image of q and the image of $1-q$ are F-subvector bundles in $\underline{F}^n$ satisfying $\mathrm{im}(q) \oplus \mathrm{im}(1-q) = \underline{F}^n$. One easily checks $C(\mathrm{im}(q)) \cong_{C(X)} P$. □

One may summarize Theorem 2.27 by saying that we obtain an equivalence of $C(X)$-additive categories from the category of F-vector bundles over X to the category of finitely generated projective $C(X)$-modules by sending ξ to $C(\xi)$.

Example 2.28 ($C(TS^n)$). Consider the n-dimensional sphere S^n. Let TS^n be its tangent bundle. Then $C(TS^n)$ is a finitely generated projective $C(S^n)$-module. It is free if and only if TS^n is trivial. This is equivalent to the condition that $n = 1, 3, 7$, see [146]. On the other hand $C(TS^n)$ is always stably finitely generated free as a $C(S^n)$-module since TS^n is stably finitely generated free as an F-vector bundle because the direct sum of TS^n and the normal bundle $\nu(S^n, \mathbb{R}^{n+1})$ of the standard embedding $S^n \subseteq \mathbb{R}^{n+1}$ is $T\mathbb{R}^{n+1}|_{S^n}$ and both F-vector bundles $\nu(S^n, \mathbb{R}^{n+1})$ and $T\mathbb{R}^{n+1}|_{S^n}$ are trivial.

Exercise 2.29. Consider an integer $n \geq 1$. Show that there exists a $C(S^n)$-module M with $C(TS^n) \cong_{C(S^n)} C(S^n) \oplus M$ if and only if S^n admits a nowhere vanishing vector field. (This is equivalent to requiring that $\chi(S^n) = 0$, or, equivalently, that n is odd.)

Remark 2.30 (Topological K-theory in dimension 0). Let X be a compact space. Let $\mathrm{Vect}_F(X)$ be the abelian semigroup of isomorphism classes of F-vector bundles over X where the addition comes from the Whitney sum. Let $K^0(X)$ be the abelian group obtained from the Grothendieck construction to it. It is called the *0-th topological K-group* of X. If $f\colon X \to Y$ is a map of compact spaces, the pullback construction yields a homomorphism $K^0(f)\colon K^0(Y) \to K^0(X)$. Thus we obtain a contravariant functor K^0 from the category of compact spaces to the category of abelian groups. Since the pullback of a vector bundle with two homotopic maps yields isomorphic vector bundles, $K^0(f)$ depends only on the homotopy class of f. Actually there is a sequence of such homotopy invariants covariant functors K^n for $n \in \mathbb{Z}$ that constitutes a generalized cohomology theory K^* called *topological K-theory.* It is 2-periodic if $F = \mathbb{C}$, i.e., there are natural so-called *Bott isomorphism* $K^n(X) \xrightarrow{\cong} K^{n+2}(X)$ for $n \in \mathbb{Z}$. If $F = \mathbb{R}$, it is 8-periodic. We will give further explanations and generalization of topological K-theory later in Section 10.2

Swan's Theorem 2.27 yields an identification

(2.31) $$K^0(X) \cong K_0(C(X)) \quad [\xi] \mapsto [C^0(\xi)].$$

Exercise 2.32. Let $f\colon X \to Y$ be a map of compact spaces. Composition with f yields a ring homomorphism $C(f)\colon C(Y) \to C(X)$. Show that under the identification (2.31) the maps $K^0(f)\colon K^0(Y) \to K^0(X)$ and $C(f)_*\colon K_0(C(Y)) \to K_0(C(X))$ coincide.

Exercise 2.33. Compute $K_0(C(D^n))$ for the n-dimensional disk D^n for $n \geq 0$.

2.5 Wall's Finiteness Obstruction

We now discuss the geometric relevance of $\widetilde{K}_0(\mathbb{Z}G)$.

Let X be a CW-complex. It is called *finite*, if it consists of finitely many cells. This is equivalent to the condition that X is compact. We call X *finitely dominated*, if there exists a *finite domination* (Y, i, r), i.e., a finite CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. If X is finitely dominated, its set of path components $\pi_0(X)$ is finite and the fundamental group $\pi_1(C)$ of each component C of X is finitely presented, see Lemma 2.42.

While studying existence problems for compact manifolds with prescribed properties (like for example the existence of certain group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex with the desired property within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent finite CW-complex. If the goal is to construct a compact manifold, this is a necessary step in the construction. Wall's finiteness obstruction, which we will explain below, decides this question.

An example of such a geometric problem is the *Spherical Space Form Problem* 9.197, i.e., the classification of closed manifolds M whose universal coverings are diffeomorphic or homeomorphic to the standard sphere. Such examples arise as unit sphere in unitary representations of finite groups, but there are also examples that do not occur in this way. This problem initiated not only the theory of the finiteness obstruction, but also surgery theory for closed manifolds with non-trivial fundamental group. We refer to the survey articles [270] and [679] for more information about the Spherical Space Form problem. It was finally solved by Madsen-Thomas-Wall [686, 687].

The finiteness obstruction also appears in the Ph.D.-thesis [898] of Siebenmann who dealt with the problem whether a given smooth or topological manifold can be realized as the interior of a compact manifold with boundary.

Next we explain the definition and the main properties of the finiteness obstruction illustrating that it is a kind of Euler characteristic, but now counting elements in the projective class group instead of counting ranks of finitely generated free modules.

2.5.1 Chain Complex Version of the Finiteness Obstruction

Definition 2.34 (Types of chain complexes). We call an R-chain complex *finitely generated*, *free*, or *projective* respectively, if each R-chain module is finitely generated, free, or projective respectively. It is called *positive*, if $C_n = 0$ for $n \le -1$. It is called *finite dimensional*, if there exists a natural number N such that $C_n = 0$ for $|n| \le N$. It is called *finite*, if it is finite dimensional and finitely generated.

For the remainder of this section all chain complexes C_* are understood to be positive. Let R be a ring and C_* be an R-chain complex. A *finite domination* (F_*, i_*, p_*) of C_* consists of a finite free R-chain complex F_* and R-chain maps $i_*\colon C_* \to F_*$ and $r_*\colon F_* \to C_*$ such that $r_* \circ i_* \simeq \mathrm{id}_{C_*}$ holds. The existence of a finite domination is equivalent to the existence of a finite projective R-chain complex P_* which is R-chain homotopy equivalence to C_*. For a proof of this claim we refer for instance to [635, Proposition 11.11 on page 222], or to the explicite construction in Subsection 24.7.5. For any such choice of P_*, define the *finiteness obstruction* $o(C_*) \in K_0(R)$ to be

$$o(C_*) := \sum_{n \ge 0} (-1)^n \cdot [P_n]. \tag{2.35}$$

The *reduced finiteness obstruction* $\widetilde{o}(C_*) \in \widetilde{K}_0(R)$ is the image of $o(C_*)$ under the projection $K_0(R) \to \widetilde{K}_0(R)$. The definition is indeed independent of the choice of P_* since for two finite projective R-chain complexes P_* and Q_* coming with an R-chain homotopy equivalence $f_*\colon P_* \xrightarrow{\simeq} Q_*$ the mapping cone $\mathrm{cone}_*(f_*)$, see Definition 3.29, is contractible and hence we obtain an R-isomorphism

$$P_{\mathrm{odd}} \oplus Q_{\mathrm{ev}} \xrightarrow{\cong} P_{\mathrm{ev}} \oplus Q_{\mathrm{odd}}.$$

from the to one another inverse isomorphism (3.30) and (3.31).

Lemma 2.36. *(i) If the two R-chain complexes C_* and D_* are R-chain homotopy equivalent and one of them is finitely dominated, then both are finitely dominated and we get*

$$o(C_*) = o(D_*);$$

(ii) Let $0 \to C_* \to D_* \to E_* \to 0$ be an exact sequence of R-chain complexes. If two of the R-chain complexes C_*, D_*, and E_* are finitely dominated, then all three are finitely dominated and we get

$$o(D_*) = o(C_*) + o(E_*);$$

(iii) Let C_* be a finitely dominated R-chain complex. Then it is R-chain homotopy equivalent to a finite free R-chain complex if and only if $\widetilde{0}(C_*)$ vanishes.

Proof. (i) This follows directly from the definitions.

(ii) One can construct a commutative diagram of R-chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \end{array}$$

such that the rows are exact, the upper row consists of finite projective R-chain complexes, and the vertical maps are R-chain homotopy equivalences, see for instance [629, Lemma 11.6 on page 216].

(iii) Suppose that $\widetilde{o}(C_*) = 0$. Choose a finite projective R-chain complex P_* which is R-chain homotopy equivalent to C_*. An *elementary R-chain complex* E_* over an R-module M is an R-chain complex which is concentrated in two consecutive dimensions and its only non-trivial differential is given by $\mathrm{id}_M \colon M \to M$. By adding elementary R-chain complexes over finitely generated free R-modules, one can arrange that P_* is of the shape $\cdots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_0$ such that P_i is finitely generated free for $i \leq n-1$. Since $\widetilde{o}(C_*) = (-1)^n \cdot [P_n] = 0$ holds in $\widetilde{K}_0(R)$, the R-module P_n is stably free. Hence, by adding one further elementary chain complexes over a finitely generated free R-module, one can arrange that P_* is finite free. $\square$

2.5.2 Space Version of the Finiteness Obstruction

In the sequel we ignore base point questions. This is not a real problem since an inner automorphism of a group G induces the identity on $K_0(RG)$.

Given a finitely dominated connected CW-complex X with fundamental group π, we consider its universal covering $\widetilde{X}$ and the associated cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$. Given a finite domination (Y, i, r), we regard the π-covering $\overline{Y}$ over Y associated to the epimorphism $r_* \colon \pi_1(Y) \to \pi_1(X)$. The pullback construction yields a π-covering $i^*\overline{Y}$ over X. Then $F_* = C_*(i^*\overline{Y})$ is a finite free $\mathbb{Z}\pi$-chain complex. The maps i and r yield $\mathbb{Z}\pi$-chain maps $r_* \colon F_* \to C_*(\widetilde{X})$ and $i_* \colon C_*(\widetilde{X}) \to F_*$ such that $r_* \circ i_*$ is $\mathbb{Z}\pi$-chain homotopic to the identity on $C_*(\widetilde{X})$. Thus (F_*, i_*, r_*) is a finite domination of the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$. We have defined $o(C_*(\widetilde{X})) \in K_0(\mathbb{Z}\pi)$ in (2.35). Now define the *unreduced finiteness obstruction*

$$o(X) := o(C_*(\widetilde{X})) \in K_0(\mathbb{Z}\pi). \tag{2.37}$$

Define the *finiteness obstruction*

$$\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}\pi) \tag{2.38}$$

to be the image of $o(X)$ under the canonical projection $K_0(\mathbb{Z}\pi) \to \widetilde{K}_0(\mathbb{Z}\pi)$. Obviously $\widetilde{o}(X) = 0$, if X is homotopy equivalent to a finite CW-complex Z since in this case we can take $P_* = C_*(\widetilde{Z})$ and $C_*(\widetilde{Z})$ is a finite free $\mathbb{Z}\pi$-chain complex. The next result is due to Wall, see [964] and [965].

Theorem 2.39 (Properties of the Finiteness Obstruction). *Let X be a finitely dominated connected CW-complex.*

(i) The space X is homotopy equivalent to a finite CW-complex if and only if $\widetilde{o}(X)$ vanishes;
(ii) Every element in $K_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $o(X)$ of a finitely dominated connected 3-dimensional CW-complex X with $G = \pi_1(X)$, provided that G is finitely presented.

Theorem 2.39 illustrates why it is important to study the algebraic object $\widetilde{K}_0(\mathbb{Z}\pi)$ when one is dealing with geometric or topological questions. The favorite case is when $\widetilde{K}_0(\mathbb{Z}\pi)$ vanishes because then the finiteness obstruction is obviously zero and one does not have to go to a specific computation.

Exercise 2.40. Let X be a finitely dominated connected CW-complex with fundamental group π. Define a homomorphism of abelian groups

$$\psi\colon K_0(\mathbb{Z}\pi) \to \mathbb{Z}, \quad [P] \;\mapsto\; \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}\pi} P).$$

Show that ψ sends $o(X)$ to the Euler characteristic $\chi(X)$.

Remark 2.41. One can extend the finiteness obstruction also to not necessarily connected CW-complexes. If X is be a (not necessarily connected) finitely dominated CW-complex, we define

$$\begin{aligned} K_0(\mathbb{Z}[\pi_1(X)]) &:= \bigoplus_{C \in \pi_0(X)} K_0(\mathbb{Z}[\pi_1(C)]); \\ \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) &:= \bigoplus_{C \in \pi_0(X)} \widetilde{K}_0(\mathbb{Z}[\pi_1(C)]), \end{aligned}$$

and the unreduced finite obstruction and the finiteness obstruction to be

$$\begin{aligned} o(X) &:= \{o(C) \mid C \in \pi_0(X)\} \in K_0(\mathbb{Z}[\pi_1(X)]); \\ \widetilde{o}(X) &:= \{\widetilde{o}(C) \mid C \in \pi_0(X)\} \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]). \end{aligned}$$

Note that $K_0(\mathbb{Z}[\pi_1(X)])$ and $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ are covariant functors in X in the obvious way.

For more information about the finiteness obstruction we refer for instance to [365, 367, 627, 654, 725, 728, 744, 822, 947, 964, 965].

2.5.3 Outline of the Proof of the Obstruction Property

In this subsection we outline the proof of Theorem 2.39.

The elementary proofs of the next two lemmascan be found in [964, Lemma 1.3] and [629, Lemma 14.8 on page 280].

Lemma 2.42. *Let G be a finitely presented group. Let $i\colon H \to G$ and $r\colon G \to H$ be group homomorphisms with $r \circ i = \mathrm{id}_H$. Then H is finitely presented.*

Lemma 2.43. *Let H be a finitely generated group and G be a finitely presented group. Then the kernel of any epimorphism $H \to G$ is finitely generated.*

The next Lemma 2.44 follows from Lemma 2.42 and Lemma 2.43.

Lemma 2.44. *Let (Y, i, r) be a finite domination of the CW-complex X. Then we can arrange by attaching finitely many cells to Y that the map $\pi_1(r)\colon \pi_1(Y) \to \pi_1(X)$ is bijective and hence r is 2-connected.*

Lemma 2.45. *Let Y be a finitely dominated connected CW-complex whose finiteness obstruction $\widetilde{o}(Y)$ vanishes. Then there are:*

(i) A finite 2-dimensional connected CW-complex Z;
(ii) A 2-connected map $h\colon Z \to Y$;
(iii) A finite free $\mathbb{Z}\pi$-chain complex C_* with $C_*|_2 = C_*(\widetilde{Z})$ and a $\mathbb{Z}\pi$-chain homotopy equivalence $f_*\colon C_* \to C_*(\widetilde{Y})$ with $f_*|_2 = C_*(\widetilde{h})$, where here and in the sequel we identify $\pi = \pi_1(Z) = \pi_1(Y)$ using the isomorphism $\pi_1(h)\colon \pi_1(Z) \xrightarrow{\cong} \pi_1(Y)$.

Proof. By Lemma 2.44 we obtain a finite domination (Y, i, r) such that $r\colon Y \to X$ is 2-connected. Take Z to be the 2-skeleton Y_2 of Y and $h\colon Z \to X$ to be the restriction of r to Z.

Since h is 2-connected, the induced $\mathbb{Z}\pi$-chain map $C_*(\widetilde{h})\colon C_*(\widetilde{Z}) \to C_*(\widetilde{Y})$ is 2-connected and hence $H_n(\mathrm{cone}_*(C_*(\widetilde{h}))) = 0$ for $n \geq 2$. Let P_* be the $\mathbb{Z}\pi$-subchain complex of $\mathrm{cone}_*(C_*(\widetilde{h}))$ given by

$$\ldots \xrightarrow{c_5} \mathrm{cone}_4(C_*(\widetilde{h})) \xrightarrow{c_4} \mathrm{cone}_3(C_*(\widetilde{h})) \xrightarrow{c_3} \ker(c_2) \to 0 \to 0 \to 0$$

where c_* is the differential of $\mathrm{cone}(C_*(\widetilde{h}))$. Because of the exact sequence

$$0 \to \ker(c_2) \to \mathrm{cone}_2(C_*(\widetilde{h})) \xrightarrow{c_2} \mathrm{cone}_1(C_*(\widetilde{h})) \xrightarrow{c_1} \mathrm{cone}_0(C_*(\widetilde{h})) \to 0$$

the $\mathbb{Z}\pi$-chain complex P_* is projective. The inclusion $i_*\colon P_* \to \mathrm{cone}_*(C_*(\widetilde{h}))$ is a homology equivalence of projective $\mathbb{Z}\pi$-chain complexes and hence a $\mathbb{Z}\pi$-chain homotopy equivalence. Put $Q_* = \Sigma^{-3}P_*$. Then Q_* is a positive projective $\mathbb{Z}\pi$-chain complex such that $\Sigma^3 Q_*$ is $\mathbb{Z}\pi$-chain homotopy equivalent to $\mathrm{cone}_*(C_*(\widetilde{h}))$.

The mapping cylinder $\operatorname{cyl}(C_*(\widetilde{h}))$, see Definition 3.29, is $\mathbb{Z}\pi$-chain homotopy equivalent to $C_*(\widetilde{Y})$ and there is an obvious short exact sequence of $\mathbb{Z}\pi$-chain complexes

$$0 \to C_*(\widetilde{Z}) \to \operatorname{cyl}_*(C_*(\widetilde{h})) \to \operatorname{cone}(C_*(\widetilde{h})) \to 0.$$

Since $C_*(\widetilde{Z})$ is finite free and $C_*(\widetilde{Y})$ is finitely dominated, we conclude from Lemma 2.36 (i) and (ii) that Q_* is finitely dominated and that we get in $\widetilde{K}_0(\mathbb{Z}\pi)$

$$\begin{aligned}\widetilde{o}(Q_*) = -\widetilde{o}(P_*) = -\widetilde{o}(\operatorname{cone}_*(C_*(\widetilde{h}))) &= \widetilde{o}(\operatorname{cyl}_*(C_*(\widetilde{h}))) - \widetilde{o}(C_*(\widetilde{Z})) \\ &= \widetilde{o}(C_*(\widetilde{Y})) - \widetilde{o}(C_*(\widetilde{Z})) = 0 - 0 = 0.\end{aligned}$$

Lemma 2.36 (iii) implies that Q_* is $\mathbb{Z}\pi$-chain homotopy equivalent to a finite free positive $\mathbb{Z}\pi$-chain complex F_*. Choose a $\mathbb{Z}\pi$-chain homotopy equivalence $g_*\colon \Sigma^3 F_* \to \operatorname{cone}_*(C_*(\widetilde{h}))$. We get a commutative diagram of $\mathbb{Z}\pi$-chain complexes with exact rows and $\mathbb{Z}\pi$-chain homotopy equivalences as vertical arrows

$$\begin{tikzcd}
0 \arrow[r] & C_*(\widetilde{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & C_* \arrow[r] \arrow[d, "g'_*"] & \Sigma^3 F_* \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & C_*(\widetilde{Z}) \arrow[r] & \operatorname{cyl}_*(C_*(\widetilde{h})) \arrow[r] & \operatorname{cone}_*(C_*(\widetilde{h})) \arrow[r] & 0
\end{tikzcd}$$

by requiring that the right square is a pull back. Now define the desired $\mathbb{Z}\pi$-chain map $f_*\colon C_* \to C_*(\widetilde{Y})$ to be the composite of g'_* with the canonical $\mathbb{Z}\pi$-chain homotopy equivalence $\operatorname{cyl}_*(C_*(\widetilde{h})) \to C_*(\widetilde{Y})$. □

Next we present the main tool to pass from chain complexes to CW-complexes. Its proof can be found in [965, Theorem 2] or in the more general equivariant setting in [629, Theorem 13.19 on page 268].

Theorem 2.46 (Realization Theorem). *Let $h\colon Z \to Y$ be a map between connected CW-complexes such that $\pi_1(h)\colon \pi_1(Z) \to \pi_1(Y)$ is an isomorphism. In the sequel identity $\pi = \pi_1(Y)$ with $\pi_1(Z)$ using $\pi_1(h)$. Put $d = \dim(Z)$ and suppose $d \geq 2$. Assume the existence of a free $\mathbb{Z}\pi$-chain complex C_* with a preferred $\mathbb{Z}\pi$-basis and a $\mathbb{Z}\pi$-chain homotopy equivalence $f_*\colon C_* \to C_*(\widetilde{Y})$ such that the restriction $C_*|_d$ to dimensions $0, 1, \ldots, d$ agrees with $C_*(\widetilde{Z})$ and $f_*|_d = C_*(\widetilde{h})$.*

Then we can construct a CW-complex X such that its d-skeleton X_d agrees with Z and a cellular homotopy equivalence $g\colon X \to Z$ satisfying under the obvious identification $\pi = \pi_1(X) = \pi_1(Y) = \pi_1(Z)$:

(i) We have $g|_Z = h$;

(ii) There is a $\mathbb{Z}\pi$-chain isomorphism $u_*\colon C \xrightarrow{\cong} C_*(\widetilde{X})$ such that the given $\mathbb{Z}\pi$-basis on C_* is mapped bijectively to the cellular $\mathbb{Z}\pi$-basis of $\widetilde{X}$;
(iii) We have $C_*(g) \circ u_* = f_*$.

Remark 2.47. Note that there is no absolute version of the Realization Theorem 2.46 in the sense that, for a d-dimensional CW-complex Z with fundamental group π and dimension $d \geq 2$ and a based free $\mathbb{Z}\pi$-chain complex C_* with $C_*|_d = C_*(\widetilde{Z})$, we can find a CW-complex X with $X_d = Z$ and $C_*(\widetilde{X}) = C_*$. Moreover, the assumption $\dim(Z) \geq 2$ cannot be dropped in the Realization Theorem 2.46.

Lemma 2.48. *Let X be a connected CW-complex. Then it is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the $\mathbb{Z}[\pi_1(X)]$-chain complex $C_*(\widetilde{X})$ is finitely dominated.*

Proof. This follows essentially from Theorem 2.46; details of the proof can be found in [965, Corollary 5.1] or in the more general equivariant setting in [629, Proposition 14.6 (a) on page 282]. □

Next we can give the proof of Theorem 2.39.

Proof of Theorem 2.39. (i) If the finitely dominated connected CW-complex Y is homotopy equivalent to a finite CW-complex, we get $\widetilde{o}(Y) = 0$ directly from the definitions. Now suppose that Y is a finitely dominated connected CW-complex with $\widetilde{o}(Y) = 0$. We conclude from Lemma 2.45 and Theorem 2.45 that Y is homotopy equivalent to a CW-complex X for which its cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$ is finite free. The latter implies that X is finite.

(ii). Since G is finitely presented, we can choose a connected finite 2-dimensional CW-complex Z with $\pi_1(Z) = G$. Consider any element $\xi \in \widetilde{K}_0(\mathbb{Z}\pi)$. Choose a finitely generated projective R-module P and a natural number n such that $\xi = [P] - [\mathbb{Z}\pi^n]$ holds. Choose an exact sequence $0 \to \bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \to P \to 0$. Now consider $X' = X \vee \bigvee_{i_2 \in I} S^2$. For each $i_3 \in I_3$ we attach a 3-cell to X' with an attaching map $q_{i_3}\colon S^2 \to X'$ such that $[q_{i_3}] \in \pi_2(X')$ corresponds to the image of the basis element in $\bigoplus_{I_3} \mathbb{Z}\pi$ associated to i_3 under the composite

$$\bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \xrightarrow{j} \pi_2(X')$$

where j sends the basis element associated to $i_2 \in I_2$ to the element in $\pi_2(X')$ given by the obvious inclusion of $S^2 \to X'$ associated to j_2. Call the resulting 3-dimensional CW-complex Y. Note that we can identify $\pi = \pi_1(Y)$. We obtain an exact sequence of free $\mathbb{Z}\pi$-chain complexes

$$0 \to C_*(\widetilde{X}) \to C_*(\widetilde{Y}) \to C_*(\widetilde{Y}, \widetilde{X}) \to 0.$$

The $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y}, \widetilde{X})$ is concentrated in dimensions 2 and 3 and its third differential is u. This implies that $C_*(\widetilde{Y}, \widetilde{X})$ is $\mathbb{Z}\pi$-chain homotopy equivalent to the $\mathbb{Z}\pi$-chain complex concentrated in dimension 2 with P as second $\mathbb{Z}\pi$-chain module. Hence $C_*(\widetilde{Y}, \widetilde{X})$ is finitely dominated and $o(C_*(\widetilde{Y}, \widetilde{X})) = [P]$ by Lemma 2.36 (i). Lemma 2.36 (ii) implies that $C_*(\widetilde{Y})$ is finitely dominated. Then Y is finitely dominated as a CW-complex by Lemma 2.48. Lemma 2.36 (ii) implies that we get for some integer m

$$o(C_*(\widetilde{Y})) = o(C_*(\widetilde{Z})) + o(C_*(\widetilde{Y}, \widetilde{X})) = m \cdot [\mathbb{Z}\pi] + [P].$$

By attaching to Y finitely many trivial 2 and 3-cells, we can arrange that Y is a finitely dominated connected CW-complex with $\pi_1(Y) = G$ and $o(Y) = [P] - [\mathbb{Z}\pi^n] = \xi$. □

Exercise 2.49. Let

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

be a cellular pushout, i.e., the diagram is a pushout, the map i_1 is an inclusion of CW-complexes, the map i_2 is cellular and X carries the induced CW-structure. Suppose that X_0, X_1, X_2 are finitely dominated.

Then X is finitely dominated and we get in $K_0(\mathbb{Z}[\pi_1(X)])$

$$o(X) = (j_1)_*(o(X_1)) + (j_2)_*(o(X_2)) - (j_0)_*(o(X_1)).$$

2.6 Geometric Interpretation of Projective Class Group and Finiteness Obstruction

Next we give a geometric construction of $\widetilde{K}_0(\mathbb{Z}\pi)$ that is in the spirit of the well-known interpretation of the Whitehead group in terms of deformation retractions which we will present later in Section 3.4. The material of this section is taken from [627] where more information and details of the proofs can be found.

Given a space Y, we want to define an abelian group $\mathrm{Wa}(Y)$. The underlying set is the set of equivalence classes of an equivalence relation $\sim$ defined on the set of maps $f\colon X \to Y$ with finitely dominated CW-complexes as source and the given space Y as target. We call $f_0\colon X_0 \to Y$ and $f_4\colon X_4 \to Y$ equivalent if there exists a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "i_0"] \arrow[drr, "f_0"'] & X_1 \arrow[r, "j_1"] \arrow[dr, "f_1"] & X_2 \arrow[d, "f_2"] & X_3 \arrow[l, "j_3"'] \arrow[dl, "f_3"'] & X_4 \arrow[l, "i_4"'] \arrow[dll, "f_4"] \\
& & Y & &
\end{tikzcd}$$

such that j_1 and j_3 are homotopy equivalences and i_0 and i_4 are inclusions of CW-complexes with the property that the larger one is obtained from the smaller one by attaching finitely many cells. Obviously this relation is symmetric and reflexive. It needs some work to show transitivity and hence that it is an equivalence relation. The addition in $\mathrm{Wa}(Y)$ is given by the disjoint sum, i.e., define the sum of the class of $f_0\colon X_0 \to Y$ and $f_1\colon X_1 \to Y$ to be the class of $f_0 \coprod f_1\colon X_0 \coprod X_1 \to Y$. It is easy to check that this is compatible with the equivalence relation. The neutral element is represented by $\emptyset \to Y$. The inverse of the class $[f]$ of $f\colon X \to Y$ is constructed as follows. Choose a finite domination (Z, i, r) of X. Construct a map $F\colon \mathrm{cyl}(i) \to X$ from the mapping cylinder of i to Y such that $F|_X = \mathrm{id}_X$ and $F|_Z = r$. Then an inverse of $[f]$ is given by the class $[f']$ of the composite

$$f'\colon \mathrm{cyl}(i) \cup_X \mathrm{cyl}(i) \xrightarrow{F \cup_{\mathrm{id}_X} F} X \xrightarrow{f} Y.$$

This finishes the definition of the abelian group $\mathrm{Wa}(Y)$. A map $f\colon Y_0 \to Y_1$ induces a homomorphism of abelian groups $\mathrm{Wa}(f)\colon \mathrm{Wa}(Y_0) \to \mathrm{Wa}(Y_1)$ by composition. Thus Wa defines a functor from the category of spaces to the category of abelian groups.

Exercise 2.50. Show that $[f] + [f'] = 0$ holds for the composite f' above.

Given a finitely dominated CW-complex X, define its *geometric finiteness obstruction* $o_{\mathrm{geo}}(X) \in \mathrm{Wa}(X)$ by the class of id_X.

Theorem 2.51 (The geometric finiteness obstruction). *Let X be a finitely dominated CW-complex. Then X is homotopy equivalent to a finite CW-complex if and only if $o_{\mathrm{geo}}(X) = 0$ in $\mathrm{Wa}(X)$.*

Proof. Obviously $o_{\mathrm{geo}}(X) = 0$ if X is homotopy equivalent to a finite CW-complex. Suppose $o_{\mathrm{geo}}(X) = 0$. Hence there are a CW-complex Y, a map $r\colon Y \to X$ and a homotopy equivalence $h\colon Y \to Z$ to a finite CW-complex Z such that Y is obtained from X by attaching finitely many cells and $r \circ i = \mathrm{id}_X$ holds for the inclusion $i\colon X \to Y$. The mapping cylinder $\mathrm{cyl}(r)$ is built from the mapping cylinder $\mathrm{cyl}(i)$ by attaching a finite number of cells and is homotopy equivalent to X. Choose a homotopy equivalence $g\colon \mathrm{cyl}(i) \to Z$. Consider the push-out

$$
\begin{tikzcd}
\operatorname{cyl}(i) \arrow[r, "i"] \arrow[d, "g"'] & \operatorname{cyl}(r) \arrow[d, "g'"] \\
Z \arrow[r, "i'"'] & Z'
\end{tikzcd}
$$

where i is the inclusion. Since g is a homotopy equivalence, the same is true for g'. Hence X is homotopy equivalent to the finite CW-complex Z'. □

Theorem 2.52 (Identifying the finiteness obstruction with its geometric counterpart). *Let Y be a space. Then there is a natural isomorphism of abelian groups*

$$\Phi\colon \operatorname{Wa}(Y) \xrightarrow{\cong} \bigoplus_{C \in \pi_0(Y)} \widetilde{K_0}(\mathbb{Z}\pi_1(C)).$$

Proof. We only explain the definition of Φ. Consider an element $[f] \in \operatorname{Wa}(Y)$ represented by a map $f\colon X \to Y$ from a finitely dominated CW-complex X to Y. Given a path component C of X, let C_f be the path component of Y containing $f(C)$. The map f induces a map $f|_C\colon C \to C_f$ and hence a map $(f|_C)_*\colon \widetilde{K}_0(\mathbb{Z}\pi_1(C)) \to \widetilde{K}_0(\mathbb{Z}\pi_1(C_f))$. Since X is finitely dominated, every path component C of X is finitely dominated, and we can consider its finiteness obstruction $\widetilde{o}(C) \in \widetilde{K}_0(\mathbb{Z}\pi_1(C))$. Let $\phi([f])_C$ be the image of $\widetilde{o}(C)$ under the composite

$$\widetilde{K}_0(\mathbb{Z}\pi_1(C)) \xrightarrow{(f|_C)_*} \widetilde{K}_0(\mathbb{Z}\pi_1(C_f)) \to \bigoplus_{C \in \pi_0(Y)} \widetilde{K_0}(\mathbb{Z}\pi_1(C)).$$

Since $\pi_0(X)$ is finite, we can define

$$\phi([f]) := \sum_{C \in \pi_0(X)} \phi([f])_C.$$

We omit the proof that this is compatible with the equivalence relation appearing in the definition of $\operatorname{Wa}(Y)$, that ϕ is a homomorphism of abelian groups and that Theorem 2.39 implies that Φ is bijective. □

2.7 Universal Functorial Additive Invariants

In this section we describe the pair $(K_0(\mathbb{Z}\pi_1(X)), o(X))$ by an abstract property.

Definition 2.53 (Functorial additive invariant for finitely dominated CW-complexes). A *functorial additive invariant for finitely dominated CW-complexes* consists of a covariant functor A from the category of finitely dominated CW-complexes to the category of abelian groups together with an assignment a that associates to every finitely dominated CW-complex X an element $a(X) \in A(X)$ such that the following axioms are satisfied:

- Homotopy invariance of A
  If $f, g\colon X \to Y$ are homotopic maps between finitely dominated CW-complexes, then $A(f) = A(g)$;
- Homotopy invariance of $a(X)$
  If $f\colon X \to Y$ is a homotopy equivalence of finitely dominated CW-complexes, then $A(f)(a(X)) = a(Y)$;
- Additivity
  Let

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

  be a *cellular pushout,* i.e., the diagram is a pushout, the map i_1 is an inclusion of CW-complexes, the map i_2 is cellular and X carries the induced CW-structure. Suppose that X_0, X_1, X_2 are finitely dominated.
  Then X is finitely dominated and

$$a(X) \;=\; A(j_1)(a(X_1)) + A(j_2)(a(X_2)) - A(j_0)(a(X_0));$$

- Normalization
  $a(\emptyset) \;=\; 0.$

Example 2.54 (Componentwise Euler characteristic). Let A be the covariant functor sending a finitely dominated CW-complex X to $H_0(X;\mathbb{Z}) = \bigoplus_{C \in \pi_0(X)} \mathbb{Z}$. Let $a(X) \in A(X)$ be the componentwise Euler characteristic, i.e., the collection of integers $\{\chi(C) \mid C \in \pi_0(X)\}$. Then (A, a) is a functorial additive invariant for finitely dominated CW-complexes.

Definition 2.55 (Universal functorial additive invariant for finitely dominated CW-complexes). A *universal functorial additive invariant for finitely dominated CW-complexes* (U, u) is a functorial additive invariant with the property that for any functorial additive invariant (A, a) there is precisely one natural transformation $T\colon U \to A$ with the property that $T(X)(u(X)) \;=\; a(X)$ holds for every finitely dominated CW-complex X.

Exercise 2.56. Show that the functorial additive invariant defined in Example 2.54 is the universal one if we restrict to finite CW-complexes.

Obviously the universal additive functorial invariant is unique (up to unique natural equivalence) if it exists. It is also easy to construct it. However, it turns out that there exists a concrete model, namely, the following theorem is proved in [627, Theorem 4.1].

Theorem 2.57 (The finiteness obstruction is the universal functorial additive invariant). *The covariant functor $X \mapsto \bigoplus_{C\in\pi_0(X)} K_0(\mathbb{Z}\pi_1(C))$ together with the componentwise finiteness obstruction $\{o(C) \mid C \in \pi_0(X)\}$ is the universal functorial additive invariant for finitely dominated CW-complexes.*

Exercise 2.58. (i) Construct for finitely dominated CW-complexes X and Y a natural pairing

$$P(X,Y)\colon U(X) \otimes_{\mathbb{Z}} U(Y) \to U(X \times Y)$$

sending $u(X) \otimes u(Y)$ to $u(X \times Y)$ where (U,u) is the universal functorial additive invariant for finitely dominated CW-complexes;

(ii) Let X be a finitely dominated CW-complex. Let Y be a finite CW-complex such that $\chi(C) = 0$ for every component C of Y. Show that $X \times Y$ is homotopy equivalent to a finite CW-complex.

2.8 Variants of the Farrell-Jones Conjecture for $K_0(RG)$

In this section we state variants of the Farrell-Jones Conjecture for $K_0(RG)$, where RG, sometimes also written as $R[G]$, is the *group ring* of a group G with coefficients in an associative ring R with unit. Elements in RG are given by formal finite sums $\sum_{g\in G} r_g \cdot g$, and addition and multiplication is given by

$$\left(\sum_{g\in G} r_g \cdot g\right) + \left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g\in G} (r_g + s_g) \cdot g;$$
$$\left(\sum_{g\in G} r_g \cdot g\right) \cdot \left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g\in G} \left(\sum_{\substack{h,k\in G,\\ g=hk}} r_h \cdot s_k\right) \cdot g.$$

The Farrell-Jones Conjecture itself will give a complete answer for arbitrary groups and rings, but to formulate the full version some additional effort will be needed. If one assumes that R is regular and G is torsionfree or that R is regular and $\mathbb{Q} \subseteq R$, then the conjecture reduces to easy to formulate statements, which we will present next. Moreover, these special cases are already very interesting.

Definition 2.59 (Projective resolution). Let M be an R-module. A *projective resolution* (P_*, ϕ_*) of M is a positive projective R-chain complex P_* with $H_n(P_*) = 0$ for $n \geq 1$ together with an R-isomorphism $\phi\colon H_0(P_*) \xrightarrow{\cong} M$. It is called *finite*, *finitely generated*, *free*, *finite dimensional*, or *d-dimensional* if the R-chain complex P_* has this property.

A ring R is *Noetherian* if any submodule of a finitely generated R-module is again finitely generated. A ring R is called *regular* if it is Noetherian and any finitely generated R-module has a finite dimensional projective resolution. Any principal ideal domain such as $\mathbb{Z}$, any field, and, more generally, any Dedekind domain is regular, see Theorem 2.21.

Conjecture 2.60 (Farrell-Jones Conjecture for $K_0(R)$ for torsionfree G and regular R). Let G be a torsionfree group and let R be a regular ring. Then the map induced by the inclusion of the trivial group into G

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective.

In particular we get for any principal ideal domain R and torsionfree G

$$\widetilde{K}_0(RG) = 0.$$

Remark 2.61 (Relevance of Conjecture 2.60). In view of Remark 2.8 Conjecture 2.60 is equivalent to the statement that for a torsionfree group G and a regular ring R every finitely generated projective RG-module is stably finitely generated free. This is the algebraic relevance of this conjecture. Its geometric meaning comes from the following conclusion of Theorem 2.39. Namely, if $R = \mathbb{Z}$ and G is a finitely presented torsionfree group, it is equivalent to the statement that every finitely dominated CW-complex with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW-complex.

Definition 2.62 (Family of subgroups). A *family $\mathcal{F}$ of subgroups* of a group G is a set of subgroups that is closed under conjugation with elements of G and under passing to subgroups.

Our main examples of families are listed below

Notation 2.63.

notation	subgroups
$\mathcal{TR}$	trivial group
$\mathcal{FCY}$	finite cyclic subgroups
$\mathcal{FIN}$	finite subgroups
$\mathcal{CYC}$	cyclic subgroups
$\mathcal{VCY}$	virtually cyclic subgroups
$\mathcal{ALL}$	all subgroups

Definition 2.64 (Orbit category). The *orbit category* $\mathsf{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G-maps. Given a family $\mathcal{F}$ of subgroups of G, let the *$\mathcal{F}$-restricted orbit category* $\mathsf{Or}_{\mathcal{F}}(G)$ be the full subcategory of $\mathsf{Or}(G)$ whose objects are homogeneous spaces G/H with $H \in \mathcal{F}$.

Definition 2.65 (Subgroup category). The *subgroup category* $\mathsf{Sub}(G)$ has as objects subgroups H of G. For $H, K \subseteq G$, let $\operatorname{conhom}_G(H, K)$ be the set of all group homomorphisms $f\colon H \to K$ for which there exists a group element $g \in G$ such that f is given by conjugation with g. The group of inner automorphisms $\operatorname{inn}(K)$ consists of those automorphisms $K \to K$ that are given by conjugation with an element $k \in K$. It acts on $\operatorname{conhom}(H,K)$ from the left by composition. Define the set of morphisms in $\mathsf{Sub}(G)$ from H to K to be $\operatorname{inn}(K)\backslash \operatorname{conhom}(H,K)$. Composition of group homomorphisms defines the composition of morphisms in $\mathsf{Sub}(G)$.

Given a family $\mathcal{F}$, define the *$\mathcal{F}$-restricted category of subgroups* $\mathsf{Sub}_{\mathcal{F}}(G)$ to be the full subcategory of $\mathsf{Sub}(G)$ that is given by objects H belonging to $\mathcal{F}$.

Exercise 2.66. Show that $\mathsf{Sub}_{\mathcal{F}}(G)$ is a quotient category of $\mathsf{Or}_{\mathcal{F}}(G)$.

Note that there is a morphism from H to K only if H is conjugated to a subgroup of K. Clearly $K_0(R(-))$ yields a functor from $\mathsf{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group K induce the identity on $K_0(RK)$. Using the inclusions into G, one obtains a map

$$\operatorname{colim}_{H\in \mathsf{Sub}_{\mathcal{F}}(G)} K_0(RH) \to K_0(RG).$$

We briefly recall the notion of a colimit of a covariant functor $F\colon \mathcal{C} \to \mathbb{Z}\text{-}\mathsf{MOD}$ from a small category $\mathcal{C}$ into the category of abelian groups, where *small* means that the objects of $\mathcal{C}$ form a set. Given an abelian group A, let C_A be the constant functor $\mathcal{C} \to \mathbb{Z}\text{-}\mathsf{MOD}$ that sends every object in $\mathcal{C}$ to A and every morphism in $\mathcal{C}$ to id_A. Given a homomorphism $f\colon A \to B$ of abelian groups, let $C_f\colon C_A \to C_B$ be the obvious transformation. The *colimit*, or sometimes also called *direct limit*, of F consists of an abelian group $\operatorname{colim}_{\mathcal{C}} F$ together with a transformation $T_F\colon F \to C_{\operatorname{colim}_{\mathcal{C}} F}$ such that for any abelian group B and transformation $T\colon F \to C_B$ there exists precisely one homomorphism of abelian groups $\phi\colon \operatorname{colim}_{\mathcal{C}} F \to B$ satisfying $C_\phi \circ T_F = T$. The colimit is unique (up to unique isomorphism) and always exists. If we replace abelian group by ring or by R-module respectively, we get the notion of a *colimit* or sometimes also called *direct limit* of functors from a small category to rings or R-modules respectively.

Conjecture 2.67 (Farrell-Jones Conjecture for $K_0(RG)$ for regular R with $\mathbb{Q} \subseteq R$). Let R be a regular ring with $\mathbb{Q} \subseteq R$ and let G be a group.

Then the homomorphism

$$I_{\mathcal{FIN}}(G,F)\colon \operatorname{colim}_{H\in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG) \tag{2.68}$$

coming from the various inclusions of finite subgroups of G into G is a bijection.

One can also ask for the following stronger version of Conjecture 2.67 which also encompasses Conjecture 2.60.

Conjecture 2.69 (Farrell-Jones Conjecture for $K_0(RG)$ for regular R). Let R be a regular ring and let G be a group. Let $\mathcal{P}(G,R)$ be the set of primes which are not invertible in R and for which G contains an element of order p.

Then the homomorphism

$$I_{\mathcal{FIN}}(G,F)\colon \operatorname{colim}_{H\in\mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH)\to K_0(RG)$$

coming from the various inclusions of finite subgroups of G into G is an $\mathcal{P}(G,R)$-isomorphism, i.e., an isomorphism after inverting all prime in $\mathcal{P}(G,R)$.

We mention that the surjectivity of the map $I_{\mathcal{FIN}}(G,F)$ is equivalent to the surjectivity of the map induced by the various inclusions of subgroups $H\in\mathcal{FIN}$ into G

$$\bigoplus_{H\in\mathcal{FIN}} K_0(RH) \;\to\; K_0(RG),$$

because this map factorizes as

$$\bigoplus_{H\in\mathcal{FIN}} K_0(RH) \xrightarrow{\psi} \operatorname{colim}_{H\in\mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \xrightarrow{I_{\mathcal{FIN}}(G,F)} K_0(RG)$$

where the first map ψ is surjective.

Remark 2.70 (Module-theoretic relevance of Conjecture 2.67). Conjecture 2.67 implies that for a regular ring R with $\mathbb{Q}\subseteq R$ every finitely generated projective R-module is up to adding finitely generated free RG-modules a direct sum of finitely many RG-modules of the shape $RG\otimes_{RH} P$ for a finite subgroup $H\subseteq G$ and a finitely generated projective RH-module P. So it predicts the (stable) structure of finitely generated projective RG-modules in the most elementary way. We mention, however, that the situation is much more complicated in the case where we drop the assumption that R is regular and $\mathbb{Q}\subseteq R$. In particular for $R=\mathbb{Z}$ new phenomena will occur as explained later that are related to so-called negative K-groups and Nil-groups. For instance, the obvious inclusion $\mathbb{Z}/6\to\mathbb{Z}\times\mathbb{Z}/6$ does not induce a surjection $K_0(\mathbb{Z}[\mathbb{Z}/6])\to K_0(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/6])$ since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6])=0$ and $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/6])\cong\mathbb{Z}$, whereas by $K_0(\mathbb{Q}[\mathbb{Z}/6])\to K_0(\mathbb{Q}[\mathbb{Z}\times\mathbb{Z}/6])$ is known to be bijective as predicted by Conjecture 2.67.

Remark 2.71 (Conjecture 2.67 and the Atiyah Conjecture). Conjecture 2.67 plays a role in a program aiming at a proof of the Atiyah Conjecture about L^2-Betti numbers as explained in [635, Section 10.2]. Atiyah defined the n-th *L^2-Betti number* of the universal covering $\widetilde{M}$ of a closed Riemannian manifold M to be the non-negative real number

$$b_n^{(2)}(\widetilde{M}) := \lim_{t\to\infty}\int_{\mathcal{F}} \operatorname{tr}\bigl(e^{-t\Delta_n(\widetilde{x},\widetilde{x})}\bigr)\,d\widetilde{x}$$

where $\mathcal{F}$ is a fundamental domain for the $\pi_1(M)$-action and $e^{-t\Delta_n(\widetilde{x},\widetilde{x})}$ denotes the heat kernel on $\widetilde{M}$. The version of the *Atiyah Conjecture*, which we are interested in and which is at the time of writing open, says that $d \cdot b_n^{(2)}(\widetilde{M})$ is an integer if d is an integer such that the order of any finite subgroup of $\pi_1(M)$ divides d. In particular $b_n^{(2)}(\widetilde{M})$ is expected to be an integer if $\pi_1(M)$ is torsionfree. This gives an interesting connection between the analysis of heat kernels and the projective class group of complex group rings $\mathbb{C}G$.

If one drops the condition that there exists a bound on the order of finite subgroups of $\pi_1(M)$, then also transcendental real numbers can occur as L^2-Betti number of the universal covering $\widetilde{M}$ of a closed Riemannian manifold M, see [58, 416, 793].

An R-module M is called *Artinian* if for any descending series of submodules $M_1 \supseteq M_2 \supseteq \ldots$ there exists an integer k such that $M_k = M_{k+1} = M_{k+2} = \ldots$ holds. An R-module M is called *simple* or *irreducible* if $M \neq \{0\}$ and M contains only $\{0\}$ and M as submodules. A ring R is called *Artinian* if both R considered as left R-module is Artinian and R considered as right R-module is Artinian, or, equivalently, every finitely generated left R-module and every finitely generated right R-module is Artinian. Skew-fields and finite rings are Artinian, whereas $\mathbb{Z}$ is not Artinian.

Conjecture 2.72 (Farrell-Jones Conjecture for $K_0(RG)$ for an Artinian ring R). Let G be a group and R be an Artinian ring.

Then the canonical map

$$I_{\mathcal{FIN}}(G,R)\colon \operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

is an isomorphism

2.9 Kaplansky's Idempotent Conjecture

In this section we discuss

Conjecture 2.73 (Kaplansky's Idempotent Conjecture). Let R be an integral domain and let G be a torsionfree group. Then all idempotents of RG are trivial, i.e., equal to 0 or 1.

Remark 2.74 (Kaplansky's Idempotent Conjecture for prime characteristic). There is a reasonable more general version of Conjecture 2.73 where one replaces the condition that G is torsionfree by the weaker condition that any prime p, which divides the order of some finite subgroup $H \subseteq G$ is not invertible in the integral domain R. If R is a skew-field of prime characteristic p, then this condition reduces to the condition that any finite subgroup H of G is a p-group.

The version of Kaplansky's Idempotent Conjecture 2.73 described in Remark 2.74 is consistent with the observation that the only known idempotents in a group ring RG come from idempotents in R or by the following construction.

Example 2.75 (Construction of idempotents). Let G be a group and $g \in G$ be an element of finite order. Suppose that the order $|g|$ is invertible in R. Define an element $x := |g|^{-1} \cdot \sum_{i=1}^{|g|} g^i$. Then $x^2 = x$, i.e., x is an idempotent in RG.

Exercise 2.76. Show that the version of Kaplansky's Idempotent Conjecture of Remark 2.74 holds for $G = \mathbb{Z}/2$.

Exercise 2.77. Consider the ring $R = \mathbb{Z}[x]/(2x^2 - 3x + 1)$. In the sequel denote by $\overline{u}$ the class of $u \in \mathbb{Z}[x]$ in R. Show:

(i) 2 is not invertible in R;
(ii) There are precisely two non-trivial idempotents in R, namely $\overline{2 - 2x}$ and $\overline{-1 + 2x}$;
(iii) The element $\overline{x} + (1 - \overline{x}) \cdot t$ is a non-trivial idempotent in $R[\mathbb{Z}/2]$.

Remark 2.78 (Sofic groups). In the next theorem we will use the notion of a *sofic group* that was introduced by Gromov and originally called *subamenable group*. Every residually amenable group is sofic but the converse is not true. The class of sofic groups is closed under taking subgroups, direct products, amalgamated free products, colimits and inverse limits, and, if H is a sofic normal subgroup of G with amenable quotient G/H, then G is sofic. To the authors' knowledge there is no example of a group that is not sofic. There is a note by Dave Witte Morris [735] following Deligne [286] where a central extension $1 \to \mathbb{Z} \to G \to SP(2n, \mathbb{R}) \to 1$ is constructed such that G is not residually finite. The group G is viewed as a candidate for a group which is not sofic. It is unknown but likely to be true that all hyperbolic groups are sofic. For more information about the notion of a sofic group we refer to [318].

Definition 2.79 (Directly finite). An R-module M is called *directly finite* if every R-module N satisfying $M \cong_R M \oplus N$ is trivial. A ring R is called *directly finite* (or *von Neumann finite*) if it is directly finite as a module over itself, or, equivalently, if $r, s \in R$ satisfy $rs = 1$, then $sr = 1$. A ring is called *stably finite* if the matrix algebra $\mathrm{M}_n(R)$ is directly finite for all $n \geq 1$.

Remark 2.80 (Stable finiteness). Stable finiteness for a ring R is equivalent to the following statement. Every finitely generated projective R-module P whose class in $K_0(R)$ is zero is already the trivial module, i.e., $0 = [P] \in K_0(R)$ implies $P \cong 0$.

If F is a field of characteristic zero, then FG is stably finite for every group G. This is proved by Kaplansky [527], see also Passman [775, Corollary 1.9 on page 38]. If R is a skew-field and G is a sofic group, then RG is stably finite. This is proved for free-by-amenable groups by Ara-Meara-Perera [35] and extended to sofic groups by Elek-Szabo [317, Corollary 4.7]. These results have been extended to extensions with a finitely generated residually finite groups as kernel and a sofic finitely generated group as quotient by Berlai [121].

The next theorem is taken from [87, Theorem 1.12].

Theorem 2.81 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture). *Let G be a group. Let R be a ring whose idempotents are all trivial. Suppose that*

$$K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Then 0 and 1 are the only idempotents in RG if one of the following conditions is satisfied:

(i) RG is stably finite;
(ii) R is a field of characteristic zero;
(iii) R is a skew-field and G is sofic.

Remark 2.82 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture). Theorem 2.81 implies that for a skew-field D of characteristic zero and a torsionfree group G Kaplansky's Idempotent Conjecture 2.73 is true for DG, provided that Conjecture 2.60 holds and that D is commutative or G is sofic.

Remark 2.83 (The Farrell-Jones Conjecture and the Kaplansky's Idempotent Conjecture for prime characteristic). Suppose that D is a skew-field of prime characteristic p, that Conjecture 2.72 holds for G and D, and that all finite subgroups of G are p-groups. Then $K_0(D) \xrightarrow{\cong} K_0(DG)$ is an isomorphism since for a finite p-group H the group ring DH is a local ring, see [257, Theorem 5.24 on page 114], and hence $\widetilde{K}_0(DH) = 0$ by Lemma 2.122. If we furthermore assume that G is sofic, then Theorem 2.81 implies that all idempotents in DG are trivial.

Remark 2.84 (Reducing the case of a field of characteristic zero to $\mathbb{C}$). Let F be a field of characteristic zero and let $u = \sum_{g \in G} x_g \cdot g \in FG$ be an element. Let K be the finitely generated field extension of $\mathbb{Q}$ given by $K = \mathbb{Q}(x_g \mid g \in G) \subset F$. Then u is already an element in KG. The field K embeds into $\mathbb{C}$ since K is finitely generated, it is a finite algebraic extension of a transcendental extension K' of $\mathbb{Q}$, see [602, Theorem 1.1 on p. 356], and K' has finite transcendence degree over $\mathbb{Q}$. Since the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is infinite, there exists an embedding $K' \hookrightarrow \mathbb{C}$ induced by an injection of a transcendence basis of K over $\mathbb{Q}$ into a transcendence basis of $\mathbb{C}$ over $\mathbb{Q}$. It extends to an embedding $K \hookrightarrow \mathbb{C}$ because $\mathbb{C}$ is algebraically closed. Hence u can be viewed as an element in $\mathbb{C}G$. This reduces the case of fields F of characteristic zero to the case $F = \mathbb{C}$.

Next we mention some further results.

Formanek [383, Theorem 9], see also [176, Proposition 4.2], has shown that all idempotents of FG are trivial, provided that F is a field of characteristic zero and there are infinitely many primes p for which there do not exist an element $g \in G, g \neq 1$ and an integer $k \geq 1$ such that g and g^{p^k} are conjugate. Torsionfree hyperbolic groups satisfy these conditions. Hence Formanek's results imply that all idempotents in FG are trivial if G is torsionfree hyperbolic and F is a field of characteristic zero.

Delzant [287] has proved the Kaplansky's Idempotent Conjecture 2.73 for all integral domains R for a torsionfree hyperbolic group G, provided that G admits an appropriate action with large enough injectivity radius. Delzant actually deals with zero-divisors and units as well.

2.10 The Bass Conjectures

2.10.1 The Bass Conjecture for Fields of Characteristic Zero as Coefficients

Let G be a group. Let $\operatorname{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\operatorname{con}(G)_f$ the subset of $\operatorname{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let R be a commutative ring. Let $\operatorname{class}(G, R)$ and $\operatorname{class}(G, R)_f$ be the free R-module with the set $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ as basis. This is the same as the R-module of R-valued functions on $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ with finite support. Define the *universal R-trace*

$$\operatorname{tr}^u_{RG} \colon RG \to \operatorname{class}(G, R), \quad \sum_{g \in G} r_g \cdot g \ \mapsto \ \sum_{g \in G} r_g \cdot (g). \tag{2.85}$$

It extends to a function $\operatorname{tr}^u_{RG} \colon \mathrm{M}_n(RG) \to \operatorname{class}(G, R)$ on (n, n)-matrices over RG by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective RG-module. Choose a matrix $A \in \mathrm{M}_n(RG)$ such that $A^2 = A$ and the image of the RG-map $r_A \colon RG^n \to RG^n$ given by right multiplication with A is RG-isomorphic to P. Define the *Hattori-Stallings rank* of P to be

$$\operatorname{HS}_{RG}(P) = \operatorname{tr}^u_{RG}(A) \in \operatorname{class}(G, R). \tag{2.86}$$

The Hattori-Stallings rank depends only on the isomorphism class of the RG-module P. It induces an R-homomorphism, the *Hattori-Stallings homomorphism*, (2.87) $\quad \mathrm{HS}_{RG} \colon K_0(RG) \otimes_{\mathbb{Z}} R \to \mathrm{class}(G,R), \quad [P] \otimes r \mapsto r \cdot \mathrm{HS}_{RG}(P).$

Let F be a field of characteristic zero. Fix an integer $m \geq 1$. Let $F(\zeta_m) \supset F$ be the Galois extension given by adjoining the primitive m-th root of unity ζ_m to F. Denote by $\Gamma(m,F)$ the Galois group of this extension of fields, i.e., the group of automorphisms $\sigma \colon F(\zeta_m) \to F(\zeta_m)$ that induce the identity on F. It can be identified with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds. Let g_1 and g_2 be two elements of G of finite order. We call them *F-conjugated* if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$ there exists an element σ in the Galois group $\Gamma(m,F)$ with the property that $g_1^{u(\sigma)}$ and g_2 are conjugated. Two elements g_1 and g_2 are F-conjugated for $F = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, respectively, if the cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are conjugated, if g_1 and g_2 or g_1 and g_2^{-1} are conjugated, or if g_1 and g_2 are conjugated, respectively.

Denote by $\mathrm{con}_F(G)_f$ the set of F-conjugacy classes $(g)_F$ of elements $g \in G$ of finite order. Let $\mathrm{class}_F(G)_f$ be the F-vector space with the set $\mathrm{con}_F(G)_f$ as basis, or, equivalently, the F-vector space of functions $\mathrm{con}_F(G)_f \to F$ with finite support. There are obvious inclusions of F-modules

$$\mathrm{class}_F(G)_f \subseteq \mathrm{class}(G,F)_f \subseteq \mathrm{class}(G,F).$$

Lemma 2.88. *Suppose that F is a field of characteristic zero and H is a finite group. Then the Hattori-Stallings homomorphism, see* (2.87)*, induces an isomorphism*

$$\mathrm{HS}_{FH} \colon K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{\cong} \mathrm{class}_F(H)_f.$$

Proof. Since H is finite, an FH-module is a finitely generated projective FH-module if and only if it is a (finite dimensional) H-representation with coefficients in F and $K_0(FH)$ is the same as the representation ring $R_F(H)$. The Hattori-Stallings rank $\mathrm{HS}_{FH}(V)$ and the character χ_V of a G-representation V with coefficients in F are related by the formula

(2.89) $$\chi_V(h^{-1}) = |C_G\langle h \rangle| \cdot \mathrm{HS}_{FH}(V)(h)$$

for $h \in H$ where $C_G\langle h \rangle$ is the centralizer of h in G. Hence Lemma 2.88 follows from representation theory, see for instance [892, Corollary 1 in Chapter 12 on page 96]. ∎

Exercise 2.90. Prove formula (2.89).

The following conjecture is the obvious generalization of Lemma 2.88 to infinite groups.

Conjecture 2.91 (Bass Conjecture for fields of characteristic zero as coefficients). Let F be a field of characteristic zero and let G be a group. The Hattori-Stallings homomorphism of (2.87) induces an isomorphism

$$\mathrm{HS}_{FG}\colon K_0(FG)\otimes_{\mathbb{Z}} F \to \mathrm{class}_F(G)_f.$$

Lemma 2.92. *Suppose that F is a field of characteristic zero and G is a group. Then the composite*

$$\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(FH)\otimes_{\mathbb{Z}} F \xrightarrow{I_{\mathcal{FIN}}(G,F)\otimes_{\mathbb{Z}}\mathrm{id}_F} K_0(FG)\otimes_{\mathbb{Z}} F \xrightarrow{\mathrm{HS}_{FG}} \mathrm{class}(G,F) \tag{2.93}$$

is injective and has as image $\mathrm{class}_F(G)_f$ *where* $I_{\mathcal{FIN}}(G,F)$ *is the map defined in* (2.68).

Proof. This follows from the following commutative diagram, compare [631, Lemma 2.15 on page 220].

$$\begin{tikzcd}
\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(FH)\otimes_{\mathbb{Z}} F \arrow[rr, "I_{\mathcal{FIN}}(G,F)\otimes_{\mathbb{Z}}\mathrm{id}_F"] \arrow[d, "\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} \mathrm{HS}_{FH}"', "\cong"] & & K_0(FG)\otimes_{\mathbb{Z}} F \arrow[d, "\mathrm{HS}_{FG}"] \\
\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} \mathrm{class}_F(H)_f \arrow[r, "j", "\cong"'] & \mathrm{class}_F(G)_f \arrow[r, "i"] & \mathrm{class}(G,F).
\end{tikzcd}$$

Here the isomorphism j is the direct limit over the obvious maps $\mathrm{class}_F(H)_f \to \mathrm{class}_F(G)_f$ given by extending a class function in the trivial way and the map i is the natural inclusion and in particular injective. $\square$

Exercise 2.94. Let F be a field of characteristic zero. Show that the group G must be torsionfree if $\widetilde{K}_0(FG)$ is a torsion group.

Theorem 2.95 (The Farrell-Jones Conjecture and the Bass Conjecture for fields of characteristic zero). *The Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R and $\mathbb{Q}\subseteq R$ implies the Bass Conjecture 2.91 for fields of characteristic zero as coefficients.*

Proof. This follows from Lemma 2.92. $\square$

The Bost Conjecture 14.23 implies the Bass Conjecture 2.91 for fields of characteristic zero as coefficients, provided that $F=\mathbb{C}$, see [124, Theorem 1.4 and Lemma 1.5].

Exercise 2.96. Let F be field of characteristic zero and let G be a group. Suppose that the Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R and $\mathbb{Q}\subseteq R$ holds for $R=F$. Consider any finitely generated projective FG-module P. Then the Hattori-Stallings rank $\mathrm{HS}(P)$ evaluated at the unit $e\in G$ belongs to $\mathbb{Q}\subseteq F$.

Remark 2.97 (Zalesskii's Theorem). Zalesskii [1012], see also [176, Theorem 3.1], has shown for every field F, every group G, and every idempotent $x\in FG$ that $\mathrm{HS}(P)$ evaluated at the unit $e\in G$ belongs to the prime field of F.

2.10.2 The Bass Conjecture for Integral Domains as Coefficients

Conjecture 2.98 (Bass Conjecture for integral domains as coefficients). Let R be a commutative integral domain and let G be a group. Let $g \in G$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.

Then for every finitely generated projective RG-module the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

Sometimes the Bass Conjecture 2.98 for integral domains as coefficients, is called the *Strong Bass Conjecture*, see [98, 4.5]. The *Weak Bass Conjecture*, see [98, 4.4], states for a finitely generated projective $\mathbb{Z}G$-module P that the evaluation of its Hattori-Stallings rank at the unit $HS(P)(1)$ agrees with $\dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P)$.

Exercise 2.99. Show that the Weak Bass Conjecture follows from the Bass Conjecture 2.98 for integral domains as coefficients.

The Bass Conjecture 2.98 can be interpreted topologically. Namely, the Bass Conjecture 2.98 is true for a finitely presented group G in the case $R = \mathbb{Z}$ if and only if every homotopy idempotent selfmap of an oriented smooth closed manifold whose dimension is greater than 2 and whose fundamental group is isomorphic to G, is homotopic to one that has precisely one fixed point, see [125]. The Bass Conjecture 2.98 for G in the case $R = \mathbb{Z}$ (or $R = \mathbb{C}$) also implies for a finitely dominated CW-complex with fundamental group G that its Euler characteristic agrees with the L^2-Euler characteristic of its universal covering, see [313, 0.3].

The next results follows from the argument in [357, Section 5].

Theorem 2.100 (The Farrell-Jones Conjecture and the Bass Conjecture for integral domains). *Let G be a group. Suppose that*

$$I(G,F) \otimes_{\mathbb{Z}} \mathbb{Q} \colon \operatorname{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic.

Then the Bass Conjecture 2.98 is satisfied for G and every commutative integral domain R.

In particular the Bass Conjecture 2.98 follows from the Farrell-Jones Conjecture 2.72.

For finite G and R an integral domain such that no prime dividing the order of $|G|$ is a unit in R, Conjecture 2.98 was proved by Swan [919, Theorem 8.1], see also [98, Corollary 4.2]. The Bass Conjecture 2.98 has been proved by Bass [98, Proposition 6.2 and Theorem 6.3] for $R = \mathbb{C}$ and G a torsionfree linear group and by Eckmann [311, Theorem 3.3] for $R = \mathbb{Q}$, provided that G has at most cohomological dimension 2 over $\mathbb{Q}$.

The following result is due to Linnell [617, Lemma 4.1].

Theorem 2.101 (The Bass Conjecture for integral domains and elements of finite order). *Let G be a group.*

(i) Let p be a prime, and let P be a finitely generated projective $\mathbb{Z}_{(p)}G$-module. Suppose for $g \in G$ that $\mathrm{HS}(P)(g) \neq 0$. Then there exists an integer $n \geq 1$ such that g and g^{p^n} are conjugated in G and we get for the Hattori-Stallings rank $\mathrm{HS}(P)(g) = \mathrm{HS}(P)(g^{p^n})$;

(ii) Let P be a finitely generated projective $\mathbb{Z}G$-module. Suppose for $g \in G$ that $g \neq 1$ and $\mathrm{HS}(P)(g) \neq 0$. Then there exist subgroups C, H of G such that $g \in C$, $C \subseteq H$, C is isomorphic to the additive group $\mathbb{Q}$, H is finitely generated, and the elements of C lie in finitely many H-conjugacy classes. In particular the order of g is infinite.

More information about the Bass Conjectures can be found in [97, 124, 126, 176, 222, 322, 323, 324, 529, 635, 772, 876, 877].

2.11 The Passage from the Integral to the Rational Group Ring

The following conjecture is taken from [657, Conjecture 85 on page 754].

Conjecture 2.102 (The rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture). The change of ring maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Z}G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Q}G)$$

is trivial.

If G satisfies the Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R with $\mathbb{Q} \subseteq R$, then it satisfies the rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture 2.102, see [657, Proposition 87 on page 754].

Remark 2.103. The question whether an integral version of Conjecture 2.102 holds, i.e., whether the change of ring maps

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$$

is trivial, is discussed in [657, Remark 89 on page 756].

The answer is no in general. Counterexamples have been constructed by Lehner [610], who actually carefully analyzes the image of the map $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$. The group $G = QD_{32} *_{Q_{16}} QD_{32}$ is a counterexample, where QD_{32} is the quasi-dihedral group of order 32, and Q_{16} is the generalized quaternion group of order 16, see [610, Theorem 1.5].

2.12 Survey on Computations of $K_0(RG)$ for Finite Groups

In this section we give a brief survey about computations of $K_0(RG)$ for finite groups G and certain rings R. The upshot will be that the reduced projective class group $\widetilde{K}_0(\mathbb{Z}G)$ is a finite abelian group, but in most cases it is non-trivial and unknown, and that for F a field of characteristic zero $K_0(FG)$ is a well-known finitely generated free abelian group.

The following result is due to Swan [919, Theorem 8.1 and Proposition 9.1].

Theorem 2.104 ($\widetilde{K}_0(RG)$ is finite for finite G and R the ring of integers in an algebraic number field). *Let G be a finite group. Let R be the ring of algebraic integers in an algebraic number field, e.g., $R = \mathbb{Z}$. Then $\widetilde{K}_0(RG)$ is finite.*

A proof of the next theorem will be given in Section 3.8. It was originally proved by Rim [836].

Theorem 2.105 (Rim's Theorem). *Let p be a prime number. The homomorphism induced by the ring homomorphism $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\exp(2\pi i/p)]$ sending the generator of $\mathbb{Z}/p$ to the primitive p-th root of unity $\exp(2\pi i/p)$*

$$K_0(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{\cong} K_0(\mathbb{Z}[\exp(2\pi i/p)])$$

is a bijection.

Example 2.106 ($\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$). Let p be a prime. We have already mentioned in Remark 2.23 that $\mathbb{Z}[\exp(2\pi i/p)]$ is the ring of integers in the algebraic number field $\mathbb{Q}[\exp(2\pi i/p)]$ and hence a Dedekind domain and that the structure of its ideal class group $C(\mathbb{Z}[\exp(2\pi i/p)])$ is only known for a few primes. Thus the message of Rim's Theorem 2.105 is that we know the structure of the finite abelian group $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ only for a few primes. Here is a table taken from [712, page 30] or [971, Tables §3 on page 352ff].

p	$\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$
≤ 19	$\{0\}$
23	$\mathbb{Z}/3$
29	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
31	$\mathbb{Z}/9$
37	$\mathbb{Z}/37$
41	$\mathbb{Z}/11 \oplus \mathbb{Z}/11$
43	$\mathbb{Z}/211$
47	$\mathbb{Z}/5 \oplus \mathbb{Z}/139$

Remark 2.107 (Strategy to study $\widetilde{K}_0(\mathbb{Z}G)$ for finite G). A *$\mathbb{Z}$-order Λ* is a $\mathbb{Z}$-algebra that is finitely generated projective over $\mathbb{Z}$. Its *locally free class group* is defined as the subgroup of $K_0(\Lambda)$

$$Cl(\Lambda) := \{[P] - [Q] \mid P_{(p)} \cong_{\Lambda_{(p)}} Q_{(p)} \text{ for all primes } p\} \tag{2.108}$$

where (p) denotes localization at the prime p. This is the part of $K_0(\Lambda)$ that can be described by localization sequences. Its significance for $\Lambda = \mathbb{Z}G$ lies in the result of Swan [919], see also Curtis-Reiner [257, Theorem 32.11 on page 676] and [258, (49.12 on page 221], that $\widetilde{K}_0(\mathbb{Z}G) \cong Cl(\mathbb{Z}G)$ for every finite group G. Now fix a maximal $\mathbb{Z}$-order $\mathbb{Z}G \subseteq \mathcal{M} \subseteq \mathbb{Q}G$. Such a maximal order has better ring properties than $\mathbb{Z}G$, namely, it is a hereditary ring. The map $i_*\colon Cl(\mathbb{Z}G) \to Cl(\mathcal{M})$ induced by the inclusion $i\colon \mathbb{Z}G \to \mathcal{M}$ is surjective. Define

$$D(\mathbb{Z}G) = \ker\left(i_*\colon Cl(\mathbb{Z}G) \to Cl(\mathcal{M})\right). \tag{2.109}$$

The definition of $D(\mathbb{Z}G)$ is known to be independent of the choice of the maximal order $\mathcal{M}$. Thus the study of $\widetilde{K}_0(\mathbb{Z}G)$ splits into the study of $D(\mathbb{Z}G)$ and $Cl(\mathcal{M})$. The analysis of $Cl(\mathcal{M})$ can be intractable and involves studying cyclotomic fields, whereas the analysis of $D(\mathbb{Z}G)$ essentially uses p-adic logarithms.

Remark 2.110 (Finiteness obstructions and $D(\mathbb{Z}G)$). Often calculations concerning finiteness obstructions are done by first showning that its image in $Cl(\mathcal{M}) = \widetilde{K}_0(\mathbb{Z}G)/D(\mathbb{Z}G)$ is trivial, and then determining it in $D(\mathbb{Z}G)$. For instance, Mislin [724] proved that the finiteness obstruction for every finitely dominated homologically nilpotent space with the finite group G as fundamental group lies in $D(\mathbb{Z}G)$, but that not every element in $D(\mathbb{Z}G)$ occurs this way. Questions concerning the Spherical Space Form Problem involve direct computations in $D(\mathbb{Z}G)$, see for instance Bentzen [115], Bentzen-Madsen [116], and Milgram [704]. The group $D(\mathbb{Z}G)$ enters also in the work of Oliver on actions of finite groups on disks, see [754, 755].

For computations of $D(\mathbb{Z}G)$ for finite p-groups we refer to Oliver [757, 758] and Oliver-Taylor [761].

A survey on $D(\mathbb{Z}G)$ and the methods of its computations can be found in Oliver [759].

Theorem 2.111 (Vanishing results for $D(\mathbb{Z}G)$).

(i) Let G be a finite abelian group G. Then $D(\mathbb{Z}G) = 0$ holds if and only if G satisfies one of the conditions:

(a) G has prime order;
(b) G is cyclic of order 4, 6, 8, 9, 10, 14;
(c) G is $\mathbb{Z}/2 \times \mathbb{Z}/2$;

(ii) If G is a finite group that is not abelian and satisfies $D(\mathbb{Z}G) = 0$, then it is D_n for $n \geq 6$, or A_4, A_5 or S_4;
(iii) One has $D(\mathbb{Z}G) = 0$ if G is A_4, A_5 or S_4;
(iv) $D(\mathbb{Z}D_n) = 0$ for $n < 120$ and $D(\mathbb{Z}D_{120}) = \mathbb{Z}/2$;

(v) $D(\mathbb{Z}D_n) = 0$ if n satisfies one of the following conditions:

(a) $n/2$ is an odd prime;
(b) $n/2$ is a power of a regular odd prime;
(c) $n/2$ is a power of 2.

Proof. (i) This is proved by Cassou-Nogués [206], see also [258, Theorem 50.16 on page 253].

(ii) This is proved in Endo-Hironaka [325], see also [258, Theorem 50.29 on page 266].

(iii) This follows from Reiner-Ulom [833], see also [258, Theorem 50.29 on page 266].

(iv) This is proved in Endo-Miyata [326], see [258, Theorem 50.30 on page 266].

(v) See [258, Theorem 50.29 on page 266]. □

Theorem 2.112 (Finite groups with vanishing $\widetilde{K}_0(\mathbb{Z}G)$).

(i) Let G be a finite abelian group G. Then $\widetilde{K}_0(\mathbb{Z}G) = 0$ holds If and only if G satisfies one of the conditions:

(a) G is cyclic of order n for $1 \leq n \leq 11$;
(b) G is cyclic of order 13, 14, 17, 19;
(c) G is $\mathbb{Z}/2 \times \mathbb{Z}/2$;

(ii) If G is a non-abelian finite group with $\widetilde{K}_0(\mathbb{Z}G) = 0$, then G is D_n for $n \geq 6$ or A_4, A_5 or S_4;
(iii) We have $\widetilde{K}_0(\mathbb{Z}G) = 0$ for $G = A_4, S_4, D_6, D_8, D_{12}$.

Proof. (i) This is proved by Cassou-Nogués [206], see also [258, Corollary 50.17 on page 253].

(ii) This follows from Theorem 2.111 (ii).

(iii) The cases $G = A_4, S_4, D_6, D_8$ are already treated in [832, Theorem 6.4 and Theorem 8.2]. Because of Theorem 2.111 (iii) it suffices to show for the maximal order $\mathcal{M}$ for the groups $G = A_4, S_4, D_6, D_8, D_{12}$ that $Cl(\mathcal{M}) = 0$. This follows from the fact that $\mathbb{Q}G$ is a products of matrix algebras over $\mathbb{Q}$ and hence the maximal $\mathbb{Z}$-order $\mathcal{M}$ is a products of matrix rings over $\mathbb{Z}$. □

Exercise 2.113. Determine all finite groups G of order ≤ 9 for which $\widetilde{K}_0(\mathbb{Z}G)$ is non-trivial.

Theorem 2.114 ($K_0(RG)$ for finite G and an Artinian ring R). *Let R be an Artinian ring. Let G be a finite group. Then RG is also an Artinian ring. There are only finitely many isomorphism classes $[P_1]$, $[P_2]$, …,*

$[P_n]$ of irreducible finitely generated projective RG-modules, and we obtain an isomorphism of abelian groups

$$\mathbb{Z}^n \xrightarrow{\cong} K_0(RG), \quad (k_1, k_2, \ldots k_n) \mapsto \sum_{i=1}^{n} k_i \cdot [P_i].$$

Proof. This follows from [257, Proposition 16.7 on page 406 and the paragraph after Corollary 6.22 on page 132]. ☐

Let F be a field of characteristic zero or a prime number p not dividing $|G|$. Then $K_0(FG)$ is the same as the representation ring $R_F(G)$ of G with coefficients in the field F since the ring FG is *semisimple* i.e., every submodule of a module is a direct summand. If F is a field of characteristic zero, then representations are detected by their characters, see Lemma 2.88. For more information about modules over FG for a finite group G and a field F we refer for instance to Curtis-Reiner [257, Chapter 1 and Chapter 2] and Serre [892].

Exercise 2.115. Compute $K_0(FD_8)$ for $F = \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

2.13 Survey on Computations of $K_0(C_r^*(G))$ and $K_0(\mathcal{N}(G))$

Let G be a group. Let $\mathcal{B}(L^2(G))$ denote the bounded linear operators on the Hilbert space $L^2(G)$ whose orthonormal basis is G. The *reduced group C^*-algebra $C_r^*(G)$* is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \to \mathcal{B}(L^2(G))$ that sends an element $u \in \mathbb{C}G$ to the (left) G-equivariant bounded operator $L^2(G) \to L^2(G)$ given by right multiplication with u. The *group von Neumann algebra $\mathcal{N}(G)$* is the closure in the weak topology. There is an identification $\mathcal{N}(G) = \mathcal{B}(L^2(G))^G$. One has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{N}(G) \subseteq \mathcal{B}(L^2(G)).$$

We have $\mathbb{C}G = C_r^*(G) = \mathcal{N}(G)$ if and only if G is finite. If $G = \mathbb{Z}$, then the Fourier transform gives identifications $C_r^*(\mathbb{Z}) = C(S^1)$ and $\mathcal{N}(\mathbb{Z}) = L^\infty(S^1)$.

Remark 2.116 ($K_0(C_r^*(G))$ versus $K_0(\mathbb{C}G)$). We will later see that the study of $K_0(C_r^*(G))$ is not done by its algebraic nature. Instead we will introduce and analyze the topological K-theory of $C_r^*(G)$ and explain that in dimension 0 the algebraic and the topological K-theory of $C_r^*(G)$ agree. In order to explain the different flavour of $K_0(C_r^*(G))$ in comparison with $K_0(\mathbb{C}G)$, we mention the conclusion of the Baum-Connes Conjecture for torsionfree groups 10.44 that for torsionfree G there exists an isomorphism

$$\bigoplus_{n\geq 0} H_{2n}(BG;\mathbb{Q}) \xrightarrow{\cong} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The space BG is the *classifying space of the group* G, which is up to homotopy characterized by the property that it is a CW-complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. We denote by $H_*(X,R)$ *the singular or cellular homology* of a space or CW-complex X with coefficient in a commutative ring R. We can identify $H_*(BG;R)$ with the group homology of G with coefficients in R.

We see that $K_0(C_r^*(G))$ can be huge also for torsionfree groups, whereas $K_0(\mathbb{C}G) \cong \mathbb{Z}$ for torsionfree G is a conclusion of the Farrell-Jones Conjecture 2.60 for $K_0(R)$ for torsionfree G and regular R. We see already here a homological behavior of $K_0(C_r^*(G))$, which is not yet evident in the case of group rings so far and will become clear later.

Remark 2.117 ($K_0(\mathcal{N}(G))$). The projective class group $K_0(\mathcal{A})$ can be computed for any von Neumann algebra $\mathcal{A}$ using the center-valued universal trace, see for instance [635, Section 9.2]. In particular one gets for a finitely generated group G that does not contain $\mathbb{Z}^n$ as subgroup of finite index an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}.$$

Here $\mathcal{Z}(\mathcal{N}(G))$ is the center of the group von Neumann algebra and the $\mathbb{Z}/2$-action comes from taking the adjoint of an operator in $\mathcal{B}(L^2(G))$, see [635, Example 9.34 on page 353]. If G is a finitely generated group that does not contain $\mathbb{Z}^n$ as subgroup of finite index and for which the conjugacy class (g) of an element g different from the unit is always infinite, then $\mathcal{Z}(\mathcal{N}(G)) = \mathbb{C}$ and one obtains an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathbb{R}.$$

A pleasant feature of $\mathcal{N}(G)$ is that there is no difference between stably isomorphic and isomorphic in the sense that for three finitely generated projective $\mathcal{N}(G)$-modules P_0, P_1 and Q we have $P_0 \oplus Q \cong_{\mathcal{N}(G)} P_1 \oplus Q$ if and only if $P_0 \cong_{\mathcal{N}(G)} P_1$.

We see that in the case of the group von Neumann algebra we can compute $K_0(\mathcal{N}(G))$ completely, but the answer does not show any homological behavior in G. In fact, the Farrell-Jones Conjecture and the Baum-Connes Conjecture have no analogues for group von Neumann algebras.

Exercise 2.118. Let G be an torsionfree hyperbolic group that is not cyclic. Prove $K_0(\mathcal{N}(G)) \cong \mathbb{R}$.

Remark 2.119 (Change of rings homomorphisms for $\widetilde{K}_0$ for $\mathbb{Z}G \to \mathbb{C}G \to C_r^*(G) \to \mathcal{N}(G)$). We summarize what is conjectured or known about the string of change of rings homomorphism

$$\widetilde{K}_0(\mathbb{Z}G) \xrightarrow{i_1} \widetilde{K}_0(\mathbb{C}G) \xrightarrow{i_2} \widetilde{K}_0(C_r^*(G)) \xrightarrow{i_3} \widetilde{K}_0(\mathcal{N}(G))$$

coming from the various inclusion of rings. The first map i_1 is conjectured to be rationally trivial, see [657, Conjecture 85 on page 754], but is not integrally trivial, see [610, Theorem 5.1]. The second map i_2 is conjectured to be rationally injective, compare [634, Theorem 0.5], but is not surjective in general. The map i_3 is in general not injective, not surjective, and not trivial. It is known that the composite $i_3 \circ i_2 \circ i_1$ is trivial, see for instance [635, Theorem 9.62 on page 362]..

2.14 Notes

Algebraic K-theory is compatible with direct limits as explained for the projective class group next. A *directed set* I is a non-empty set with a partial ordering $\leq$ such that for two elements i_0 and i_1 there exists an element i with $i_0 \leq i$ and $i_1 \leq i$. A *directed system of rings* is a set of rings $\{R_i \mid i \in I\}$ indexed by a directed set I together with a choice of a ring homomorphism $\phi_{i,j} \colon R_i \to R_j$ for $i, j \in I$ with $i \leq j$ such that $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$ holds for $i, j, k \in I$ with $i \leq j \leq k$ and $\phi_{i,i} = \mathrm{id}$ holds for $i \in I$. The *colimit*, sometimes also called *direct limit*, of $\{R_i \mid i \in I\}$ is a ring denoted by $\mathrm{colim}_{i\in I} R_i$ together with ring homomorphisms $\psi_j \colon R_i \to \mathrm{colim}_{i\in I} R_i$ for every $j \in I$ such that $\psi_j \circ \phi_{i,j} = \psi_i$ holds for $i, j \in I$ with $i \leq j$ and the following universal property is satisfied: For every ring S and every system of ring homomorphisms $\{\mu_i \colon R_i \to S \mid i \in I\}$ such that $\mu_j \circ \phi_{i,j} = \mu_i$ holds for $i, j \in I$ with $i \leq j$, there is precisely one ring homomorphism $\mu \colon \mathrm{colim}_{i\in I} R_i \to S$ satisfying $\mu \circ \psi_i = \mu_i$ for every $i \in I$. If we replace ring by group or module respectively everywhere, we get the notion of directed system and direct limit of groups or modules respectively. This is a special case of the direct limit of a functor, namely, consider I as category with the set I as objects and precisely one morphism from i to j if $i \leq j$, and no other morphisms.

Remark 2.120 (Filtered categories). One consider instead of a directed set a *filtered category*, i.e, a nonempty category I such that for every two objects i and j there is an object k together with two morphisms $i \to k$ and $j \to k$ and for two morphism $f, g \colon i \to j$ with the same source and target there is a morphism $h \colon j \to k$ with $hj \circ f = h \circ k$, and all the results about colimits over a directed sets stay true if one considers colimits over filtered categories. Then one talks about filtered systems instead of filtered sets.

Let $\{R_i \mid i \in I\}$ be a direct system of rings. For every $i \in I$, we obtain a change of rings homomorphism $(\psi_i)_* \colon K_0(R_i) \to K_0(R)$. The universal property of the direct limit yields a homomorphism

$$\mathrm{colim}_{i\in I}(\psi_i)_* \colon \mathrm{colim}_{i\in I} K_0(R_i) \xrightarrow{\cong} K_0(R), \tag{2.121}$$

which turns out to be an isomorphism, see [844, Theorem 1.2.5].

We denote by $R^\times$ the group of units in R. A ring R is called *local* if the set $I := R - R^\times$ forms a (left) ideal. If I is a left ideal, it is automatically a two-sided ideal and it is maximal both as a left ideal and as a right ideal. A ring R is local if and only if it has a unique maximal left ideal and a unique maximal right ideal and these two coincide. An example of a local ring is the ring of formal power series $F[[t]]$ with coefficients in a field F. If R is a commutative ring and I is a prime ideal, then the localization R_I of R at I is a local ring.

Theorem 2.122 ($K_0(R)$ of local rings). *Let R be a local ring. Then every finitely generated projective R-module is free and $K_0(R)$ is infinite cyclic with $[R]$ as generator.*

Proof. See for instance [712, Lemma 1.2 on page 5] or [844, Theorem 1.3.11 on page 14]. □

Its proof is based on *Nakayama's Lemma* which says for a ring R and a finitely generated R-module M that $\operatorname{rad}(R)M = M \iff M = 0$ holds. Here $\operatorname{rad}(R)$ is the *radical*, or *Jacobson radical*, i.e., the two sided ideal that is given by the intersection of all maximal left ideals, or, equivalently, of all maximal right ideals of R. The radical is the same as the set of elements $r \in R$ for which there exists $s \in S$ such that $1 - rs$ has a left inverse in R.

If R is a commutative ring and *spec*(R) is its *spectrum* consisting of its prime ideals and equipped with the Zariski topology, then we obtain for every finitely generated projective R-module P a continuous rank function $\operatorname{Spec}(R) \to \mathbb{Z}$ by sending a prime ideal I to the rank of the finitely generated free R_I-module $P_I = P \otimes_R R_I$. This makes sense because of Theorem 2.122 since R_I is local. If R is a commutative integral domain, this rank function is constant. For more details we refer for instance to [844, Proposition 1.3.12 on page 15].

Exercise 2.123. Prove for an integer $n \geq 1$ that $K_0(\mathbb{Z}/n)$ is the free abelian group whose rank is the number of prime numbers dividing n.

A ring is called *semilocal* if $R/\operatorname{rad}(R)$ is Artinian, or, equivalently, $R/\operatorname{rad}(R)$ is semisimple. If R is commutative, then R is semilocal if and only if it has only finitely many maximal ideas, see [899, page 69]. For a semilocal ring R, the projective class group $K_0(R)$ is a finitely generated free abelian group, see [899, Proposition 14 on page 28]. More information about semilocal rings can be found for instance in [594, § 20].

Lemma 2.124. *For any ring R and nilpotent two-sided ideal $I \subseteq R$, the map $K_0(R) \to K_0(R/I)$ induced by the projection $R \to R/I$ is bijective.*

Proof. See [979, Lemma 2.2 in Section II.2 on page 70]. □

Given two groups G_1 and G_2, let $G_1 * G_2$ by the amalgamated free product. Then the natural maps $G_k \to G_0 * G_1$ for $k = 1, 2$ induce an isomorphism, see [405, Theorem 1.1],

$$\widetilde{K}_0(\mathbb{Z}[G_1]) \oplus \widetilde{K}_0(\mathbb{Z}[G_1]) \;\cong\; \widetilde{K}_0(\mathbb{Z}[G_1 * G_2]). \tag{2.125}$$

This is a first glimpse of a homological behavior of K_0 if one compares this with the corresponding isomorphism of group homology

$$\widetilde{H}_n(G_1) \oplus \widetilde{H}_n(G_1) \;\cong\; \widetilde{H}_n(G_1 * G_2).$$

Exercise 2.126. Show that the projections $\mathrm{pr}_k \colon G_1 \times G_2 \to G_k$ for $k = 1, 2$ do *not* in general induce isomorphisms

$$\widetilde{K}_0(\mathbb{Z}[G_1 \times G_2]) \to \widetilde{K}_0(\mathbb{Z}[G_1]) \times \widetilde{K}_0(\mathbb{Z}[G_2]).$$

There are also equivariant versions of the finiteness obstructions, see for instance [32], [627], and [629, Chapter 3 and 11]. Finiteness obstructions for categories are investigated in [376, 375].

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Chapter 3
The Whitehead Group

3.1 Introduction

This chapter is devoted to the *first K-group* $K_1(R)$ of a ring R and the *Whitehead group* $\mathrm{Wh}(G)$ of a group G.

We give two equivalent definitions of $K_1(R)$, namely, as the universal determinant and in terms of invertible matrices. We explain some elementary calculations of $K_1(R)$ for rings with Euclidian algorithm, local rings, and rings of integers in algebraic number fields.

We introduce the *Whitehead group* of a group and the *Whitehead torsion* of a homotopy equivalence of finite CW-complexes algebraically and geometrically. The relevance of these notions are illustrated by the *s-Cobordism Theorem* and its applications to the classification of manifolds and by the classification of *lens spaces* by *Reidemeister torsion*.

The next topic is the *Bass-Heller-Swan decomposition* and the *long exact sequence* associated to a pullback of rings and to a two-sided ideal. These are important tools for computations and relate $K_0(R)$ and $K_1(R)$.

We discuss *Swan homomorphisms* and *free homotopy representations*. Thus we provide a link between torsion invariants and finite obstructions.

We explain the variant of the *Farrell-Jones Conjecture* that for a torsionfree group G the reduced projective class group $\widetilde{K}_0(\mathbb{Z}G)$ and the Whitehead group $\mathrm{Wh}(G)$ vanish. It implies that any h-cobordism with torsionfree fundamental group and dimension ≥ 6 is trivial.

Finally, we give a survey of computations of $K_1(\mathbb{Z}G)$ for finite groups G and of the algebraic K_1-group of commutative Banach algebras, commutative C^*-algebras, and of some group von Neumann algebras.

3.2 Definition and Basic Properties of $K_1(R)$

Definition 3.1 (K_1-group $K_1(R)$). Let R be a ring. Define the *K_1-group of a ring* $K_1(R)$ to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Additivity
  Given a commutative diagram of finitely generated projective R-modules

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "i"] \arrow[d, "f_1"] & P_2 \arrow[r, "p"] \arrow[d, "f_2"] & P_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "i"] & P_2 \arrow[r, "p"] & P_3 \arrow[r] & 0 \end{tikzcd}$$

with exact rows and automorphisms as vertical arrows, we get

$$[f_1] + [f_3] = [f_2];$$

- Composition formula
  Given automorphisms $f, g\colon P \to P$ of a finitely generated projective R-module P, we get
  $$[g \circ f] = [f] + [g].$$

Define $G_1(R)$ analogously but replacing finitely generated projective by finitely generated everywhere.

Given a ring homomorphism $f\colon R \to S$, we obtain a change of rings homomorphism

(3.2) $f_* = K_1(f)\colon K_1(R) \to K_1(S),\ [g\colon P \to P] \mapsto [f_*g\colon f_*P \to f_*P]$

analogously as we have defined it for the projective class group in (2.2). Thus K_1 becomes a covariant functor from the category of rings to the category of abelian groups.

Exercise 3.3. Show that $K_1(R) = 0$ holds for the ring R appearing in Example 2.17.

Remark 3.4 (The universal property of $K_1(R)$). One should view $K_1(R)$ together with the assignment sending an automorphism $f\colon P \to P$ of a finitely generated projective R-module P to its class $[f] \in K_1(R)$ as the *universal determinant*. Namely, for any abelian group A and assignment a that sends the automorphism f of a finitely generated projective R-module to $a(f) \in A$ such that (A, a) satisfies additivity and the composition formula appearing in Definition 3.1, there exists precisely one homomorphism of abelian groups $\phi\colon K_1(R) \to A$ such that $\phi([f]) = a(f)$ holds for every automorphism f of a finitely generated projective R-module.

We always have the following map of abelian groups

(3.5) $$i\colon R^\times/[R^\times, R^\times] \ \to\ K_1(R), \quad [x] \ \mapsto [r_x\colon R \to R]$$

where r_x right multiplication is with x. It is neither injective nor surjective in general. However, we have

Theorem 3.6 ($K_1(F)$ of skew-fields). *The map i defined in* (3.5) *is an isomorphism if R is a skew-field or, more generally, a local ring. It is surjective (with an explicitly described kernel) if R is a semilocal ring.*

Proof. See for instance [899, Corollary 43 on page 133], [844, Corollary 2.2.6 on page 69], and [899, Proposition 53 on page 140]. □

Exercise 3.7. Let $\mathbb{H}$ be the skew-field of quaternions $\{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. Since $\mathbb{H}$ is a 4-dimensional vector space, there is an embedding $\mathrm{GL}_n(\mathbb{H}) \to \mathrm{GL}_{4n}(\mathbb{R})$. Its composite with the determinant over $\mathbb{R}$ yields a homomorphism $\mu_n \colon \mathrm{GL}_n(\mathbb{H}) \to \mathbb{R}^{>0}$ to the multiplicative group of positive real numbers. Show that the system of homomorphisms μ_n induces an isomorphism

$$\mu \colon K_1(\mathbb{H}) \xrightarrow{\cong} \mathbb{R}^{>0}.$$

The proof of the next two results is analogous to the one of Theorem 2.10 and Lemma 2.12.

Theorem 3.8 (Morita equivalence for $K_1(R)$). *For every ring R and integer $n \geq 1$, there is natural isomorphism*

$$\mu \colon K_1(R) \xrightarrow{\cong} K_1(M_n(R)).$$

Lemma 3.9. *Let R_0 and R_1 be rings. Denote by $\mathrm{pr}_i \colon R_0 \times R_1 \to R_i$ for $i = 0, 1$ the projection. Then we obtain an isomorphism*

$$(\mathrm{pr}_0)_* \times (\mathrm{pr}_1)_* \colon K_1(R_0 \times R_1) \xrightarrow{\cong} K_1(R_0) \times K_1(R_1).$$

Lemma 3.10. *Define the abelian group $K_1^f(R)$ analogous to $K_1(R)$ but with finitely generated projective replaced by finitely generated free everywhere. Then the canonical homomorphism*

$$\alpha \colon K_1^f(R) \xrightarrow{\cong} K_1(R), \quad [f] \mapsto [f]$$

is an isomorphism.

Proof. Given an automorphism $f \colon P \to P$ of a finitely generated projective R-module P, we can choose a finitely generated projective R-module Q, a finitely generated free R-module F and an R-isomorphism $\phi \colon P \oplus Q \xrightarrow{\cong} F$. We obtain an automorphism $\phi \circ (f \oplus \mathrm{id}_Q) \circ \phi^{-1} \colon F \to F$ and thus an element $[\phi \circ (f \oplus \mathrm{id}_Q) \circ \phi^{-1}] \in K_1^f(R)$. One easily checks that it is independent of the choice of Q and ϕ and then that it depends only on $[f] \in K_1(R)$. Thus we obtain a homomorphism of abelian groups $\beta \colon K_1(R) \to K_1^f(R)$. One easily checks that α and β are inverse to one another. □

Next we give a matrix description of $K_1(R)$. Denote by $E_n(i,j)$ for $n \geq 1$ and $1 \leq i, j \leq n$ the (n,n)-matrix whose entry at (i,j) is one and is zero elsewhere. Denote by I_n the identity matrix of size n. An *elementary (n,n)-matrix* is a matrix of the form $I_n + r \cdot E_n(i,j)$ for $n \geq 1$, $1 \leq i, j \leq n$, $i \neq j$ and $r \in R$. Let A be an (n,n)-matrix. The matrix $B = A \cdot (I_n + r \cdot E_n(i,j))$ is obtained from A by adding the i-th column multiplied with r from the right to the j-th column. The matrix $C = (I_n + r \cdot E_n(i,j)) \cdot A$ is obtained from A by adding the j-th row multiplied with r from the left to the i-th row. Let $\mathrm{E}(R) \subset \mathrm{GL}(R)$ be the subgroup generated by all elements in $\mathrm{GL}(R)$ that are represented by elementary matrices.

Lemma 3.11. *The subgroup* $\mathrm{E}(R)$ *of* $\mathrm{GL}(R)$ *coincides with the commutator subgroup* $[\mathrm{GL}(R), \mathrm{GL}(R)]$.

Proof. For $n \geq 3$, pairwise distinct numbers $1 \leq i,j,k \leq n$, and $r \in R$, we can write $I_n + r \cdot E_n(i,k)$ as a commutator in $\mathrm{GL}_n(R)$, namely,

$$\begin{aligned} & I_n + r \cdot E_n(i,k) \\ = \; & (I_n + r \cdot E_n(i,j)) \cdot (I_n + E_n(j,k)) \cdot (I_n + r \cdot E_n(i,j))^{-1} \cdot (I_n + E_n(j,k))^{-1}. \end{aligned}$$

This implies $\mathrm{E}(R) \subset [\mathrm{GL}(R), \mathrm{GL}(R)]$.

Let A and B be two elements in $\mathrm{GL}_n(R)$. Let $[A]$ and $[B]$ be the elements in $\mathrm{GL}(R)$ represented by A and B. Given two elements x and y in $\mathrm{GL}(R)$, we write $x \sim y$ if there are elements e_1 and e_2 in $\mathrm{E}(R)$ with $x = e_1 y e_2$, in other words, if the classes of x and y in $\mathrm{E}(R) \backslash \mathrm{GL}(R) / \mathrm{E}(R)$ agree. One easily checks

$$[AB] \sim \left[\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} AB & A \\ 0 & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & A \\ -B & I_n \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right]$$

since each step is given by multiplication from the right or left with a block matrix of the form $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$ or $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$ and such a block matrix is obviously obtained from I_{2n} by a sequence of column and row operations and hence its class in $\mathrm{GL}(R)$ belongs to $\mathrm{E}(R)$. Analogously we get

$$[BA] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

Since the element in $\mathrm{GL}(R)$ represented by $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ belongs to $\mathrm{E}(R)$, we conclude

$$\left[\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix}\right] \sim \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] \sim \left[\begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right].$$

and hence

$$[AB] \;\sim\; [BA].$$

This implies for any element $x \in \mathrm{GL}(R)$ and $e \in \mathrm{E}(R)$ that $xex^{-1} \sim ex^{-1}x = e$ and hence $xex^{-1} \in \mathrm{E}(R)$. Therefore $\mathrm{E}(R)$ is normal. Given a commutator $xyx^{-1}y^{-1}$ for $x, y \in \mathrm{GL}(R)$, we conclude for appropriate elements e_1, e_2, e_3 in $\mathrm{E}(R)$

$$xyx^{-1}y^{-1} = e_1 yxe_2 x^{-1}y^{-1} = e_1(yx)e_2(yx)^{-1} = e_1e_3 \in \mathrm{E}(R).$$

□

Theorem 3.12 ($K_1(R)$ equals $\mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)]$). *There is a natural isomorphism*

$$\mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)] \xrightarrow{\cong} K_1(R).$$

Proof. Because of Lemma 3.10 it suffices to construct to one another inverse homomorphisms of abelian groups $\alpha\colon \mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)] \to K_1^f(R)$ and $\beta\colon K_1^f(R) \to \mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)]$. The map α sends the class $[A]$ of $A \in \mathrm{GL}_n(R)$ to the class $[r_A]$ of $r_A\colon R^n \to R^n$. This is a well-defined homomorphism of abelian groups since $[r_{AB}] = [r_A]+[r_B]$, $[r_{A\oplus I_1}] = [r_A]$ holds for all $n \in \mathbb{Z}, n \geq 1$ and $A, B \in \mathrm{GL}_n(R)$, and $K_1(R)$ is abelian. The map β sends the class $[f]$ of an automorphism f of a finitely generated free R-module F to the class $[A(f,B)]$ of the invertible (n,n)-matrix $A(f,B)$ associated to f after a choice of some R-basis B for F. This class is independent of the choice of B since for another choice of a bases B' there exists $U \in \mathrm{GL}_n(R)$ with $UA(f,B)U^{-1} = A(f,B')$, which implies $[A(f,B')] = [UA(f,B)U^{-1}] = [U][A(f,B)][U]^{-1} = [U][U]^{-1}[A(f,B)] = [A(f,B)]$. Thus we have defined β on generators. It remains to check the relations. Obviously the composition formula is satisfied. Additivity is satisfied because of the following calculation in $\mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)]$ for $A \in \mathrm{GL}_m(R)$, $B \in \mathrm{GL}_n(R)$ and $C \in M_{m,n}(R)$ based on Lemma 3.11

$$\left[\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\right] = \left[\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ C^{-1}B & I_n \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}\right] \cdot \left[\begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix}\right] \cdot \left[\begin{pmatrix} I_m & 0 \\ C^{-1}B & I_n \end{pmatrix}\right] = [A]\cdot[C]\cdot[I_{m+n}] = [A]\cdot[C].$$

One easily checks that α and β are inverse to one another. □

Remark 3.13 (What $K_1(R)$ measures). We conclude from Lemma 3.11 and Theorem 3.12 that two matrices $A \in \mathrm{GL}_m(R)$ and $B \in \mathrm{GL}_n(R)$ represent the same class in $K_1(R)$ if and only if B can be obtained from A by a sequence of the following operations:

(i) Elementary row operation
B is obtained from A by adding the k-th row multiplied with r from the left to the l-th row for $r \in R$ and $k \neq l$;
(ii) Elementary column operation
B is obtained from A by adding the k-th column multiplied with r from the right to the l-th row for $r \in R$ and $k \neq l$;
(iii) Stabilization
B is obtained by taking the direct sum of A and I_1, i.e., B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

(iv) Destabilization
A is the direct sum of B and I_1. (This is the inverse operation to (iii).)

Since multiplication from the left or right with an elementary matrix corresponds to the operation (i) or the operation (ii), the abelian group $K_1(R)$ is trivial if and only if any invertible matrix $A \in \mathrm{GL}_n(R)$ can be reduced by a sequence of the operations above to the empty matrix.

One could delete the operation (i) or the operation (ii) from the list above without changing the conclusion. This follows from the fact that $\mathrm{E}(R)$ is a normal subgroup of $\mathrm{GL}(R)$.

The elementary proof of the next lemma is left to the reader.

Lemma 3.14. *Let R be a commutative ring. Then the determinant defines a homomorphism of abelian groups*

$$\det\colon K_1(R) \to R^{\times}, \quad [f] \mapsto \det(f).$$

It satisfies $\det \circ i = \mathrm{id}_{R^{\times}}$ *for the map i defined in* (3.5)*. In particular the map* $\det$ *is surjective.*

Definition 3.15 ($SK_1(R)$ of a commutative ring R). Let R be a commutative ring. Define

$$SK_1(R) := \ker\big(\det\colon K_1(R) \to R^{\times}\big).$$

We will see in Section 3.12 that there are commutative group rings $\mathbb{Z}G$ for which the surjective map $\det\colon K_1(\mathbb{Z}G) \to \mathbb{Z}G^{\times}$ is not injective, or, equivalently, with non-trivial $SK_1(\mathbb{Z}G)$. Here is another example.

Example 3.16. The following example is taken from [100, Example 4.4], see also [844, Exercise 2.3.11 on page 82]. Let Λ be obtained from the polynomial ring $\mathbb{R}[x,y]$ by dividing out the ideal generated by x^2+y^2-1. This is a Dedekind domain. The matrix

$$M := \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{SL}_2(\Lambda)$$

represents a non-trivial element in $SK_1(\Lambda)$. The proof uses Mennicke symbols and is based on the observation that the function

$$S^1 \to \mathrm{SL}_n(\mathbb{R}), \quad (x,y) \mapsto \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

represents a non-trivial element in $\pi_1(\mathrm{SL}_n(\mathbb{R})) \cong \pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ for $n \geq 3$.

Theorem 3.17 ($K_1(R) = R^{\times}$ for commutative rings with Euclidean algorithm). *Let R be a commutative ring with Euclidean algorithm in the sense of [844, 2.3.1 on page 74], for instance a field or $\mathbb{Z}$.*

Then the determinant induces an isomorphism

$$\det\colon K_1(R) \xrightarrow{\cong} R^\times.$$

Proof. Because of Lemma 3.14 it suffices to show for $A \in \mathrm{GL}_n(R)$ with $\det(A) = 1$ that it can be reduced to the empty matrix by a sequence of operations appearing in Remark 3.13. But this is a well-known result of elementary algebra, see for instance [844, Theorem 2.3.2 on page 74]. □

Exercise 3.18. Prove $K_1(\mathbb{Z}[i]) \cong \{1, -1, i, -i\} \cong \mathbb{Z}/4$.

Remark 3.19 ($K_1(R)$ of principal ideal domains). There exists principal ideal domains R such that $\det\colon K_1(R) \to R^\times$ is not bijective. For instance Grayson [418] gives such an example, namely, take $\mathbb{Z}[x]$ and invert x and all polynomials of the shape $x^m - 1$ for $m \geq 1$. Other examples can be found in Ischebeck [500].

Theorem 3.20 (Vanishing of SK_1 of ring of integers in an algebraic number field). *Let R be the ring of integers in an algebraic number field. Then the determinant induces an isomorphism*

$$\det\colon K_1(R) \xrightarrow{\cong} R^\times.$$

Proof. See [100, page 77] or [712, Corollary 16.3 on page 159]. □

The proof of the next classical result can be found for instance in [843, Theorem 2.3.8 on page 79].

Theorem 3.21 (Dirichlet Unit Theorem). *Let R be the ring of integers in an algebraic number field F. Let r_1 be the number of distinct embeddings of F into $\mathbb{R}$ and let r_2 be the number of distinct conjugate pairs of embeddings of F into $\mathbb{C}$ with image not contained in $\mathbb{R}$. Then:*

(i) $r_1 + 2r_2$ is the degree $[F : \mathbb{Q}]$ of the extension $\mathbb{Q} \subseteq F$;
(ii) The abelian group $R^\times$ is finitely generated;
(iii) The torsion subgroup of $R^\times$ is the finite cyclic group of roots of unity in F;
(iv) The rank of $R^\times$ is $r_1 + r_2 - 1$.

Exercise 3.22. Let R be the ring of integers in an algebraic number field F. Then $K_1(R)$ is finite if and only if F is $\mathbb{Q}$ or an imaginary quadratic field.

3.3 Whitehead Group and Whitehead Torsion

In this section we will assign to a homotopy equivalence $f\colon X \to Y$ of finite CW-complexes its Whitehead torsion $\tau(f)$ in the Whitehead group

$\mathrm{Wh}(\pi(Y))$ associated to Y. A basic feature is that the Whitehead torsion can distinguish manifolds or spaces that are homotopy equivalent. The notion of Whitehead torsion goes back to the papers by J.H.C. Whitehead [988, 989, 990].

The *reduced* K_1*-group* $\widetilde{K}_1(R)$ is defined to be the cokernel of the map $K_1(\mathbb{Z}) \to K_1(R)$ induced by the unique ring homomorphism $\mathbb{Z} \to R$. The homomorphism $\det\colon K_1(\mathbb{Z}) \to \{\pm 1\}$ is a bijection, because $\mathbb{Z}$ is a ring with Euclidean algorithm, see Theorem 3.17. Hence $\widetilde{K}_1(R)$ is the same as the quotient of $K_1(R)$ by the cyclic subgroup of at most order two generated by the class of the $(1,1)$-matrix (-1).

Definition 3.23 (Whitehead group). Define the *Whitehead group* $\mathrm{Wh}(G)$ of a group G to be the cokernel of the map $G \times \{\pm 1\} \to K_1(\mathbb{Z}G)$ that sends $(g, \pm 1)$ to the class of the invertible $(1,1)$-matrix $(\pm g)$.

Obviously a group homomorphism $u\colon G \to H$ induces a homomorphism of abelian groups

$$u_* = \mathrm{Wh}(u)\colon \mathrm{Wh}(G) \to \mathrm{Wh}(H). \tag{3.24}$$

Exercise 3.25. Using the ring homomorphism $f\colon \mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$ that sends the generator of $\mathbb{Z}/5$ to $\exp(2\pi i/5)$ and the norm of a complex number, define a homomorphism of abelian groups

$$\phi\colon \mathrm{Wh}(\mathbb{Z}/5) \to \mathbb{R}^{>0}.$$

Show that $1-t-t^{-1}$ is a unit in $\mathbb{Z}[\mathbb{Z}/5]$ whose class in $\mathrm{Wh}(\mathbb{Z}/5)$ is an element of infinite order. (Actually $\mathrm{Wh}(\mathbb{Z}/5)$ is an infinite cyclic group with this class as generator).

For a ring R and a group G we denote by

$$A_0 = K_0(i)\colon K_0(R) \to K_0(RG) \tag{3.26}$$

the map induced by the inclusion $i\colon R \to RG$. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG-automorphism $R[G] \otimes_R P \to R[G] \otimes_R P,\ u \otimes x \mapsto ug^{-1} \otimes x$ defines a map $\Phi\colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \to K_1(RG)$. Define a homomorphism

$$A_1 := \Phi \oplus K_1(i)\colon\ (G/[G,G] \otimes_{\mathbb{Z}} K_0(R)) \oplus K_1(R) \to K_1(RG). \tag{3.27}$$

Definition 3.28 (Generalized Whitehead group). For a regular ring R and a group G we define the *generalized Whitehead group* $\mathrm{Wh}_1^R(G)$ as the cokernel of the map A_1. Denote by $\mathrm{Wh}_0^R(G)$ the cokernel of the map A_0.

Note that the abelian group $\mathrm{Wh}_1^{\mathbb{Z}}(G)$ of Definition 3.28 agrees with the abelian group $\mathrm{Wh}(G)$ of Definition 3.23.

Next we will define torsion invariants on the level of chain complexes.

We begin with some input about chain complexes. Let $f_*\colon C_* \to D_*$ be a chain map of R-chain complexes for some ring R. Define $\mathrm{cyl}_*(f_*)$ to be the chain complex with n-th differential

$$C_{n-1} \oplus C_n \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 & 0 \\ -\mathrm{id} & c_n & 0 \\ f_{n-1} & 0 & d_n \end{pmatrix}} C_{n-2} \oplus C_{n-1} \oplus D_{n-1}.$$

Define $\mathrm{cone}_*(f_*)$ to be the quotient of $\mathrm{cyl}_*(f_*)$ by the obvious copy of C_*. Hence the n-th differential of $\mathrm{cone}_*(f_*)$ is

$$C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}.$$

Given a chain complex C_*, define ΣC_* to be the quotient of $\mathrm{cone}_*(\mathrm{id}_{C_*})$ by the obvious copy of C_*, i.e., the chain complex with n-th differential

$$C_{n-1} \xrightarrow{-c_{n-1}} C_{n-2}.$$

Definition 3.29 (Mapping cylinder and mapping cone). For a chain map $f_* \colon C_* \to D_*$, we call $\mathrm{cyl}_*(f_*)$ the *mapping cylinder* and $\mathrm{cone}_*(f_*)$ the *mapping cone*. For a chain complex C_*, we call ΣC_* the *suspension*.

These algebraic notions of mapping cylinder, mapping cone, and suspension are modelled on their geometric counterparts. Namely, the cellular chain complex of a mapping cylinder of a cellular map f of CW-complexes is the mapping cylinder of the chain map induced by f. As suggested already from the geometric picture, there exists obvious exact sequences such as $0 \to C_* \to \mathrm{cyl}_*(f_*) \to \mathrm{cone}_*(f_*) \to 0$ and $0 \to D_* \to \mathrm{cone}_*(f_*) \to \Sigma C_* \to 0$.

A *chain contraction* γ_* for an R-chain complex C_* is a collection of R-homomorphisms $\gamma_n \colon C_n \to C_{n+1}$ for $n \in \mathbb{Z}$ satisfying $c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$. We call a finite free R-chain complex *based free* if each R-chain module C_n comes with a preferred (finite ordered) basis. Suppose that C_* is a finite based free R-chain complex which is *contractible*, i.e., which possesses a chain contraction. Put $C_{\mathrm{odd}} = \oplus_{n \in \mathbb{Z}} C_{2n+1}$ and $C_{\mathrm{ev}} = \oplus_{n \in \mathbb{Z}} C_{2n}$. Let γ_* and δ_* be two chain contractions. Define R-homomorphisms

$$(c_* + \gamma_*)_{\mathrm{odd}} : C_{\mathrm{odd}} \to C_{\mathrm{ev}}; \tag{3.30}$$

$$(c_* + \delta_*)_{\mathrm{ev}} : C_{\mathrm{ev}} \to C_{\mathrm{odd}}. \tag{3.31}$$

Let A be the matrix of $(c_* + \gamma_*)_{\mathrm{odd}}$ with respect to the given bases. Let B be the matrix of $(c_* + \delta_*)_{\mathrm{ev}}$ with respect to the given bases. We define $\mu_n := (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n$ and $\nu_n := (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n$. One easily checks that the endomorphisms $(\mathrm{id} + \mu_*)_{\mathrm{odd}}$, $(\mathrm{id} + \nu_*)_{\mathrm{ev}}$, $(c_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (c_* + \delta_*)_{\mathrm{ev}}$, and $(c_* + \delta_*)_{\mathrm{ev}} \circ (\mathrm{id} + \nu_*)_{\mathrm{ev}} \circ (c_* + \gamma_*)_{\mathrm{odd}}$ are given by upper triangular matrices whose diagonal entries are identity maps. Hence A and B are invertible and their classes $[A], [B] \in \widetilde{K}_1(R)$ satisfy $[A] = -[B]$. Since $[B]$ is independent of the choice of γ_*, the same is true for $[A]$. Thus we can associate to a finite based free contractible R-chain complex C_* an element

$$\tau(C_*) := [A] \quad \in \widetilde{K}_1(R). \tag{3.32}$$

Let $f_*\colon C_* \to D_*$ be a homotopy equivalence of finite based free R-chain complexes. Its mapping cone $\operatorname{cone}(f_*)$ is a contractible finite based free R-chain complex. Define the *Whitehead torsion* of f_* by

$$\tau(f_*) := \tau(\operatorname{cone}_*(f_*)) \quad \in \widetilde{K}_1(R). \tag{3.33}$$

Now we can pass to CW-complexes. Let $f\colon X \to Y$ be a cellular homotopy equivalence of connected finite CW-complexes. Let $p_X\colon \widetilde{X} \to X$ and $p_Y\colon \widetilde{Y} \to Y$ be the universal coverings. Identify $\pi_1(Y)$ with $\pi_1(X)$ using the isomorphism induced by f. (We ignore base point questions here and in the sequel. This can be done since an inner automorphisms of a group G induces the identity on $K_1(\mathbb{Z}G)$ and hence also on $\operatorname{Wh}(G)$.) There is a lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ which is $\pi_1(Y)$-equivariant. It induces a $\mathbb{Z}\pi_1(Y)$-chain homotopy equivalence $C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$. The CW-structure defines a basis for each $\mathbb{Z}\pi_1(Y)$-chain module $C_n(\widetilde{X})$ and $C_n(\widetilde{Y})$ which is unique up to multiplying each basis element with a unit of the form $\pm g \in \mathbb{Z}\pi_1(Y)$ and permuting the elements of the basis. Pick such a *cellular basis* for each chain module. We can apply (3.33) to it and thus obtain an element in $\widetilde{K}_1(\mathbb{Z}\pi_1(Y))$. Its image under the projection $\widetilde{K}_1(\mathbb{Z}\pi_1(Y)) \to \operatorname{Wh}(\pi_1(Y))$ is denoted by

$$\tau(f) \in \operatorname{Wh}(\pi_1(Y)). \tag{3.34}$$

Since we consider $\tau(f)$ in $\operatorname{Wh}(\pi_1(Y))$, the choice of the cellular basis does not matter anymore.

Given a (not necessarily cellular) homotopy equivalence of connected finite CW-complexes $f\colon X \to Y$, we can defines its Whitehead torsion $\tau(f)$ as follows. We can choose by the Cellular Approximation Theorem a cellular map $f'\colon X \to Y$ that is homotopic to f, and define the Whitehead torsion $\tau(f)$ by $\tau(f')$. Since the Whitehead torsion of two cellular maps which are homotopic and hence even cellularly homotopic by the Cellular Approximation Theorem, agrees, this is independent of the choice of f'.

If $f\colon X \to Y$ is a homotopy equivalence of finite CW-complexes, then define $\operatorname{Wh}(\pi_1(Y)) := \bigoplus_{C \in \pi_0(Y)} \operatorname{Wh}(\pi_1(C))$ and $\tau(f) \in \operatorname{Wh}(\pi_1(Y))$ by the collection of the Whitehead torsions of the homotopy equivalences induce between path components. Obviously a map $g\colon Y_1 \to Y_2$ induces a homomorphism of abelian groups $g_*\colon \operatorname{Wh}(\pi_1(Y_1)) \to \operatorname{Wh}(\pi_1(Y_2))$ by the homomorphisms between the various fundamental groups of the path components induced by g.

Definition 3.35 (Whitehead torsion). We call $\tau(f)$ the (algebraic) *Whitehead torsion* of the homotopy equivalence $f\colon X \to Y$ of finite CW-complexes.

Exercise 3.36. Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ be an exact sequence of projective R-chain complexes. Suppose that E_* is contractible. Construct an R-chain map $s_* \colon E_* \to D_*$ such that $p_* \circ s_* = \mathrm{id}_{E_*}$. Show that $i_* \oplus s_* \colon C_* \oplus E_* \to D_*$ is an isomorphism of R-chain complexes. Give a counterexample to the conclusion if one drops the condition that E_* is contractible.

The basic properties of this invariant are summarized in the following theorem whose proof can be found for instance in [233, (22.1), (22.4), (23.1), and (23.2)], [652, Chapter 3] or [633, Chapter 2].

Theorem 3.37 (Basic properties of Whitehead torsion).

(i) Sum formula

Let the following two diagrams be pushouts of finite CW-complexes

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} Y_0 \arrow[r, "k_1"] \arrow[d, "k_2"'] & Y_1 \arrow[d, "l_1"] \\ Y_2 \arrow[r, "l_2"'] & Y \end{tikzcd}$$

where the left vertical arrows are inclusions of CW-complexes, the upper horizontal maps are cellular, and X and Y are equipped with the induced CW-structure. Let $f_i \colon X_i \to Y_i$ be homotopy equivalences for $i = 0, 1, 2$ satisfying $f_1 \circ i_1 = k_1 \circ f_0$ and $f_2 \circ i_2 = k_2 \circ f_0$. Put $l_0 = l_1 \circ k_1 = l_2 \circ k_2$. Denote by $f \colon X \to Y$ the map induced by f_0, f_1, and f_2 and the pushout property.

Then f is a homotopy equivalence and

$$\tau(f) \;=\; (l_1)_*\tau(f_1) + (l_2)_*\tau(f_2) - (l_0)_*\tau(f_0);$$

(ii) Homotopy invariance

Let $f \simeq g \colon X \to Y$ be homotopic maps of finite CW-complexes. Then the homomorphisms $f_*, g_* \colon \mathrm{Wh}(\pi_1(X)) \to \mathrm{Wh}(\pi_1(Y))$ agree. If additionally f and g are homotopy equivalences, then we obtain

$$\tau(g) \;=\; \tau(f);$$

(iii) Composition formula

Let $f \colon X \to Y$ and $g \colon Y \to Z$ be homotopy equivalences of finite CW-complexes. Then we get

$$\tau(g \circ f) \;=\; g_*\tau(f) + \tau(g);$$

(iv) Product formula

Let $f \colon X' \to X$ and $g \colon Y' \to Y$ be homotopy equivalences of connected finite CW-complexes. Then

$$\tau(f \times g) \;=\; \chi(X) \cdot j_* \tau(g) + \chi(Y) \cdot i_* \tau(f)$$

where $\chi(X), \chi(Y) \in \mathbb{Z}$ denote the Euler characteristics, $j_* \colon \mathrm{Wh}(\pi_1(Y)) \to \mathrm{Wh}(\pi_1(X \times Y))$ is the homomorphism induced by $j \colon Y \to X \times Y, y \mapsto (y, x_0)$ for some base point $x_0 \in X$ and i_* is defined analogously.

Let X be a finite simplicial complex. Let X' be its barycentric subdivision. Then one can show $\tau(f) = 0$ for the map $f \colon X \to X'$ whose underlying map of spaces is the identity. However, if X_1 and X_2 are two finite CW-complexes with the same underlying space, it is not at all clear that $\tau(f) = 0$ holds for the map $f \colon X_1 \to X_2$ whose underlying map of spaces is the identity. This problem is solved by the following (in comparison with the other statements above much deeper) result due to Chapman [215], [216], see also [233, Appendix] and [785, Section 5].

Theorem 3.38 (Topological invariance of Whitehead torsion). *The Whitehead torsion of a homeomorphism $f \colon X \to Y$ of finite CW-complexes vanishes.*

3.4 Geometric Interpretation of Whitehead Group and Whitehead Torsion

In this section we introduce the concept of a simple homotopy equivalence $f \colon X \to Y$ of finite CW-complexes geometrically. We will show that the obstruction for a homotopy equivalence $f \colon X \to Y$ of finite CW-complexes to be simple is the Whitehead torsion.

We have the inclusion of spaces $S^{n-2} \subset S^{n-1}_+ \subset S^{n-1} \subset D^n$ where $S^{n-1}_+ \subset S^{n-1}$ is the upper hemisphere. The pair (D^n, S^{n-1}_+) carries an obvious relative CW-structure. Namely, attach an $(n-1)$-cell to S^{n-1}_+ by the attaching map $\mathrm{id} \colon S^{n-2} \to S^{n-2}$ to obtain S^{n-1}. Then we attach to S^{n-1} an n-cell by the attaching map $\mathrm{id} \colon S^{n-1} \to S^{n-1}$ to obtain D^n. Let X be a CW-complex. Let $q \colon S^{n-1}_+ \to X$ be a map satisfying $q(S^{n-2}) \subset X_{n-2}$ and $q(S^{n-1}_+) \subset X_{n-1}$. Let Y be the space $D^n \cup_q X$, i.e., the pushout

$$\begin{tikzcd}
S^{n-1}_+ \arrow[r, "q"] \arrow[d, "i"'] & X \arrow[d, "j"] \\
D^n \arrow[r, "g"'] & Y
\end{tikzcd}$$

where i is the inclusion. Then Y inherits a CW-structure by putting $Y_k = j(X_k)$ for $k \leq n-2$, $Y_{n-1} = j(X_{n-1}) \cup g(S^{n-1})$ and $Y_k = j(X_k) \cup g(D^n)$ for $k \geq n$. Note that Y is obtained from X by attaching one $(n-1)$-cell and one n-cell. Since the map $i \colon S^{n-1}_+ \to D^n$ is a homotopy equivalence and cofibration, the map $j\colon X \to Y$ is a homotopy equivalence and a cofibration. We call j an *elementary expansion* and say that Y is obtained from X by an elementary expansion. There is a map $r\colon Y \to X$ with $r \circ j = \mathrm{id}_X$. This map is unique up to homotopy relative $j(X)$. We call any such map an *elementary collapse* and say that X is obtained from Y by an elementary collapse.

Definition 3.39 (Simple homotopy equivalence). Let $f\colon X \to Y$ be a map of finite CW-complexes. We call it a simple *homotopy equivalence* if there is a sequence of maps

$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each f_i is an elementary expansion or elementary collapse and f is homotopic to the composition of the maps f_i.

Remark 3.40 (Combinatorial meaning of simple homotopy equivalence). The idea of the definition of a simple homotopy equivalence is that such a map can be written as a composition of elementary maps, namely, elementary expansions and collapses, which are obviously homotopy equivalences and in some sense the smallest and most elementary steps to pass from one finite CW-complex to another without changing the homotopy type. If one works with simplicial complexes, an elementary map has a purely combinatorial description. An elementary collapse means to delete a simplex and one of its faces that is not shared by another simplex. So one can describe the passage from one finite simplicial complex to another coming from a simple homotopy equivalence by finitely many combinatorial data. This does not work for two finite simplicial complexes that are homotopy equivalent but not simple homotopy equivalent.

This approach is similar to the idea in knot theory that two knots are equivalent if one can pass from one knot to the other by a sequence of elementary moves, the so-called Reidemeister moves. A Reidemeister move obviously does not change the equivalence class of a knot and, indeed, it turns out that one can pass from one knot to a second knot by a sequence of Reidemeister moves if and only if the two knots are equivalent, see for instance [174, Chapter 1] or [970]. The analogous statement is not true for homotopy equivalences $f\colon X \to Y$ of finite CW-complexes because there is an obstruction for f to be simple, namely, its Whitehead torsion.

Exercise 3.41. Consider the simplicial complex X with four vertices v_0, v_1, v_2, and v_3, the edges $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\{v_0, v_2\}$ and $\{v_2, v_3\}$ and one 2-simplex $\{v_0, v_1, v_2\}$. Describe a sequence of elementary collapses and expansions transforming it to the one-point-space $\{\bullet\}$.

Recall that the *mapping cylinder* $\mathrm{cyl}(f)$ of a map $f\colon X \to Y$ is defined by the pushout

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & \mathrm{cyl}(f). \end{array}$$

There are natural inclusions $i_X : X = X \times \{1\} \to \mathrm{cyl}(f)$ and $i_Y : Y \to \mathrm{cyl}(f)$ and a natural projection $p\colon \mathrm{cyl}(f) \to Y$. Note that i_X is a cofibration and $p \circ i_X = f$ and $p_Y \circ i_Y = \mathrm{id}_Y$. Define *the mapping cone* $\mathrm{cone}(f)$ by the quotient $\mathrm{cyl}(f)/i_X(X)$.

Lemma 3.42. *Let $f\colon X \to Y$ be a cellular map of finite CW-complexes and $A \subset X$ be a CW-subcomplex. Then the inclusion $\mathrm{cyl}(f|_A) \to \mathrm{cyl}(f)$ and (in the case $A = \emptyset$) $i_Y : Y \to \mathrm{cyl}(f)$ is a composition of elementary expansions and hence a simple homotopy equivalence.*

Proof. It suffices to treat the case where X is obtained from A by attaching an n-cell by an attaching map $q\colon S^{n-1} \to X$. Then there is an obvious pushout

$$\begin{array}{ccc} S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} & \longrightarrow & \mathrm{cyl}(f|_A) \\ \downarrow & & \downarrow \\ D^n \times [0,1] & \longrightarrow & \mathrm{cyl}(f) \end{array}$$

and an obvious homeomorphism

$$(D^n \times [0,1], S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\}) \to (D^{n+1}, S^n_+).$$

□

Lemma 3.43. *A map $f\colon X \to Y$ of finite CW-complexes is a simple homotopy equivalence if and only if $i_X\colon X \to \mathrm{cyl}(f)$ is a simple homotopy equivalence.*

Proof. This follows from Lemma 3.42 since a composition of simple homotopy equivalence and a homotopy inverse of a simple homotopy equivalence is again a simple homotopy equivalence. □

We only sketch the proof of the next result. More details can be found for instance in [233, (22.2)] or [633, Chapter 2]. However, we try to give enough information about its proof to illustrate why the geometric problem to decide whether a homotopy equivalence is simple, is equivalent to a question about an invertible matrix A, which has a positive answer precisely if the class of A vanishes in the Whitehead group. Then the key will be Remark 3.13.

Theorem 3.44 (Whitehead torsion and simple homotopy equivalences).

(i) Let X be a finite CW-complex. Then for any element $x \in \mathrm{Wh}(\pi_1(X))$ there is an inclusion $i\colon X \to Y$ of finite CW-complexes such that i is a homotopy equivalence and $i_*^{-1}(\tau(i)) = x$;

(ii) Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Then its Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ vanishes if and only if f is a simple homotopy equivalence.

Proof. (i) We can assume without loss of generality that X is connected. Put $\pi = \pi_1(X)$. Choose an element $A \in \mathrm{GL}_n(\mathbb{Z}\pi)$ representing $x \in \mathrm{Wh}(\pi)$. Choose $n \geq 2$. In the sequel we fix a zero-cell in X as base point. Put $X' = X \vee \vee_{j=1}^n S^n$. Let $b_j \in \pi_n(X')$ be the element represented by the inclusion of the j-th copy of S^n into X' for $j = 1, 2 \ldots, n$. Recall that $\pi_n(X')$ is a $\mathbb{Z}\pi$-module. Choose for $i = 1, 2 \ldots, n$ a map $f_i\colon S^n \to X'_n$ such that $[f_i] = \sum_{j=1}^n a_{i,j} \cdot b_j$ holds in $\pi_n(X')$. Attach to X' for each $i \in \{1, 2 \ldots, n\}$ an $(n+1)$-cell by $f_i\colon S^n \to X'_n$. Let Y be the resulting CW-complex and $i\colon X \to Y$ be the inclusion. Then i is an inclusion of finite CW-complexes and induces an isomorphism on the fundamental groups. In the sequel we identify π and $\pi_1(Y)$ by $\pi_1(i)$. The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y}, \widetilde{X})$ is concentrated in dimensions n and $(n+1)$ and its $(n+1)$-differential is given by the matrix A with respect to the cellular basis. Hence $C_*(\widetilde{Y}, \widetilde{X})$ is a contractible finite based free $\mathbb{Z}\pi$-chain complex with $\tau(C_*(\widetilde{Y}, \widetilde{X})) = [A]$ in $\mathrm{Wh}(\pi)$. This implies that $i\colon X \to Y$ is a homotopy equivalence with $i_*^{-1}(\tau(i)) = x$.

(ii) Suppose that f is a simple homotopy equivalence. We want to show $\tau(f) = 0$. Because of Theorem 3.37 (iii) it suffices to prove for an elementary expansion $j\colon X \to Y$ that its Whitehead torsion $\tau(j) \in \mathrm{Wh}(Y)$ vanishes. We can assume without loss of generality that Y is connected. In the sequel we write $\pi = \pi_1(Y)$ and identify $\pi = \pi_1(X)$ by $\pi_1(f)$. The following diagram of based free finite $\mathbb{Z}\pi$-chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(\widetilde{X}) \arrow[r, "C_*(\widetilde{j})"] & C_*(\widetilde{Y}) \arrow[r, "\mathrm{pr}_*"] & C_*(\widetilde{Y}, \widetilde{X}) \arrow[r] & 0 \\
0 \arrow[r] & C_*(\widetilde{X}) \arrow[r, "\mathrm{id}_*"] \arrow[u, "\mathrm{id}_*"] & C_*(\widetilde{X}) \arrow[r, "\mathrm{pr}_*"] \arrow[u, "C_*(\widetilde{j})"'] & 0 \arrow[r] \arrow[u, "0_*"] & 0
\end{tikzcd}$$

has based exact rows and $\mathbb{Z}\pi$-chain homotopy equivalences as vertical arrows. Elementary facts about chain complexes, in particular the conclusion from Exercise 3.36, imply

$$\begin{aligned}
\tau\bigl(C_*(\widetilde{j})\bigr) &= \tau\bigl(\mathrm{id}_*\colon C_*(\widetilde{X}) \to C_*(\widetilde{X})\bigr) + \tau\bigl(0_*\colon 0 \to C_*(\widetilde{Y}, \widetilde{X})\bigr) \\
&= 0 + \tau\bigl(C_*(\widetilde{Y}, \widetilde{X})\bigr) = \tau\bigl(C_*(\widetilde{Y}, \widetilde{X})\bigr).
\end{aligned}$$

The $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y}, \widetilde{X})$ is concentrated in two consecutive dimensions and its only non-trivial differential is $\mathrm{id}\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ if we identify the two non-trivial $\mathbb{Z}\pi$-chain modules with $\mathbb{Z}\pi$ using the cellular basis. This implies $\tau(C_*(\widetilde{Y},\widetilde{X})) = 0$ and hence $\tau(j) := \tau(C_*(\widetilde{j})) = 0$.

Now suppose that $\tau(f) = 0$. We want to show that f is simple. We can assume without loss of generality that X is connected, otherwise treat each path component separately. Because of Lemma 3.43 we can assume that f is an inclusion $i\colon X \to Y$ of connected finite CW-complexes which is a homotopy equivalence. We have to show that we can achieve by a sequence of elementary collapses and expansions that $Y = X$, i.e., we must get rid of all the cells in $Y - X$.

Since $\chi(X) = \chi(Y)$, it is clear that one cannot remove a single cell, this always has to be done in pairs. In the first step one shows for an n-dimensional cell e_n that one can attach one new $(n+1)$-cell e_{n+1} and a new $(n+2)$-cell e_{n+2} by an elementary expansion and then get rid of e_n and e_{n+1} by an elementary collapse. The outcome is that one can replace an n-cell by a $(n+2)$-cell. Analogously one can show that one can replace an $(n+2)$-cell by a n-cell. Thus one can arrange for some integer $n \geq 2$ that Y is obtained from X by attaching k cells of dimension n trivially and then attaching k cells of dimension $(n+1)$. Hence the cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ is concentrated in dimension n and $(n+1)$. After we have picked a cellular basis, its $(n+1)$-differential is given by an invertible (k,k)-matrix A. By definition $\tau(f)$ is the class of this matrix in $\mathrm{Wh}(\pi)$. In Remark 3.13 we have described what $\tau(f) = [A] = 0$ means, namely, there is a sequence of operations that transform A to the empty matrix. Note that $X = Y$ holds if and only if A is the empty matrix. Now the main idea is to show that each of this operations can be realized by elementary expansions and collapses. □

Next we describe the Whitehead group geometrically. Fix a finite CW-complex X. Consider two pairs of finite CW-complexes (Y,X) and (Z,X) such that the inclusions of X into Y and Z are homotopy equivalences. We call them equivalent if there is a chain of pairs of finite CW-complexes

$$(Y,X) = (Y[0],X),(Y[1],X),(Y[2],X),\ldots,(Y[n],X) = (Z,X)$$

such that for each $k \in \{1,2,\ldots,n\}$ either $Y[k]$ is obtained from $Y[k-1]$ by an elementary expansion or $Y[k-1]$ is obtained from $Y[k]$ by an elementary expansion. Denote by $\mathrm{Wh}^{\mathrm{geo}}(X)$ the equivalence classes $[Y,X]$ of such pairs (Y,X). This becomes an abelian group under the addition $[Y,X]+[Z,X] := [Y \cup_X Z, X]$. The zero element is given by $[X,X]$. The inverse of $[Y,X]$ is constructed as follows. Choose a map $r\colon Y \to X$ with $r_X = \mathrm{id}$. Let $p\colon X \times [0,1] \to X$ be the projection. Then $[(\mathrm{cyl}(r) \cup_p X) \cup_r X, X] + [Y,X] = 0$. A map $g\colon X \to X'$ induces a homomorphism $g_*\colon \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}^{\mathrm{geo}}(X')$ by sending $[Y,X]$ to $[Y \cup_g X', X']$. We obviously have $\mathrm{id}_* = \mathrm{id}$ and $(g \circ h)_* = g_* \circ h_*$. In other words, we obtain a covariant functor on the category of finite CW-complexes with values in abelian groups. More information about this construction can be found for instance in [233, § 6 in Chapter II].

Given a homotopy equivalence of finite CW-complexes $f\colon X \to Y$, define its *geometric Whitehead torsion* $\tau^{\mathrm{geo}}(f) \in \mathrm{Wh}^{\mathrm{geo}}(X)$ to be the class of $(\mathrm{cyl}(f), X)$. Because of Lemma 3.43 we have $\tau^{\mathrm{geo}}(f) = 0$ if and only f is a simple homotopy equivalence

The next result is essentially a consequence of Theorem 3.44. Details of its proof can be found in [233, §21].

Theorem 3.45 (Geometric and algebraic Whitehead groups).

(i) Let X be a finite CW-complex. The map

$$\tau\colon\ \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}(\pi_1(X))$$

sending $[Y, X]$ to $i_*^{-1}\tau(i)$ for the inclusion $i\colon X \to Y$ is a natural isomorphism of abelian groups.
It sends $\tau^{\mathrm{geo}}(f)$ to $f_*^{-1}\tau(f)$ for a homotopy equivalence $f\colon X \to Y$ of finite CW-complexes.

(ii) A homotopy equivalence $f\colon X \to Y$ is a simple homotopy equivalence if and only if $\tau(f) \in \mathrm{Wh}(Y)$ vanishes.

Exercise 3.46. Let Y be a simply connected finitely dominated CW-complex. Show that there exists a finite CW-complex X and a homotopy equivalence $f\colon X \to Y$. Prove that for any two finite CW-complexes X_0 and X_1 and homotopy equivalences $f_i\colon X_i \to Y$ for $i = 0, 1$ there exists a simple homotopy equivalence $g\colon X_0 \to X_1$ with $f_1 \circ g \simeq f_0$.

3.5 The s-Cobordism Theorem

One of the main applications of Whitehead torsion is the theorem below.

Theorem 3.47 (s-Cobordism Theorem). *Let M_0 be a connected closed manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M_0)$. Then:*

(i) Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ vanishes;

(ii) For any $x \in \mathrm{Wh}(\pi)$ there is an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 with $\tau(W, M_0) = x \in \mathrm{Wh}(\pi)$;

(iii) The function assigning to an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of h-cobordisms over M_0 to the Whitehead group $\mathrm{Wh}(\pi)$.

Here are some explanations. An *n-dimensional cobordism* (sometimes also called just bordism) $(W; M_0, f_0, M_1, f_1)$ consists of a compact n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0 and M_1, a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ of the boundary ∂W of W, and diffeomorphisms $f_0 \colon M_0 \to \partial W_0$ and $f_1 \colon M_1 \to \partial W_1$. If we want to specify M_0, we say that W is a *cobordism over* M_0. If f_0 and f_1 are obvious from the context, we briefly write $(W; \partial_0 W, \partial_1 W)$. We call a cobordism $(W; M_0, f_0, M_1, f_1)$ an *h-cobordism*, if the inclusions $\partial_i W \to W$ for $i = 0, 1$ are homotopy equivalences. Two cobordisms $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0, f_0', M_1', f_1')$ over M_0 are *diffeomorphic relative* M_0 if there is a diffeomorphism $F \colon W \to W'$ with $F \circ f_0 = f_0'$. We call an h-cobordism over M_0 *trivial* if it is diffeomorphic relative $M_0 = M_0 \times \{0\}$ to the trivial h-cobordism $(M_0 \times [0,1]; M_0 \times \{0\}, (M_0 \times \{1\}))$. Note that the choice of the diffeomorphisms f_i do play a role although they are often suppressed in the notation.

The *Whitehead torsion* of an h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0)) \tag{3.48}$$

is defined to be the preimage of the Whitehead torsion, see Definition 3.35,

$$\tau\left(M_0 \xrightarrow{f_0} \partial_0 W \xrightarrow{i_0} W\right) \in \mathrm{Wh}(\pi_1(W))$$

under the isomorphism $(i_0 \circ f_0)_* \colon \mathrm{Wh}(\pi_1(M_0)) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(W))$ where the map $i_0 \colon \partial_0 W \to W$ is the inclusion. Here we use the fact that each smooth closed manifold has a CW-structure, which comes for instance from a smooth triangulation, or that each closed topological manifold of dimension different from 4 has a CW-structure, which comes from a handlebody decomposition, and that the choice of CW-structure does not matter by the topological invariance of the Whitehead torsion, see Theorem 3.38.

The idea of the proof of Theorem 3.47 is analogous to the one of Theorem 3.44, but now one uses a handlebody decomposition instead of a CW-structure and carries out the manipulation for handlebodies instead of cells. Here a handlebody of index k corresponds to a k-dimensional cell. More details can be found for instance in [652, Chapter 2].

The h-Cobordism Theorem 3.50 is due to Smale [903]. The s-Cobordism Theorem 3.47 is due to Barden, Mazur, Stallings, see [65, 696]. In the PL category proofs can be found in [861, 6.19 on page 88]. Its topological version follows from Kirby and Siebenmann [562, Conclusion 7.4 on page 320]. More information about the s-Cobordism Theorem can be found for instance in [558], [710], [861, page 87-90]. The s-Cobordism Theorem is known to be false for $\dim(M_0) = 4$ in general, by the work of Donaldson [298], but it is true for $n = \dim(M_0) = 4$ for good fundamental groups in the topological category by results of Quinn and Freedman [112, 386, 387, 388]. Counterexamples in the case $\dim(M_0) = 3$ are constructed by Matsumoto and Siebenmann [695] and Cappell and Shaneson [194] where the relevant 4-dimensional s-cobordism is a topological manifold. It is not known whether one can choose the s-cobordism to be smooth in these counterexamples. It follows from Kwasik and Schultz [581] and Perelman's proof of the Thurston

Geometrization Conjecture, see [563, 734], that every h-cobordism between two orientable closed 3-manifolds is an s-cobordism.

Exercise 3.49. Show for $n \geq 6$ that there exists an n-dimensional h-cobordism $(W; M_0, M_1)$ which is not trivial such that the h-cobordism $(W \times S^3; M_0 \times S^3, M_1 \times S^3)$ is trivial.

Since the Whitehead group of the trivial group vanishes, see Theorem 3.17, the s-Cobordism Theorem 3.47 implies, see also [710],

Theorem 3.50 (h-Cobordism Theorem). *Let M_0 be a simply connected closed n-dimensional manifold with $\dim(M_0) \geq 5$. Then every h-cobordism $(W; M_0, f_0, M_1, f_1)$ over M_0 is trivial.*

Theorem 3.51 (Poincaré Conjecture). *The Poincaré Conjecture is true for a closed n-dimensional manifold M with $\dim(M) \geq 5$, namely, if M is simply connected and its homology $H_p(M)$ is isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$, then M is homeomorphic to S^n.*

Proof. We only give the proof for $\dim(M) \geq 6$. Since M is simply connected and $H_*(M) \cong H_*(S^n)$, one can conclude from the Hurewicz Theorem and Whitehead Theorem [987, Theorem IV.7.13 on page 181 and Theorem IV.7.17 on page 182] that there is a homotopy equivalence $f\colon M \to S^n$. Let $D_i^n \subset M$ for $i = 0, 1$ be two embedded disjoint disks. Let W be obtained from M by removing the interior of the two disks D_0^n and D_1^n. Then W turns out to be a simply connected h-cobordism. Hence we can find because of Theorem 3.50 a homeomorphism $F\colon (\partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$ that is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$ and induces some (unknown) homeomorphism $f_1\colon \partial D_0^n \times \{1\} \to \partial D_1^n$. By the *Alexander trick* one can extend $f_1\colon \partial D_0^n = \partial D_0^n \times \{1\} \to \partial D_1^n$ to a homeomorphism $\overline{f_1}\colon D_0^n \to D_1^n$. Namely, any homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f}\colon D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h\colon D_0^n \times \{0\} \cup_{i_0} \partial D_0^n \times [0,1] \cup_{i_1} D_0^n \times \{1\} \to M$ for the canonical inclusions $i_k\colon \partial D_0^n \times \{k\} \to \partial D_0^n \times [0,1]$ for $k = 0, 1$ by $h|_{D_0^n \times \{0\}} = \mathrm{id}$, $h|_{\partial D_0^n \times [0,1]} = F$ and $h|_{D_0^n \times \{1\}} = \overline{f_1}$. Since the source of h is obviously homeomorphic to S^n, Theorem 3.51 follows.

In the case $\dim(M) = 5$ one uses the fact that M is the boundary of a contractible 6-dimensional manifold W and applies Theorem 3.50 to W with an embedded disc removed. □

The Poincaré Conjecture, see Theorem 3.51, is known in all dimensions, where dimension 3 is due to the work of Perelman, see [563, 733, 734, 787, 788, 789], and dimension 4 is due to Freedman, see [112, 386, 387, 388]. The first poof of the Poincaré Conjecture in the topological category in dimension ≥ 5 was given by Newman [741] using engulfing theory. The smooth version of the Poincaré Conjecture holds in dimensions ≤ 3, is open in dimension 4, and is discussed in dimensions ≥ 5 for instance in [652, Remark 12.26 on page 455].

Remark 3.52 (Exotic Spheres). Note that the proof of the Poincaré Conjecture in Theorem 3.51 works only in the topological category but not in the smooth category. In other words, we cannot conclude the existence of a diffeomorphism $h\colon S^n \to M$. The proof in the smooth case breaks down when we apply the Alexander trick. The construction of $\overline{f}$ given by coning f yields only a homeomorphism $\overline{f}$ and not a diffeomorphism, even if we start with a diffeomorphism f. The map $\overline{f}$ is smooth outside the origin of D^n but not necessarily at the origin. Indeed, not every diffeomorphism $f\colon S^{n-1} \to S^{n-1}$ can be extended to a diffeomorphism $D^n \to D^n$ and there exist so-called *exotic spheres*, i.e., closed manifolds that are homeomorphic to S^n but not diffeomorphic to S^n. The classification of these exotic spheres is one of the early very important achievements of surgery theory and one motivation for its further development. For more information about exotic spheres we refer for instance to [559], [595], [612], [633, Chapter 6] and [652, Chapter 12].

Remark 3.53 (The Surgery Program). In some sense the s-Cobordism Theorem 3.47 is one of the first theorems where diffeomorphism classes of certain manifolds are determined by an algebraic invariant, namely, the Whitehead torsion. Moreover, the Whitehead group $\mathrm{Wh}(\pi)$ depends only on the fundamental group $\pi = \pi_1(M_0)$, whereas the diffeomorphism classes of h-cobordisms over M_0 a priori depend on M_0 itself. The s-Cobordism Theorem 3.47 is one step in a program to decide whether two closed manifolds M and N are diffeomorphic what is in general a very hard question. The idea is to construct an h-cobordism $(W; M, f, N, g)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M what implies that M and N are diffeomorphic. So the *Surgery Program* is:

(i) Construct a simple homotopy equivalence $f\colon M \to N$;
(ii) Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id})\colon (W; M, N) \to (N \times [0,1], N \times \{0\}, N \times \{1\})$;
(iii) Modify W and F relative boundary by so-called surgery such that F becomes a simple homotopy equivalence and thus W becomes an h-cobordism whose Whitehead torsion is trivial.

The advantage of this approach will be that it can be reduced to problems in homotopy theory and algebra, which can sometimes be handled by well-known techniques. In particular one will get sometimes computable obstructions for two homotopy equivalent manifolds to be diffeomorphic. Often surgery theory has proved to be very useful when one wants to distinguish two closed manifolds which have very similar properties. The classification of homotopy spheres is one example. Moreover, surgery techniques can be applied to problems that are of different nature than of diffeomorphism or homeomorphism classifications, for instance for the construction of group actions.

More information about surgery theory will be given in Chapter 9.

3.6 Reidemeister Torsion and Lens Spaces

In this section we briefly deal with Reidemeister torsion, which was defined earlier than Whitehead torsion and motivated the definition of Whitehead torsion. Reidemeister torsion was the first invariant in algebraic topology that could distinguish between spaces which are homotopy equivalent but not homeomorphic. Namely, it can be used to classify lens spaces up to homeomorphism, see Reidemeister [831]. We will give no proofs. More information and complete proofs can be found in [233, Chapter V], [633, Section 2.4], and [652, Section 3.5].

Let X be a finite CW-complex with fundamental group π. Let U be an orthogonal finite dimensional π-representation. Denote by $H_p(X;U)$ the homology of X with coefficients in U, i.e., the homology of the $\mathbb{R}$-chain complex $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$. Suppose that X is *U-acyclic*, i.e., $H_n(X;U) = 0$ for all $n \geq 0$. If we fix a cellular basis for $C_*(\widetilde{X})$ and some orthogonal $\mathbb{R}$-basis for U, then $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ is a contractible based free finite $\mathbb{R}$-chain complex and yields an element $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$, see (3.32). Define the *Reidemeister torsion*

$$\rho(X;U) \in \mathbb{R}^{>0} \tag{3.54}$$

to be the image of $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ under the homomorphism $\widetilde{K}_1(\mathbb{R}) \to \mathbb{R}^{>0}$ sending the class $[A]$ of $A \in \mathrm{GL}_n(\mathbb{R})$ to $|\det(A)|$. Note that for any trivial unit $\pm\gamma$ the automorphism of U given by multiplication with $\pm\gamma$ is orthogonal and that the absolute value of the determinant of any orthogonal automorphism of U is 1. Therefore $\rho(X;U) \in \mathbb{R}^{>0}$ is independent of the choice of cellular basis for $C_*(\widetilde{X})$ and the orthogonal basis for U, and hence is an invariant of the CW-complex X and the orthogonal representation U.

We state without proof the next result, which essential says that Whitehead torsion of a homotopy equivalence is related to the difference of Reidemeister torsion of the target and the source when defined.

Lemma 3.55. *Let $f\colon X \to Y$ be a homotopy equivalence of connected finite CW-complexes and let U be an orthogonal finite dimensional $\pi = \pi_1(Y)$-representation. Suppose that Y is U-acyclic. Let f^*U be the orthogonal $\pi_1(X)$-representation obtained from U by restriction with the isomorphism $\pi_1(f)$. Let $d_U\colon \mathrm{Wh}(\pi(Y)) \to \mathbb{R}^{>0}$ be the map sending the class $[A]$ of $A \in \mathrm{GL}_n(\mathbb{Z}\pi_1(Y))$ to $|\det(\mathrm{id}_U \otimes_{\mathbb{Z}\pi} r_A\colon U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n \to U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n)|$.*

Then X is f^*U-acyclic and we get

$$\frac{\rho(Y;U)}{\rho(X;f^*U)} = d_U(\tau(f)).$$

Next we introduce lens spaces. Let G be a cyclic group of finite order $|G|$. Let V be a unitary finite dimensional G-representation. Define its *unit sphere*

SV and its *unit disk* DV to be the G-subspaces $SV = \{v \in V \mid ||v|| = 1\}$ and $DV = \{v \in V \mid ||u|| \leq 1\}$ of V. Note that a complex finite dimensional vector space has a preferred orientation as real vector space, namely, the one given by the $\mathbb{R}$-basis $\{b_1, ib_1, b_2, ib_2, \ldots, b_n, ib_n\}$ for any $\mathbb{C}$-basis $\{b_1, b_2, \ldots, b_n\}$. Any $\mathbb{C}$-linear automorphism of a complex finite dimensional vector space preserves this orientation. Thus SV and DV are oriented compact Riemannian manifolds with isometric orientation preserving G-action. We call a unitary G-representation V *free* if the induced G-action on its unit sphere SV is free. Then $SV \to G\backslash SV$ is a covering and the quotient space $L(V) := G\backslash SV$ inherits from SV the structure of an oriented closed Riemannian manifold.

Definition 3.56 (Lens space). We call the closed oriented Riemannian manifold $L(V)$ the *lens space* associated to the free finite dimensional unitary representation V of the finite cyclic group G.

Exercise 3.57. Show that the 3-dimensional real projective space $\mathbb{RP}^3$ is a lens space. Let $\mathbb{R}^-$ be the non-trivial orthogonal $\mathbb{Z}/2$-representation. Show that $\mathbb{RP}^3$ is $\mathbb{R}^-$-acyclic and compute the Reidemeister torsion $\rho(\mathbb{RP}^3; \mathbb{R}^-)$.

One can specify these lens spaces also by numbers as follows.

Notation 3.58. Let $\mathbb{Z}/t$ be the cyclic group of order $t \geq 2$. The 1-dimensional unitary representation V_k for $k \in \mathbb{Z}/t$ has as underlying vector space $\mathbb{C}$ and $l \in \mathbb{Z}/t$ acts on it by multiplication with $\exp(2\pi ikl/t)$. Note that V_k is free if and only if $k \in (\mathbb{Z}/t)^\times$, and is trivial if and only if $k = 0$ in $\mathbb{Z}/t$. Define the lens space $L(t; k_1, \ldots, k_c)$ for an integer $c \geq 1$ and elements $k_1, \ldots, k_c$ in $(\mathbb{Z}/t)^\times$ by $L(\oplus_{i=1}^c V_{k_i})$.

Lens spaces form a very interesting family of manifolds, which can be completely classified as we will see. Two lens spaces $L(V)$ and $L(W)$ of the same dimension $n \geq 3$ have the same homotopy groups, namely, their fundamental group is G and their p-th homotopy group is isomorphic to $\pi_p(S^n)$ for $p \geq 2$. They also have the same homology with integral coefficients, namely $H_p(L(V)) \cong \mathbb{Z}$ for $p \in \{0, n\}$, $H_p(L(V)) \cong G$ for p odd and $1 \leq p < n$, and $H_p(L(V)) = 0$ for all other values of p. Also their cohomology groups agree. Nevertheless not all of them are homotopy equivalent. Moreover, there are homotopy equivalent lens spaces that are not diffeomorphic, see Example 3.62.

We state without proof the following result.

Theorem 3.59 (Homotopy Classification of Lens Spaces). *The lens spaces $L(t; k_1, \ldots, k_c)$ and $L(t; l_1, \ldots, l_c)$ are homotopy equivalent if and only if there are $e \in (\mathbb{Z}/t)^\times$ and $\epsilon \in \{\pm 1\}$ satisfying $\prod_{i=1}^c k_i = \epsilon \cdot e^c \cdot \prod_{i=1}^c l_i$ in $(\mathbb{Z}/t)^\times$.*

The lens spaces $L(t; k_1, \ldots, k_c)$ and $L(t; l_1, \ldots, l_c)$ are oriented homotopy equivalent if and only if there is $e \in (\mathbb{Z}/t)^\times$ satisfying $\prod_{i=1}^c k_i = e^c \cdot \prod_{i=1}^c l_i$ in $(\mathbb{Z}/t)^\times$.

Theorem 3.60 (Diffeomorphism Classification of Lens Spaces).

(i) Let G be a finite cyclic group. Let $L(V)$ and $L(W)$ be two lens spaces of the same dimension $n \geq 3$. Then the following statements are equivalent:

(a) There is an automorphism $\alpha\colon G \to G$ such that V and α^*W are isomorphic as orthogonal G-representations;
(b) There is an isometric diffeomorphism $L(V) \to L(W)$;
(c) There is a diffeomorphism $L(V) \to L(W)$;
(d) There is a homeomorphism $L(V) \to L(W)$;
(e) There is a simple homotopy equivalence $L(V) \to L(W)$;
(f) There is an automorphism $\alpha\colon G \to G$ such that for any orthogonal finite dimensional representation U with $U^G = 0$

$$\rho(L(W); U) = \rho(L(V); \alpha^* U)$$

holds;
(g) There is an automorphism $\alpha\colon G \to G$ such that for any non-trivial 1-dimensional unitary G-representation U

$$\rho(L(W); \operatorname{res} U) = \rho(L(V); \alpha^* \operatorname{res} U)$$

holds where the orthogonal representation $\operatorname{res} U$ is obtained from U by restricting the scalar multiplication from $\mathbb{C}$ to $\mathbb{R}$;

(ii) Two lens spaces $L(t; k_1, \ldots, k_c)$ and $L(t; l_1, \ldots, l_c)$ are homeomorphic if and only if there are $e \in (\mathbb{Z}/t)^\times$, signs $\epsilon_i \in \{\pm 1\}$ and a permutation $\sigma \in \Sigma_c$ such that $k_i = \epsilon_i \cdot e \cdot l_{\sigma(i)}$ holds in $(\mathbb{Z}/t)^\times$ for $i = 1, 2, \ldots, c$.

Proof. We give only a sketch of the proof of assertion (i). Assertion (ii) is a direct consequence of assertion (i).

The implications (ia) $\Rightarrow$ (ib) $\Rightarrow$ (ic) $\Rightarrow$ (id) and (if) $\Rightarrow$ (ig) are obvious. The implication (id) $\Rightarrow$ (ie) follows from Theorem 3.38. The implication (ie) $\Rightarrow$ (if) follows from Lemma 3.55. The hard part of the proof is the implication (ig) $\Rightarrow$ (ia). It involves proving the formula

$$\rho(L(V \oplus W); \operatorname{res} U) \;=\; \rho(L(V); \operatorname{res} U) \cdot \rho(L(W); \operatorname{res} U)$$

for two free unitary G-representations V and W and then directly computing $\rho(L(V); \operatorname{res} U)$ for every free 1-dimensional unitary representation V. Finally one has to show that the values of the Reidemeister torsion do distinguish the unitary representations V and W up to automorphisms of G. This proof is based on the number theoretic result mentioned below, whose proof can be found for instance in [280] or [384]. $\square$

Lemma 3.61 (Franz' independence Lemma). *Let $t \geq 2$ be an integer and $S = \{j \in \mathbb{Z} \mid 0 < j < t, (j,t) = 1\}$. Let $(a_j)_{j \in S}$ be a sequence of integers indexed by S such that $\sum_{j \in S} a_j = 0$, $a_j = a_{t-j}$ for $j \in S$ and*

$\prod_{j \in S}(\zeta^j - 1)^{a_j} = 1$ *holds for every* t*-th root of unity* $\zeta \neq 1$. *Then* $a_j = 0$ *for* $j \in S$.

Example 3.62. We conclude from Theorem 3.59 and Theorem 3.60 (ii) the following facts:

(i) Any homotopy equivalence $L(7; k_1, k_2) \to L(7; k_1, k_2)$ has degree 1. Thus $L(7; k_1, k_2)$ possesses no orientation reversing selfdiffeomorphism;
(ii) $L(5; 1, 1)$ and $L(5; 2, 1)$ have the same homotopy groups, homology groups and cohomology groups, but they are not homotopy equivalent;
(iii) $L(7; 1, 1)$ and $L(7; 2, 1)$ are homotopy equivalent, but not homeomorphic.

Example 3.63 (h-cobordisms between lens spaces). The rigidity of lens spaces is illustrated by the following fact. Let (W, L, L') be an h-cobordism of lens spaces that is compatible with the orientations and the identifications of $\pi_1(L)$ and $\pi_1(L')$ with G. Then W is diffeomorphic relative L to $L \times [0, 1]$ and L and L' are diffeomorphic, see [711, Corollary 12.13 on page 410].

Remark 3.64 (Differential geometric characterization of lens spaces). Lens spaces with their preferred Riemannian metric have constant positive sectional curvature. A closed Riemannian manifold with constant positive sectional curvature and cyclic fundamental group is isometrically diffeomorphic to a lens space after possibly rescaling the Riemannian metric with a constant [999].

Remark 3.65 (de Rham's Theorem). The results above when interpreted as statements about unit spheres in free representations are generalized by De Rham's Theorem [279], see also [624, Proposition 3.2 on page 478], [630, page 317], and [860, section 4], as follows. It says for a finite group G and two orthogonal G-representations V and W whose unit spheres SV and SW are G-diffeomorphic that V and W are isomorphic as orthogonal G-representations. This remains true if one replaces G-diffeomorphic by G-homeomorphic provided that G has odd order, see [487], [684], but not for any finite group G, see [193, 195, 443, 445, 446].

We refer to [233],[633, Chapter 2] and [711] for more information about Reidemeister torsion and lens spaces.

Remark 3.66 (Further appearance of Reidemeister torsion). The Alexander polynomial of a knot can be interpreted as a kind of Reidemeister torsion of the canonical infinite cyclic covering of the knot complement, see [709], [941]. Reidemeister torsion appears naturally in surgery theory [680]. Counterexamples to the (polyhedral) Hauptvermutung that two homeomorphic simplicial complexes are already PL-homeomorphic are given by Milnor [708], see also [826], and detected by Reidemeister torsion. Seiberg-Witten invariants for 3-manifolds are closely related to torsion invariants, see Turaev [940].

Remark 3.67 (Analytic Reidemeister torsion). Ray-Singer [829] defined the analytic counterpart of topological Reidemeister torsion using a regularization of the zeta-function. Ray and Singer conjectured that the analytic and topological Reidemeister torsion agree. This conjecture was proved independently by Cheeger [224] and Müller [736]. Manifolds with boundary and manifolds with symmetries, sum (= glueing) formulas and fibration formulas are treated in [168, 259, 624, 630, 666, 948]. For a survey on analytic and topological torsion we refer for instance to [646]. There are also L^2-versions of these notions, see for instance [177, 198, 622], [635, Chapter 3], [664, 692].

3.7 The Bass-Heller-Swan Theorem for K_1

In the section we want to compute $K_1(R[\mathbb{Z}])$ for a ring R. This computation, the so-called Bass-Heller-Swan decomposition, marks the beginning of the (long) way towards the final formulation of the Farrell-Jones Conjecture for algebraic K-theory.

3.7.1 The Bass-Heller-Swan Decomposition for K_1

We need some preparation to formulate it. In the sequel we write $R[\mathbb{Z}]$ as the ring $R[t,t^{-1}]$ of finite Laurent polynomials in t with coefficients in R. Obviously the ring $R[t]$ of polynomials in t with coefficients in R is a subring of $R[t,t^{-1}]$. Define the ring homomorphisms

$$\begin{array}{lllllll} \mathrm{ev}_0\colon R[t] & \to & R, & \sum_{n\in\mathbb{Z}} r_n t^n & \mapsto & r_0 \\ i'\colon R & \to & R[t], & r & \mapsto & r\cdot t^0; \\ i\colon R & \to & R[t,t^{-1}], & r & \mapsto & r\cdot t^0. \end{array}$$

Definition 3.68 ($NK_n(R)$). Define for $n = 0, 1$

$$NK_n(R) := \ker\big((\mathrm{ev}_0)_*\colon K_n(R[t]) \to K_n(R)\big).$$

Example 3.69. Let F be a field. Put $R = F[t]/(t^2)$. Every element in R can be uniquely written as $a+bt$ for $a, b \in F$. We have $(1+bt)\cdot(1-bt) = 1-b^2t^2 = 1$ in R. Hence the element $a+bt \in R$ is a unit if and only if $a \neq 0$. We conclude that R is a local ring with $(t) = \{bt \mid b \in F\}$ as the unique maximal ideal. Since R is commutative, the homomorphism

$$i_R\colon R^\times \xrightarrow{\cong} K_1(R), \quad [x] \mapsto [r_x\colon R \to R]$$

is bijective by Theorem 3.6. Let $\mathrm{ev}_0\colon R \to F$ be the ring homomorphism sending $a + bt$ to a. Its kernel is (t). It induces a group homomorphism

$R[x]^\times \to F[x]^\times$. Since $F[x]^\times$ is the multiplicative group of non-trivial polynomials over F of degree 0 and $(1+tvx)\cdot(1-tvx) = 1 - v^2t^2x^2 = 1$ holds in $R[x]$ for all $v \in F[x]$, we obtain an isomorphism of abelian groups

$$\phi\colon R^\times \oplus F[x] \xrightarrow{\cong} R[x]^\times, \quad (u,v) \mapsto u\cdot(1+tvx).$$

Since $R[x]$ is commutative, the map $i_{R[x]}\colon R[x]^\times \xrightarrow{\cong} K_1(R[x])$ is injective, a retraction is given by the determinant. We conclude that the following composite is an injection of abelian groups

$$F[x] \xrightarrow{\phi|_{F[x]}} \operatorname{coker}\left(R^\times \to R[x]^\times\right) \xrightarrow{i} \operatorname{coker}\left(K_1(R) \to K_1(R[x])\right) \cong NK_1(R)$$

where i is the map induced by i_R and $i_{R[x]}$. This implies that $NK_1(R)$ is an abelian group which is not finitely generated.

Example 3.69 illustrates the following fact. If R is any ring, then $NK_1(R)$ is either trivial or infinitely generated as abelian group, see Theorem 6.20. So in general $NK_1(R)$ is hard to compute. At least we have the following useful results. If R is a ring of finite characteristic N, then we get $NK_n(R)[1/N] = 0$ for $n = 0,1$, see Theorem 6.17. If $NK_n(R) = 0$ and G is finite, then $NK_n(RG)[1/|G|] = 0$ for $n = 0,1$, see Theorem 6.18.

Recall that an endomorphism $f\colon P \to P$ of an R-module P is called *nilpotent* if there exists a positive integer n with $f^n = 0$.

Definition 3.70 (Nil-group $\mathrm{Nil}_0(R)$). Define the 0-th Nil-group $\mathrm{Nil}_0(R)$ to be the abelian group whose generators are conjugacy classes $[f]$ of nilpotent endomorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relation. Given a commutative diagram of finitely generated projective R-modules

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "i"] \arrow[d, "f_1"] & P_2 \arrow[r, "p"] \arrow[d, "f_2"] & P_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "i"] & P_2 \arrow[r, "p"] & P_3 \arrow[r] & 0 \end{tikzcd}$$

with exact rows and nilpotent endomorphisms as vertical arrows, we get

$$[f_1] + [f_3] = [f_2].$$

Let $\iota\colon K_0(R) \to \mathrm{Nil}_0(R)$ be the homomorphism sending the class $[P]$ of a finitely generated projective R-module P to the class $[0\colon P \to P]$ of the trivial endomorphism of P.

Definition 3.71 (Reduced Nil-group $\widetilde{\mathrm{Nil}}_0(R)$). Define the reduced 0-th Nil-groups $\widetilde{\mathrm{Nil}}_0(R)$ to be the cokernel of the map ι.

The homomorphism $\mathrm{Nil}_0(R) \to K_0(R)$, $[f\colon P \to P] \mapsto [P]$ is a retraction of the map ι. So we get a natural splitting

$$\mathrm{Nil}_0(R) \xrightarrow{\cong} \widetilde{\mathrm{Nil}}_0(R) \oplus K_0(R).$$

Denote by

$$j\colon NK_1(R) \to K_1(R[t])$$

the inclusion. Let

$$l_\pm\colon R[t] \to R[t,t^{-1}]$$

be the inclusion of rings sending t to $t^{\pm 1}$. Define

$$j_\pm := K_1(l_\pm) \circ j\colon NK_1(R) \to K_1(R[t,t^{-1}]).$$

The homomorphism

$$B\colon K_0(R) \to K_1(R[t,t^{-1}])$$

sends the class $[P]$ of a finitely generated projective R-module P to the class $[r_t \otimes_R \mathrm{id}_P]$ of the $R[t,t^{-1}]$-automorphism $r_t \otimes_R \mathrm{id}_P\colon R[t,t^{-1}] \otimes_R P \to R[t,t^{-1}] \otimes_R P$ that maps $u \otimes p$ to $ut \otimes p$. The homomorphism

$$N'\colon \mathrm{Nil}_0(R) \to K_1(R[t])$$

sends the class $[f]$ of the nilpotent endomorphism $f\colon P \to P$ of the finitely generated projective R-module P to the class $[\mathrm{id} - r_t \otimes_R f]$ of the $R[t]$-automorphism

$$\mathrm{id} - r_t \otimes_R f\colon R[t] \otimes_R P \to R[t] \otimes_R P, \quad u \otimes p \mapsto u \otimes p - ut \otimes f(p).$$

This is indeed an automorphism. Namely, if $f^{n+1} = 0$, then an inverse is given by $\sum_{k=0}^{n} (r_t \otimes_R f)^k$. The composite of N' with both $(\mathrm{ev}_0)_*\colon K_1(R[t]) \to K_1(R)$ and $\iota\colon K_0(R) \to \mathrm{Nil}_0(R)$ is trivial. Hence N' induces a homomorphism

$$N\colon \widetilde{\mathrm{Nil}}_0(R) \to NK_1(R).$$

The proof of the following theorem can be found for instance in [99] (for regular rings), [96, Chapter XII], [844, Theorem 3.2.22 on page 149] and [979, 3.6 in Section III.3 on page 205].

Theorem 3.72 (Bass-Heller-Swan decomposition for K_1). *The following maps are isomorphisms of abelian groups, natural in R,*

$$N\colon \widetilde{\mathrm{Nil}}_0(R) \xrightarrow{\cong} NK_1(R);$$
$$j \oplus K_1(i')\colon NK_1(R) \oplus K_1(R) \xrightarrow{\cong} K_1(R[t]);$$
$$B \oplus K_1(i) \oplus j_+ \oplus j_-\colon K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R) \xrightarrow{\cong} K_1(R[t,t^{-1}]).$$

One easily checks that Theorem 3.72 applied to $R = \mathbb{Z}G$ implies the following reduced version

Theorem 3.73 (Bass-Heller-Swan decomposition for $\mathrm{Wh}(G\times\mathbb{Z})$). *Let G be a group. Then there is an isomorphism of abelian groups, natural in G*

$$\overline{B}\oplus\mathrm{Wh}(i)\oplus\overline{j_+}\oplus\overline{j_-}\colon \widetilde{K}_0(\mathbb{Z}G)\oplus\mathrm{Wh}(G)\oplus NK_1(\mathbb{Z}G)\oplus NK_1(\mathbb{Z}G) \xrightarrow{\cong} \mathrm{Wh}(G\times\mathbb{Z}).$$

Example 3.74 ($\widetilde{K}_0(\mathbb{Z}G)$ affects $\mathrm{Wh}(G)$). The Whitehead group $\mathrm{Wh}(S_n)$ of the symmetric group S_n is trivial, see Theorem 3.116 (iii), whereas $\widetilde{K}_0(\mathbb{Z}[S_n])$ is a finite non-trivial group for $n \geq 5$, see Theorem 2.112 (ii). In the sequel we let $n \geq 5$. We conclude from Theorem 3.73 that $\mathrm{Wh}(S_n \times \mathbb{Z})$ is non-trivial for $n \geq 5$, whereas the obvious map

$$\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(S_n\times\mathbb{Z})} \mathrm{Wh}(H) \to \mathrm{Wh}(S_n \times \mathbb{Z})$$

is the zero map and hence not surjective. Also the map

$$\operatorname{colim}_{H\in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_1(\mathbb{Z}H) \to K_1(\mathbb{Z}[S_n \times \mathbb{Z}])$$

cannot be surjective. Hence there is no hope that a formula, which computes $K_n(RG)$ in terms of the values $K_n(RH)$ for all finite or all virtually cyclic subgroups H of G, (such as appearing in Conjecture 2.67), is true in general. The general picture will be that a computation of a K or L-group of RG in dimension n does involve K- and L-groups of RH in all dimensions $\leq n$ where H runs through all virtually cyclic subgroups of G.

Denote by

$$k_\pm\colon R \to R[t^{\pm 1}]$$

the ring homomorphism sending r to $r \cdot t^0$. Obviously $\tau_\pm \circ k_\pm = i$. Define a map

$$C\colon K_1(R[t,t^{-1}]) \to K_0(R)$$

by sending the class $[f]$ of an $R[t,t^{-1}]$-automorphism $f\colon R[t,t^{-1}]^n \to R[t,t^{-1}]^n$ to the element $[P(f,l)] - l\cdot[R]$ where l is a large enough positive integer and $P(f,l)$ is the finitely generated projective R-module $f(t^{l-1}\cdot R[t^{-1}]) \cap R[t]$. We omit the proof that $P(f,l)$ is a finitely generated projective R-module for large enough l, that the class $[P(f,l)] - l\cdot[R]$ is independent of l and depends only on $[f]$, and that the map C is a well-defined homomorphism of abelian groups.

Theorem 3.75 (Fundamental Theorem of K-theory in dimension 1). *There is a sequence which is natural in R and exact*

$$\begin{aligned} 0 \to K_1(R) &\xrightarrow{K_1(k_+)\oplus -K_1(k_-)} K_1(R[t]) \oplus K_1(R[t^{-1}]) \\ &\xrightarrow{K_1(l_+)_*\oplus K_1(l_-)} K_1(R[t,t^{-1}]) \xrightarrow{C} K_0(R) \to 0 \end{aligned}$$

where k_+, k_-, l_+, and l_- are the obvious inclusions.

If we regard it as an acyclic $\mathbb{Z}$-chain complex, there exists a chain contraction, natural in R.

Proof. One checks $C \circ B = \mathrm{id}_{K_0(R)}$ and $C \circ i_* = C \circ j_- = C \circ j_+ = 0$. Now apply Theorem 3.72. □

3.7.2 The Grothendieck Decomposition for G_0 and G_1

There is also a G-theory version of the Bass-Heller-Swan decomposition, which is due to Grothendieck. Its proof can be found in [99] or [844, Theorem 3.2.12 on page 141, Theorem 3.2.16 on page 143 and Theorem 3.2.19 on page 147].

Theorem 3.76 (Grothendieck decomposition for G_0 and G_1). *Let R be a Noetherian ring.*

(i) The inclusions $R \to R[t]$ and $R \to R[t, t^{-1}]$ induce isomorphisms of abelian groups

$$G_0(R) \xrightarrow{\cong} G_0(R[t]);$$
$$G_0(R) \xrightarrow{\cong} G_0(R[t, t^{-1}]);$$

(ii) There are natural isomorphism

$$i'_* \colon G_1(R) \xrightarrow{\cong} G_1(R[t]);$$
$$B \oplus i_* \colon G_0(R) \oplus G_1(R) \xrightarrow{\cong} G_1(R[t, t^{-1}]),$$

where i'_*, B, and i_* are defined analogously to the maps appearing in Theorem 3.72.

Exercise 3.77. Show that the map $\mathbb{Z} \xrightarrow{\cong} G_0(R[\mathbb{Z}^n])$ sending n to $n \cdot [R[\mathbb{Z}^n]]$ is an isomorphism for a principal ideal domain R and $n \geq 0$.

3.7.3 Regular Rings

Theorem 3.78 (Hilbert Basis Theorem). *If R is Noetherian, then $R[t]$ and $R[t, t^{-1}]$ are Noetherian.*

Proof. See for instance [844, Theorem 3.2.1 on page 133 and Corollary 3.2.2 on page 134]. □

Let (P) be a property of groups, e.g., being finite or being cyclic. A group G is called *virtually (P)* if G contains a subgroup $H \subset G$ of finite index such that H has property (P). A group G is *poly-(P)* if there is a finite sequence of subgroups $\{1\} = G_0 \subset G_1 \subset G_2 \subset \ldots G_r = G$ such that G_i is normal in G_{i+1} and the quotient G_{i+1}/G_i has property (P) for $i = 0, 1, 2, \ldots, r-1$. Thus the notions of *virtually finitely generated abelian*, *virtually free*, *virtually nilpotent*, *poly-cyclic*, *poly-$\mathbb{Z}$*, and *virtually poly-cyclic* are defined, where poly-$\mathbb{Z}$ stands of poly-(infinite cyclic).

Theorem 3.79 (Noetherian group rings). *If R is a Noetherian ring and G is a virtually poly-cyclic group, then RG is Noetherian.*

Proof. See for instance [635, Lemma 10.55 on page 397]. □

No counterexample is known to the conjecture that $\mathbb{C}G$ is Noetherian if and only if G is virtually poly-cyclic.

Theorem 3.80 (Regular group rings).

(i) The rings $R[t]$ and $R[t, t^{-1}]$ are regular if R is regular;
(ii) The ring RG is regular if R is regular and G is poly-$\mathbb{Z}$;
(iii) The ring RG is regular if R is regular, $\mathbb{Q} \subseteq R$ and G is virtually poly-cyclic;

Proof. (i) This is proved for instance in [844, Theorem 3.2.3 on page 134 and Corollary 3.2.4 on page 136].

(ii) This is follows from [864, Theorem 8.2.2 on page 533 and Theorem 8.2.18 on page 537] in the case where R is a field.

(iii) This is follows from [864, Theorem 8.2.2 on page 533 and Theorem8.2.20 on page 538] in the case where R is a field. □

A ring is called *semihereditary*, if every finitely generated ideal is projective, or, equivalently, if every finitely generated submodule of a projective R-module is projective, see [203, Proposition 6.2 in Chapter I.6 on page 15].

Theorem 3.81 (Bass-Heller-Swan decomposition for K_1 for regular rings). *Suppose that R is semihereditary or regular. Then we get*

$$\widetilde{Nil}_0(R) = NK_1(R) = 0,$$

and the Bass-Heller-Swan decomposition of Theorem 3.72 reduces to the isomorphism

$$B \oplus i_* \colon K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]).$$

Proof. The proof for regular R can be found for instance in [844, Exercise 3.2.25 on page 152] or [922, Corollary 16.5 on page 226].

Suppose that R is semihereditary. Consider a nilpotent endomorphism $f\colon P \to P$ of the finitely generated projective R-module P. Define $I_1(f) = \operatorname{im}(f)$ and $K_1(f) = \ker(f)$. Let $f|_{I_1(f)}\colon I_1(f) \to I_1(f)$ be the endomorphism induced by f. Since R is semihereditary, $I_1(f)$ is a finitely generated projective R-module. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_1(f) \arrow[r, "i"] \arrow[d, "0"] & P \arrow[r, "f"] \arrow[d, "f"] & I_1(f) \arrow[r] \arrow[d, "f|_{I_1(f)}"] & 0 \\ 0 \arrow[r] & K_1(f) \arrow[r, "i"] & P \arrow[r, "f"] & I_1(f) \arrow[r] & 0 \end{tikzcd}$$

with exact rows and nilpotent endomorphisms of finitely generated projective R-modules as vertical arrows. Hence we get $[f\colon P \to P] \;=\; [I_1(f)\colon I_1(f) \to I_1(f)]$ in $\widetilde{\mathrm{Nil}}_0(R)$. Define inductively $I_{n+1}(f) = I_1\left(f|_{I_n(f)}\right)$. Hence we get for all $n \geq 1$

$$[f\colon P \to P] \;=\; [f|_{I_n(f)}\colon I_n(f) \to I_n(f)].$$

Since f is nilpotent, there exists some n with $I_n(f) = 0$. This implies $[f] = 0$ in $\widetilde{\mathrm{Nil}}_0(R)$. Now apply Theorem 3.72. $\square$

Exercise 3.82. Prove that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^n]) = \mathrm{Wh}(\mathbb{Z}^n) = 0$ for all $n \geq 0$.

Remark 3.83 (Glimpse of a homological behavior of K-theory). In the case that R is regular, Theorem 3.81 shades some homological flavour on K-theory. Just observe the analogy between the two formulas

$$\begin{aligned} K_1(R[\mathbb{Z}]) &\cong K_0(R[\{1\}]) \oplus K_1(R[\{1\}]); \\ H_1(\mathbb{Z}; A) &\cong H_0(\{1\}; A) \oplus H_1(\{1\}; A), \end{aligned}$$

where in the second line we consider group homology with coefficients in some abelian group A, which corresponds to the role of R in the first line.

Remark 3.84 (Von Neumann algebras are semihereditary but not Noetherian). Note that any von Neumann algebra is semihereditary. This follows from the facts that any von Neumann algebra is a Baer $*$-ring and hence in particular a Rickart C^*-algebra [117, Definition 1, Definition 2 and Proposition 9 in Chapter 1.4] and that a C^*-algebra is semihereditary if and only if it is Rickart [34, Corollary 3.7 on page 270]. The group von Neumann algebra $\mathcal{N}(G)$ is Noetherian if and only if G is finite, see [635, Exercise 9.11 on page 367].

Lemma 3.85. *If R is regular, then the canonical homomorphism*

$$f\colon K_0(R) \xrightarrow{\cong} G_0(R), \quad [P] \mapsto [P]$$

is a bijection.

Proof. We have to define an inverse homomorphism

$$r\colon G_0(R) \to K_0(R).$$

Given a finitely generated R-module M, we can choose a finite projective resolution $P_* = (P_*, \phi)$ since R is by assumption regular. We want to define

$$r([M]) := \sum_{n \geq 0} (-1)^n \cdot [P_n].$$

The *Fundamental Lemma of Homological Algebra* implies for two projective resolutions P_* and Q_* of M the existence of an R-chain homotopy equivalence $f_*\colon P_* \to Q_*$, see for instance [978, Comparison Theorem 2.2.6 on page 35]. We conclude from Lemma 2.36 (i)

$$\sum_{n \geq 0} (-1)^n \cdot [P_n] = o(P_*) = o(Q_*) = \sum_{n \geq 0} (-1)^n \cdot [Q_n].$$

Hence the choice of projective resolution does not matter in the definition of $r([M])$. It remains to show for an exact sequence of finitely generated R-modules $0 \to M \to M' \to M'' \to 0$ that $r(M) - r(M') + r(M'') = 0$ holds. This follows from Lemma 2.36 (ii) since we can construct from finite projective R-resolutions P_* of M and P''_* of M'' a finite projective R-resolution P'_* of M' such that there exists a short exact sequence of R-chain complexes $0 \to P_* \to P'_* \to P''_* \to 0$, see [629, Lemma 11.6 on page 216]. Hence r is well-defined. One easily checks that r and f are inverse to one another. □

3.8 The Mayer-Vietoris *K*-Theory Sequence of a Pullback of Rings

Theorem 3.86 (Mayer-Vietoris sequence for middle K-theory of a pullback of rings). *Consider a pullback of rings*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & R_0 \end{tikzcd}$$

such that j_1 or j_2 is surjective. Then there exists a natural exact sequence of six terms

$$K_1(R) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_1(R_1) \oplus K_1(R_2) \xrightarrow{(j_1)_* - (j_2)_*} K_1(R_0)$$
$$\xrightarrow{\partial_1} K_0(R) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_0(R_1) \oplus K_0(R_2) \xrightarrow{(j_1)_* - (j_2)_*} K_0(R_0).$$

Its construction and its proof requires some preparation. In particular we need the following basic construction due to Milnor [712, page 20]. Let $\overline{j_k}\colon P_k \to (j_k)_*P_k$ be the map sending $x \in P_k$ to $1 \otimes x \in R_0 \otimes_{j_k} P_k$ for $k = 1, 2$. Define a ring homomorphism $i_0 = j_1 \circ i_1 = j_2 \circ i_2 \colon R \to R_0$. Given R_k-modules P_k for $k = 0, 1, 2$ and isomorphisms of R_0-modules $f_k \colon (j_k)_*P_k \xrightarrow{\cong} P_0$ for $k = 1, 2$, define an R-module $M = M(P_1, P_2, f_1, f_2)$ by the pullback of abelian groups

$$\begin{tikzcd} M \arrow[r] \arrow[d] & P_1 \arrow[d, "f_1 \circ \overline{j_1}"] \\ P_2 \arrow[r, "f_2 \circ \overline{j_2}"'] & P_0 \end{tikzcd}$$

together with the R-multiplication on M induced by the R-actions on P_k that comes from the ring homomorphisms $i_k \colon R \to R_k$ for $k = 0, 1, 2$.

Lemma 3.87. *(i) The R-module M is projective if P_0 and P_1 are projective. The R-module M is finitely generated projective if P_0 and P_1 are finitely generated projective;*
(ii) Every projective R-module P can be realized up to isomorphism as M for appropriate projective R_k-modules P_k for $k = 0, 1, 2$ and isomorphisms of R_0-modules $f_k \colon (j_k)_*P_k \xrightarrow{\cong} P_0$ for $k = 1, 2$;
(iii) The R_k-modules $(i_k)_*M$ and P_k are isomorphic for $k = 1, 2$.

Proof. This is proved in Milnor [712, Theorems 2.1, 2.2 and 2.3 on page 20] or in[899, Proposition 59 on page 155, Proposition 60 on page 157, Proposition 61 on page 158]. □

Now we can give the proof of Theorem 3.86

Proof. The main step is to construct the boundary homomorphism ∂_1. Given an element $x \in K_1(R_0)$, we can find an automorphism $f \colon R_0^n \xrightarrow{\cong} R_0^n$ of a finitely generated free R-module with $x = [f]$, see Lemma 3.10. The R_0-module $M(R_1^n, R_2^n, \mathrm{id}_{R_0^n}, f)$ is a finitely generated projective R_0-module by Lemma 3.87 (i). Define

$$\partial_1(x) \ := \ [M(R_1^n, R_2^n, \mathrm{id}_{R_0^n}, f)] - [R_0^n].$$

This is a well-defined homomorphism of abelian groups, see [899, page 164]. The elementary proof of the exactness of the sequence of six terms can be found in [899, Proposition 63 on page 164]. □

Now we are ready to give the promised proof of Rim's Theorem 2.105.

Proof. Consider the pullback of rings

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}/p] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}[\exp(2\pi i/p)] \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"'] & \mathbb{F}_p \end{tikzcd}$$

where here and in the sequel $\mathbb{F}_q$ denotes the field with q elements, i_1 sends the generator of $\mathbb{Z}/p$ to $\exp(2\pi i/p)$, the map i_2 sends the generator of $\mathbb{Z}/p$ to $1 \in \mathbb{Z}$, the map j_2 is the projection and the homomorphism j_1 sends $\exp(2\pi i/p)$ to 1. Obviously j_1 and j_2 are surjective. Hence we get from Theorem 3.86 an exact sequence

$$K_1(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{(i_1)_*\oplus(i_2)_*} K_1(\mathbb{Z}[\exp(2\pi i/p)]) \oplus K_1(\mathbb{Z}) \xrightarrow{(j_1)_*-(j_2)_*} K_1(\mathbb{F}_p) \xrightarrow{\partial_1}$$
$$K_0(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{(i_1)_*\oplus(i_2)_*} K_0(\mathbb{Z}[\exp(2\pi i/p)]) \oplus K_0(\mathbb{Z}) \xrightarrow{(j_1)_*-(j_2)_*} K_0(\mathbb{F}_p).$$

The map $(j_2)_*\colon K_0(\mathbb{Z}) \to K_0(\mathbb{F}_p)$ is bijective by Example 2.4. Hence it remains to prove that $(j_1)_*\colon K_1(\mathbb{Z}[\exp(2\pi i/p)]) \to K_1(\mathbb{F}_p)$ is surjective. Because of Theorem 3.17 we have to find for each integer k with $1 \le k \le p-1$ a unit $u \in \mathbb{Z}[\exp(2\pi i/p)]^\times$ satisfying $j_1(u) = \overline{k}$. Put $\xi = \exp(2\pi i/p)$. Choose an integer l such that $kl = 1 \mod p$. Define

$$\begin{aligned} u &:= 1 + \xi + \xi^2 + \cdots + \xi^{k-1}; \\ v &:= 1 + \xi^k + \xi^{2k} + \cdots + \xi^{(l-1)k}. \end{aligned}$$

Since $(\xi - 1)u = \xi^k - 1$ and $(\xi^k - 1) \cdot v = \xi - 1$ and $\mathbb{Z}[\exp(2\pi i/p)]$ is an integral domain, we get $uv = 1$ and hence $u \in \mathbb{Z}[\exp(2\pi i/p)]^\times$. Obviously $j_1(u) = \overline{k}$. □

3.9 The K-Theory Sequence of a Two-Sided Ideal

Let $I \subseteq R$ be a two-sided ideal in the ring R. The *double of the ring R along the ideal I* is the subring $D(R,I)$ of $R \times R$ consisting of pairs (r_1, r_2) satisfying $r_1 - r_2 \in I$. Let $p_k\colon D(R,I) \to R$ send (r_1, r_2) to r_k for $k = 1, 2$.

Definition 3.88 ($K_n(R,I)$). Define for $n = 0, 1$ the abelian group $K_n(R,I)$ to be the kernel of the homomorphism

$$(p_1)_*\colon K_n(D(R,I)) \to K_n(R).$$

Theorem 3.89 (Exact sequence of a two-sided ideal for middle K-theory). *We obtain an exact sequence, natural in $I \subseteq R$*

$$K_1(R,I) \xrightarrow{j_1} K_1(R) \xrightarrow{\mathrm{pr}_1} K_1(R/I) \xrightarrow{\partial_1} K_0(R,I) \xrightarrow{j_1} K_0(R) \xrightarrow{\mathrm{pr}_0} K_0(R/I).$$

Proof. We obtain a pullback of rings

$$\begin{tikzcd} D(R,I) \arrow[r, "p_1"] \arrow[d, "p_2"'] & R \arrow[d, "\mathrm{pr}"] \\ R \arrow[r, "\mathrm{pr}"'] & R/I \end{tikzcd}$$

such that pr is surjective. We get from Theorem 3.86 the exact sequence

$$K_1(D(R,I)) \xrightarrow{(p_1)_* \oplus (p_2)_*} K_1(R) \oplus K_1(R) \xrightarrow{-\mathrm{pr}_* + \mathrm{pr}_*} K_1(R/I) \xrightarrow{\partial}$$
$$K_0(D(R,I)) \xrightarrow{(p_1)_* \oplus (p_2)_*} K_0(R) \oplus K_0(R) \xrightarrow{-\mathrm{pr}_* + \mathrm{pr}_*} K_0(R/I).$$

This yields the desired exact sequence if we define $j_n \colon K_n(R,I) \to K_n(R)$ to be the restriction of $(p_2)_* \colon K_n(D(R,I)) \to K_n(R)$ to $K_n(R,I)$ for $n = 0, 1$ and let ∂_1 be the map induced by ∂. □

Next we give alternative descriptions of $K_0(R,I)$.

Let S be a ring, but now for some time we do not require that it has a unit. If we want to emphasize that we do not require this, we say that S is a ring without unit although it may have one. The point is that a homomorphism of rings without units $f \colon S \to S'$ is a map compatible with the abelian group structure and the multiplication but no requirement about the unit is made. The *ring obtained from S by adjoining a unit S_+* has as underlying group $S \oplus \mathbb{Z}$. The multiplication is given by

$$(s_1, n_1) \cdot (s_2, n_2) \;:=\; (s_1 s_2 + n_1 s_2 + n_2 s_1, n_1 n_2).$$

The unit in S_+ is given by $(0,1)$. We obtain a natural embedding $i_S \colon S \to S_+$ by sending s to $(s,0)$. Let $p_S \colon S_+ \to \mathbb{Z}$ be the homomorphism of rings with unit sending (s,n) to n. We obtain an exact sequence of rings without unit $0 \to S \xrightarrow{i_S} S_+ \xrightarrow{p_S} \mathbb{Z} \to 0$. If $f \colon S \to S'$ is a homomorphism of rings without unit, we obtain a homomorphism $f_+ \colon S_+ \to S'_+$ of rings with unit by sending (s,n) to $(f(s),n)$. If S does has a unit 1_S, then we obtain an isomorphism of rings with unit $u_S \colon S_+ \xrightarrow{\cong} S \times \mathbb{Z}$ by sending (s,n) to $(s + n \cdot 1_S, n)$.

Definition 3.90 ($K_n(S)$ for rings without unit). Let S be a ring without unit. Define for $n = 0, 1$

$$K_n(S) \;:=\; \ker\left((p_S)_* \colon K_n(S_+) \to K_n(\mathbb{Z})\right).$$

Given a homomorphism $f \colon S \to S'$ of rings without unit, the homomorphism $(f_+)_* \colon K_n(S_+) \to K_n(S'_+)$ induces a homomorphism of abelian groups

$f_* \colon K_n(S) \to K_n(S')$. Thus we obtain a covariant functor from the category of rings without unit to the category of abelian groups by sending S to $K_n(S)$.

If S happens to have already a unit, we get back the old definition (up to natural isomorphism). Namely, the isomorphism $K_0(u_S) \colon K_n(S_+) \xrightarrow{\cong} K_n(S \times \mathbb{Z})$ sends $\ker((p_S)_*)$ to the kernel of the map $(\mathrm{pr}_{\mathbb{Z}})_* \colon K_n(S \times \mathbb{Z}) \to K_n(\mathbb{Z})$ given by the projection $\mathrm{pr}_{\mathbb{Z}} \colon S \times \mathbb{Z} \to \mathbb{Z}$ and the inclusion $j \colon S \to S \times \mathbb{Z},\ s \mapsto (s, 0)$ induces an isomorphism of $K_n(S)$ to the kernel of the map $\mathrm{pr}_{\mathbb{Z}}$ by Theorem 2.12 and Theorem 3.9.

Lemma 3.91. *Let I be a two-sided ideal in the ring R. Let $K_0(I)$ be the projective class group of the ring I without unit, see Definition 3.90. Then there is a natural isomorphism*

$$K_0(I) \xrightarrow{\cong} K_0(R, I).$$

In particular $K_0(R, I)$ depends only on the ring without unit I but not on R.

Proof. The isomorphism is induced by the homomorphism of rings with unit $I_+ \to D(R, I)$ sending (s, n) to $(n \cdot 1_R, n \cdot 1_R + s)$. The proof that it is bijective can be found for instance in [844, Theorem 1.5.9 on page 30]. ◻

Exercise 3.92. Let n be a positive integer. Compute

$$K_0((n)) \cong \begin{cases} 0 & \text{if } n = 2; \\ (\mathbb{Z}/n)^{\times}/\{\pm 1\} & \text{if } n \geq 3, \end{cases}$$

for the ideal $(n) = \{mn \mid m \in \mathbb{Z}\} \subseteq \mathbb{Z}$. Prove for the ideal $(N_{\mathbb{Z}/2}) \subseteq \mathbb{Z}[\mathbb{Z}/2]$ generated by the norm element that $(N_{\mathbb{Z}/2})$ and $(2\mathbb{Z})$ are isomorphic as rings without unit. Conclude

$$\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) = 0.$$

Next we give an alternative description of $K_1(R, I)$. Define $\mathrm{GL}(R, I)$ to be the kernel of the map $\mathrm{GL}(R) \to \mathrm{GL}(R/I)$ induced by the projection $R \to R/I$. Let $\mathrm{E}(R, I)$ be the smallest normal subgroup of $\mathrm{E}(R)$ that contains all matrices of the shape $I_n + r \cdot E^n_{i,j}$ for $n \in \mathbb{Z}, n \geq 1,\ i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $r \in I$. Note that $\mathrm{E}(R, I) \subseteq \mathrm{GL}(R, I)$. The proof of the next result can be found for instance in [844, Theorem 2..5.3 on page 93].

Theorem 3.93 (Relative Whitehead Lemma). *Let $I \subseteq R$ be a two-sided ideal. Then:*

(i) The subgroup $\mathrm{E}(R, I)$ of $\mathrm{GL}(R)$ is normal;
(ii) There is an isomorphism, natural in (R, I)

$$\mathrm{GL}(R, I)/\mathrm{E}(R, I) \xrightarrow{\cong} K_1(R, I);$$

(iii) The center of $\mathrm{GL}(R)/\mathrm{E}(R, I)$ is $\mathrm{GL}(R, I)/\mathrm{E}(R, I)$;

(iv) We have $\mathrm{E}(R,I) = [\mathrm{E}(R), \mathrm{E}(R,I)] = [\mathrm{GL}(R), \mathrm{E}(R,I)]$.

Example 3.94 ($K_1(R,I)$ depends on R). In contrast to $K_0(R,I)$ it is not true that $K_1(R,I)$ is independent of R as shown by Swan [924, Section 1]. Let S be a ring and put

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a,b,d \in S \right\};$$
$$R' = \left\{ \begin{pmatrix} n & b \\ 0 & n \end{pmatrix} \mid n \in \mathbb{Z}, b \in S \right\};$$
$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}.$$

Then $K_1(R,I) = \{0\}$ and $K_1(R',I) \cong S$.

Remark 3.95 (Congruence Subgroup Problem). Given a commutative ring R, the *Congruence Subgroup Problem* asks if every normal subgroup of $\mathrm{GL}(R)$ is of the form $\mathrm{SL}(R,I) := \{A \in GL(R,I) \mid \det(A) = 1\}$ for some two-sided ideal $I \subseteq R$. Bass has shown that for any normal subgroup $H \subseteq \mathrm{GL}(R)$ there exists an ideal $I \subseteq R$ satisfying $\mathrm{E}(R,I) \subseteq H \subseteq \mathrm{GL}(R,I)$, see [96, Theorem 2.1 (a) on page 229] or [843, Exercise 2.5.21 on page 106]. Hence the Congruence Subgroup Problem has a positive answer if and only for every two-sided ideal $I \subseteq R$ we have $\mathrm{E}(R,I) = \mathrm{SL}(R,I)$, see [843, Exercise 2.5.21 on page 106]. More information about this problem can be found for instance in [100].

Exercise 3.96. Show that the Congruence Subgroup Problem has a positive answer for every field F.

3.10 Swan Homomorphisms

3.10.1 The Classical Swan Homomorphism

The definitions and results of this subsection are taken from Swan [920]. This paper marked the beginning of a development that finally leads to a solution of the Spherical Space Form Problem 9.197 which we have discussed also in Section 2.5. It presents a nice and illuminating interaction between geometry, group theory, and algebraic K-theory.

Let G be a finite group. Let $N_G \in \mathbb{Z}G$ be the *norm element*, i.e., $N_G := \sum_{g \in G} g$. Consider the following pullback of rings

(3.97)
$$\begin{tikzcd} \mathbb{Z}G \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}G/(N_G) \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"'] & \mathbb{Z}/|G| \end{tikzcd}$$

where $(N_G) \subseteq \mathbb{Z}G$ is the ideal generated by N_G, i_1 and j_2 are the obvious projections, i_2 is induced by the group homomorphism $G \to \{1\}$, and j_1 is the unique ring homomorphism for which the diagram above commutes. One easily checks that it is a pullback and that the maps i_1 and j_1 are surjective. Hence we can apply Theorem 3.86 and obtain a boundary homomorphism $\partial\colon K_1(\mathbb{Z}/|G|) \to K_0(\mathbb{Z}G)$. The obvious homomorphism $i\colon \mathbb{Z}/|G|^\times \to K_1(\mathbb{Z}/|G|)$ is an isomorphism by Theorem 3.6 since the commutative finite ring $\mathbb{Z}/|G|$ is a commutative semilocal ring and hence the determinant $\det\colon K_1(\mathbb{Z}/|G|) \to \mathbb{Z}/|G|^\times$ is an inverse of i.

Definition 3.98 (Swan homomorphism). The *(classical) Swan homomorphism* is the composition

$$\mathrm{sw}^G\colon \mathbb{Z}/|G|^\times \xrightarrow{i} K_1(\mathbb{Z}/|G|) \xrightarrow{\partial} K_0(\mathbb{Z}G).$$

Lemma 3.99. *Let $\overline{n} \in \mathbb{Z}/|G|^\times$ be an element represented by $n \in \mathbb{Z}$. Then the ideal $(n, N_G) \subseteq \mathbb{Z}G$ generated by n and N_G is a finitely generated projective $\mathbb{Z}G$-module and*

$$\mathrm{sw}(\overline{n}) = [(n, N_G)] - [\mathbb{Z}G].$$

Proof. Let P_1 be the $\mathbb{Z}$-module $\mathbb{Z}$, P_2 be the $\mathbb{Z}G/(N_G)$-module $\mathbb{Z}G/(N_G)$, and P_0 be the $\mathbb{Z}/|G|$-module $\mathbb{Z}/|G|$. Consider the automorphism $r_{\overline{n}}\colon \mathbb{Z}/|G| \to \mathbb{Z}/|G|$ given by multiplication with n. Define a $\mathbb{Z}G$-module P by the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "i_2"'] & \mathbb{Z}G/(N_G) \arrow[d, "r_{\overline{n}} \circ j_1"] \\ \mathbb{Z} \arrow[r, "j_2"'] & \mathbb{Z}/|G|. \end{tikzcd}$$

One easily checks that the $\mathbb{Z}G$ map $(n, N_G) \to \mathbb{Z}$ which sends n to n and N_G to $|G|$ and the $\mathbb{Z}G$ map $(n, N_G) \to \mathbb{Z}G/(N_G)$ which sends n to the class of 1 and N_G to 0 induce an isomorphism of $\mathbb{Z}G$-modules $(n, N_G) \xrightarrow{\cong} P$. We conclude from Lemma 3.87 (i) that (r, N_G) is a finitely generated projective $\mathbb{Z}G$-module and that $\mathrm{sw}(\overline{n}) = [(n, N_G)] - [\mathbb{Z}G]$. □

Remark 3.100 (Another description of the Swan homomorphism). For any $n \in \mathbb{Z}$ with $(n, |G|) = 1$, the abelian group $\mathbb{Z}/n$ with the trivial G-action is a $\mathbb{Z}G$-module that possesses a finite projective resolution P_*, see [161, Theorem VI.8.12 on page 152]. Since two finite projective resolutions of $\mathbb{Z}/n$ are $\mathbb{Z}G$-chain homotopic, their finiteness obstructions agree, Lemma 2.36 (i). Thus we can define $[\mathbb{Z}/n] \in K_0(\mathbb{Z}G)$ by $o(P_*) = \sum_{n \geq 0} (-1)^n \cdot [P_n]$ for any finite projective resolution P_*. We get

$$\mathrm{sw}(\overline{n}) = -[\mathbb{Z}/n]$$

for any integer $n \in \mathbb{Z}$ with $(n, |G|) = 1$. This follows essentially from [920, Lemma 6.2] and Lemma 3.99.

Exercise 3.101. Show that sw^G is trivial for a finite cyclic group G.

3.10.2 The Classical Swan Homomorphism and Free Homotopy Representations

Let G be a finite group. A *free d-dimensional G-homotopy representation X* is a d-dimensional CW-complex X together with a G-action such that for any open cell e we have $ge \cap e \neq \emptyset \Rightarrow g = 1$, the space X is homotopy equivalent to S^d and $G\backslash X$ is a finitely dominated CW-complex. Let $f \colon X \to Y$ be a G-map of free d-dimensional G-homotopy representations for $d \geq 2$. Let $n \geq 0$ be the integer such that the homomorphism of infinite cyclic groups $H_d(f) \colon H_d(X) \to H_d(Y)$ sends a generator of $H_d(X)$ to $\pm n$-times the generator of $H_d(Y)$. Let $o(G\backslash X), o(G\backslash Y) \in K_0(\mathbb{Z}G)$ be the finiteness obstructions of X and Y with respect to the obvious identification $G = \pi_1(X) = \pi_1(Y)$.

Lemma 3.102. *Let G be a finite group of order ≥ 3.*

(i) The G-action on $H_m(X)$ is trivial for $m \geq 0$ and d is odd;
(ii) We have $n \geq 1$, $(n, |G|) = 1$, and

$$\mathrm{sw}^G(\overline{n}) = o(G\backslash Y) - o(G\backslash X),$$

Proof. (i) Let $C_*(X)$ be the cellular $\mathbb{Z}G$-chain complex. The conditions about the G-actions imply that $C_*(X)$ is a free $\mathbb{Z}G$-chain complex and is the same as $C_*(\widetilde{G\backslash X})$. Since $G\backslash X$ is finitely dominated, we can find a finite projective $\mathbb{Z}G$-chain complex P_* that is $\mathbb{Z}G$-chain homotopy equivalent to $C_*(X)$, see [635, Proposition 11.11 on page 222] or Subsection 24.7.5. Since $\mathbb{C}G$ is semisimple, every submodule of a finitely generated $\mathbb{C}G$-module is finitely generated projective again. This implies the following equality in $K_0(\mathbb{C}G) = R_{\mathbb{C}}(G)$:

$$\sum_{m \geq 0} (-1)^m \cdot [P_m \otimes_{\mathbb{Z}G} \mathbb{C}G] = [H_0(X; \mathbb{C})] + (-1)^d \cdot [H_d(X; \mathbb{C})].$$

The Bass Conjecture for integral domains 2.98 has been proved for finite groups and $R = \mathbb{Z}$ by Swan [919, Theorem 8.1]. This implies that $P_n \otimes_{\mathbb{Z}G} \mathbb{C}G$ is a finitely generated free $\mathbb{C}G$-module for every n. Since $P_* \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq C_*(G\backslash X)$, we conclude $\sum_{m\geq 0}(-1)^m \cdot [P_m \otimes_{\mathbb{Z}G} \mathbb{C}G] = \chi(G\backslash X)\cdot[\mathbb{C}G]$. Hence we get the following equality in $R_{\mathbb{C}}(G)$

$$\chi(G\backslash X)\cdot[\mathbb{C}G] \;=\; [H_0(X;\mathbb{C})] + (-1)^d \cdot [H_d(X;\mathbb{C})].$$

Obviously $H_0(X;\mathbb{C})$ is $\mathbb{C}G$-isomorphic to the trivial 1-dimensional G-representation $[\mathbb{C}]$. Since $H_d(X) \cong \mathbb{Z}$, there is a group homomorphism $w\colon G \to \{\pm 1\}$ such that $H_d(X;\mathbb{C})$ is the 1-dimensional G-representation $\mathbb{C}^w$ for which $g \in G$ acts by multiplication with $w(g)$. Thus we get in $R_{\mathbb{C}}(G)$

$$\chi(G\backslash X)\cdot[\mathbb{C}G] \;=\; [\mathbb{C}] + (-1)^d \cdot [\mathbb{C}^w].$$

Computing the characters on both sides yields the following equalities for $g \in G$

$$\begin{aligned} \chi(G\backslash X)\cdot|G| &= 1 + (-1)^d; \\ 0 &= 1 + (-1)^d \cdot w(g) \quad \text{ for } g \neq 1. \end{aligned}$$

Since we assume $|G| \geq 3$ and $\chi(G\backslash X)$ is an integer, the first equality implies that d is odd. The second inequality implies that $w(g) = 1$ for all $g \in G$. Hence G acts trivially on $H_m(X)$ for all $m \geq 0$.

(ii) Let $C_*(X)$ and $C_*(Y)$ be the free cellular $\mathbb{Z}G$-chain complexes. Choose finite projective $\mathbb{Z}G$-chain complexes P_* and Q_* together with $\mathbb{Z}G$-chain homotopy equivalences $u_*\colon P_* \to C_*(X)$ and $v_*\colon Q_* \to C_*(Y)$. The map $f\colon X \to Y$ induces a $\mathbb{Z}G$-chain map $C_*(f)\colon C_*(X) \to C_*(Y)$. Choose a $\mathbb{Z}G$-chain map $h_*\colon P_* \to Q_*$ satisfying $v_* \circ h_* \simeq C_*(f) \circ u_*$. Let $\operatorname{cone}_* = \operatorname{cone}_*(h_*)$ be the mapping cone of h_*. It is a $(d+1)$-dimensional free $\mathbb{Z}G$-chain complex such that $H_m(\operatorname{cone}_*) = 0$ for $m \neq d$ and $H_d(\operatorname{cone}(C_*(f)))$ is $\mathbb{Z}G$-isomorphic to $\mathbb{Z}/n$ with the trivial G-action. This follows from the long exact homology sequence associated to the short exact sequence of $\mathbb{Z}G$-chain complexes $0 \to Q_* \to \operatorname{cone}(h_*) \to \Sigma P_* \to 0$ and assertion (i). Let D_* be the $\mathbb{Z}G$-chain subchain complex of cone_* such that $D_{d+1} = \operatorname{cone}_{d+1}$, D_d is the kernel of the d-th differential of cone_* and $D_k = 0$ for $k \neq d, d+1$. Then D_* is a projective $\mathbb{Z}G$-chain complex and the inclusion $D_* \to \operatorname{cone}_*$ induces an isomorphism on homology and hence is a $\mathbb{Z}G$-chain homotopy equivalence. In particular we get a short exact sequence $0 \to D_{d+1} \to D_d \to \mathbb{Z}/n \to 0$. This excludes $n = 0$ since the cohomological dimension of a non-trivial finite group is ∞. Suppose that $(n, |G|) = 1$ is not true. Then we can find a prime number p such that $\mathbb{Z}/p$ is a subgroup of G and $\mathbb{Z}/p^l$ is a direct summand in $\mathbb{Z}/n$ for some $l \geq 1$. This implies that the cohomological dimension of the trivial $\mathbb{Z}[\mathbb{Z}/p]$-module $\mathbb{Z}/p^l$ is bounded by 1. An easy computation shows that $\operatorname{Ext}^n_{\mathbb{Z}[\mathbb{Z}/p]}(\mathbb{Z}, \mathbb{Z}/p^l)$ does not vanish for all $n \geq 2$, a contradiction. Hence $(n, |G|) = 1$.

We conclude from Lemma 2.36

$$\begin{aligned}(-1)^d \cdot [\mathbb{Z}/n] = (-1)^{d+1} \cdot [D_{d+1}] + (-1)^d \cdot [D_d] &= o(D_*) = o(\mathrm{cone}_*)\\ &= [Q_*] - [P_*] = o(G\backslash Y) - o(G\backslash X).\end{aligned}$$

Since d is odd by assertion (i), we conclude $\mathrm{sw}(\overline{n}) = o(G\backslash Y) - o(G\backslash X)$ from Remark 3.100. □

Exercise 3.103. Let X be a free d-dimensional G-homotopy representation of the finite cyclic group G. Then $G\backslash X$ is homotopy equivalent to a finite CW-complex.

3.10.3 The Generalized Swan Homomorphism

In this subsection we briefly introduce the generalized Swan homomorphism. For proofs and more information we refer to [629, Chapter 19].

Fix a finite group G. Let m be its order $|G|$. We obtain a pullback of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{Z}[1/m]G \\ \downarrow & & \downarrow \\ \mathbb{Z}_{(m)}G & \longrightarrow & \mathbb{Q}G. \end{array}$$

Despite the fact that neither the right horizontal arrow nor the lower vertical arrow are surjective, one obtains a long exact sequence, which is an example of a localization sequence

$$\begin{aligned}(3.104)\quad K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}[1/m]G) \oplus K_1(\mathbb{Z}_{(m)}G) \to K_1(\mathbb{Q}G) &\xrightarrow{\partial} K_0(\mathbb{Z}G)\\ \to K_0(\mathbb{Z}[1/m]G) \oplus K_0(\mathbb{Z}_{(m)}G) &\to K_0(\mathbb{Q}G).\end{aligned}$$

We denote in the sequel by $K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$ the cokernel of the change of rings homomorphism $K_1(\mathbb{Z}_{(m)}G) \to K_1(\mathbb{Q}G)$.

Definition 3.105 (Generalized Swan homomorphism). The *generalized Swan homomorphism*

$$\overline{\mathrm{sw}}^G \colon \mathbb{Z}/m^{\times} \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$$

sends $\overline{r}$ to element in $K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$ that is given by the element in $K_1(\mathbb{Q}G)$ represented by the $\mathbb{Q}G$-automorphism $r \cdot \mathrm{id} \colon \mathbb{Q} \to \mathbb{Q}$ of the trivial $\mathbb{Q}G$-module $\mathbb{Q}$.

This is well-defined by the argument in [629, page 381]. The following result is taken from [629, Theorem 19.4 on page 381]

Theorem 3.106 (The generalized Swan homomorphism). *Let G be a finite group of order m.*

(i) The composite of the generalized Swan homomorphism $\overline{\mathrm{sw}}^G \colon \mathbb{Z}/m^\times \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$ *introduced in Definition 3.105 with the homomorphism* $\overline{\partial} \colon K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G) \to K_0(\mathbb{Z}G)$ *induced by the boundary homomorphism of the localization sequence* (3.104) *is the classical Swan homomorphism* $\mathrm{sw}^G \colon \mathbb{Z}/m^\times \to K_0(\mathbb{Z}G)$ *of Definition 3.98;*

(ii) The generalized Swan homomorphism $\overline{\mathrm{sw}}^G \colon \mathbb{Z}/m^\times \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$ *is injective.*

3.10.4 The Generalized Swan Homomorphism and Free Homotopy Representations

In this subsection we briefly discuss Reidemeister torsion for free homotopy representations. For proofs and more information we refer to [629, Chapter 20].

Let G be a finite group of order $m = |G|$. Let X be a free d-dimensional G-homotopy representation. Suppose that we have fixed an orientation, i.e., a generator of $H_d(X;\mathbb{Z})$. Then we can define a kind of Reidemeister torsion of X

$$\overline{\rho}^G(X) \in K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G) \tag{3.107}$$

as follows. The change of rings map $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Z}_{(m)}G)$ is trivial, see [919, Theorem 7.1 and Theorem 8.1]. Hence there is a finite free $\mathbb{Z}_{(m)}G$-chain complex F_* together with a $\mathbb{Z}_{(m)}G$-chain homotopy equivalence $f_* \colon F_* \to C_*(X) \otimes_{\mathbb{Z}G} \mathbb{Z}_{(m)}G$. Choose a $\mathbb{Z}_{(m)}G$-basis for F_*. Then $F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G$ is a finite based free $\mathbb{Q}G$-chain complex. Note that we have preferred isomorphisms of abelian group $H_0(X) \cong \mathbb{Z}$ and $H_d(X) \cong \mathbb{Z}$ and G acts trivially on $H_0(X)$ and $H_d(X)$. This induces preferred $\mathbb{Q}G$-isomorphisms $H_i(F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G) \cong \mathbb{Q}$ for $i = 0, d$ where we equip $\mathbb{Q}$ with the trivial G-action. This enables us to define a torsion invariant $\tau(F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G) \in \widetilde{K}_1(\mathbb{Q}G)$ although $F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G$ is not acyclic. Define $\overline{\rho}^G(X)$ to be its image under the projection $\widetilde{K}_1(\mathbb{Q}G) \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$. One easily checks that $\overline{\rho}^G(X)$ is independent of the choice of F_*, f_*, and the choice of the $\mathbb{Z}_{(m)}G$-basis for F_*. The proof of the following result is a special case of the results in [629, Theorem 20.37 on page 403 and Corollary 20.39 on page 404].

Theorem 3.108 (Torsion and free homotopy representations). *Let G be a finite group of order $m = |G| \geq 3$. Let X and Y be free oriented G-homotopy representations.*

(i) The homomorphism $\overline{\partial} \colon K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G) \to K_0(\mathbb{Z}G)$ *sends the torsion* $\overline{\rho}^G(X)$ *to the finiteness obstruction* $o(G\backslash X)$*;*

(ii) Let $f \colon X \to Y$ be a G-map, which always exists. Then its degree $\deg(f)$ *is prime to m and*

$$\overline{\mathrm{sw}}^G(\overline{\deg(f)}) = \overline{\rho}^G(Y) - \overline{\rho}^G(X);$$

(iii) The free G-homotopy representations X and Y are oriented G-homotopy equivalent if and only if $\overline{\rho}^G(X) = \overline{\rho}^G(Y)$.

Theorem 3.108 gives an interesting relation between torsion invariants and finite obstructions and generalizes the homotopy classification of lens spaces to free G-homotopy representations.

All this can be extended to not necessarily free G-homotopy representations, see [629, Section 20]. The theory of G-homotopy representations was initiated by tom Dieck-Petrie [937].

3.11 Variants of the Farrell-Jones Conjecture for $K_1(RG)$

In this section we state variants of the Farrell-Jones Conjecture for $K_1(RG)$. The Farrell-Jones Conjecture itself will give a complete answer for arbitrary rings but to formulate the full version some additional effort will be needed. If one assumes that R is regular and G is torsionfree, the conjecture reduces to an easy to formulate statement, which we will present next. Moreover, this special case is already very interesting.

Conjecture 3.109 (Farrell-Jones Conjecture for $K_0(RG)$ and $K_1(RG)$ for regular R and torsionfree G). Let G be a torsionfree group and let R be a regular ring. Then the maps defined in (3.26) and (3.27)

$$A_0\colon K_0(R) \xrightarrow{\cong} K_0(RG);$$
$$A_1\colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(RG),$$

are both isomorphisms. In particular the groups $\mathrm{Wh}_0^R(G)$ and $\mathrm{Wh}_1^R(G)$ introduced in Definition 3.28 vanish.

We mention the following important special case of Conjecture 3.109.

Conjecture 3.110 (Farrell-Jones Conjecture for $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ for torsionfree G). Let G be a torsionfree group. Then $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

We have already discussed the K_0-part of the two conjectures above in Section 2.8. The following exercise shows that we cannot expect to have an analogue for $K_1(RG)$ of the Conjecture 2.67.

Exercise 3.111. Let G be a group and let R be a ring. Suppose that the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G\times\mathbb{Z})} K_1(RH) \to K_1(R[G \times \mathbb{Z}]).$$

is surjective. Show that then $K_0(RG) = 0$ and hence $K_0(R) = 0$. In particular, R cannot be a commutative integral domain.

Remark 3.112 (Relevance of Conjecture 3.110). In view of Remark 3.13 Conjecture 3.110 predicts for a torsionfree group G that any matrix A in $\mathrm{GL}_n(\mathbb{Z}G)$ can be transformed by a sequence of the operations mentioned in Remark 3.13 to a $(1,1)$-matrix of the form $(\pm g)$ for some $g \in G$. This is the algebraic relevance of this conjecture. Its geometric meaning comes from the following conclusion of the s-Cobordism Theorem 2.39. Namely, if G is a finitely presented torsionfree group, and n an integer with $n \geq 6$, then it implies that every compact n-dimensional h-cobordism is trivial.

3.12 Survey on Computations of $K_1(\mathbb{Z}G)$ for Finite Groups

In contrast to $K_0(\mathbb{Z}G)$ for finite groups G, the Whitehead group $\mathrm{Wh}(G)$ of a finite group is very well understood. The key source for the computation of $\mathrm{Wh}(G)$ for finite groups G is the book written by Oliver [760].

Definition 3.113 ($SK_1(\mathbb{Z}G)$ and $\mathrm{Wh}'(G)$). Let G be a finite group. Define

$$\begin{aligned} SK_1(\mathbb{Z}G) &:= \ker((K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G))\,; \\ \mathrm{Wh}'(G) &= \mathrm{Wh}(G)/\operatorname{tors}(\mathrm{Wh}(G)). \end{aligned}$$

Remark 3.114 ($SK_1(\mathbb{Z}G)$ and reduced norms). Let G be a finite group. The *reduced norm* on $\mathbb{C}G$ is defined as the composite of isomorphisms of abelian groups

$$\begin{aligned} \mathrm{nr}_{\mathbb{C}G}\colon K_1(\mathbb{C}G) \xrightarrow{\phi_*} K_1\left(\prod_{i=1}^k M_{r_i}(\mathbb{C})\right) &\xrightarrow{\cong} \prod_{i=1}^k K_1(M_{r_i}(\mathbb{C})) \\ &\xrightarrow{\cong} \prod_{i=1}^k K_1(\mathbb{C}) \xrightarrow{\prod_{i=1}^k \det} \prod_{i=1}^k \mathbb{C}^\times \end{aligned}$$

where the isomorphism of rings $\phi\colon \mathbb{C}G \xrightarrow{\cong} \prod_{i=1}^k M_{r_i}(\mathbb{C})$ comes from Wedderburn's Theorem applied to the semisimple ring $\mathbb{C}G$ and the remaining three isomorphisms come from Theorem 3.6, Lemma 3.8 and Lemma 3.9. The *reduced norm* on RG for $R = \mathbb{Z}, \mathbb{Q}$ is defined as the composite

$$\mathrm{nr}_{RG}\colon K_1(RG) \xrightarrow{i_R} K_1(\mathbb{C}G) \xrightarrow{\mathrm{nr}_{\mathbb{C}G}} \prod_{i=1}^k \mathbb{C}^\times$$

where i_R is the obvious change of rings homomorphism. The map $i_{\mathbb{Q}}$ is injective, see [760, Theorem 2.5 on page 43]). Thus we can identify

$$SK_1(\mathbb{Z}G) \;=\; \ker\left(\mathrm{nr}_{\mathbb{Z}G}\colon K_1(\mathbb{Z}G) \to \prod_{i=1}^{k} \mathbb{C}^{\times}\right).$$

This identification is useful for investigating $SK_1(\mathbb{Z}G)$ and $\mathrm{Wh}'(G)$. We conclude that for abelian groups the two definitions of $SK_1(\mathbb{Z}G)$ appearing in Definition 3.15 and Definition 3.113 agree.

We denote by $r_F(G)$, the number of isomorphism classes of irreducible representations of the finite G over the field F. Recall that $r_F = |\operatorname{con}_F(G)|$ by Lemma 2.88. The proof of the next result can be found for instance in [760, Theorem 2.5 on page 48] and is based on the Dirichlet Unit Theorem 3.21.

Theorem 3.115 ($SK_1(\mathbb{Z}G) = \mathrm{tors}(\mathrm{Wh}(G))$). *Let G be a finite group. Then the abelian group $SK_1(\mathbb{Z}G)$ is finite and agrees with the torsion subgroup* $\mathrm{tors}(\mathrm{Wh}(G))$ *of* $\mathrm{Wh}(G)$. *The group* $\mathrm{Wh}'(G) = \mathrm{Wh}(G)/\mathrm{tors}(\mathrm{Wh}(G))$ *is a finitely generated free abelian group of rank* $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.

Hence the next step is to compute $SK_1(\mathbb{Z}G)$. This is done using localization sequences, see [760, Theorem 1.17 on page 36 and Section 3c], which do also involve the second algebraic K-group, see Chapter 5, and are consequences of the general result of Quillen stated in Theorem 6.49. Define

$$SK_1(\widehat{\mathbb{Z}_p}G) \;:=\; \ker\left(K_1(\widehat{\mathbb{Z}_p}G) \to K_1(\widehat{\mathbb{Q}_p}G)\right).$$

Put

$$\mathrm{Cl}_1(\mathbb{Z}G) \;:=\; \ker\left(SK_1(\mathbb{Z}G) \to \prod_{p|\ |G|} SK_1(\widehat{\mathbb{Z}_p}G)\right)$$

where p runs over all prime numbers dividing $|G|$. Then one obtains an exact sequence

$$0 \to \mathrm{Cl}_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to \prod_{p|\ |G|} SK_1(\widehat{\mathbb{Z}_p}G) \to 0.$$

The analysis of $\mathrm{Cl}_1(\mathbb{Z}G)$ and $SK_1(\widehat{\mathbb{Z}_p}G)$ is carried out independently and with different methods. Besides localization sequences p-adic logarithms play a key role. Details can be found in Oliver [760].

Given a groups G and Q the *wreath product* $G \wr Q$ is defined to be the semidirect product $\prod_Q G \rtimes Q$ where Q acts on $\prod_Q G \rtimes Q$ permuting the factors.

Theorem 3.116 (Finite groups with vanishing $\mathrm{Wh}(G)$ or $SK_1(\mathbb{Z}G)$). *Let G be a finite group.*

(i) Let p be a prime number. If the p-Sylow subgroup S_pG of G is isomorphic to $\mathbb{Z}/p^n$ or $\mathbb{Z}/p^n \times \mathbb{Z}/p$ for some $n \geq 0$, then $SK_1(\mathbb{Z}G)_{(p)} = 0$, i.e., the finite abelian group $SK_1(\mathbb{Z}G)$ contains no p-torsion;

(ii) Let G be a finite abelian group. Then $SK_1(\mathbb{Z}G) = 0$ if and only if for every prime p the p-Sylow subgroup S_pG is isomorphic to $\mathbb{Z}/p^n$ or $\mathbb{Z}/p^n \times \mathbb{Z}/p$ for some $n \geq 0$ or if $G = (\mathbb{Z}/2)^n$ for some $n \geq 1$;

(iii) Let $\mathcal{C}_{\mathrm{Wh}}$ be the smallest class of groups that is closed under finite products and wreath products with S_n for every $n \geq 2$ and contains the trivial group. Let $\mathcal{C}_{SK_1}$ be the smallest class of groups that is closed under finite products and wreath products with S_n for every $n \geq 2$ and contains the dihedral groups D_n for $n \geq 2$.
Then $\mathrm{Wh}(G) = 0$ for $G \in \mathcal{C}_{\mathrm{Wh}}$ and $SK_1(\mathbb{Z}G) = 0$ if $G \in \mathcal{C}_{SK_1}$;

(iv) We have $SK_1(\mathbb{Z}G) = 0$ if G is one of the following groups

(a) G is finite cyclic;
(b) $\mathbb{Z}/p^n \times \mathbb{Z}/p$ for some prime p and $n \geq 1$;
(c) $(\mathbb{Z}/2)^n$ for $n \geq 1$;
(d) G is any symmetric group;
(e) G is any dihedral group;
(f) G is any semidihedral 2-group.

Proof. (i) See Oliver [760, Theorem 14.2 (i) on page 330].

(ii) See Oliver [760, Theorem 14.2 (iii) on page 330].

(iii) See Oliver [760, Theorem 14.1 on page 328].

(iv) This follows essentially from the other assertions. See Oliver [760, Examples 1 and 2 on page 14]. □

The group $SK_1(\mathbb{Z}G)$ can be computed for many examples. We mention the following example taken from [760, Theorem 14.6 on page 336].

Example 3.117 ($SK_1(\mathbb{Z}[A_n])$). We have $SK_1(\mathbb{Z}[A_n]) \cong \mathbb{Z}/3$ if we can write $n = \sum_{i=1}^{r} 3^{m_i}$ such that $m_1 > m_2 > \cdots > m_r > 0$ and $\sum_{i=1}^{r} m_i$ is odd. Otherwise we get $SK_1(\mathbb{Z}[A_n]) = \{0\}$.

Exercise 3.118. Show that the Whitehead group $\mathrm{Wh}(\mathbb{Z}/m)$ of the finite cyclic group $\mathbb{Z}/m$ of order m is a free abelian group of rank $\lfloor m/2 \rfloor + 1 - \delta(m)$, where $\lfloor m/2 \rfloor$ is the greatest integer less or equal to $m/2$ and $\delta(m)$ is the number of divisors of m.

Let p be a prime. Show that $\mathrm{Wh}(\mathbb{Z}/p)$ is isomorphic to $\mathbb{Z}^{(p-1)/2-1}$ if p is odd and is trivial if $p = 2$.

3.13 Survey on Computations of Algebraic $K_1(C_r^*(G))$ and $K_1(\mathcal{N}(G))$

Define $\mathrm{SL}(R) := \{A \in \mathrm{GL}(R) \mid \det(A) = 1\}$. Let B be a commutative Banach algebra. Then $\mathrm{GL}_n(B)$ inherits a topology, namely the subspace topology for the obvious embedding $\mathrm{GL}_n(B) \subseteq M_n(B) = \prod_{i=1}^{n^2} B$. Equip $\mathrm{GL}(B) = \bigcup_{n\geq 1} \mathrm{GL}_n(B)$ with the weak topology, i.e., a subset $A \subset \mathrm{GL}(B)$ is closed if and only if $A \cap \mathrm{GL}_n(B)$ is a closed subset of $\mathrm{GL}_n(B)$ for all $n \geq 1$. Equip $\mathrm{SL}(B) \subseteq \mathrm{GL}(B)$ with the subspace topology.

The following results are due to Milnor [712, Corollary 7.2 on page 57 and Corollary 7.3 on page 58].

Theorem 3.119 ($K_1(B)$ of a commutative Banach algebra). *Let B be a commutative Banach algebra. Then there is a natural isomorphism*

$$K_1(B) \xrightarrow{\cong} B^\times \times \pi_0(\mathrm{SL}(B)).$$

Define the *infinite special orthogonal group* $\mathrm{SO} = \bigcup_{n\geq 1} \mathrm{SO}(n)$ and *infinite special unitary group* $\mathrm{SU}, = \bigcup_{n\geq 1} \mathrm{SU}(n)$ where $\mathrm{SO}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^t = I, \det(A) = 1\}$ is the special n-th orthogonal group and $\mathrm{SU}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid AA^* = I, \det(A) = 1\}$ is the special n-th unitary group. Denote by $[X, \mathrm{SO}]$ and $[X, \mathrm{SU}]$ respectively the set of homotopy classes of maps from X to SO and SU respectively.

Theorem 3.120 ($K_1(C(X))$ of a commutative C^*-algebra $C(X)$). *Let X be compact space. Then there are natural isomorphisms*

$$K_1(C(X,\mathbb{R})) \xrightarrow{\cong} C(X,\mathbb{R})^\times \times [X, \mathrm{SO}];$$
$$K_1(C(X,\mathbb{C})) \xrightarrow{\cong} C(X,\mathbb{C})^\times \times [X, \mathrm{SU}].$$

The sets $[X, \mathrm{SO}]$ and $[X, \mathrm{SU}]$ are closely related to the topological K-groups $KO^{-1}(X)$ and $K^{-1}(X)$.

If B is a group C^*-algebra $C_r^*(G)$, then not much is known about the algebraic K-group $K_1(B)$ in general. However, the algebraic K_1-group of a von Neumann algebra is fully understood, see [635, Section 9.3],[662]. We mention the special case, see [635, Example 9.34 on page 353], that for a finitely generated group G which is not virtually finitely generated abelian the Fuglede-Kadison determinant induces an isomorphism

$$K_1(\mathcal{N}(G)) \xrightarrow{\cong} \mathcal{Z}(\mathcal{N}(G))^{+,\mathrm{inv}} \tag{3.121}$$

where $\mathcal{Z}(\mathcal{N}(G))^{+,\mathrm{inv}}$ consists of the elements of the center of $\mathcal{N}(G)$ that are both positive and invertible.

The connection between the algebraic and the topological K-theory of a C^*-algebra will be discussed in Section 10.7.

3.14 Notes

A universal property describing the Whitehead group and the Whitehead torsion similar to the description of the finiteness obstruction in Section 2.7 is presented in [629, Theorem 6.11].

Geometric versions or analogues of maps related to the Bass-Heller-Swan decomposition are described in [335], [365], [629, (7.34) on page 130], and [824, § 10].

Given two groups G_1 and G_2, let $G_1 * G_2$ by the amalgamated free product. Then the natural maps $G_k \to G_0 * G_1$ for $k = 1, 2$ induce an isomorphism, see [906],

$$\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \cong \mathrm{Wh}(G_1 * G_2). \tag{3.122}$$

Compare this with the analog for the reduced projective class groups stated in (2.125).

Exercise 3.123. Show that the projections $\mathrm{pr}_k \colon G_1 \times G_2 \to G_k$ for $k = 1, 2$ do *not* in general induce an isomorphism

$$\mathrm{Wh}(G_1 \times G_2) \xrightarrow{\cong} \mathrm{Wh}(G_1) \times \mathrm{Wh}(G_2).$$

There are also equivariant versions of the Whitehead torsion, see for instance [629, Chapter 4 and Chapter 12], where more references can be found.

Next we discuss the following conjecture.

Conjecture 3.124 (Unit-Conjecture). Let R be an integral domain and G be a torsionfree group. Then every unit in RG is trivial, i.e., of the form $r \cdot g$ for some unit $r \in R^{\times}$ and $g \in G$.

For more information about it we refer for instance to [594, page 95].

Remark 3.125 (Status of the Unit Conjecture and its stable version). Actually, Gardam found an explicite counterexample to the Unit Conjecture, see [401, Theorem A]. His group G is given by the presentation

$$\langle a, b \mid ba^2b^{-1} = a^{-2}, ab^2a^{-1} = b^{-2} \rangle.$$

It can be written as a non-split extension extension $1 \to \mathbb{Z}^3 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$ and is a crystallographic group. The underlying coefficient ring is the field of two elements $\mathbb{F}_2$. Note that Gardam found his counterexample using computer algebra, but in his paper he presents a short human-readable proof. Counterexamples where the coefficient ring is a field of (arbitrary) prime characteristic were constructed by Murray [738]. Gardam [402, Theorem A] constructed counterexamples with coefficients in $\mathbb{C}$ for the same group G as above.

Note that Conjecture 3.109 does not imply the Unit Conjecture 3.124. At least the bijectivity of the map A_1 implies the stable version of the Unit Conjecture 3.124 that the class $[x] \in K_1(RG)$ of any unit $x \in RG^\times$ is represented by the class $[u]$ of some trivial unit u, or, equivalently, by a sequence of elementary row and column operation and (de-)stabilization one can transform the $(1,1)$-matrix (x) to the $(1,1)$-matrix (u), see Remark 3.13, provided that $\widetilde{K}_0(R)$ vanishes.

Note, that the map $(\mathbb{Z}G)^\times \to K_1(\mathbb{Z}G)$ sending a unit to its class in the K_1-group is in general not injective and in general not every unit is a trivial unit, as the following example shows. If G is a finite group, then a result of Hartley-Pickels [451, Theorem 2] says that exactly one of the following cases occurs:

- G is abelian and $(\mathbb{Z}G)^\times$ is abelian;
- G is a Hamiltonian 2-group and $(\mathbb{Z}G)^\times = \{\pm g \mid g \in G\}$;
- $(\mathbb{Z}G)^\times$ contains a free subgroup of rank 2.

Hence for the symmetric group S_n for $n \geq 3$, the group of units $\mathbb{Z}[S_n]^\times$ is infinite, whereas $\mathrm{Wh}(S_n)$ vanishes, see Theorem 3.116 (iii), and hence $K_1(\mathbb{Z}[S_n])$ and $\{\pm g \mid g \in S_n\}$ are finite. This implies that the map $(\mathbb{Z}[S_n])^\times \to K_1(\mathbb{Z}[S_n])$ has an infinite kernel for $n \geq 3$ and that there are infinitely many elements in $(\mathbb{Z}[S_n])^\times$ which are not trivial units.

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Chapter 4
Negative Algebraic K-Theory

4.1 Introduction

In this chapter we introduce *negative K-groups*. They are designed such that the *Bass-Heller-Swan decomposition* and the long exact sequence of a pullback of rings and of a two-sided ideal extend beyond K_0. We give a geometric interpretation of negative K-groups of group rings in terms of bounded h-cobordisms. We state variants of the *Farrell-Jones Conjecture for negative K-groups* and give a survey of computations for group rings of finite groups.

4.2 Definition and Basic Properties of Negative K-Groups

Recall that we get from Theorem 3.75 an isomorphism

$$K_0(R) = \operatorname{coker}\left(K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t,t^{-1}])\right).$$

This motivates the following definition of negative K-groups due to Bass.

Definition 4.1. Given a ring R, define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t,t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$NK_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Obviously a ring homomorphism $f \colon R \to S$ induces for $n \le -1$ a map of abelian groups

$$K_n(f) \colon K_n(R) \to K_n(S). \tag{4.2}$$

The Bass-Heller-Swan decomposition 3.72 for $K_1(R[t,t^{-1}])$ extends to negative K-theory.

Theorem 4.3 (Bass-Heller-Swan decomposition for middle and lower K-theory). *There are isomorphisms of abelian groups, natural in R, for $n = 1, 0, -1, -2, \ldots$*

$$NK_n(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t]);$$
$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]).$$

There is a sequence which is natural in R and exact for $n = 1, 0, -1, \ldots$

$$0 \to K_n(R) \xrightarrow{K_n(k_+) \oplus -K_n(k_-)} K_n(R[t]) \oplus K_n(R[t^{-1}]) \xrightarrow{K_n(l_+) \oplus K_n(l_-)_*} K_n(R[t,t^{-1}]) \xrightarrow{C_n} K_{n-1}(R) \to 0$$

where k_+, k_-, l_+, and l_- are the obvious inclusions.

If we regard it as an acyclic $\mathbb{Z}$-chain complex, there exists a chain contraction, natural in R.

Proof. We give the proof only for $n = 0$, then an iteration of the argument proves the claim for all $n \le 0$. Take $S = R[\mathbb{Z}] = R[x,x^{-1}]$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K_0(R) \arrow[r, "f_3"] \arrow[d, "K_0(k_+) \oplus -K_0(k_-)"'] & K_1(S) \arrow[d, "K_0(k_+) \oplus -K_0(k_-)"] \\
K_0(R[t]) \oplus K_0(R[t^{-1}]) \arrow[r, "f_2"] \arrow[d, "K_0(l_+) \oplus K_0(l_-)"'] & K_1(S[t]) \oplus K_1(S[t^{-1}]) \arrow[d, "K_0(l_+) \oplus K_0(l_-)"] \\
K_0(R[t,t^{-1}]) \arrow[r, "f_1"] \arrow[d, "C'"'] & K_1(S[t,t^{-1}]) \arrow[d, "C"] \\
K_{-1}(R) \arrow[r, "f_0"] \arrow[d] & K_0(S) \arrow[d] \\
0 & 0
\end{tikzcd}$$

where the right column is the exact sequence appearing in Theorem 3.75, the map C' is the canonical projection, the maps f_1, f_2, and f_3 come from the Bass-Heller-Swan decompositions for $S = R[x,x^{-1}]$, $S[t] = R[t][x,x^{-1}]$, $S[t^{-1}] = R[t^{-1}][x,x^{-1}]$, and $S[t,t^{-1}] = R[t,t^{-1}][x,x^{-1}]$, and the map f_0 is the unique map that makes the diagram commutative. There are natural retractions r_k of f_k for $k = 1,2,3$ for which the diagram remains commutative, and a natural chain contraction $\gamma = \{\gamma_k \mid k = 0,1,2\}$ of the right column, see Theorem 3.72. Let $r_0 \colon K_0(S) \to K_{-1}(R)$ be the unique map that satisfies $r_0 \circ C = C' \circ r_1$. An easy diagram shows that r_0 is well-defined since $C' \circ r_3$ sends the kernel of C to zero. One easily checks $r_0 \circ f_0 = \mathrm{id}$. We obtain a chain contraction for the left column by considering the composites $r_k \circ \gamma_k \circ f_{k+1}$ for $k = 0, 1, 2$. $\square$

Remark 4.4 (Extending exact sequences to negative K-theory). The Mayer-Vietoris sequence of a pullback of rings, see Theorem 3.86, can be extended to negative K-theory and also to K_2 as we will explain in Theorem 5.9. Similarly, the long exact sequence of a two-sided ideal appearing in Theorem 3.89 can be extended to negative K-theory and also to K_2, as we will explain in Theorem 5.12.

Exercise 4.5. Let R and S be rings. Show for $n \le 1$ that the projections induce an isomorphism

$$K_n(R \times S) \xrightarrow{\cong} K_n(R) \times K_n(S).$$

Definition 4.6. Define for $n \le 1$ inductively for $p = 0, 1, 2, \ldots$

$$\begin{aligned} N^0K_n(R) &:= K_n(R); \\ N^{p+1}K_n(R) &:= \operatorname{coker}\left(N^pK_n(R) \to N^pK_n(R[t])\right). \end{aligned}$$

Obviously $N^1K_n(R)$ agrees with $NK_n(R)$.

Theorem 4.7 (Bass-Heller-Swan decomposition for lower and middle K-theory for regular rings). *Suppose that R is regular. Then we get*

$$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \le -1; \\ N^pK_n(R) &= 0 \quad \text{for } n \le 1 \text{ and } p \ge 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition appearing in Theorem 4.3 reduces for $n \le 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]).$$

Proof. The Bass-Heller-Swan decomposition, see Theorem 4.3, applied to R and $R[t]$ together with the obvious maps $i\colon R \to R[t]$ and $\epsilon\colon R[t] \to R$ satisfying $\epsilon \circ i = \mathrm{id}_R$ yield a natural Bass-Heller-Swan decomposition

$$NK_n(R) \oplus NK_{n-1}(R) \oplus N^2K_n(R) \oplus N^2K_n(R) \xrightarrow{\cong} NK_n(R[\mathbb{Z}]). \tag{4.8}$$

Hence $NK_{n-1}(R) = 0$ if $NK_n(R[\mathbb{Z}]) = 0$. If R is regular, then $R[\mathbb{Z}]$ is regular by Theorem 3.80 (i). Hence $NK_{n-1}(R)$ vanishes for all regular rings R if $NK_n(R)$ vanishes for all regular rings. We have shown in Theorem 3.81 that $NK_1(R)$ vanishes for all regular rings R. We conclude by induction over n that $NK_n(R)$ vanishes for all regular rings R and $n \le 1$. Obviously $N^pK_n(R)$ is a direct summand in $NK_n(R[t])$ and $R[t]$ is regular by Theorem 3.80 (i). Hence $N^pK_n(R)$ vanishes for $p \ge 1$ and $n \le 1$ if R is a regular ring.

Next we show that $K_{-1}(R) = 0$ for every regular ring. It suffices to show that the obvious map $K_0(R[t]) \to K_0(R[t,t^{-1}])$ is surjective. The homomorphism

$$\alpha\colon G_0(R[t]) \to G_0(R[t,t^{-1}]), \quad [M] \to [M \otimes_{R[t]} R[t,t^{-1}]]$$

is well-defined since $R[t,t^{-1}]$ is a localization of $R[t]$ and hence flat as $R[t]$-module. Since R by assumption and hence $R[t]$ and $R[t,t^{-1}]$ by Theorem 3.80 (i) are regular, we conclude from Lemma 3.85 that it remains to prove surjectivity of α. Let M be a finitely generated $R[t,t^{-1}]$-module. Since $R[t,t^{-1}]$ is Noetherian, we can find a matrix $A \in M_{m,n}(R[t,t^{-1}])$ such that there exists an exact sequence of $R[t,t^{-1}]$-modules $R[t,t^{-1}]^m \xrightarrow{A} R[t,t^{-1}]^n \to M \to 0$. Since t is invertible in $R[t,t^{-1}]$, the sequence remains exact if we replace A by t^kA for some $k \geq 1$. Hence we can assume without loss of generality that $A \in M_{m,n}(R[t])$. Define the $R[t]$-module N to be the cokernel of $R[t]^m \xrightarrow{A} R[t]^n$. Then $N \otimes_{R[t]} R[t,t^{-1}]$ is $R[t,t^{-1}]$-isomorphic to M and hence $\alpha([N]) = [M]$.

Now $K_n(R) = 0$ follows inductively for $n \leq -1$ for every regular ring from Theorem 3.80 (i) and the Bass-Heller-Swan decomposition 4.3.

Finally apply Theorem 4.3. □

Exercise 4.9. Let R be a regular ring. Prove

$$\begin{aligned} K_1(R[\mathbb{Z}^k]) &= K_1(R) \oplus \bigoplus_{i=1}^{k} K_0(R); \\ K_0(R[\mathbb{Z}^k]) &\cong K_0(R); \\ K_n(R[\mathbb{Z}^k]) &\cong 0 \quad \text{for } n \leq -1. \end{aligned}$$

Example 4.10 ($K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ for $n \leq 0$ and a prime p). Let p be a prime number. We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0 \quad \text{for } n \leq -1 \text{ and } k \geq 0$$

and that $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is finitely generated for $k \geq 0$. Consider the pullback of rings appearing in the proof of Rim's Theorem in Section 3.8.

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}/p] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}[\exp(2\pi i/p)] \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"'] & \mathbb{F}_p. \end{tikzcd}$$

If we apply $- \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^k]$, we obtain the pullback of rings

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k] \arrow[d, "j_1"] \\
\mathbb{Z}[\mathbb{Z}^k] \arrow[r, "j_2"'] & \mathbb{F}_p[\mathbb{Z}^k].
\end{tikzcd}$$

The ring $\mathbb{Z}[\exp(2\pi i/p)]$ is a Dedekind domain, see Theorem 2.23, and in particular regular. The rings $\mathbb{Z}$ and $\mathbb{F}_p$ are regular as well. Hence the rings $\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]$, $\mathbb{Z}[\mathbb{Z}^k]$ and $\mathbb{F}_p[\mathbb{Z}^k]$ are regular by Theorem 3.80 (i). The negative K-groups of $\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]$, $\mathbb{Z}[\mathbb{Z}^k]$ and $\mathbb{F}_p[\mathbb{Z}^k]$ vanish by Theorem 4.7. The obvious maps

$$\begin{aligned}
K_0(\mathbb{Z}) &\xrightarrow{\cong} K_0(\mathbb{Z}[\mathbb{Z}^k]); \\
K_0(\mathbb{Z}[\exp(2\pi i/p)]) &\xrightarrow{\cong} K_0(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]); \\
K_0(\mathbb{F}_p) &\xrightarrow{\cong} K_0(\mathbb{F}_p[\mathbb{Z}^k]),
\end{aligned}$$

are bijective because of Theorem 4.7. Hence the associated long exact Mayer-Vietoris sequence, see Remark 4.4, implies that $K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$ holds for $n \leq -2$ and that we get the exact sequence

$$\begin{aligned}
&K_1(\mathbb{F}_p[\mathbb{Z}^k]) \to K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \\
&\quad \to K_0(\mathbb{Z}) \oplus K_0(\mathbb{Z}[\exp(2\pi i/p)]) \to K_0(\mathbb{F}_p) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to 0.
\end{aligned}$$

Since $\mathbb{F}_p$ is a field and hence $K_0(\mathbb{F}_p)$ is generated by $[\mathbb{F}_p]$, see Example 2.4, we conclude $K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$. Example 2.4, Theorem 3.17, and Theorem 4.7 imply $K_1(\mathbb{F}_p[\mathbb{Z}^k]) \cong K_1(\mathbb{F}_p) \oplus K_0(\mathbb{F}_p)^k \cong (\mathbb{F}_p)^\times \oplus \mathbb{Z}^k$. The abelian group $K_0(\mathbb{Z}) \oplus K_0(\mathbb{Z}[\exp(2\pi i/p)])$ is finitely generated by Theorem 2.23. Hence $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is finitely generated.

Exercise 4.11. Show for $k \geq 0$ and $n \leq 0$ that $\widetilde{K}_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$. Prove that $N^p K_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$ holds for $n \leq -1$ and $p \geq 0$ and for $n = 0$ and $p \geq 1$.

Example 4.12 (Negative K-theory of $\mathbb{Z}[\mathbb{Z}/6]$). We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z} & n = -1; \\ 0 & n \leq -2. \end{cases}$$

Consider the pullback of rings

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z} \arrow[d, "j_1"] \\
\mathbb{Z} \arrow[r, "j_2"'] & \mathbb{Z}/2
\end{tikzcd}$$

where i_1 sends $a+bt$ to $a-b$ and i_2 sends $a+bt$ to $a+b$ for $t \in \mathbb{Z}/2$ the generator and the two maps from $\mathbb{Z}$ to $\mathbb{Z}/2$ are the canonical projections. Since $\mathbb{Z}[\mathbb{Z}/3]$ is free as abelian group, this remains to be a pullback of rings if we apply $-\otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$. We have isomorphisms of rings $\mathbb{Z}[\mathbb{Z}/2]\otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] = \mathbb{Z}[\mathbb{Z}/6]$ and $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}[\mathbb{Z}/3] = \mathbb{Z}[\mathbb{Z}/3]$. From the pullback for $p=3$ appearing in Example 4.10 we obtain an isomorphism of rings

$$\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times (\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\exp(2\pi i/3)]).$$

The ring $\mathbb{Z}[\exp(2\pi i/3)]$ is as abelian group free with two generators 1 and $\omega = \exp(2\pi i/3)$ and the multiplication is uniquely determined by $\omega^2 = -1-\omega$. Hence $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\exp(2\pi i/3)]$ contains four elements, namely 0, 1, $1\otimes\omega$ and the sum $1+1\otimes\omega$. Since $(1\otimes\omega)\cdot(1+1\otimes\omega)=1$, it is the field $\mathbb{F}_4$ consisting of four elements. Hence we obtain a pullback of rings

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/6] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}[\mathbb{Z}/3] \arrow[d, "j_1"] \\
\mathbb{Z}[\mathbb{Z}/3] \arrow[r, "j_2"'] & \mathbb{F}_2 \times \mathbb{F}_4.
\end{tikzcd}$$

Since $K_n(\mathbb{F}_2 \times \mathbb{F}_4) \cong K_n(\mathbb{F}_2) \times K_n(\mathbb{F}_4)$ vanishes for $n \le -1$ and $K_n(\mathbb{Z}[\mathbb{Z}/3])$ vanishes for $n \le -1$ by Example 4.10, the associated long exact Mayer-Vietoris sequence, see Remark 4.4, implies that $K_n(\mathbb{Z}[\mathbb{Z}/6]) = 0$ holds for $n \le -2$ and there is an exact sequence

$$K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to 0.$$

Since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$ is trivial, see Example 2.106, and the projections induce an isomorphism $K_0(\mathbb{F}_2 \times \mathbb{F}_4) \xrightarrow{\cong} K_0(\mathbb{F}_2) \times K_0(\mathbb{F}_4) \cong \mathbb{Z}\oplus\mathbb{Z}$, we conclude $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$.

Exercise 4.13. Consider $k \in \{0,1,2,\ldots\}$. Compute

$$K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z}^{k+1} & \text{for } n = 0; \\ \mathbb{Z} & \text{for } n = -1; \\ 0 & \text{for } n \le -2, \end{cases}$$

and prove $N^pK_n(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) = 0$ for $p \ge 1$ and $n \le 0$.

The Bass-Heller-Swan decomposition can be used to show that certain results about the K-groups in a fixed degree m have implications to all the K-groups in degree $n \le m$, as illustrated by the next result.

Lemma 4.14. *Consider a ring R and $m \in \mathbb{Z}$ with $m \le 1$. Suppose that for every $k \ge 1$ the map $K_m(R) \to K_m(R[\mathbb{Z}^k])$ induced by the inclusion $R \to R[\mathbb{Z}^k]$ is bijective.*

Then $K_n(R[\mathbb{Z}^l]) = 0$ for $n \le m-1$ and $NK_n(R[\mathbb{Z}^l]) = 0$ for $n \le m$ hold for all $l \ge 0$.

Proof. Since the bijectivity of $K_m(R) \to K_m(R[\mathbb{Z}^k])$ for all $k \ge 1$ implies the bijectivity of $K_m(R[\mathbb{Z}^l]) \to K_m((R[\mathbb{Z}^l])[\mathbb{Z}^k])$ for all $k, l \ge 0$ because of the identification $(R[\mathbb{Z}^l])[\mathbb{Z}^k] = R[\mathbb{Z}^{k+l}]$, it suffices to treat the case $l = 0$.

Consider any integer $k \ge 1$. The assumptions in Lemma 4.14 imply that the map $K_m(R[\mathbb{Z}^{k-1}]) \to K_m(R[\mathbb{Z}^k])$ induced by the inclusion $R[\mathbb{Z}^{k-1}] \to R[\mathbb{Z}^k]$ is bijective. Theorem 4.3 applied to the ring $R[\mathbb{Z}^{k-1}]$ together with the identity $R[\mathbb{Z}^k] = (R[\mathbb{Z}^{k-1}])[\mathbb{Z}]$ shows that $K_{m-1}(R[\mathbb{Z}^{k-1}]) = 0$ and $NK_m(R[\mathbb{Z}^{k-1}]) = 0$. Using Theorem 4.3 and the Bass-Heller-Swan-decomposition for NK, see (4.8), one shows inductively for $i = 0, 1, \ldots, (k-1)$ that $K_{m-1-j}(R[\mathbb{Z}^{k-i-1}]) = 0$ and $NK_{m-j}(R[\mathbb{Z}^{k-i-1}]) = 0$ holds for $j = 0, 1 \ldots, i$. Then the case $i = k-1$ shows that $K_n(R) = 0$ for $m-k \le n \le m-1$ and $NK_n(R) = 0$ for $m-k+1 \le n \le m$. Since $k \ge 1$ was arbitrary, Lemma 4.14 follows. □

Exercise 4.15. Consider a ring R and $m \in \mathbb{Z}$ with $m \le 1$. Suppose that $K_m(R[\mathbb{Z}^k]) = 0$ for every $k \ge 1$. Then $K_i(R[\mathbb{Z}^l]) = NK_i(R[\mathbb{Z}^l]) = 0$ holds for $i \le m$ and $l \ge 0$.

Theorem 4.16 (The middle and lower K-theory of RG for finite G and Artinian R). *Let G be a finite group, and let R be an Artinian ring. Then:*

(i) For every $k \ge 0$ the map

$$K_0(RG) \xrightarrow{\cong} K_0(RG[\mathbb{Z}^k])$$

induced by the inclusion is bijective;

(ii) Given any $k \ge 0$, we have $K_n(RG[\mathbb{Z}^k]) = 0$ for $n \le -1$ and $NK_n(RG[\mathbb{Z}^k]) = 0$ for $n \le 0$.

Proof. (i) Denote by $J = \mathrm{rad}(RH) \subseteq RH$ the Jacobson radical of RH. Since R and hence RH are Artinian, there exists a natural number l with $JJ^l = J^l$. By Nakayama's Lemma, see [899, Proposition 8 in Chapter 2 on page 20], J^l is $\{0\}$, in other words, J is nilpotent. The ring RH/J is a semisimple Artinian ring, see [594, Definition 20.3 on page 311 and (20.3) on page 312], and in particular regular. Theorem 3.80 (ii) implies that $(RH/J)[\mathbb{Z}^k]$ is regular for all $k \ge 1$. We derive from Theorem 4.7 that $K_n((RH/J)[\mathbb{Z}^k]) = 0$ for $n \le -1$ and $NK_n((RH/J)[\mathbb{Z}^k])$ for $n \le 0$ hold for all $k \ge 0$. We conclude from Theorem 4.7 by induction over $k = 0, 1, 2 \ldots$ that the inclusion $RH/J \to (RH/J)[\mathbb{Z}^k]$ induces an isomorphism

$$K_0(RH/J) \xrightarrow{\cong} K_0((RH/J)[\mathbb{Z}^k])$$

for all $k \ge 0$.

The following diagram

$$\begin{tikzcd}
K_0(RH) \arrow[r] \arrow[d] & K_0(RH[\mathbb{Z}^k]) \arrow[d] \\
K_0(RH/J) \arrow[r] & K_0((RH/J)[\mathbb{Z}^k])
\end{tikzcd}$$

commutes. Since J is a nilpotent two-sided ideal of RH, $J[\mathbb{Z}^k]$ is a nilpotent two-sided ideal of $RH[\mathbb{Z}^k]$. Obviously $(RH/J)[\mathbb{Z}^k]$ can be identified with $(RH[\mathbb{Z}^k])/(J[\mathbb{Z}^k])$. Hence the vertical arrows in the diagram above are bijective by Lemma 2.124. Since the lower horizontal arrow is bijective for every $k \geq 1$, the upper horizontal arrow is bijective for every $k \geq 1$.

(ii) This follows from assertion (i) and Lemma 4.14 applied in the case $m = 0$ to the ring RG. $\square$

4.3 Geometric Interpretation of Negative K-Groups

One possible geometric interpretation of negative K-groups is in terms of bounded h-cobordisms.

We consider *manifolds W parametrized over* $\mathbb{R}^k$, i.e., manifolds that are equipped with a surjective proper map $p\colon W \to \mathbb{R}^k$. Recall that *proper map* means that preimages of compact subsets are compact again. We will always assume that the fundamental group(oid) is bounded, see [781, Definition 1.3]. A map $f\colon W \to W'$ between two manifolds parametrized over $\mathbb{R}^k$ is called *bounded* if $\{p' \circ f(x) - p(x) \mid x \in W\}$ is a bounded subset of $\mathbb{R}^k$.

A *bounded cobordism* $(W; M_0, f_0, M_1, f_1)$ is defined just as in Section 3.5 but compact manifolds are replaced by manifolds parametrized over $\mathbb{R}^k$ and the parametrization for M_l is given by $p_W \circ f_l$. If we assume that the inclusions $i_l\colon \partial_k W \to W$ are homotopy equivalences, then there exist deformations $r_l\colon W \times I \to W$ such that $r_l|_{W\times\{0\}} = \mathrm{id}_W$ and $r_l(W \times \{1\}) \subset \partial_l W$. A bounded cobordism is called a *bounded h-cobordism* if the inclusions i_l are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S_l = \{p_W(r_l(x,t)) - p_W(r_l(x,1)) \mid x \in W, t \in [0,1]\}$$

are bounded subsets of $\mathbb{R}^k$.

The following theorem, see [781] and [981, Appendix], contains the s-Cobordism Theorem 3.47 as a special case, gives another interpretation of elements in $\widetilde{K}_0(\mathbb{Z}\pi)$ and explains one aspect of the geometric relevance of negative K-groups.

Theorem 4.17 (Bounded h-Cobordism Theorem). *Suppose that M_0 is parametrized over $\mathbb{R}^k$ and satisfies $\dim M_0 \geq 5$. Let π be its fundamental group(oid). Equivalence classes of bounded h-cobordisms over M_0 modulo bounded diffeomorphism relative M_0 correspond bijectively to elements in $\kappa_{1-k}(\pi)$ where*

$$\kappa_{1-k}(\pi) = \begin{cases} \mathrm{Wh}(\pi) & \text{if } k = 0; \\ \widetilde{K}_0(\mathbb{Z}\pi) & \text{if } k = 1; \\ K_{1-k}(\mathbb{Z}\pi) & \text{if } k \geq 2. \end{cases}$$

4.4 Variants of the Farrell-Jones Conjecture for Negative *K*-Groups

In this section we state variants of the Farrell-Jones Conjecture for negative K-theory. The Farrell-Jones Conjecture itself will give a complete answer for arbitrary rings but to formulate the full version some additional effort will be needed. If one assumes that R is regular and G torsionfree or that $R = \mathbb{Z}$, the conjecture reduces to an easy to formulate statement, which we will present next.

Conjecture 4.18 (The Farrell-Jones Conjecture for negative K-theory and regular coefficient rings). Let R be a regular ring and G be a group such that for every finite subgroup $H \subseteq G$ the element $|H| \cdot 1_R$ of R is invertible in R. Then we get

$$K_n(RG) = 0 \quad \text{for} \quad n \leq -1.$$

Exercise 4.19. Prove that Conjecture 4.18 is true if G is finite.

Conjecture 4.20 (The Farrell-Jones Conjecture for negative K-theory of the ring of integers in an algebraic number field). Let R be the ring of integers in an algebraic number field. Then, for every group G, we have

$$K_n(RG) = 0 \quad \text{for } n \leq -2,$$

and the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_{-1}(RH) \xrightarrow{\cong} K_{-1}(RG)$$

is an isomorphism.

Conjecture 4.21 (The Farrell-Jones Conjecture for negative K-theory and Artinian rings as coefficient rings). Let G be a group, and let R be an Artinian ring. Then we have

$$K_n(RG) = 0 \quad \text{for} \quad n \leq -1.$$

4.5 Survey on Computations of Negative K-Groups for Finite Groups

The following result is due to Carter [205]. See also [96, Theorem 10.6 on page 695].

Theorem 4.22 (Negative K-theory of RG for a finite group G and a Dedekind domain of characteristic zero R). *Let R be a Dedekind domain of characteristic zero. Let k be its fraction field. For any maximal ideal P of R, let k_P be the P-adic completion. Let G be a finite group of order $n = |G|$.*

For a field F we denote by $r_F(G)$ the number of isomorphism classes of irreducible representations of G over the field F. Then:

(i) $K_m(RG) = 0$ *for* $m \leq -2$;
(ii) $K_{-1}(RG)$ *is a finitely generated group;*
(iii) Suppose that no prime divisor of n is invertible in R. Then the rank r of the finitely generated abelian group $K_{-1}(RG)$ is given by

$$r = 1 - r_k(G) + \sum_{p|nR} r_{k_P}(G) - r_{R/P}(G)$$

where the sum runs over all maximal (= non-zero prime) ideals P dividing nR;
(iv) If R is the ring of integers in an algebraic number field k, then

$$K_{-1}(RG) = \mathbb{Z}^r \oplus \mathbb{Z}/2^s$$

There is an explicit description of the integer s in terms of global and local Schur indices.
If G contains a normal abelian subgroup of odd index, then $s = 0$;
(v) Let A be a finite abelian group. Then $K_{-1}(\mathbb{Z}A)$ vanishes if and only if $|A|$ is a prime power.

If $R = \mathbb{Z}$, then $r = 1 - r_{\mathbb{Q}}(G) + \sum_{p|n} r_{\mathbb{Q}\widehat{p}}(G) - r_{\mathbb{F}_p}(G)$ where p runs through the prime numbers dividing n.

4.6 Notes

More information about $NK_n(RG)$ for all $n \in \mathbb{Z}$ will be given in Theorem 6.17, Theorem 6.18, Theorem 6.19, and Theorem 6.21.

More information about negative K-groups can be found for instance in [30, 96, 204, 205, 353, 513, 670, 671, 685, 780, 781, 809, 824, 844, 979].

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Chapter 5
The Second Algebraic K-Group

5.1 Introduction

This chapter is devoted to the *second algebraic K-group*.

We give two equivalent definitions, namely, in terms of the Steinberg group and in terms of the universal central extension of $\mathrm{E}(R)$. We extend the long exact sequence associated to a pullback of rings and to a two-sided ideal beyond K_1 to K_2. The long exact sequence associated to a pullback of rings cannot be extended to the left to higher algebraic K-groups, whereas this will be done for the long exact sequence associated to a two-sided ideal later.

We will introduce the *second Whitehead group* and state a variant of the *Farrell-Jones Conjecture* for it, namely, that it vanishes for torsionfree groups. Finally we give some information about computations of the second algebraic K-group.

5.2 Definition and Basic Properties of $K_2(R)$

Definition 5.1 (n-th Steinberg group). For $n \geq 3$ and a ring R, define its *n-th Steinberg group* $\mathrm{St}_n(R)$ to be the group given by generators and relations as follows. The set of generators is

$$\{x_{i,j}^r \mid i,j \in \{1,2,\dots,n\} \text{ and } r \in R\}.$$

The relations are

$$\begin{aligned}
&\text{(i)} & x_{i,j}^r \cdot x_{i,j}^s &= x_{i,j}^{r+s} \text{ for } i,j \in \{1,2,\dots,n\} \text{ and } r,s \in R;\\
&\text{(ii)} & [x_{i,j}^r, x_{j,k}^s] &= x_{i,k}^{rs} \text{ for } i,j,k \in \{1,2,\dots,n\} \text{ with } i \neq k \text{ and } r,s \in R;\\
&\text{(iii)} & [x_{i,j}^r, x_{k,l}^s] &= 1 \text{ for } i,j,k,l \in \{1,2,\dots,n\} \text{ with } i \neq l,\ j \neq k \text{ and } r,s \in R,
\end{aligned}$$

where $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$.

The idea behind the Steinberg group is that for every ring R the corresponding relations hold in $\mathrm{GL}_n(R)$ if we replace $x_{i,j}^r$ by the matrix $I_n + r \cdot E_n(i,j)$ appearing in Section 3.2. Hence we get a canonical group homomorphism

$$\phi_n^R \colon \mathrm{St}_n(R) \to \mathrm{GL}_n(R), \quad x_{i,j}^r \mapsto I_n + r \cdot E_n(i,j).$$

The image of ϕ_n^R is by definition the subgroup of $\mathrm{GL}_n(R)$ generated by all elements of the form $I_n + r \cdot E_n(i,j)$ for $i,j \in \{1,2,\ldots,n\}$ and $r \in R$. There is an obvious inclusion $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ sending a generator $x_{i,j}^r$ to $x_{i,j}^r$.

Definition 5.2 (Steinberg group). Define the *Steinberg group* $\mathrm{St}(R)$ to be the union of the groups $\mathrm{St}_n(R)$.

The set of maps $\{\phi_n^R \mid n \geq 3\}$ defines a homomorphism of groups

$$\phi^R \colon \mathrm{St}(R) \to \mathrm{GL}(R). \tag{5.3}$$

The image of ϕ^R is just the group $\mathrm{E}(R)$ which agrees with $[\mathrm{GL}(R), \mathrm{GL}(R)]$, see Lemma 3.11.

Definition 5.4 ($K_2(R)$). Define the *algebraic K_2-group $K_2(R)$ of a ring R* to be the kernel of the group homomorphism $\phi^R \colon \mathrm{St}(R) \to \mathrm{GL}(R)$ of (5.3).

Obviously a ring homomorphism $f \colon R \to S$ induces a map of abelian groups

$$K_2(f) \colon K_2(R) \to K_2(S). \tag{5.5}$$

Exercise 5.6. Show that there is a natural exact sequence

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 0.$$

5.3 The Steinberg Group as Universal Extension

A *central extension* of a group Q is a surjective group homomorphism $\phi \colon G \to Q$ with Q as target such that the kernel of ϕ is contained in the center $\{g \in G \mid g'g = gg' \text{ for all } g' \in G\}$ of G. A central extension $\phi \colon U \to Q$ of a group Q is called *universal* if for every central extension $\psi \colon G \to Q$ there is precisely one group homomorphism $f \colon U \to G$ with $\psi \circ f = \phi$. If a group Q admits a universal central extension, it is unique up to unique isomorphism. A group Q possesses a universal central extension if and only if it is *perfect*, i.e., it is equal to its commutator subgroup, see [712, Theorem 5.7 on page 44] or [844, Theorem 4.1.3 on page 163]. In this case the kernel of the universal central extension $\phi \colon U \to Q$ is isomorphic to the second homology $H_2(Q;\mathbb{Z})$ of Q, see [712, Corollary 5.8 on page 46] or [844, Theorem 4.1.3 on page 163]. A central extension $\phi \colon G \to Q$ of a group Q is universal if and only if G is perfect and every central extension $\psi \colon H \to G$ of G splits, i.e., there is a homomorphism $s \colon G \to H$ with $\psi \circ s = \mathrm{id}_G$, see [712, Theorem 5.3 on page 43] or [844, Theorem 4.1.3 on page 163]. A central extension $\phi \colon G \to Q$ of a perfect group Q is universal if and only if $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$, see [844, Corollary 4.1.18 on page 177]. The proof of the next result can be found in [712, Theorem 5.10 on page 47] or [844, Theorem 4.2.7 on page 190].

Theorem 5.7 ($K_2(R)$ and universal central extensions of $\mathrm{E}(R)$). *The canonical epimorphism $\phi^R\colon \mathrm{St}(R) \to \mathrm{E}(R)$ coming from the map (5.3) is the universal central extension of $\mathrm{E}(R)$.*

Exercise 5.8. Prove $K_2(R) \cong H_2(\mathrm{E}(R);\mathbb{Z})$.

5.4 Extending Exact Sequences of Pullbacks and Ideals

Theorem 5.9 (Mayer-Vietoris sequence for K-theory in degree ≤ 2 of a pullback of rings). *Consider a pullback of rings*

$$\begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R_0 \end{array}$$

such that both j_1 and j_2 are surjective. Then there exists a natural exact sequence, infinite to the right,

$$\begin{aligned} K_2(R) &\xrightarrow{(i_1)_*\oplus(i_2)_*} K_2(R_1)\oplus K_2(R_2) \xrightarrow{(j_1)_*-(j_2)_*} K_2(R_0) \\ &\xrightarrow{\partial_2} K_1(R) \xrightarrow{(i_1)_*\oplus(i_2)_*} K_1(R_1)\oplus K_1(R_2) \xrightarrow{(j_1)_*-(j_2)_*} K_1(R_0) \\ &\xrightarrow{\partial_1} K_0(R) \xrightarrow{(i_1)_*\oplus(i_2)_*} K_0(R_1)\oplus K_0(R_2) \xrightarrow{(j_1)_*-(j_2)_*} K_0(R_0) \\ &\xrightarrow{\partial_0} K_{-1}(R) \xrightarrow{(i_1)_*\oplus(i_2)_*} K_{-1}(R_1)\oplus K_{-1}(R_2) \\ &\xrightarrow{(j_1)_*-(j_2)_*} K_{-1}(R_0) \xrightarrow{\partial_{-1}} \cdots . \end{aligned}$$

Proof. See [712, Theorem 6.4 on page 55] for the extension to K_2. The extension for negative K-theory follows for example from the fact that the passage going from R to $R[\mathbb{Z}]$ sends a pullback of rings to a pullback of rings. □

Remark 5.10 (Surjectivity assumption is necessary). Swan [924, Corollary 1.2] has shown that the assumption that both j_1 and j_2 are surjective in Theorem 5.9 is necessary. It is not enough that j_1 or j_2 is surjective, in contrast to the weaker Theorem 3.86.

Remark 5.11 (No exact sequence for pullbacks in higher degrees). Swan [924, Corollary 6.9] has shown that it is not possible to define a functor K_3 so that the natural exact sequence appearing in Theorem 5.9 can be extended to K_3.

Theorem 5.12 (Exact sequence of a two-sided ideal K-theory in degree ≤ 2). *Given a two-sided ideal $I \subset R$, we obtain an exact sequence,* *natural in $I \subseteq R$ and infinite to the right*

$$\begin{aligned} K_2(R) \xrightarrow{\mathrm{pr}_*} K_2(R/I) \xrightarrow{\partial_2} K_1(R,I) \xrightarrow{j_1} K_1(R) \xrightarrow{\mathrm{pr}_*} K_1(R/I) \\ \xrightarrow{\partial_1} K_0(R,I) \xrightarrow{j_0} K_0(R) \xrightarrow{\mathrm{pr}_*} K_0(R/I) \xrightarrow{\partial_0} K_{-1}(R,I) \\ \xrightarrow{j_{-1}} K_{-1}(R) \xrightarrow{\mathrm{pr}_*} \cdots \end{aligned}$$

where $\mathrm{pr}\colon R \to R/I$ *the projection.*

Proof. See [712, Theorem 6.2 on page 54], [844, Theorem 3.3.4. on page 155 and Theorem 4.3.1 on page 200] or [979, Theorem 5.7.1 in Section III.5 on page 223]. □

Remark 5.13 (Dependence of $K_n(R, I)$ on R). The group $K_n(R, I)$ can be identified for $n \leq 0$ with $K_n(I)$, see Definition 3.90, and hence depends only on the structure of I as a ring without unit but not on the embedding $I \subseteq R$. But for $n \geq 1$ the group $K_n(R, I)$ does depend on the embedding $I \subseteq R$, see Example 3.94.

The sequence appearing in Theorem 5.12 is indeed an extension of the long exact sequence appearing in Theorem 3.89.

Often one wants to get information about K_2 in order to compute K_1-groups using for instance Theorem 5.12. This is illustrated by the following example.

Example 5.14. Let R be the ring of integers in an algebraic number field, and let P be a non-zero prime ideal. Then the exact sequence appearing in Theorem 5.12 induces an exact sequence

$$K_2(R/P) \to SK_1(R,P) \to SK_1(R) \to SK_1(R/P)$$

where $SK_1(R)$ has been defined in Definition 3.15 and we put:

$$\begin{aligned} SK_1(R,P) &:= (\mathrm{SL}(R) \cap \mathrm{GL}(R,P)) / \mathrm{E}(R,P) \\ &\cong \ker\bigl(\det\colon \mathrm{GL}(R,P) \to \{r \in R \mid r \equiv 1 \mod P\}\bigr). \end{aligned}$$

Since R/P is a finite field, $SK_1(R/P)$ and $K_2(R/P)$ vanish by Theorem 3.17 and Theorem 5.18 (v). Hence we obtain an isomorphism

$$SK_1(R,P) \xrightarrow{\cong} SK_1(R).$$

The group $SK_1(R)$ vanishes by [712, Corollary 16.3]. Hence also $SK_1(R,P)$ vanishes.

Example 5.15 ($K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ for $n \leq 1$ and a prime p). Let p be a prime number. We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0 \quad \text{for } n \leq -1 \text{ and } k \geq 0$$

and that $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ and $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ are finitely generated. All of these statements except the claim for $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ have already been proved in Example 4.10. The same method of proof applies to this case since Theorem 5.9 yields the exact sequence

$$K_2(\mathbb{F}_p[\mathbb{Z}^k]) \to K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to K_1(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k])$$

and $K_2(\mathbb{F}_p[\mathbb{Z}^k])$, $K_1(\mathbb{Z}[\mathbb{Z}^k])$, and $K_1(\mathbb{Z}[\exp(2\pi i/p)])$ are finitely generated abelian groups by Theorem 4.7, as $K_m(\mathbb{F}_p)$ for $m = 0, 1, 2$, $K_m(\mathbb{Z})$ for $m = 0, 1$, and $K_m(\mathbb{Z}[\exp(2\pi i/p)])$ for $m = 0, 1$ are finitely generated and $K_m(\mathbb{F}_p)$, $K_m(\mathbb{Z})$, and $K_m(\mathbb{Z}[\exp(2\pi i/p)])$ vanish for $m \leq -1$ because of Example 2.4, Theorem 2.23, Theorem 3.17, Theorem 3.21, Theorem 3.80 (i) Theorem 4.7, and Theorem 5.18 (iv).

5.5 Steinberg Symbols

Let R be a commutative ring and $u, v \in R^\times$. Consider the elements $d_{1,2}(u), d_{1,3}(v) \in \mathrm{E}(R)$ given by the invertible $(3,3)$-matrices

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}.$$

Let $\widetilde{d}_{1,2}(u)$ and $\widetilde{d}_{1,3}(v)$ be any preimages of $d_{1,2}(u)$ and $d_{1,3}(v)$ under the canonical map $\phi^R\colon \mathrm{St}(R) \to \mathrm{E}(R)$. Then the commutator $[\widetilde{d}_{1,2}(u), \widetilde{d}_{1,3}(v)]$ in $\mathrm{St}(R)$ defines an element in the kernel of $\phi^R\colon \mathrm{St}(R) \to \mathrm{E}(R)$ and hence in $K_2(R)$. It depends only on u and v. The proof of the facts above can be found for instance in [844, page 192].

Definition 5.16 (Steinberg symbol). Let R be a commutative ring and $u, v \in R^\times$. The element in $K_2(R)$ given by the construction above is called the *Steinberg symbol* of u and v and is denoted by $\{u, v\}$.

Exercise 5.17. Prove that the Steinberg symbol of Definition 5.16 is well-defined.

Theorem 5.18 (Properties of the Steinberg symbol). *Let R be a commutative ring. Then:*

(i) The Steinberg symbol defines a bilinear skew-symmetric pairing

$$R^\times \times R^\times \to K_2(R), \quad (u, v) \mapsto \{u, v\},$$

i.e., $\{u_1 \cdot u_2, v\} = \{u_1, v\} + \{u_2, v\}$ *and* $\{u, v\} = -\{v, u\}$ *for all* u_1, u_2, u, v *in* $R^\times$*;*

(ii) *For* $u \in R^\times$ *we have* $\{u, -u\} = 0$*;*

(iii) *If for* $u \in R^\times$ *also* $1 - u \in R^\times$*, then* $\{u, 1-u\} = 0$*;*

(iv) *(Matsumoto's Theorem) If* F *is a field, then* $K_2(F)$ *is isomorphic to the abelian group given by the generators* $\{u, v\}$ *for* $u, v \in F^\times$ *and the relations:*

(a) $\{u, 1-u\} = 0$ *for* $u \in F$ *with* $u \neq 0, 1$*;*
(b) $\{u_1 \cdot u_2, v\} = \{u_1, v\} + \{u_2, v\}$ *for* $u_1, u_2, v \in F^\times$*;*
(c) $\{u, v_1 \cdot v_2\} = \{u, v_1\} + \{u, v_2\}$ *for* $u, v_1, v_2 \in F^\times$*;*

(v) *If* F *is a finite field, then* $K_2(F) = 0$*;*

(vi) *We have* $K_2(\mathbb{Z}) = \mathbb{Z}/2$*. A generator is given by the Steinberg symbol* $\{-1, -1\}$*;*

(vii) *Let* $m \geq 2$ *be an integer. If* $m \neq 0 \mod 4$*, then* $K_2(\mathbb{Z}/m) = \{0\}$*. If* $m = 0 \mod 4$*, then* $K_2(\mathbb{Z}/m) = \mathbb{Z}/2$ *and a generator is given by the Steinberg symbol* $\{1, 1\}$*;*

(viii) *(Tate) We have* $K_2(\mathbb{Q}) = \mathbb{Z}/2 \times \prod_p \mathbb{F}_p^\times$ *where* p *runs through the odd prime numbers;*

(ix) *(Bass, Tate) Let* R *be a Dedekind domain with quotient field* F*. Then there is an exact sequence*

$$\begin{aligned} K_2(F) \to \bigoplus_P K_1(R/P) \to K_1(R) \to K_1(F) \\ \to \bigoplus_P K_0(R/P) \to K_0(R) \to K_0(F) \to 0, \end{aligned}$$

where P *runs through the maximal ideals of* R*.*

Proof. (i) See [712, Theorem 8.2 on page 64] or [844, Lemma 4.2.14 on page 194].

(ii) and (iii) See [712, Theorem 9.8 on page 74] or [844, Theorem 4.2.17 on page 197].

(iv) See [712, Theorem 11.1 on page 93] or [844, Theorem 4.3.15 on page 214].

(v) See [712, Theorem 9.13 on page 78] or [844, Theorem 4.3.13 and Remark 4.3.14 on page 213].

(vi) See [712, Corollary 10.2 on page 81].

(vii) See [712, Corollary 10.8 on page 92], [293, Theorem 5.1], and [844, Exercise 4.3.19 on page 217].

(viii) See [712, Theorem 11.6 on page 101].

(ix) See [712, Corollary 13.1 on page 123] and [96, pages 702, 323]. □

5.6 The Second Whitehead Group

Let R be a ring. Consider $u \in R^\times$ and integers $i, j \geq 1$. If $x_{i,j}^u$ is the canonical generator of $\mathrm{St}(R)$, see Definition 5.1, then define

$$w_{i,j}^u := x_{i,j}^u x_{j,i}^{-u^{-1}} x_{ij}^u \quad \in \mathrm{St}(R).$$

Let G be a group. Let W_G be the subgroup of $\mathrm{St}(\mathbb{Z}G)$ generated by all elements of the shape $w_{i,j}^g$ for $g \in G$ and integers $i, j \geq 1$. Recall that we can think of $K_2(\mathbb{Z}G)$ as a subgroup of $\mathrm{St}(\mathbb{Z}G)$.

Definition 5.19 (The second Whitehead group). Let G be a group. Define the *second Whitehead group of* G by

$$\mathrm{Wh}_2(G) := K_2(\mathbb{Z}G)/(K_2(\mathbb{Z}G) \cap W_G).$$

Exercise 5.20. Show that the second Whitehead group of the trivial group vanishes using the fact, see [844, Example 4.2.19 on page 198], that $w_{1,2}(1)^4 = \{-1, -1\}$ holds in $\mathrm{St}(\mathbb{Z})$.

Let I denote the unit interval $[0,1]$. Let M be a closed smooth manifold. A smooth *pseudoisotopy* of M is a diffeomorphism $h\colon M \times I \to M \times I$ that restricted to $M \times \{0\} \subseteq M \times I$ is the obvious inclusion. The group $P^{\mathrm{Diff}}(M)$ of smooth pseudoisotopies is the group of all such diffeomorphisms under composition. Pseudoisotopies play an important role if one tries to understand the homotopy type of the topological group $\mathrm{Diff}(M)$ of selfdiffeomorphisms of M. Two selfdiffeomorphisms $f_0, f_1\colon M \to M$ are called *isotopic* if there is a smooth map $h\colon M \times [0,1] \to M$ called isotopy such that $h_t\colon M \to M, x \mapsto h(x,t)$ is a selfdiffeomorphism for each $t \in [0,1]$ and $h_k = f_k$ for $k = 0, 1$. They are called *pseudoisotopic* if there exists a diffeomorphism $H\colon M \times [0,1] \to M \times [0,1]$ such that $H(x,k) = (f_k(x), k)$ for all $x \in M$ and $k = 0, 1$. If h is an isotopy, then we obtain a pseudoisotopy by $H(x,k) = (h(x,k), k)$. Hence isotopic selfdiffeomorphisms are pseudoisotopic. The converse is not true in general, there is no reason why a pseudoisotopy is level preserving, i.e., sends $M \times \{t\}$ to $M \times \{t\}$ for every $t \in [0,1]$.

In order to decide whether two selfdiffeomorphisms are isotopic, it is often very useful to firstly decide whether they are pseudoisotopic what is in general easier.

The set of path components $\pi_0(\mathrm{Diff}(M))$ of the space $\mathrm{Diff}(M)$ agrees with the set of isotopy classes of selfdiffeomorphisms of M. The group $P^{\mathrm{DIFF}}(M)$ acts on $\mathrm{Diff}(M)$ by $h \cdot f := h_1 \circ f$. If $P^{\mathrm{DIFF}}(M)$ is path-connected, then two pseudoisotopic diffeomorphisms $M \to M$ are isotopic since the orbit of the identity $\mathrm{id}_M\colon M \to M$ under the $P^{\mathrm{DIFF}}(M)$-action consists of the diffeomorphisms $M \to M$ that are pseudoisotopic to the identity. If M is simply connected, $P^{\mathrm{DIFF}}(M)$ is known to be path connected by a result of Cerf [207, 208] if $\dim(M) \geq 5$.

The relevance of the second Whitehead group comes from the following result, see [452, 453].

Theorem 5.21 (Pseudoisotopy and the second Whitehead group). *Let M be a smooth closed manifold of dimension ≥ 5. Then there is an epimorphism*

$$\pi_0(P^{\mathsf{DIFF}}(M)) \to \mathrm{Wh}_2(\pi_1(M)).$$

More information about pseudoisotopy and its relation to algebraic K-theory will be given in Chapter 7. The Farrell-Jones Conjecture for pseudoisotopy will be stated as Conjecture 15.61.

5.7 A Variant of the Farrell-Jones Conjecture for the Second Whitehead group

Conjecture 5.22 (Farrell-Jones Conjecture for $\mathrm{Wh}_2(G)$ for torsionfree G). Let G be a torsionfree group. Then $\mathrm{Wh}_2(G)$ vanishes.

5.8 The Second Whitehead Group of Some Finite Groups

We give some information about $K_2(\mathbb{Z}G)$ and $\mathrm{Wh}_2(G)$ for some finite groups.

The group $K_2(RG)$ is finite for every finite group G and every ring of integer R in a number field, see [580, Theorem 1.1]. In particular $K_2(\mathbb{Z}G)$ and $\mathrm{Wh}_2(G)$ are finite for any finite group G.

We have

$$\begin{aligned}
\mathrm{Wh}_2(G) &= 0, \quad \text{for } G = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4;\\
|\,\mathrm{Wh}_2(\mathbb{Z}/6)| &\leq 2;\\
\mathrm{Wh}_2(D_6) &\cong \mathbb{Z}/2,
\end{aligned}$$

where D_6 is the dihedral group of order six. The claim for the finite cyclic groups follow from [304, page 482] and [910, pages 218 and 221]. We get $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$ from [910, Theorem 3.1]. This implies $\mathrm{Wh}_2(D_6) \cong \mathbb{Z}/2$ as explained in [667, Theorem 3.2.d.iii].

Given a prime p, the p-rank of an abelian group A is $\dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{Z}} A)$. The 2-rank of the finite abelian group $\mathrm{Wh}_2\bigl((\mathbb{Z}/2)^n\bigr)$ is at least $(n-1)\cdot 2^n - \frac{(n+2)(n-1)}{2}$ by [288, Corollary 7]. If p is an odd prime, then the p-rank of the finite abelian group $\mathrm{Wh}_2\bigl((\mathbb{Z}/p)^n\bigr)$ is at least $(n-1)\cdot(p^n-1) - \binom{p+n-1}{p} - \frac{n(n-1)}{2}$ by [288, Corollary 8]. In particular $\mathrm{Wh}_2\bigl((\mathbb{Z}/p)^n\bigr)$ is non-trivial for a prime p and $n \geq 2$.

Some information about $K_2(\mathbb{F}_pG)$ for finite groups can be found in [689].

Exercise 5.23. Determine all integers $n \geq 1$ for which $\widetilde{K}_i(\mathbb{Z}[\mathbb{Z}/n])$ for all $i \leq 0$, $\mathrm{Wh}(\mathbb{Z}/n)$, and $\mathrm{Wh}_2(\mathbb{Z}/n)$ vanish.

5.9 Notes

We have already mentioned that often computations involving K_1 use information about K_2 since there are various long exact sequences relating K-groups of different rings. Examples of such sequences have been given in Theorem 5.9, Theorem 5.12 and Theorem 5.18 (ix). Another important class of such exact sequences are given by localization sequences, see [760, Chapter 3].

The second algebraic K-group of fields plays also a role in number theory, as for instance explained in [712, Chapters 11, 15, 16], [905, Chapter 8] and [843, Chapter 4, Section 4]. Keywords are Hilbert symbols, Gauss' laws of quadratic reciprocity, Brauer groups, and the Mercurjev-Suslin Theorem. Relations to operator theory are discussed in [712, Chapter 7], and [843, Chapter 4, Section 4].

Further references to K_2 and the second Whitehead group are [23, 289, 290, 291, 292, 293, 453, 688, 911, 979].

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Chapter 6
Higher Algebraic K-Theory

6.1 Introduction

In this chapter we extend the definition of the *algebraic K-groups* $K_n(R)$ to all integers $n \in \mathbb{Z}$.

We first present the plus-construction to define *higher algebraic K-theory* and record the basic properties. We introduce *algebraic K-theory with coefficients in $\mathbb{Z}/k$*. We discuss other constructions of K-theory that apply to more general situations such as to exact categories or Waldhausen categories. The previous constructions lead only to spaces and one can find deloopings which result in spectra whose homotopy groups are the algebraic K-groups also in negative degrees. We present the *K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings.* We introduce Mayer-Vietoris sequences for amalgamated free products and Wang sequence for HNN extensions for the algebraic K-theory of group rings. The appearance of Nil-*terms* in these exact sequences is responsible for some complications concerning algebraic K-theory and the Farrell-Jones Conjecture that do not occur in the Baum-Connes setting. We discuss *homotopy K-theory*, a theory that is on the one hand close to algebraic K-theory and on the other hand is free of Nil-phenomena. We briefly explain relations between algebraic K-theory and cyclic homology.

6.2 The Plus-Construction

Let R be a ring. So far the algebraic K-groups $K_n(R)$ for $n \leq 2$ have been described in a purely algebraic fashion by generators and relations. The definition of the higher algebraic K-groups $K_n(R)$ for $n \geq 3$ has been achieved topologically, namely, one assigns to a ring R a space $K(R)$ and defines $K_n(R)$ by the n-th homotopy group $\pi_n(K(R))$ for $n \geq 0$. This will coincide with the previous definition for $n = 0, 1, 2$. There are various definitions of the space $K(R)$ that extend to more general settings as explained below and are appropriate in different situations. We briefly recall the technically less demanding one, the plus-construction.

A space Z is called *acyclic* if it has the homology of a point, i.e., the singular homology with integer coefficients $H_n(Z)$ vanishes for $n \geq 1$ and is isomorphic to $\mathbb{Z}$ for $n = 0$.

Exercise 6.1. Prove that an acyclic space is path connected and that its fundamental group π is perfect and satisfies $H_2(\pi;\mathbb{Z}) = 0$.

In the following we will suppress choices of and questions about base points. The *homotopy fiber* $\operatorname{hofib}(f)$ of a map $f\colon X \to Y$ of path connected spaces has the property that it is the fiber of a fibration $p_f\colon X \to E_f$ which comes with a homotopy equivalence $h\colon E_f \to X$ satisfying $p_f = f \circ h$, see [987, Theorem 7.30 in Chapter I.7 on page 42]. The *long exact homotopy sequence associated to* f, see [987, Corollary 8.6 in Chapter IV.8 on page 187], looks like

$$(6.2) \quad \cdots \xrightarrow{\partial_3} \pi_2(\operatorname{hofib}(f)) \xrightarrow{\pi_2(i)} \pi_2(X) \xrightarrow{\pi_2(f)} \pi_2(Y) \xrightarrow{\partial_2} \pi_1(\operatorname{hofib}(f)) \\ \xrightarrow{\pi_1(i)} \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y) \xrightarrow{\partial_1} \pi_0(\operatorname{hofib}(f)) \to \{\{\bullet\}\}.$$

Definition 6.3 (Acyclic map). Let X and Y be path connected CW-complexes. A map $f\colon X \to Y$ is called *acyclic* if its homotopy fiber $\operatorname{hofib}(f)$ is acyclic.

We conclude for an acyclic map $f\colon X \to Y$ from the long exact homotopy sequence (6.2) that $f_1\colon \pi_1(X) \to \pi_1(Y)$ is surjective and its kernel is a perfect subgroup P of $\pi_1(X)$ since P is a quotient of the perfect group $\pi_1(\operatorname{hofib}(f))$ and $\pi_0(\operatorname{hofib}(f))$ consists of one element. Obviously a space Z is acyclic if and only if the map $Z \to \{\bullet\}$ is acyclic.

Definition 6.4 (Plus-construction). Let X be a connected CW-complex and $P \subseteq \pi_1(X)$ be a perfect subgroup. A map $f\colon X \to X^+$ to a CW-complex is called a *plus-construction of* X *relative to* P if f is acyclic and the kernel of $f_1\colon \pi_1(X) \to \pi_1(X^+)$ is P.

The next result is due to Quillen. A proof can be found for instance in [844, Theorem 5.2.2 on page 266 and Proposition 5.2.4 on page 268].

Theorem 6.5 (Properties of the plus-construction). *Let X be a connected CW-complex and let $P \subseteq \pi_1(X)$ be a perfect subgroup. Then:*

(i) There exists a plus-construction $f\colon X \to X^+$ relative to P. (One can construct X^+ by attaching 2- and 3-cells to X);
(ii) Let $f\colon X \to X^+$ be a plus-construction relative to P, and let $g\colon X \to Y$ be a map such that the kernel of $\pi_1(g)\colon \pi_1(X) \to \pi_1(Y)$ contains P. Then there is a map $\overline{g}\colon X^+ \to Y$ which is up to homotopy uniquely determined by the property that $\overline{g} \circ f$ is homotopic to g;
(iii) If $f_1\colon X \to X_1^+$ and $f_2\colon X \to X_2^+$ are two plus-constructions for X relative to P, then there exists a homotopy equivalence $g\colon X_1^+ \to X_2^+$ which is up to homotopy uniquely determined by the property $g \circ f_1 \simeq f_2$;
(iv) If $f\colon X \to X^+$ is a plus-construction relative to P, then $\pi_1(f)\colon \pi_1(X) \to \pi_1(X^+)$ can be identified with the canonical projection $\pi_1(X) \to \pi_1(X)/P$;

(v) If $f\colon X \to X^+$ is a plus-construction, then $H_n(f;M)\colon H_n(X;f^*M) \to H_n(X^+;M)$ is bijective for all $n \geq 0$ and all local coefficient systems M on X^+.

Remark 6.6 (Perfect radical). Every group G has a unique largest perfect subgroup $P \subseteq G$, called the *perfect radical* of G. In the following we will always use the perfect radical of G for P unless explicitly stated otherwise.

Exercise 6.7. Show that every group has a unique largest perfect subgroup.

Exercise 6.8. Show that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ is the perfect radical of $\mathrm{GL}(R)$.

Definition 6.9 (Higher algebraic K-groups of a ring). Let $B\mathrm{GL}(R) \to B\mathrm{GL}(R)^+$ be a plus-construction in the sense of Definition 6.4 for the classifying space $B\mathrm{GL}(R)$ of $\mathrm{GL}(R)$ (relative to the perfect radical of $\mathrm{GL}(R)$, which is $E(R)$). Define the *K-theory space associated to R*

$$K(R) := K_0(R) \times B\mathrm{GL}(R)^+$$

where we view $K_0(R)$ with the discrete topology. Define the n-th algebraic K-group

$$K_n(R) := \pi_n(K(R)) \quad \text{for } n \geq 0.$$

This definition makes sense because of Theorem 6.5 (i) and (iii). Note that for $n \geq 1$ we have $K_n(R) = \pi_n(B\mathrm{GL}(R)^+)$.

Exercise 6.10. Show that the Definition 6.9 of $K_n(R)$ for $n = 0, 1$ is compatible with the one of Definitions 2.1 and 3.1.

For $n = 0, 1, 2$ Definition 6.9 is compatible with the previous Definitions 2.1, 3.1, and 5.4, and we have $K_3(R) \cong H_3(\mathrm{St}(R))$ and $K_n(R) = \pi_n(B\mathrm{St}(R)^+)$ for $n \geq 3$, see [843, Corollary 5.2.8 on page 273], [407].

A ring homomorphism $f\colon R \to S$ induces a group homomorphism $\mathrm{GL}(R) \to \mathrm{GL}(S)$ and hence maps $B\mathrm{GL}(R) \to B\mathrm{GL}(S)$ and $B\mathrm{GL}(R)^+ \to B\mathrm{GL}(S)^+$. We have a already defined a map $K_0(f)\colon K_0(R) \to K_0(S)$ in (2.2). Therefore f induces a map $K(f)\colon K(R) \to K(S)$ and hence for every $n \geq 0$ a map of abelian groups $K_n(f)\colon K_n(R) \to K_n(S)$. This turns out to be compatible with the previous definitions for $n = 0, 1, 2$ in (2.2), (3.2), and (5.5). We have also defined $K_n(f)\colon K_n(R) \to K_n(S)$ for $n \leq -1$ in (4.2). Hence we get a covariant functor from the category of rings to the category of abelian groups by $K_n(-)$ for $n \in \mathbb{Z}$.

Definition 6.11 (Relative K-groups). Define for a two-sided ideal $I \subseteq R$ and $n \geq 0$

$$K_n(R, I) := \pi_n\big(\mathrm{hofib}(K(\mathrm{pr})\colon K(R) \to K(R/I))\big)$$

for $\mathrm{pr}\colon R \to R/I$ the projection.

The long exact homotopy sequence (6.2) associated to $K(\mathrm{pr})\colon K(R) \to K(R/I)$ together with Theorem 5.12 implies

Theorem 6.12 (Long exact sequence of a two-sided ideal for higher algebraic K-theory). *Let $I \subseteq R$ be a two sided ideal. Then there is a long exact sequence, infinite to both sides,*

$$\begin{aligned} \cdots \xrightarrow{\partial_3} K_2(R,I) \xrightarrow{j_2} K_2(R) \xrightarrow{K_2(\mathrm{pr})} K_2(R/I) \xrightarrow{\partial_2} K_1(R,I) \xrightarrow{j_1} K_1(R) \\ \xrightarrow{K_1(\mathrm{pr})} K_1(R/I) \xrightarrow{\partial_1} K_0(R,I) \xrightarrow{j_0} K_0(R) \xrightarrow{K_0(\mathrm{pr})} K_0(R/I) \\ \xrightarrow{\partial_0} K_{-1}(R,I) \xrightarrow{j_{-1}} K_{-1}(R) \xrightarrow{K_{-1}(\mathrm{pr})} K_{-1}(R/I) \xrightarrow{\partial_{-1}} \cdots . \end{aligned}$$

The existence of the long exact sequence of a two-sided ideal of Theorem 6.12 has been one important requirement of an extension of middle and lower algebraic K-theory to higher degrees. It is indeed an extension of the long exact sequences appearing in Theorem 3.89 and Theorem 5.12.

For more information about the plus-construction we refer for instance to [123], [843, Chapter 5], [905, Chapter 2].

6.3 Survey on Main Properties of Algebraic K-Theory of Rings

6.3.1 Compatibility with Finite Products

The basic idea of the proof of the following result for $n \geq 1$ can be found in [805, (8) in §2 on page 20]. The case $n \leq 1$ follows from Lamma 2.12, Lemma 3.9, and by inspecting Definition 4.1, see also Exercise 4.5.

Theorem 6.13 (Algebraic K-theory and finite products). *Let R_0 and R_1 be rings. Denote by $\mathrm{pr}_i\colon R_0 \times R_1 \to R_i$ for $i = 0, 1$ the projection. Then we obtain for $n \in \mathbb{Z}$ isomorphisms*

$$K_n(\mathrm{pr}_0) \times K_n(\mathrm{pr}_1)\colon K_n(R_0 \times R_1) \xrightarrow{\cong} K_n(R_0) \times K_n(R_1).$$

6.3.2 Morita Equivalence

The idea of the proof of the next result is essentially the same as of Theorem 2.10.

Theorem 6.14 (Morita equivalence for algebraic K-theory). *For every ring R and integer $k \geq 1$ there are for all $n \in \mathbb{Z}$ natural isomorphisms*

$$\mu_n \colon K_n(R) \xrightarrow{\cong} K_n(M_k(R)).$$

6.3.3 Compatibility with Colimits over Directed Sets

We conclude from [805, (12) in §2 on page 20], (at least in the connective setting) and [884, Lemma 6 in Section 7].

Theorem 6.15 (Algebraic K-theory and colimits over directed sets). *Let $\{R_i \mid i \in I\}$ be a directed system of rings. Then the canonical map*

$$\operatorname{colim}_{i\in I} K_n(R_i) \xrightarrow{\cong} K_n\bigl(\operatorname{colim}_{i\in I} R_i\bigr)$$

is bijective for $n \in \mathbb{Z}$.

Actually, one may consider more generally filtered colimits.

6.3.4 The Bass-Heller-Swan Decomposition

We have already explained the following result for $n \le 1$ in Theorem 3.72 and Theorem 4.3. Definition 3.68 of $NK_n(R)$ makes sense for every $n \in \mathbb{Z}$. The proof for higher algebraic K-theory can be found in [905, Theorem 9.8 on page 207], see also [843, Theorem 5.3.30 on page 295]. More general versions where twistings are allowed and additive categories are considered are presented in [417, 419, 440, 514, 516, 593, 671].

Theorem 6.16 (Bass-Heller-Swan decomposition for algebraic K-theory).

(i) There are isomorphisms of abelian groups, natural in R, for $n \in \mathbb{Z}$

$$NK_n(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t]);$$
$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]).$$

There is a sequence, which is natural in R and exact, for $n \in \mathbb{Z}$

$$0 \to K_n(R) \xrightarrow{K_n(k_+)\oplus -K_n(k_-)} K_n(R[t]) \oplus K_n(R[t^{-1}])$$
$$\xrightarrow{K_n(l_+)\oplus K_n(l_-)} K_n(R[t,t^{-1}]) \xrightarrow{C_n} K_{n-1}(R) \to 0$$

where k_+, k_-, l_+, and l_- are the obvious inclusions.
If we regard it as an acyclic $\mathbb{Z}$-chain complex, there exists a chain contraction, natural in R;

(ii) If R is regular, then

$$NK_n(R) = \{0\} \quad \text{for } n \in \mathbb{Z};$$
$$K_n(R) = \{0\} \quad \text{for } n \leq -1.$$

6.3.5 Some Information about NK-groups

The proof of the next result can be found in Weibel [976, Corollary 3.2].

Theorem 6.17 ($NK_n(R)[1/N]$ vanishes for characteristic N). *Let R be a ring of finite characteristic N. Then we get for $n \in \mathbb{Z}$*

$$NK_n(R)[1/N] \;=\; 0.$$

Theorem 6.18 (Vanishing criterion of $NK_n(RG)$ for finite groups). *Let R be a ring and let G be a finite group. Fix $n \in \mathbb{Z}$. Suppose $NK_n(R) = 0$. Then we get*

$$NK_n(RG)[1/|G|] = 0.$$

Proof. This follows from Hambleton-Lück [440, Theorem A]. □

The following result is taken from Hambleton-Lück [440, Corollary B].

Theorem 6.19 (p-elementary induction for $NK_n(RG)$). *Let R be a ring and let G be a finite group. For all $n \in \mathbb{Z}$, the sum of the induction maps*

$$\bigoplus_E NK_n(RE)_{(p)} \to NK_n(RG)_{(p)},$$

is surjective where E runs through all p-elementary subgroups.

The following theorem due to Prasolov [798] is an extension of a result due to Farrell [336] for $n = 1$ to $n \geq 1$.

Theorem 6.20 ($NK_n(R)$ is trivial or infinitely generated for $n \geq 1$). *Let R be a ring. Then $NK_n(R)$ is either trivial or infinitely generated as abelian group for $n \geq 1$.*

Theorem 6.21 (Vanishing of $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k])$ for $n \leq 1$, $k \geq 0$ and finite G of square-free order). *Let G be a finite group whose order is square-free. Then $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k]) = 0$ for $n \leq 1$ and $k \geq 0$.*

Proof. Fix a prime p. We know from Example 5.15 that $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is finitely generated for every $k \leq 0$. We conclude from Theorem 6.16 that $K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is finitely generated for every $n \leq 1$ and $k \geq 0$ and hence that $NK_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is finitely generated for every $n \leq 1$ and $k \geq 0$.

Theorem 6.20 implies that $NK_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$ is trivial every $n \leq 1$ and $k \geq 0$.

We conclude from [440, Theorem A] that for any ring R, any finite group G, and any prime number p, there is a surjection

$$\bigoplus_P NK_n(RP)_{(p)} \to NK_n(RG)_{(p)},$$

where P runs through the p-subgroups of G. This implies that $NK_n(RG)$ vanishes if $NK_n(RP)_{(p)}$ vanishes for every prime p and every p-subgroup P of G. In particular $NK_n(RG)$ vanishes for a finite group G of square-free order, if $NK_n(R[\mathbb{Z}/p])_{(p)}$ vanishes for every prime number p. Put $R = \mathbb{Z}[\mathbb{Z}^k]$. Then $R[\mathbb{Z}/p] = \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]$ and $RG = \mathbb{Z}[G \times \mathbb{Z}^k]$, and we know already that $NK_n(R[\mathbb{Z}/p])_{(p)}$ vanishes for every prime number p, $n \leq 1$ and $k \geq 0$. Hence $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k]) = NK_n(RG)$ vanishes for $n \leq 1$ and $k \geq 0$, if G is a finite group of square-free order. □

Theorem 6.21 has been proved in the case $k = 0$ by Harmon [450].

Exercise 6.22. Let G be a finite group of square-free order. Show for all $k \geq 1$

$$K_n(\mathbb{Z}[G \times \mathbb{Z}^k]) = \begin{cases} K_1(\mathbb{Z}G) \oplus K_0(\mathbb{Z}G)^k \oplus K_{-1}(\mathbb{Z}G)^{k(k-1)/2} & \text{if } n = 1; \\ K_0(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G)^k & \text{if } n = 0; \\ K_{-1}(\mathbb{Z}G) & \text{if } n = -1; \\ \{0\} & \text{if } n \leq -2. \end{cases}$$

6.3.6 Algebraic K-Theory of Finite Fields

The following result has been proved by Quillen [804].

Theorem 6.23 (Algebraic K-theory of finite fields). *Let $\mathbb{F}_q$ be a finite field of order q. Then $K_n(\mathbb{F}_q)$ vanishes if $n = 2k$ for some integer $k \geq 1$, and is a finite cyclic group of order $q^k - 1$ if $n = 2k - 1$ for some integer $k \geq 1$.*

Recall that $K_0(F) \cong \mathbb{Z}$ and $K_n(F) = \{0\}$ for $n \leq -1$ if F is a field, see Example 2.4 and Theorem 4.7.

6.3.7 Algebraic K-Theory of the Ring of Integers in a Number Field

The computation of the higher algebraic K-groups of $\mathbb{Z}$ or, more generally, of the ring of integers R in an algebraic number field F, is very hard.

Quillen [804] showed that these are finitely generated as abelian group. Their ranks as abelian groups have been determined by Borel [143].

Theorem 6.24 (Rational Algebraic K-theory of ring of integers of number fields). *Let R be a ring of integers in an algebraic number field. Let r_1 be the number of distinct embeddings of F into $\mathbb{R}$ and let r_2 be the number of distinct conjugate pairs of embeddings of F into $\mathbb{C}$ with image not contained in $\mathbb{R}$. Then*

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \{0\} & n \leq -1; \\ \mathbb{Q} & n = 0; \\ \mathbb{Q}^{r_1+r_2-1} & n = 1; \\ \mathbb{Q}^{r_1+r_2} & n \geq 2 \quad \textit{and} \quad n = 1 \mod 4; \\ \mathbb{Q}^{r_2} & n \geq 2 \quad \textit{and} \quad n = 3 \mod 4; \\ \{0\} & n \geq 2 \quad \textit{and} \quad n = 0 \mod 2. \end{cases}$$

We have $K_n(\mathbb{Z}) = \{0\}$ for $n \leq -1$ and the first values of $K_n(\mathbb{Z})$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$ are given by $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, $\mathbb{Z}/48$, $\{0\}$, $\mathbb{Z}$, $\{0\}$, $\mathbb{Z}/240$.

The Lichtenbaum-Quillen Conjecture makes a prediction about the torsion, see [614, 615], relating the algebraic K-groups to number theory via the zeta-function. We refer to the survey article of Weibel [975], where a complete picture about the algebraic K-theory of ring of integers in algebraic number fields and in particular of $K_*(\mathbb{Z})$ is given and a list of relevant references can be found. See also Weibel [979, Section VI.10 on pages 527ff].

An outline how the next corollary follows from Theorem 6.49 can be found in [805, page 29] and [843, page 294]. It is a basic tool for computations.

Corollary 6.25. Let R be a Dedekind domain with quotient field F. Then there is an exact sequence

$$\cdots \to K_{n+1}(F) \to \bigoplus_P K_n(R/P) \to K_n(R) \to K_n(F) \to \bigoplus_P K_{n-1}(R/P)$$
$$\cdots \to K_1(F) \to \bigoplus_P K_0(R/P) \to K_0(R) \to K_0(F) \to 0$$

where P runs through the maximal ideals of R.

Exercise 6.26. Consider the part of the sequence

$$K_1(\mathbb{Z}) \to K_1(\mathbb{Q}) \xrightarrow{\partial_1} \bigoplus_p K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}) \to 0$$

of Corollary 6.25 for $R = \mathbb{Z}$. Compute the five terms appearing in it. Guess what the map ∂_1 is and determine the others.

Exercise 6.27. Show that the map $K_n(\mathbb{Z}) \to K_n(\mathbb{Q})$ is injective if $n = 2k$ for $k \geq 1$, is surjective if $n = 2k-1$ for $k \geq 2$, and rationally bijective for $n \geq 2$.

6.4 Algebraic *K*-Theory with Coefficients

By invoking the Moore space associated to $\mathbb{Z}/k$, one can introduce *K-theory mod k*, denoted by $K_n(R;\mathbb{Z}/k)$, for any integer $k \geq 2$ and every $n \in \mathbb{Z}$. Its main feature is that there exists a long exact sequence

$$(6.28) \quad \cdots \to K_{n+1}(R;\mathbb{Z}/k) \to K_n(R) \xrightarrow{k\cdot\mathrm{id}} K_n(R) \to K_n(R;\mathbb{Z}/k) \\ \to K_{n-1}(R) \xrightarrow{k\cdot\mathrm{id}} K_{n-1}(R) \to K_{n-1}(R;\mathbb{Z}/k) \to \cdots .$$

The next theorem is due to Suslin [916].

Theorem 6.29 (Algebraic K-theory mod k of algebraically closed fields). *The inclusion of algebraically closed fields induces isomorphisms on $K_*(-;\mathbb{Z}/k)$.*

Let p be a prime number. Quillen [804] has computed the algebraic K-groups for any algebraic extension of the field $\mathbb{F}_p$ of p elements for every prime p. One can determine $K_n(\overline{\mathbb{F}_p};\mathbb{Z}/k)$ for the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$ from (6.28). Hence one obtains $K_n(F;\mathbb{Z}/k)$ for any algebraically closed field of prime characteristic p by Suslin's Theorem 6.29.

The next theorem is due to Suslin [917]. We will explain the topological K-groups $K_n^{\mathsf{TOP}}(\mathbb{R})$ and $K_n^{\mathsf{TOP}}(\mathbb{C})$ of the C^*-algebras $\mathbb{R}$ and $\mathbb{C}$ in Subsection 10.3.2. There are mod k versions $K_n^{\mathsf{TOP}}(\mathbb{R};\mathbb{Z}/k)$ and $K_n^{\mathsf{TOP}}(\mathbb{C};\mathbb{Z}/k)$, for which a long exact sequence analogous to the one of (6.28) exists.

Theorem 6.30 (Algebraic and topological K-theory mod k for $\mathbb{R}$ and $\mathbb{C}$). *The comparison map from algebraic to topological K-theory induces for all integers $k \geq 2$ and all $n \geq 0$ isomorphisms*

$$K_n(\mathbb{R};\mathbb{Z}/k) \xrightarrow{\cong} K_n^{\mathsf{TOP}}(\mathbb{R};\mathbb{Z}/k);$$
$$K_n(\mathbb{C};\mathbb{Z}/k) \xrightarrow{\cong} K_n^{\mathsf{TOP}}(\mathbb{C};\mathbb{Z}/k).$$

Generalizations of Theorem 6.30 to C^*-algebras will be discussed in Section 10.7.

Since $K_n^{\mathsf{TOP}}(\mathbb{C})$ is $\mathbb{Z}$ for n even and vanishes for n odd and for every algebraically closed field F of characteristic 0 we have an injection $\overline{\mathbb{Q}} \to F$ for the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, Theorem 6.29 and Theorem 6.30 imply for every algebraically closed field F of characteristic zero.

$$K_n(F;\mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \geq 0, n \text{ even;} \\ \{0\} & n \geq 1, n \text{ odd.} \\ \{0\} & n \leq -1. \end{cases}$$

Exercise 6.31. Using the fact that $K_n^{\mathsf{TOP}}(\mathbb{R})$ is 8-periodic and its values for $n = 0, 1, 2, 3, 4, 5, 6, 7$ are given by $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, $\{0\}$, $\mathbb{Z}$, $\{0\}$, $\{0\}$, $\{0\}$, compute $K_n(\mathbb{R};\mathbb{Z}/k)$ and $K_n^{\mathsf{TOP}}(\mathbb{R};\mathbb{Z}/k)$ for $n \in \mathbb{Z}$ and $k \geq 3$ an odd natural number.

6.5 Other Constructions of Connective Algebraic *K*-Theory

The plus-construction works for rings and finitely generated free or projective modules. However, it turns out that it is important to consider more general situations where one can feed in categories with certain extra structures. The main examples are Quillen's Q-construction, see [805, §2], [843, Chapter 5], [905, Chapter 4], designed for exact categories, the group completion construction, see [417, 890], designed for symmetric monoidal categories, and Waldhausen's $wS_\bullet$-construction, see [960] and Subsection 7.3.2, designed for categories with cofibrations and weak equivalences. Given a ring R, the category of finitely generated projective R-modules yields examples of the type of categories above and the appropriate construction yields always the same, namely, the K-groups as defined by the plus-construction above. The Q-construction and exact categories can be used to define K-theory for the category of finitely generated R-modules (dropping projective) or the category of locally free $\mathcal{O}_X$-modules of finite rank over a scheme X. One important feature is that the notion of exact sequences can be different from the one given by split exact sequences, or, equivalently, by direct sums. Whereas in Quillen's setting one needs exact structures in an algebraic sense, Waldhausen's $wS_\bullet$-construction is also suitable for categories where the input are spaces and one can replace isomorphisms by weak equivalences.

We briefly recall the setup of exact categories beginning with additive categories. A category $\mathcal{C}$ is called *small* if its objects form a set. An *additive category* $\mathcal{A}$ is a small category $\mathcal{A}$ such that for two objects A and B the morphism set $\operatorname{mor}_{\mathcal{A}}(A,B)$ has the structure of an abelian group, there exists a zero-object, i.e., an object which is both initial and terminal, the direct sum $A \oplus B$ of two objects A and B exists, and the obvious compatibility conditions hold, e.g., composition of morphisms is bilinear. A *functor of additive categories* $F\colon \mathcal{A}_0 \to \mathcal{A}_1$ is a functor respecting the zero-objects such that for two objects A and B in $\mathcal{A}_0$ the map $\operatorname{mor}_{\mathcal{A}_0}(A,B) \to \operatorname{mor}_{\mathcal{A}_1}(F(A),F(B))$ sending f to $F(f)$ respects the abelian group structures and $F(A \oplus B)$ is a model for $F(A) \oplus F(B)$.

A *skeleton* $\mathcal{D}$ of a category $\mathcal{C}$ is a full subcategory such that $\mathcal{D}$ is small and the inclusion $\mathcal{D} \to \mathcal{C}$ is an equivalence of categories, or, equivalently, for every object $C \in \mathcal{C}$ there is an object D in $\mathcal{D}$ together with an isomorphism $C \xrightarrow{\cong} D$ in $\mathcal{C}$.

Definition 6.32 (Exact category). An *exact category* $\mathcal{P}$ is a full additive subcategory of some abelian category $\mathcal{A}$ with the following properties:

- $\mathcal{P}$ is closed under extensions in $\mathcal{A}$, i.e., for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ in $\mathcal{A}$ with P_0, P_2 in $\mathcal{P}$ we have $P_1 \in \mathcal{P}$;
- $\mathcal{P}$ has a *small skeleton*.

An *exact functor* $F\colon \mathcal{P}_0 \to \mathcal{P}_1$ is a functor of additive categories that sends exact sequences to exact sequences.

Examples of exact categories are abelian categories possessing a small skeleton, the category of finitely generated projective R-modules, the category of finitely generated R-modules, the category of vector bundles over a compact space, the category of algebraic vector bundles over a projective algebraic variety, and the category of locally free sheaves of finite rank on a scheme.

An additive category becomes an exact category in the sense of Quillen with respect to split exact sequences. On the other hand there are interesting exact categories where the exact sequences are not necessarily split exact sequences.

The *Q-construction*, see [805, §2], [843, Chapter 5], [905, Chapter 4], assigns to any exact category $\mathcal{P}$ its *K-theory space* $K(\mathcal{P})$ and one defines $K_n(\mathcal{P}) := \pi_n(K(\mathcal{P}))$ for $n \geq 0$. If $\mathcal{P}$ is the category of finitely generated projective R-modules, this definition coincides with the Definition 6.9 of $K_n(R)$ coming from the plus-construction.

The Q-construction allows to define algebraic K-theory for objects naturally appearing in algebraic geometry, arithmetic geometry and number theory, since these give exact categories as described above.

Example 6.33 (The category of nilpotent endomorphism). Let $\mathbf{NIL}(R)$ be the exact category whose objects are pairs (P, f) of finitely generated projective R-modules together with nilpotent endomorphisms $f\colon P \to P$. Its K-theory $\operatorname{Nil}_n(R) := K_n(\mathbf{NIL}(R))$ splits as $K_n(R) \oplus \widetilde{\operatorname{Nil}}_n(R)$ for $n \geq 0$ where $\widetilde{\operatorname{Nil}}_n(R)$ is the cokernel of the homomorphism $K_n(R) \to K_n(\mathbf{NIL}(R))$ induced by the obvious functor sending a finitely generated projective R-module P to $0\colon P \to P$. We get for $n \geq 1$

$$NK_n(R) = \widetilde{\operatorname{Nil}}_{n-1}(R).$$

This has been considered for $n = 1$ already in Theorem 3.72. A proof, which works also for the more general context of non-connective K-theory of additive categories where a twist with an automorphism is allows, can be found in [671, Theorem 0.4], see also [419].

6.6 Non-Connective Algebraic K-Theory of Additive Categories

The approaches mentioned in Section 6.5 will always yield spaces $K(R)$ such that the algebraic K-groups are defined to be its homotopy groups. Since a space has no negative homotopy groups, this definition will not encompass the negative algebraic K-groups. In order to take these into account, one has to find appropriate deloopings.

So the task is to replace the space $K(R)$ by a (non-connective) spectrum $\mathbf{K}(R)$ such that one can define $K_n(R)$ by $\pi_n(\mathbf{K}(R))$ for $n \in \mathbb{Z}$ and this definition coincides with the other definitions for all $n \in \mathbb{Z}$. For rings this has been achieved by Gersten [406] and Wagoner [954].

We would like to feed in additive categories.

The category of spectra $\mathsf{SPECTRA}$ will be introduced in Section 12.4. Denote by ADDCAT the category of additive categories. There is an obvious notion of the direct sum of two additive categories. We will use a construction of Pedersen-Weibel [783], see also Schlichting [197], or Lück-Steimle [668] of a functor

$$\mathbf{K}\colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}, \quad \mathcal{A} \mapsto \mathbf{K}(\mathcal{A}). \tag{6.34}$$

Definition 6.35 (Algebraic K-groups of additive categories). We call $\mathbf{K}(\mathcal{A})$ the *non-connective K-theory spectrum associated to an additive category.* Define for $n \in \mathbb{Z}$ the *n-th algebraic K-group of an additive category $\mathcal{A}$* by

$$K_n(\mathcal{A}) := \pi_n(\mathbf{K}(\mathcal{A})).$$

Definition 6.36 (Flasque and Eilenberg swindle). An additive category $\mathcal{A}$ is called *flasque* if there exists a functor of additive categories $S\colon \mathcal{A} \to \mathcal{A}$ together with a natural equivalence $T\colon \operatorname{id}_{\mathcal{A}} \oplus S \xrightarrow{\cong} S$. Sometimes the pair (S,T) is called an *Eilenberg swindle.*

We conclude from Pedersen-Weibel [783], see also [197], or from Lück-Steimle [668]

Theorem 6.37 (Properties of $\mathbf{K}(\mathcal{A})$).

(i) If R is a ring and $\mathcal{A}$ is the additive category of finitely generated projective R-modules, then $K_n(\mathcal{A})$ coincides with $K_n(R)$ for $n \in \mathbb{Z}$;

(ii) Let F_1 and F_2 be functors of additive categories. If there exists a natural equivalence of such functors from F_1 to F_2, then the maps of spectra $\mathbf{K}(F_1)$ and $\mathbf{K}(F_2)$ are homotopic;

In particular a functor $F\colon \mathcal{A} \to \mathcal{A}'$ of additive categories which is an equivalence of categories induces a homotopy equivalence $\mathbf{K}(F)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}')$;

(iii) If $\mathcal{A}$ is flasque, then $\mathbf{K}(\mathcal{A})$ is weakly contractible.

Exercise 6.38. Give a definition of $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ as abelian groups in terms of generators and relations such that in the case where R is a ring and $\mathcal{A}$ is the category of finitely generated projective R-modules, this definition coincides with the ones appearing in Definitions 2.1 and 3.1. Show that $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ are trivial if $\mathcal{A}$ is flasque.

Exercise 6.39. Let $\mathcal{A}$ be the category of countably generated projective R-modules. Show that $K_n(\mathcal{A}) = 0$ for all $n \in \mathbb{Z}$.

Remark 6.40 (Non-connective K-theory spectra for exact categories). Schlichting [884] has constructed for an exact category $\mathcal{P}$ a delooping of the space $K(\mathcal{P})$. Thus he can assign to an exact category $\mathcal{P}$ a (non-connective) spectrum $\mathbf{K}(\mathcal{P})$ and define $K_n(\mathcal{P}) := \pi_n(\mathbf{K}(\mathcal{P}))$ for $n \in \mathbb{Z}$. If $\mathcal{P}$ is the category of finitely generated projective R-modules, this definition coincides with our previous definition of $K_n(R)$. If the exact sequences in $\mathcal{P}$ are given by split exact sequences, this definition agrees with the one of Definition 6.35 when we consider $\mathcal{P}$ as an additive category.

We will use later the following construction for additive categories.

Given an additive category $\mathcal{A}$, its *idempotent completion* $\operatorname{Idem}(\mathcal{A})$ is defined to be the following additive category. Objects are morphisms $p\colon A \to A$ in $\mathcal{A}$ satisfying $p \circ p = p$. A morphism f from $p_1\colon A_1 \to A_1$ to $p_2\colon A_2 \to A_2$ is a morphism $f\colon A_1 \to A_2$ in $\mathcal{A}$ satisfying $p_2 \circ f \circ p_1 = f$. The identity of an object (A, p) is given by the morphism $p\colon (A,p) \to (A,p)$. The structure of an additive category on A induces the structure of an additive category on $\operatorname{Idem}(\mathcal{A})$ in the obvious way. A functor of additive categories $F\colon \mathcal{A} \to \mathcal{A}'$ induces a functor $\operatorname{Idem}(F)\colon \operatorname{Idem}(\mathcal{A}) \to \operatorname{Idem}(\mathcal{A}')$ of additive categories by sending (A,p) to $(F(A), F(p))$.

There is a obvious embedding

$$\eta(\mathcal{A})\colon \mathcal{A} \to \operatorname{Idem}(\mathcal{A})$$

sending an object A to $\mathrm{id}_A\colon A \to A$ and a morphism $f\colon A \to B$ to the morphism given by f again. A unital additive category $\mathcal{A}$ is called *idempotent complete*, if $\eta(\mathcal{A})\colon \mathcal{A} \to \operatorname{Idem}(\mathcal{A})$ is an equivalence of additive categories, or, equivalently, if for every idempotent $p\colon A \to A$ in A there exists objects B and C and an isomorphism $f\colon A \xrightarrow{\cong} B \oplus C$ in $\mathcal{A}$ such that $f \circ p \circ f^{-1}\colon B \oplus C \to B \oplus C$ is given by $\begin{pmatrix} \mathrm{id}_B & 0 \\ 0 & 0 \end{pmatrix}$. The idempotent completion $\operatorname{Idem}(\mathcal{A})$ of a unital additive category $\mathcal{A}$ is idempotent complete.

Theorem 6.41. *The map η induces an equivalence*

$$\mathbf{K}(\eta)\colon \mathbf{K}(\mathcal{A}) \xrightarrow{\simeq} \mathbf{K}(\operatorname{Idem}(\mathcal{A}))$$

on the non-connective K-theory spectra.

Proof. This follows from [931, Theorem A.9.1] and [668, Corollary 3.7]. □

Note that Theorem 6.41 is not true for the standard construction of the connective K-theory of an additive category. Therefore we always replaces in the construction of the connective K-theory spectrum $\mathcal{A}$ by its idempotent completion $\mathrm{Idem}(\mathcal{A})$. This passage does not change $K_n(\mathcal{A})$ for $n \geq 1$, but $K_0(\mathcal{A})$, see [931, Theorem A.9.1]. This is analogous to the fact that in previous constructions of the connective K-theory of a ring we had taking the cross product with $K_0(\mathcal{A})$, see Definition 6.9.

Let R be a ring. Let $R\text{-}\mathsf{MOD}_{\mathrm{fgf}}$ and $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ respectively be the unital additive category of finitely generated free R-modules and of finitely generated projective R-modules respectively. We obtain an equivalence of unital additive categories $\mathrm{Idem}(R\text{-}\mathsf{MOD}_{\mathrm{fgf}}) \xrightarrow{\simeq} R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ by sending an object (F,p) to $\mathrm{im}(p)$. Let $\underline{R}_{\oplus}$ be the additive category which has a objects the natural numbers $0, 1, 2, \ldots$ and morphisms from m to n are given by (m,n)-matrices over R. The composition is given by multiplication of matrices, more precisely, given morphisms $A\colon l \to m$ and $B\colon m \to n$, their composite is $AB\colon l \to m$. The direct sum of of two objects m and n is the object $m+n$ and the direct sum of morphisms is given by the block sum of matrices. We have the obvious equivalence of additive categories

$$\underline{R}_{\oplus} \xrightarrow{\simeq} R\text{-}\mathsf{MOD}_{\mathrm{fgf}} \tag{6.42}$$

which sends an object m to R^m and a morphism $A\colon m \to n$ to the R-linear homomorphism $r_A\colon R^m \to R^n,\ (s_1, \ldots, s_m) \mapsto (s_1, \ldots, s_m)A$ given by right multiplication with A. Thus we obtain an equivalence of unital additive categories, natural in the unital ring R,

$$\Theta_R\colon \mathrm{Idem}\bigl(\underline{R}_{\oplus}\bigr) \xrightarrow{\simeq} R\text{-}\mathsf{MOD}_{\mathrm{fgp}}. \tag{6.43}$$

Note that $\mathrm{Idem}\bigl(\underline{R}_{\oplus}\bigr)$ is small, in contrast to $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$. The non-connective K-theory spectrum of a ring $\mathbf{K}(R)$ is defined to be $\mathbf{K}(\underline{R}_{\oplus})$ for $\mathbf{K}$ defined in (6.34). Then $\pi_n(\mathbf{K}(R))$ can be identified with all other definitions of $K_n(R)$ above for every $n \in \mathbb{Z}$.

6.7 Survey on Main Properties of Algebraic K-Theory of Exact Categories

Next we state some basic and important general results about algebraic K-theory of exact categories. **Comment 1 (by W.)**: This section may have to be adapted to Chapter 8 when it is written.

6.7.1 Additivity

For a proof of the next result we refer for instance to [805, Corollary 1 in §3 on page 22], [905, Corollary 4.3 on page 41], [979, Theorem 1.2 in Section V.I on page 366] (at least in the connective setting), and [884, Corollary 4 in Section 7].

Theorem 6.44 (Additivity Theorem for exact categories). *Let $0 \to F_0 \xrightarrow{i} F_1 \xrightarrow{p} F_2 \to 0$ be an exact sequence of functors $F_k \colon \mathcal{P}_1 \to \mathcal{P}_2$ of exact categories $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., i and p are natural transformations such that for each object P the sequence $0 \to F_0(P) \xrightarrow{i(P)} F_1(P) \xrightarrow{p(P)} F_2(P) \to 0$ is exact. Then we get for the induced morphisms $K_n(F_k) \colon K_n(\mathcal{P}_1) \to K_n(\mathcal{P}_2)$ for every $n \in \mathbb{Z}$*

$$K_n(F_1) = K_n(F_0) + K_n(F_2).$$

6.7.2 Resolution Theorem

Let $\mathcal{M}$ and $\mathcal{P}$ be exact categories which are contained in the same abelian category $\mathcal{A}$. Suppose that $\mathcal{P}$ is a full subcategory of $\mathcal{M}$. A *finite resolution* of an object M of $\mathcal{M}$ by objects in $\mathcal{P}$ is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ for some natural number n. We say that $\mathcal{P}$ is *closed under extensions in* $\mathcal{M}$ if for any exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ in $\mathcal{M}$ with M_0, M_2 in $\mathcal{P}$ we have $M_1 \in \mathcal{P}$. For a proof of the next Theorem we refer for instance to [805, Corollary 1 in §4 on page 25] or [905, Theorem 4.6 on page 41], [979, Theorem 3.1 in Section V.3 on page 385] (at least in the connective setting), and [884].

Theorem 6.45 (Resolution Theorem). *Let $\mathcal{M}$ and $\mathcal{P}$ be exact categories which are contained in the same abelian category $\mathcal{A}$. Suppose that $\mathcal{P}$ is a full subcategory of $\mathcal{M}$ and is closed under extensions in $\mathcal{M}$. Suppose that every object in $\mathcal{M}$ has a finite resolution by objects in $\mathcal{P}$.*

Then the inclusion $\mathcal{P} \to \mathcal{M}$ induces for every $n \in \mathbb{Z}$ an isomorphism

$$K_n(\mathcal{P}) \xrightarrow{\cong} K_n(\mathcal{M}).$$

Exercise 6.46. Let R be a regular ring. Show that for every $n \in \mathbb{Z}$ the canonical map $K_n(R) \to G_n(R)$ is bijective.

6.7.3 Devissage

For a proof of the next result we refer for instance to [805, Theorem 4 in §5 on page 28], [905, Theorem 4.8 on page 42], or [979, Theorem 4.1 in Section V.4 on page 400].

Theorem 6.47 (Devissage). *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B}$ be a full abelian subcategory of $\mathcal{A}$ which is closed under taking subobjects, quotients, and finite products in $\mathcal{A}$. Suppose that each object A in $\mathcal{A}$ has a finite filtration in $\mathcal{A}$*

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$$

such that A_i/A_{i-1} is isomorphic to an object in $\mathcal{B}$ for $i = 1, 2, \ldots, n$.

Then the inclusion of exact categories $i\colon \mathcal{B} \to \mathcal{A}$ induces an isomorphism for $n \geq 0$.

Note that in Theorem 6.47 the condition $n \geq 0$ appears. To the author's knowledge it is not known whether Theorem 6.47 holds also for $n \leq -1$. If $\mathcal{A}$ is a Noetherian abelian category, then its negative K-groups vanish and Theorem 6.47 holds also for negative K-groups of trivial reasons, see [884, Theorem 7].

An object N in an abelian category is called *simple* if $N \neq 0$ and any monomorphism $M \to N$ is the zero-homomorphism or an isomorphism. For a simple object M its ring of automorphisms $\operatorname{end}_{\mathcal{A}}(M)$ is a skew-field (Schur's Lemma). An object N in an abelian category is called *semisimple* if it is isomorphic to a finite direct sum of simple objects. A zero object is called an object of length 0. Call the simple objects of an abelian category objects of length ≤ 1. We define inductively for $l \geq 2$ an object M to be *of length* $\leq l$ if there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ for an object M_1 of length ≤ 1 and an object M_2 of length $\leq (l-1)$. An object is *of finite length* if it has length $\leq l$ for some natural number l. For a proof of the following corollary of Theorem 6.47 we refer to [805, Corollary 1 in §5 on page 28].

Corollary 6.48. Let $\mathcal{A}$ be an abelian category. Suppose that there is a subset S of the set of objects of $\mathcal{A}$ with the property that any simple object in $\mathcal{A}$ is isomorphic to precisely one object in S. Let $\mathcal{A}_{ss}$ be the full subcategory of $\mathcal{A}$ consisting of semisimple objects and let $\mathcal{A}_{fl}$ be the full subcategory consisting of objects of finite length. Then we obtain for every $n \in \mathbb{Z}, n \geq 0$ an isomorphism

$$\bigoplus_{M \in S} K_n(\operatorname{end}_{\mathcal{A}}(M)) \xrightarrow{\cong} K_n(\mathcal{A}_{ss}) \xrightarrow{\cong} K_n(\mathcal{A}_{fl}).$$

In particular we get in the situation of Corollary 6.48 from Example 2.4 and Theorem 3.6.

$$K_0(\mathcal{A}_{fl}) \cong \bigoplus_S \mathbb{Z};$$
$$K_1(\mathcal{A}_{fl}) \cong \prod_S \operatorname{end}_{\mathcal{A}}(S)^{\times}/[\operatorname{end}_{\mathcal{A}}(S)^{\times}, \operatorname{end}_{\mathcal{A}}(S)^{\times}].$$

6.7.4 Localization

Theorem 6.49 (Localization). *Let $\mathcal{A}$ be a small abelian category and let $\mathcal{B}$ be an additive subcategory such that for any exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ in $\mathcal{A}$ the object M_1 belongs to $\mathcal{B}$ if and only if both M_0 and M_2 belong to $\mathcal{B}$. Then there exists a well-defined quotient abelian category $\mathcal{A}/\mathcal{B}$. It has the same objects as $\mathcal{A}$, and its morphisms are obtained from those in $\mathcal{A}$ by formally inverting morphisms whose kernel and cokernel belong to $\mathcal{B}$.*

Then there are obvious functors $\mathcal{B} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ that induce a long exact sequence

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to \cdots .$$

The full description of $\mathcal{A}/\mathcal{B}$ can be found in [905, Appendix B.3] or [979, Section II.6 on page 119] A proof of the last theorem is given in [805, Theorem 5 in §5 on page 29], [905, Theorem 4.9 on page 42], [979, Theorem 5.1 in Section V.5 on page 402] (at least in the connective setting), and [884, Theorem 1].

The next example is taken from [979, Application 6.1 in Section V.6 on page 406]

Example 6.50. Let R be a Noetherian ring and s be an element in the center of R which is different from 0. Then one can consider the subcategory of finitely generated s-torsion modules of the abelian category of finitely generated R-modules and the localization sequence of Theorem 6.49 reduces to a long exact sequence

$$\begin{aligned}\cdots \to G_{n+1}(R[s^{-1}]) \to G_n(R/(s)) \to G_n(R) \to G_n(R[s^{-1}])\\ \cdots \to G_1(R[s^{-1}]) \to G_0(R/(s)) \to G_0(R) \to G_0(R[s^{-1}]) \to 0\end{aligned}$$

where, roughly speaking, $R[s^{-1}]$ is obtained from R by inverting s.

Exercise 6.51. Let p be a prime number. Then we obtain a long exact sequence

$$\begin{aligned}\cdots \to K_{n+1}(\mathbb{Z}[p^{-1}]) \to K_n(\mathbb{F}_p) \to K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[p^{-1}])\\ \cdots \to K_1(\mathbb{Z}[p^{-1}]) \to K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[p^{-1}]) \to 0.\end{aligned}$$

6.7.5 Filtered Colimits

For a proof of the next Theorem we refer for instance to [805, (9) in §2 on page 20] or [905, Lemma 3.8 on page 35], [979, (6.4) in Section IV.6 on page 321] (at least in the connective setting), and [884, Corollary 5].

Theorem 6.52 (K-theory and directed colimits).
Let $\mathcal{A}$ be an exact category. Let $\{\mathcal{A}_i \mid i \in I\}$ be a directed set of exact subcategories of $\mathcal{A}$, directed by inclusion such that $\mathcal{A}$ is the union of the categories $\mathcal{A}$ in the sense that for every object A in $\mathcal{A}$ and every morphism $f\colon A \to A'$ there is $i \in I$ with $A \in \mathcal{A}$ and $f \in \mathcal{A}_i$. Then the canoncial map

$$\operatorname{colim}_{i\in I} K_n(\mathcal{A}_i) \to K_n(\mathcal{A})$$

is bijective for $n \in \mathbb{Z}$.

Theorem 6.52 holds more generally for filtered colimits.

6.8 The K-Theoretic Farrell-Jones Conjecture for Torsionfree Groups and Regular Rings

The Farrell-Jones Conjecture for algebraic K-theory, which we will formulate in full generality in Conjecture 13.1, reduces for a torsionfree group and a regular ring to the following conjecture. Under the additional assumption that there is a finite model for BG it appears already in [486].

Conjecture 6.53 (Farrell-Jones Conjecture for torsionfree groups and regular rings for K-theory). Let G be a torsionfree group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Here $H_*(-; \mathbf{K}(R))$ denotes the homology theory that is associated to the (non-connective) K-spectrum $\mathbf{K}(R)$. Recall that $H_n(\{\bullet\}; \mathbf{K}(R))$ is $K_n(R)$ for $n \in \mathbb{Z}$, where here and elsewhere $\{\bullet\}$ denotes the space consisting of one point. The space BG is the *classifying space of the group G*, which is up to homotopy characterized by the property that it is a CW-complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. The technical details of the construction of $H_n(-; \mathbf{K}(R))$ and the assembly map will be explained in a more general setting in Sections 12.4 and 12.5.

The point of Conjecture 6.53 is that on the right-hand side of the assembly map we have the group $K_n(RG)$ we are interested in, whereas the left-hand side is a homology theory and hence much easier to compute. A basic tool for the computation of a homology theory is the Atiyah-Hirzebruch spectral sequence, which in our case has as E^2-term $E^2_{p,q} = H_p(BG; K_q(R))$ and converges to $H_{p+q}(BG; \mathbf{K}(R))$.

Remark 6.54 (The conditions appearing in Conjecture 6.53 are necessary). The condition that G is torsionfree and that R is regular are necessary in Conjecture 6.53. If one drops one of these conditions, one obtains counterexamples as follows.

If G is a finite group, then we obtain an isomorphism

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(\{\bullet\}; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Hence Conjecture 6.53 would predict for a finite group that the change of rings homomorphism $K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$ is bijective. This contradicts for instance Lemma 2.88.

In view of the Bass-Heller-Swan decomposition 6.16, Conjecture 6.53 is true for $G = \mathbb{Z}$ in degree n only if $NK_n(R)$ vanishes.

Exercise 6.55. Let R be a regular ring. Let $G = G_1 *_{G_0} G_2$ be an amalgamated free product of torsionfree groups, where G_0 is a common subgroup of G_1 and G_2. Suppose that Conjecture 6.53 is true for G_0, G_1, G_2, and G with coefficients in the ring R. Show that then there exists a long exact Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to K_n(RG_0) \to K_n(RG_1) \oplus K_n(RG_2) \to K_n(RG) \\ \to K_{n-1}(RG_0) \to K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \to \cdots \end{aligned}$$

Exercise 6.56. Let R be a regular ring. Let $\phi\colon G \to G$ be an automorphism of the torsionfree group G. Suppose that Conjecture 6.53 is true for G and the semidirect product $G \rtimes_\phi \mathbb{Z}$ with coefficients in the ring R. Show that then there exists a long exact Wang sequence

$$\begin{aligned} \cdots \to K_n(RG) \xrightarrow{\mathrm{id} - K_n(\phi)} K_n(RG) \to K_n(R[G \rtimes_\phi \mathbb{Z}]) \\ \to K_{n-1}(RG) \xrightarrow{\mathrm{id} - K_{n-1}(\phi)} K_{n-1}(RG) \to \cdots . \end{aligned}$$

Remark 6.57 ($K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ for torsionfree G). Rationally the Atiyah-Hirzebruch spectral sequence always collapses and the homological Chern character gives an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah-Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Subsection 12.6.1 and Section 12.7.

Because of Theorem 6.24 the left hand side of the isomorphism described in Remark 6.57 specializes for $R = \mathbb{Z}$ to $H_n(BG;\mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG;\mathbb{Q})$. Hence Conjecture 6.53 predicts for a torsionfree group G

$$K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(BG;\mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG;\mathbb{Q}). \tag{6.58}$$

Conjecture 6.59 (Nil-groups for regular rings and torsionfree groups). Let G be a torsionfree group and let R be a regular ring. Then we get

$$NK_n(RG) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Exercise 6.60. Show that a torsionfree group G satisfies Conjecture 6.59 for all regular rings R if it satisfies Conjecture 6.53 for all regular rings R.

6.9 Mayer-Vietoris Sequences for Amalgamated Free Products and Wang Sequences for HNN-Extensions

We have seen in the introduction that for the topological K-theory of reduced group C^*-algebras there exist Mayer-Vietoris sequences associated to amalgamated free products, see (1.4), and long exact Wang sequences for semidirect products of the shape $G = H \rtimes_\phi \mathbb{Z}$, see (1.5). These lead to the final formulation of the Baum-Connes Conjecture 1.1. Because of Exercises 6.55 and 6.56 one can expect similar long exact sequences to exists for algebraic K-theory of group rings for torsionfree groups and regular rings, but not in general, as one can derive for instance from the Bass-Heller-Swan decomposition 6.16.

We want to explain the more complicated general answer for algebraic K-theory of group rings, which is given by Waldhausen [956] and [957].

A ring R is called *regular coherent* if every finitely presented R-module possesses a finite projective resolution. A ring R is regular if and only if it is regular coherent and Noetherian. A group G is called *regular* or *regular coherent* respectively if for any regular ring R the group ring RG is regular or regular coherent respectively. If $G = G_1 *_{G_0} G_2$ for regular coherent groups G_1 and G_2 and a regular group G_0 or if $G = H \rtimes_\phi \mathbb{Z}$ for a regular group H, then G is regular coherent. In particular $\mathbb{Z}^n$ is regular and regular coherent, whereas a non-abelian finitely generated free group is regular coherent but not regular. For proofs of the claims above and for more information about regular coherent groups we refer to [957, Theorem 19.1].

The maps of spectra appearing in the theorem below are all induced by obvious functors between categories.

Theorem 6.61 (Waldhausen's cartesian squares for non-connective algebraic K-theory). *Let $G = G_1 *_{G_0} G_2$ be an amalgamated free product and let R be a ring.*

(i) The exists a homotopy cartesian square of spectra

$$\begin{tikzcd} \mathbf{Nil}(RG_0; RG_1, RG_2) \arrow[r, "\mathbf{j}"] \arrow[d, "\mathbf{i}"'] & \mathbf{K}(RG_1) \vee \mathbf{K}(RG_2) \arrow[d, "\mathbf{k}_1"] \\ \mathbf{K}(RG_0) \arrow[r, "\mathbf{k}_0"'] & \mathbf{K}(RG) \end{tikzcd}$$

where $\mathbf{Nil}(RG_0; RG_1, RG_2)$ is a certain non-connective Nil*-spectrum associated to $G = G_1 *_{G_0} G_2$ and R and $\mathbf{K}$ is the (non-connective) K-theory spectrum;*

(ii) There is a map $\mathbf{f}\colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \to \mathbf{Nil}(RG_0; RG_1, RG_2)$ and for $k = 1, 2$ a map $\mathbf{g}_k\colon \mathbf{Nil}(RG_0; RG_1, RG_2) \to \mathbf{K}(RG_0)$ with the following properties. The composite $\mathbf{g}_k \circ \mathbf{f}\colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \to \mathbf{K}(RG_0)$ is the projection to the k-th summand, the composite

$$\mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \xrightarrow{\mathbf{f}} \mathbf{Nil}(RG_0; RG_1, RG_2) \xrightarrow{\mathbf{l}} \mathbf{K}(RG_1) \vee \mathbf{K}(RG_2)$$

is homotopic to $\mathbf{K}(j_1) \vee \mathbf{K}(j_2)$ for $j_k\colon G_0 \to G_k$ the canonical inclusion, and $\mathbf{i} \circ \mathbf{f}$ is homotopic to $\mathrm{id} \vee \mathrm{id}\colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \to \mathbf{K}(RG_0)$;

(iii) If R is regular and G_0 is regular coherent, then $\mathbf{f}\colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \to \mathbf{Nil}(RG_0; RG_1, RG_2)$ is a weak homotopy equivalence;

(iv) The composite of the map $\Omega\mathbf{K}(RG) \to \mathbf{Nil}(RG_0; RG_1, RG_2)$ associated to the homotopy cartesian square of assertion (i) *with the canonical map from $\mathbf{Nil}(RG_0; RG_1, RG_2)$ to the homotopy cofiber of the map $\mathbf{f}$ induces a split surjection on homotopy groups.*

Proof. All these claims are proved for connective K-theory in Waldhausen [957, 11.2, 11.3, 11.6]. In [74, Section 9 and 10] the definitions and assertions are extended to the non-connective version except for assertion (iv). Assertion (iv) can be derived from the connective version by using the Bass-Heller-Swan decomposition 6.16. ☐

Theorem 6.62 (Mayer-Vietoris sequence of an amalgamated free product for algebraic K-theory). *Let $G = G_1 *_{G_0} G_2$ be an amalgamated free product and let R be a ring. Denote by $i_k\colon G_0 \to G_k$ and $j_k\colon G_k \to G$ the obvious inclusions. Define $NK_n(RG_0; RG_1, RG_2)$ to be the $(n-1)$-homotopy group of the homotopy cofiber of the map $\mathbf{f}$ appearing in Theorem 6.61* (ii)*. Let $p_n\colon K_n(RG) \to NK_n(RG_0; RG_1, RG_2)$ be the split surjection coming from Theorem 6.61* (iv)*. Then:*

(i) We obtain a splitting

$$K_n(RG) \cong \ker(p_n) \oplus NK_n(RG_0; RG_1, RG_2);$$

(ii) There exists a long exact Mayer-Vietoris sequence sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) \xrightarrow{K_n(i_1)\oplus K_n(i_2)} K_n(RG_1) \oplus K_n(RG_2)$$
$$\xrightarrow{K_n(j_1)-K_n(j_2)} \ker(p_n) \xrightarrow{\partial_n} K_{n-1}(RG_0)$$
$$\xrightarrow{K_{n-1}(i_1)\oplus K_{n-1}(i_2)} K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \xrightarrow{K_{n-1}(j_1)-K_{n-1}(j_2)} \cdots;$$

(iii) If G_0 is regular coherent and R is regular, then

$$NK_n(RG_0; RG_1, RG_2) = 0 \quad \textit{for } n \in \mathbb{Z}$$

and the sequence of assertion (ii) *reduces to the long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) \xrightarrow{K_n(i_1)\oplus K_n(i_2)} K_n(RG_1) \oplus K_n(RG_2)$$
$$\xrightarrow{K_n(j_1)-K_n(j_2)} K_n(RG) \xrightarrow{\partial_n} K_{n-1}(RG_0)$$
$$\xrightarrow{K_{n-1}(i_1)\oplus K_{n-1}(i_2)} K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \xrightarrow{K_{n-1}(j_1)-K_{n-1}(j_2)} \cdots.$$

Exercise 6.63. Show that Theorem 6.61 implies Theorem 6.62.

Analogously one gets from Waldhausen [956] and [957] using [74, Section 9 and 10]

Theorem 6.64 (Wang sequence associated to an HNN-extension for algebraic K-theory). *Let $\alpha, \beta\colon H \to K$ be two injective group homomorphisms. Let G be the associated HNN-extension and let $j\colon K \to G$ be the canonical inclusion. Then there are certain* Nil*-groups $NK_n(RH, RK, \alpha, \beta)$ and homomorphisms $p_n\colon K_n(RG) \to NK_n(RH, RK, \alpha, \beta)$ such that the following holds:*

(i) There is a long exact Wang sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RH) \xrightarrow{K_n(\alpha)-K_n(\beta)} K_n(RK) \xrightarrow{K_n(j)} \ker(p_n)$$
$$\xrightarrow{\partial_n} K_{n-1}(RH) \xrightarrow{K_{n-1}(\alpha)-K_{n-1}(\beta)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots;$$

(ii) The map $p_n\colon K_n(RG) \to NK_n(RH, RK, \alpha, \beta)$ is split surjective;
(iii) If R is regular and H is regular coherent, then $NK_n(RH, RK, \alpha, \beta)$ vanishes for all $n \in \mathbb{Z}$. In this case the Wang sequence reduces to

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RH) \xrightarrow{K_n(\alpha)-K_n(\beta)} K_n(RK) \xrightarrow{K_n(j)} K_n(RG)$$
$$\xrightarrow{\partial_n} K_{n-1}(RH) \xrightarrow{K_{n-1}(\alpha)-K_{n-1}(\beta)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots.$$

Remark 6.65 (Wang sequence of a semidirect product $G = K \rtimes_\phi \mathbb{Z}$ for algebraic K-theory). A semidirect product $G = K \rtimes_\phi \mathbb{Z}$ for a group automorphism $\phi\colon K \to K$ is a special case of an HNN-extensions, namely take $H = K$, $\alpha = \mathrm{id}$ and $\beta = \phi$. In this case the Wang sequence appearing in Theorem 6.64 (i) takes the form

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RK) \xrightarrow{\mathrm{id} - K_n(\phi)} K_n(RK) \xrightarrow{K_n(j)} \ker(p_n) \xrightarrow{\partial_n} K_{n-1}(RK) \xrightarrow{\mathrm{id} - K_{n-1}(\phi)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots$$

and we get an isomorphism

$$N_+K_n(RK,\phi) \oplus N_-K_n(RK,\phi) \xrightarrow{\cong} NK_n(RK,RK,\mathrm{id},\phi).$$

Here $N_\pm K_n(RK,\phi)$ is the kernel of the split surjection $K_n(RK_\phi[t^{\pm 1}]) \to K_n(RK)$ that is induced by the homomorphism $RK_\phi[t^{\pm 1}] \to RK$ obtained by evaluation at $t = 0$.

Such a Wang sequence is established more generally for additive categories in [671, Theorem 0.1].

We mention the following computation from [651, Corollary 1.14].

Theorem 6.66 (Vanishing of $NK_n(RK,\phi)$). *Let R be a regular ring. Let $\phi\colon K \xrightarrow{\cong} K$ be an automorphism of the finite group K. Let $\mathcal{P}(K,R)$ be the set of primes which divide the order of K and are not invertible in R.*

Then for every $n \in \mathbb{Z}$ the abelian group $N_\pm K_n(RK,\phi)$ vanishes after inverting all primes in $\mathcal{P}(K,R)$. In particular we get $N_\pm K_n(RK,\phi) \otimes_\mathbb{Z} \mathbb{Q} = 0$ for all $n \in \mathbb{Z}$.

6.10 Homotopy Algebraic K-Theory

Homotopy algebraic K-theory has been introduced for rings by Weibel [977]. He constructs for a ring R a spectrum $\mathbf{KH}(R)$ and defines

$$KH_n(R) := \pi_n(\mathbf{KH}(R)) \quad \text{for } n \in \mathbb{Z}. \tag{6.67}$$

The main feature of homotopy K-theory is that it is *homotopy invariant*, i.e., for every ring R and every $n \in \mathbb{Z}$ the canonical inclusion induces an isomorphism [977, Theorem 1.2 (i)]

$$KH_n(R) \xrightarrow{\cong} KH_n(R[t]). \tag{6.68}$$

Note that homotopy invariance does not hold for algebraic K-theory unless R is regular, see Theorem 6.16.

A consequence of homotopy invariance is that we get for every ring R and $n \in \mathbb{Z}$ isomorphisms, see [977, Theorem 1.2 (iii)],

$$KH_n(R) \oplus KH_{n-1}(R) \xrightarrow{\cong} KH_n(R\mathbb{Z}). \tag{6.69}$$

Hence the are no Nil-terms appearing for the trivial HNN-extension $G \times \mathbb{Z}$. It turns out that there are no Nil-phenomena concerning amalgamated free products and HNN-extensions in general. Namely, we conclude from [74, Theorem 11.3]

Theorem 6.70 (Mayer-Vietoris sequence of an amalgamated free product for homotopy K-theory). *Let $G = G_1 *_{G_0} G_2$ be an amalgamated free product and let R be a ring. Denote by $i_k\colon G_0 \to G_k$ and $j_k\colon G_k \to G$ the obvious inclusions.*

Then there exists a Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial_{n+1}} KH_n(RG_0) \xrightarrow{KH_n(i_1)\oplus KH_n(i_2)} KH_n(RG_1) \oplus KH_n(RG_2)$$
$$\xrightarrow{KH_n(j_1)-KH_n(j_2)} KH_n(RG) \xrightarrow{\partial_n} KH_{n-1}(RG_0)$$
$$\xrightarrow{KH_{n-1}(i_1)\oplus KH_{n-1}(i_2)} KH_{n-1}(RG_1)\oplus KH_{n-1}(RG_2) \xrightarrow{KH_{n-1}(j_1)-KH_{n-1}(j_2)} \cdots .$$

Theorem 6.71 (Wang sequence associated to an HNN-extension for homotopy K-theory). *Let $\alpha, \beta\colon H \to K$ be two injective group homomorphisms. Let G be the associated HNN-extension and let $j\colon K \to G$ be the canonical inclusion. Then there is a long exact Wang sequence*

$$\cdots \xrightarrow{\partial_{n+1}} KH_n(RH) \xrightarrow{KH_n(\alpha)-KH_n(\beta)} KH_n(RK) \xrightarrow{KH_n(j)} KH_n(RG)$$
$$\xrightarrow{\partial_n} KH_{n-1}(RH) \xrightarrow{KH_{n-1}(\alpha)-KH_{n-1}(\beta)} KH_{n-1}(RK) \xrightarrow{KH_{n-1}(j)} \cdots .$$

There is a natural map of (non-connective) spectra $\mathbf{K}(R) \to \mathbf{KH}(R)$ and hence one obtains natural homomorphisms

$$K_n(R) \to KH_n(R) \quad \text{for } n \in \mathbb{Z}. \tag{6.72}$$

This map is in general neither injective nor surjective. It is bijective if R is regular by Theorem 6.16. In some sense homotopy algebraic K-theory is the best approximation of algebraic K-theory by a homotopy invariant functor.

Exercise 6.73. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$ be a graded ring. Show that the inclusion $i\colon R_0 \to R$ induces isomorphisms $KH_n(R_0) \xrightarrow{\cong} KH_n(R)$ for $n \in \mathbb{Z}$.

The same discussion as for the Baum Conjecture in Subsection 1.2.3 leads to the following conjecture.

Conjecture 6.74 (Farrell-Jones Conjecture for torsionfree groups for homotopy K-theory). Let G be a torsionfree group. Then the assembly map

$$H_n(BG; \mathbf{KH}(R)) \to KH_n(RG)$$

is an isomorphism for every $n \in \mathbb{Z}$ and every ring R.

The next result is taken from [87, Lemma 2.11].

Lemma 6.75. *(i) Let R be a ring of finite characteristic N. Then the canonical map from algebraic K-theory to homotopy K-theory induces an isomorphism*

$$K_n(R)[1/N] \xrightarrow{\cong} KH_n(R)[1/N]$$

for all $n \in \mathbb{Z}$;

(ii) Let H be a finite group. Then the canonical map from algebraic K-theory to homotopy K-theory induces an isomorphism

$$K_n(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} KH_n(\mathbb{Z}[H]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for all $n \in \mathbb{Z}$.

Conjecture 6.76 (Comparison of algebraic K-theory and homotopy K-theory for torsionfree groups). Let R be a regular ring and let G be a torsionfree group. Then the canonical map

$$K_n(RG) \to KH_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Note that Conjecture 6.76 follows from Conjecture 6.53 and Conjecture 6.74.

6.11 Algebraic K-Theory and Cyclic Homology

Fix a commutative ring k, referred to as the *ground ring*. Let R be a k-algebra. We denote by $HH_*^{\otimes_k}(R)$ the *Hochschild homology* of R relative to the ground ring k, and similarly by $HC_*^{\otimes_k}(R)$, $HP_*^{\otimes_k}(R)$, and $HN_*^{\otimes_k}(R)$ the *cyclic*, the *periodic cyclic*, and the *negative cyclic homology* of R relative to k. Hochschild homology receives a map from the algebraic K-theory, which is known as the *Dennis trace map*. There are variants of the Dennis trace taking values in cyclic, periodic cyclic, and negative cyclic homology (sometimes called *Chern characters*), as displayed in the following commutative diagram.

(6.77)
$$\begin{tikzcd}
 & HN_*^{\otimes_k}(R) \arrow[r] \arrow[d, "h"] & HP_*^{\otimes_k}(R) \arrow[d] \\
K_*(R) \arrow[ur, "\mathrm{ntr}"] \arrow[r, "\mathrm{dtr}"] & HH_*^{\otimes_k}(R) \arrow[r] & HC_*^{\otimes_k}(R).
\end{tikzcd}$$

For the definition of these maps, see [621, Chapters 8 and 11] and [658, Section 5]. In [658] the question is investigated, which parts of $K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$

can be detected by using the linear traces above. Here is an example, see [658, Theorem 0.7].

Theorem 6.78 (Detection Result for $\mathbb{Q}$ and $\mathbb{C}$ as coefficients). *For every group G and every integer $n \geq 0$, there exist injective homomorphisms*

$$\bigoplus_{(C) \in (\mathcal{FCY})} H_*(BN_GC;\mathbb{Q}) \to K_*(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q};$$
$$\bigoplus_{(g) \in \operatorname{con}(G), |g|<\infty} H_*(BC_G\langle g\rangle;\mathbb{C}) \to K_*(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C},$$

where we denote by $(\mathcal{FCY})$ the conjugacy classes of finite cyclic subgroups of G, by $\operatorname{con}(G)$ the set of conjugacy classes (g) of elements $g \in G$, by N_GC the normalizer of $C \subseteq G$, and by $C_G\langle g\rangle$ the centralizer of $g \in G$.

Remark 6.79. In [141], Bökstedt, Hsiang and Madsen define the *cyclotomic trace*, a map out of K-theory, which takes values in *topological cyclic homology*. The cyclotomic trace map can be thought of as an even more elaborate refinement of the Dennis trace map. In contrast to the Dennis trace, the cyclotomic trace has the potential to detect almost all of the rationalized K-theory of an integral group ring. This question is investigated in detail by Lück-Rognes-Reich-Varisco [659, 660]. More information will be given in Subsection 15.11.2.

6.12 Notes

A good source of survey articles about algebraic K-theory is the handbook of K-theory, edited by Friedlander and Grayson [391]. There the relevance of higher algebraic K-theory for algebra, topology, arithmetic geometry, and number theory is explained. Other good sources are the books by Rosenberg [844], Srinivas [905], and Weibel [979].

The relation of the exact sequences for amalgamated free products and HNN-extensions appearing in Sections 6.9 and 6.10 to the Farrell-Jones Conjecture is explained in Section 15.7.

The exact sequences for amalgamated free products and HNN-extensions appearing in Sections 6.9 and 6.10 are the main ingredients in the proof that Conjecture 6.53 holds for a certain class of groups $\mathcal{CL}$, see [957, Theorem 19.4 on page 249] in the connective case and [74, Corollary 0.12] in general. The class $\mathcal{CL}$ is described and analyzed in [957, Definition 19.2 on page 248 and Theorem 17.5 on page 250] and [74, Definition 0.10]. It is closed under taking subgroups and contains for instance all torsionfree one-relator groups.

We remark that algebraic K-theory does commute with infinite products for additive categories, see [200] and also [556, Theorem 1.2], but not with infinite products of rings.

The question, under which condition the long exact sequence associated to a pullback of rings, see Remark 4.4 and Remark 5.11, can be extended to higher algebraic K-theory, is investigated by Land-Tamme [601], actually for ring spectra.

The group $K_{2n}(RG)$ is finite for every finite group G, every ring of integer R in a number field, and every $n \geq 1$, see [580, Theorem 1.1].

In Chapter 8 we will deal with (non-connective) K-theory in the more general setting of higher categories. A non-connective K-theory spectrum associated to homotopical Waldhausen categories wil be briefly discussed in Section 21.4.

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Chapter 7
Algebraic *K*-Theory of Spaces

7.1 Introduction

We give a brief introduction to the *K-theory of spaces* called *A-theory*. This theory was initialized by Waldhausen. Its benefit is that it allows to study interesting spaces of geometric structures such as groups of diffeomorphisms or homeomorphism of manifolds, pseudoisotopy spaces, spaces of h-cobordisms, and Whitehead spaces. It is the instance of a very successful strategy in topology to extend algebraic notions to spaces. Other examples of this type are topological Hochschild homology and topological cyclic homology.

7.2 Pseudoisotopy

Let I denote the unit interval $[0,1]$. A topological *pseudoisotopy* of a compact manifold M is a homeomorphism $h\colon M \times I \to M \times I$ that restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion. The space $P(M)$ of pseudoisotopies is the group of all such homeomorphisms, where the group structure comes from composition. If we allow M to be non-compact, we will demand that h has compact support, i.e., there is a compact subset $C \subseteq M$ such that $h(x,t) = (x,t)$ for all $x \in M - C$ and $t \in [0,1]$.

Pseudoisotopies play an important role if one tries to understand the homotopy type of the topological group $\operatorname{Top}(M)$ of selfhomeomorphisms of a closed manifold M. We will see in Section 9.21 how the results about pseudoisotopies discussed in this section combined with surgery theory lead to quite explicit results about the homotopy groups of $\operatorname{Top}(M)$ for an aspherical closed manifold M.

There is a stabilization map $P(M) \to P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I and the stable pseudoisotopy space is defined as $\mathcal{P}(M) = \operatorname{hocolim}_{j \to \infty} P(M \times I^j)$. There exist also smooth versions $P^{\mathsf{DIFF}}(M)$ and $\mathcal{P}^{\mathsf{DIFF}}(M) = \operatorname{hocolim}_{j \to \infty} P^{\mathsf{DIFF}}(M \times I^j)$. The PL-version agrees for closed manifolds of dimension ≥ 6 with the topological version, see [178].

The natural inclusions $P(M) \to \mathcal{P}(M)$ and $P^{\mathsf{DIFF}}(M) \to \mathcal{P}^{\mathsf{DIFF}}(M)$ induce isomorphisms on the i-th homotopy group if the dimension n of M is large compared to i, roughly for $i \leq n/3$, see [179, 452, 454, 495]. Meanwhile one has more information about this range by Goodwillie, Krannich, Kupers, Randal-Williams, and others, see for instance [413, 566, 567]. A consequence of their work, is that for a closed Spin-manifold M of dimension $d \geq 10$, the rational connectivity of $P^{\mathsf{DIFF}}(M) \to \mathcal{P}^{\mathsf{DIFF}}(M)$ is at most $d-4$ and is equal to $d-4$ if M is simply connected. If M is the disk D^d for $d \geq 10$, then the rational connectivity of $P^{\mathsf{DIFF}}(M) \to \mathcal{P}^{\mathsf{DIFF}}(M)$ is $d-4$.

Next we want to define a delooping of $P(M)$. Let $p\colon M \times \mathbb{R}^k \times I \to \mathbb{R}^k$ denote the natural projection. For a manifold M the space $P_b(M;\mathbb{R}^k)$ of bounded pseudoisotopies is the space of all selfhomeomorphism $h\colon M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$ satisfying: (i) The restriction of h to $M \times \mathbb{R}^k \times \{0\} \cup \partial M \times \mathbb{R} \times [0,1]$ is the inclusion, (ii) the map h is bounded in the $\mathbb{R}^i$-direction, i.e., the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of $\mathbb{R}^k$, and (iii) the map h has compact support in the M-direction, i.e., there is a compact subset $C \subseteq M$ such that $h(x,y,t) = (x,y,t)$ for all $x \in M - C$, $y \in \mathbb{R}^i$ and $t \in [0,1]$. There is an obvious stabilization map $P_b(M;\mathbb{R}^k) \to P_b(M \times I;\mathbb{R}^k)$ and a stable bounded pseudoisotopy space $\mathcal{P}_b(M;\mathbb{R}^k) = \operatorname{hocolim}_{j\to\infty} P_b(M \times I^j;\mathbb{R}^k)$. There is a homotopy equivalence $\mathcal{P}_b(M;\mathbb{R}^k) \to \Omega\mathcal{P}_b(M;\mathbb{R}^{k+1})$, see [455, Appendix II]. Hence the sequences of spaces $\mathcal{P}_b(M;\mathbb{R}^k)$ for $k = 0,1,2,\ldots$ and $\Omega^{-i}\mathcal{P}_b(M)$ for $i = 0,-1,-2,\ldots$ define an Ω-spectrum $\mathbf{P}(M)$. Analogously one defines the differentiable bounded pseudoisotopies $\mathcal{P}_b^{\mathsf{DIFF}}(M;\mathbb{R}^k)$ and an Ω-spectrum $\mathbf{P}^{\mathsf{DIFF}}(M)$.

Definition 7.1 ((Non-connective) pseudoisotopy spectrum). We call the Ω-spectra $\mathbf{P}(X)$ and $\mathbf{P}^{\mathsf{DIFF}}(X)$ associated to a topological space X the (non-connective) pseudoisotopy spectrum and the smooth (non-connective) pseudoisotopy spectrum of X.

Remark 7.2 (Strict Functoriality). A priori the pseudoisotopy space and its non-connective version are only homotopy functors in the following sense. They assign to a map between manifolds only a homotopy class of maps between the pseudoisotopy spaces and not a specific map. At least the homotopy class of maps between the pseudoisotopy spaces depends only on the homotopy class of the map between manifolds we started with. The homotopy class of the identity is sent to the homotopy class of the identity and the construction is compatible with composition up to homotopy. Moreover, it is a priori not clear what the values of the pseudoisotopy space on general topological spaces are.

There are several places in the literature where a construction as a strict functor from the category of topological spaces to the category of non-connective spectra is indicated, but it seems to be the case that the only places where all the details of this non-trivial extensions are carried out in the smooth, topological and PL category are the PhD-theses of Enkelmann [329] and Pieper [794]. This is important for the construction of the assembly map appearing in the Farrell-Jones Conjecture for pseudoisotopy spaces 15.61, since we want the pseudoisotopy functor to digest for instance classifying spaces of groups and groupoids, which obviously are not compact manifolds in general, and to construct the assembly map we need strict functoriality.

Theorem 7.3 (Pseudoisotopy is a homotopy-invariant functor). *Let $f\colon X \to Y$ be a weak homotopy equivalence. Then the induced maps*

$$\begin{gathered}\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(Y);\\ \mathbf{P}^{\mathsf{DIFF}}(f)\colon \mathbf{P}^{\mathsf{DIFF}}(X) \to \mathbf{P}^{\mathsf{DIFF}}(Y),\end{gathered}$$

are weak homotopy equivalences.

Proof. See [455, Proposition 1.3]. □

Remark 7.4. There is also a PL-version $\mathbf{P}^{\mathsf{PL}}(X)$ of $\mathbf{P}(X)$. Since the canoncial map $\mathbf{P}^{\mathsf{PL}}(X) \to \mathbf{P}(X)$ is a weak homotopy equivalence, we do not consider it further.

7.3 Whitehead Spaces and A-Theory

7.3.1 Categories with Cofibrations and Weak Equivalences

The following definition is a generalization of the notion of an exact category of Definition 6.32 in the sense of Quillen. It allows to deal with spaces instead of algebraic objects such as modules. It is due to Waldhausen.

A category $\mathcal{C}$ is called *pointed* if it comes with a distinguished *zero-object*, i.e., an object that is both initial and terminal.

Definition 7.5 (Category with cofibrations and weak equivalences). A *category with cofibrations and weak equivalences* is a small pointed category with a subcategory $co\mathcal{C}$, called *category of cofibrations*, in $\mathcal{C}$ and a subcategory $w\mathcal{C}$, called *category of weak equivalences*, in $\mathcal{C}$ such that the following axioms are satisfied:

(i) The isomorphisms in $\mathcal{C}$ are cofibrations, i.e., belong to $co\mathcal{C}$;
(ii) For every object C the map $* \to C$ is a cofibration, where $*$ is the distinguished zero-object;
(iii) If in the diagram $A \overset{i}{\leftarrowtail} B \xrightarrow{f} C$ the left arrow is a cofibration, the pushout

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "\overline{f}"] \\ C \arrow[r, tail, "\overline{i}"] & D \end{tikzcd}$$

exists and $\overline{i}$ is a cofibration;
(iv) The isomorphisms in $\mathcal{C}$ are contained in $w\mathcal{C}$;
(v) If in the commutative diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \longrightarrow & C \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ B' & \leftarrowtail & A' & \longrightarrow & C' \end{array}$$

the horizontals arrow on the left are cofibrations, and all vertical arrows are weak equivalences, then the induced map on the pushout of the upper row to the pushout of the lower row is a weak homotopy equivalence.

Example 7.6 (Exact categories are categories with cofibrations and weak equivalences). Let $\mathcal{P} \subseteq \mathcal{A}$ be an exact category in the sense of Definition 6.32. The zero-object is just a zero-object in the abelian category $\mathcal{A}$. A cofibration in $\mathcal{P}$ is a morphism $i\colon A \to B$ that occurs in an exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{P}$. The weak equivalences are given by the isomorphisms.

Exercise 7.7. Let $\mathcal{C}$ be the category of finite projective R-chain complexes. Define cofibrations to be chain maps $i_*\colon C_* \to D_*$ such that $i_n\colon C_n \to D_n$ is split injective for all $n \geq 0$. Define weak equivalences to be homology equivalences. Show that $\mathcal{C}$ is a category with cofibrations and weak equivalences in the sense of Definition 7.5 ignoring the fact that $\mathcal{C}$ is not small.

Example 7.8 (The category $\mathcal{R}(X)$ of retractive spaces). Let X be a space. A retractive space over X is a triple (Y, r, s) consisting of a space Y and maps $s\colon X \to Y$ and $r\colon Y \to X$ such that s is a cofibration and $r \circ s = \mathrm{id}_X$. A morphism from (Y, r, s) to (Y', r', s') is a map $f\colon X \to X'$ satisfying $r' \circ f = r$ and $f \circ s = s'$. The zero-object is $(X, \mathrm{id}_X, \mathrm{id}_X)$. A morphism $f\colon (Y, r, s) \to (Y', r', s')$ is declared to be a cofibration if the underlying map of spaces $f\colon Y \to Y'$ is a cofibration. Now there are several possibilities to define weak equivalences. One may require that $f\colon Y \to Y'$ is a homeomorphism, a homotopy equivalence, weak homotopy equivalence, or a homology equivalence with respect to some fixed homology theory. Then one obtains a category $\mathcal{R}(X)$ with cofibrations and weak equivalences in the sense of Definition 7.5 except that $\mathcal{R}(X)$ is not small.

To achieve that $\mathcal{R}(X)$ is small and later to get interesting K-theory, one may for instance require that (Y, X) is a relative CW-complex which is relatively finite, and $s\colon X \to Y$ is the inclusion and morphisms to be cellular maps. Denote this category with cofibrations and weak equivalences by $\mathcal{R}^f(X)$ where we choose all weak homotopy equivalences as weak equivalences and inclusion of relative CW-complexes as cofibrations.

7.3.2 The $wS_\bullet$-Construction

Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Next we briefly recall Waldhausen's $wS_\bullet$-construction, see [960, Section 1.3].

For an integer $n \geq 0$ let $[n]$ be the ordered set $\{0, 1, 2, \ldots, n\}$. Let Δ be the category whose set of objects is $\{[n] \mid n = 0, 1, 2 \ldots\}$ and whose set of morphisms from $[m]$ to $[n]$ consists of the order preserving maps. A simplicial category is a contravariant functor from Δ to the category CAT of categories. Analogously, a simplicial category with cofibrations and weak equivalences is a contravariant functor from Δ to the category $\mathsf{CAT}_{\mathrm{cof,weq}}$ of categories with cofibrations and weak equivalences. Now we assign to $\mathcal{C}$ a simplicial category with cofibrations and weak equivalences $S_\bullet\mathcal{C}$ as follows. Define $S_n\mathcal{C}$ to be the category for which an object is a sequence of cofibrations $A_{0,1} \xrightarrow{k_{0,1}} A_{0,2} \xrightarrow{k_{0,2}} \cdots \xrightarrow{k_{0,n-1}} A_{0,n}$ together with explicit choices of quotient objects $\mathrm{pr}_{i,j} \colon A_{0,j} \to A_{i,j} = A_{0,j}/A_{0,i}$ for $i, j \in \{1, 2, \ldots, n\}, i < j$, i.e., we fix pushouts

$$\begin{tikzcd} A_{0,i} \arrow[r, "k_{0,j-1}\circ\cdots\circ k_{0,i}"] \arrow[d] & A_{0,,j} \arrow[d, "\mathrm{pr}_{i,j}"] \\ 0 \arrow[r] & A_{i,j}. \end{tikzcd}$$

Morphisms are given by a collection of morphisms $\{f_{i,j}\}$ which make the obvious diagram commute.

With these explicit choices of quotient objects, it is easy to define the relevant face and degeneracy maps. For instance the face map $d_i \colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is given for $i \geq 1$ by dropping $A_{0,i}$ and for $i = 0$ by passing to $A_{0,2}/A_{0,1} \to A_{0,3}/A_{0,1} \to \cdots \to A_{0,n}/A_{0,1}$. An arrow in $S_n\mathcal{C}$ is declared to be a cofibration if each arrow $A_{i,j} \to A'_{i,j}$ is a cofibration and analogously for weak equivalences.

We obtain a simplicial category $wS_\bullet\mathcal{C}$ by considering the category of weak equivalences of $S_\bullet\mathcal{C}$. Let $|wS_\bullet\mathcal{C}|$ be the geometric realization of the simplicial category $wS_\bullet\mathcal{C}$, which is the geometric realization of the bisimplicial set obtained by the composite of the functor nerve of a category with $wS_\bullet\mathcal{C}$.

Definition 7.9 (Algebraic K-theory space of a category with cofibrations and weak equivalences). Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Its *algebraic K-theory space* $K(\mathcal{C})$ is defined by

$$K(\mathcal{C}) := \Omega|wS_\bullet\mathcal{C}|.$$

The 1-skeleton in the $S_\bullet$ direction of $|wS_\bullet\mathcal{C}|$ is obtained from $|wS\mathcal{C}| \times [0,1] = |wS_1\mathcal{C}| \times \Delta_1$ by collapsing $\{*\} \times [0,1] \cup |wS\mathcal{C}| \times \{0\}$ to a point because of $|wS_0| = \{\bullet\}$. Hence there is a canonical map $|w\mathcal{C}| \to \Omega|wS_\bullet\mathcal{C}|$ that is the adjoint of the obvious identification of the 1-skeleton in the $S_\bullet$-direction of $|wS_\bullet\mathcal{C}|$ with the reduced suspension $|w\mathcal{C}| \wedge S^1$. If we apply the construction to $S_n\mathcal{C}$, we obtain a map of spaces $|wS_n\mathcal{C}| \to \Omega|wS_\bullet S_n\mathcal{C}|$. The collection of these maps for $n \geq 0$ yields a map of simplicial spaces and hence by geometric realization a map of spaces $|wS_\bullet\mathcal{C}| \to \Omega|wS_\bullet S_\bullet\mathcal{C}|$. By iterating this construction, we obtain a sequence of maps

$$|w\mathcal{C}| \to \Omega|wS_\bullet\mathcal{C}| \to \Omega\Omega|wS_\bullet S_\bullet\mathcal{C}| \to \Omega\Omega\Omega|wS_\bullet S_\bullet S_\bullet\mathcal{C}| \to \cdots$$

such that all maps except the first one are weak homotopy equivalences. So $K(\mathcal{C})$ is an infinite loop space beyond the first term.

7.3.3 A-Theory

Next we recall Waldhausen's definition of A-theory of a topological space, see [960, Chapter 2].

Definition 7.10 (Connective A-theory). Let X be a topological space. Let $\mathcal{R}^f(X)$ be the category with cofibrations and weak equivalences defined in Example 7.8. Define the A-theory space $A(X)$ associated to X to be the algebraic K-theory space $K(\mathcal{R}^f(X))$ in the sense of Definition 7.9.

Remark 7.11 (The $wS_\bullet$-construction encompasses the Q-construction). Waldhausen's construction encompasses the Q-construction of Quillen, see [960, Section 1.9].

As in the case of algebraic K-theory of rings or pseudoisotopy, it will be crucial for us to consider a non-connective version. Vogell [949] has defined a delooping of $A(X)$ yielding a non-connective Ω-spectrum $\mathbf{A}(X)$ for a topological space. The idea is similar to the construction of the (non-connective) pseudoisotopy spectrum in Section 7.2, where one considers parametrizations over $\mathbb{R}^n$ and imposes control conditions. This construction actually yields a covariant functor from the category of topological spaces to the category of Ω-spectra

$$\mathbf{A}\colon \mathsf{TOP} \to \Omega\text{-}\mathsf{SPECTRA}. \tag{7.12}$$

Definition 7.13 (Non-connective A-theory). We call $\mathbf{A}(X)$ the (non-connective) A-theory spectrum associated to the topological space X. We write for $n \in \mathbb{Z}$

$$A_n(X) := \pi_n(\mathbf{A}(X)).$$

Note that $A_n(X)$ agrees with $\pi_n(A(X))$ for $n \geq 1$ if $A(X)$ is the space appearing in Definition 7.10. Actually there is a map of spectra, natural in X,

$$\mathbf{i}(X)\colon A(X) \to \mathbf{A}(X) \tag{7.14}$$

which induces isomorphisms $\pi_n(\mathbf{i}(X))\colon \pi_n(A(X)) \xrightarrow{\cong} \pi_n(\mathbf{A}(X))$ for $n \geq 1$.

Remark 7.15 ($\pi_0(A(X))$). If X is path connected, then $A_0(X) \cong \mathbb{Z}$. The isomorphism comes from taking the Euler characteristic of a relatively finite relative CW-complex (Y, X).

One may replace in the definition of $A(X)$ the category $\mathcal{R}^f(X)$ by the full subcategory of $\mathcal{R}(X)$ of those triples (Y,r,s) such that (Y,X) is a relative CW-complex consisting of countably many cells, $s\colon X \to Y$ is the inclusion and the object (Y,r,s) is up to homotopy the retract of an object (Y',r',s') such that (Y',X) is a relatively finite relative CW-complex. Then $\pi_n(A(X))$ is unchanged for $n \geq 1$, whereas $\pi_0(A(X))$ can now be identified with $K_0(\mathbb{Z}[\pi_1(X)])$ if X is path connected. The identification comes from taking an appropriate finiteness obstruction. With this new definition the map $\pi_0(\mathbf{i})\colon \pi_0(A(X)) \to \pi_0(\mathbf{A}(X))$ is bijective.

For the proof of the next result see [960, Proposition 2.1.7].

Theorem 7.16 (A-theory is a homotopy-invariant functor). *Let $f\colon X \to Y$ be a weak homotopy equivalence. Then the induced maps*

$$A(f)\colon A(X) \to A(Y);$$
$$\mathbf{A}(f)\colon \mathbf{A}(X) \to \mathbf{A}(Y),$$

are weak homotopy equivalences.

Let X be a connected space with fundamental group $\pi = \pi_1(X)$, which admits a universal covering $p_X\colon \widetilde{X} \to X$. Consider an object in $\mathcal{R}^f(X)$. Recall that it is given by a relatively finite relative CW-complex (Y,X) together with a map $r\colon Y \to X$ satisfying $r|_X = \mathrm{id}_X$. Let $\widetilde{Y} \to Y$ be the π-covering obtained from $p_X\colon \widetilde{X} \to X$ by the pullback construction applied to $r\colon Y \to X$. The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{Y},\widetilde{X})$ of the relative free π-CW-complex $(\widetilde{Y},\widetilde{X})$ is a finite free $\mathbb{Z}\pi$-chain complex. This yields a functor of categories with cofibrations and weak equivalences from $\mathcal{R}^f(X)$ to the category of finite free $\mathbb{Z}\pi$-chain complexes. The algebraic K-theory of the category of finite free $\mathbb{Z}\pi$-chain complexes agrees with the one of the finitely generated free $\mathbb{Z}\pi$-modules. Hence we get a natural map of spectra called *linearization map*

$$\mathbf{L}(X)\colon \mathbf{A}(X) \to \mathbf{K}(\mathbb{Z}\pi_1(X)). \tag{7.17}$$

The next result follows by combining [950, Section 4] and [959, Proposition 2.2 and Proposition 2.3].

Theorem 7.18 (Connectivity of the linearization map). *Let X be a connected CW-complex. Then:*

(i) The linearization map $\mathbf{L}(X)$ of (7.17) is 2-connected, i.e., the map

$$L_n(X) := \pi_n(\mathbf{L}(X))\colon A_n(X) \to K_n(\mathbb{Z}\pi_1(X))$$

is bijective for $n \leq 1$ and surjective for $n = 2$;

(ii) Rationally the map $L_n(X)$ is bijective for all $n \in \mathbb{Z}$ provided that X is aspherical.

Exercise 7.19. Show that the canonical map of spectra $A(\{\bullet\}) \to \mathbf{A}(\{\bullet\})$ is a weak homotopy equivalence.

Remark 7.20. We obtain from the transformation $\mathbf{L}$ of (7.17) for every group G and every $n \in \mathbb{Z}$ a commutative diagram

(7.21)
$$\begin{tikzcd} H_n(BG;\mathbf{A}(\{\bullet\})) \arrow[r] \arrow[d, "H_n(\mathrm{id}_{BG};\mathbf{L}(\{\bullet\}))"'] & A_n(BG) \arrow[d, "L_n(BG)"] \\ H_n(BG;\mathbf{K}(\mathbb{Z})) \arrow[r] & K_n(\mathbb{Z}G) \end{tikzcd}$$

whose horizontal arrows are assembly maps. We conclude from Theorem 7.18 that its vertical arrows are bijective for $n \le 1$, surjective for $n = 2$, and rationally bijective for all $n \in \mathbb{Z}$. Hence the upper horizontal arrow is rationally bijective if and only if the lower horizontal arrow is rationally bijective. Recall that Conjecture 6.53 says that the lower horizontal map is bijective. So one may wonder whether the upper horizontal is always bijective. The answer is no, already for $G = \mathbb{Z}$ the assembly map

$$H_n(B\mathbb{Z};\mathbf{A}(\{\bullet\})) = A_{n-1}(\{\bullet\}) \oplus A_n(\{\bullet\}) \to A_n(B\mathbb{Z}) = \mathbf{A}(S^1)$$

is known to be not surjective by the following consideration.

Let $NA_n(\{\bullet\})$ be the Nil-term occurring in the Bass-Heller-Swan-isomorphisms for non-connective A-theory, see [493, 494],

$$A_n(S^1) = A_n(\{\bullet\}) \oplus A_{n-1}(\{\bullet\}) \oplus NA_n(\{\bullet\}) \oplus NA_n(\{\bullet\}). \tag{7.22}$$

We conclude $NA_n(\{\bullet\}) = \{0\}$ for $n \le 1$ and $NA_n(\{\bullet\}) \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$ for $n \in \mathbb{Z}$ from (7.21) and [669, Theorem 0.3]. On the other hand, $NA_n(\{\bullet\})$ for $n = 2, 3$ is an infinite-dimensional $\mathbb{F}_2$–vector space. For more information about $NA_n(\{\bullet\})$ we refer to Grunewald-Klein-Macko [428] and Hesselholt [461].

Exercise 7.23. Show that the map linearization map

$$L_2(S^1)\colon A_2(S^1) \to K_2(\mathbb{Z}\pi_1(S^1))$$

is not injective using the fact that $\mathrm{Wh}_2(\mathbb{Z})$ vanishes.

7.3.4 Whitehead Spaces

Waldhausen [959, 960] defines the functor $\mathrm{Wh}(X)$ from spaces to infinite loop spaces, which can be viewed as connective Ω-spectra, and a fibration sequence

$$X_+ \wedge A(\{\bullet\}) \to A(X) \to \mathrm{Wh}(X). \tag{7.24}$$

Here $X_+ \wedge A(\{\bullet\}) \to A(X)$ is an assembly map. After taking homotopy groups, it can be compared with the algebraic K-theory assembly map that appears in Conjecture 6.53 via a commutative diagram

$$\begin{tikzcd}
\pi_n(X_+ \wedge A(\{\bullet\})) \arrow[rr] \arrow[d, "\cong"'] & & \pi_n(A(X)) \arrow[d] \\
\pi_n(X_+ \wedge \mathbf{A}(\{\bullet\})) = H_n(X;\mathbf{A}(\{\bullet\})) \arrow[rr] \arrow[d, "H_n(X;\mathbf{L})"] & & \pi_n(\mathbf{A}(X)) \arrow[d, "\pi_n(\mathbf{L})"] \\
H_n(B\pi_1(X);\mathbf{K}(\mathbb{Z})) \arrow[rr] & & K_n(\mathbb{Z}\pi_1(X)).
\end{tikzcd} \tag{7.25}$$

Here the vertical arrows from the first row to the second row come from the map $\mathbf{i}$ of (7.14). The left one of these is bijective for $n \in \mathbb{Z}$ by Exercise 7.19 and the right one of these is bijective for $n \geq 1$. As already discussed in Remark 7.20, the lower vertical arrows from the second row to the third row come from the linearization map $\mathbf{L}$ of (7.17) and because of Theorem 7.18 the left lower vertical arrow is bijective for $n \leq 1$ and rationally bijective for $n \in \mathbb{Z}$. In the case where X is aspherical, the lower right vertical map $\pi_n(L)$ is bijective for $n \leq 1$ and rationally bijective for all $n \in \mathbb{Z}$ because of Theorem 7.18. Because of (7.24) and the fact that

$$\Omega^2 \operatorname{Wh}(X) \simeq \mathcal{P}(X), \tag{7.26}$$

see [307, Section 9] and [962], Conjecture 6.53 implies rational vanishing results for the groups $\pi_n(\mathcal{P}(M))$ if M is an aspherical closed manifold.

Theorem 7.27 (Homotopy groups of $\mathrm{Wh}(BG)$ and $\mathcal{P}(BG)$ rationally for torsionfree G). *Let G be a torsionfree group. Suppose that Conjecture 6.53 holds for G and $R = \mathbb{Z}$. Then we get for all $n \geq 0$*

$$\begin{aligned}
\pi_n(\mathrm{Wh}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} &= 0;\\
\pi_n(\mathcal{P}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} &= 0.
\end{aligned}$$

Exercise 7.28. Show that $\pi_1(\mathrm{Wh}(BG))$ is $\mathrm{Wh}(G)$.

There is also a smooth version of the Whitehead space $\mathrm{Wh}^{\mathsf{DIFF}}(X)$ defined as homotopy cofiber

$$\Sigma^\infty(X_+) \to A(X) \to \mathrm{Wh}^{\mathsf{DIFF}}(X) \tag{7.29}$$

where $\Sigma^\infty(X_+) \to A(X)$ factors as the unit map $\Sigma^\infty(X_+) = X_+ \wedge \mathbf{S} \to \Sigma^\infty(X_+) \wedge A(\{\bullet\})$ and the assembly map $\Sigma^\infty(X_+) \wedge A(\{\bullet\}) \to A(X)$. We have

$$\Omega^2 \operatorname{Wh}^{\mathsf{DIFF}}(X) \simeq \mathcal{P}^{\mathsf{DIFF}}(X). \tag{7.30}$$

Again there is a close relation to A-theory via the natural splitting of connective spectra due to Waldhausen [959, 961, 962]

$$A(X) \simeq \Sigma^\infty(X_+) \vee \operatorname{Wh}^{\mathsf{DIFF}}(X). \tag{7.31}$$

Here $\Sigma^\infty(X_+)$ denotes the suspension spectrum associated to X_+. Since for every space $\pi_n(\Sigma^\infty(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X;\mathbb{Q})$, Conjecture 6.53 combined with Remark 6.57 and Theorem 7.18 yields the following result.

Theorem 7.32 (Homotopy groups of $\operatorname{Wh}^{\mathsf{DIFF}}(BG)$ and $\mathcal{P}^{\mathsf{DIFF}}(BG)$ rationally for torsionfree G). *Let G be a torsionfree group. Suppose that Conjecture 6.53 holds for $R = \mathbb{Z}$ and G. Then we get for all $n \geq 0$*

$$\begin{aligned}
\pi_n(\operatorname{Wh}^{\mathsf{DIFF}}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(BG;\mathbb{Q});\\
\pi_n(\mathcal{P}^{\mathsf{DIFF}}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(BG;\mathbb{Q}).
\end{aligned}$$

Note that Theorem 7.27 and Theorem 7.32 is a key ingredient in the computation of the homotopy groups of $\operatorname{Top}(M)$ and $\operatorname{Diff}(M)$ for a closed (smooth) manifolds M, as they appear in Theorem 9.192 and Theorem 9.193.

Exercise 7.33. Show that there is no connected closed manifold M such that the homomorphism induced by the forgetful map $\pi_n(\operatorname{Wh}^{\mathsf{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(\operatorname{Wh}(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is bijective for all $n \geq 0$. Use the fact that the composite of the obvious inclusion of $\operatorname{Wh}^{\mathsf{DIFF}}(X)$ into $\Sigma^\infty(X_+) \vee \operatorname{Wh}^{\mathsf{DIFF}}(X)$ with the inverse of the splitting (7.31) and the map $A(X) \to \operatorname{Wh}(X)$ of (7.24) is up to homotopy the obvious forgetful map $\operatorname{Wh}^{\mathsf{DIFF}}(M) \to \operatorname{Wh}(M)$.

Remark 7.34. There are also non-connective versions $\mathbf{Wh}$ of the Whitehead space Wh defined by the homotopy fibration sequence of non-connective spectra

$$X_+ \wedge \mathbf{A}(\{\bullet\}) \to \mathbf{A}(X) \to \mathbf{Wh}(X) \tag{7.35}$$

and $\mathbf{Wh}^{\mathsf{DIFF}}(X)$ of the smooth Whitehead space $\operatorname{Wh}^{\mathsf{DIFF}}(X)$ defined to be the homotopy cofiber in the sequence of non-connective spectra

$$\Sigma^\infty(X_+) \to \mathbf{A}(X) \to \mathbf{Wh}^{\mathsf{DIFF}}(X) \tag{7.36}$$

such that the result above have non-connective versions working for all $n \in \mathbb{Z}$.

Integral computations of the homotopy groups of Whitehead spaces are much harder. We at least state one example, which follows directly from [330, Theorem 1.3].

Theorem 7.37 (Homotopy groups of $\mathbf{Wh}(BG)$ of a torsionfree hyperbolic group G). *Let G be a torsionfree hyperbolic group. Then we get for $n \in \mathbb{Z}$ an isomorphism*

$$\pi_n(\mathbf{Wh}(BG)) \cong \bigoplus_{(C)} NA_n(\{\bullet\}) \oplus NA_n(\{\bullet\})$$

where (C) ranges over the conjugacy classes of maximal infinite cyclic subgroups C of G and $NA_n(\{\bullet\})$ has been introduced in (7.22).

In particular, $\pi_n(\mathbf{Wh}(BG)) = 0$ for $n \leq 1$.

7.4 Notes

One of the basic tools to investigate algebraic K-theory of spaces is the Additivity Theorem, see [697] and [960, Theorem 1.4.2]. If $\mathcal{C}$ is a category with cofibrations and weak equivalences, we can assign to it a category with cofibrations and weak equivalences $E(\mathcal{C})$ whose objects are exact sequences $A \xrightarrow{i} B \xrightarrow{p} C$, where exact means that the map i is a cofibration and the following diagram is a pushout

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d, "p"] \\ * \arrow[r] & C \end{tikzcd}$$

Theorem 7.38 (Additivity Theorem for categories with cofibrations and weak equivalences). *Let F_1 and F_3 respectively be the functors $E(\mathcal{C}) \to \mathcal{C}$ of categories with cofibrations and weak equivalences sending an object $A \xrightarrow{i} B \xrightarrow{p} C$ to A and C respectively. Then we obtain a weak homotopy equivalence*

$$K(F_1) \times K(F_3) \colon K(E(\mathcal{C})) \xrightarrow{\simeq} K(\mathcal{C}) \times K(\mathcal{C}).$$

Further useful tools are the Approximation Theorem, see [960, Theorem 1.6.7], the Fibration Theorem, see [960, Theorem 1.6.4], and the Cofinality Theorem, see [960, Theorem 1.5.9], which give criterions to decide when a functor of categories with cofibrations and weak equivalences induces a weak homotopy equivalence on the K-theory spaces.

To the author's knowledge, it is not known how to define a non-connective K-theory spectrum for an arbitrary Waldhausen category. If we restrict ourselves to homotopical Waldhausen category, a non-connective K-theory spectrum has been defined by Bunke-Kasprowski-Winges [173, Definition 2.37], which we will recall in Section 21.4, where one can also find the statements of the Approximation Theorem, the Fibration Theorem, and the Cofinality

Theorem in the non-connective case. The usefulness of these theorems become clear when one inspects the proof Theorem 21.5.

There is also a *space of parametrized h-cobordisms* $H(M)$ for a closed topological manifold M. Roughly speaking, the space is designed such that a map $N \to H(M)$ is the same as a bundle over N whose fibers are h-cobordisms over M. The set of path component $\pi_0(H(M))$ agrees with the isomorphism classes of h-cobordisms over M. In particular the s-Cobordism Theorem 3.47 is equivalent to the statement that for $\dim(M) \geq 5$ we obtain a bijection $\pi_0(H(M)) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(M))$ coming from taking the Whitehead torsion, or, equivalently, that we obtain a bijection $\pi_0(H(M)) \xrightarrow{\cong} \pi_0(\Omega\,\mathrm{Wh}(M))$. There is also a stable version, *the space of stable parametrized h-cobordisms* $\mathcal{K}(M) = \operatorname{hocolim}_{j\to\infty} H(M \times I^j)$.

Theorem 7.39 (The stable parametrized h-cobordism Theorem). *If M is a closed topological manifold, then there is a homotopy equivalence*

$$\mathcal{K}(M) \xrightarrow{\simeq} \Omega\,\mathrm{Wh}(M).$$

There is also a smooth version of the result above. For the proof and more information about the stable parametrized h-cobordism Theorem we refer to [962].

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Chapter 8
Algebraic K-theory of Higher Categories

8.1 Introduction

Comment 2 (by W.): This chapter should be filled in by Christoph.

8.2 To do

Comment 3 (by W.): The list below contains certain things which should be addressed in this chapter:

- Introduce the notion of a *right-exact* ∞*-category*. We will denote them by $\mathcal{C}$. Recall that $\mathcal{A}$ is reserved for additive categories (and exact categories).
  Comment 4 (by W.): Add a list of references where one can find basics about higher categories, for instance [230, 456, 597, 674, 675, 676, 835].
  Comment 5 (by W.): Add a list of references where one can find basics about the algebraic K-theory of higher categories, for instance [136].
- Add a discussion what the advantage of the passage to right-exact ∞-category are.
- We neet to explain the (non-connective) algebraic K-theory spectrum associated to a right-exact ∞-category. Or more generally, we should consider any *lax monoidal finitary localising invariant* H or just of a *localizing invariant*.
- Describe the universal property of the algebraic K-theory of right-exact ∞-categories.
- Introduce the notion of a *right-exact* G*-*∞*-category*;
- Construct for a right-exact G-∞-category the relevant contravariant functor to the category CATREX of right-exact ∞-category

$$\mathsf{GROUPOIDS} \downarrow G \to \mathsf{CATREX}, \quad \mathrm{pr}\colon \mathcal{G} \to I(G) \;\mapsto\; \int_{\mathcal{G}} \mathcal{C} \circ \mathrm{pr}\,.$$

  Composing it with the functor sending a right-exact ∞-category to its non-connective K-theory spectrum, yields the desired functor

$$\mathbf{K}_{\mathcal{C}}\colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA}, \tag{8.1}$$

  which will give the equivariant homology theory $H_*^?(-;\mathbf{K}_{\mathcal{C}})$, see Theorem 12.30 and Subsection 13.3.1.

- We have to explain why an additive category $\mathcal{A}$ defines a right-exact ∞-category $\mathcal{C}(\mathcal{A})$ and the functor (8.1) for $\mathcal{C}(\mathcal{A})$ agrees with the functor (13.10). The candidate is
$$\mathcal{C}(\mathcal{A}) = \mathrm{Ch}^{\mathrm{b}}(\mathcal{A})[h^{-1}].$$
- The construction has to be compared with Schlichting's definition of non-connective K-theory for exact categories, see [884] and Remark 6.40.
- In which sense can one define module categories (of perfect modules) over group rings with ring spectra as coefficients and interprete them in term of higher categories? This is related to the question whether one can formulate a Farrell-Jones Conjecture with coefficients in ring spectra.
- Explain why Waldhausen's A theory functor corresponds to the algebraic K-theory of an appropriate right-exact ∞-category; **Comment 6 (by W.)**: When this is done, we should update the references appearing in the proof of Theorem 15.63 (iii).
- Introduce the Swan group in the setting of right-exact G-∞-categories and explain why the results of Bob Oliver [755, 756] lead to interesting induction results, see [942].
  Later we have to figure out which of the reduction theorems for the families in Section 13.10 carry over in a possibly modified form to the setting of right-exact G-∞-categories.
- We shall discuss some basics, the localisation theorem, devissage, the theorems of the heart, Bass-Heller-Swan decomposition, and so on.
- Shall we mention for instance [598] [601]?

8.3 Why Should One Consider Hhigher Categories in the Farrell-Jones-Conjecture

It is out of question that higher categories has become a very important tool in many branches of mathematics in the last years, including K-theory. Here we want briefly to discuss its relevance in connection with the Isomorphism Conjectures.

If one goes through the list of applications of the Farrell-Jones Conjecture in Section 13.12, one sees that for all them except the applications to the computations of homotopy groups of automorphisms of aspherical closed manifolds it suffices to consider rings as coefficients. For the computations of homotopy groups of automorphisms of aspherical closed manifolds, it is enough to know the Farrell-Jones Conjecture for A-theory, which has already been treated in detail in [330]. The passage from rings to additive categories is not directly relevant for applications, but is clearly motivated by the facts that it allows to handle also twisted group rings and orientation characters and ensures all the useful inheritance properties. At the time of writing we know no application of the Farrell-Jones Conjecture to prominent problems in algebra, geometry, group theory, or topology where it does not suffice to deal with rings as coefficients or with the A-theory version and one is forced to consider higher categories. We also think that computations using cyclotomic traces have come to their limit concerning the detection of the algebraic K-theory of integral group rings.

Nevertheless, we expect that in the future the version of the Farrell-Jones Conjecture for higher categories will become important. Actually there are already instances where the passage to higher categories was necessary to get information about the classical setting. The constructions of natural transformations from the topological K-theory to the algebraic L-theory of C^*-algebra have been poorly (and incorrectly) treated in the classical setting, but in very satisfactory way using higher category theory in [599, 600] and thus open the door to a link between the Farrell-Jones Conjecture and the baum-Connes Conjecture as explained in Subsection 15.14.4. Even for computations of the topological K-theory of C^*-algebras methods from higher category theory are useful and actually needed, see for instance [600, Theorem B].

So far the L-theoretic version of the Farrell-Jones Conjecture has only been established for additive categories with involution. Christoph Winges and the author of the book expect that it can be generalized to the setting of higher categories and proved for all Dress-Farrell-Hsiang-Jones groups.

Another related topic is Hermitian K-theory in the setting of higher categories and all its applications, see for instance Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus and Steimle [183, 181, 182, 184].

8.4 Leftover

Comment 7 (by W.): This is a section which Christoph has already written some time ago. Some parts of it may be used in the chapter.

The language of homotopical algebra (and ∞-categories in particular) provides a framework in which the algebraic K-theory of group rings (or additive categories) and Waldhausen's A-theory can be treated on equal footing. The plus-construction described in Section 6.2 can be generalized to apply to E_1-ring spectra [320, Section VI.7]. The resulting K-theory functor reduces to the functor of of Definition 6.9 by precomposing with the functor that sends an ordinary ring to its associated Eilenberg-MacLane spectrum. It is a folk theorem that the A-theory of a path-connected space X is equivalent to the algebraic K-theory of the spherical group ring $\mathbb{S}[\Omega X]$, see for example [616]. In particular, there is an equivalence $A(BG) \simeq K(\mathbb{S}[G])$ for every group G. We ignore the question of a non-connective algebraic K-theory functor for the moment since we are going to switch to a slightly different perspective in a moment.

Just as in the case of ordinary rings, one can only formulate a Farrell-Jones Conjecture for group rings over an E_1-ring spectrum R after promoting the assignment $G/H \mapsto K(\mathbb{R}[H])$ to a functor on the orbit category of G. In analogy to Section 13.3 however, it is worthwhile to pass directly to an even more general setting in which the Farrell-Jones Conjecture enjoys the same inheritance properties as the Full Farrell-Jones Conjecture formulated in Section 13.7.

We find such a setting by considering algebraic K-theory as a functor defined on stable ∞-categories. See [674, Chapter 1] for fundamentals on stable ∞-categories and [136, Section 9] for a description of non-connective algebraic K-theory as a functor $\mathbf{K}\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathsf{SPECTRA}$ from the ∞-category of stable ∞-categories to the ∞-category of spectra. In the sequel, we assume some familiarity with the basics of higher category theory.

In slightly greater generality, one can also allow right-exact ∞-categories, i.e. pointed and finitely cocomplete ∞-categories, as input for the algebraic K-theory functor $\mathbf{K}$. The inclusion functor from $\mathrm{Cat}_\infty^{\mathrm{st}}$ to the ∞-category $\mathrm{Cat}_{\infty,*}^{\mathrm{rex}}$ of right-exact ∞-categories admits a left adjoint Stab. When applied to the ∞-category of finite spaces, this construction specialises to the (∞-category associated to) the classical Spanier-Whitehead category of finite spectra. The unit of this adjunction induces an equivalence $\mathbf{K}(\mathcal{C}) \xrightarrow{\sim} \mathbf{K}(\mathrm{Stab}(\mathcal{C}))$ for every right-exact ∞-category, so the right-hand term might as well be considered a definition of $\mathbf{K}$ as a functor on $\mathrm{Cat}_{\infty,*}^{\mathrm{rex}}$.

To formulate the Farrell-Jones Conjecture with coefficients in right-exact ∞-categories, we proceed as follows. Denote by BG the ∞-category given by the one-object groupoid with automorphism group G. Consider a right-exact ∞-category $\mathcal{C}$ with a (right) G-action, i.e. a functor $\mathcal{C}\colon BG^{\mathrm{op}} \to \mathrm{Cat}_{\infty,*}^{\mathrm{rex}}$. There is a canonical functor $j\colon BG^{\mathrm{op}} \to \mathsf{Or}(G)$ that sends the unique object of BG to the transitive G-set G and each element $g \in G$ to the G-equivariant map $r_g\colon G \to G$ given by right multiplication with g. Since $\mathrm{Cat}_{\infty,*}^{\mathrm{rex}}$ is cocomplete, we can take the left Kan extension $j_!\mathcal{C}\colon \mathsf{Or}(G) \to \mathrm{Cat}_{\infty,*}^{\mathrm{rex}}$ of $\mathcal{C}$ along j and compose this functor with $\mathbf{K}$ to obtain the $\mathsf{Or}(G)$-spectrum

$$\mathbf{K}_{\mathcal{C}}\colon \mathsf{Or}(G) \xrightarrow{j_!\mathcal{C}} \mathrm{Cat}_{\infty,*}^{\mathrm{rex}} \xrightarrow{\mathbf{K}} \mathsf{SPECTRA}.$$

The universal property of $\mathcal{P}(\mathsf{Or}(G))$ implies that the $\mathsf{Or}(G)$-spectrum $\mathbf{K}_{\mathcal{C}}$ is the same as a colimit-preserving functor

$$H^G(-;\mathbf{K}_{\mathcal{C}})\colon \mathcal{P}(\mathsf{Or}(G)) \to \mathsf{SPECTRA}.$$

By Elmendorf's theorem, the ∞-category $\mathcal{P}(\mathsf{Or}(G)) := \mathrm{Fun}(\mathsf{Or}(G)^{\mathrm{op}}, \mathsf{Spc})$ of presheaves on $\mathsf{Or}(G)$ is a model for the ∞-category of G-spaces. Under this identification, $H^G(-;\mathbf{K}_{\mathcal{C}})$ is precisely the G-homology theory associated to the $\mathsf{Or}(G)$-spectrum $\mathbf{K}_{\mathcal{C}}$. Moreover, we use this identification to consider the classifying space $E_{\mathcal{F}}(G)$ of G for a family $\mathcal{F}$ of subgroups as an object in $\mathcal{P}(\mathsf{Or}(G))$.

Consequently, we may say that a group G *satisfies the K-theoretic Farrell-Jones Conjecture with coefficients in right-exact ∞-categories* if the essentially unique map $E_{\mathcal{VCY}}(G) \to *$ to the final object $*$ of $\mathcal{P}(\mathsf{Or}(G))$ induces for every right-exact ∞-category $\mathcal{C}$ with G-action an equivalence

$$H^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{C}}) \to H^G(*; \mathbf{K}_{\mathcal{C}})$$

in $\mathsf{SPECTRA}$. Similarly, there is a *Full K-theoretic Farrell-Jones Conjecture with right-exact coefficients* which asks that every wreath product $G \wr F$ with a finite group F satisfies the Farrell-Jones Conjecture with coefficients in right-exact ∞-categories.

As promised, this formulation of the K-theoretic Farrell-Jones conjecture encompasses both the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories and the Farrell-Jones Conjecture for A-theory:

(i) For an additive G-category $\mathcal{A}$, we obtain a stable ∞-category

$$\mathcal{A} := \mathrm{Ch}^{\mathrm{b}}(\mathcal{A})[h^{-1}]$$

by localizing the category $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$ at the collection of chain homotopy equivalences h. This construction is functorial, so $\mathcal{A}$ is a stable ∞-category with G-action.
If we apply the above construction to $\mathcal{C}$, the resulting G-homology theory $H^G(-; \mathbf{K}_{\mathcal{C}})$ is equivalent to the G-homology theory $H^G(-, \mathbf{K}_{\mathcal{A}})$ considered in Conjecture 13.11, see [169, Example 1.6].

(ii) Let Z be a free G-CW-complex where we assume for convenience that G acts on the right. Then we may regard Z as an object in the functor category $\mathrm{Fun}(BG^{\mathrm{op}}, \mathsf{Spc})$. Since $\mathrm{Cat}^{\mathrm{rex}}_{\infty,*}$ is cocomplete, it is tensored over spaces. Denoting the right-exact ∞-category of pointed and compact (i.e. finitely dominated) spaces by $\mathsf{Spc}^{\omega}_{*}$, we obtain a right-exact ∞-category $Z \otimes \mathsf{Spc}^{\omega}_{*}$ with G-action. The G-homology theory $H^G(-; \mathbf{K}_{Z \otimes \mathsf{Spc}^{\omega}_{*}})$ is equivalent to the G-homology theory $H^G(-; \mathbf{A}^G_Z)$ considered in Conjecture, see [169, Example 1.9 & Corollary 7.71].

Comment 8 (by W.): The next theorem will become a consequence of Theorem 15.63 (iii), see Remark 13.21.

Theorem 8.2 (The Farrell-Jones Conjecture for coefficients in right-exact ∞-categories). *Theorem 16.1 holds verbatim for the Full K-theoretic Farrell-Jones Conjecture with right-exact coefficients.*

Proof. See [172, Theorem 1.6 and Section 8]. □

In addition, the evident analogs of Theorem 13.33 and of the split injectivity results from Section 16.6 hold for the assembly map associated to right-exact ∞-categories with G-action [169, Theorem 6.52 & Theorem 1.1].

The recent work of Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus and Steimle [183, 181, 182, 184] provides a framework in which these results can be further generalized to treat the K-theoretic and L-theoretic Farrell-Jones Conjectures in a unified way.

Exercise 8.3. Test. (There is a section with solutions to the exercises.)

8.5 Notes

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Chapter 9
Algebraic *L*-Theory

9.1 Introduction

In Remark 3.53 we have briefly discussed the *Surgery Program*. Starting with a map of degree one of connected closed manifolds $f\colon M \to N$, the goal is to modify it by surgery steps so that it becomes a homotopy equivalence. This will change the source but not the target, and can only be carried out if the map f is covered by bundle data. With the bundle data, one is able to make the map highly connected, but in the last step towards a homotopy equivalence an obstruction, the *surgery obstruction*, occurs, whose appearance is among other things due to *Poincaré duality*. This surgery obstruction takes values in the *algebraic L-groups* $L_n(\mathbb{Z}G)$ for $G = \pi_1(N)$. An introduction to the surgery obstruction and the algebraic L-groups will be given in this chapter. These are the key tools for the classification of manifolds besides the s-Cobordism Theorem 3.47. All this will be carried out in Sections 9.2 to 9.5 in the even dimensional case and in Sections 9.6 to 9.8 in the odd dimensional case.

We will also consider normal maps between compact manifolds with boundary that induce homotopy equivalences on the boundary. Here we want to achieve a homotopy equivalence by surgery on the interior, see Section 9.9.

Since the Whitehead torsion appears in the s-Cobordism Theorem 3.47, it will be important to achieve a simple homotopy equivalence and not only a homotopy equivalence by surgery. This leads to the *simple surgery obstruction* and *decorated L-groups*, see Section 9.10. The various decorated L-groups are linked by *Rothenberg sequences*. The L-theoretic analogue of the Bass-Heller-Swan decomposition for K-theory is the *Shaneson splitting*.

We will present the *L-theoretic Farrell-Jones Conjecture for torsionfree groups 9.114*, which relates the algebraic L-groups $L_n(\mathbb{Z}G)$ to the homology of BG with coefficient in the L-theory spectrum, analogous to the Farrell-Jones Conjecture for torsionfree groups and regular rings for K-theory 6.53. This together with the *Surgery Exact Sequence* of Section 9.12 opens the door to many applications. We will discuss the *Novikov Conjecture 9.137* about the homotopy invariance of higher signatures and the *Borel Conjecture 9.160* about the topologically rigidity of aspherical closed manifolds. Moreover, we deal with the problems whether a given finitely presented *Poincaré duality group* occurs as the fundamental group of an aspherical closed manifold, see Section 9.17, of which *hyperbolic groups* have spheres as their *boundary*, see Section 9.18, the *stable Cannon Conjecture*, see Section 9.19, and when does a *product decomposition* of the fundamental group of an aspherical closed manifold already implies a product decomposition of the manifold itself, see Section 9.20. *Automorphism groups* of aspherical closed manifolds are treated in Section 9.21. A brief survey on computations of L-theory of group rings of finite groups is presented in Section 9.22.

This chapter is an extract of [652].

9.2 Symmetric and Quadratic Forms

9.2.1 Symmetric Forms

Definition 9.1 (Ring with involution). A *ring with involution* R is an associative ring R with unit together with an *involution of rings*

$$\bar{\ }\colon R \to R, \quad r \mapsto \overline{r},$$

i.e., a map satisfying $\overline{\overline{r}} = r$, $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$, and $\overline{1} = 1$ for $r, s \in R$.

If R is commutative, we can equip it with the trivial involution $\overline{r} = r$.

Below we fix a ring R with involution. Module is to be understood as left module unless explicitly stated otherwise.

Example 9.2 (Involutions on groups rings). Let $w\colon G \to \{\pm 1\}$ be a group homomorphism. Then the group ring RG inherits an involution, the so-called *w-twisted involution*, that sends $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} w(g) \cdot \overline{r_g} \cdot g^{-1}$.

Remark 9.3 (Dual modules). The main purpose of the involution is to ensure that the dual of a left R-module can be viewed as a left R-module again. Namely, let M be a left R-module. Then $M^* := \hom_R(M, R)$ carries a canonical right R-module structure given by $(fr)(m) = f(m) \cdot r$ for a homomorphism of left R-modules $f\colon M \to R$ and $m \in M$. The involution allows us to view $M^* = \hom_R(M, R)$ as a left R-module, namely, define rf for $r \in R$ and $f \in M^*$ by $(rf)(m) := f(m) \cdot \overline{r}$ for $m \in M$.

Notation 9.4. Given a finitely generated projective R-module P, denote by $e(P)\colon P \xrightarrow{\cong} (P^*)^*$ the canonical isomorphism of (left) R-modules that sends $p \in P$ to the element in $(P^*)^*$ given by $P^* \to R,\ f \mapsto \overline{f(p)}$.

We will often use the following elementary fact. Let $f\colon P \to Q$ be a homomorphism of finitely generated projective R-modules. Then the following diagram commutes

(9.5)
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "e(P)"', "\cong"] & Q \arrow[d, "\cong"', "e(Q)"] \\ (P^*)^* \arrow[r, "(f^*)^*"'] & (Q^*)^*. \end{tikzcd}$$

Exercise 9.6. Show that the map $e(P)\colon P \to (P^*)^*$ of Notation (9.4) is a well-defined isomorphism of finitely generated projective R-modules, is compatible with direct sums and is natural, i.e., the diagram (9.5) commutes.

Definition 9.7 (Non-singular ϵ-symmetric form). Let $\epsilon \in \{\pm 1\}$. An *ϵ-symmetric form* (P, ϕ) over an associative ring R with involution is a finitely generated projective (left) R-module P together with an R-map $\phi\colon P \to P^*$ such that the composite $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{\phi^*} P^*$ agrees with $\epsilon \cdot \phi$.

A morphism $f\colon (P, \phi) \to (P', \phi)$ of ϵ-symmetric forms is an R-homomorphism $f\colon P \to P'$ satisfying $f^* \circ \phi' \circ f = \phi$.

We call an ϵ-symmetric form (P, ϕ) *non-singular* if ϕ is an isomorphism.

If ϵ is 1 or -1 respectively, we often replace ϵ-symmetric by symmetric or skew-symmetric respectively. The direct sum of two ϵ-symmetric forms is defined in the obvious way. The direct sum of two non-singular ϵ-symmetric forms is again a non-singular ϵ-symmetric form.

Remark 9.8 (ϵ-symmetric forms as pairings). We can rewrite an ϵ-symmetric form (P, ϕ) as a pairing

$$\lambda\colon P \times P \to R, \qquad (p, q) \mapsto \phi(p)(q).$$

The conditions that ϕ is R-linear and that $\phi(p)$ is R-linear for all $p \in P$ translates to

$$\begin{aligned} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}. \end{aligned}$$

The condition $\phi = \epsilon \cdot \phi^* \circ e(P)$ translates to $\lambda(q, p) = \epsilon \cdot \overline{\lambda(p, q)}$.

If we consider the real numbers $\mathbb{R}$ as a ring with involution by the trivial involution, then a non-singular 1-symmetric form ϕ on a finite dimensional $\mathbb{R}$-vector space V such that $\phi(x)(x) \geq 0$ holds for all $x \in \mathbb{R}^n$ is the same as a scalar product on V. If we consider the complex numbers $\mathbb{C}$ as a ring with involution by taking complex conjugation, then the corresponding statement holds for a finite dimensional complex vector space.

Definition 9.9 (The standard hyperbolic ϵ-symmetric form). Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-symmetric form* $H^\epsilon(P)$ is given by the R-module $P \oplus P^*$ and the R-isomorphism

$$\phi\colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* \xrightarrow{\rho} (P \oplus P^*)^*$$

where ρ is the obvious R-isomorphism.

If we write the standard hyperbolic ϵ-symmetric form $H^\epsilon(P)$ as a pairing, see Remark 9.8, we obtain

$$(P \oplus P^*) \times (P \oplus P^*) \to R, \qquad ((p,\alpha),(p',\alpha')) \mapsto \alpha'(p) + \epsilon \cdot \alpha(p').$$

9.2.2 The Signature

Consider a non-singular symmetric bilinear pairing $s\colon V \times V \to \mathbb{R}$ for a finite dimensional real vector space V, or, equivalently, a non-singular symmetric form of finitely generated free $\mathbb{R}$-modules. Choose a basis for V and let A be the square matrix describing s with respect to this basis. Since s is symmetric and non-singular, A is symmetric and invertible. Hence A can be diagonalized by an orthogonal matrix U to a diagonal matrix whose entries on the diagonal are non-zero real numbers. Let n_+ be the number of positive entries and n_- be the number of negative entries on the diagonal. These two numbers are independent of the choice of the basis and the orthogonal matrix U. Namely n_+ is the maximum of the dimensions of subvector spaces $W \subset V$, on which s is positive-definite, and analogous for n_-. Obviously $n_+ + n_- = \dim_{\mathbb{R}}(V)$.

Definition 9.10 (Signature). Define the *signature* of the non-singular symmetric bilinear pairing $s\colon V \times V \to \mathbb{R}$ for a finite dimensional real vector space V to be the integer

$$\operatorname{sign}(s) := n_+ - n_-.$$

Define the signature of a non-singular symmetric form over $\mathbb{Z}$ to be the signature of the associated non-singular symmetric form over $\mathbb{R}$.

Lemma 9.11. *Let $s\colon V \times V \to \mathbb{R}$ be a non-singular symmetric bilinear pairing for a finite dimensional real vector space V. Then* $\operatorname{sign}(s) = 0$ *if and only if there exists a subvector space $L \subset V$ such that* $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{R}}(L)$ *and $s(a,b) = 0$ for $a, b \in L$.*

Proof. Suppose that $\operatorname{sign}(s) = 0$. Then one can find an orthogonal (with respect to s) basis $\{b_1, b_2, \ldots, b_{n_+}, c_1, c_2, \ldots, c_{n_-}\}$ such that $s(b_i, b_i) = 1$ and $s(c_j, c_j) = -1$ holds. Since $0 = \operatorname{sign}(s) = n_+ - n_-$, we can define L to be the subvector space generated by $\{b_i + c_i \mid i = 1, 2, \ldots, n_+\}$. One easily checks that L has the desired properties.

Suppose such an $L \subset V$ exists. Choose subvector spaces V_+ and V_- of V such that s is positive-definite on V_+ and negative-definite on V_- and that

V_+ and V_- are maximal with respect to this property. Then $V_+ \cap V_- = \{0\}$ and $V = V_+ \oplus V_-$. Obviously $V_+ \cap L = V_- \cap L = \{0\}$. From

$$\dim_{\mathbb{R}}(V_\pm) + \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(V_\pm \cap L) \leq \dim_{\mathbb{R}}(V),$$

we conclude $\dim_{\mathbb{R}}(V_\pm) \leq \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(L)$. Since $2 \cdot \dim_{\mathbb{R}}(L) = \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V_+) + \dim_{\mathbb{R}}(V_-)$ holds, we get $\dim_{\mathbb{R}}(V_\pm) = \dim_{\mathbb{R}}(L)$. This implies

$$\operatorname{sign}(s) = \dim_{\mathbb{R}}(V_+) - \dim_{\mathbb{R}}(V_-) = \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(L) = 0.$$

□

If M is an orientable connected closed manifold of dimension d, then $H_d(M)$ is infinite cyclic. An *orientation* on M is equivalent to a choice of generator $[M] \in H_d(M)$ called *fundamental class*. This definition extends to a (not necessarily connected) orientable closed manifold M of dimension d by defining $[M] \in H_d(M)$ to be the image of $\{[C] \mid C \in \pi_0(M)\}$ under the canonical isomorphism $\bigoplus_{C \in \pi_0(M)} H_{\dim(M)}(C) \xrightarrow{\cong} H_{\dim(M)}(M)$.

Example 9.12 (Intersection pairing). Let M be a closed oriented manifold of even dimension $2n$. Then we obtain a $(-1)^n$-symmetric form on the finitely generated free $\mathbb{R}$-module $H^n(M;\mathbb{R})$

$$i \colon H^n(M;\mathbb{R}) \times H^n(M;\mathbb{R}) \to \mathbb{R}$$

by sending $([x],[y])$ for $x, y \in H^n(M;\mathbb{R})$ to $\langle x \cup y, [M]_{\mathbb{R}} \rangle$ where $\langle u, v \rangle$ denotes the *Kronecker pairing* and $[M]_{\mathbb{R}}$ is the image of the fundamental class $[M]$ under the change of rings homomorphism $H_n(M;\mathbb{Z}) \to H_n(M;\mathbb{R})$. It is non-singular by Poincaré duality.

Next we define a fundamental invariant of a closed oriented manifold, namely, its signature. This is the first kind of surgery obstruction we will encounter.

Definition 9.13 (Signature of a closed oriented manifold). Let M be a closed oriented manifold of dimension n. If n is divisible by four, then the signature $\operatorname{sign}(M)$ of M is defined to be the signature of its intersection pairing. If n is not divisible by four, define $\operatorname{sign}(M) = 0$.

One easily checks $\operatorname{sign}(M) = \sum_{C \in \pi_0(M)} \operatorname{sign}(C)$.

Exercise 9.14. Let M be an oriented closed $4k$-dimensional manifold. Let $\chi(M)$ be its Euler characteristic. Show $\operatorname{sign}(M) \equiv \chi(M) \mod 2$.

The signature can also be defined for oriented compact manifolds with possibly non-empty boundary, see for instance [652, Definition 5.84 on page 138], and has the following properties.

Theorem 9.15 (Properties of the signature of oriented compact manifolds).

(i) The signature is an oriented bordism invariant, i.e., if M is a $(4k+1)$-dimensional oriented compact manifold with boundary ∂M, then

$$\operatorname{sign}(\partial M) = 0;$$

(ii) Let M and N be oriented compact manifolds and $f\colon \partial M \to \partial N$ be an orientation reversing diffeomorphism. Then $M \cup_f N$ inherits an orientation from M and N and

$$\operatorname{sign}(M \cup_f N) = \operatorname{sign}(M) + \operatorname{sign}(N);$$

(iii) Let M and N be oriented compact manifolds. Then we get

$$\operatorname{sign}(M \times N, \partial(M \times N)) = \operatorname{sign}(M, \partial M) \cdot \operatorname{sign}(N, \partial N);$$

(iv) Let $p\colon \overline{M} \to M$ be a finite covering with d sheets of oriented closed manifolds. Then

$$\operatorname{sign}(\overline{M}) = d \cdot \operatorname{sign}(M);$$

(v) If the oriented oriented connected closed manifolds M and N are oriented homotopy equivalent, then

$$\operatorname{sign}(M) = \operatorname{sign}(N);$$

(vi) If M is an oriented closed manifold and M^- is obtained from M by reversing the orientation, then

$$\operatorname{sign}(M^-) = -\operatorname{sign}(M).$$

Proof. (i) Let $i\colon \partial M \to M$ be the inclusion. Then the following diagram commutes

$$\begin{tikzcd}
H^{2k}(M;\mathbb{R}) \arrow[r, "H^{2k}(i)"] \arrow[d, "-\cap[M,\partial M]_{\mathbb{R}}"', "\cong"] & H^{2k}(\partial M;\mathbb{R}) \arrow[r, "\delta^{2k}"] \arrow[d, "-\cap\partial_{4k+1}([M,\partial M]_{\mathbb{R}})"', "\cong"] & H^{2k+1}(M,\partial M;\mathbb{R}) \arrow[d, "-\cap[M,\partial M]_{\mathbb{R}}"', "\cong"] \\
H_{2k+1}(M,\partial M;\mathbb{R}) \arrow[r, "\partial_{2k+1}"] & H_{2k}(\partial M;\mathbb{R}) \arrow[r, "H_{2k}(i)"] & H_{2k}(M;\mathbb{R}).
\end{tikzcd}$$

This implies $\dim_{\mathbb{R}}(\ker(H_{2k}(i))) = \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i)))$. Since $\mathbb{R}$ is a field, we get from the Kronecker pairing an isomorphism $H^{2k}(M;\mathbb{R}) \cong (H_{2k}(M;\mathbb{R}))^*$ and analogously for ∂M. Under these identifications $H^{2k}(i)$ becomes $(H_{2k}(i))^*$. Hence $\dim_{\mathbb{R}}(\operatorname{im}(H_{2k}(i))) = \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i)))$. From

$$\dim_{\mathbb{R}}(H_{2k}(\partial M;\mathbb{R})) = \dim_{\mathbb{R}}(\ker(H_{2k}(i))) + \dim_{\mathbb{R}}(\operatorname{im}(H_{2k}(i)))$$

we conclude

$$\dim_{\mathbb{R}}(H^{2k}(\partial M;\mathbb{R})) = 2 \cdot \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i))).$$

We have for $x, y \in H^{2k}(M;\mathbb{R})$

$$\begin{aligned}
\langle H^{2k}(i)(x) \cup H^{2k}(i)(y), \partial_{4k+1}([M,\partial M]_{\mathbb{R}})\rangle \\
&= \langle H^{4k}(i)(x \cup y), \partial_{4k+1}([M,\partial M]_{\mathbb{R}})\rangle \\
&= \langle x \cup y, H_{4k}(i) \circ \partial_{4k+1}([M,\partial M]_{\mathbb{R}})\rangle \\
&= \langle x \cup y, 0\rangle \\
&= 0.
\end{aligned}$$

If we apply Lemma 9.11 to the non-singular symmetric bilinear pairing

$$H^{2k}(\partial M;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(\partial M;\mathbb{R}) \xrightarrow{\cup} H^{4k}(\partial M;\mathbb{R}) \xrightarrow{\langle -, \partial_{4k+1}([M,\partial M]_{\mathbb{R}})\rangle} \mathbb{R}$$

with L the image of $H^{2k}(i)\colon H^{2k}(M;\mathbb{R}) \to H^{2k}(\partial M;\mathbb{R})$, we see that the signature of this pairing is zero.

(ii) This is due to Novikov. For a proof see for instance [53, Proposition 7.1 on page 588].

(iii) See for instance [652, Lemma 5.85 (ii) on page 139].

(iv) For a smooth manifold M this follows from Atiyah's L^2-index theorem [46, (1.1)]. Topological closed manifolds are treated in [875, Theorem 8].

(v) The two intersection pairings are isomorphic and hence have the same signatures.

(vi) This follows from $[M^-] = -[M]$. □

Exercise 9.16. Compute for $n \geq 1$ the signature of:

(i) the complex projective space $\mathbb{CP}^n$;
(ii) the total space STM of the sphere tangent bundle of an oriented closed n-dimensional manifold M;
(iii) an oriented closed n-dimensional manifold M admitting an orientation reversing selfdiffeomorphism.

9.2.3 Quadratic Forms

Next we introduce quadratic forms, which are refinements of symmetric forms.

Notation 9.17. For a finitely generated projective R-module P define an involution of R-modules

$$T = T(P)\colon \hom_R(P, P^*) \to \hom_R(P, P^*), \quad u \mapsto u^* \circ e(P).$$

Notation 9.18. Let P be a finitely generated projective R-module. Define abelian groups

$$Q^\epsilon(P) := \ker\left((1-\epsilon\cdot T)\colon\ \hom_R(P,P^*) \to \hom_R(P,P^*)\right);$$
$$Q_\epsilon(P) := \operatorname{coker}\left((1-\epsilon\cdot T)\colon\ \hom_R(P,P^*) \to \hom_R(P,P^*)\right).$$

An R-homomorphism $f\colon P \to Q$ induces a homomorphism of abelian groups

$$Q^\epsilon(f)\colon Q^\epsilon(Q) \to Q^\epsilon(P), \quad u \mapsto f^* \circ u \circ f;$$
$$Q_\epsilon(f)\colon Q_\epsilon(Q) \to Q_\epsilon(P), \quad [u] \mapsto [f^* \circ u \circ f].$$

Let

$$(1+\epsilon\cdot T)\colon Q_\epsilon(P) \to Q^\epsilon(P)$$

be the homomorphism that sends the class $[u]$ represented by $u\colon P \to P^*$ to the element $u + \epsilon \cdot T(u)$.

Definition 9.19 (Non-singular ϵ-quadratic form). Let $\epsilon \in \{\pm 1\}$. An *ϵ-quadratic form* (P,ψ) is a finitely generated projective R-module P together with an element $\psi \in Q_\epsilon(P)$. It is called *non-singular* if the *associated ϵ-symmetric form* $(P,(1+\epsilon\cdot T)(\psi))$ is non-singular, i.e. $(1+\epsilon\cdot T)(\psi)\colon P \to P^*$ is bijective.

A morphism $f\colon (P,\psi) \to (P',\psi')$ of two ϵ-quadratic forms is an R-homomorphism $f\colon P \xrightarrow{\cong} P'$ such that the induced map $Q_\epsilon(f)\colon Q_\epsilon(P') \to Q_\epsilon(P)$ sends ψ' to ψ.

Given a non-singular ϵ-symmetric form (P,ϕ), a *quadratic refinement* is a non-singular ϵ-quadratic form (P,ψ) with $\phi = (1+\epsilon\cdot T)(\psi)$.

There is an obvious notion of a direct sum of two ϵ-quadratic forms. The direct sum of two non-singular ϵ-quadratic forms is a non-singular ϵ-quadratic form.

Consider the pairing

$$\rho\colon R \times Q_{(-1)^k}(R) \to Q_{(-1)^k}(R), \quad (r,[s]) \mapsto [rs\overline{r}]. \tag{9.20}$$

It is well defined since for $r,s,t \in R$ we get if we put $t' = rt\overline{r}$,

$$r\bigl(s + (t - (-1)^k \cdot \overline{t})\bigr)\overline{r} = rs\overline{r} + \bigl(t' - (-1)^k \cdot \overline{t'}\bigr).$$

It is additive in the second variable, i.e., $\rho(r,[s_1]-[s_2]) = \rho(r,[s_1]) - \rho(r,[s_2])$, but it is not additive in the first variable and in particular ρ does *not* give the structure of a left R-module on $Q_{(-1)^k}(R)$. Nevertheless, sometimes in the literature $\rho(r,[s])$ is denoted by $r[s]\overline{r}$.

Remark 9.21 (Writing a quadratic form as a triple (P,λ,μ)). We can rewrite this as follows. An ϵ-quadratic form (P,ψ) is equivalent to a triple (P,λ,μ) consisting of a pairing

$$\lambda\colon P \times P \to R$$

satisfying

$$\lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2) = r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2);$$
$$\lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) = \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2};$$
$$\lambda(q, p) = \epsilon \cdot \overline{\lambda(p, q)},$$

and a map

$$\mu\colon P \to Q_\epsilon(R) = R/\{r - \epsilon \cdot \overline{r} \mid r \in R\}$$

satisfying

$$\mu(rp) = \rho(r, \mu(p));$$
$$\mu(p+q) - \mu(p) - \mu(q) = \mathrm{pr}(\lambda(p, q));$$
$$\lambda(p, p) = (1 + \epsilon \cdot T)(\mu(p)),$$

where the pairing ρ was introduced in (9.20), $\mathrm{pr}\colon R \to Q_\epsilon(R)$ is the projection and $(1+\epsilon\cdot T)\colon Q_\epsilon(R) \to R$ the map sending the class of r to $r+\epsilon\cdot\overline{r}$. Namely, put

$$\lambda(p, q) = \big((1 + \epsilon \cdot T)(\psi)(p)\big)(q);$$
$$\mu(p) = \mathrm{pr}(\psi(p)(p)).$$

These two descriptions of an ϵ-quadratic form are equivalent, see [967, Theorem 1].

Definition 9.22 (The standard hyperbolic ϵ-quadratic form). Let P be a finitely generated projective R-module. The *standard hyperbolic ϵ-quadratic form* $H_\epsilon(P)$ is given by the R-module $P \oplus P^*$ and the class in $Q_\epsilon(P \oplus P^*)$ of the R-homomorphism

$$\phi\colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* \xrightarrow{\rho} (P \oplus P^*)^*$$

where ρ is the obvious R-isomorphism.

If we write the standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$ as a pairing, see Remark 9.21, we obtain

$$\begin{array}{ll} \lambda\colon (P \oplus P^*) \times (P \oplus P^*) \to R, & ((p,\alpha),(p',\alpha')) \mapsto \alpha'(p) + \epsilon \cdot \alpha(p'); \\ \mu\colon P \oplus P^* \to Q_\epsilon(R), & (p,\alpha) \mapsto \mathrm{pr}(\alpha(p)). \end{array}$$

In particular the ϵ-symmetric form associated to the standard ϵ-quadratic form $H_\epsilon(P)$ is just the standard ϵ-symmetric form $H^\epsilon(P)$.

Exercise 9.23. Let $\lambda\colon P \times P \to \mathbb{Z}$ be a non-singular symmetric $\mathbb{Z}$-bilinear pairing on the finitely generated free $\mathbb{Z}$-module P. Show that it has, when considered as a non-singular symmetric form, a quadratic refinement if and only if $\lambda(x, x)$ is even for all $x \in P$.

Remark 9.24. Suppose that $1/2 \in R$. Then the homomorphism

$$(1 + \epsilon \cdot T)\colon Q_\epsilon(P) \to Q^\epsilon(P), \quad [u] \mapsto [u + \epsilon \cdot T(u)]$$

is bijective. The inverse sends v to $[v/2]$. Hence any ϵ-symmetric form carries a unique ϵ-quadratic structure. Therefore there is no difference between the symmetric and the quadratic setting if 2 is invertible in R.

9.3 Even Dimensional L-groups

Next we define even dimensional L-groups. Below R is an associative ring with involution.

Definition 9.25 (L-groups in even dimensions). For an even integer $n = 2k$ define the abelian group $L_n(R)$, called the *n-th quadratic L-group,* of R to be the abelian group of equivalence classes $[P, \psi]$ of non-singular $(-1)^k$-quadratic forms (P, ψ) whose underlying R-module P is a finitely generated free R-module, with respect to the following equivalence relation: We call (P, ψ) and (P', ψ') equivalent if and only if there exists integers $u, u' \geq 0$ and an isomorphism of non-singular $(-1)^k$-quadratic forms

$$(P, \psi) \oplus H_\epsilon(R)^u \cong (P', \psi') \oplus H_\epsilon(R)^{u'}.$$

Addition is given by the direct sum of two $(-1)^k$-quadratic forms. The zero element is represented by $[H_{(-1)^k}(R)^u]$ for any integer $u \geq 0$. The inverse of $[P, \psi]$ is given by $[P, -\psi]$.

A morphism $u\colon R \to S$ of rings with involution induces homomorphisms $u_*\colon L_n(R) \to L_n(S)$ and $u_*\colon L^n(R) \to L^n(S)$ for even $n \in \mathbb{Z}$ by induction satisfying $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 0, 2$.

Next we will present a criterion for an ϵ-quadratic form (P, ψ) to represent zero in $L_{1-\epsilon}(R)$. Let (P, ψ) be an ϵ-quadratic form. A *sublagrangian* $L \subset P$ is an R-submodule such that the inclusion $i\colon L \to P$ is split injective, the image of ψ under the map $Q_\epsilon(i)\colon Q_\epsilon(P) \to Q_\epsilon(L)$ is zero, and L is contained in its *annihilator* $L^\perp$, that is by definition the kernel of

$$P \xrightarrow{(1+\epsilon \cdot T)(\psi)} P^* \xrightarrow{i^*} L^*.$$

A sublagrangian $L \subset P$ is called *lagrangian* if $L = L^\perp$. Equivalently, a lagrangian $L \subset P$ is an R-submodule L with inclusion $i\colon L \to P$ such that the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} L^* \to 0.$$

is exact.

Lemma 9.26. *Let (P, ψ) be an ϵ-quadratic form. Let $L \subset P$ be a sublagrangian. Then L is a direct summand in $L^{\perp}$ and ψ induces the structure of a non-singular ϵ-quadratic form $(L^{\perp}/L, \psi^{\perp}/\psi)$. Moreover, the inclusion $i\colon L \to P$ extends to an isomorphism of ϵ-quadratic forms*

$$H_{\epsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi) \xrightarrow{\cong} (P, \psi).$$

In particular a non-singular ϵ-quadratic form (P, ψ) is isomorphic to $H_{\epsilon}(Q)$ if and only if it contains a lagrangian $L \subset P$ which is isomorphic as R-module to Q.

Proof. See for instance [652, Lemma 8.95 on page 265]. □

Exercise 9.27. Show for a non-singular ϵ-quadratic form (P, ψ) that $(P, \psi) \oplus (P, -\psi)$ is isomorphic to $H_{\epsilon}(P)$ and hence an inverse of $[P, \psi]$ in $L_{1-\epsilon}(R)$ is given by $[P, -\psi]$.

Exercise 9.28. Show that the signature defines an isomorphism $L_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z}$.

Finally we state the computation of the even dimensional L-groups of the ring of integers $\mathbb{Z}$. Consider an element (P, ψ) in $L_0(\mathbb{Z})$. By tensoring over $\mathbb{Z}$ with $\mathbb{R}$ and only taking the symmetric structure into account, we obtain a non-singular symmetric $\mathbb{R}$-bilinear pairing $\lambda\colon \mathbb{R} \otimes_{\mathbb{Z}} P \times \mathbb{R} \otimes_{\mathbb{Z}} P \to \mathbb{R}$. It turns out that its signature is always divisible by eight. A proof the following classical result can be found for instance in [652, Subsection 8.5.2], see also [713].

Theorem 9.29 (L-groups of the ring of integers in dimension $4n$). *The signature defines for $n \in \mathbb{Z}$ an isomorphism*

$$\frac{1}{8} \cdot \operatorname{sign}\colon L_{4n}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \qquad [P, \psi] \mapsto \frac{1}{8} \cdot \operatorname{sign}(\mathbb{R} \otimes_{\mathbb{Z}} P, \lambda).$$

Consider a non-singular quadratic form (P, ψ) over the field $\mathbb{F}_2$ of two elements. Write (P, ψ) as a triple (P, λ, μ) as explained in Remark 9.21. Choose any symplectic basis $\{b_1, b_2, \ldots, b_{2m}\}$ for P, where symplectic means that $\lambda(b_i, b_j)$ is 1 if $i - j = m$ and 0 otherwise. Such a symplectic basis always exists. Define the *Arf invariant* of (P, ψ) by

$$\operatorname{Arf}(P, \psi) := \sum_{i=1}^{m} \mu(b_i) \cdot \mu(b_{i+m}) \qquad \in \mathbb{Z}/2. \tag{9.30}$$

It turns out that the Arf invariant of (P, ψ) is 1 if and only if μ sends a (strict) majority of the elements of P to 1, see [159, Corollary III.1.9 on page 55]. (Because of this property sometimes the Arf invariant is called the democratic invariant.) This description shows that (9.30) is independent of the choice of symplectic basis.

Exercise 9.31. Let V be a two-dimensional $\mathbb{F}_2$-vector space. Classify all non-singular quadratic forms on V up to isomorphism and compute their Arf invariants.

The Arf invariant defines an isomorphism

$$\mathrm{Arf}\colon L_{2n}(\mathbb{F}_2) \xrightarrow{\cong} \mathbb{Z}/2,$$

essentially, since two non-singular quadratic forms over $\mathbb{F}_2$ on the same finite dimensional $\mathbb{F}_2$-vector space are isomorphic if and only if they have the same Arf invariant, see [159, Theorem III.1.12 on page 55]. The change of rings homomorphism $\mathbb{Z} \to \mathbb{F}_2$ induces an isomorphism,

$$L_{4n+2}(\mathbb{Z}) \xrightarrow{\cong} L_{4n+2}(\mathbb{F}_2).$$

This implies, see for instance [652, Subsection 8.5.3],

Theorem 9.32 (L-groups of the ring of integers in dimension $4n+2$). *The Arf invariant defines for $n \in \mathbb{Z}$ an isomorphism*

$$\mathrm{Arf}\colon L_{4n+2}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2, \qquad [P,\psi] \mapsto \mathrm{Arf}(\mathbb{F}_2 \otimes_{\mathbb{Z}} (P,\psi)).$$

For more information about forms over the integers and the Arf invariant we refer for instance to [159, 713]. Implicitly the computation of $L_n(\mathbb{Z})$ is already in [559].

9.4 Intersection and Selfintersection Pairings

The notions of an ϵ-symmetric form as presented in Remark 9.8 and of an ϵ-quadratic form as presented in Remark 9.21 are best motivated by considering intersections and selfintersection pairings. When trying to solve a surgery problem in even dimensions, one faces in the final step, namely, when dealing with the middle dimension, the problem to decide whether we can change an immersion $f\colon S^k \to M$ within its regular homotopy class to an embedding where M is a compact manifold of dimension $n = 2k$. This problem leads in a natural way to selfintersection pairings and ϵ-quadratic forms as explained next.

9.4.1 Intersections of Immersions

Let $k \geq 2$ be a natural number, and let M be a connected compact smooth manifold of dimension $n = 2k$. We fix base points $s \in S^k$ and $b \in M$. We will consider *pointed immersions* (f, w), i.e., an immersion $f\colon S^k \to M$ together with a path w from b to $f(s)$ in M. A *regular homotopy* $h\colon M \times [0,1] \to N$ from an immersion $q_0\colon M \to N$ to an immersion $q_1\colon M \to N$ is a (continuous, but not necessarily smooth) homotopy $h\colon M \times [0,1] \to N$ such that $h_0 = q_0$, $h_1 = q_1$, $h_t\colon M \to N$ is a (smooth) immersion for each $t \in [0,1]$, and the derivatives $Th_t\colon TM \to TN$ of h_t fit together to define a (continuous) homotopy of bundle monomorphisms

$$TM \times [0,1] \to TN, \quad (v,t) \mapsto Th_t(v)$$

between Tq_0 and Tq_1. A *pointed regular homotopy* from (f_0, w_0) to (f_1, w_1) is a regular homotopy $h\colon S^k \times [0,1] \to M$ from $h_0 = f_0$ to $h_1 = f_1$ such that $w_0 * h(s,-)$ and w_1 are homotopic paths relative end points. Here $h(s,-)$ is the path from $f_0(s)$ to $f_1(s)$ given by restricting h to $\{s\} \times [0,1]$. Denote by $I_k(M)$ the set of pointed regular homotopy classes $[f,w]$ of of pointed immersions (f,w) from S^k to M. We need the paths to define the structure of an abelian group on $I_k(M)$. The sum of $[f_0,w_0]$ and $[f_1,w_1]$ is given by the connected sum along the path $w_0^- * w_1$ from $f_0(s)$ to $f_1(s)$. The zero element is given by the composite of the standard embedding $S^k \to \mathbb{R}^{k+1} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{k-1} = \mathbb{R}^n$ with some embedding $\mathbb{R}^n \subset M$ and any path w from b to the image of s. The inverse of the class of (f,w) is the class of $(f \circ a, w)$ for any base point preserving diffeomorphism $a\colon S^k \to S^k$ of degree -1.

The fundamental group $\pi = \pi_1(M,b)$ operates on $I_k(M)$ by composing the path w with a loop at b. Thus $I_k(M)$ inherits the structure of a $\mathbb{Z}\pi$-module.

Next we want to define the *intersection pairing*

$$\lambda\colon I_k(M) \times I_k(M) \to \mathbb{Z}\pi. \tag{9.33}$$

For this purpose we will have to fix an orientation of T_bM at b. Consider $\alpha_0 = [f_0,w_0]$ and $\alpha_1 = [f_1,w_1]$ in $I_k(M)$. Choose representatives (f_0,w_0) and (f_1,w_1). We can arrange without changing the pointed regular homotopy classes that $D = \operatorname{im}(f_0) \cap \operatorname{im}(f_1)$ is finite, for any $y \in D$ both the preimage $f_0^{-1}(y)$ and the preimage $f_1^{-1}(y)$ consists of precisely one point, and, for any two points x_0 and x_1 in S^k with $f_0(x_0) = f_1(x_1)$, we have $T_{x_0}f_0(T_{x_0}S^k) + T_{x_1}f_1(T_{x_1}S^k) = T_{f_0(x_0)}M$. Consider $d \in D$. Let x_0 and x_1 in S^k be the points uniquely determined by $f_0(x_0) = f_1(x_1) = d$. Let u_i be a path in S^k from s to x_i. Then we obtain an element $g(d) \in \pi$ by the loop at b given by the composite $w_1 * f_1(u_1) * f_0(u_0)^- * w_0^-$. Recall that we have fixed an orientation of T_bM. The fiber transport along the path $w_0 * f(u_0)$ yields an isomorphism $T_bM \xrightarrow{\cong} T_dM$ that is unique up to isotopy. Hence T_dM inherits an orientation from the given orientation of T_bM. The standard orientation of S^k determines an orientation on $T_{x_0}S^k$ and $T_{x_1}S^k$. We have the isomorphism of oriented vector spaces

$$T_{x_0}f_0 \oplus T_{x_1}f_1\colon T_{x_0}S^k \oplus T_{x_1}S^k \xrightarrow{\cong} T_dM.$$

Define $\epsilon(d) = 1$ if this isomorphism respects the orientations and $\epsilon(d) = -1$ otherwise. The elements $g(d) \in \pi$ and $\epsilon(d) \in \{\pm 1\}$ are independent of the choices of u_0 and u_1 since S^k is simply connected for $k \geq 2$. Define

$$\lambda(\alpha_0, \alpha_1) := \sum_{d \in D} \epsilon(d) \cdot g(d).$$

Lift $b \in M$ to a base point $\widetilde{b} \in \widetilde{M}$. Let $\widetilde{f_i} \colon S^k \to \widetilde{M}$ be the unique lift of f_i determined by w_i and $\widetilde{b}$ for $i = 0, 1$. Let $\lambda_{\mathbb{Z}}(\widetilde{f_0}, \widetilde{f_1})$ be the $\mathbb{Z}$-valued intersection number of $\widetilde{f_0}$ and $\widetilde{f_1}$. This is the same as the algebraic intersection number of the classes in the k-th homology with compact support defined by $\widetilde{f_0}$ and $\widetilde{f_1}$, which obviously depends only on the homotopy classes of $\widetilde{f_0}$ and $\widetilde{f_1}$; the proof in [153, Theorem 11.9 in Chapter VI on page 372] can be extended to our setting. Then

$$\lambda(\alpha_0, \alpha_1) = \sum_{g \in \pi} \lambda_{\mathbb{Z}}(\widetilde{f_0}, l_{g^{-1}} \circ \widetilde{f_1}) \cdot g \tag{9.34}$$

where $l_{g^{-1}}$ denotes left multiplication with g^{-1}. This shows that $\lambda(\alpha_0, \alpha_1)$ depends only on the pointed regular homotopy classes of (f_0, w_0) and (f_1, w_1).

Below we use the $w_1(M)$-twisted involution on $\mathbb{Z}\pi$ that sends $\sum_{g \in \pi} a_g \cdot g$ to $\sum_{g \in \pi} w_1(M)(g) \cdot a_g \cdot g^{-1}$, where $w_1(M) \colon \pi \to \{\pm 1\}$ is the first Stiefel-Whitney class of M. The elementary proof of the next lemma is left to the reader.

Lemma 9.35. *For $\alpha, \beta, \beta_1, \beta_2 \in I_k(M)$ and $u_1, u_2 \in \mathbb{Z}\pi$ we have*

$$\lambda(\alpha, \beta) = (-1)^k \cdot \overline{\lambda(\beta, \alpha)};$$
$$\lambda(\alpha, u_1 \cdot \beta_1 + u_2 \cdot \beta_2) = u_1 \cdot \lambda(\alpha, \beta_1) + u_2 \cdot \lambda(\alpha, \beta_2).$$

Remark 9.36 (Intersection pairing and $(-1)^k$-symmetric forms). Lemma 9.35 says that the pair $(I_k(M), \lambda)$ satisfies all the requirements appearing in Remark 9.8 except that $I_k(M)$ may not be finitely generated free over $\mathbb{Z}\pi$.

Remark 9.37 (The intersection pairing as necessary obstruction for finding an embedding). Suppose that the normal bundle of the immersion $f \colon S^k \to M$ has a nowhere vanishing section. (In the typical situation that appears in surgery theory it actually will be trivial.) Suppose that f is regular homotopic to an embedding g. Then the normal bundle of g has a nowhere vanishing section σ. Let g' be the embedding obtained by moving g a little bit in the direction of this normal vector field σ. Choose a path w_f from $f(s)$ to b. Then for appropriate paths w_g and $w_{g'}$ we get pointed embeddings (g, w_g) and $(g', w_{g'})$ such that the pointed regular homotopy classes of (f, w), (g, w_g) and $(g', w_{g'})$ agree. Since g and g' have disjoint images, we conclude

$$\lambda([f, w], [f, w]) = 0.$$

Hence the vanishing of $\lambda([f,w],[f,w])$ is a necessary condition for finding an embedding in the regular homotopy class of f, provided that the normal bundle of f has a nowhere vanishing section. It is not a sufficient condition. To get a sufficient condition we have to consider selfintersections what we will do next.

9.4.2 Selfintersections of Immersions

Let $\alpha \in I_k(M)$ be an element. Let (f,w) be a pointed immersion representing α. Recall that we have fixed base points $s \in S^k$, $b \in M$, and an orientation of T_bM. Since we can find arbitrarily close to f an immersion which is in general position, we can assume without loss of generality that f itself is in general position. This means that there is a finite subset D of $\operatorname{im}(f)$ such that $f^{-1}(y)$ consists of precisely two points for $y \in D$ and of precisely one point for $y \in \operatorname{im}(f) - D$ and that for two points x_0 and x_1 in S^k with $x_0 \neq x_1$ and $f(x_0) = f(x_1)$ we have $T_{x_0}f(T_{x_0}S^k) + T_{x_1}f(T_{x_1}S^k) = T_{f_0(x_0)}M$. Now fix for any $d \in D$ an ordering $x_0(d), x_1(d)$ of $f^{-1}(d)$. Analogously to the construction above one defines $\epsilon(x_0(d),x_1(d)) \in \{\pm 1\}$ and $g(x_0(d),x_1(d)) \in \pi = \pi_1(M,b)$. Consider the element $\sum_{d \in D} \epsilon(x_0(d),x_1(d)) \cdot g(x_0(d),x_1(d))$ of $\mathbb{Z}\pi$. It does not only depend on f, but also on the choice of the ordering of $f^{-1}(d)$ for $d \in D$. One easily checks that the change of ordering of $f^{-1}(d)$ has the following effect for $w = w_1(M)\colon \pi \to \{\pm 1\}$

$$\begin{aligned}
g(x_1(d),x_0(d)) &= g(x_0(d),x_1(d))^{-1};\\
w(g(x_1(d),x_0(d))) &= w(g(x_0(d),x_1(d)));\\
\epsilon(x_1(d),x_0(d)) &= (-1)^k \cdot w(g(x_0(d),x_1(d))) \cdot \epsilon(x_0(d),x_1(d));\\
\epsilon(x_1(d),x_0(d)) \cdot g(x_1(d),x_0(d)) &= (-1)^k \cdot \epsilon(x_0(d),x_1(d)) \cdot \overline{g(x_0(d),x_1(d))}.
\end{aligned}$$

We have defined the abelian group $Q_{(-1)^k}(\mathbb{Z}\pi,w)$ in Notation 9.18. Define the *selfintersection element* for $\alpha \in I_k(M)$

$$\mu(\alpha) := \left[\sum_{d \in D} \epsilon(x_0(d),x_1(d)) \cdot g(x_0(d),x_1(d))\right] \in Q_{(-1)^k}(\mathbb{Z}\pi,w). \tag{9.38}$$

The passage from $\mathbb{Z}\pi$ to $Q_{(-1)^k}(\mathbb{Z}\pi,w)$ ensures that the definition is independent of the choice of the order on $f^{-1}(d)$ for $d \in D$. It remains to show that it depends only on the pointed regular homotopy class of (f,w). Let h be a pointed regular homotopy from (f,w) to (g,v). We can arrange that h is in general position. In particular each immersion h_t is in general position and comes with a set D_t. The collection of the D_t-s yields a bordism W from the finite set D_0 to the finite set D_1. Since W is a compact one-dimensional manifold, it consists of circles and arcs joining points in $D_0 \cup D_1$ to points in

$D_0 \cup D_1$. Suppose that the point e and the point e' in $D_0 \cup D_1$ are joint by an arc. Then one easily checks that their contributions to

$$\mu(f,w) - \mu(g,w) := \left[\sum_{d_0 \in D_0} \epsilon(x_0(d_0), x_1(d_0)) \cdot g(x_0(d_0), x_1(d_0)) \right.$$
$$\left. - \sum_{d_1 \in D_1} \epsilon(x_0(d_1), x_1(d_1)) \cdot g(x_0(d_1), x_1(d_1))\right]$$

cancel out. This implies $\mu(f,w) = \mu(g,w)$.

Consider the pairing

(9.39) $$\rho\colon \mathbb{Z}\pi \times Q_{(-1)^k}(\mathbb{Z}\pi, w) \to Q_{(-1)^k}(\mathbb{Z}\pi, w), \quad (u,[v]) \mapsto [uv\overline{u}].$$

It is additive in the second variable, i.e., $\rho(x,[y_1]-[y_2]) = \rho(x,[y_1]) - \rho(x,[y_2])$, but it is not additive in the first variable, and in particular ρ does *not* give the structure of a left $\mathbb{Z}\pi$-module on $Q_{(-1)^k}(\mathbb{Z}\pi, w)$. Sometimes in the literatur $\rho(x,[y])$ is denoted by $x[y]\overline{x}$, but this is a little bit misleading since it might lead to the wrong impression that $Q_{(-1)^k}(\mathbb{Z}\pi, w)$ is a left or right $\mathbb{Z}\pi$-module.

Lemma 9.40. *Let $\mu\colon I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ be the map given by the self-intersection element, see* (9.38)*, and let $\lambda\colon I_k(M) \times I_k(M) \to \mathbb{Z}\pi$ be the intersection pairing, see* (9.33)*. Then:*

(i) Let $(1 + (-1)^k \cdot T)\colon Q_{(-1)^k}(\mathbb{Z}\pi, w) \to \mathbb{Z}\pi$ be the homomorphism of abelian groups that sends $[u]$ to $u + (-1)^k \cdot \overline{u}$. Denote for $\alpha \in I_k(M)$ by $\chi(\alpha) \in \mathbb{Z}$ the Euler number of the normal bundle $\nu(f)$ for any representative (f,w) of α with respect to the orientation of $\nu(f)$ given by the standard orientation on S^k and the orientation on f^*TM given by the fixed orientation on T_bM and w. Then:

$$\lambda(\alpha,\alpha) = (1 + (-1)^k \cdot T)(\mu(\alpha)) + \chi(\alpha) \cdot 1;$$

(ii) We get for $\mathrm{pr}\colon \mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$ the canonical projection and $\alpha, \beta \in I_k(M)$

$$\mu(\alpha + \beta) - \mu(\alpha) - \mu(\beta) = \mathrm{pr}(\lambda(\alpha,\beta));$$

(iii) For $\alpha \in I_k(M)$ and $u \in \mathbb{Z}\pi$, we get

$$\mu(x \cdot \alpha) = \rho(x, \mu(\alpha))$$

where ρ is defined in (9.39)*.*

Proof. (i) Represent $\alpha \in I_k(M)$ by a pointed immersion (f,w) which is in general position. Choose a section σ of $\nu(f)$ which meets the zero section transversally. Note that then the Euler number satisfies

$$\chi(f) = \sum_{y \in N(\sigma)} \epsilon(y)$$

where $N(\sigma)$ is the (finite) set of zero points of σ and $\epsilon(y)$ is a sign that depends on the local orientations. We can arrange that no zero of σ is the preimage of an element in the set of double points D_f of f. Now move f a little bit in the direction of this normal field σ. We obtain a new immersion $g\colon S^k \to M$ with a path v from b to $g(s)$ such that (f, w) and (g, v) are pointed regularly homotopic.

We want to compute $\lambda(\alpha, \alpha)$ using the representatives (f, w) and (g, v). Note that any point in $d \in D_f$ corresponds to two distinct points $x_0(d)$ and $x_1(d)$ in the set $D = \operatorname{im}(f) \cap \operatorname{im}(g)$ and any element $n \in N(\sigma)$ corresponds to one point $x(n)$ in D. Moreover any point in D occurs as $x_i(d)$ or $x(n)$ in a unique way. Now the contribution of d to $\lambda([f, w], [g, v])$ is $\epsilon(d) \cdot g(d) + (-1)^k \cdot \epsilon(d) \cdot \overline{g(d)}$ and the contribution of $n \in N(\sigma)$ is $\epsilon(n) \cdot 1$. Now assertion (i) follows.

(ii) and (iii) The proof of these assertions are left to the reader. $\square$

Remark 9.41 (Selfintersection pairing and $(-1)^k$-quadratic forms). Lemma 9.40 says that the triple $(I_k(M), \lambda, \mu)$ satisfies all the requirements appearing in Remark 9.21 except that $I_k(M)$ may not be finitely generated free over $\mathbb{Z}\pi$ and we have to require $\chi(\alpha) = 0$, which will be satisfied in the cases of interest.

The following theorem of Wall is taken from [968, Theorem 5.2 on page 45].

Theorem 9.42 (Selfintersections and embeddings). *Let M be a connected compact manifold of even dimension $n = 2k$. Fix base points $s \in S^k$ and $b \in M$ and an orientation of T_bM. Let (f, w) be a pointed immersion of S^k in M. Suppose that $k \geq 3$. Then (f, w) is pointed homotopic to a pointed immersion (g, v) for which $g\colon S^k \to M$ is an embedding if and only $\mu(f, w) = 0$.*

Proof. If f is represented by an embedding, then $\mu(f, w) = 0$ by definition. Suppose that $\mu(f, w) = 0$. We can assume without loss of generality that f is in general position. Since $\mu(f, w) = 0$, we can find d and e in the set of double points D_f of f and a numeration $x_0(d), x_1(d)$ of $f^{-1}(d)$ and $x_0(e), x_1(e)$ of $f^{-1}(e)$ such that

$$\begin{aligned}
\epsilon(x_0(d), x_1(d)) &= -\epsilon(x_0(e), x_1(e));\\
g(x_0(d), x_1(d)) &= g(x_0(e), x_1(e)).
\end{aligned}$$

Therefore we can find arcs u_0 and u_1 in S^k such that $u_0(0) = x_0(d)$, $u_0(1) = x_0(e)$, $u_1(0) = x_1(d)$, and $u_1(1) = x_1(e)$ hold, the path u_0 and u_1 are disjoint from one another, $f(u_0((0,1)))$ and $f(u_1((0,1)))$ do not meet D_f, and $f(u_0)$ and $f(u_1)$ are homotopic relative endpoints. We can change u_0 and u_1 without destroying the properties above and find a smooth map $U\colon D^2 \to M$ whose restriction to S^1 is an embedding (ignoring corners on the boundary) and is given on the upper hemisphere S^1_+ by u_0 and on the lower hemisphere S^1_- by u_1 and which meets $\operatorname{im}(f)$ transversally. There is a compact neighborhood K of S^1 such that $U|_K$ is an embedding. Since $k \geq 3$ we can find arbitrarily close to U an embedding which agrees with U on K. Hence we can assume without loss of generality that U itself is an embedding. The Whitney trick, see [710, Theorem 6.6 on page 71], [993], allows to change f within its pointed regular homotopy class to a new pointed immersion (g,v) such that $D_g = D_f - \{d,e\}$ and $\mu(g,v) = 0$. By iterating this process we achieve $D_f = \emptyset$. $\square$

Remark 9.43 (The dimension assumption $\dim(M) \geq 5$). The condition $\dim(M) \geq 5$, which arises in high-dimensional manifold theory, ensures in the proof of Theorem 9.42 that $k \geq 3$ and hence we can arrange U to be an embedding. If $k = 2$, one can achieve that U is an immersion but not necessarily an embedding. This is the technical reason, why surgery in dimension 4 is much more complicated than in dimensions ≥ 5.

Exercise 9.44. Let $f\colon S^k \to M$ be an immersion into a compact $2k$-dimensional manifold. Suppose that it is in general position and the set of double points consists of precisely one element. Show that f is not regular homotopic to an embedding.

Exercise 9.45. Construct an immersion $f\colon M \to N$ of connected closed manifolds which is homotopic but not regularly homotopic to an embedding.

9.5 The Surgery Obstruction in Even Dimensions

We give a brief introduction to the surgery obstruction in even dimension to motivate the relevance of the L-groups for topology. We will use the sign conventions for chain complexes as they appear in [652, Section 14.4].

9.5.1 Poincaré Duality Spaces

Consider a connected finite CW-complex X with fundamental group π and a group homomorphism $w\colon \pi \to \{\pm 1\}$. Below we use the w-twisted involution on $\mathbb{Z}\pi$. Denote by $C_*(\widetilde{X})$ the cellular $\mathbb{Z}\pi$-chain complex of the universal covering. It is a finite free $\mathbb{Z}\pi$-chain complex. The product $\widetilde{X} \times \widetilde{X}$ equipped with the diagonal π-action is again a π-CW-complex. The diagonal map $D\colon \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ sending $\widetilde{x}$ to $(\widetilde{x}, \widetilde{x})$ is π-equivariant but not cellular. By the Equivariant Cellular Approximation Theorem, see for instance [629, Theorem 2.1 on page 32], there is up to cellular π-homotopy precisely one cellular

π-map $\overline{D}\colon \widetilde{X} \to \widetilde{X} \times \widetilde{X}$ which is π-homotopic to D. It induces a $\mathbb{Z}\pi$-chain map unique up to $\mathbb{Z}\pi$-chain homotopy

$$C_*(\overline{D})\colon C_*(\widetilde{X}) \to C_*(\widetilde{X} \times \widetilde{X}). \tag{9.46}$$

There is a natural isomorphism of $\mathbb{Z}\pi$-chain complexes

$$i_*\colon C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(\widetilde{X}) \xrightarrow{\cong} C_*(\widetilde{X} \times \widetilde{X}). \tag{9.47}$$

Definition 9.48 (Dual chain complex). Given an R-chain complex of left R-modules C_* and $n \in \mathbb{Z}$, we define its *dual chain complex* C^{n-*} to be the chain complex of left R-modules whose p-th chain module is $\hom_R(C_{n-p}, R)$ and whose p-th differential is given by

$$\begin{aligned}(-1)^{n-p+1} \cdot \hom_R(c_{n-p+1}, \mathrm{id})\colon (C^{n-*})_p &= \hom_R(C_{n-p}, R) \\ &\to (C^{n-*})_{p-1} = \hom_R(C_{n-p+1}, R).\end{aligned}$$

Denote by $\mathbb{Z}^w$ the $\mathbb{Z}\pi$-module whose underlying abelian group is $\mathbb{Z}$ and on which $g \in \pi$ acts by $w(g) \cdot \mathrm{id}$. Given two projective $\mathbb{Z}\pi$-chain complexes C_* and D_*, we obtain a natural $\mathbb{Z}$-chain map unique up to $\mathbb{Z}$-chain homotopy

$$s\colon \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} (C_* \otimes_{\mathbb{Z}} D_*) \to \hom_{\mathbb{Z}\pi}(C^{-*}, D_*) \tag{9.49}$$

by sending $1 \otimes x \otimes y \in \mathbb{Z} \otimes C_p \otimes D_q$ to

$$s(1 \otimes x \otimes y)\colon \hom_{\mathbb{Z}\pi}(C_p, \mathbb{Z}\pi) \to D_q, \quad (\phi\colon C_p \to \mathbb{Z}\pi) \mapsto (-1)^{|x|\cdot|y|+|x|} \cdot \overline{\phi(x)} \cdot y.$$

The composite of the chain map (9.49) for $C_* = D_* = C_*(\widetilde{X})$, the inverse of the chain map (9.47) tensored with $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} -$, and the chain map (9.46) tensored with $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} -$ yield a $\mathbb{Z}$-chain map

$$\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) \to \hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X})).$$

Note that the n-th homology of $\hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X}))$ is the set of $\mathbb{Z}\pi$-chain homotopy classes $[C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi}$ of $\mathbb{Z}\pi$-chain maps from $C^{n-*}(\widetilde{X})$ to $C_*(\widetilde{X})$. Define $H_n(X; \mathbb{Z}^w) := H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$. Taking the n-th homology group yields a well-defined $\mathbb{Z}$-homomorphism

$$\cap\colon H_n(X; \mathbb{Z}^w) \to [C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi} \tag{9.50}$$

that sends a class $x \in H_n(X; \mathbb{Z}^w) = H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$ to the $\mathbb{Z}\pi$-chain homotopy class of a $\mathbb{Z}\pi$-chain map denoted by $- \cap x\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$.

Definition 9.51 (Poincaré complex). A *connected finite n-dimensional Poincaré complex* is a connected finite CW-complex of dimension n together with a group homomorphism $w = w_1(X)\colon \pi_1(X) \to \{\pm 1\}$ called *orientation* *homomorphism* if there exists an element $[X] \in H_n(X;\mathbb{Z}^w)$ called *fundamental class* such that the $\mathbb{Z}\pi$-chain map $- \cap [X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence. We will call it the *Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence*.

Exercise 9.52. Show that the orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ is uniquely determined by the homotopy type of the finite n-dimensional Poincaré complex X.

Obviously there are two possible choices for $[X]$ since it has to be a generator of the infinite cyclic group $H_n(X,\mathbb{Z}^w) \cong H^0(X;\mathbb{Z}) \cong \mathbb{Z}$. A choice of $[X]$ is called *w-orientation* on X. We call X *w-oriented* if we have chosen a w-orientation.

A map $f\colon Y_1 \to Y_2$ of w-oriented connected Poincaré complexes *has degree one* if $w_1(Y_2) \circ \pi_1(f) = w_1(Y_2)$ and the map $H_n(Y_1,\mathbb{Z}^{w_1(Y_1)}) \to H_n(Y_2,\mathbb{Z}^{w_1(Y_2)})$ induced by f sends $[Y_1]$ to $[Y_2]$.

Theorem 9.53. *Let M be a connected closed manifold of dimension n. Then M carries the structure of a connected finite n-dimensional Poincaré complex.*

For a proof we refer for instance to [968, Theorem 2.1 on page 23].

Below a *w-orientation* of a connected closed manifold M of dimension n is a choice of a generator $[M]$ of the infinite cyclic group $H_n(M;\mathbb{Z}^{w_1(M)})$. We call M *w-oriented* if we have chosen a w-orientation. Note that w-oriented does *not* necessarily mean that $w_1(M)$ is trivial. Following the standard conventions, we say that M is *orientable* if $w_1(M)$ is trivial, and we call M *oriented* if $w_1(M)$ is trivial and we have chosen a fundamental class $[M] \in H_n(M;\mathbb{Z})$.

Remark 9.54 (Poincaré duality as obstruction for being homotopy equivalent to a closed manifold). Theorem 9.53 gives us the first obstruction for a topological space X to be homotopy equivalent to a connected closed n-dimensional manifold. Namely, X must be homotopy equivalent to a connected finite n-dimensional Poincaré complex.

9.5.2 Normal Maps and the Surgery Step

Definition 9.55 (Normal map of degree one). Let X be a w-oriented connected finite n-dimensional Poincaré complex together with an m-dimensional vector bundle $\xi\colon E \to X$. A *normal m-map* $(M,i,f,\overline{f})$ with (X,ξ) as target consists of a w-oriented connected closed manifold M of dimension n together with an embedding $i\colon M \to \mathbb{R}^{n+m}$ and a bundle map $(\overline{f},f)\colon \nu(M) \to \xi$ where $\nu(M)$ denotes the normal bundle $\nu(i)$ of the embedding $i\colon M \to \mathbb{R}^{m+n}$. A *normal map of degree one* is a normal map such that the degree of $f\colon M \to X$ is one.

Remark 9.56. We are somewhat sloppy here since we ignore the problem that the choices of the fundamental classes and the bundle data have to be consistent with one another. This is an issue that has been overlooked at many places. This is explained in detail and fixed in [652, Section 7.4, Example 7.44 on page 217 and Remark 7.45 on page 218]. However, to keep this exposition comprehensible, we ignore this issue and also will not treat the notion of an intrinsic fundamental class of [652, Section 5.5].

Given a normal map $(M, i, f, \overline{f})$ with (X, ξ) as target, we obtain for $k \geq 1$ a normal map $(M, i, f, \overline{f}')$ with $(X, \xi \oplus \underline{\mathbb{R}^k})$ as target as follows. Let $i' \colon M \to \mathbb{R}^{n+m+k}$ be the composite of the embedding $i \colon M \to \mathbb{R}^{n+m}$ with the standard inclusion $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m+k}$. Then $\nu(i')$ is the Whitney sum $\nu(i) \oplus \underline{\mathbb{R}^k}$ where $\underline{\mathbb{R}^k}$ is the trivial k-dimensional bundle. Let $\overline{f}' \colon \nu(i') \to \xi \oplus \underline{\mathbb{R}^k}$ be the stabilization of $\overline{f}$. We call $(M, i', f, \overline{f}')$ a *stabilization* of $(M, i, f, \overline{f})$.

The next result is due to Whitney [992, 993].

Theorem 9.57 (Whitney's Approximation Theorem). *Let M and N be closed manifolds of dimensions m and n. Then any map $f \colon M \to N$ is arbitrarily close and in particular homotopic to an immersion, provided that $2m \leq n$, and arbitrarily close and in particular homotopic to an embedding, provided that $2m < n$.*

Remark 9.58 (Existence of a normal map of degree one as obstruction for being homotopy equivalent to a closed manifold). Given a connected finite n-dimensional Poincaré complex X, the existence of a normal map of degree one with (X, ξ) as target for some vector bundle ξ over X (for some appropriate choice of w-orientations) is necessary for X to be homotopy equivalent to a closed manifold. Namely, if $f \colon M \to X$ is such a homotopy equivalence, choose a homotopy inverse $g \colon X \to M$ and put $\xi = g^*\nu(i)$ for some embedding $i \colon M \subseteq \mathbb{R}^{n+m}$. Such an embedding exists always for $n < m$ by Theorem 9.57. Obviously f can be covered by a bundle map $\overline{f} \colon \nu(M) \to \xi$ and f has degree one (for some appropriate choice of w-orientations).

Note that an orientation of a compact manifold W induces an orientation of its boundary ∂W, see for instance [652, Remark 5.37 on page 121]. In the special case $W = M \times [0, 1]$ for closed M, the induced orientations on $M = M \times \{0\}$ and $M = M \times \{1\}$ are inverse to one another.

Definition 9.59 (Normal bordism). Consider two normal maps of degree one $(M_k, i_k, f_k, \overline{f}_k)$ with the same target (X, ξ) for $k = 0, 1$. A *normal bordism* from $(\overline{f_0}, f_0)$ to $(\overline{f_1}, f_1)$ consists of a w-oriented connected compact manifold W with boundary ∂W, an embedding $j \colon (W, \partial W) \to (\mathbb{R}^{n+m} \times [0, 1], \mathbb{R}^{n+m} \times \{0, 1\})$, a map $(F, \partial F) \colon (W, \partial W) \to (X \times [0, 1], X \times \{0, 1\})$ of degree one covered by a bundle map $\overline{F} \colon \nu(W) \to \xi$, and an orientation preserving diffeomorphism $u \colon \partial W \xrightarrow{\cong} M_0 \amalg M_1$ satisfying the obvious compatibility conditions.

We call $(M_0, i_0, f_0, \overline{f}_0)$ and $(M_1, i_1, f_1, \overline{f}_1)$ *normally bordant* if after stabilization there exists a normal bordism between them.

Note Definition 9.59 corresponds in [652] to the notion of a normal bordism with cylindrical ends, see [652, Definition 716 on page 206].

Exercise 9.60. Let $(M, i_0, f, \overline{f}_0)$ be a normal map of degree one with target (X, ξ). Let $i_1 \colon M \to \mathbb{R}^{n+k}$ be an embedding. Show that there exists a normal map of degree one $(M, i_1, f, \overline{f}_1)$ with target (X, ξ) which is normally bordant to $(M, i_0, f, \overline{f}_0)$.

Below we will often suppress the embedding $i \colon M \to \mathbb{R}^{n+m}$ in the notation.

9.5.3 The Surgery Step

So the question is whether we can modify a normal map of degree one with (X, ξ) as target (without changing the target) so that the underlying map f is a homotopy equivalence. There is a procedure in the world of CW-complexes to turn a map into a weak homotopy equivalence, namely, by attaching cells. If $f \colon Y_1 \to Y_2$ is already k-connected, we can attach $(k+1)$ cells to Y_1 to obtain an extension $f' \colon Y_1' \to Y_2$ of f which is $(k+1)$-connected. In principle we want to do the same for a normal map of degree one with target (X, ξ). However, there are two fundamental difficulties. First of all we have to keep the manifold structure on the source and cannot therefore just attach cells. Moreover, by Poincaré duality any modification in dimension k will cause a dual modification in dimension $n-k$ if n is the dimension of X so that one encounters at any rate problems when n happens to be $2k$.

Consider a normal map $(\overline{f}, f) \colon \nu(M) \to \xi$ such that $f \colon M \to X$ is a k-connected map. Consider an element $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d, "j"'] & M \arrow[d, "f"] \\ D^{k+1} \arrow[r, "Q"'] & X. \end{tikzcd}$$

We cannot attach a single cell to M without destroying the manifold structure. But one can glue two manifolds together along a common boundary such that the result is a manifold. Suppose that the map $q \colon S^k \to M$ extends to an embedding $\overline{q} \colon S^k \times D^{n-k} \to M$. (This assumption will be justified later.) Let $\operatorname{int}(\operatorname{im}(\overline{q}))$ be the interior of the image of $\overline{q}$. Then $M - \operatorname{int}(\operatorname{im}(\overline{q}))$ is a manifold with boundary $\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})$. We can get rid of the boundary by attaching $D^{k+1} \times S^{n-k-1}$ along $\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})$. Call the result

$$M' := D^{k+1} \times S^{n-k-1} \cup_{\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})} (M - \operatorname{int}(\operatorname{im}(\overline{q}))) .$$

Here and elsewhere we apply without further mentioning the technique of straightening the angle in order to get a well-defined smooth structure, see [157, Definition 13.11 on page 145 and (13.12) on page 148] and [480, Chapter 8, Section 2]. Choose a map $\overline{Q}\colon D^{k+1}\times D^{n-k}\to X$ which extends Q and $f\circ\overline{q}$. The restriction of f to $M-\mathrm{int}(\mathrm{im}(\overline{q}))$ extends to a map $f'\colon M'\to X$ using $\overline{Q}|_{D^{k+1}\times S^{n-k}}$. Note that the inclusion $M-\mathrm{int}(\mathrm{im}(\overline{q}))\to M$ is $(n-k-1)$-connected since $S^k\times S^{n-k-1}\to S^k\times D^{n-k}$ is $(n-k-1)$-connected. So the passage from M to $M-\mathrm{int}(\mathrm{im}(\overline{q}))$ will not affect $\pi_j(f)$ for $j<n-k-1$. All in all we see that $\pi_l(f)=\pi_l(f')$ for $l\le k$ and that there is an epimorphism $\pi_{k+1}(f)\to\pi_{k+1}(f')$ whose kernel contains ω, provided that $2(k+1)\le n$. The condition $2(k+1)\le n$ can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality forces also a change in dimension $(n-l)$. This phenomenon will cause surgery obstructions to appear.

Note that $f\colon M\to X$ and $f'\colon M'\to X$ are bordant. The relevant bordism is given by $W=D^{k+1}\times D^{n-k}\cup_{\overline{q}} M\times[0,1]$ where we think of $\overline{q}$ as an embedding $S^k\times D^{n-k}\to M\times\{1\}$. In other words, W is obtained from $M\times[0,1]$ by attaching a handle $D^{k+1}\times D^{n-k}$ to $M\times\{1\}$. Then M appears in W as $M\times\{0\}$ and M' as other part of the boundary of W. Define $F\colon W\to X$ by $f\times\mathrm{id}_{[0,1]}$ and $\overline{Q}$. Then F restricted to M and M' is f and f'.

Why can we assume that the map $q\colon S^k\to M$ extends to an embedding $\overline{q}\colon S^k\times D^{n-k}\to M$? This will be ensured by the bundle data in the case $2k+1<n$ by the following argument.

Because of Theorem 9.57 we can arrange that q is an embedding. The extension $\overline{q}$ exists if and only if the normal bundle $\nu(q)$ of the embedding $q\colon S^k\to M$ is trivial. Since D^{k+1} is contractible, every vector bundle over D^{k+1} is trivial. Hence $Q^*\xi$ is a trivial vector bundle over D^{k+1}. Recall that $i\colon M\to\mathbb{R}^{m+n}$ is a fixed embedding and $\nu(M)$ is define to be the normal bundle $\nu(i)$ of i. Pullbacks of trivial vector bundles are trivial again. This implies that $q^*\nu(M)\cong q^*f^*\xi\cong j^*Q^*\xi$ is a trivial vector bundle over S^k. Since $\nu(q)\oplus q^*\nu(M)\cong\nu(i\colon S^k\to\mathbb{R}^{n+m})$ is trivial, $\nu(q)$ is a stably trivial $(n-k)$-dimensional vector bundle over S^k. Since $2k+1\le n$, this implies that $\nu(q)$ itself is trivial.

So we see that the bundle data are needed to carry out the desired surgery step. Note that the construction yields a map $f'\colon M'\to X$ of degree one and a bundle map $\overline{f'}\colon\nu(M')\to\xi$ covering f' so that we end up with a normal map of degree one with target X again. Hence we are able to repeat this surgery step over and over again in dimensions $2k-1\le n$. Actually, also the bordism W together with the map $F\colon W\to X$ come by a bundle map $\overline{F}\colon\nu(W)\to\xi$ covering F and is therefore a normal bordism in the sense of Definition 9.59. In particular surgery does not change the normal bordism class.

For the proof of the next lemma we refer for instance to [652, Theorem 7.41 on page 217].

Lemma 9.61. *Consider a normal map of degree one* $(\overline{f}, f)\colon \nu(M) \to \xi$ *covering* $f\colon M \to X$ *where* M *is a* w*-oriented connected closed manifold of dimension* n *and* X *is a connected finite Poincaré complex of dimension* n*. Let* k *be the natural number given by* $n = 2k$ *or* $n = 2k+1$*.*

Then we can always arrange by finitely many surgery steps that for the resulting normal map of degree one $(\overline{f'}, f')\colon \nu(M') \to \xi$ *its underlying map* $f'\colon M' \to X$ *is* k*-connected.*

Now assume that n is even, let us say $n = 2k$. As mentioned above, we can arrange that f is k-connected. If we can achieve that f is $(k+1)$-connected, then by Poincaré duality the map f is a homotopy equivalence.

But in this last step we encounter a problem which actually leads to the surgery obstruction in the even dimensional case. Namely, in the argument above we used at one point that the map $q\colon S^k \to M$ can be arranged to be an embedding by general position if $2k+1 \le n$ and that certain normal bundle are trivial. In the situation $n = 2k$ we can arrange q to be an immersion by Theorem 9.57 and simultaneously ensure that the bundle data carry over to the desired normal bordism, essentially, because of a systematic use of Theorem 9.63 below. However, the latter fixes the regular homotopy class of the immersion q. Hence one open problem is to ensure that we can change q to an embedding within its regular homotopy class. We have already introduced the main obstruction for that, the selfintersection element in (9.38). We also encounter the problem that by Poincaré duality any change in the homology of the middle dimension comes with a dual change and one has to ensure that there two have the desired effect and do not disturb one another. Next we explain how this leads to the so-called surgery obstruction in $L_{2k}(\mathbb{Z}\pi_1(X))$ with respect to the $w_1(X)$-twisted involution on $\mathbb{Z}\pi$.

9.5.4 The Even Dimensional Surgery Obstruction

For the rest of this subsection we fix a normal map $(\overline{f}, f)\colon \nu(M) \to \xi$ of degree one covering $f\colon M \to X$ where M is a w-oriented connected closed manifold of dimension n and X is a w-oriented connected finite Poincaré complex of dimension n. Suppose that f induces an isomorphism on the fundamental groups. Fix a base point $b \in M$ together with lifts $\widetilde{b} \in \widetilde{M}$ of b and $\widetilde{f(b)} \in \widetilde{X}$ of $f(b)$. We identify $\pi = \pi_1(M,b) = \pi(X, f(b))$ by $\pi_1(f,b)$. The choices of $\widetilde{b}$ and $\widetilde{f(b)}$ determine π-operations on $\widetilde{M}$ and on $\widetilde{X}$ and a lift $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$ which is π-equivariant.

Definition 9.62 (Surgery kernels). Let $K_k(\widetilde{M})$ be the kernel of the $\mathbb{Z}\pi$-map $H_k(\widetilde{f})\colon H_k(\widetilde{M}) \to H_k(\widetilde{X})$. Denote by $K^k(\widetilde{M})$ the cokernel of the $\mathbb{Z}\pi$-map $H^k(\widetilde{f})\colon H^k(\widetilde{X}) \to H^k(\widetilde{M})$ which is the $\mathbb{Z}\pi$-map induced by $C^*(\widetilde{f})\colon C^*(\widetilde{X}) \to C^*(\widetilde{M})$. We call $K_k(\widetilde{M})$ the *surgery kernel.*

Given two vector bundles $\xi\colon E \to M$ and $\eta\colon F \to N$, we have so far only considered bundle maps $(\overline{f}, f)\colon \xi \to \eta$ which are fiberwise isomorphisms. We need to consider, at least for the next theorem, more generally bundle monomorphisms, i.e., we only will require that the map is fiberwise injective. Consider two bundle monomorphism $(\overline{f}_0, f_0), (\overline{f}_1, f_1)\colon \xi \to \eta$. Let $\xi \times [0,1]$ be the vector bundle $\xi \times \mathrm{id}\colon E \times [0,1] \to M \times [0,1]$. A homotopy of bundle monomorphisms $(\overline{h}, h)$ from $(\overline{f}_0, f_0)$ to $(\overline{f}_1, f_1)$ is a bundle monomorphism $(\overline{h}, h)\colon \xi \times [0,1] \to \eta$ whose restriction to $X \times \{j\}$ is $(\overline{f}_j, f_j)$ for $j = 0, 1$. Denote by $\pi_0(\mathrm{Mono}(\xi, \eta))$ the set of homotopy classes of bundle monomorphisms.

For a proof of the following result we refer to Haefliger-Poenaru [434], Hirsch [479], and Smale [903]. Denote by $\pi_0(\mathrm{Imm}(M, N))$ the set of regular homotopy classes of immersions from M to N.

Theorem 9.63 (Immersions and Bundle Monomorphisms). *Let M be an m-dimensional and N be an n-dimensional closed manifold.*

(i) Suppose that $1 \le m < n$. Then taking the differential of an immersion yields a bijection

$$T\colon \pi_0(\mathrm{Imm}(M, N)) \xrightarrow{\cong} \pi_0(\mathrm{Mono}(TM, TN));$$

(ii) Suppose that $1 \le m \le n$ and that M has a handlebody decomposition consisting of q-handles for $q \le n - 2$. Then taking the differential of an immersion yields a bijection

$$T\colon \pi_0(\mathrm{Imm}(M, N)) \xrightarrow{\cong} \operatorname{colim}_{a \to \infty} \pi_0(\mathrm{Mono}(TM \oplus \underline{\mathbb{R}^a}, TN \oplus \underline{\mathbb{R}^a}))$$

where the colimit is given by stabilization.

Lemma 9.64. *(i) The cap product with $[M]$ induces isomorphisms*

$$- \cap [M]\colon K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

(ii) Suppose that f is k-connected. Then there is the composite of natural $\mathbb{Z}\pi$-isomorphisms

$$h_k\colon \pi_{k+1}(f) \xrightarrow{\cong} \pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M});$$

(iii) Suppose that f is k-connected and $n = 2k$. Then there is a natural $\mathbb{Z}\pi$-homomorphism

$$t_k\colon \pi_{k+1}(f) \to I_k(M).$$

Proof. (i) The following diagram commutes and has isomorphisms as vertical arrows

(9.65)
$$\begin{tikzcd} H^{n-k}(\widetilde{M}) \arrow[d, "-\cap[M]"', "\cong"] & H^{n-k}(\widetilde{X}) \arrow[l, "H^{n-k}(\widetilde{f})"'] \\ H_k(\widetilde{M}) \arrow[r, "H_k(\widetilde{f})"'] & H_k(\widetilde{X}). \arrow[u, "\cong", "-\cap[X]"'] \end{tikzcd}$$

Hence the composite $K_k(\widetilde{M}) \to H_k(\widetilde{M}) \xrightarrow{(-\cap[M])^{-1}} H^{n-k}(\widetilde{M}) \to K^{n-k}(\widetilde{M})$ is bijective.

(ii) The commutative square (9.65) above implies that $H_l(\widetilde{f})\colon H_l(\widetilde{M}) \to H_l(\widetilde{X})$ is split surjective for all l. We conclude from the long exact sequence of $C_*(\widetilde{f})$ that the boundary map

$$\partial\colon H_{k+1}(\widetilde{f}) := H_{k+1}(\operatorname{cone}(C_*(\widetilde{f}))) \to H_k(\widetilde{M})$$

induces an isomorphism

$$\partial_{k+1}\colon H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M}).$$

Since f and hence $\widetilde{f}$ is k-connected, the Hurewicz homomorphism

$$\pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f})$$

is bijective [987, Corollary IV.7.10 on on page 181]. The canonical map

$$\pi_{k+1}(\widetilde{f}) \to \pi_{k+1}(f)$$

is bijective. The composite of the maps above or their inverses yields a natural isomorphism $h_k\colon \pi_{k+1}(f) \to K_k(\widetilde{M})$.

(iii) Note that an element in $\pi_{k+1}(f, b)$ is given by a commutative diagram

$$\begin{tikzcd} S^k \arrow[r, "q"] \arrow[d] & M \arrow[d] \\ D^{k+1} \arrow[r, "Q"'] & X \end{tikzcd}$$

together with a path w from b to $f(s)$ for a fixed base point $s \in S^k$. We leave the details of the rest of the proof, which is based on Theorem 9.63 (ii), to the reader. The necessary bundle monomorphisms come from the bundle data of $(\overline{f}, f)$, the stable triviality of TS^k, and the fact that any vector bundle over D^{k+1} is trivial. $\square$

Suppose that $n = 2k$. The Kronecker pairing $\langle\ ,\ \rangle\colon H^k(\widetilde{M}) \times H_k(\widetilde{M}) \to \mathbb{Z}\pi$ is induced by the evaluation pairing $\hom_{\mathbb{Z}\pi}(C_p(\widetilde{M}), \mathbb{Z}\pi) \times C_p(\widetilde{M}) \to \mathbb{Z}\pi$ which sends (β, x) to $\beta(x)$. It induces a pairing

$$\langle\ ,\ \rangle\colon K^k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Together with the isomorphism

$$- \cap [M]\colon K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M})$$

of Theorem 9.64 (i) it yields the *intersection pairing*

$$s\colon K_k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi. \tag{9.66}$$

We get from Lemma 9.64 (ii) and (iii) a $\mathbb{Z}\pi$-homomorphism

$$\alpha\colon K_k(\widetilde{M}) \to I_k(\widetilde{M}). \tag{9.67}$$

The elementary proof of the next lemma is left to the reader.

Lemma 9.68. *The following diagram commutes*

$$\begin{tikzcd}
K_k(\widetilde{M}) \times K_k(\widetilde{M}) \arrow[r, "s"] \arrow[d, "\alpha\times\alpha"'] & \mathbb{Z}\pi \arrow[d, "\mathrm{id}"] \\
I_k(M) \times I_k(M) \arrow[r, "\lambda"'] & \mathbb{Z}\pi
\end{tikzcd}$$

where the upper pairing is defined in (9.66)*, the lower pairing in* (9.33)*, and the left vertical arrows in* (9.67)*.*

Exercise 9.69. Let $f\colon X \to Y$ be a map of connected finite Poincaré complexes of dimension $n \geq 4$. Suppose that f has degree one and that f is $(k+1)$-connected where k is given by $n = 2k$ if n is even, and by $n = 2k+1$ if n is odd. Show that then f is a homotopy equivalence.

Recall that an R-module V is called *stably finitely generated free* if for some non-negative integer l the R-module $V \oplus R^l$ is a finitely generated free R-module.

Lemma 9.70. *If $f\colon X \to Y$ is k-connected for $n = 2k$ or $n = 2k+1$, then $K_k(\widetilde{M})$ is stably finitely generated free.*

Proof. See for instance [652, Lemma 8.55 (ii) on page 252]. □

Example 9.71 (Effect of trivial surgery). Consider the normal map $(\overline{f}, f)\colon \nu(M) \to \xi$ covering the k-connected map of degree one $f\colon M \to X$ for a w-oriented connected closed n-dimensional manifolds M for $n = 2k$. If we do surgery on the zero element in $\pi_{k+1}(f)$, then the effect on M is that M is replaced by the connected sum $M' = M\sharp(S^k \times S^k)$. The effect on $K_k(\widetilde{M})$ is that it is replaced by $K_k(\widetilde{M'}) = K_k(\widetilde{M}) \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)$. The intersection pairing on this new kernel is the sum of the given intersection pairing on $K_k(\widetilde{M})$ together with the standard hyperbolic symmetric form $H^{(-1)^k}(\mathbb{Z}\pi)$. Moreover, taking the selfintersections into account, the non-singular $(-1)^k$-quadratic form on the new kernel is direct sum of the one of the old kernel and the standard hyperbolic $(-1)^k$-quadratic form $H_{(-1)^k}(\mathbb{Z}\pi)$. In particular we can arrange by finitely many surgery steps on the trivial element in $\pi_{k+1}(f)$ that $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module.

Remark 9.72. Let $(\overline{f}, f)\colon \nu(M) \to \xi$ be a normal map of degree one covering $f\colon M \to X$ where M is a w-oriented connected closed manifold of dimension n and X is a w-oriented connected finite Poincaré complex of dimension n. Suppose that $n = 2k$ and f is k-connected.

By Lemma 9.70 and Example 9.71, we can do finitely many trivial surgery steps to achieve that the kernel $K_k(\widetilde{M})$ is a finitely generated free $\mathbb{Z}\pi$-module. By the intersection pairing s of (9.66), we obtain a non-singular $(-1)^k$-symmetric form $(K_k(\widetilde{M}), s)$, see Remark 9.8.

So far we have not used the bundle data. They come now into play, when we want to refine $(K_k(\widetilde{M}), s)$ to a non-singular $(-1)^k$-quadratic form. Because of Remark 9.21 we have to specify a map $t\colon K_k(\widetilde{M}) \to Q_{(-1)^k}(\mathbb{Z}\pi)$. We will take the composite

$$K_k(\widetilde{M}) \xrightarrow{h_k^{-1}} \pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi)$$

where μ has been defined (9.38) and the isomorphism h_k and the map t_k have been introduced in Lemma 9.64. This is indeed a quadratic refinement by Lemma 9.40 and Lemma 9.68.

Definition 9.73 (Even dimensional surgery obstruction). Consider a normal map of degree one $(\overline{f}, f)\colon \nu(M) \to \xi$ covering $f\colon M \to X$ where M is a w-oriented connected closed manifold of even dimension $n = k$ and X is a connected finite Poincaré complex of dimension n with fundamental group π. Perform surgery below the middle dimension and trivial surgery in the middle dimension so that we obtain a k-connected normal map of degree one $(\overline{f'}, f')\colon \nu(M) \to \xi$ such that $K_k(\widetilde{M'})$ is finitely generated free $\mathbb{Z}\pi$-module. Define the *surgery obstruction* of $(\overline{f}, f)\colon \nu(M) \to \xi$

$$\sigma(\overline{f}, f) \in L_{2k}(\mathbb{Z}\pi, w_1(X))$$

by the class of the non-singular $(-1)^k$-quadratic form $(K_k(\widetilde{M'}), s, t)$ of Remark 9.72.

We omit the proof that this element is well-defined, e.g., independent of the previous surgery steps; details of the proof can be found in [652, Section 8.6.3]

Theorem 9.74 (Surgery obstruction in even dimensions). *Consider a normal map of degree one $(\overline{f}, f)\colon \nu(M) \to \xi$ covering $f\colon M \to X$ where*

M is a w-oriented connected closed manifold of even dimension $n = 2k$ and X is a w-oriented connected finite Poincaré complex of dimension n with fundamental group π. Then:

(i) Suppose $k \geq 3$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w_1(X))$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f')\colon \nu(M') \to \xi$ which covers a homotopy equivalence $f'\colon M' \to X$;
(ii) The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.

Proof. We only give the proof of assertion (i). More details can be found in [652, Theorem 8.112 on page 274] or [968, Chapter 5]. By Lemma 9.61, Example 9.71, and the definition of $L_{2k}(\mathbb{Z}\pi, w)$, we can arrange by finitely many surgery steps that the non-singular $(-1)^k$-quadratic form $(K_k(\widetilde{M}), s, t)$ is isomorphic to $H_{(-1)^k}(\mathbb{Z}\pi^v)$. Thus we can choose for some natural number v a $\mathbb{Z}\pi$-basis $\{b_1, b_2, \ldots, b_v, c_1, c_2, \ldots, c_v\}$ for $K_k(\widetilde{M})$ such that

$$\begin{array}{lll} s(b_i, c_i) &= 1 & i \in \{1, 2, \ldots, v\}; \\ s(b_i, c_j) &= 0 & i, j \in \{1, 2, \ldots, v\}, i \neq j; \\ s(b_i, b_j) &= 0 & i, j \in \{1, 2, \ldots, v\}; \\ s(c_i, c_j) &= 0 & i, j \in \{1, 2, \ldots, v\}; \\ t(b_i) &= 0 & i \in \{1, 2, \ldots, v\}. \end{array}$$

Note that f is a homotopy equivalence if and only if the number v is zero. Hence it suffices to explain how we can lower the number v to $(v-1)$ by a surgery step on an element in $\pi_{k+1}(f)$. Of course our candidate is the element ω in $\pi_{k+1}(f)$ which corresponds under the isomorphism $h\colon \pi_{k+1}(f) \to K_k(\widetilde{M})$, see Lemma 9.64 (ii), to the element b_v. By construction the composite

$$\pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w)$$

of the maps defined in (9.38) and Lemma 9.64 (iii) sends ω to zero. Now Theorem 9.42 ensures that we can perform surgery on ω. Note that the assumption $k \geq 3$ and the quadratic structure on the kernel become relevant exactly at this point. Finally it remains to check whether the effect on $K_k(\widetilde{M})$ is the desired one, namely, that we get rid of one of the hyperbolic summands $H_\epsilon(\mathbb{Z}\pi)$, or equivalently, v is lowered to $v-1$.

We have explained earlier that doing surgery yields not only a new manifold M', but also a bordism from M to M'. Namely, take $W = M \times [0,1] \cup_{S^k \times D^{n-k}} D^{k+1} \times D^{n-k}$ where we attach $D^{k+1} \times D^{n-k}$ by an embedding $S^k \times D^{n-k} \to M \times \{1\}$, and $M' := \partial W - M$ using the identification $M = M \times \{0\}$. The manifold W comes with a map $F\colon W \to X \times [0,1]$ whose restriction to M is the given map $f\colon M = M \times \{0\} \to X = X \times \{0\}$ and whose restriction to M' is a map $f'\colon M' \to X \times \{1\}$. The definition of the kernels makes also sense for pair of maps. We obtain an exact braid

$$
\begin{tikzcd}[column sep=tiny]
0 \arrow[rr, bend left] \arrow[dr] & & K_{k+1}(\widetilde{W},\widetilde{M}) \arrow[rr, bend left] \arrow[dr] & & K_k(\widetilde{M}) \arrow[rr, bend left] \arrow[dr] & & K_k(\widetilde{W},\widetilde{M'}) \arrow[rr, bend left] \arrow[dr] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & K_{k+1}(\widetilde{W},\widetilde{\partial W}) \arrow[ur] \arrow[dr] & & K_k(\widetilde{W}) \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
& & 0 \arrow[ur] \arrow[rr, bend right] & & K_k(\widetilde{M'}) \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] & &
\end{tikzcd}
$$

which combines the various long exact sequences of pairs.

The $(k+1)$-handle $D^{k+1} \times D^{n-k}$ defines an element ϕ^{k+1} in $K_{k+1}(\widetilde{W},\widetilde{M})$ and the associated dual k-handle defines an element $\psi^k \in K_k(\widetilde{W},\widetilde{M'})$. These elements constitute a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{M}) \cong \mathbb{Z}\pi$ and $K_k(\widetilde{W},\widetilde{M'}) \cong \mathbb{Z}\pi$. The $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_k(\widetilde{M})$ maps ϕ to b_v. The $\mathbb{Z}\pi$-homomorphism $K_k(\widetilde{M}) \to K_k(\widetilde{W},\widetilde{M'})$ sends x to $s(b_v, x) \cdot \psi^k$. Hence we can find elements b'_1, b'_2, $\ldots$, b'_v and c'_1, c'_2, $\ldots$, c'_{v-1} in $K_{k+1}(\widetilde{W},\widetilde{\partial W})$ uniquely determined by the property that b'_i is mapped to b_i and c'_i to c_i under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M})$. Moreover, these elements form a $\mathbb{Z}\pi$-basis for $K_{k+1}(\widetilde{W},\widetilde{\partial W})$, and the element ϕ^{k+1} is mapped to b'_v under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{M}) \to K_{k+1}(\widetilde{W},\widetilde{\partial W})$. Define b''_i and c''_i for $i = 1, 2, \ldots, (v-1)$ to be the image of b'_i and c'_i under the $\mathbb{Z}\pi$-homomorphism $K_{k+1}(\widetilde{W},\widetilde{\partial W}) \to K_k(\widetilde{M'})$. Then

$$\{b''_i \mid i = 1, 2 \ldots, (v-1)\} \coprod \{c''_i \mid i = 1, 2 \ldots, (v-1)\}$$

is a $\mathbb{Z}\pi$-basis for $K_k(\widetilde{M'})$. One easily checks for the quadratic structure (s', t') on $K_k(\widetilde{M'})$

$$
\begin{array}{llll}
s'(b''_i, c''_i) = s(b_i, c_i) & = 1 & & i \in \{1, 2, \ldots, (v-1)\}; \\
s'(b''_i, c''_j) = s(b_i, c_j) & = 0 & & i, j \in \{1, 2, \ldots, (v-1)\}, i \neq j; \\
s'(b''_i, b''_j) = s(b_i, b_j) & = 0 & & i, j \in \{1, 2, \ldots, (v-1)\}; \\
s'(c''_i, c''_j) = s(c_i, c_j) & = 0 & & i, j \in \{1, 2, \ldots, (v-1)\}; \\
t'(b''_i) \quad\;\; = t(b_i) & = 0 & & i \in \{1, 2, \ldots, (v-1)\}.
\end{array}
$$

This finishes the proof of assertion (i) of Theorem 9.74. $\square$

Exercise 9.75. Let M be a stably framed manifold of dimension $(4k+2)$, i.e., a closed $(4k+2)$-dimensional manifold together with a choice of a stable trivialization of its tangent bundle. Assign to it an element $\alpha(M) \in \mathbb{Z}/2$ such that $\alpha(M) = \alpha(N)$ depends only on the stably framed bordism class of M. (The easy solutions that α is constant is not what we have in mind).

9.6 Formations

In this subsection we explain the algebraic objects, so-called formations, which describe the surgery obstruction and will be the typical elements in the surgery obstruction group in odd dimensions. Throughout this section R will be an associative ring with involution and $\epsilon \in \{\pm 1\}$.

Definition 9.76 (Formation). An *ϵ-quadratic formation* $(P, \psi; F, G)$ is a non-singular ϵ-quadratic form (P, ψ) together with two lagrangians F and G.

An isomorphism $f\colon (P, \psi; F, G) \to (P', \psi'; F,', G')$ of ϵ-quadratic formations is an isomorphism $f\colon (P, \psi) \to (P', \psi')$ of non-singular ϵ-quadratic forms such that $f(F) = F'$ and $f(G) = G'$ holds.

Definition 9.77 (Trivial formation). The *trivial ϵ-quadratic formation* associated to a finitely generated projective R-module P is the formation $(H_\epsilon(P); P, P^*)$. A formation $(P, \psi; F, G)$ is called *trivial* if it isomorphic to the trivial ϵ-quadratic formation associated to some finitely generated projective R-module. Two formations are *stably isomorphic* if they become isomorphic after taking the direct with trivial formations.

Remark 9.78 (Formations and automorphisms). We conclude from Lemma 9.26 that any ϵ-quadratic formation is isomorphic to an ϵ-quadratic formation of the type $(H_\epsilon(P); P, F)$ for some lagrangian $F \subset P \oplus P^*$. Given an automorphism $v\colon H_\epsilon(P) \xrightarrow{\cong} H_\epsilon(P)$ of the standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$ for some finitely generated projective R-module P, we get a formation by $(H_\epsilon(P); P, v(P))$.

Consider an ϵ-quadratic formation $(P, \psi; F, G)$ such that P, F, and G are finitely generated free and suppose that R has the property that R^n and R^m are R-isomorphic if and only if $n = m$. Then $(P, \psi; F, G)$ is stably isomorphic to $(H_\epsilon(Q); Q, v(Q))$ for some finitely generated free R-module Q and automorphism v of $H_\epsilon(Q)$ by the following argument. Because of Lemma 9.26 we can choose isomorphisms of non-singular ϵ-quadratic forms $f\colon H_\epsilon(F) \xrightarrow{\cong} (P, \psi)$ and $g\colon H_\epsilon(G) \xrightarrow{\cong} (P, \psi)$ such that $f(F) = F$ and $g(G) = G$. Since $F \cong R^a$ and $G \cong R^b$ by assumption and $R^{2a} \cong F \oplus F^* \cong P \cong G \oplus G^* \cong R^{2b}$, we conclude $a = b$. Hence we can choose an R-isomorphism $u\colon F \to G$. Then we obtain an automorphism of non-singular ϵ-quadratic forms by the composite

$$v\colon H_\epsilon(F) \xrightarrow{H_\epsilon(u)} H_\epsilon(G) \xrightarrow{g} (P, \psi) \xrightarrow{f^{-1}} H_\epsilon(F)$$

and an isomorphism of ϵ-quadratic formations

$$f\colon (H_\epsilon(F); F, v(F)) \xrightarrow{\cong} (P, \psi; F, G).$$

Recall that $K_1(R)$ is defined in terms of automorphisms of finitely generated free R-modules. Hence it is plausible that the odd dimensional L-groups will be defined in terms of formations, which is essentially the same as in terms of automorphisms of the standard hyperbolic form over a finitely generated free R-module.

Definition 9.79 (Boundary formation). Let (P, ψ) be a (not necessarily non-singular) $(-\epsilon)$-quadratic form. Define its *boundary* $\partial(P, \psi)$ to be the ϵ-quadratic formation $(H_\epsilon(P); P, \Gamma_\psi)$ where Γ_ψ is the lagrangian given by the image of the R-homomorphism

$$P \to P \oplus P^*, \qquad x \mapsto (x, (1 - \epsilon \cdot T)(\psi)(x)) .$$

One easily checks that Γ_ψ appearing in Definition 9.79 is indeed a lagrangian. Two lagrangians F, G of a non-singular ϵ-quadratic form (P, ψ) are called *complementary* if $F \cap G = \{0\}$ and $F + G = P$.

Lemma 9.80. *Let $(P, \psi; F, G)$ be an ϵ-quadratic formation. Then:*

(i) *$(P, \psi; F, G)$ is trivial if and only F and G are complementary to one another;*
(ii) *$(P, \psi; F, G)$ is isomorphic to a boundary if and only if there is a lagrangian $L \subset P$ such that L is a complement of both F and G;*
(iii) *There is an ϵ-quadratic formation $(P', \psi'; F', G')$ such that $(P, \psi; F, G) \oplus (P', \psi'; F', G')$ is a boundary;*
(iv) *An $(-\epsilon)$-quadratic form (Q, μ) is non-singular if and only if its boundary is trivial.*

Proof. See for instance [652, Lemma 9.13 on page 337]. □

9.7 Odd Dimensional *L*-groups

Now we can define the odd dimensional L-groups.

Definition 9.81 (Odd dimensional L-groups). Let R be an associative ring with involution. For an odd integer $n = 2k + 1$ define the abelian group $L_n(R)$, called the *n-th quadratic L-group*, of R to be the abelian group of equivalence classes $[P, \psi; F, G]$ of $(-1)^k$-quadratic formations $(P, \psi; F, G)$ such that P, F and G are finitely generated free R-modules with respect to the following equivalence relation. We call $(P, \psi; F, G)$ and $(P', \psi'; F', G')$ equivalent if and only if there exist $(-(-1)^k)$-quadratic forms (Q, μ) and (Q', μ') for finitely generated free R-modules Q and Q' and finitely generated free R-modules S and S' together with an isomorphism of $(-1)^k$-quadratic formations

$$\begin{aligned}(P, \psi; F, G) &\oplus \partial(Q, \mu) \oplus (H_\epsilon(S); S, S^*) \\ &\cong (P', \psi'; F', G') \oplus \partial(Q', \mu') \oplus (H_\epsilon(S'); S', (S')^*).\end{aligned}$$

Addition is given by the sum of two $(-1)^k$-quadratic formations. The zero element is represented by $\partial(Q,\mu) \oplus (H_{(-1)^k}(S); S, S^*)$ for any $(-(-1)^k)$-quadratic form (Q,μ) for any finitely generated free R-module Q and any finitely generated free R-module S. The inverse of $[P,\psi;F,G]$ is represented by $(P,-\psi;F',G')$ for any choice of lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary.

A morphism $u\colon R \to S$ of rings with involution induces homomorphisms $u_*\colon L_k(R) \to L_k(S)$ for $k = 1,3$ by induction satisfying $(u \circ v)_* = u_* \circ v_*$ and $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$ for $k = 1,3$.

Theorem 9.82 (Vanishing of the odd dimensional L-groups of the ring of integers). *We have $L_{2k+1}(\mathbb{Z}) = 0$ for all $k \in \mathbb{Z}$.*

Proof. See for instance [652, Subsection 9.2.4]. □

Remark 9.83 (Four-periodicity of the L-groups). Obviously the L-groups are four-periodic, i.e., $L_n(R) = L_{n+4k}(R)$ holds for all $k,n \in \mathbb{Z}$.

9.8 The Surgery Obstruction in Odd Dimensions

Next we very briefly treat the odd dimensional surgery obstruction. Consider a normal map of degree one $(\overline{f}, f)\colon \nu(M) \to \xi$ covering $f\colon M \to X$ where M is a w-oriented closed manifold of dimension n and X is a w-oriented connected finite Poincaré complex of dimension n for odd $n = 2k+1$. Put $\pi = \pi_1(X)$. To these data one can assign the *surgery obstruction* of $(\overline{f}, f)$

$$\sigma(\overline{f}, f) \in L_{2k+1}(\mathbb{Z}\pi, w), \tag{9.84}$$

Its construction and the proof of the following result can be found in [652, Section 9.3] or [968, Chapter 6].

Theorem 9.85 (Surgery obstruction in odd dimensions). *We get under the conditions above:*

(i) Suppose $k \geq 2$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps to obtain a normal map $(\overline{f'}, f')\colon \nu(M') \to \xi$ covering a homotopy equivalence $f'\colon M' \to X$;
(ii) The surgery obstruction $\sigma(\overline{f}, f)$ depends only on the normal bordism class of $(\overline{f}, f)$.

Example 9.86 (The surgery obstruction in the simply connected case). Consider a normal map of degree one $(\overline{f}, f)\colon \nu(M) \to \xi$ covering $f\colon M \to X$ where M is a w-oriented connected closed manifold of dimension n and X is a w-oriented connected finite Poincaré complex of dimension n. Suppose that X is simply connected.

If n is odd, $L_n(\mathbb{Z})$ is trivial and hence $\sigma(\overline{f}, f) = 0$. In particular we can arrange by finitely many surgery steps that the underlying map is a homotopy equivalence, provided $n \geq 5$.

If n is divisible by four, we obtain an isomorphism $L_n(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$ by sending a quadratic form to its signature divided by eight, see Theorem 9.29. It turns out that under this isomorphism we get

$$\sigma(\overline{f}, f) = \frac{\operatorname{sign}(X) - \operatorname{sign}(M)}{8}.$$

Note that in this case the surgery obstruction depends only on M and X, but not on f and $\overline{f}$. This is not true in general.

If n is even, but not divisible by four, then the Arf invariant yields an isomorphism $L_n(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$. It turns out that $\sigma(\overline{f}, f)$ depends not only on f but also on the bundle data $\overline{f}$. For instance, for different framings of T^2 one obtains different invariants $\alpha(T^2)$ in Exercise 9.75.

More details can be found in [652, Subsection 8.7.6].

9.9 Surgery Obstructions for Manifolds with Boundary

Next we deal with manifolds with boundary.

Definition 9.87 (Poincaré pairs). The notion of a Poincaré complex can be extended to pairs as follows. Let X be a connected finite n-dimensional CW-complex with fundamental group π together with a subcomplex $A \subset X$ of dimension $(n-1)$. Denote by $\widetilde{A} \subset \widetilde{X}$ the preimage of A under the universal covering $\widetilde{X} \to X$. We call (X, A) a finite n-dimensional Poincaré pair with respect to the orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$ if there is a fundamental class $[X, A] \in H_n(X, A; \mathbb{Z}^w)$ such that the $\mathbb{Z}\pi$-chain maps $- \cap [X, A]\colon C^{n-*}(\widetilde{X}, \widetilde{A}) \to C_*(\widetilde{X})$ and $- \cap [X, A]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{A})$ are $\mathbb{Z}\pi$-chain homotopy equivalences.

We call (X, A) *simple* if the Whitehead torsions of these $\mathbb{Z}\pi$-chain homotopy equivalences vanish.

If M is a connected compact manifold of dimension n with boundary ∂M, then $(M, \partial M)$ is a simple finite n-dimensional Poincaré pair.

We want to extend the notion of a normal map from closed manifolds to manifolds with boundary. The underlying map f is a map of pairs $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ where M is a w-oriented compact manifold with boundary ∂M and $(X, \partial X)$ is a w-oriented finite Poincaré pair, the degree of f is one and $\partial f\colon \partial M \to \partial X$ is required to be a homotopy equivalence. The bundle data are unchanged, they consist of a vector bundle ξ over X and a bundle map $\overline{f}\colon \nu(M) \to \xi$.

The notion of a normal bordism for manifolds with boundaries is rather complicated, but also obvious. We will at least explain what happens for the underlying spaces and maps, more details can be found in [652, Subsection 8.8.2].

Consider two normal maps in dimension n whose underlying maps are $(f_m, \partial f_m)\colon (M_m, \partial M_m) \to (X_m, \partial X_m)$ such that ∂f_m is a homotopy equivalence. A normal bordism between them is defined a follows. As in the closed case W is a w-oriented compact $(n+1)$-dimensional manifold with boundary ∂W, but now the boundary is the union of three pieces

$$\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$$

where $\partial_m W$ is a codimension zero submanifold of ∂W possibly with non-empty boundary $\partial\partial_m W$ for $m = 0, 1, 2$ satisfying

$$\begin{aligned} \partial_0 W \cap \partial_1 W &= \emptyset; \\ \partial_2 W \cap \partial_m W &= \partial\partial_m W \quad \text{for } m = 0, 1; \\ \partial\partial_2 W &= \partial\partial_0 W \amalg \partial\partial_1 W. \end{aligned}$$

We have an $(n+1)$-dimensional finite Poincaré pair $(Y, \partial Y)$ with a decomposition of ∂Y into three n-dimensional finite CW-subcomplexes

$$\partial Y = \partial_0 Y \cup \partial_1 Y \cup \partial_2 Y,$$

such that for appropriate $(n-1)$-dimensional finite CW-subcomplexes $\partial\partial_m Y \subseteq \partial_m Y$ for $m = 0, 1, 2$ we have

$$\begin{aligned} \partial_0 Y \cap \partial_1 Y &= \emptyset; \\ \partial_2 Y \cap \partial_m Y &= \partial\partial_m Y \quad \text{for } m = 0, 1; \\ \partial\partial_2 Y &= \partial\partial_0 Y \amalg \partial\partial_1 Y. \end{aligned}$$

The map $F\colon W \to Y$ is required to induce maps $\partial_m F\colon \partial_m W \to \partial_m Y$ for $m = 0, 1, 2$ and $\partial_2 F\colon \partial_2 W \to \partial_2 Y$ is required to be a homotopy equivalence. The various identifications $M_m \xrightarrow{\cong} \partial_m W$ and $X_m \to \partial_m Y$ for $m = 0, 1$ in the closed case are now required to be identifications $(M_m, \partial M_m) \xrightarrow{\cong} (\partial_m W, \partial\partial_m W)$ and $(X_m, \partial X_m) \xrightarrow{\cong} (\partial_m Y, \partial Y_m)$ for $m = 0, 1$.

The definition and the main properties of the surgery obstruction carry over from normal maps for closed manifolds to normal maps for compact manifolds with boundary. The main reason is that we require $\partial f\colon \partial M \to \partial X$ to be a homotopy equivalence so that the surgery kernels "do not feel the boundary". All arguments such as making a map highly connected by surgery steps and intersection pairings and selfintersection can be carried out in the interior of M without affecting the boundary. Thus we get, see [652, Theorem 8.186 on page 302 and Theorem 9.109 on page 381],

Theorem 9.88. (Surgery Obstruction for Manifolds with Boundary). *Let $(\overline{f}, f)$ be a normal map of degree one with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ such that ∂f is a homotopy equivalence. Put $n = \dim(M)$ and $\pi = \pi_1(X)$. Then:*

(i) We can associate to it its surgery obstruction

$$\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w);$$

(ii) The surgery obstruction depends only on the normal bordism class of $(\overline{f}, f)$;
(iii) Suppose $n \geq 5$. Then $\sigma(\overline{f}, f) = 0$ in $L_n(\mathbb{Z}\pi, w)$ if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map $(\overline{f'}, f')$ which covers a homotopy equivalence of pairs $(f', \partial f')\colon (M', \partial M') \to (X, \partial X)$ with $\partial M' = \partial M$ and $\partial f' = \partial f$.

More details can be found in [652, Sections 8.8 and 9.5].

9.10 Decorations

Next we want to modify the L-groups and the surgery obstruction so that the surgery obstruction is the obstruction to achieve a simple homotopy equivalence. This will force us to study L-groups with decorations.

9.10.1 L-groups with K_1-Decorations

We begin with the L-groups. It is clear that this requires to take equivalence classes of bases into account. Suppose that we have specified a subgroup $U \subset K_1(R)$ such that U is closed under the involution on $K_1(R)$ coming from the involution of R and contains the image of the change of rings homomorphism $K_1(\mathbb{Z}) \to K_1(R)$.

Two bases B and B' for the same finitely generated free R-module V are called *U-equivalent*, if the change of basis matrix defines an element in $K_1(R)$ which belongs to U. Note that the U-equivalence class of a basis B is unchanged if we permute the order of elements of B. We call an R-module V *U-based* if V is finitely generated free and we have chosen a U-equivalence class of bases.

Let V be a stably finitely generated free R-module. A *stable basis* for V is a basis B for $V \oplus R^u$ for some integer $u \geq 0$. Denote for any integer v the direct sum of the basis B and the standard basis S^a for R^a by $B \coprod S^a$ which is a basis for $V \oplus R^{u+a}$. Let C be a basis for $V \oplus R^v$. We call the stable basis B and C *stably U-equivalent* if and only if there is an integer $w \geq u, v$ such that $B \coprod S^{w-u}$ and $C \coprod S^{w-v}$ are U-equivalent basis. We call an R-module V *stably U-based* if V is stably finitely generated free and we have specified a stable U-equivalence class of stable basis for V.

Let V and W be stably U-based R-modules. Let $f\colon V \oplus R^a \xrightarrow{\cong} W \oplus R^b$ be an R-isomorphism. Choose a non-negative integer c together with basis for $V \oplus R^{a+c}$ and $W \oplus R^{b+c}$ which represent the given stable U-equivalence classes of basis for V and W. Let A be the matrix of $f \oplus \mathrm{id}_{R^c}\colon V \oplus R^{a+c} \xrightarrow{\cong} W \oplus R^{b+c}$ with respect to these bases. It defines an element [A] in $K_1(R)$. Define the *U-torsion*

$$\tau^U(f) \in K_1(R)/U \tag{9.89}$$

by the class represented by $[A]$. It is easy to prove that $\tau^U(f)$ is independent of the choices of c and the basis and depends only on f and the stable U-basis for V and W. Moreover, one easily checks

$$\begin{aligned}
\tau^U(g \circ f) &= \tau^U(g) + \tau^U(f);\\
\tau^U\begin{pmatrix} f & 0 \\ u & v \end{pmatrix} &= \tau^U(f) + \tau^U(v);\\
\tau^U(\mathrm{id}_V) &= 0,
\end{aligned}$$

for R-isomorphisms $f\colon V_0 \xrightarrow{\cong} V_1$, $g\colon V_1 \xrightarrow{\cong} V_2$, and $v\colon V_3 \xrightarrow{\cong} V_4$ and an R-homomorphism $u\colon V_0 \to V_4$ of stably U-based R-modules V_i. Let C_* be a contractible stably U-based finite R-chain complex, i.e., a contractible R-chain complex C_* of stably U-based R-modules which satisfies $C_i = 0$ for $|i| > N$ for some integer N. The definition of Whitehead torsion in (3.32) carries over to the definition of the *U-torsion*

$$\tau^U(C_*) = [A] \qquad \in K_1(R)/U. \tag{9.90}$$

Analogously we can associate to an R-chain homotopy equivalence $f\colon C_* \to D_*$ of stably U-based finite R-chain complexes its *U-torsion*, cf. (3.33),

$$\tau^U(f_*) := \tau(\mathrm{cone}_*(f_*)) \qquad \in K_1(R)/U. \tag{9.91}$$

We will consider *stably U-based ϵ-quadratic forms* (P, ψ), i.e., non-singular ϵ-quadratic forms whose underlying R-module P is a stably U-based R-module such that the U-torsion of the isomorphism $(1 + \epsilon \cdot T)(\psi)\colon P \xrightarrow{\cong} P^*$ is zero in $K_1(R)/U$. An isomorphism $f\colon (P, \psi) \to (P', \psi')$ of stably U-based ϵ-quadratic forms is *U-simple* if the U-torsion of $f\colon P \to P'$ vanishes in $K_1(R)/U$. Note that for a stably U-based R-module P the ϵ-quadratic form $H_\epsilon(P)$ is a stably U-based ϵ-quadratic form. The sum of two stably U-based ϵ-quadratic forms is again a stably U-based ϵ-quadratic form. It is worthwhile to mention the following U-simple version of Lemma 9.26.

Lemma 9.92. *Let (P, ψ) be a stably U-based ϵ-quadratic form. Let $L \subset P$ be a lagrangian such that L is a stably U-based R-module and the U-torsion of the following 2-dimensional stably U-based finite R-chain complex*

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} L^* \to 0$$

vanishes in $K_1(R)/U$. Then the inclusion $i\colon L \to P$ extends to a U-simple isomorphism of stably U-based ϵ-quadratic forms

$$H_\epsilon(L) \xrightarrow{\cong} (P, \psi).$$

Next we give the simple version of the even dimensional L-groups.

Definition 9.93. Let R be an associative ring with involution. For $\epsilon \in \{\pm 1\}$ define $L^U_{1-\epsilon}(R)$ to be the abelian group of equivalence classes $[P, \psi]$ of stably U-based non-singular ϵ-quadratic forms (P, ψ) with respect to the following equivalence relation. We call (P, ψ) and (P', ψ') equivalent if and only if there exists integers $u, u' \geq 0$ and a U-simple isomorphism of stably U-based non-singular ϵ-quadratic forms

$$(P, \psi) \oplus H_\epsilon(R^u) \cong (P', \psi') \oplus H_\epsilon(R^{u'}).$$

Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $[H_\epsilon(R^u)]$ for any integer $u \geq 0$. The inverse of $[P, \psi]$ is given by $[P, -\psi]$.

For an even integer n define the abelian group $L^U_n(R)$, called the *n-th U-decorated quadratic L-group*, of R by

$$L^U_n(R) := \begin{cases} L^U_0(R) & \text{if } n \equiv 0 \mod 4; \\ L^U_2(R) & \text{if } n \equiv 2 \mod 4. \end{cases}$$

A *stably U-based ϵ-quadratic formation* $(P, \psi; F, G)$ consists of an ϵ-quadratic formation $(P, \psi; F, G)$ such that (P, ψ) is a stably U-based ϵ-quadratic form, the lagrangians F and G are stably U-based R-modules, and the U-torsion of the following two contractible stably U-based finite R-chain complexes

$$0 \to F \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} F^* \to 0$$

and

$$0 \to G \xrightarrow{j} P \xrightarrow{j^* \circ (1+\epsilon \cdot T)(\psi)} G^* \to 0$$

vanish in $K_1(R)/U$ where $i\colon F \to P$ and $j\colon G \to P$ denote the inclusions. An isomorphism $f\colon (P, \psi; F, G) \to (P', \psi'; F', G')$ of U-based ϵ-quadratic formations is *U-simple* if the U-torsion of the induced R-isomorphisms $P \xrightarrow{\cong} P'$, $F \xrightarrow{\cong} F'$ and $G \xrightarrow{\cong} G'$ vanishes in $K_1(R)/U$. Note that for a U-stably based R-module P the trivial ϵ-quadratic formation $(H_\epsilon(P); P, (P)^*)$ has the structure of a stably based ϵ-quadratic formation. Given a stably U-based $(-\epsilon)$-

quadratic form (Q, ψ), its boundary $\partial(Q, \psi)$ is a stably U-based ϵ-quadratic formation. Obviously the sum of two stably U-based ϵ-quadratic formations is again a stably U-based ϵ-quadratic formation. Next we give the simple version of the odd dimensional L-groups.

Definition 9.94. Let R be an associative ring with involution. For $\epsilon \in \{\pm 1\}$ define $L^U_{2-\epsilon}(R)$ to be the abelian group of equivalence classes $[P, \psi; F, G]$ of stably U-based ϵ-quadratic formations $(P, \psi; F, G)$ with respect to the following equivalence relation. We call two stably U-based ϵ-quadratic formations $(P, \psi; F, G)$ and $(P', \psi'; F', G')$ equivalent if and only if there exists stably U-based $(-\epsilon)$-quadratic forms (Q, μ) and (Q', μ') and non-negative integers u and u' together with a U-simple isomorphism of stably U-based ϵ-quadratic formations

$$\begin{aligned}(P, \psi; F, G) &\oplus \partial(Q, \mu) \oplus (H_\epsilon(R^u); R^u, (R^u)^*) \\ &\cong (P', \psi'; F', G') \oplus \partial(Q', \mu') \oplus (H_\epsilon(R^{u'}); R^{u'}, (R^{u'})^*).\end{aligned}$$

Addition is given by the sum of two stably U-based ϵ-quadratic forms. The zero element is represented by $\partial(Q, \mu) \oplus (H_\epsilon(R^u); R^u, (R^u)^*)$ for any stably U-based $(-\epsilon)$-quadratic form (Q, μ) and non-negative integer u. The inverse of $[P, \psi; F, G]$ is represented by $(P, -\psi; F', G')$ for any choice of stably U-based lagrangians F' and G' in $H_\epsilon(P)$ such that F and F' are complementary and G and G' are complementary and the U-torsion of the obvious isomorphism $F \oplus F' \xrightarrow{\cong} P$ and $G \oplus G' \xrightarrow{\cong} P$ vanishes in $K_1(R)/U$.

For an odd integer n define the abelian group $L^U_n(R)$ called the *n-th U-decorated quadratic L-group* of R

$$L^U_n(R) := \begin{cases} L^U_1(R) & \text{if } n \equiv 1 \mod 4; \\ L^U_3(R) & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Notation 9.95. Consider the case of a group ring $R\pi$ with the w-twisted involution. For a group G denote by $\mathrm{Wh}^R_n(G)$ the n-th Whitehead group of RG, which is the $(n-1)$-th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \to \mathbf{K}(RG)$. Then we define $L^s_n(R\pi, w)$ by $L^U_n(R\pi)$ for U the kernel of the map $K_1(R\pi) \to \mathrm{Wh}^R_1(\pi)$. Observe that $L^s_n(R\pi)$ depends on the pair (R, π). Sometimes one denotes $L^s_n(R\pi, w)$ also by $L^{\langle 2 \rangle}_n(R\pi, w)$.

If $R = \mathbb{Z}\pi$ with the w-twisted involution, then $U \subseteq K_1(\mathbb{Z}\pi)$ reduces to the abelian group $V \subseteq K_1(\mathbb{Z}G)$ of elements of the shape $(\pm g)$ for $g \in \pi$. So we get the *simple quadratic L-groups*

$$L^s_n(\mathbb{Z}\pi, w) = L^{\langle 2 \rangle}_n(\mathbb{Z}\pi, w) = L^V_n(\mathbb{Z}\pi, w).$$

9.10.2 The Simple Surgery Obstruction

Let $(\overline{f}, f)$ be a normal map of degree one with $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ as underlying map such that $(X, \partial X)$ is a simple finite Poincaré complex and ∂f is a simple homotopy equivalence. Then the definition of the surgery obstruction appearing in Theorem 9.88 (i) can be modified to the simple setting. Note that the difference between the L-groups $L_n^h(\mathbb{Z}\pi, w)$ and the simple L-groups $L_n^s(\mathbb{Z}\pi, w)$ is the additional structure of a U-basis. The definition of the simple surgery obstruction

$$\sigma(\overline{f}, f) \in L_n^s(\mathbb{Z}\pi, w). \tag{9.96}$$

is the same as the one appearing in Theorem 9.88 (i) except that we must explain how the various surgery kernels inherit a stable U-basis.

The elementary proof of the following lemma is left to the reader. Note that for any stably U-based R-module V and element $x \in K_1(R)/U$ we can find another stable U-basis C for V such that the U-torsion $\tau^U(\mathrm{id} \colon (V, B) \to (V, C))$ is x. This is not true in the unstable setting. For instance, there exists a ring R with an element $x \in K_1(R)/U$ for U the image of $K_1(\mathbb{Z}) \to K_1(R)$ such that x cannot be represented by a unit in R, in other words x is not the U-torsion of any R-automorphism of R.

Lemma 9.97. *Let C_* be a contractible finite stably free R-chain complex and r be an integer. Suppose that each chain module C_i with $i \neq r$ comes with a stable U-basis. Then C_r inherits a preferred stable U-basis which is uniquely defined by the property that the U-torsion of C_* vanishes in $K_1(R)/U$.*

We have the following version of Lemma 9.70

Lemma 9.98. *If $f \colon X \to Y$ is k-connected for $n = 2k$ or $n = 2k+1$, then $K_k(\widetilde{M})$ is stably finitely generated free and inherits a preferred stable U-basis.*

Proof. See [652, Lemma 10.27 (i) on page 414]. □

Next we can give the simple version of the surgery obstruction theorem. For its proof see for instance [652, Theorem 10.30 on page 415]. Note that simple normal bordism class means that in the definition of normal nullbordisms the pairs $(Y, \partial Y)$, $(\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$ and $(\partial_1 Y, \partial_0 Y \cap \partial_1 Y)$ are required to be simple finite Poincaré pairs and the map $\partial_2 F \colon \partial_2 M \to \partial_2 Y$ is required to be a simple homotopy equivalence.

Theorem 9.99. (Simple surgery obstruction for manifolds with boundary) *Let $(\overline{f}, f)$ be a normal map of degree one whose underlying map is $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ such that $(X, \partial X)$ is a simple finite Poincaré complex and ∂f is a simple homotopy equivalence. Put $n = \dim(M)$ and $\pi = \pi_1(X)$. Then:*

(i) The simple surgery obstruction depends only on the simple normal bordism class of $(\overline{f}, f)$*;*

(ii) Suppose $n \geq 5$*. Then* $\sigma(\overline{f}, f) = 0$ *in* $L_n^s(\mathbb{Z}\pi, w)$ *if and only if we can do a finite number of surgery steps on the interior of* M *leaving the boundary fixed to obtain a normal map* $(\overline{f'}, f') \colon \nu M' \to \xi$ *which covers a simple homotopy equivalence of pairs* $(f', \partial f') \colon (M', \partial M') \to (X, \partial X)$ *with* $\partial M' = \partial M$ *and* $\partial f' = \partial f$*.*

Exercise 9.100. Let W be a compact manifold of dimension n whose boundary is the disjoint union $M \amalg N$. Let $(\overline{f}, f)$ be a normal map such that the underlying maps of pairs is of the shape $f \colon (W, \partial W) \to (X \times [0,1], X \times \{0,1\})$ for some closed manifold X and induces a simple homotopy equivalence $\partial W \to X \times \{0,1\}$. Show that M and N are diffeomorphic provided that the simple surgery obstruction $\sigma(\overline{f}, f)$ of (9.96) vanishes and $n \geq 6$.

9.10.3 Decorated *L*-Groups

L-groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in.

The L-group $L_n(R)$ of Definitions 9.25 and 9.81 are also denoted by $L_n^{\langle 1 \rangle}(R)$ or by $L_n^h(R)$. If one works with finitely generated projective modules instead of finitely generated free R-modules in Definitions 9.25 and 9.81, one obtains *projective quadratic* L*-groups* $L_n^p(R)$ which are also denoted by $L_n^{\langle 0 \rangle}(R)$. The negative decorations $L_n^{\langle j \rangle}(R)$ for $j \in \mathbb{Z}, j \leq -1$ can be obtained using suitable categories of modules parametrized over $\mathbb{R}^k$. There are forgetful maps $L_n^{\langle j+1 \rangle}(R) \to L_n^{\langle j \rangle}(R)$ for $j \in \mathbb{Z}, j \leq 1$. The group $L_n^{\langle -\infty \rangle}(R)$ is defined as the colimit over these maps. For details the reader can consult [817, 824].

Let us summarize the decorations for integral group rings. We have already introduced $L_n^s(\mathbb{Z}\pi, w) = L_n^{\langle 2 \rangle}(\mathbb{Z}\pi, w)$ in Notation 9.95, we get

$$L_n^h(\mathbb{Z}\pi, w) = L_n^{\langle 1 \rangle}(\mathbb{Z}\pi, w) = L_n(\mathbb{Z}\pi, w);$$
$$L_n^p(\mathbb{Z}\pi, w) = L_n^{\langle 0 \rangle}(\mathbb{Z}\pi, w),$$

and have furthermore $L_n^{\langle j \rangle}(\mathbb{Z}\pi)$ for $j \in \mathbb{Z}, j \leq -1$ and $L_n^{\langle -\infty \rangle}(\mathbb{Z}\pi)$.

For the Farrell-Jones Conjecture we will have to take the decoration $\langle -\infty \rangle$ where for applications the decorations h and s will be relevant. So we have to understand how one can compare them.

9.10.4 The Rothenberg Sequence

Next we explain how decorated L-groups can be computed from one another for a ring with involution. We have the long exact *Rothenberg sequence* [821, Proposition 1.10.1 on page 104], [824, 17.2] for $j \in \{0, -1, -2, \ldots\} \amalg \{-\infty\}$ and $n \in \mathbb{Z}$

$$(9.101) \quad \cdots \to L_n^{\langle j+1 \rangle}(R) \to L_n^{\langle j \rangle}(R) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R)) \\ \to L_{n-1}^{\langle j+1 \rangle}(R) \to L_{n-1}^{\langle j \rangle}(R) \to \cdots .$$

Here $\widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R))$ is the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$-module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R.

For a group ring $R\pi$ with the w-twisted involution and elements j in $\{1, 0, -1, \ldots\} \amalg \{-\infty\}$ and n in $\mathbb{Z}$, we get the long exact sequence

$$(9.102) \quad \cdots \to L_n^{\langle j+1 \rangle}(R\pi, w) \to L_n^{\langle j \rangle}(R\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j^R(\pi)) \\ \to L_{n-1}^{\langle j+1 \rangle}(R\pi, w) \to L_{n-1}^{\langle j \rangle}(R\pi, w) \to \cdots .$$

Over the integral group ring $\mathrm{Wh}_1^{\mathbb{Z}}(\pi)$ agrees with $\mathrm{Wh}(\pi)$ and $\mathrm{Wh}_j^{\mathbb{Z}}(\pi)$ agrees with $\widetilde{K}_j(\mathbb{Z}\pi)$ for $j \leq 0$. Hence (9.101) reduces for $R = \mathbb{Z}$ and $j = 1$ to

$$(9.103) \quad \cdots \to L_n^{\langle s \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle h \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}(\pi)) \\ \to L_{n-1}^{\langle s \rangle}(R) \to L_{n-1}^{\langle h \rangle}(R) \to \cdots ,$$

and for $R = \mathbb{Z}$ and $j \leq 0$ to

$$(9.104) \quad \cdots \to L_n^{\langle j+1 \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle j \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathbb{Z}\pi)) \\ \to L_{n-1}^{\langle j+1 \rangle}(\mathbb{Z}\pi, w) \to L_{n-1}^{\langle j \rangle}(\mathbb{Z}\pi, w) \to \cdots .$$

In particular we get

$$(9.105) \quad \ldots \to L_n^{\langle h \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle p \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}\pi)) \\ \to L_{n-1}^{\langle h \rangle}(\mathbb{Z}\pi, w) \to L_{n-1}^{\langle p \rangle}(\mathbb{Z}\pi, w) \to \cdots .$$

Theorem 9.106 (Independence of decorations). *Let G be a group such that $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G)$ for all $n \in \mathbb{Z}, n \leq -1$ vanish. Then for every $j \in \mathbb{Z}, j \leq -1$ and every $n \in \mathbb{Z}$ the forgetful maps induce isomorphisms*

$$L_n^s(\mathbb{Z}G) \xrightarrow{\cong} L_n^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^p(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

Proof. This follows from the various Rothenberg sequences. □

Exercise 9.107. Show that for every group G, every $j \in \mathbb{Z}, j \leq -1$, and every $n \in \mathbb{Z}$ the forgetful maps induces isomorphisms after inverting 2

$$L^s(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^h(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^p(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^{\langle j\rangle}(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^{\langle -\infty\rangle}(\mathbb{Z}G)[1/2].$$

9.10.5 The Shaneson Splitting

The Bass-Heller-Swan decomposition in K-theory, see Theorem 6.16, has the following analogue for the algebraic L-groups.

Theorem 9.108 (Shaneson splitting). *For every group G, every ring with involution R, every $j \in \mathbb{Z}, j \leq 2$ and $n \in \mathbb{Z}$ there is a natural isomorphism*

$$L_n^{\langle j\rangle}(RG) \oplus L_{n-1}^{\langle j-1\rangle}(RG) \xrightarrow{\cong} L_n^{\langle j\rangle}(R[G \times \mathbb{Z}])$$

and we have the natural isomorphism

$$L_n^{\langle -\infty\rangle}(RG) \oplus L_{n-1}^{\langle -\infty\rangle}(RG) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(R[G \times \mathbb{Z}]). \tag{9.109}$$

The map appearing in the theorem above comes from the map $L_n^{\langle j\rangle}(RG) \to L_n^{\langle j\rangle}(R[G\times\mathbb{Z}])$ induced by the inclusion $G \to G\times\mathbb{Z}$ and a map $L_{n-1}^{\langle j-1\rangle}(RG) \to L_n^{\langle j\rangle}(R[G \times \mathbb{Z}])$ which is essentially given by the cartesian product with S^1. The latter explains the raise from $(n-1)$ to n. But why does the decoration raises from $j-1$ to j? The reason is the product formula for Whitehead torsion, see Theorem 3.37 (iv). It predicts for any (not necessarily simple) homotopy equivalence $f\colon X \to Y$ of finite CW-complexes that the homotopy equivalence $f \times \mathrm{id}_{S^1}\colon X \times S^1 \to Y \times S^1$ is a simple homotopy equivalence. There is also a product formula for the finiteness obstruction which predicts for a finitely dominated (not necessarily up to homotopy finite) CW-complex X that $X \times S^1$ is homotopy equivalent to a finite CW-complex. The original proof of the Shaneson splitting for the case $j = 2$ and $R = \mathbb{Z}$ i.e., for the isomorphism

$$L_n^s(\mathbb{Z}G) \oplus L_{n-1}^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^s(\mathbb{Z}[G \times \mathbb{Z}])$$

can be found in [896]. The proof for arbitrary j and R is given in [824, 17.2]. Note that for $j = 1$ we obtain an isomorphism

$$L_n^h(RG) \oplus L_{n-1}^p(RG) \xrightarrow{\cong} L_n^h(R[G \times \mathbb{Z}]) \tag{9.110}$$

Remark 9.111 (UNil-groups). Even though in the Shaneson splitting (9.109) above there are no terms analogous to the Nil-terms in the Bass-Heller-

Swan decomposition in K-theory, see Theorem 6.16, such Nil-phenomena do also occur in L-theory, as soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They vanish if one inverts 2, see [192]. For more information about the UNil-groups we refer for instance to [62, 189, 190, 240, 241, 244, 337, 825]. How the Farrell-Jones Conjecture predicts exact Mayer-Vietoris sequences for amalgamated free products after inverting 2 is explained in Section 15.7.

Exercise 9.112. Compute $L_n^{\langle j \rangle}(\mathbb{Z}[\mathbb{Z}])$.

9.11 The Farrell-Jones Conjecture for Algebraic L-Theory for Torsionfree Groups

The Farrell-Jones Conjecture for algebraic L-theory, which will later be formulated in full generality in Chapter 13, reduces for a torsionfree group to the following conjecture. Given a ring with involution R, there exists an *L-spectrum associated to R with decoration $\langle -\infty \rangle$*

$$\mathbf{L}^{\langle -\infty \rangle}(R) \tag{9.113}$$

with the property that $\pi_n\big(\mathbf{L}^{\langle -\infty \rangle}(R)\big) = L_n^{\langle -\infty \rangle}(R)$ holds for $n \in \mathbb{Z}$.

Conjecture 9.114 (Farrell-Jones Conjecture for torsionfree groups for L-theory). Let G be a torsionfree group. Let R be any ring with involution.

Then the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$.

We get for every $j \in \{1, 0, -1, \ldots\} \amalg \{-\infty\}$

$$\begin{aligned} H_n(B\mathbb{Z}; \mathbf{L}^{\langle j \rangle}(R)) \cong H_n(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(R)) \oplus H_{n-1}(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(R)) \\ \cong L_n^{\langle j \rangle}(R) \oplus L_{n-1}^{\langle j \rangle}(R). \end{aligned}$$

In view of the Shaneson splitting of Theorem 9.108 it is now obvious, why we have passed to the decoration $j = -\infty$ in Conjecture 9.114.

Exercise 9.115. Let F_g be the closed orientable surface of genus g. Compute $L_n^{\langle j \rangle}(\mathbb{Z}[\pi_1(F_g)])$ for all $j \in \mathbb{Z}, j \leq 2$ and $n \in \mathbb{Z}$ using the fact that Conjecture 9.114 holds for $G = \pi_1(F_g)$.

Lemma 9.116. *Let X be a CW-complex.*

(i) If X is finite and we localize at the prime 2, we obtain a natural isomorphism

$$H_n(X;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))_{(2)} \cong \prod_{j\in\mathbb{Z}} \big(H_{n+4j}(X;\mathbb{Z}_{(2)}) \times H_{n+4j-2}(X;\mathbb{Z}/2)\big);$$

(ii) If we invert 2, we obtain a natural isomorphism

$$H_n(X;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))[1/2] \cong KO_n(X)[1/2].$$

Proof. (i) The L-theory spectrum $\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})_{(2)}$ localized at (2) is a infinite product of Eilenberg-Mac-Lane spectra by [928, Theorem A].

(ii) This follows from the more general case discussed in Subsection 15.14.4, which is based on [599, 600]. □

9.12 The Surgery Exact Sequence

In this section we introduce the Surgery Exact Sequence. It is the realization of the Surgery Program, which we have explained in Remark 3.53. The Surgery Exact Sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater than or equal to five.

9.12.1 The Structure Set

Definition 9.117 (Simple structure set). Let X be a closed manifold of dimension n. We call two simple homotopy equivalences $f_i\colon M_i \to X$ from closed manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there exists a diffeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0. The *simple structure set* $\mathcal{S}^s_n(X)$ of X is the set of equivalence classes of simple homotopy equivalences $M \to X$ from closed manifolds of dimension n to X. This set has a preferred base point, namely, the class of the identity $\mathrm{id}\colon X \to X$.

The simple structure set $\mathcal{S}^s_n(X)$ is the basic object in the study of manifolds which are diffeomorphic to X. Note that a simple homotopy equivalence $f\colon M \to X$ is homotopic to a diffeomorphism if and only if it represents the base point in $\mathcal{S}^s_n(X)$. A manifold M is diffeomorphic to N if and only if for some simple homotopy equivalence $f\colon M \to N$ the class of $[f]$ agrees with the preferred base point. Some care is necessary since it may be possible that a given simple homotopy equivalence $f\colon M \to N$ is not homotopic to a diffeomorphism, although M and N are diffeomorphic. Hence it does not suffice to compute $\mathcal{S}^s_n(N)$, one also has to understand the operation of the group of homotopy classes of simple selfequivalences of N on $\mathcal{S}^s_n(N)$. This can be rather complicated in general. But it will be no problem in the case $N = S^n$ because any selfhomotopy equivalence $S^n \to S^n$ is homotopic to a selfdiffeomorphism.

There is also a version of the structure set which does not take Whitehead torsion into account.

Definition 9.118 (Structure set). Let X be a closed manifold of dimension n. We call two homotopy equivalences $f_i \colon M_i \to X$ from closed manifolds M_i of dimension n to X for $i = 0, 1$ equivalent if there is a manifold triad $(W; \partial_0 W, \partial_1 W)$ with $\partial_0 W \cap \partial_1 W = \emptyset$ and a homotopy equivalence of triads $(F; \partial_0 F, \partial_1 F) \colon (W; \partial_0 W, \partial_1 W) \to (X \times [0,1]; X \times \{0\}, X \times \{1\})$ together with diffeomorphisms $g_0 \colon M_0 \to \partial_0 W$ and $g_1 \colon M_1 \to \partial_1 W$ satisfying $\partial_i F \circ g_i = f_i$ for $i = 0, 1$. The *structure set* $\mathcal{S}_n^h(X)$ of X is the set of equivalence classes of homotopy equivalences $M \to X$ from a closed manifold M of dimension n to X. This set has a preferred base point, namely the class of the identity $\mathrm{id} \colon X \to X$.

Remark 9.119 (The simple structure set and s-cobordisms). If we require in Definition 9.118 the homotopy equivalences F, f_0, and f_1 to be simple homotopy equivalences, we get the simple structure set $\mathcal{S}_n^s(X)$ of Definition 9.117, provided that $n \geq 5$. We have to show that the two equivalence relations are the same. This follows from the s-Cobordism Theorem 3.47. Namely, W appearing in Definition 9.118 is an h-cobordism and is even an s-cobordism if we require F, f_0, and f_1 to be simple homotopy equivalences, see Theorem 3.37. Hence there is a diffeomorphism $\Phi \colon \partial_0 W \times [0,1] \to W$ inducing the obvious identification $\partial_0 W \times \{0\} \to \partial_0 W$ and some diffeomorphism $\phi_1 \colon (\partial_0 W) = (\partial_0 W \times \{1\}) \to \partial_1 W$. Then $\phi \colon M_0 \to M_1$ given by $g_1^{-1} \circ \phi_1 \circ g_0$ is a diffeomorphism such that $f_1 \circ \phi$ is homotopic to f_0. The other implication is obvious.

9.12.2 Realizability of Surgery Obstructions

In this section we explain that any element in the L-group $L_n(\mathbb{Z}\pi, w)$ for $n \geq 5$ can be realized as the surgery obstruction of a normal map $(\overline{f}, f)$ covering a map $(f, \partial f) \colon (M, \partial M) \to (X, \partial X)$ of compact manifolds if we require that X has non-empty boundary ∂X and that ∂f is a (simple) homotopy equivalence.

Theorem 9.120 (Realizability of the surgery obstruction). *Suppose $n \geq 5$. Consider a w-oriented connected compact manifold X with non-empty boundary ∂X. Let π be its fundamental group and let $w \colon \pi \to \{\pm 1\}$ be its orientation homomorphism. Consider an element $x \in L_n(\mathbb{Z}\pi, w)$.*

Then we can find a normal map of degree one $(\overline{f}, f)$ covering a map of triads

$$f = (f; \partial_0 f, \partial_1 f) \colon (M; \partial_0 M, \partial_1 M) \to (X \times [0,1]; X \times \{0\} \cup \partial X \times [0,1], X \times \{1\})$$

with the following properties:

(i) $\partial_0 f$ *is a diffeomorphism and* $\overline{f}|_{\partial_0 M}$ *is a stabilization of* $T(\partial f_0)$*;*
(ii) $\partial_1 f$ *is a homotopy equivalence;*
(iii) *The surgery obstruction* $\sigma(\overline{f}, f)$ *in* $L_n(\mathbb{Z}\pi, w)$*, see* (9.84)*, is the given element* x*.*

The analogous statement holds for $x \in L_n^s(\mathbb{Z}\pi, w)$ *if we require* $\partial_1 f$ *to be a simple homotopy equivalence and we consider the simple surgery obstruction, see* (9.96)*.*

Proof. See [652, Theorem 8.192 on page 308 and Theorem 9.111 on page 382]. □

Remark 9.121 (Surgery obstructions of closed manifolds). It is not true that for any w-oriented closed manifold N of dimension n with fundamental group π and orientation homomorphism $w\colon \pi \to \{\pm 1\}$ and any element $x \in L_n(\mathbb{Z}\pi, w)$ there is a normal map $(\overline{f}, f)$ covering a map of w-oriented closed manifolds $f\colon M \to N$ of degree one such that $\sigma(\overline{f}, f) = x$. Note that in Theorem 9.120 the target manifold $X \times [0,1]$ is not closed. The same remark holds for $L_n^s(\mathbb{Z}\pi, w)$. These questions are discussed in in [437, 442, 705, 706], see also [652, Remark 8.199 on page 209 and Ramark 9.117 on page 401].

9.12.3 The Surgery Exact Sequence

Now we can establish one of the main tools in the classification of manifolds, the Surgery Exact Sequence. We have already extended the notion of a normal map for closed manifolds to manifolds with boundary and explained the notion of a normal bordism for normal maps of pairs in Section 9.9. In this Subsection 9.12.3, we will consider only normal maps with the same target $(X, \partial X)$ whose underlying maps are diffeomorphisms on the boundary, and we call two of them with the same target normally bordant if there is a normal bordism between them in the sense of Definition 9.59 whose underlying map induces a diffeomorphism $\partial_1 W \to \partial X \times [0,1]$.

Definition 9.122. Let $(X, \partial X)$ be a w-oriented compact manifold of dimension n with boundary ∂X. Define the *set of normal maps* to $(X, \partial X)$

$$\mathcal{N}_n(X, \partial X)$$

to be the set of normal bordism classes of normal maps of degree one $(\overline{f}, f)$ with underlying map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$, for which $\partial f\colon \partial M \to \partial X$ is a diffeomorphism.

Let X be a closed w-oriented connected manifold of dimension $n \geq 5$. Denote by π its fundamental group and by $w\colon \pi \to \{\pm 1\}$ its orientation homomorphism. Let $\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ and $\mathcal{N}_n(X)$ be the set of normal maps of degree one as introduced in Definition 9.122. Let $\mathcal{S}^s_n(X)$ be the structure set of Definition 9.117. Denote by $L^s_n(\mathbb{Z}\pi, w)$ the simple surgery obstruction group, see Notation 9.95. Denote by

$$\sigma^s_{n+1} \colon \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi, w); \tag{9.123}$$
$$\sigma^s_n \colon \mathcal{N}_n(X) \to L^s_n(\mathbb{Z}\pi, w), \tag{9.124}$$

the maps that assign to the normal bordism class of a normal map of degree one its simple surgery obstruction, see (9.96). This is well-defined by Theorem 9.88 (ii). Let

$$\eta^s_n \colon \mathcal{S}^s_n(X) \to \mathcal{N}_n(X) \tag{9.125}$$

be the map that sends the class $[f] \in \mathcal{S}^s_n(X)$ represented by a simple homotopy equivalence $f \colon M \to X$ to the normal bordism class of the following normal map of degree one. Choose a homotopy inverse $f^{-1} \colon X \to M$ and a homotopy $h \colon \mathrm{id}_M \simeq f^{-1} \circ f$. Put $\xi = (f^{-1})^*TM$. Up to isotopy of bundle maps there is precisely one bundle map $(\overline{h}, h) \colon TM \times [0,1] \to TM$ covering $h \colon M \times [0,1] \to M$ whose restriction to $TM \times \{0\}$ is the identity map $TM \times \{0\} \to TM$. The restriction of $\overline{h}$ to $X \times \{1\}$ induces a bundle map $\overline{f} \colon TM \to \xi$ covering $f \colon M \to X$. Put $\eta([f]) := [(\overline{f}, f)]$. One easily checks that the normal bordism class of $(\overline{f}, f)$ depends only on $[f] \in \mathcal{S}^s_n(X)$ and hence that η is well-defined.

Next we define an action of the abelian group $L^s_{n+1}(\mathbb{Z}\pi, w)$ on the structure set $\mathcal{S}^s_n(X)$

$$\rho^s_n \colon L^s_{n+1}(\mathbb{Z}\pi, w) \times \mathcal{S}^s_n(X) \to \mathcal{S}^s_n(X). \tag{9.126}$$

Fix $x \in L^s_{n+1}(\mathbb{Z}\pi, w)$ and $[f] \in \mathcal{S}^s_n(X)$ represented by a simple homotopy equivalence $f \colon M \to X$. By Theorem 9.120 we can find a normal map $(\overline{F}, F)$ covering a map of triads $(F; \partial_0 F, \partial_1 F) \colon (W; \partial_0 W, \partial_1 W) \to (M \times [0,1]; M \times \{0\}, M \times \{1\})$ such that $\partial_0 F$ is a diffeomorphism and $\partial_1 F$ is a simple homotopy equivalence and $\sigma(\overline{F}, F) = x$. Then define $\rho^s_n(x, [f])$ by the class $[f \circ \partial_1 F \colon \partial_1 W \to X]$. We have to show that this is independent of the choice of $(\overline{F}, F)$. Let $(\overline{F'}, F')$ be a second choice. We can glue W' and W^- together along the diffeomorphism $(\partial_0 F)^{-1} \circ \partial_0 F' \colon \partial_0 W' \to \partial_0 W$ and obtain a normal bordism from $(\overline{F}|_{\partial_1 W}, \partial_1 F)$ to $(\overline{F'}|_{\partial_1 W'}, \partial_1 F')$. The simple obstruction of this normal bordism is

$$\sigma(\overline{F'}, F') - \sigma(\overline{F}, F) = x - x = 0.$$

Because of Theorem 9.99 (ii) we can perform surgery relative boundary on this normal bordism to arrange that the reference map from it to $X \times [0,1]$ is a simple homotopy equivalence. In view of Remark 9.119 this shows that $f \circ \partial_1 F$ and $f \circ \partial_1 F'$ define the same element in $\mathcal{S}^s_n(X)$. One easily checks that this defines a group action since the surgery obstruction is additive under stacking normal bordisms together. The next result is the main result of this chapter and follows from the definitions and Theorem 9.99 (ii).

Theorem 9.127 (The Surgery Exact Sequence). *Under the conditions and in the notation above the so-called* Surgery Exact Sequence

$$\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma^s_{n+1}} L^s_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial^s_{n+1}} \mathcal{S}^s_n(X) \xrightarrow{\eta^s_n} \mathcal{N}_n(X) \xrightarrow{\sigma^s_n} L^s_n(\mathbb{Z}\pi, w)$$

is exact for $n \geq 5$ in the following sense. An element $z \in \mathcal{N}_n(X)$ lies in the image of η^s_n if and only if $\sigma^s_n(z) = 0$. Two elements $y_1, y_2 \in \mathcal{S}^s_n(X)$ have the same image under η^s_n if and only if there exists an element $x \in L^s_{n+1}(\mathbb{Z}\pi, w)$ with $\rho^s_n(x, y_1) = y_2$. For two elements x_1, x_2 in $L^s_{n+1}(\mathbb{Z}\pi)$ we have $\rho^s_n(x_1, [\mathrm{id}\colon X \to X]) = \rho^s_n(x_2, [\mathrm{id}\colon X \to X])$ if and only if there is $u \in \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\})$ with $\sigma^s_{n+1}(u) = x_1 - x_2$.

There is an analogous Surgery Exact Sequence

$$\mathcal{N}^h_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma^h_{n+1}} L^h_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial^h_{n+1}} \mathcal{S}^h(X) \xrightarrow{\eta^h_n} \mathcal{N}_n(X) \xrightarrow{\sigma^h_n} L^h_n(\mathbb{Z}\pi, w)$$

where $\mathcal{S}^h(X)$ is the structure set of Definition 9.118 and $L^h_n(\mathbb{Z}\pi, w) := L_n(\mathbb{Z}\pi, w)$ has been introduced in Definitions 9.25 and 9.81.

Remark 9.128 (Extending the Surgery Exact Sequence to the left). The Surgery Exact Sequence of Theorem 9.127 can be extended to infinity to the left. In the range far enough to the left it is a sequence of abelian groups.

9.13 Surgery Theory in the PL and in the Topological Category

One can also develop surgery theory in the PL (=piecewise linear) category or in the topological category [562]. This requires to carry over the notions of vector bundles and tangent bundles to these categories. There are analogs of the sets of normal invariants $\mathcal{N}^{\mathsf{PL}}_n(X)$ and $\mathcal{N}^{\mathsf{TOP}}_n(X)$ and the structure sets $\mathcal{S}^{\mathsf{PL},h}_n(X)$, $\mathcal{S}^{\mathsf{PL},s}_n(X)$ $\mathcal{S}^{\mathsf{TOP},h}_n(X)$ and $\mathcal{S}^{\mathsf{TOP},s}_n(X)$. There are analogs PL and TOP of the group $\mathrm{O} = \operatorname{colim}_{n\to\infty} \mathrm{O}_n$. The topological group TOP is the limit of the groups $\mathrm{TOP}(k)$ that are the groups of homeomorphisms of $\mathbb{R}^k$ fixing the origin:

$$\mathrm{TOP} = \operatorname{colim}_{k\to\infty} \mathrm{TOP}(k).$$

The definition of PL is more elaborate and therefore omitted. Let $\mathrm{G} = \operatorname{colim}_{n\to\infty} \mathrm{G}(n)$ where $\mathrm{G}(n)$ is the monoid of self homotopy equivalences of S^n. There are classifying spaces BPL (resp. BTOP), which classify stable isomorphism classes of PL (resp. TOP) $\mathbb{R}^k$ bundles and which are infinite loop spaces with multiplication corresponding to the Whitney sum of bundles. The space BG is the classifying space for spherical fibrations. There are also canonical maps $\mathrm{BPL} \to \mathrm{BG}$ (resp. $\mathrm{BTOP} \to \mathrm{BG}$) which classify the passage to strong fiber homotopy equivalence classes of stable spherical fibrations. The homotopy fibres of these maps are denoted G/PL (resp. G/TOP) and have infinite loop space structures so that the canonical maps $\mathrm{G/PL} \to \mathrm{BPL}$ and $\mathrm{G/TOP} \to \mathrm{BTOP}$ are maps of infinite loop spaces. Define G/O as the homotopy fiber of the map $\mathrm{BO} \to \mathrm{BG}$.

Theorem 9.129 (The set of normal maps and G/O, G/PL and G/TOP). *Let X be a connected compact n-dimensional manifold. Then there is a canonical group structure on the set $[X, \mathrm{G/O}]$ $[X, \mathrm{G/PL}]$, or $[X, \mathrm{G/TOP}]$ respectively, and a transitive free operation of this group on $\mathcal{N}_n(X)$, $\mathcal{N}_n^{\mathsf{PL}}(X)$, or $\mathcal{N}_n^{\mathsf{TOP}}(X)$ respectively. In particular we get bijections*

$$\begin{aligned}[]
[X/\partial X, \mathrm{G/O}] &\xrightarrow{\cong} \mathcal{N}_n(X);\\
[X/\partial X, \mathrm{G/PL}] &\xrightarrow{\cong} \mathcal{N}_n^{\mathsf{PL}}(X);\\
[X/\partial X, \mathrm{G/TOP}] &\xrightarrow{\cong} \mathcal{N}_n^{\mathsf{TOP}}(X),
\end{aligned}$$

respectively.

There are analogs of the Surgery Exact Sequence, see Theorem 9.127, for the PL category and the topological category.

Theorem 9.130 (The Surgery Exact Sequence for the PL and the topological category). *There is a Surgery Exact Sequence*

$$\begin{aligned}
\mathcal{N}_{n+1}^{\mathsf{PL}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^s} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^s} \mathcal{S}_n^{\mathsf{PL},s}(X)&\\
\xrightarrow{\eta_n^s} \mathcal{N}_n^{\mathsf{PL}}(X) \xrightarrow{\sigma_n^s} L_n^s(\mathbb{Z}\pi, w)&
\end{aligned}$$

which is exact for $n \geq 5$ in the sense of Theorem 9.127. There is an analogous Surgery Exact Sequence

$$\begin{aligned}
\mathcal{N}_{n+1}^{\mathsf{PL}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^h} L_{n+1}^h(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^h} \mathcal{S}_n^{\mathsf{PL},h}(X)&\\
\xrightarrow{\eta_n^h} \mathcal{N}_n^{\mathsf{PL}}(X) \xrightarrow{\sigma_n^h} L_n^h(\mathbb{Z}\pi, w).&
\end{aligned}$$

The analogous sequences exists in the topological category, namely there is a surgery sequence

$$\mathcal{N}^{\mathsf{TOP}}_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma^s_{n+1}} L^s_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial^s_{n+1}} \mathcal{S}^{\mathsf{TOP},s}_n(X)$$
$$\xrightarrow{\eta^s_n} \mathcal{N}^{\mathsf{TOP}}_n(X) \xrightarrow{\sigma^s_n} L^s_n(\mathbb{Z}\pi, w)$$

which is exact for $n \geq 5$ in the sense of Theorem 9.127, and an analogous Surgery Exact Sequence

$$\mathcal{N}^{\mathsf{TOP}}_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma^h_{n+1}} L^h_{n+1}(\mathbb{Z}\pi, w) \xrightarrow{\partial^h_{n+1}} \mathcal{S}^{\mathsf{TOP},h}_n(X)$$
$$\xrightarrow{\eta^h_n} \mathcal{N}^{\mathsf{TOP}}_n(X) \xrightarrow{\sigma^h_n} L^h_n(\mathbb{Z}\pi, w).$$

Note that the surgery obstruction groups are the same in the smooth category, PL category, and topological category. Only the set of normal invariants and the structure sets are different. The set of normal invariants in the smooth category, PL category or topological category do not depend on the decoration h and s, whereas the structure sets and the surgery obstruction groups depend on the decoration h and s. In particular the structure set depends on both the choice of category and choice of decoration.

As in the smooth setting the surgery sequence above can be extended to infinity to the left.

Some interesting constructions can be carried out in the topological category, which do not have smooth counterparts.

Remark 9.131 (The total surgery obstruction). Given a finite Poincaré complex X of dimension ≥ 5, a single obstruction, the so-called *total surgery obstruction*, is constructed in [823, §17], see also [579]. It vanishes if and only if X is homotopy equivalent to a closed topological manifold. It combines the two stages of the classical obstructions, namely, the problem whether the Spivak normal fibration has a reduction to a TOP-bundle (which is equivalent to the condition that $\mathcal{N}^{\mathsf{TOP}}(X)$ is non-empty) and whether the surgery obstruction of the associated normal map is trivial.

Remark 9.132 (Group structures on the surgery sequence). An algebraic surgery sequence is constructed in [823, § 14, § 18] and identified with the geometric surgery sequence above in the topological category. Moreover, in the topological situation one can find abelian group structures on $\mathcal{S}^{\mathsf{TOP},s}_n(X)$, $\mathcal{S}^{\mathsf{TOP},h}_n(X)$ and $\mathcal{N}^{\mathsf{TOP}}_n(X)$ such that the surgery sequence becomes a sequence of abelian groups. The main point is to find the right addition on G/TOP.

There cannot be a group structure in the smooth category for $\mathcal{S}^h_n(X)$ and $\mathcal{N}_n(X)$ such that $\mathcal{S}^h_n(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi, w)$ is a sequence of groups (and analogous for the simple version), see [249]. Note that the composite, see Theorem 9.129,

$$[X; \mathrm{G/O}] \cong \mathcal{N}_n(X) \xrightarrow{\sigma^s_n} L^s_n(\mathbb{Z}\pi, w)$$

is a map whose source and target come with canonical group structures but it is not a homomorphism of abelian groups in general, see [968, page 114]. The same problem arises with the decoration h. More information about this topic can be found for instance in [652, Sections 11.8 and 17.6].

Remark 9.133 (The homotopy type of G/TOP **and** TOP/PL**).** The computation of the homotopy type of the space G/TOP (and also of G/PL) due to Sullivan [915] is explained in detail in [682, Chapter 4]. One obtains homotopy equivalences

$$\begin{aligned} \mathrm{G/TOP}\left[\frac{1}{2}\right] &\simeq BO\left[\frac{1}{2}\right]; \\ \mathrm{G/TOP}_{(2)} &\simeq \prod_{j\geq 1} K(\mathbb{Z}_{(2)}, 4j) \times \prod_{j\geq 1} K(\mathbb{Z}/2, 4j-2), \end{aligned}$$

where $K(A,i)$ denotes the *Eilenberg-MacLane space* of type (A,i), i.e., a CW-complex such that $\pi_n(K(A,i))$ is trivial for $n \neq i$ and is isomorphic to A if $n = i$, the subscript $_{(2)}$ stands for localizing at (2), i.e., all primes except 2 are inverted, and $\left[\frac{1}{2}\right]$ stands for localization of 2, i.e. 2 is inverted. In particular we get for a space X isomorphisms

$$\begin{aligned} [X, \mathrm{G/TOP}]\left[\frac{1}{2}\right] &\cong \widetilde{KO}^0(X)\left[\frac{1}{2}\right]; \\ [X, \mathrm{G/TOP}]_{(2)} &\cong \prod_{j\geq 1} H^{4j}(X; \mathbb{Z}_{(2)}) \times \prod_{j\geq 1} H^{4j-2}(M; \mathbb{Z}/2), \end{aligned}$$

where KO^* is K-theory of real vector bundles, see Subsection 10.2.2.

The various groups G, TOP, and PL, and their (homotopy theoretic) quotients G/PL, PL/O and G/PL fit into a braid by inspecting long exact sequences of fibrations. This braid can be interpreted geometrically in terms of L-groups, bordism groups, and homotopy groups of exotic spheres in dimensions ≥ 5, see for instance [652, Chapter 12].

Kirby and Siebenmann [562, Theorem 5.5 in Essay V on page 251], see also [867], have proved

Theorem 9.134. *The space* TOP/PL *is an Eilenberg MacLane space of type* $(\mathbb{Z}/2, 3)$.

More information about the homotopy type of G/O, G/PL, and G/TOP can be found for instance in [652, Chapter 17].

9.14 The Novikov Conjecture

In this section we introduce the Novikov Conjecture in its original form in terms of higher signatures and make a first link to surgery theory. It follows from both the Baum-Connes Conjecture and the Farrell-Jones Conjecture and has been an important interface between topology and non-commutative geometry.

9.14.1 The Original Formulation of the Novikov Conjecture

Let G be a (discrete) group. Let $u\colon M \to BG$ be a map from an oriented closed smooth manifold M to BG. Let

$$\mathcal{L}(M) \in \bigoplus_{k\in\mathbb{Z},k\geq 0} H^{4k}(M;\mathbb{Q}) \tag{9.135}$$

be the *L-class of M*. Its k-th entry $\mathcal{L}(M)_k \in H^{4k}(M;\mathbb{Q})$ is a certain homogeneous polynomial of degree k in the rational Pontrjagin classes $p_i(M;\mathbb{Q}) \in H^{4i}(M;\mathbb{Q})$ for $i = 1,2,\ldots,k$ such that the coefficient s_k of the monomial $p_k(M;\mathbb{Q})$ is different from zero. It is defined in terms of multiplicative sequences, see for instance [714, § 19]. We mention at least the first values

$$\begin{aligned}
\mathcal{L}(M)_1 &= \frac{1}{3} \cdot p_1(M;\mathbb{Q});\\
\mathcal{L}(M)_2 &= \frac{1}{45} \cdot \left(7 \cdot p_2(M;\mathbb{Q}) - p_1(M;\mathbb{Q})^2\right);\\
\mathcal{L}(M)_3 &= \frac{1}{945} \cdot \left(62 \cdot p_3(M;\mathbb{Q}) - 13 \cdot p_1(M;\mathbb{Q}) \cup p_2(M;\mathbb{Q}) + 2 \cdot p_1(M;\mathbb{Q})^3\right).
\end{aligned}$$

The L-class $\mathcal{L}(M)$ is determined by all the rational Pontrjagin classes and vice versa. Recall that the *k-th rational Pontrjagin class* $p_k(M,\mathbb{Q}) \in H^{4k}(M;\mathbb{Q})$ is defined as the image of *k-th Pontrjagin class* $p_k(M)$ under the obvious change of coefficients map $H^{4k}(M;\mathbb{Z}) \to H^{4k}(M;\mathbb{Q})$. The L-class depends on the tangent bundle and thus on the differentiable structure of M. For $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$ define the *higher signature of M associated to x and u* to be

$$\operatorname{sign}_x(M,u) := \langle \mathcal{L}(M) \cup u^*x, [M]_{\mathbb{Q}}\rangle \qquad \in \mathbb{Q} \tag{9.136}$$

where $[M]_{\mathbb{Q}}$ denotes the image of the fundamental class $[M]$ of an oriented closed d-dimensional manifold M under the change of coefficients map $H_d(M;\mathbb{Z}) \to H_d(M;\mathbb{Q})$. Recall that for $\dim(M) = 4n$ the *signature* $\operatorname{sign}(M)$ of M is the signature of the non-singular bilinear symmetric pairing on the middle cohomology $H^{2n}(M;\mathbb{R})$ given by the intersection pairing $(a,b) \mapsto \langle a \cup b, [M]_{\mathbb{R}}\rangle$. Obviously $\operatorname{sign}(M)$ depends only on the oriented homotopy type of M. We say that sign_x for $x \in H^*(BG;\mathbb{Q})$ is *homotopy invariant*, if for two oriented closed closed smooth manifolds M and N with reference maps $u\colon M \to BG$ and $v\colon N \to BG$ we have

$$\operatorname{sign}_x(M,u) = \operatorname{sign}_x(N,v),$$

whenever there is an orientation preserving homotopy equivalence $f\colon M \to N$ such that $v \circ f$ and u are homotopic.

Conjecture 9.137 (Novikov Conjecture). The group G satisfies the *Novikov Conjecture* if sign_x is homotopy invariant for all elements x of $\prod_{k\in\mathbb{Z},k\geq 0} H^k(BG;\mathbb{Q})$.

This conjecture appears for the first time in the paper by Novikov [748, § 11]. A survey about its history can be found in [368].

9.14.2 Invariance Properties of the *L*-Class

One motivation for the Novikov Conjecture comes from the Signature Theorem due to Hirzebruch [481, 482].

Theorem 9.138 (Signature Theorem). *Let M be an oriented closed manifold of dimension n. Then the higher signature $\operatorname{sign}_1(M,u) = \langle \mathcal{L}(M), [M]_{\mathbb{Q}}\rangle$ associated to $1 \in H_0(M)$ and some map $u\colon M \to BG$ coincides with the signature $\operatorname{sign}(M)$ of M if $\dim(M) = 4n$, and is zero, if $\dim(M)$ is not divisible by four.*

The Signature Theorem 9.138 leads to the question whether the Pontrjagin classes or the L-classes are homotopy invariants or homeomorphism invariants. They are obviously invariants of the diffeomorphism type. However, the Pontrjagin classes $p_k(M) \in H^{4k}(M;\mathbb{Z})$ for $k \geq 2$ are not homeomorphism invariants, see for instance [570, Theorem 4.8 on page 31]. On the other hand, there is the following deep result due to Novikov [745, 746, 747].

Theorem 9.139 (Topological invariance of rational Pontrjagin classes). *The rational Pontrjagin classes $p_k(M,\mathbb{Q}) \in H^{4k}(M;\mathbb{Q})$ are topological invariants, i.e., for a homeomorphism $f\colon M \to N$ of closed smooth manifolds we have*

$$H^{4k}(f;\mathbb{Q})(p_k(N;\mathbb{Q})) = p_k(M;\mathbb{Q})$$

for all $k \geq 0$ and in particular $H^*(f;\mathbb{Q})(\mathcal{L}(N)) = \mathcal{L}(M)$.

Example 9.140 (The L-class is not a homotopy invariant). The rational Pontrjagin classes and the L-class are not homotopy invariants as the following example shows. There exists for $k \geq 1$ and large enough $j \geq 0$ a $(j+1)$-dimensional vector bundle $\xi\colon E \to S^{4k}$ with Riemannian metric whose k-th Pontrjagin class $p_k(\xi)$ is not zero and which is trivial as a fibration. The total space SE of the associated sphere bundle is a closed $(4k+j)$-dimensional manifold which is homotopy equivalent to $S^{4k} \times S^j$ and satisfies

$$p_k(SE) = -p_k(\xi) \neq 0;$$
$$\mathcal{L}(SE)_k = s_k \cdot p_k(SE) \neq 0,$$

where $s_k \neq 0$ is the coefficient of p_k in the polynomial defining the L-class. But $p_k(S^{4k} \times S^j)$ and $\mathcal{L}(S^{4k} \times S^j)_k$ vanish since the tangent bundle of $S^{4k} \times S^j$ is stably trivial. In particular SE and $S^{4k} \times S^j$ are simply connected homotopy equivalent closed manifolds which are not homeomorphic. This example is taken from [825, Proposition 2.9] and attributed to Dold and Milnor there. See also [825, Proposition 2.10] or [714, Section 20].

Remark 9.141 (The homological version of the Novikov Conjecture). One may understand the Novikov Conjecture as an attempt to figure out, how much of the L-class is a homotopy invariant of M. If one considers the oriented homotopy type and the simply connected case, it is just the expression $\langle \mathcal{L}(M), [M]_{\mathbb{Q}} \rangle$ or, equivalently, the top component of $\mathcal{L}(M)$. In the Novikov Conjecture one asks the same question, but now taking the fundamental group into account by remembering the classifying map $u_M \colon M \to B\pi_1(M)$, or, more generally, a reference map $u \colon M \to BG$. The Novikov Conjecture can also be rephrased by saying that for any group G and any pair (M, u) consisting of an oriented closed manifold M of dimension n together with a reference map $u \colon M \to BG$ the term

$$u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) \in \bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q})$$

depends only on the oriented homotopy type of the pair (M, u). This follows from the elementary computation for $x \in H^*(BG; \mathbb{Q})$

$$\langle \mathcal{L}(M) \cup u^* x, [M]_{\mathbb{Q}} \rangle = \langle u^* x, \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \rangle = \langle x, u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) \rangle$$

and the fact that the Kronecker pairing $\langle -, - \rangle$ for rational coefficients is non-singular. Note that $- \cap [M]_{\mathbb{Q}} \colon H^{n-i}(M; \mathbb{Q}) \to H_i(M; \mathbb{Q})$ is an isomorphism for all $i \geq 0$ by Poincaré duality. Hence $\mathcal{L}(M) \cap [M]_{\mathbb{Q}}$ carries the same information as $\mathcal{L}(M)$.

Exercise 9.142. Let $f \colon M \to N$ be an orientation preserving homotopy equivalence of oriented closed manifolds which are aspherical. Assume that the Novikov Conjecture 9.137 holds for $G = \pi_1(M)$. Show that then $\mathcal{L}(M) = f^*\mathcal{L}(N)$ must be true.

9.14.3 The Novikov Conjecture and Surgery Theory

Remark 9.143 (The Novikov Conjecture and assembly map). There exists an assembly map

$$\mathrm{asmb}_n^G\colon \bigoplus_{k\in\mathbb{Z}} H_{n+4k}(BG;\mathbb{Q}) \to L_n^h(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q},$$

which fits into the following commutative diagram

$$\begin{tikzcd}
\mathcal{S}_n^{\mathsf{TOP},h}(M) \arrow[r, "\eta_n^h"] \arrow[ddr, "s"'] & \mathcal{N}_n^{\mathsf{TOP}}(M) \arrow[r, "\sigma_n^s"] \arrow[d, "b"] & L_n^h(\mathbb{Z}\pi_1(M)) \arrow[d, "u_*"] \\
& {[M, \mathrm{G/TOP}]} \arrow[d, "c"] & L_n^h(\mathbb{Z}G) \arrow[d, "i"] \\
& \bigoplus_{k\in\mathbb{Z}} H_{n+4k}(BG;\mathbb{Q}) \arrow[r, "\mathrm{asmb}_n^G"] & L_n^h(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}.
\end{tikzcd}$$

The map i is the obvious map and u_* is the homomorphism coming from $\pi_1(u)\colon \pi_1(M)\to\pi_1(BG)=G$. The bijection b is taken from Theorem 9.129. The map c comes from the rational version of the homotopy equivalences describing G/TOP appearing in Remark 9.133, and Poincaré duality. The composite $c\circ b$ sends the class of a normal map $(\overline{f},f)$ with underlying map $f\colon N\to M$ of degree one to $(u\circ f)_*(\mathcal{L}(N)\cap[N]_{\mathbb{Q}}) - u_*(\mathcal{L}(M)\cap[M]_{\mathbb{Q}})$. This fact is for instance explained in [611, page 728]. The map s is defined analogously, it sends the class $[f]$ of a homotopy equivalence $f\colon N\to M$ to the difference $(u\circ f)u_*(\mathcal{L}(N)\cap[N]_{\mathbb{Q}}) - u_*(\mathcal{L}(M)\cap[M]_{\mathbb{Q}})$ where we choose $[N]$ such that the map f has degree one. We conclude from Remark 9.141 that the Novikov Conjecture 9.137 is equivalent to the statement that s is trivial. The upper row is part of the Surgery Exact Sequence of Theorem 9.130. This implies that the composite

$$\mathcal{S}_n^{\mathsf{TOP},h}(M) \xrightarrow{s} \bigoplus_{k\in\mathbb{Z}} H_{n+4k}(BG;\mathbb{Q}) \xrightarrow{\mathrm{asmb}_n^G} L_n^h(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is trivial.

Thus we can conclude that the group G satisfies the Novikov Conjecture 9.137 if the map $\mathrm{asmb}_n^G\colon \bigoplus_{k\in\mathbb{Z}} H_{n+4k}(BG;\mathbb{Q}) \to L_n^h(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}$ is injective. See also Kaminker-Miller [524] or [570, Proposition 15.4 on page 112]. Note that the last map involves only G. This conclusion will be a key ingredient in the proof that the L-theoretic Farrell-Jones Conjecture for G implies the Novikov Conjecture 9.137, see Theorem 13.62 (xi).

Remark 9.144 (The converse of the Novikov Conjecture). A kind of converse to the Novikov Conjecture 9.137 is the following result. Let N be an oriented connected closed smooth manifold of dimension $n\geq 5$. Let $u\colon N\to BG$ be a map inducing an isomorphism on the fundamental groups. Consider any element $l\in\prod_{i\geq 0} H^{4i}(N;\mathbb{Q})$ such that $u_*(l\cap[N]_{\mathbb{Q}})=0$ holds in $H_*(BG;\mathbb{Q})$. Then there exists a non-negative integer K such that for any multiple k of K there is a homotopy equivalence $f\colon M\to N$ of oriented closed smooth manifolds satisfying

$$f^*(\mathcal{L}(N) + k \cdot l) = \mathcal{L}(M).$$

A proof can be found for instance in [263, Theorem 6.5]. This shows that the top dimension part of the L-class $\mathcal{L}(M)$ is essentially the only homotopy invariant rational characteristic class for simply connected closed $4k$-dimensional manifolds.

More information about the Novikov Conjecture can be found for instance in [369, 370, 570, 849, 1010]. An algebraic geometric and an equivariant version of the Novikov Conjecture is introduced in [847] and [854].

9.15 Topologically Rigidity and the Borel Conjecture

In this section we deal with the Borel Conjecture and how it follows from the Farrell-Jones Conjecture in dimensions ≥ 5.

9.15.1 Aspherical Spaces

Definition 9.145 (Aspherical). A space X is called *aspherical*, if it is path connected and all its higher homotopy groups vanish, i.e., $\pi_n(X)$ is trivial for $n \geq 2$.

Remark 9.146 (Homotopy classification of aspherical CW-complexes). A CW-complex is aspherical if and only if it is connected and its universal covering is contractible. Given two aspherical CW-complexes X and Y, the map from the set of homotopy classes of maps $X \to Y$ to the set of group homomorphisms $\pi_1(X) \to \pi_1(Y)$ modulo inner automorphisms of $\pi_1(Y)$ given by the map induced on the fundamental groups is a bijection. In particular, two aspherical CW-complexes are homotopy equivalent if and only if they have isomorphic fundamental groups and every isomorphism between their fundamental groups comes from a homotopy equivalence.

Remark 9.147 (Classifying space of a group). An aspherical CW-complex X with fundamental group π is the same as an Eilenberg Mac-Lane space $K(\pi, 1)$ of type $(\pi, 1)$ and the same as the classifying space $B\pi$ for the group π.

Exercise 9.148. Let $F \to E \to B$ be a fibration. Suppose that F and B are aspherical. Show that then E is aspherical.

Exercise 9.149. Let X be an aspherical CW-complex of finite dimension. Show that $\pi_1(X)$ is torsionfree.

Example 9.150 (Examples of aspherical manifolds).

(i) A connected closed 1-dimensional manifold is homeomorphic to S^1 and hence aspherical;

(ii) Let M be a connected closed 2-dimensional manifold. Then M is either aspherical or homeomorphic to S^2 or $\mathbb{RP}^2$;

(iii) A connected closed 3-manifold M is called *prime* if for any decomposition as a connected sum $M \cong M_0 \sharp M_1$ one of the summands M_0 or M_1 is homeomorphic to S^3. It is called *irreducible*, if any embedded sphere S^2 bounds a disk D^3. Every irreducible closed 3-manifold is prime. A prime closed 3-manifold is either irreducible or an S^2-bundle over S^1. The following statements are equivalent for a closed 3-manifold M:

- M is aspherical;
- M is irreducible and its fundamental group is infinite and contains no element of order 2;
- The fundamental group $\pi_1(M)$ cannot be written in a non-trivial way as an amalgamated free product of two groups, is infinite, different from $\mathbb{Z}$, and contains no element of order 2.
- The universal covering of M is homeomorphic to $\mathbb{R}^3$.

(iv) Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a discrete torsionfree subgroup. Then $M = G\backslash L/K$ is an aspherical closed manifold with fundamental group G since its universal covering L/K is diffeomorphic to $\mathbb{R}^n$ for appropriate n;

(v) Every closed Riemannian (smooth) manifold with non-positive sectional curvature has a universal covering which is diffeomorphic to $\mathbb{R}^n$ and is in particular aspherical.

Exercise 9.151. Classify all simply connected aspherical closed manifolds.

Exercise 9.152. Suppose that M is a connected sum $M_1 \sharp M_2$ of two closed manifolds M_1 and M_2 of dimension $n \geq 3$, which are not homotopy equivalent to a sphere. Show that M is not aspherical.

There exists exotic aspherical manifolds as the following results illustrate. The following theorem is due to Davis-Januszkiewicz [276, Theorem 5a.4].

Theorem 9.153 (Non-PL-example).
For every $n \geq 4$ there exists an aspherical closed topological n-manifold that is not homotopy equivalent to a PL*-manifold*

The following result is proved by Davis-Fowler-Lafont [275] using the work of Manolescu [691, 690].

Theorem 9.154 (Non-triangulable aspherical closed manifolds). *There exists for each $n \geq 6$ an n-dimensional aspherical closed topological manifold that cannot be triangulated. One can arrange that the fundamental group is hyperbolic.*

The proof of the following theorem can be found in [273], [276, Theorem 5b.1].

Theorem 9.155 (Exotic universal covering of aspherical closed manifolds). *For each $n \geq 4$ there exists an aspherical closed n-dimensional manifold such that its universal covering is not homeomorphic to $\mathbb{R}^n$.*

By the Hadamard-Cartan Theorem, see [398, 3.87 on page 134], the manifold appearing in Theorem 9.155 above cannot be homeomorphic to a smooth manifold with Riemannian metric with non-positive sectional curvature.

The following theorem is proved in [276, Theorem 5c.1 and Remark on page 386].

Theorem 9.156 (Exotic aspherical closed manifolds with hyperbolic fundamental group).

For every $n \geq 5$, there exists an aspherical closed smooth n-dimensional manifold M that is homeomorphic to a strictly negatively curved polyhedron and has in particular a hyperbolic fundamental group such that the universal covering is homeomorphic to $\mathbb{R}^n$, but M is not homeomorphic to a smooth manifold with Riemannian metric with negative sectional curvature.

The next results are due to Belegradek [114, Corollary 5.1], Mess [700], and Weinberger, see [272, Section 13].

Theorem 9.157 (Aspherical closed manifolds with exotic fundamental groups).

(i) *For every $n \geq 4$, there is an aspherical closed topological manifold of dimension n whose fundamental group contains an infinite divisible abelian group;*
(ii) *For every $n \geq 4$, there is an aspherical closed* PL *manifold of dimension n whose fundamental group has an unsolvable word problem and whose simplicial volume in the sense of Gromov [421] is non-zero.*

More information about fundamental groups of aspherical closed manifolds with unusual properties can be found for instance in [871].

The question, when the isometry group of the universal covering of an aspherical closed manifold is non-discrete, is studied by Farb-Weinberger [332].

Remark 9.158 (S^1-actions on aspherical closed manifolds). If S^1 acts on an aspherical closed manifold, then the orbit circle is a non-trivial element in the center by a result of Borel, see for instance [235, Lemma 5.1 on page 242]. There is the conjecture of Conner-Raymond [235, page 229] stating that the converse is true, namely, if the fundamental group of an aspherical closed manifold has nontrivial center, then the manifold has a circle action, such that the orbit circle is a nontrivial central element of the fundamental group. A counterexample in dimensions ≥ 6 was constructed by Cappell-Weinberger-Yan [196].

It is an open question whether the conjecture of Conner-Raymond above is true if one allows the passage to a finite covering.

Another interesting open question is whether the center of the fundamental group of an aspherical closed manifold is finitely generated.

For more information about aspherical closed manifolds we refer for instance to [645].

9.15.2 Formulation and Relevance of the Borel Conjecture

Definition 9.159 (Topologically rigid). We call a closed topological manifold N *topologically rigid* if any homotopy equivalence $M \to N$ with a closed topological manifold M as source is homotopic to a homeomorphism.

Conjecture 9.160 (Borel Conjecture (for a group G in dimension n)). The *Borel Conjecture for a group G in dimension n* predicts for two aspherical closed topological manifolds M and N of dimensions n with $\pi_1(M) \cong \pi_1(N) \cong G$, that M and N are homeomorphic and any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

The *Borel Conjecture* says that every aspherical closed topological manifold is topologically rigid.

Remark 9.161 (The Borel Conjecture in low dimensions). The Borel Conjecture is true in dimension ≤ 2. It is true in dimension 3 if Thurston's Geometrization Conjecture is true. This follows from results of Waldhausen, see Hempel [460, Lemma 10.1 and Corollary 13.7], and Turaev, see [939], as explained for instance in [571, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [563, 734] following ideas of Perelman. Some information in dimension 4 can be found in Davis [264].

Remark 9.162 (Topological rigidity for non-aspherical manifolds). Topological rigidity phenomenons do hold also for some non-aspherical closed manifolds. For instance the sphere S^n is topologically rigid by the Poincaré Conjecture. The Poincaré Conjecture is known to be true in all dimensions. This follows in high dimensions from the h-cobordism theorem, in dimension four from the work of Freedman [386, 387], in dimension three from the work of Perelman as explained in [563, 733], and and in dimension two from the classification of surfaces.

Many more examples of classes of manifolds which are topologically rigid are given and analyzed in Kreck-Lück [571]. For instance, the connected sum of closed manifolds of dimension ≥ 5 which are topologically rigid and whose fundamental groups do not contain elements of order two is again topologically rigid. The product $S^k \times S^n$ is topologically rigid if and only if k and n are odd. An integral homology sphere of dimension $n \geq 5$ is topologically rigid if and only if the inclusion $\mathbb{Z} \to \mathbb{Z}[\pi_1(M)]$ induces an isomorphism of simple L-groups $L^s_{n+1}(\mathbb{Z}) \to L^s_{n+1}\big(\mathbb{Z}[\pi_1(M)]\big)$. Every 3-manifold with torsionfree fundamental group is topologically rigid.

Exercise 9.163. Give an example of a closed orientable 3-manifold with finite fundamental group that is not topologically rigid.

Exercise 9.164. Give an example of two topologically rigid orientable closed smooth manifolds whose cartesian product is not topologically rigid.

Remark 9.165 (The Borel Conjecture does not hold in the smooth category). The Borel Conjecture 9.160 is false in the smooth category, i.e., if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The torus T^n for $n \geq 5$ is an example, see [968, 15A]. Other counterexample involving negatively curved manifolds are constructed by Farrell-Jones [347, Theorem 0.1].

Remark 9.166 (The Borel Conjecture versus Mostow rigidity). The examples of Farrell-Jones [347, Theorem 0.1] give actually more. Namely, they yield for given $\epsilon > 0$ a closed Riemannian manifold M_0 whose sectional curvature lies in the interval $[1-\epsilon, -1+\epsilon]$ and a closed hyperbolic manifold M_1 such that M_0 and M_1 are homeomorphic but not diffeomorphic. The idea of the construction is essentially to take the connected sum of M_1 with exotic spheres. Note that by definition M_0 were hyperbolic if we could take $\epsilon = 0$. Hence this example is remarkable in view of *Mostow rigidity*, which predicts for two closed hyperbolic manifolds N_0 and N_1 that they are isometrically diffeomorphic if and only if $\pi_1(N_0) \cong \pi_1(N_1)$ and any homotopy equivalence $N_0 \to N_1$ is homotopic to an isometric diffeomorphism.

One may view the Borel Conjecture as the topological version of Mostow rigidity. The conclusion in the Borel Conjecture is weaker, one gets only homeomorphisms and not isometric diffeomorphisms, but the assumption is also weaker since there are many more aspherical closed topological manifolds than hyperbolic closed manifolds.

Remark 9.167 (The work of Farrell-Jones). Farrell-Jones have made deep contributions to the Borel Conjecture. They have proved it in dimension ≥ 5 for non-positively curved closed Riemannian manifolds, for compact complete affine flat manifolds, and for aspherical closed manifolds whose fundamental group is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold that is A-regular. Relevant references are [348, 349, 352, 354, 355].

The Borel Conjecture for higher dimensional graph manifolds is studied by Frigerio-Lafont-Sisto [392].

More information about the Borel Conjecture can be found in [652, Chapter 19] and [980].

9.15.3 The Farrell-Jones and the Borel Conjecture

Theorem 9.168 (The Farrell-Jones and the Borel Conjecture). *Let G be a finitely presented group. Suppose that it satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$.*

Then every aspherical closed manifold of dimension ≥ 5 with G as fundamental group is topologically rigid, in other words, the Borel Conjecture 9.160 holds for G in dimensions ≥ 5.

For its proof we need the following lemma.

Lemma 9.169. *Let M be a closed topological manifold with $\mathrm{Wh}(\pi_1(M)) = 0$. Then M is topologically rigid if and only if the simple topological structure set $\mathcal{S}^{\mathsf{TOP},s}(M)$ consists of precisely on element, namely the class of id_M.*

Proof. Suppose that M is topologically rigid. Consider any element in $\eta \in \mathcal{S}^{\mathsf{TOP},s}(M)$. Choose a simple homotopy equivalence $f\colon N \to M$ representing η. Since M is topologically rigid, f is homotopic to a homeomorphism $h\colon N \to M$. Hence $\mathrm{id}_M \circ h \simeq f$. This implies that η is represented by id_M.

Suppose that $\mathcal{S}^{\mathsf{TOP},s}(M)$ consists only of one class, the one represented by id_M. Consider any homotopy equivalence $f\colon N \to M$. Since $\mathrm{Wh}(\pi_1(M)) = 0$ holds by assumption, f is a simple homotopy equivalence and thus represents an element in $\mathcal{S}^{\mathsf{TOP},s}(M)$. Since it represents the same class as id_M by assumption, there exists a homeomorphism $h\colon N \to M$ such that $h = \mathrm{id}_M \circ h$ is homotopic to f. $\square$

Lemma 9.170. *Let M be a closed topological manifold of dimension $n \geq 5$. Let $w\colon \pi := \pi_1(M) \to \{\pm 1\}$ be given by its first Stiefel-Whitney class. Suppose $\mathrm{Wh}(\pi_1(M)) = 0$. Assume that the homomorphism of abelian groups $\sigma^s_{n+1}\colon \mathcal{N}^{\mathsf{TOP}}_{n+1}(M \times [0,1], M \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi, w)$ of (9.123) is surjective and that the preimage of 0 under the map $\sigma^s_n\colon \mathcal{N}^{\mathsf{TOP}}_n(X) \to L^s_n(\mathbb{Z}\pi, w)$ of (9.124) consists of one point.*

Then M is topologically rigid.

Proof. This follows from the simple topological Surgery Exact Sequence of Theorem 9.130 and Lemma 9.169. $\square$

Now we can give a sketch of the proof of Theorem 9.168.

Sketch of the proof of Theorem 9.168. We deal for simplicity with the orientable case, i.e., $w_1 = 0$, only. Let $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$ be the L-theory spectrum appearing in the version of the L-theoretic Farrell-Jones Conjecture 9.114. Since it holds by assumption, the so-called assembly map

$$\mathrm{asmb}^{\langle -\infty \rangle}_k\colon H_k(B\pi; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L^{\langle -\infty \rangle}_k(\mathbb{Z}\pi)$$

is bijective for all k. Let $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle$ be the 1-connected cover of $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$. This spectrum comes with a map of spectra $\mathbf{i}\colon \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle \to \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$ such that $\pi_k(\mathbf{i})$ is bijective for $k \geq 1$ and $\pi_k(\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle) = 0$ for $k \leq 0$. For $k \geq 1$ there is a connective version of the assembly map asmb_k above

$$\operatorname{asmb}_k^{\langle -\infty \rangle}\langle 1 \rangle\colon H_k\bigl(B\pi; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle\bigr) \to L_k^{\langle -\infty \rangle}(\mathbb{Z}\pi)$$

such that $\pi_k(\mathbf{i}) \circ \operatorname{asmb}_k\langle 1 \rangle = \operatorname{asmb}_k$ holds. A comparison argument of the Atiyah-Hirzebruch spectral sequence shows that the bijectivity of $\operatorname{asmb}_k^{\langle -\infty \rangle}$ for $k = n, n+1$ implies that $\operatorname{asmb}_{n+1}^{\langle -\infty \rangle}\langle 1 \rangle$ is bijective and in particular surjective and $\operatorname{asmb}_n^{\langle -\infty \rangle}\langle 1 \rangle$ is injective if n is the dimension of the aspherical closed manifold under consideration. Because by assumption Conjectures 3.110 and 4.20 hold for π, we conclude from Theorem 9.106 that the simple versions of the 1-connective assembly maps

$$\operatorname{asmb}_k^s\langle 1 \rangle\colon H_k\bigl(B\pi; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle\bigr) \to L_k^s(\mathbb{Z}\pi)$$

agree with the maps $\operatorname{asmb}_k^{\langle -\infty \rangle}\langle 1 \rangle$. One can identify the map $\operatorname{asmb}_{n+1}^s\langle 1 \rangle$ with the map $\sigma_{n+1}^s\colon \mathcal{N}_{n+1}^{\mathsf{TOP}}(M \times [0,1], M \times \{0,1\}) \to L_{n+1}^s(\mathbb{Z}\pi)$ of (9.123) and the map $\operatorname{asmb}_n^s\langle 1 \rangle$ with the map $\sigma_n^s\colon \mathcal{N}_n^{\mathsf{PL}}(X) \to L_n^s(\mathbb{Z}\pi)$ of (9.124), see [823, Theorem 18.5 on page 198], [818], [579] using Remark 18.18, Remark 18.19, and Example 18.23.

Now Theorem 9.168 follows from Lemma 9.170. □

Remark 9.171 (Dimension 4). The conclusion of Theorem 9.168 hold also in dimension 4, provided that the fundamental group is good in the sense of Freedman, see [386, 387]. Groups of subexponential growth are good, see [389, 577].

Remark 9.172 (The Novikov Conjecture implies a stable version of the Borel Conjecture). For a group G that satisfies the Novikov Conjecture 9.137, the following stable version of the Borel Conjecture holds: For any homotopy equivalence $f\colon M \to N$ of aspherical closed manifolds of dimension ≥ 5 whose fundamental groups are isomorphic to G, the map $f \times \operatorname{id}_{\mathbb{R}^3}\colon M \times \mathbb{R}^3 \to N \times \mathbb{R}^3$ is homotopic to a homeomorphism. See [504, Proposition 2.8], where the proof is attributed to Shmuel Weinberger, see also [341, Proof of Corollary B on page 207].

Remark 9.173 (Homology-ANR-manifolds). If one works in the category of homology ANR-manifolds, one does not have to pass to the 1-connective cover, see [164, Main Theorem].

9.16 Homotopy Spheres

An oriented closed smooth manifold is called a *homotopy sphere* if it is homotopy equivalent to the standard sphere. By the Poincaré Conjecture a homotopy sphere is always homeomorphic to a standard sphere and actually topologically rigid. However, it may not be diffeomorphic to a standard sphere, and in this case it is called an *exotic homotopy sphere.*

The classification of homotopy spheres due to Kervaire-Milnor[559] marks the beginning of surgery theory. In order to understand the surgery machinery and in particular the long exact surgery sequence, we recommend to the reader to study the classification of homotopy spheres, which boils down to compute $\mathcal{S}_n^s(S^n)$. Moreover, there are some beautiful constructions of exotic spheres and results about the curvature properties of Riemannian metric on an exotic sphere. We refer for instance to the following survey articles [509], [595], [612], and [633, Chapter 6], and to [652, Chapter 12].

9.17 Poincaré Duality Groups

The following definition is due to Johnson-Wall [511].

Definition 9.174 (Poincaré duality group).
A group G is called a *Poincaré duality group of dimension* n if the following conditions holds:

(i) The group G is of type FP, i.e., the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ possesses a finite dimensional projective $\mathbb{Z}G$-resolution by finitely generated projective $\mathbb{Z}G$-modules;
(ii) We get an isomorphism of abelian groups

$$H^i(G;\mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

Recall that a CW-complex X is called *finitely dominated* if there exists a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ with $r \circ i \simeq \mathrm{id}_X$.

A metric space X is called an *absolute neighborhood retract* or briefly ANR if, for every embedding $i\colon X \to Y$ as a closed subspace into a metric space Y, there is an open neighbourhood U of $\mathrm{im}(i)$ together with a retraction $r\colon U \to \mathrm{im}(i)$, or, equivalently, for every metric space Z, every closed subset $Y \subseteq Z$, and every (continuous) map $f\colon Y \to X$, there exists an open neighborhood U of Y in Z together with an extension $F\colon U \to X$ of f to U. Every ANR is locally contractible, see [488, Theorem 7.1 in Chapter III on page 96]. A metrizable space of finite dimension is an ANR if and only if it is locally contractible, see [488, Theorem 7.1 in Chapter V on page 168]. Being an ANR is a local property, see [488, Theorem 8.1 in Chapter III on page 98]. Every finite CW-complex and every topological manifold is an ANR. Another good source about ANR-s is the book by Borsuk [145].

A *compact n-dimensional homology* ANR-*manifold* X is a compact absolute neighborhood retract such that it has a countable basis for its topology, has finite topological dimension, see Definition 23.35, and for every $x \in X$ the abelian group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$. A closed n-dimensional topological manifold is an example of a compact n-dimensional homology ANR-manifold, see [261, Corollary 1A in V.26 page 191].

Exercise 9.175. Show that the product of two Poincaré duality groups is again a Poincaré duality group.

Theorem 9.176 (Homology ANR-manifolds and finite Poincaré complexes). *Let M be a closed topological manifold, or more generally, a compact homology* ANR*-manifold of dimension n. Then M is homotopy equivalent to a finite n-dimensional Poincaré complex.*

Proof. A closed topological manifold, and more generally a compact ANR, has the homotopy type of a finite CW-complex, see [562, Theorem 2.2], [985]. The usual proof of Poincaré duality for closed manifolds carries over to homology ANR-manifolds. □

Theorem 9.177 (Poincaré duality groups). *Let G be a group and $n \geq 1$ be an integer. Then:*

(i) The following assertions are equivalent:

- *(a) G is finitely presented and a Poincaré duality group of dimension n;*
- *(b) There exists a finitely dominated n-dimensional aspherical Poincaré complex with G as fundamental group;*

(ii) Suppose that $\widetilde{K}_0(\mathbb{Z}G) = 0$. Then the following assertions are equivalent:

- *(a) G is finitely presented and a Poincaré duality group of dimension n;*
- *(b) There exists a finite n-dimensional aspherical Poincaré complex with G as fundamental group;*

(iii) A group G is a Poincaré duality group of dimension 1 if and only if $G \cong \mathbb{Z}$;
(iv) A group G is a Poincaré duality group of dimension 2 if and only if G is isomorphic to the fundamental group of an aspherical closed surface;

Proof. (i) Every finitely dominated CW-complex has a finitely presented fundamental group since every finite CW-complex has a finitely presented fundamental group and a group that is a retract of a finitely presented group is again finitely presented, see [964, Lemma 1.3]. If there exists a CW-model for BG of dimension n, then the cohomological dimension of G satisfies $\operatorname{cd}(G) \leq n$ and the converse is true, provided that $n \geq 3$, see [161, Theorem 7.1 in Chapter VIII.7 on page 205], [316], [964], and [965]. This implies that the implication (ib) $\implies$ (ia) holds for all $n \geq 1$ and that the implication (ia) $\implies$ (ib) holds for $n \geq 3$. For more details we refer to [511, Theorem 1]. The remaining part to show the implication (ia) $\implies$ (ib) for $n = 1, 2$, follows from assertions (iii) and (iv).

(ii) This follows in dimension $n \geq 3$ from assertion (i) and Wall's results about the finiteness obstruction, which decides whether a finitely dominated CW-complex is homotopy equivalent to a finite CW-complex, and takes values in $\widetilde{K}_0(\mathbb{Z}\pi)$, see [367, 725, 964, 965] or Section 2.5. The implication (iib) $\implies$ (iia) holds for all $n \geq 1$. The remaining part to show the implication (iia) $\implies$ (iib) holds, follows from assertions (iii) and (iv).

(iii) Since $S^1 = B\mathbb{Z}$ is a 1-dimensional closed manifold, $\mathbb{Z}$ is a finite Poincare duality group of dimension 1 by Theorem 9.176. We conclude from the (easy) implication (ib) $\implies$ (ia) appearing in assertion (i) that $\mathbb{Z}$ is a Poincaré duality group of dimension 1. Suppose that G is a Poincaré duality group of dimension 1. Since the cohomological dimension of G is 1, it has to be a free group, see [907, 923]. Since the homology group of a group of type FP is finitely generated, G is isomorphic to a finitely generated free group F_r of rank r. Since $H^1(BF_r) \cong \mathbb{Z}^r$ and $H_0(BF_r) \cong \mathbb{Z}$, Poincaré duality can only hold for $r = 1$, i.e., G is $\mathbb{Z}$.

(iv) This is proved in [314, Theorem 2]. See also [131, 132, 312, 315]. $\square$

Conjecture 9.178 (Manifold structures on aspherical Poincaré complexes). Every finitely dominated aspherical Poincaré complex is homotopy equivalent to a closed topological manifold.

Remark 9.179 (Existence and uniqueness part of the Borel Conjecture). Conjecture 9.178 can be viewed as the existence part of the Borel Conjecture 9.160, namely, the question whether an aspherical finite Poincaré complex carries up to homotopy the structure of a closed topological manifold. The Borel Conjecture 9.160 as stated above is the uniqueness part.

Conjecture 9.180 (Poincaré duality groups). A finitely presented group is an n-dimensional Poincaré duality group if and only if it is the fundamental group of an aspherical closed n-dimensional topological manifold.

The *disjoint disk property* says that for any $\epsilon > 0$ and maps $f, g\colon D^2 \to M$ there are maps $f', g'\colon D^2 \to M$ so that the distance between f and f' and the distance between g and g' are bounded by ϵ and $f'(D^2) \cap g'(D^2) = \emptyset$.

Theorem 9.181 (Poincaré duality groups and aspherical compact homology ANR-manifolds). *Suppose that the torsionfree group G is a finitely presented Poincaré duality group of dimension $n \geq 6$ and satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$. Let X be some aspherical finite Poincaré complex with $\pi_1(X) \cong G$. (It exists because of Theorem 9.177* (ii)*.) Suppose that the Spivak normal fibration of X admits a* TOP*-reduction.*

Then BG is homotopy equivalent to an aspherical compact homology ANR-*manifold satisfying the disjoint disk property.*

Proof. See [823, Remark 25.13 on page 297], [164, Main Theorem on page 439 and Section 8] and [166, Theorem A and Theorem B]. □

Remark 9.182. Note that in Theorem 9.181 the condition appears that for some aspherical finite Poincaré complex X with $\pi_1(X) \cong G$ the Spivak normal fibration of X admits a TOP-reduction. This condition does not appear in earlier versions. The reason is that there seems to be a mistake in [164] as explained in the Erratum [165]. The problem was pointed out by Hebestreit-Land-Winges, see [459]. The problem is that the proof that any compact homology ANR-manifold has a TOP-reduction of its Sprivak normal fibration is not correct. In the applications of [164] to results appearing in this book one has either to assume that the TOP-reduction exists or to prove its existence. This is the reason why this extra assumption in Theorem 9.181 appears.

As pointed out in [165], Theorem 9.185 and 9.189 remain true with adding any further hypothesis. This is also true for Theorem 9.191 by the following argument. Let X_1 and X_2 be connected finite Poincare complexes. Let $p_1\colon E_1 \to X_1$ and $p_2\colon E_2 \to X_2$ be spherical fibrations representing their Spivak normal fibration. Then the fibration $p_1 * p_2\colon E_1 * E_2 \to X_1 \times X_2$ is a representative of the Spivak normal fibration of $X_1 \times X_2$, where the fiber over $(x_1, x_2) \in X_1 \times X_2$ is the join $p_1^{-1}(x_1) * p_2^{-1}(x_2)$ of the fibers of p_1 over x_1 and p_2 over x_2. Now suppose that $p_1 * p_2$ has a TOP reduction after possibly stabilization. Then $i^*(p_1 * p_2)$ has also a TOP-reduction for the inclusion $i\colon X_1 \to X_1 \times X_2$ sending x_1 to (x_1, x_2) for some fixed $x_2 \in X_2$. But $i^*(p_1 * p_2)$ is a stabilization of p_1. Hence the Spivak normal fibration of X_1 has a TOP-reduction. Analogously one sees that the Spivak normal fibration of X_2 has a TOP-reduction.

Remark 9.183 (Compact homology ANR**-manifolds versus closed topological manifolds).** In the following all manifolds have dimension ≥ 6. One would prefer that in the conclusion of Theorem 9.181 one could replace "compact homology ANR-manifold" by "closed topological manifold". The problem is that in the geometric exact surgery sequence one has to work with the 1-connective cover $\mathbf{L}\langle 1\rangle$ of the L-theory spectrum $\mathbf{L}$, whereas in the assembly map appearing in the Farrell-Jones setting one uses the L-theory spectrum $\mathbf{L}$. The L-theory spectrum $\mathbf{L}$ is 4-periodic, i.e., $\pi_n(\mathbf{L}) \cong \pi_{n+4}(\mathbf{L})$ for $n \in \mathbb{Z}$. The 1-connective cover $\mathbf{L}\langle 1\rangle$ comes with a map of spectra $\mathbf{f}\colon \mathbf{L}\langle 1\rangle \to \mathbf{L}$ such that $\pi_n(\mathbf{f})$ is an isomorphism for $n \geq 1$ and $\pi_n(\mathbf{L}\langle 1\rangle) = 0$ for $n \leq 0$. Since $\pi_0(\mathbf{L}) \cong \mathbb{Z}$, one misses a part involving $L_0(\mathbb{Z})$ of the so-called *total surgery obstruction* due to Ranicki, i.e., the obstruction for a finite Poincaré complex to be homotopy equivalent to a closed topological manifold. If one deals with the periodic L-theory spectrum $\mathbf{L}$, one picks up only the obstruction for a finite Poincaré complex to be homotopy equivalent to a compact homology

ANR-manifold, the so-called *four-periodic total surgery obstruction.* The difference of these two obstructions is related to the *resolution obstruction* of Quinn, which takes values in $L_0(\mathbb{Z})$. Any element of $L_0(\mathbb{Z})$ can be realized by an appropriate compact homology ANR-manifold as its *resolution obstruction.* There are compact homology ANR-manifolds, that are not homotopy equivalent to closed manifolds. But no example of an aspherical compact homology ANR-manifold that is not homotopy equivalent to a closed topological manifold is known. For an aspherical compact homology ANR-manifold M, the total surgery obstruction and the resolution obstruction carry the same information. So we could replace in the conclusion of Theorem 9.181 "compact homology ANR-manifold" by "closed topological manifold" if and only if every aspherical compact homology ANR-manifold with the disjoint disk property admits a resolution.

We refer for instance to [164, 366, 810, 811, 823] for more information about this topic.

Question 9.184 (Vanishing of the resolution obstruction in the aspherical case). Is every aspherical compact homology ANR-manifold homotopy equivalent to a closed manifold?

9.18 Boundaries of Hyperbolic Groups

If G is the fundamental group of an n-dimensional closed Riemannian (smooth) manifold with negative sectional curvature, then G is a hyperbolic group in the sense of Gromov, see for instance [149], [155], [408], and [423]. Moreover, such a group is torsionfree and its boundary ∂G is homeomorphic to a sphere. This leads to the natural question whether a torsionfree hyperbolic group with a sphere as boundary occurs as fundamental group of an aspherical closed manifold, see Gromov [424, page 192]. In high dimensions this question is answered by the following two theorems taken from Bartels-Lück-Weinberger [89]. For the notion of and information about the boundary of a hyperbolic group and its main properties we refer for instance to [528].

Theorem 9.185 (Hyperbolic groups with spheres as boundary). *Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then:*

(i) The following statements are equivalent:

  (a) The boundary ∂G is homeomorphic to S^{n-1};
  (b) There is an aspherical closed topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n;

(ii) The aspherical closed topological manifold M appearing in the assertion above is unique up to homeomorphism.

Theorem 9.186 (Hyperbolic groups with Čech-homology spheres as boundary). *Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then*

(i) The following statements are equivalent:

(a) The boundary ∂G has the integral Čech cohomology of S^{n-1};
(b) G is a Poincaré duality group of dimension n;
(c) There exists a compact homology ANR-*manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$;*

(ii) If the statements in assertion (i) *hold, then the compact homology* ANR-*manifold M appearing there is unique up to s-cobordism of compact* ANR-*homology manifolds.*

One of the main ingredients in the proof of the two theorems above is the fact that both the K-theoretic and the L-theoretic the Farrell-Jones Conjecture hold for hyperbolic groups, see [77] and [86].

9.19 The Stable Cannon Conjecture

Tremendous progress in the theory of 3-manifolds has been made during the last decade. A proof of Thurston's Geometrization Conjecture is given in [563], [734] following ideas of Perelman. The Virtually Fibering Conjecture was settled by the work of Agol, Liu, Przytycki-Wise, and Wise [20, 21, 618, 800, 801, 997, 998].

However, the following famous conjecture, taken from [185, Conjecture 5.1], is still open.

Conjecture 9.187 (Cannon Conjecture). Let G be a hyperbolic group. Suppose that its boundary is homeomorphic to S^2.

Then G acts properly cocompactly and isometrically on the 3-dimensional hyperbolic space.

In the torsionfree case it boils down to

Conjecture 9.188 (Cannon Conjecture in the torsionfree case). Let G be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to S^2.

Then G is the fundamental group of a closed hyperbolic 3-manifold.

More information about Conjecture 9.187 and its status can be found for instance in [362, Section 2],and [142].

The following theorem is taken from [362, Theorem 2]. It is a stable version of the Conjecture 9.188 above. Its proof is based on high-dimensional surgery theory and the theory of homology ANR-manifolds.

Theorem 9.189 ((Stable Cannon Conjecture)). *Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth,* PL*, or topological manifold respectively, that is closed and whose dimension is ≥ 2.*

Then there is a closed smooth, PL*, or topological manifold M and a normal map of degree one*

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\underline{f}"] \arrow[d] & \xi \times TN \arrow[d] \\ M \arrow[r, "f"] & BG \times N \end{tikzcd}$$

satisfying

(i) The map f is a simple homotopy equivalence;

(ii) Let $\widehat{M} \to M$ be the G-covering associated to the composite of the isomorphism $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ with the projection $G \times \pi_1(N) \to G$. Suppose additionally that N is aspherical, $\dim(N) \geq 3$*, and $\pi_1(N)$ satisfies the Full Farrell-Jones Conjecture 13.27. (Its status is discussed in Theorem 16.1.)*

Then $\widehat{M}$ is homeomorphic to $\mathbb{R}^3 \times N$. Moreover, there is a compact topological manifold $\overline{\widehat{M}}$ whose interior is homeomorphic to $\widehat{M}$ and for which there exists a homeomorphism of pairs $(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$.

If we could choose $N = \{\bullet\}$ in Theorem 9.189, it would imply Conjecture 9.188.

Exercise 9.190. Show that the manifold M appearing in Theorem 9.189 is unique up to homeomorphism if N is aspherical and $\pi_1(N)$ satisfies the Full Farrell-Jones Conjecture 13.27.

9.20 Product Decompositions

In this section we show that, roughly speaking, an aspherical closed topological manifold M is a product $M_1 \times M_2$ if and only if its fundamental group is a product $\pi_1(M) = G_1 \times G_2$ and that such a decomposition is unique up to homeomorphism.

Theorem 9.191 (Product decompositions of aspherical closed manifolds). *Let M be an aspherical closed topological manifold of dimension n with fundamental group $G = \pi_1(M)$. Suppose we have a product decomposition*

$$p_1 \times p_2\colon G \xrightarrow{\cong} G_1 \times G_2.$$

Suppose that G, G_1, and G_2 satisfy the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$.

Then G, G_1 and G_2 are Poincaré duality groups whose cohomological dimensions satisfy

$$n = \mathrm{cd}(G) = \mathrm{cd}(G_1) + \mathrm{cd}(G_2).$$

Suppose in the following:

- *the cohomological dimension $\mathrm{cd}(G_i)$ is different from 3, 4 and 5 for $i = 1, 2$;*
- *$n \neq 4$ or ($n = 4$ and G is good in the sense of Freedman).*

Then:

(i) There are aspherical closed topological manifolds M_1 and M_2 together with isomorphisms

$$v_i\colon \pi_1(M_i) \xrightarrow{\cong} G_i$$

and maps

$$f_i\colon M \to M_i$$

for $i = 1, 2$ such that

$$f = f_1 \times f_2\colon M \to M_1 \times M_2$$

is a homeomorphism and $v_i \circ \pi_1(f_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$;

(ii) Suppose we have another such choice of aspherical closed topological manifolds M_1' and M_2' together with isomorphisms

$$v_i'\colon \pi_1(M_i') \xrightarrow{\cong} G_i$$

and maps

$$f_i'\colon M \to M_i'$$

for $i = 1, 2$ such that the map $f' = f_1' \times f_2'$ is a homotopy equivalence and $v_i' \circ \pi_1(f_i') = p_i$ (up to inner automorphisms) for $i = 1, 2$. Then there are for $i = 1, 2$ homeomorphisms $h_i\colon M_i \to M_i'$ such that $h_i \circ f_i \simeq f_i'$ and $v_i \circ \pi_1(h_i) = v_i'$ holds for $i = 1, 2$.

Proof. The case $n \neq 3$ is proved in [645, Theorem 6.1]. The case $n = 3$ is done as follows. One gets from [460, Theorem 11.1 on page 100] that G_1 and G_2 are the fundamental groups of compact 2-manifolds. This implies that $G_1 \cong \mathbb{Z} \cong \pi_1(S^1)$ and G_2 is the fundamental group $\pi_1(F)$ of a closed surface or the other way around. Now use the fact that the Borel Conjecture is true in dimensions ≤ 3. □

9.21 Automorphisms of Manifolds

We record the following two results that deduce information about the homotopy groups of the automorphism group of an aspherical closed manifold from the Farrell-Jones Conjecture and the material from Chapter 7 about pseudoisotopy spaces.

Theorem 9.192 (Homotopy Groups of $\mathrm{Top}(M)$ **rationally for closed aspherical** M**).** *Let M be an aspherical orientable closed topological manifold of dimension > 10 with fundamental group G. Suppose the L-theory assembly map*

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K-theory assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is an isomorphism for $n \le 1$ and a rational isomorphism for $n \ge 2$. Then for $1 \le i \le (\dim M - 7)/3$ one has

$$\pi_i(\mathrm{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

In the differentiable case one additionally needs to study involutions on the higher K-theory groups.

Theorem 9.193 (Homotopy Groups of $\mathrm{Diff}(M)$ **rationally for closed aspherical** M**).** *Let M be an aspherical orientable closed smooth manifold of dimension > 10 with fundamental group G. Then under the same assumptions as in Theorem 9.192 we have for $1 \le i \le (\dim M - 7)/3$*

$$\pi_i(\mathrm{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even}. \end{cases}$$

For a proof see for instance [339], [349, Section 2] and [338, Lecture 5]. For a survey on automorphisms of manifolds we refer to [983].

Remark 9.194. We get also some information about the cohomology of $\mathrm{BTop}(M)^\circ$, where $\mathrm{Top}(M)^\circ$ denotes the component of the identity of $\mathrm{Top}(M)$. There is a canonical map

$$\pi_1(\mathrm{BTop}(M), \mathrm{id}) \to G_1(M) \subseteq \pi_1(M).$$

onto Gottliebs subgroups $G_1(M)$ of $\pi_1(M)$, see [415]. Suppose from now on that M is an aspherical orientable closed topological manifold of dimension > 10. Then $G_1(M) = \mathrm{center}(G)$ and the induced map

$$\mathrm{BTop}(M)^\circ \to K(\mathrm{center}(G), 2)$$

is a map of simply connected spaces inducing isomorphism on the rationalized homotopy groups in a range. This implies that in this range we get an isomorphism

$$H^*(K(\operatorname{center}(G),2);\mathbb{Q}) \xrightarrow{\cong} H^*(\operatorname{BTop}(M)^\circ;\mathbb{Q}).$$

Integral computations of the homotopy groups of automorphisms are much harder. We mention at least the following result taken from [330, Theorem 1.3].

Theorem 9.195 (Homotopy groups of $\operatorname{Top}(M)$ for closed aspherical M with hyperbolic fundamental group).
Let M be a smoothable aspherical closed topological manifold of dimension ≥ 10 whose fundamental group π is hyperbolic.

Then there is a $\mathbb{Z}/2$-action on $\mathbf{Wh}^{\mathrm{TOP}}(B\pi)$ such that we obtain for every n satisfying $1 \leq n \leq \min\{(\dim M - 7)/2, (\dim M - 4)/3\}$ isomorphisms

$$\pi_n(\mathrm{TOP}(M)) \cong \pi_{n+2}\Big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathbf{Wh}^{\mathrm{TOP}}(BC)\big)\Big)$$

and an exact sequence

$$1 \to \pi_2\Big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_{(C)} \mathbf{Wh}^{\mathrm{TOP}}(BC)\big)\Big) \to \pi_0(\mathrm{TOP}(M)) \to \operatorname{Out}(\pi) \to 1$$

where (C) ranges over the conjugacy classes (C) of maximal infinite cyclic subgroups C of π.

The methods described in this book about automorphisms groups of closed manifold apply only to aspherical closed manifolds. It is of course also essential to study automorphism groups of disks. The techniques used to analyze them are quite different. There has been tremendous progress on this topic during the recent years. We refer to the survey article by Randal-Williams [816], where also further references in the literatur about this topic are given.

Moreover, there has been tremendous progress on the moduli spaces of manifolds in the recent years, which gives informations about the cohomology of the classifying space $B\mathrm{Diff}(M)$ of closed smooth manifolds M such as the connected sum of several copies of a product of spheres. We refer to the survey article by Galatius and Randal-Williams [397], where also further references in the literatur about this topic are given.

9.22 Survey on Computations of L-Theory of Group Rings of Finite Groups

Theorem 9.196 (Algebraic L-theory of $\mathbb{Z}G$ for finite groups). *Let G be a finite group. Then* *(i) The groups* $L_n^{\langle j\rangle}(\mathbb{Z})$ *are independent of the decoration* j *and given by* $\mathbb{Z}$, $\{0\}$, $\mathbb{Z}/2$, $\{0\}$ *for* $n \equiv 0,1,2,3 \mod (4)$;

(ii) For every $n \in \mathbb{Z}$, *the groups* $L_n^{\langle s\rangle}(\mathbb{Z}G)$, $L_n^{\langle h\rangle}(\mathbb{Z}G)$, $L_n^{\langle p\rangle}(\mathbb{Z}G)$, $L_n^{\langle -\infty\rangle}(\mathbb{Z}G)$, *and* $L_n^{\langle j\rangle}(\mathbb{Z}G)$ *for every* $j \le 1$ *are finitely generated as abelian groups and contain no* p*-torsion for odd primes* p. *Moreover, they all are finite for odd* n;

(iii) Let $r(G)$ *be the number of isomorphisms classes of irreducible real* G*-representations. Let* $r_{\mathbb{C}}(G)$ *be the number of isomorphisms classes of irreducible real* π*-representations* V *that are of complex type. For every decoration* $\langle j\rangle$ *we have*

$$L_n^{\langle j\rangle}(\mathbb{Z}G)[1/2] \cong L_n^{\langle j\rangle}(\mathbb{Q}G)[1/2] \cong L_n^{\langle j\rangle}(\mathbb{R}G)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r(G)} & n \equiv 0 \ \ (4); \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \ \ (4); \\ 0 & n \equiv 1,3 \ \ (4); \end{cases}$$

(iv) If G *has odd order and* n *is odd, then* $L_n^{\varepsilon}(\mathbb{Z}G) = 0$ *for* $\varepsilon = p,h,s$ *and* $L_n^{\langle j\rangle} \cong (\mathbb{Z}G) = \mathbb{Z}/2^r$ *for* $j \in \{-1,-2,\dots,\} \amalg \{-\infty\}$ *where* r *is the number appearing in Theorem 4.22* (iii);

(v) If G *is a cyclic group of odd order, then the kernel of the split epimorphism* $L_n^s(\mathbb{Z}G) \to L_n^s(\mathbb{Z})$ *is torsionfree. In particular* $\operatorname{tors}(L_n^s(\mathbb{Z}G))$ *is* $\mathbb{Z}/2$ *if* $n \equiv 2 \mod 4$ *and trivial otherwise.*

Proof. (i) See for instance [652, Theorem 16.8 (i) on page 687].

(ii) See [968, Theorem 13.A.4 (i) on page 177], [448] for the decoration s. Now the claim follows for all decorations from the Rothenberg sequences, see Subsection 9.10.4 since the relevant K-groups of $\mathbb{Z}G$ are all finitely generated abelian groups.

(iii) See [823, Proposition 22.34 on page 253].

(iv) See [59], [448, Theorem 10.1] for $\epsilon \in \{s,p,h\}$. Note that $K_n(\mathbb{Z}G) = 0$ for $n \le -2$ and $K_{-1}(\mathbb{Z}G) = \mathbb{Z}^r$ by Theorem 4.22. The involution on $K_{-1}(\mathbb{Z}G) = \mathbb{Z}^r$ is given by $-\mathrm{id}$. Hence $\widehat{H}^0(Z/2, K_{-1}(ZG)) = 0$ and $H^1(Z/2, K_{-1}(ZG)) = (Z/2)^r$. Since $L_n^p(ZG) = 0$ for odd n and $L_n^p(ZG)$ is known to be torsionfree for even n, the claim follows from the Rothenberg sequence (9.101). See also [441, Section 3].

(v) See [968, Theorem 13.A.4 (ii) on page 177], [448, Section 10]. □

9.23 Notes

The next problem is meanwhile solved and triggered surgery theory for non-simply connected manifolds. It is a kind of generalization of the *Space Form*

Problem asking which finite groups occur as fundamental groups of closed Riemannian manifolds with constant positive sectional curvature.

Problem 9.197 (Spherical Space Form Problem). Which finite groups can act freely (topologically or smoothly) on a standard sphere, or, equivalently, occur as fundamental groups of closed manifolds whose universal covering is (homeomorphic or diffeomorphic to) a standard sphere.

More information about this interesting problem and its solution can be found in [270] and [679].

For a survey of the classification of fake spaces such as fake product of spheres, fake projective spaces, fake lens spaces, and fake tori, and the literature about them, we refer to [652, Chapter 18].

Our definition of the L-groups follows the original approach due to Wall. A much more satisfactory and elegant approach via chain complexes is due to Mishchenko and Ranicki and is of fundamental importance for many applications and generalizations, see for instance [652, 719, 720, 721, 819, 820, 821, 823].

We mention that a different approach to surgery has been developed by Kreck. A survey about his approach is given in [569]. Its advantage is that one does not have to get a complete homotopy classification first. The price to pay is that the L-groups are much more complicated, they are not necessarily abelian groups any more. This approach is in particular successful when the manifolds under consideration are already highly connected. See for instance [572, 573, 912].

More information about surgery theory can be found for instance in [159, 187, 188, 214, 652, 633, 827, 968].

We will relate the algebraic L-theory of C^*-algebras to their topological K-theory in Theorem 10.78. In particular we get for all $n \in \mathbb{Z}$ natural isomorphisms

$$\begin{aligned} L_n(C_r^*(G,\mathbb{R}))[1/2] &\cong K_n^{\mathrm{TOP}}(C_r^*(G;\mathbb{R}))[1/2]; \\ L_n(C_r^*(G,\mathbb{C}))[1/2] &\cong K_n^{\mathrm{TOP}}(C_r^*(G;\mathbb{C}))[1/2]. \end{aligned}$$

We mention already here Conjecture 15.87, which deals with the passage for L-theory from $\mathbb{Q}G$ to $\mathbb{R}G$ to $C_r^*(G;\mathbb{R})$. Its connection to the Baum-Connes Conjecture and the Farrell-Jones Conjecture is analyzed in Lemma 15.88. For more information about the algebraic L-theory of C^*-algebras we refer to [600].

There is also a version of the Borel Conjecture for manifolds with boundary, which is implied by the Farrell-Jones Conjecture, see for instance [368, page 17 and page 31].

Another survey article about topological rigidity is [538].

There is an equivariant version of the Borel Conjecture where one replaces EG with the classifying space for proper G-actions $\underline{E}G$, see Definition 11.18. One may ask whether there is a compact closed G-manifold which is a model for $\underline{E}G$ and whether, for two compact proper topological G-manifolds M and N that both are models for $\underline{E}G$, any G-homotopy equivalence between them is G-homotopic to a G-homeomorphism. This version is not true in general and investigated for instance in [237, 238, 239, 243, 269, 650, 980].

The vanishing of κ-classes for aspherical closed manifolds is analyzed in [458] using the Farrell-Jones Conjecture.

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Chapter 10
Topological K-Theory

10.1 Introduction

In this chapter we deal with *topological K-theory* of reduced group C^*-algebras, which is the target of the *Baum-Connes Conjecture*, in contrast to algebraic K- and L-theory of group rings, which is the target of the Farrell-Jones Conjecture. We begin with reviewing the topological K-theory of spaces and its equivariant version for proper actions of possibly infinite discrete groups. Then we pass to its generalization to C^*-algebras. We discuss the *Baum-Connes Conjecture for torsionfree groups 10.44* and present two applications, namely, to the *Trace Conjecture* about the integrality of the trace map and to the *Kadison Conjecture* about idempotents in reduced group C^*-algebras of torsionfree groups. Then we briefly state the main properties of *Kasparov's KK-theory* and its equivariant version (without explaining its construction). This will later be needed in Chapter 14 to explain the *analytic Baum-Connes assembly* map and state the *Baum-Connes Conjecture for arbitrary groups and with coefficients* in a G-C^*-algebra.

10.2 Topological K-Theory of Spaces

10.2.1 Complex Topological K-Theory of Spaces

Complex topological K-theory of spaces, sometimes also called *complex topological K-cohomology of spaces*, is a generalized cohomology theory, i.e., it assigns to a pair of CW-complexes (X, A) a $\mathbb{Z}$-graded abelian group $K^*(X, A)$ and a homomorphism of degree one $\delta^*\colon K^*(A) \to K^{*+1}(X, A)$ and to a map $f\colon (X, A) \to (Y, B)$ of such pairs a homomorphism $K^*(f)\colon K^*(Y, B) \to K^*(X, A)$ of $\mathbb{Z}$-graded abelian groups such that the Eilenberg-Steenrod axioms of a cohomology theory are satisfied, i.e., one has naturality, homotopy invariance, the long exact sequence of a pair, and excision. Moreover, the disjoint union axiom holds, see Definition 12.1. In contrast to singular cohomology the dimension axiom is not satisfied, actually $K^n(\{\bullet\})$ is $\mathbb{Z}$ if n is even and is trivial if n is odd. A very important feature is that topological complex K-theory satisfies *Bott periodicity*, i.e., there is a natural isomorphism of degree two compatible with the boundary map in the long exact sequence of pairs

$$\beta^*(X,A)\colon K^*(X,A) \xrightarrow{\cong} K^{*+2}(X,A).$$

Topological complex K-theory comes with a multiplicative structure.

It can be constructed by the so-called *complex topological K-theory spectrum* $\mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}}$ that is the following Ω-spectrum. (Spectra will be defined in Section 12.4.) The n-th space is $\mathbb{Z} \times BU$ for even n and $\Omega(\mathbb{Z} \times BU)$ for odd n. The n-th structure map is given by the identity $\mathrm{id}\colon \Omega(\mathbb{Z} \times BU) \to \Omega(\mathbb{Z} \times BU)$ for odd n and by an explicit homotopy equivalence due to Bott $\mathbb{Z} \times BU \xrightarrow{\simeq} \Omega^2(\mathbb{Z} \times BU)$ for even n. As usual, associated to this spectrum is also a generalized homology theory $K_*(X,A)$, called *topological complex K-homology of spaces*, such that $K_n(\{\bullet\})$ is $\mathbb{Z}$ if n is even and is trivial if n is odd. A proof of a universal coefficient theorem for complex K-theory can be found in [27] and [1004, (3.1)], the homological version then follows from [12, Note 9 and 15].

Rationally one can compute complex topological K-theory by Chern characters. (Equivariant versions will be explained in Section 12.7.) Namely, we get for any pair of CW-complexes (X,A) a natural $\mathbb{Q}$-isomorphism

$$\bigoplus_{p\in\mathbb{Z}, p\equiv n(2)} H_p(X,A;\mathbb{Q}) \xrightarrow{\cong} K_n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{10.1}$$

and for any pair of finite CW-complexes (X,A) a natural $\mathbb{Q}$-isomorphism

$$K^n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{p\in\mathbb{Z}, p\equiv n(2)} H^p(X,A;\mathbb{Q}). \tag{10.2}$$

The condition that (X,A) is finite is needed in (10.2). The cohomological Chern character (10.2) is compatible with the multiplicative structures.

For integral computations one has to use the Atiyah-Hirzebruch spectral sequence, which does not collapse in general.

Exercise 10.3. Let X be a finite CW-complex. Show for its Euler characteristic

$$\chi(X) = \mathrm{rk}_{\mathbb{Z}}(K^0(X)) - \mathrm{rk}_{\mathbb{Z}}(K^1(X)) = \mathrm{rk}_{\mathbb{Z}}(K_0(X)) - \mathrm{rk}_{\mathbb{Z}}(K_1(X)).$$

The groups $K^*(BG)$ can be computed explicitly for all finite groups G using the Completion Theorem due to Atiyah-Segal [43, 51], see for instance [641, Theorem 0.3]. Namely, if for a prime p we denote by $r(p)$ the number of conjugacy classes (g) of elements $g \in G$ whose order $|g|$ is p^d for some integer $d \geq 1$, and by $\widehat{\mathbb{Z}_p}$ the p-adic integers, then there are isomorphisms of abelian groups

$$K^0(BG) \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\widehat{\mathbb{Z}_p})^{r(p)}; \tag{10.4}$$

$$K^1(BG) \cong 0. \tag{10.5}$$

One can also figure out the multiplicative structure on $K^0(BG)$ in (10.4). This shows how accessible topological K-theory is, for instance, one does *not* know the group cohomology $H^*(BG)$ of all finite groups G.

If X is a finite CW-complex, $K^*(X)$ can be described in terms of vector bundles. For instance, $K^0(X)$ is the Grothendieck group associated to the abelian monoid of isomorphism classes of (finite dimensional complex) vector bundles over X under the Whitney sum. Naturality comes from the pullback construction, the multiplicative structure from the tensor product of vector bundles.

There are a *Thom isomorphism* and a *Künneth Theorem* for finite CW-complexes for topological complex K-cohomology, see [48, Corollary 2.7.12 on page 111 and Corollary 2.7.15 on page 113].

Using exterior powers one can construct the so-called *Adams operations* on topological complex K-cohomology. They were a key ingredient in the work of Adams on the Hopf invariant one problem, see [3, 14], and on linear independent vector fields on spheres, see [4, 5, 6]. Atiyah [44] introduced the groups $J(X)$ where vector bundles are considered up to fiber homotopy equivalence. They were studied by Adams [8, 9, 10, 11].

Complex topological K-theory is one of the first generalized cohomology theories. There are other generalized (coh)omology theories such as *bordism*, see for instance [963], *complex bordism*, see for instance [828], *Morava K-theory*, see for instance [1002], *elliptic cohomology*, see for instance [677, 929], and *topological modular forms* tmf, see for instance [484, 485, 677], which have been of great interest in algebraic topology over the last decades.

The connection between topological K-theory and spaces of Fredholm operators was explained by Jänich [502]. Namely, there exists a natural bijection of abelian groups for finite CW-complexes X

$$[X, \mathrm{Fred}] \cong K^0(X) \tag{10.6}$$

where Fred is the space of *Fredholm operators*, i.e., bounded operators with finite dimensional kernel and cokernel. This shows that there is a relation between topological K-theory and index theory. For instance, we get from (10.6) applied to $X = \{\bullet\}$ an isomorphism $\pi_0(\mathrm{Fred}) = K^0(\{\bullet\}) \cong \mathbb{Z}$ that sends a Fredholm operator to its classical index which is the difference of the dimension of its kernel and the dimension of its cokernel. The bijection of (10.6) assigns to a map $X \to \mathrm{Fred}$ which can be interpreted as a family of Fredholm operators parametrized by X, its family index which is essentially the difference of the class of the vector bundle over X whose fiber over x is the kernel of the Fredholm operator associated to $x \in X$ and the vector bundle over X whose fiber over x is the cokernel of the Fredholm operator associated to $x \in X$. Good introductions to index theory are the seminal papers [50, 52, 53, 55, 56]. Other references about index theory are [135, 837, 1003].

10.2.2 Real Topological K-Theory of Spaces

There is also *real topological K-theory of spaces*, sometimes also called *real topological KO-cohomology of spaces*, $KO^*(X,A)$ and *real topological K-homology* $KO_*(X,A)$ where one considers real vector bundles instead of complex vector bundles and BO instead of BU. One uses a specific homotopy equivalence $\mathbb{Z} \times BO \xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times BO)$ to construct so the called *real K-theory spectrum* $\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}$. A much more sophisticated and structured symmetric spectrum representing real K-theory in terms of Fredholm operators was constructed by Joachim [506, 507] and Mitchener [729] based on ideas of Atiyah-Singer [54].

The main difference between the real and the complex version is that KO_* is 8-periodic and $KO_n(\{\bullet\}) = KO^{-n}(\{\bullet\})$ is given by $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ for $n = 0,1,2,3,4,5,6,7$. There are natural transformations $i^*\colon KO^*(X) \to K^*(X)$ and $r^*\colon K^*(X) \to KO^*(X)$ that corresponds to assigning to a real vector bundle its complexification and to a complex vector bundle its restriction to a real vector bundle. They satisfy $r \circ i = 2 \cdot \mathrm{id}$. They also exist on K-homology. It is sometimes useful to consider the real topological K-theory instead of the complex version since one does loose information when passing to the complex topological version. On the other hand computations for the real topological K-theory are harder than for the complex topological K-theory since the real version is 8-periodic and its value at $\{\bullet\}$ contains 2-torsion, whereas the complex version is 2-periodic and its evaluation at $\{\bullet\}$ is much simpler than for the real version.

Rationally we get again a Chern character, namely, for any pair of CW-complexes (X,A) a natural $\mathbb{Q}$-isomorphism

$$\bigoplus_{p \in \mathbb{Z}, p \equiv n(4)} H_p(X,A;\mathbb{Q}) \xrightarrow{\cong} KO_n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{10.7}$$

and for any pair of finite CW-complexes (X,A) a natural $\mathbb{Q}$-isomorphism

$$KO^n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{p \in \mathbb{Z}, p \equiv n(4)} H^p(X,A;\mathbb{Q}). \tag{10.8}$$

There is a natural transformation of homology theories called *KO-orientation of Spin bordism* due to Atiyah-Bott-Shapiro [49] which can be interpreted by sending a Spin manifold to the index class of the associated Dirac operator

$$D\colon \Omega_n^{\mathrm{Spin}}(X) \to KO_n(X). \tag{10.9}$$

It plays an important role for the question when a closed spin manifold admits a Riemannian metric of positive sectional curvature, see Subsection 14.8.2.

A relation of KO-theory to surgery theory has already been explained in Remark 9.133.

Another variant of topological K-theory denoted by $KR^*(X,A)$ was defined by Atiyah [45]. *Twisted topological K-theory* has been studied intensively, see instance [41, 42, 385, 535].

More information about topological K-theory of spaces can be found for instance in [7, 40, 47, 48, 491, 492, 534, 605].

10.2.3 Equivariant Topological K-Theory of Spaces

Equivariant topological K-theory has been considered for compact topological groups acting on compact spaces, see for instance [48, 889]. For our purpose it will be important to treat the more general case of a proper action of a not necessarily compact group. It suffices for our purposes to consider discrete groups G and proper G-CW-complexes, or, equivalently, CW-complexes with a G-action such that all isotropy groups are finite and for every open cell e of X with $g \cdot e \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$. This is difficult enough, but not as hard as the much less understood case of a topological group acting properly on locally compact Hausdorff space.

If G is a discrete group, G-cohomology theories K_G^* and KO_G^* are constructed by Lück-Oliver [655] for pairs of proper G-CW-complexes (X,A) using classifying spaces for G-vector bundles. More precisely, for every pair of proper G-CW-complexes (X,A) one obtains $\mathbb{Z}$-graded abelian groups $K_G^*(X,A)$ and $KO_G^*(X,A)$ such that one has naturality, G-homotopy invariance, long exact sequence of pairs, excision, and the disjoint union axiom holds, see Definition 12.1. The complex version K_G^* is 2-periodic, the real version is 8-periodic.

Let $H \subseteq G$ be a finite subgroup, Then

$$K_G^n(G/H) = \begin{cases} \operatorname{Rep}_{\mathbb{C}}(H) & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases} \tag{10.10}$$

There is a decomposition of the real group ring $\mathbb{R}H$ as a direct product $\prod_{i=0}^r M_{n_i,n_i}(D_i)$ of matrix algebras over skew-fields D_i where D_i is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then one obtains a decomposition for each $n \in \mathbb{Z}$

$$KO_G^{-n}(G/H) = \prod_{i=1}^r KO_G^{-n}(G/H)_i \tag{10.11}$$

where

$$KO_G^{-n}(G/H)_i = \begin{cases} KO_n(\{\bullet\}) & \text{if } D_i = \mathbb{R}; \\ K_n(\{\bullet\}) & \text{if } D_i = \mathbb{C}; \\ KO_{n+4}(\{\bullet\}) & \text{if } D_i = \mathbb{H}. \end{cases}$$

There is a natural *external multiplicative structure*, i.e., there is a natural pairing

$$(10.12) \qquad K_G^m(X,A) \otimes_{\mathbb{Z}} K_H^n(X,B) \to K_{G\times H}^{m+n}((X,A)\times(Y,B))$$

for discrete groups G and H and a pair (X,A) of proper G-CW-complexes and a pair (Y,B) of proper H-CW-complexes. There exists a natural *restriction homomorphism* for any inclusion $i\colon H \to G$ of discrete groups

$$(10.13) \qquad i^*\colon K_G^*(X,A) \to K_H^*(i^*(X,A))$$

where (X,A) is a pair of proper G-CW-complexes and $i^*(X,A)$ is its restriction to H. Applying this to the diagonal map $G \to G\times G$ and the external product and using the diagonal embedding $X \to X\times X$, one obtains a natural *internal multiplicative structure*, i.e., natural pairings

$$(10.14) \qquad K_G^m(X,A) \otimes_{\mathbb{Z}} K_G^n(X,B) \to K_G^{m+n}(X,A\cup B)$$

for a discrete group G and a proper G-CW-complex X with G-CW-subcomplexes A and B. In particular $K_G^*(X)$ becomes a $\mathbb{Z}$-graded algebra for any proper G-CW-complex X. Given a group homomorphism $\alpha\colon H \to G$, there is an *induction homomorphism*

$$(10.15) \qquad \mathrm{ind}_\alpha\colon K_H^*(X,A) \to K_G^*(\mathrm{ind}_\alpha(X,A))$$

where (X,A) is a proper H-CW-complex and $\mathrm{ind}_\alpha(X,A)$ is the proper G-CW-complex $G\times_\alpha (X,A)$. If $\ker(\alpha)$ acts freely on (X,A), the map ind_α is bijective.

All the constructions and results above are carried out in [655], and the corresponding statements do hold also for the real version KO_G^*. If G is finite, they all reduce to the classical constructions and results.

One can give a description for pairs (X,A) of finite proper G-CW-complexes for a discrete group G in terms of G-vector bundles such that for instance $K_G^0(X)$ and $KO_G^0(X)$ respectively agree with the Grothendieck groups of isomorphism classes of G-equivariant complex and real respectively vector bundles over the finite proper G-CW-complex X. This follows from [656, Theorem 3.2 and Theorem 3.15] and [655, Proposition 1.5]. (A C^*-theoretic analogue of this result is discussed in [109, Section 6].) However, the interpretation of $K_G^0(X)$ in terms of vector bundles does not hold if G is a Lie group, as explained in [656, Section 5]. A description in terms of infinite dimensional G-vector bundles is discussed by Phillips [790]. The question how the Grothendieck group of isomorphism classes of G-vector bundles over a classifying space BG of a compact Lie group G looks like and is related to $K^0(BG)$ is treated in [501]. (Note that this is a non-trivial question already for finite groups since BG does not have a finite dimensional CW-model for non-trivial finite groups.)

Let G be a discrete group. For any cyclic group $C \subseteq G$ of order $n < \infty$ we denote by $\mathbb{Z}[\zeta_C] \subseteq \mathbb{Q}(\zeta_C)$ the cyclotomic ring and field generated by the n-th roots of unity. We regard them as quotient rings of the group rings $\mathbb{Z}[\hom(C, \mathbb{C}^*)] \subseteq \mathbb{Q}[\hom(C, \mathbb{C}^*)]$. In other words, we fix an identification of the n-th roots of unity in $\mathbb{Q}(\zeta_C)$ with the irreducible characters of C. Let $\mathcal{C}(G)$ be a set of conjugacy class representatives for the cyclic subgroups $C \subseteq G$ of finite order. Denote by $C_G C$ the centralizer and by $N_G C$ the normalizer of C in G. Then for any pair of finite proper G-complexes (X, A), there is the following version of an *equivariant Chern character*, namely, a natural isomorphism of rings

$$(10.16)\quad K_G^*(X;A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{C \in \mathcal{C}(G)} \left(H^*((X,A)^C/C_G C; \mathbb{Q}(\zeta_C))\right)^{N_G C/C_G C}$$

where $N_G C/C_G C$ acts via the conjugation action on $\mathbb{Q}(\zeta_C)$ and on $X^C/C_G C$ in terms of the given G-action on X.

Equivariant Chern characters can be used to compute $K^*(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$ for infinite groups possessing a finite G-CW-model for its classifying space for proper G-actions, i.e., for instance for hyperbolic groups G or compact lattices G in connected Lie groups, see [641] and also [15, 16]. More information about $K^*(BG)$ for infinite groups can be found in [508, Theorem 0.1], and about cohomological Chern characters in [638].

Exercise 10.17. Let G be an abelian group. Let X be a finite proper G-CW-complex. Show that there is a $\mathbb{Q}$-isomorphism

$$K_G^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{C \in \mathcal{C}(G)} H^*(X^C/G; \mathbb{Q})^{\varphi(|C|)}$$

for the Euler Phi-function φ. We will construct in Section 10.6 the equivariant K-homology K_*^G that is a G-homology theory defined for pairs of proper G-CW-complexes for discrete groups G and satisfies the disjoint union axiom.

An equivariant *Universal Coefficient Theorem for equivariant complex K-theory* for discrete groups G and finite proper G-CW-complexes X is given in [508, Theorem 0.3], namely, there are short exact sequences, natural in X,

$$(10.18)\quad 0 \to \operatorname{Ext}_{\mathbb{Z}}(K_{*-1}^G(X), \mathbb{Z}) \to K_G^*(X) \to \hom_{\mathbb{Z}}(K_*^G(X), \mathbb{Z}) \to 0;$$
$$(10.19)\quad 0 \to \operatorname{Ext}_{\mathbb{Z}}(K_G^{*+1}(X), \mathbb{Z}) \to K_*^G(X) \to \hom_{\mathbb{Z}}(K_G^*(X), \mathbb{Z}) \to 0.$$

It reduces for a finite group G to the one of Bökstedt [137] as explained in [508, Remark 5.21],

An *external Künneth Theorem for complex K-theory* relating $K_{G\times H}^*(X \times Y)$ to $K_G^*(X)$ and $K_H^*(Y)$ is given in [715] for compact Lie groups G and H and finite G-CW-complexes X and Y, namely, there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} K_G^i(X) \otimes_{\mathbb{Z}} K_H^j(Y) \to K_{G\times H}^n(X \times Y) \to \bigoplus_{i+j=n+1} \operatorname{Tor}_{\mathbb{Z}}(K_G^i(X), K_H^j(Y)) \to 0. \tag{10.20}$$

The situation is much more complicated and much less understood if one wants to relate $K_G^*(X \times Y)$ to $K_G^*(X)$ and $K_G^*(Y)$ for a finite group G and finite G-CW-complexes X and Y, see [483, 848]. This complication is not surprising since it is related to the difficult question to compute $K_G^*(X \times Y)$ for the diagonal G-action on $X \times Y$ from $K_{G\times G}^*(X \times Y)$ for a finite group G and finite G-CW-complexes X and Y.

Exercise 10.21. Let G and H be discrete groups. Let (X, A) be a pair of finite proper G-CW-complexes, and let (Y, B) be a pair of finite proper H-CW-complexes. Suppose that either $K_G^i(X)$ is torsionfree for $i \in \mathbb{Z}$ or that $K_H^j(Y)$ is torsionfree for all $j \in \mathbb{Z}$.

Then the external multiplicative structure induces for every $n \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{i+j=n} K_G^i(X, A) \otimes_{\mathbb{Z}} K_H^j(Y, B) \xrightarrow{\cong} K_{G\times H}^n((X, A) \times (Y, B)).$$

Consider a discrete group G and a complex G-vector bundle $p\colon E \to X$ with Hermitian metric over a finite proper G-CW-complex. Let $p_{DE}\colon DE \to X$ be the *disk bundle* and $p_{SE}\colon SE \to X$ be the *sphere bundle* associated to p whose fiber over $x \in X$ is the disk and sphere in $p^{-1}(x)$. Then there exists a *Thom class* $\lambda_E \in K_G^0(DE, SE)$ and the composite

$$T_E\colon K_G^*(X) \xrightarrow{K_G^*(p_{DE})} K_G^*(DE) \xrightarrow{-\cup\lambda_E} K_G^*(DE, SE) \tag{10.22}$$

is an isomorphism of $\mathbb{Z}$-graded abelian groups called *Thom isomorphism*, see [656, Theorem 3.14].

Exercise 10.23. For a discrete group G and a complex G-vector bundle $p\colon E \to X$ over a finite proper G-CW-complex define its *Euler class* $e(p) \in K_0^G(X)$ to be the image of the Thom class under the composite $K_G^0(DE, SE) \xrightarrow{K_G^0(j)} K_G^0(DE) \xrightarrow{K_G^0(p_{DE})^{-1}} K_G^0(X)$ for $j\colon DE \to (DE, SE)$ the inclusion. Show that there exists a long exact *Gysin sequence*

$$\cdots \xrightarrow{\delta^{n-1}} K_G^n(X) \xrightarrow{-\cup e(p)} K_G^n(X) \xrightarrow{K_G^n(p_{SE})} K_G^n(SE) \xrightarrow{\delta^n} K_G^{n+1}(X) \xrightarrow{-\cup e(p)} K_G^{n+1}(X) \xrightarrow{K_G^{n+1}(p_{SE})} \cdots. \tag{10.24}$$

A *Completion Theorem* for complex and real topological K-theory allowing families of subgroups is proved in [655, Theorem 6.5] for a discrete group

G and a finite proper G-CW-complex X in terms of isomorphisms of pro-systems, see also [656, Theorem 4.3]. Let $p\colon EG \to BG$ be the universal covering of BG, or, equivalently, the universal principal G-bundle. Up to G-homotopy EG is uniquely characterized by the property that it is a fre G-CW-complex which is (after forgetting the group action) contractible. A consequence of the Completion Theorem is that the inverse system

$$\bigl\{K^*\bigl((EG\times_G X)^{(n)}\bigr)\bigr\}_{n\geq 0}$$

satisfies the Mittag-Leffler condition and we obtain isomorphisms

$$K_G^*(X)\widehat{_I} \cong K^*(EG\times_G X) \cong \operatorname{invlim}_{n\to\infty} K^*\bigl((EG\times_G X)^{(n)}\bigr). \tag{10.25}$$

Here $K_G^*(X)\widehat{_I}$ is the completion of $K_G^*(X)$ with respect to the so-called augmentation ideal I that is the kernel of the dimension map $K_G^0(\underline{E}G) \to \mathbb{Z}$ for $\underline{E}G$ the classifying space for proper G-actions, and we have to assume that there is a finite dimensional model for $\underline{E}G$. If G is finite and we take $X = \{\bullet\}$, this reduces to the classical Atiyah-Segal Completion Theorem predicting an isomorphism

$$K^n(BG) = \begin{cases} \operatorname{Rep}_{\mathbb{C}}(G)\widehat{_I} & n \text{ even;} \\ 0 & n \text{ odd,} \end{cases}$$

where I is the augmentation ideal, i.e., kernel of the map given by taking complex dimension $\operatorname{Rep}_{\mathbb{C}}(G) \to \mathbb{Z}$. There is also a version for the real topological K-theory.

A *Cocompletion Theorem for the topological complex K-homology* for discrete groups and finite proper G-CW-complexes is proved in [508, Theorem 0.2]. It assigns to a finite proper G-CW-complex X a short exact sequence

$$\begin{aligned} 0 \to \operatorname{colim}_{n\geq 1} \operatorname{Ext}^1_{\mathbb{Z}}(K_G^{*+1}(X)/I^n \cdot K_G^{*+1}(X),\mathbb{Z}) &\to K_*(EG\times_G X) \\ \to \operatorname{colim}_{n\geq 1} \hom_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X),\mathbb{Z}) &\to 0. \end{aligned} \tag{10.26}$$

The Completion and Cocompletion Theorems are not only interesting in its own right, they are needed in the computation of the topological K-theory of certain group C^*-algebras, see for instance [267, 268, 604, 969].

Another important tool for equivariant K-theory over compact Lie groups is the *Localization Theorem for equivariant topological complex K-theory* of Segal [889, Proposition 4.1]. Given a prime ideal $\mathcal{P}$ of $\operatorname{Rep}_{\mathbb{C}}(G) = K_G^0(\{\bullet\})$, there is a topologically cyclic group S associated to $\mathcal{P}$, its so-called support. If X is a finite G-CW-complex, let $X^{(S)}$ be the G-CW-subcomplex $G\cdot X^S$. Then after localization at $\mathcal{P}$ the inclusion $X^{(S)} \to X$ induces an isomorphism

$$K_G^*(X)_{(\mathcal{P})} \xrightarrow{\cong} K_G^*(X^{(S)})_{(\mathcal{P})}. \tag{10.27}$$

Localization for equivariant cohomology theories for compact Lie groups is treated in general in [934, Chapter 7] and [935, III.3 and III.4].

Equivariant topological K-theory was designed for and is a key ingredient when one considers indices of equivariant operators. See for instance [50, 52, 53] where also applications such as Lefschetz Theorems, Riemann-Roch Theorems, and G-Signature Theorems are treated for compact Lie groups.

The K_G*-degree* of G-maps between spheres of unitary G-representations for a compact Lie group G is an important tool, see [935, II.5].

A discussion about equivariant K-theory and orbifold K-theory can be found in [18, Chapter 3].

A geometric description of equivariant K-homology for proper actions in term cycles built by proper cocompact G-Spin^c-manifolds and smooth complex G-vector bundles over them is given in [109], extending the non-equivariant versions of [104, 108].

10.3 Topological K-Theory of C^*-Algebras

10.3.1 Basics about C^*-algebras

For this section let F be $\mathbb{R}$ or $\mathbb{C}$. For $\lambda \in F$, denote by $\overline{\lambda}$ the complex conjugate of λ.

A *Banach algebra* over F is an associative F-algebra $A = (A, +, \cdot)$ together with a norm $|| \; ||$ for the underlying F-vector space such that the underlying F-vector space is complete with respect to the given norm and we have the inequality $||a \cdot b|| \leq ||a|| \cdot ||b||$ for all elements $a, b \in A$.

A *Banach $*$-algebra* is a Banach algebra together with an involution $*\colon A \to A,\ a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a \cdot b)^* = b^* \cdot a^*$, $(\lambda \cdot a + \mu \cdot b)^* = \overline{\lambda} \cdot a^* + \overline{\mu} \cdot b^*$, and $||a^*|| = ||a||$ for $a, b \in A$ and $\lambda, \mu \in F$. If G is a discrete group, $L^1(G, F)$ carries the structure of a Banach $*$-algebra coming from the convolution product, the L^1-norm, and the involution sending $\sum_{g \in G} \lambda_g \cdot g$ to $\sum_{g \in G} \overline{\lambda_g} \cdot g^{-1}$.

A *C^*-algebra* is a Banach $*$-algebra A that satisfies additionally *the C^*-identity* $||a^*a|| = ||a||^2$ for all $a \in A$. A *homomorphism of C^*-algebras* $f\colon A \to B$ is a homomorphism of F-algebras in the algebraic sense that respects the involutions. A consequence of the C^*-identity is that a homomorphism of C^*-algebras $f\colon A \to B$ automatically satisfies $||f(a)|| \leq ||a||$ for all $a \in A$ and is in particular continuous. Moreover, any injective homomorphism of C^*-algebras $f\colon A \to B$ is automatically isometric, i.e., satisfies $||f(a)|| = ||a||$ for all $a \in A$, and two C^*-algebras which are isomorphic as F-algebras with involutions in the purely algebraic sense are automatically isomorphic as C^*-algebras. Two homomorphisms $f, g\colon A \to B$ are *homotopic* if there is a path $\{\gamma_t \mid t \in [0,1]\}$ of homomorphisms of C^*-algebras $\gamma_t\colon A \to B$ such that

$\gamma_0 = f$ and $\gamma_1 = g$ and for every a the evaluation map $[0,1] \to B$, $t \mapsto \gamma_t(a)$ is continuous with respect to the C^*-norm on B. Equivalently, there is a homomorphism of C^*-algebras $\gamma\colon A \to C([0,1], B)$ to the C^*-algebra of continuous functions from $[0,1]$ to B under the supremum norm such that its composition with the evaluation maps at $t = 0$ and $t = 1$ from $C([0,1], B)$ to B are f and g.

If H is a Hilbert F-space, then the algebra of bounded operators $\mathcal{B}(H)$ with the involution given by taking adjoint operators and the operator norm is a C^*-algebra. Any subalgebra $A \subseteq \mathcal{B}(H)$ that is closed in the norm topology and closed under taking adjoints inherits the structure of a C^*-algebra, and any C^*-algebra is isomorphic as C^*-algebra to such A.

We are not requiring a unit for the multiplication. If the Banach algebra or C^*-algebra A has a unit for the multiplication, we call A a *unital Banach algebra* or *unital C^*-algebra*.

Given a C^*-algebra A, an *ideal* in A is a two-sided ideal of the underlying F-algebra that is closed in the norm topology. It is automatically closed under the involution and hence inherits the structure of a C^*-algebra. The quotient A/I inherits the structure of a C^*-algebra by the obvious F-algebra structure and the norm $||a + I||_{A/I} := \inf\{||a + i||_A \mid i \in I\}$. Kernels of C^*-homomorphisms $f\colon A \to B$ are ideals A and each ideal in A is the kernel of some homomorphism of C^*-algebras with A as source, namely, of the projection $A \to A/I$.

Fix an infinite dimensional separable F Hilbert space H. Let $\mathcal{B}$ be the unital C^*-algebra of bounded operators $H \to H$. An element $T \in \mathcal{B}(H)$ is *compact* if for any bounded subset $B \subseteq H$ the closure of $T(B)$ is a compact subset of H. The compact operators form an ideal $\mathcal{K}$ in $\mathcal{B}$. The *Calkin algebra* is the unital C^*-algebra $\mathcal{B}/\mathcal{K}$

Let X be a locally compact Hausdorff space. Denote by $C_0(X, F)$ the C^*-algebra of continuous functions $f\colon X \to F$ that vanish at infinity, i.e., for every $\epsilon > 0$ there exists a compact subset $C \subseteq X$ such that $|f(x)| \le \epsilon$ holds for all $x \in X \setminus C$. If F is clear from the context, we often abbreviate $C_0(X) = C_0(X, F)$. Define an involution $*\colon C_0(X, F) \to C_0(X, F)$ by sending f to the function mapping $x \in X$ to $\overline{f(x)}$. Equip $C_0(X, F)$ with the supremum norm. Then $C_0(X, F)$ is a C^*-algebra. If X is compact, the constant function on X with value 1 is a unit. Moreover, $C_0(X, F)$ is unital if and only if X is compact.

Example 10.28 (One-point and Stone-Čech compactification). If X is a locally compact Hausdorff space, then we can assign to it two compactifications, the one-point compactification X_+ and the Stone-Čech compactification βX, see [737, page 183 and Section 5.3]. Then $C_0(X_+, F)$ agrees with $C_0(X, F)_+$ and $C(\beta X, F)$ agrees with $C_b(X, F)$, the C^*-algebra of bounded continuous functions $X \to F$. (Actually $C_b(X, F)$ is the so-called multiplier algebra of $C_0(X, F)$.) See for instance [972, Example 2.1.2 on page 28 and Example .2.2.4 on page 32].

Let $L^2(G, F)$ be the Hilbert F-space whose orthonormal basis is G. If F is clear from the context, we often abbreviate $L^2(G) = L^2(G, F)$. Let $\mathcal{B}(L^2(G, F))$ denote the bounded linear operators on the Hilbert F-space $L^2(G, F)$. The *reduced group C^*-algebra* $C_r^*(G, F)$ is the closure in the norm topology of the image of the regular representation $FG \to \mathcal{B}(L^2(G, F))$ that sends an element $u \in FG$ to the (left) G-equivariant bounded operator $L^2(G, F) \to L^2(G, F)$ given by right multiplication with u. Let $L^1(G, F)$ be the Banach $*$-algebra of formal sums $\sum_{g \in G} \lambda_g \cdot g$ with coefficients in F such that $\sum_{g \in G} |\lambda_g| < \infty$. If F is clear from the context, we often abbreviate $L^1(G) = L^1(G, F)$. There are natural inclusions

$$FG \subseteq L^1(G, F) \subseteq C_r^*(G, F) \subseteq \mathcal{B}(L^2(G, F))^G \subseteq \mathcal{B}(L^2(G, F)).$$

Exercise 10.29. Show for a discrete group G that $L^1(G, F)$ is a C^*-algebra if and only if G is trivial or (G has order 2 and $F = \mathbb{R}$).

For a group G let $C_m^*(G, F)$ be its *maximal group C^*-algebra* that is the norm closure of the image of the so-called universal representation $FG \to \mathcal{B}(H_u)$, compare [786, 7.1.5 on page 229]. The maximal group C^*-algebra has the advantage that every homomorphism of groups $\phi \colon G \to H$ induces a homomorphism $C_m^*(G, F) \to C_m^*(H, F)$ of C^*-algebras. This is not true for the reduced group C^*-algebra $C_r^*(G, F)$. Here is a counterexample: since $C_r^*(G, F)$ is a simple algebra if G is a non-abelian free group [797], there is no unital algebra homomorphism $C_r^*(G, F) \to C_r^*(\{1\}, F) = F$. There is a canonical homomorphism of C^*-algebras $C_m^*(G, F) \to C_r^*(G, F)$ which is an isomorphism of C^*-algebra if and only if G is amenable, see [786, Theorem 7.3.9 on page 243].

If F is clear from the context, we often abbreviate $C_r^*(G) = C_r^*(G, F)$ and $C_m^*(G) = C_m^*(G, F)$.

Given a discrete group G, a *G-C^*-algebra* A is a C^*-algebra together with a G-action $\rho \colon G \to \operatorname{aut}(A)$ by C^*-automorphisms. One can associate to a G-C^*-algebra A two new C^*-algebras, its *reduced crossed product* $A \rtimes_r G$ and its *maximal crossed product* $A \rtimes_m G$, see [786, 7.6.5 on page 257 and 7.7.4 on page 262]. There is a canonical homomorphism from the maximal crossed product to the reduced crossed product which is an isomorphism if G is amenable, see [786, Theorem 7.7.7. on page 263]. If we take $A = F$ with the trivial G-action, then $F \rtimes_r G$ and $F \rtimes_m G$ are just $C_r^*(G, F)$ and $C_m^*(G, F)$.

Let $\{A_i \mid i \in I\}$ be a directed system of C^*-algebras. Then its *colimit*, often also called *inductive limit*, or *direct limit*, is a C^*-algebra denoted by $\operatorname{colim}_{i \in I} A_i$, together with homomorphisms of C^*-algebras $\psi_j \colon A_i \to \operatorname{colim}_{i \in I} A_i$ for every $j \in I$ such that $\psi_j \circ \phi_{i,j} = \psi_i$ holds for $i, j \in I$ with $i \leq j$ and the following universal property is satisfied: For every C^*-algebra B and every system of homomorphisms of C^*-algebras $\{\mu_i \colon A_i \to B \mid i \in I\}$ such that $\mu_j \circ \phi_{i,j} = \mu_i$ holds for $i, j \in I$ with $i \leq j$, there is precisely one homomorphism of C^*-algebras $\mu \colon \operatorname{colim}_{i \in I} A_i \to B$ satisfying $\mu \circ \psi_i = \mu_i$ for every $i \in I$. The colimit exists and is unique up to isomorphism of C^*-algebras.

An extensive discussions about tensor products $A\widehat{\otimes}B$ of C^*-algebras can be found in [972, Appendix T]. There are various ways for two C^*-algebras A and B to complete their algebraic tensor product $A\otimes_F B$ to a new C^*-algebra $A\widehat{\otimes}B$. One is the spatial norm which turns out to be the minimal norm and leads to the *spatial tensor product*, sometimes also called *the minimal tensor product*. A second is the maximal norm which leads to the *maximal tensor product*. Any C^*-norm on the algebraic tensor product lies between the minimal and the maximal norm. The favorite situation is the case where A is a so-called *nuclear* C^*-algebra, i.e., the minimal and the maximal norm on the algebraic tensor product $A \otimes_F B$ agree for any C^*-algebra B. Then for any C^*-algebra B there exists only one C^*-norm on the algebraic tensor product $A\otimes_F B$ and hence there is a unique tensor product C^*-algebra $A\widehat{\otimes}B$. Commutative C^*-algebras and finite dimensional C^*-algebras are nuclear. The class of nuclear C^*-algebras is closed under taking colimits limits over directed systems and extensions. In particular the C^*-algebra of compact operators $\mathcal{K}$ is nuclear. Ideals in and quotients of nuclear C^*-algebras are again nuclear. The reduced group C^*-algebra of G is nuclear if and only if G is amenable.

Given a C^*-algebra A, define $\mathrm{M}_n(A) = A\widehat{\otimes}\,\mathrm{M}_n(F)$ which is well-defined since $\mathrm{M}_n(F) = \mathcal{B}(F^n)$ is nuclear. Actually, the underlying F-algebra of $\mathrm{M}_n(A)$ is the algebraic tensor product $A \otimes_F \mathrm{M}_n(F)$ itself, one does not have to complete.

The C^*-algebra $\mathcal{K}$ of compact operators on an infinite dimensional separable Hilbert F-space is the colimit of the directed system $\mathrm{M}_1(F) \to \mathrm{M}_2(F) \to \mathrm{M}_3(F) \to \cdots$ where the structure maps are given by taking the block sum with the $(1,1)$-zero matrix (0). Given a C^*-algebra A, the tensor product $A\widehat{\otimes}\mathcal{K}$ is the colimit of the directed system $\mathrm{M}_1(A) \to \mathrm{M}_2(A) \to \mathrm{M}_3(A) \to \cdots$.

A C^*-algebra is called *separable* if its underlying topological space is separable, i.e., contains a dense countable subset.

A C^*-algebra SA is called *stable* if A is isomorphic as C^*-algebra to $A\widehat{\otimes}\mathcal{K}$. Since $\mathcal{K}\widehat{\otimes}\mathcal{K}$ is isomorphic to $\mathcal{K}$, the tensor product $A\widehat{\otimes}\mathcal{K}$ is a stable C^*-algebra for every C^*-algebra A.

More information about C^*-algebras can be found for instance in [38, 134, 236, 262, 296, 373, 522, 523, 786].

10.3.2 Basic Properties of the Topological K-Theory of C^*-Algebras

Topological K-theory assigns to any (not necessarily unital) C^*-algebra A a $\mathbb{Z}$-graded abelian group $K_*(A)$ such that the following properties hold:

(i) *Functoriality*

A homomorphism $f\colon A \to B$ of C^*-algebras induces a map of $\mathbb{Z}$-graded abelian groups $K_*(f)\colon K_*(A) \to K_*(B)$. If $g\colon B \to C$ is another homomorphism of C^*-algebras, we have $K_*(g \circ f) = K_*(g) \circ K_*(f)$. Moreover $K_*(\mathrm{id}_A) = \mathrm{id}_{K_*(A)}$;

(ii) *Homotopy invariance*
Homotopic homomorphisms of C^*-algebras induce the same map on the topological K-theory;

(iii) *Finite direct products*
If A and B are C^*-algebras, their direct product $A \times B$ inherits the structure of a C^*-algebra by $||(a,b)|| = \max\{||a||, ||b||\}$. The projections to the factors are homomorphisms of C^*-algebras and induce a natural isomorphism of $\mathbb{Z}$-graded abelian groups

$$K_*(A \times B) \xrightarrow{\cong} K_*(A) \times K_*(B);$$

(iv) *Compatibility with colimits over directed systems*
Let $\{A_i \mid i \in I\}$ be a directed system of C^*-algebras. Then the canonical map of $\mathbb{Z}$-graded abelian groups is an isomorphism

$$\operatorname{colim}_{i \in I} K_*(A_i) \xrightarrow{\cong} K_*\bigl(\operatorname{colim}_{i \to I} A_i\bigr);$$

(v) *Morita equivalence*
There are canonical isomorphisms $K_*(A) \to K_*(\mathrm{M}_n(A))$;

(vi) *Stabilization*
The canonical inclusion $F = \mathrm{M}_1(F) \to \mathcal{K}$ yields an inclusion $i_A\colon A \to A \widehat{\otimes} \mathcal{K}$. The induced map of $\mathbb{Z}$-graded abelian groups $K_*(i_A)\colon K_*(A) \to K_*(A \widehat{\otimes} \mathcal{K})$ is an isomorphism;

(vii) *Long exact sequence of an ideal*
Let I be a (two-sided closed) ideal in the C^*-algebra A. Denote by $i\colon I \to A$ the inclusion and by $p\colon A \to A/I$ the projection. Then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} K_n(I) \xrightarrow{K_n(i)} K_n(A) \xrightarrow{K_n(p)} K_n(A/I) \xrightarrow{\partial_n} K_{n-1}(I)$$
$$\xrightarrow{K_{n-1}(i)} K_{n-1}(A) \xrightarrow{K_{n-1}(p)} K_{n-1}(A/I) \xrightarrow{\partial_{n-1}} \cdots;$$

(viii) *Bott periodicity*
For any C^*-algebra A over F there exists an isomorphism of degree $b(F)$

$$\beta_*(A)\colon K_*(A) \xrightarrow{\cong} K_{*+b(F)}(A),$$

which is natural in A, and compatible with the boundary operator ∂_* of the long exact sequence of an ideal where $b(F) = 2$ if $F = \mathbb{C}$ and $b(F) = 8$ if $F = \mathbb{R}$;

(ix) *Commutative C^*-algebras*
Let X be a finite CW-complex (or more generally, compact Hausdorff

space). Then there are isomorphisms of $\mathbb{Z}$-graded abelian groups, natural in X,

$$K^*(X) \xrightarrow{\cong} K_*(C(X,\mathbb{C}));$$
$$KO^*(X) \xrightarrow{\cong} K_*(C(X,\mathbb{R})),$$

from the topological complex or real K-theory of X to the topological K-theory of the unital C^*-algebra $C(X,F)$ of continuous functions $X \to F$.

Of course the last property about commutative C^*-algebras is closely related to the material in Section 2.4 about Swan's Theorem 2.27.

Notation 10.30 (K and KO). If one considers a real C^*-algebra, one often writes $KO_*(A)$ instead of $K_*(A)$ to indicate that the C^*-algebra under consideration lives over $\mathbb{R}$.

The 0-th topological K-group $K_0(A)$ of a C^*-algebra A agrees with the projective class group $K_0(A)$ of the underlying ring (possibly without unit) in the sense of Definition 3.90. In contrast to $K_0(A)$ the topology of A enters in the definition of $K_1(A)$ as explained next.

If A is a C^*-algebra (with or without unit), then we define the unital C^*-algebra A_+ as follows. The underling unital F-algebra is $A \oplus F$ with the addition $(a,\lambda)+(b,\mu) = (a+b,\lambda+\mu)$, multiplication $(a,\lambda)\cdot(b,\mu) = (a\cdot b+\lambda\cdot b+\mu\cdot a,\lambda\cdot\mu)$, and unit $(0,1)$. The involutions sends (a,λ) to $(a^*,\overline{\lambda})$. The C^*-norm is explained for instance in [786, 1.1.3 on page 1] or [972, Proposition 2.1.7 on page 30]. Let $p\colon A_+ \to F$ be the canonical projection sending (a,λ) to λ. It induces maps $\mathrm{M}_n(A_+) \to \mathrm{M}_n(F)$ and $\mathrm{GL}_n(A_+) \to \mathrm{GL}_n(F)$, denoted again by p. Define

$$\mathrm{GL}_n^+(A) := \{B \in \mathrm{GL}_n(A_+) \mid p(B) = 1\}. \tag{10.31}$$

This becomes a topological group by the subspace topology with respect to the inclusion $\mathrm{GL}_n^+(A) \subseteq \mathrm{M}_n(A_+)$. There is an obvious directed system of topological groups

$$\mathrm{GL}_1^+(A) \subseteq \mathrm{GL}_2^+(A) \subseteq \mathrm{GL}_3^+(A) \subseteq \cdots$$

coming from embedding $\mathrm{M}_n(A_+)$ into $\mathrm{M}_{n+1}(A_+)$ by taking the block sum with the $(1,1)$-identity matrix (1). Its colimit is a topological group denoted by $\mathrm{GL}^+(A)$. Let $\mathrm{GL}^+(A)^0$ be the path component of the unit element in $\mathrm{GL}^+(A)$. Then we get

$$K_1(A) = \mathrm{GL}^+(A)/\mathrm{GL}^+(A)_0 = \pi_0(\mathrm{GL}^+(A)). \tag{10.32}$$

More generally, we have

$$K_n(A) = \pi_{n-1}(\mathrm{GL}^+(A)) \quad \text{for } n \geq 1. \tag{10.33}$$

If A is unital, then one defines the topological group $\mathrm{GL}(A) = \operatorname{colim}_{n\to\infty} \mathrm{GL}_n(A)$ and obtains a canonical isomorphism

$$K_n(A) \cong \pi_{n-1}(\mathrm{GL}(A)) \quad \text{for } n \geq 1. \tag{10.34}$$

Exercise 10.35. Compute for $F = \mathbb{C}$ the topological K-theory of $\mathcal{B}$, $\mathcal{K}$ and $\mathcal{B}/\mathcal{K}$.

Remark 10.36 (Six term sequence of an ideal). Let $F = \mathbb{C}$ in this Remark 10.36. Since K_* is two-periodic, one thinks often about it as a $\mathbb{Z}/2$-graded theory. The long exact sequence of an extension $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$ becomes the *six-term exact sequence of an ideal*

$$\begin{tikzcd} K_1(I) \arrow[r, "K_1(i)"] & K_1(A) \arrow[r, "K_1(p)"] & K_1(A/I) \arrow[d, "\partial_1"] \\ K_0(A/I) \arrow[u, "\partial_0"] & K_0(A) \arrow[l, "K_0(p)"] & K_0(I) \arrow[l, "K_0(i)"] \end{tikzcd}$$

Remark 10.37 (Topological K-theory in terms of unitary groups). Let $F = \mathbb{C}$ in this Remark 10.37. Let $U_n(A)$ be the group of unitary (n,n)-matrices over A, i.e., (n,n)-matrices U that are invertible and satisfy $U^{-1} = U^*$ where U^* is defined by transposing and applying to each entry the involution on A. Define $U_n^+(A) := \{U \in U_n(A_+) \mid p(U) = 1\}$. Put $U(A) = \operatorname{colim}_{n\to\infty} U_n(A)$ and $U^+(A) := \operatorname{colim}_{n\to\infty} U_n^+(A)$. Then then we have isomorphisms of groups, see [972, Proposition 4.2.6 on page 77],

$$\begin{aligned} K_1(A) = \mathrm{GL}^+(A)/\mathrm{GL}^+(A)_0 &\cong \mathrm{GL}(A^+)/\mathrm{GL}(A^+)_0 \\ &\cong U^+(A)/U^+(A)_0 \cong U(A^+)/U(A^+)_0. \end{aligned}$$

Example 10.38 (On the boundary map and indices). Let $F = \mathbb{C}$ in this Example 10.38. Let A be a unital C^*-algebra, $I \subseteq A$ be an ideal, and $p\colon A \to A/I$ be the projection. Let u be a unitary element in A/I. Let $a \in A$ be any element in A with $p(a) = u$ and $||a|| = 1$. Consider the $(2,2)$-matrices over A

$$\begin{aligned} P &:= \begin{pmatrix} aa^* & a(1a^*a)^{1/2} \\ a^*(1-aa^*)^{1/2} & 1 - a^*a \end{pmatrix}; \\ Q &:= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where $(1-aa^*)^{1/2}$ is uniquely determined by the properties that it is positive, i.e., of the form b^*b for some $b \in A$, and satisfies $(1-aa^*)^{1/2} \cdot (1-aa^*)^{1/2} = 1-aa^*$, and analogously for $(1-a^*a)^{1/2}$. Then P is a projection, i.e., $P^2 = P$ and $P^* = P$, and Q is a projection. Moreover, $P - Q$ lies in $\mathrm{M}_2(I)$. Define matrices in $\mathrm{M}_2(I_+)$ by

$$P_+ := \begin{pmatrix} (aa^* - 1, 1) & (a(1a^*a)^{1/2}, 0) \\ (a^*(1 - aa^*)^{1/2}, 0) & (1 - a^*a, 0) \end{pmatrix};$$
$$Q := \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}.$$

One easily checks $P_+^2 = P_+$ and $Q_+^2 = Q_+$ and $P_+ - Q_+ \in I$. Hence P_+ and Q_+ determine elements $[P_+], [Q_+] \in K_0(I_+)$ such that the difference $[P_+] - [Q_+]$ is mapped under the canonical projection $K_0(I_+) \to K_0(\mathbb{C})$ to zero. Hence $[P_+] - [Q_+]$ defines an element in $K_0(I)$. It turns out that the image $\partial_1([u])$ of the class $[u] \in K_1(A)$ under the boundary homomorphism $\partial_1 \colon K_1(A/I) \to K_0(I)$ is the class $[P_+] - [Q_+]$, see [474, Proposition 4.8.10 (a) on page 109].

If we can additionally arrange that a is a partial isometry, i.e., a^*a is a projection, then $1 - a^*a$ and $1 - aa^*$ lie in I and are projections, and we obtain an element $[1 - a^*a] - [1 - aa^*]$ in $K_0(I)$ which agrees with $\partial_1([u])$, see [474, Proposition 4.8.10 (b) on page 109].

Now we apply this to $A = \mathcal{B} = \mathcal{B}(H)$ and $I = \mathcal{K} = \mathcal{K}(H)$ for an infinite dimensional separable Hilbert space H. Let $a \in \mathcal{B}$ be a Fredholm operator such that a is a partial isometry. Then $1 - a^*a$ is the orthogonal projection onto the kernel of a and $1 - aa^*$ is the orthogonal projection onto the cokernel of a. Hence the element $[1 - a^*a] - [1 - aa^*] \in K_0(\mathcal{K})$ becomes under the standard identification $K_0(\mathcal{K}) \cong \mathbb{Z}$ the difference of the dimension of the kernel of a and the dimension of the cokernel of a which is by definition the classical index of the Fredholm operator a. This shows that $\partial_1 \colon K_1(\mathcal{B}/\mathcal{K}) \to K_0(\mathcal{K}) \cong \mathbb{Z}$ sends the class of $[a]$ to the classical index of a.

It will often occur in many more general and important situations that ∂_1 can be viewed as an index map.

Example 10.39 (Suspensions and cones). The *suspension* of a C^*-algebra A is the C^*-algebra ΣA of continuous functions $f \colon [0,1] \to A$ with $f(0) = f(1) = 1$ equipped with the obvious algebra structure and involution and the supremums norm inherited from A. Denote by $\Sigma^n(A)$ the n-fold suspension. It can be identified with the tensor product of C^*-algebras $A \widehat{\otimes} C_0(\mathbb{R}^n)$. The *cone* is defined analogously as the C^*-algebra $\mathrm{cone}(A)$ of continuous functions $f \colon [0,1] \to A$ with $f(0) = 0$. It can be identified with the tensor product of C^*-algebras $A \widehat{\otimes} C_0((0,1])$. There is an obvious exact sequence of C^*-algebras $0 \to \Sigma A \to \mathrm{cone}(A) \to A \to 0$. Moreover, the C^*-algebra $\mathrm{cone}(A)$ is *contractible*, i.e., the zero and the identity endomorphism are homotopic. The desired homotopy is given by the formula $f_t(s) := f(ts)$. Hence $K_*(\mathrm{cone}(A))$ is trivial and the boundary operator in the associated long exact sequence induces isomorphisms

$$\partial_n \colon K_n(A) \xrightarrow{\cong} K_{n-1}(\Sigma A).$$

For complex C^*-algebras A and B for which A lies in the so-called bootstrap category $\mathcal{N}$ a *Künneth Theorem*, i.e., an exact sequence $0 \to K_*(A) \otimes K_*(B) \to K_*(A \widehat{\otimes} B) \to \operatorname{Tor}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$ is established in [887]. The case of real C^*-algebras is treated in [138].

Remark 10.40 (Topological K-theory and the classification of C^*-algebras). One prominent feature is that for certain classes of C^*-algebras their isomorphism type is determined by their topological K-theory, sometimes taking the order structure on $K_0(A)$ coming from the positive cone of those elements that are represented by finitely generated projective modules into account. If one considers the topological K-theory of spaces such nice classification results are not available.

One example is the class of *AF-algebras*, i.e., C^*- algebras that occur as a colimit of a sequence of finite dimensional C^*-algebras, due to Elliot, see [319], [840, Chapter 7], [972, 12.1]. The index n of the *Cuntz C^*-algebra* $\mathcal{O}_n$ is determined by the topological K-theory since $K_0(\mathcal{O}_n) \cong \mathbb{Z}/n$ and $K_1(\mathcal{O}_n) = 0$, see [252], [972, 12.2]. A very important result about the classification of so-called *Kirchberg C^*-algebras* in terms of their topological K-theory is due to Kirchberg, see for instance [839, Chapter 8].

Remark 10.41 (Topological K-theory and generalized index theory). One important motivation to study the topological K-theory of C^*-algebras is index theory and its generalizations. A first introduction how one can assign to a Fredholm operator over a C^*-algebra A an element in $K_0(A)$ is given in [972, Chapter 17] following Mingo [718]. There are many other index theorems taking values in the topological K-theory of C^*-algebras. Often they are generalizations of the classical family index theorem for families of operators parametrized over a closed manifold M which take values in $K^*(M) = K_*(C(M))$.

One can attach to geometric or topological situations new C^*-algebras and consider their topological K-theory and indices of appropriate operators where it is not possible anymore to work with topological spaces. Examples are foliations and coarse geometry. There are also plenty other generalizations of the classical index theorems using topological K-theory of C^*-algebras. For information about these topics we refer for instance to [236, 256, 474, 723].

More information about the topological K-theory of C^*-algebras can be found for instance in [133, 236, 256, 474, 840, 972].

10.4 The Baum-Connes Conjecture for Torsionfree Groups

Let G be a group. Then there exist for all $n \in \mathbb{Z}$ assembly maps

$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_n \colon K_n(BG) \to K_n(C_r^*(G,\mathbb{C})); \tag{10.42}$$

$$\operatorname{asmb}^{G,\mathbb{R}}(BG)_n \colon KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})). \tag{10.43}$$

Conjecture 10.44 (Baum-Connes Conjecture for torsionfree groups). The assembly maps appearing in (10.42) and (10.43) are isomorphisms for all $n \in \mathbb{Z}$, provided that G is torsionfree.

It is crucial for the Baum-Connes Conjecture to work with the reduced group C^*-algebra, it is definitely not true for the maximal group C^* algebra in general. Moreover, Conjecture 10.44 in general fails for groups with torsion. The general version which makes sense for all groups will be discussed in Chapter 14.

Exercise 10.45. Show for a finite group G that the following statements are equivalent:

(i) $K_0(BG)$ and $K_0(C_r^*(G))$ are rationally isomorphic;
(ii) $KO_0(BG)$ and $KO_0(C_r^*(G))$ are rationally isomorphic;
(iii) G is trivial.

One benefit of Conjecture 10.44 is that the right side is of great interest because of index theory but hard to compute, whereas the left side is accessible to standard methods from algebraic topology.

Example 10.46 (Three-dimensional Heisenberg group). Let $\operatorname{Hei}(\mathbb{R})$ be the *three-dimensional Heisenberg group* that is the subgroup of $\operatorname{GL}_3(\mathbb{R})$ consisting of upper triangular matrices whose diagonal entries are all equal to 1. The *three-dimensional discrete Heisenberg group* Hei is the intersection of $\operatorname{Hei}(\mathbb{R})$ with $\operatorname{GL}_3(\mathbb{Z})$. Obviously Hei is a torsionfree discrete subgroup of the contractible Lie group $\operatorname{Hei}(\mathbb{R})$. Hence $\operatorname{Hei}\backslash\operatorname{Hei}(\mathbb{R})$ is a model for $B\operatorname{Hei}$ which is an orientable closed 3-manifold.

Define elements in Hei

$$u := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad v := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad w := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we get the presentation

$$\operatorname{Hei} = \langle u, v, w \mid [u,w] = v, [u,v] = 1, [w,v] = 1 \rangle.$$

Therefore we have a central extension $1 \to \mathbb{Z} \xrightarrow{i} \operatorname{Hei} \xrightarrow{p} \mathbb{Z}^2 \to 1$, where i sends the generator of $\mathbb{Z}$ to v and p sends v to $(0,0)$, u to $(1,0)$ and w to $(0,1)$. Hence the map $H_1(B\operatorname{Hei}) \to H_1(B\mathbb{Z}^2)$ is an isomorphism. Using Poincaré duality we conclude

$$H_n(B\operatorname{Hei}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 3; \\ \mathbb{Z}^2 & \text{if } n = 1, 2. \end{cases}$$

We conclude from the Chern character (10.1) for every $n \in \mathbb{Z}$.

$$K_n(B\mathrm{Hei}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^3.$$

Next we consider the Atiyah-Hirzebruch spectral sequence converging to $K_{p+q}(B\mathrm{Hei})$ whose E^2-term is $E^2_{p,q} = H_p(B\mathrm{Hei}; K_q(\{\bullet\}))$. Its E^2-page looks as follows

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Each entry is a finitely generated free $\mathbb{Z}$-module and we have for every $n \in \mathbb{Z}$

$$\sum_{p+q=n} \dim_{\mathbb{Q}}(E^2_{p,q}) \otimes_{\mathbb{Z}} \mathbb{Q} = 3 = K_n(B\mathrm{Hei}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This implies that all differentials must vanish and we get for every $n \in \mathbb{Z}$

$$K_n(B\mathrm{Hei}) \cong \mathbb{Z}^3.$$

Conjecture 10.44 is known to be true for Hei and hence we conclude for every $n \in \mathbb{Z}$

$$K_n(C^*_r(\mathrm{Hei})) \cong \mathbb{Z}^3.$$

Exercise 10.47. Let G be the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$ where the generator of $\mathbb{Z}$ acts on $\mathbb{Z}$ by $-\mathrm{id}$. Compute $K_*(C^*_r(G))$ using the fact that Conjecture 10.44 is known to be true for G.

Next we discuss some consequences of the Baum-Connes Conjecture for torsionfree groups 10.44.

10.4.1 The Trace Conjecture in the Torsionfree Case

The assembly map appearing in the Baum-Connes Conjecture has an interpretation in terms of index theory. Namely, an element $\eta \in K_0(BG)$ can be represented by a pair (M, P^*) consisting of a cocompact free proper smooth G-manifold M with a G-invariant Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M, see [104]. To such a pair one can assign an index $\operatorname{ind}_{C_r^*(G)}(M, P^*)$ in $K_0(C_r^*(G))$, see [723] that is the image of η under the assembly map $K_0(BG) \to K_0(C_r^*(G))$ appearing in Conjecture 10.44. With this interpretation the surjectivity of the assembly map for a torsionfree group says that any element in $K_0(C_r^*(G))$ can be realized as an index. This allows to apply index theorems to get interesting information. It is of the same significance as the interpretation of the L-theoretic assembly map as the map σ appearing in the exact surgery sequence discussed in the proof of Theorem 9.168.

Here is a prototype of such an argument. The *standard trace*

$$\operatorname{tr}_{C_r^*(G)} \colon C_r^*(G) \to \mathbb{C} \tag{10.48}$$

sends an element $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(1), 1\rangle_{l^2(G)}$. Applying the trace to idempotent matrices yields a homomorphism

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}.$$

Let $\operatorname{pr} \colon BG \to \{\bullet\}$ be the projection. For a group G the following diagram commutes

$$\begin{tikzcd} K_0(BG) \arrow[r, "\operatorname{asmb}^{G,\mathbb{C}}(BG)_*"] \arrow[d, "K_0(\operatorname{pr})"'] & K_0(C_r^*(G)) \arrow[r, "\operatorname{tr}_{C_r^*(G)}"] & \mathbb{R} \\ K_0(\{\bullet\}) \arrow[r, "\cong"] & K_0(\mathbb{C}) \arrow[r, "\cong", "\operatorname{tr}_{\mathbb{C}}"'] & \mathbb{Z} \arrow[u, "i"'] \end{tikzcd} \tag{10.49}$$

where $i \colon \mathbb{Z} \to \mathbb{R}$ is the inclusion. This non-trivial statement follows from Atiyah's L^2-index theorem [46]. Atiyah's theorem says that the L^2-index $\operatorname{tr}_{C_r^*(G)} \circ \operatorname{asmb}_*(\eta)$ of an element $\eta \in K_0(BG)$, which is represented by a pair (M, P^*), agrees with the ordinary index of $(G\backslash M; G\backslash P^*)$, which is given by $\operatorname{tr}_{\mathbb{C}} \circ K_0(\operatorname{pr})(\eta) \in \mathbb{Z}$.

The following conjecture is taken from [102, page 21].

Conjecture 10.50 (Trace Conjecture for torsionfree groups). For a torsionfree group G the image of

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

The commutativity of diagram (10.49) shows

Lemma 10.51. *If the Baum-Connes assembly map* $K_0(BG) \to K_0(C_r^*(G))$ *of* (10.42) *is surjective, then the Trace Conjecture for Torsionfree Groups 10.50 holds for* G.

A Modified Trace Conjecture for not necessarily torsionfree groups is discussed in Subsection 14.8.1.

10.4.2 The Kadison Conjecture

Conjecture 10.52 (Kadison Conjecture). If G is a torsionfree group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.

Lemma 10.53. *The Trace Conjecture for Torsionfree Groups 10.50 implies the Kadison Conjecture 10.52.*

Proof. Assume that $p \in C_r^*(G)$ is an idempotent different from 0 or 1. From p one can construct a non-trivial projection $q \in C_r^*(G)$, i.e. $q^2 = q$, $q^* = q$, with $\operatorname{im}(p) = \operatorname{im}(q)$ and hence with $0 < q < 1$. Since the standard trace $\operatorname{tr}_{C_r^*(G)}$ is faithful, we conclude $\operatorname{tr}_{C_r^*(G)}(q) \in \mathbb{R}$ with $0 < \operatorname{tr}_{C_r^*(G)}(q) < 1$. Since $\operatorname{tr}_{C_r^*(G)}(q)$ is by definition the image of the element $[\operatorname{im}(q)] \in K_0(C_r^*(G))$ under $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$, we get a contradiction to the assumption $\operatorname{im}(\operatorname{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$. □

Remark 10.54 (The Kadison Conjecture 10.52 and Kaplansky's Idempotent Conjecture 2.73). Obviously the Kadison Conjecture 10.52 implies Kaplansky's Idempotent Conjecture 2.73 in the case that R can be embedded in $\mathbb{C}$. Because of Remark 2.84 the Kadison Conjecture 10.52 implies Kaplansky's Idempotent Conjecture 2.73 if R is any field of characteristic zero. The Bost Conjecture 14.23 implies that there are no non-trivial idempotents in $L^1(G)$ and hence the Kaplansky's Idempotent Conjecture 2.73 for fields of characteristic zero, see [124, Corollary 1.6].

10.4.3 The Zero-in-the-Spectrum Conjecture

The following Conjecture is due to Gromov [422, page 120].

Conjecture 10.55 (Zero-in-the-spectrum Conjecture). Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure

$$(\Delta_p)_{\min} \colon L^2\Omega^p(\widetilde{M}) \supset \operatorname{dom}(\Delta_p)_{\min} \to L^2\Omega^p(\widetilde{M})$$

for some $p \in \{0, 1, \ldots, \dim M\}$ where Δ_p denotes the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Theorem 10.56 (The strong Novikov Conjecture implies the Zero-in-the-spectrum Conjecture). *Suppose that M is an aspherical closed Riemannian manifold with fundamental group G, then the injectivity of the assembly map*

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-spectrum Conjecture 10.55 for $\widetilde{M}$.

Proof. We give a sketch of the proof. More details can be found in [623, Corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on $\widetilde{M}$, yields a contradiction in the case that $n = \dim(M)$ is even. Namely, this assumption implies that the $C_r^*(G)$-valued index of the signature operator twisted with the flat bundle $\widetilde{M} \times_G C_r^*(G) \to M$ in $K_0(C_r^*(G))$ is zero where $G = \pi_1(M)$. This index is the image of the class $[S]$ defined by the signature operator in $K_0(BG)$ under the assembly map $K_0(BG) \to K_0(C_r^*(G))$. Since by assumption the assembly map is rationally injective, this implies $[S] = 0$ in $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that M is aspherical by assumption and hence $M = BG$. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} H_{2p}(M; \mathbb{Q})$$

that sends $[S]$ to the Poincaré dual $\mathcal{L}(M) \cap [M]_{\mathbb{Q}}$ of the Hirzebruch L-class $\mathcal{L}(M) \in \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{Q})$. This implies that $\mathcal{L}(M) \cap [M]_{\mathbb{Q}} = 0$ and hence $\mathcal{L}(M) = 0$. This contradicts the fact that the component of $\mathcal{L}(M)$ in $H^0(M; \mathbb{Q})$ is 1. $\square$

More information about the Zero-in-the-Spectrum Conjecture 10.55 can be found for instance in [623] and [635, Section 12].

10.5 Kasparov's *KK*-Theory

Kasparov introduced the bivariant *KK-theory* that assigns to two separable C^*-algebras A and B a $\mathbb{Z}$-graded abelian group $KK_*(A, B)$. We give a very brief summary of it. In the sequel all C^*-algebras are assumed to be separable.

10.5.1 Basic Properties of KK-theory for C^*-Algebras

(i) *Bi-functoriality*
A homomorphism $f\colon A \to B$ of C^*-algebras induces homomorphisms of $\mathbb{Z}$-graded abelian groups

$$KK_*(f, \mathrm{id}_D)\colon KK^*(B, D) \to KK_*(A, D);$$
$$KK_*(\mathrm{id}_D, f)\colon KK_*(D, A) \to KK_*(D, B).$$

If $g\colon B \to C$ is another homomorphism of C^*-algebras, we have

$$KK_*(g \circ f, \mathrm{id}) = KK_*(f, \mathrm{id}_D) \circ K_*(g, \mathrm{id}_D);$$
$$KK_*(\mathrm{id}_D, g \circ f) = KK_*(\mathrm{id}_D, g) \circ K_*(\mathrm{id}_D, f).$$

Moreover $K_*(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{KK_*(A,B)}$. In particular $KK_*(-, D)$ is a contravariant and $KK_*(D, -)$ is a covariant functor from the category of C^*-algebras to the category of $\mathbb{Z}$-graded abelian groups;

(ii) *Homotopy invariance*
If $f, g\colon A \to B$ are homotopic homomorphisms of C^*-algebras, then $KK_*(f, \mathrm{id}_C) = KK_*(g, \mathrm{id}_C)$ and $KK_*(\mathrm{id}_C, f) = KK_*(\mathrm{id}_C, g)$;

(iii) *Finite direct products*
If A and B are C^*-algebras, there are a natural isomorphism of $\mathbb{Z}$-graded abelian groups

$$KK_*(A \times B, C) \xrightarrow{\cong} KK_*(A, C) \times KK_*(B, C);$$
$$KK_*(C, A \times B) \xrightarrow{\cong} KK_*(C, A) \times KK_*(C, B);$$

(iv) *Countable direct sums in the first variable*
If $A = \bigoplus_{i=0}^{\infty} A_i$ is a countable direct sum of C^*-algebras, then there is a natural isomorphism

$$KK_n\Big(\bigoplus_{i=0}^{\infty} A_i, B\Big) \xrightarrow{\cong} \prod_{i=0}^{\infty} K_n(A_i, B);$$

(v) *Morita equivalence*
For any integers $m, n \geq 1$ there are natural isomorphisms of $\mathbb{Z}$-graded abelian groups $KK_*(A, B) \xrightarrow{\cong} KK_*(\mathrm{M}_m(A), \mathrm{M}_n(B))$;

(vi) *Stabilization*
There are natural isomorphisms of $\mathbb{Z}$-graded abelian groups

$$KK_*(A, B) \xrightarrow{\cong} KK_*(A \widehat{\otimes} \mathcal{K}, B);$$
$$KK_*(A, B) \xrightarrow{\cong} KK_*(A, B \widehat{\otimes} \mathcal{K});$$

(vii) *Long exact sequence of an ideal*
Let $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$ be an extensions of (separable) C^*-algebras. If it is semisplit in the sense of [133, Definition 19.5.1. on page 195] (what is automatically true if A is nuclear,) then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

$$\begin{aligned} \cdots \xrightarrow{\delta_{n-1}} KK_n(A/I, B) \xrightarrow{KK_n(p,\mathrm{id}_B)} KK_n(A, B) \xrightarrow{KK_n(i,\mathrm{id}_B)} KK_n(I, B) \\ \xrightarrow{\delta_n} KK_{n+1}(A/I, B) \xrightarrow{KK_{n+1}(p,\mathrm{id}_B)} KK_{n+1}(A, B) \\ \xrightarrow{KK_{n+1}(i,\mathrm{id}_B)} KK_{n+1}(I, B,) \xrightarrow{\delta_{n+1}} \cdots . \end{aligned}$$

If the extension is semisplit or if B is nuclear, then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

$$\begin{aligned} \cdots \xrightarrow{\partial_{n+1}} KK_n(B, I) \xrightarrow{KK_n(\mathrm{id}_B,i)} KK_n(B, A) \xrightarrow{KK_n(\mathrm{id}_B,p)} KK_n(B, A/I) \\ \xrightarrow{\partial_n} KK_{n-1}(B, I) \xrightarrow{KK_{n-1}(\mathrm{id}_B,i)} KK_{n-1}(B, A) \\ \xrightarrow{KK_{n-1}(\mathrm{id}_B,p)} KK_{n-1}(B, A/I) \xrightarrow{\partial_{n-1}} \cdots ; \end{aligned}$$

(viii) *Bott periodicity*
There exists an isomorphism of degree $b(F)$

$$\beta_*(A) \colon KK_*(A, B) \xrightarrow{\cong} KK_{*+b(F)}(A, B),$$

which is natural in A and B and compatible with the boundary operators ∂_* of the long exact sequence of an ideal where $b(F) = 2$ if $F = \mathbb{C}$ and $b(F) = 8$ if $F = \mathbb{R}$;

(ix) *Connection to topological K-theory*
There is a natural isomorphism of $\mathbb{Z}$-graded abelian groups

$$K_*(A) \xrightarrow{\cong} KK_*(F, A);$$

(x) *Homomorphisms of C^*-algebras*
A homomorphism $f \colon A \to B$ of C^*-algebras defines an element $[f]$ in $KK_*(A, B)$.

Remark 10.57 (Some failures). The second variable is in general not compatible with countable direct sums and in particular not with colimits over directed sets. However, in the special case $A = \mathbb{C}$, this is the case since then $KK_*(\mathbb{C}, B)$ is just the topological K-theory of B.

The conditions about the existence of long exact sequence of an ideal such as semisplit or B being nuclear are needed.

10.5.2 The Kasparov's Intersection Product

One of the basic features of KK-theory is *Kasparov's intersection product* which is a bilinear pairing of $\mathbb{Z}$-graded abelian groups

$$\widehat{\otimes}_B \colon KK_*(A,B) \otimes KK_*(B,C) \to KK_*(A,C). \tag{10.58}$$

It has the following properties

(i) *Naturality*
It is natural in A, B and C;
(ii) *Associativity*
It is associative;
(iii) *Composition of homomorphisms*
If $f \colon A \to B$ and $g \colon B \to C$ are homomorphisms of C^*-algebras, then we get for the associated elements $[f] \in KK_0(A,B)$, $[g] \in KK_0(B,C)$ and $[g \circ f] \in KK_0(A,C)$
$$[g \circ f] = [f]\widehat{\otimes}_B[g];$$
(iv) *Units*
There is a unit $1_A := [\mathrm{id}_A]$ in $KK_0(A,A)$ for the intersection product.

Remark 10.59 (KK-equivalence). One of the basic features of the product is that an element x in $KK_0(A,B)$ induces a homomorphism

$$-\widehat{\otimes}_B x \colon K_n(A) = KK_n(F,A) \to K_n(B) = KK_n(F,B).$$

Of course $-\widehat{\otimes}_B[f]$ agrees with $K_n(f)$ if $f \colon A \to B$ is a homomorphism of C^*-algebras. An element $x \in KK_0(A,B)$ is called a KK-equivalence if there exists an element $y \in KK_0(B,A)$ satisfying $x\widehat{\otimes}_B y = 1_A$ and $y\widehat{\otimes}_A x = 1_B$. The basic feature of a KK-equivalence is that

$$-\widehat{\otimes}_B x \colon K_n(A) = KK_n(F,A) \to K_n(B) = KK_n(F,B)$$

is automatically an isomorphism, the inverse is $-\widehat{\otimes}_B y$.

Remark 10.60 (K-homology of C^*-algebras). One can define *topological K-homology of a C^*-algebra* $K^*(A)$ by $K^n(A) := KK_{-n}(A,F)$. It is in some sense dual to the topological K-theory $K_*(A)$. Moreover, the intersection product yields the *index pairing*

$$K_n(A) \otimes_{\mathbb{Z}} K^n(A) \to KK_0(F,F) = \mathbb{Z}, \quad (x,y) \mapsto \langle x,y \rangle := x\widehat{\otimes}_A y.$$

If we take $n = 0$ and $A = C(M)$ for a smooth closed Riemannian manifold M, then an appropriate elliptic operator P over M defines an element in $[P]$ in $K^0(C(M)) = K_0(M)$ and a vector bundle ξ over M defines an element in $K_0(C(M)) = K^0(M)$ and the pairing $\langle [\xi], [P] \rangle$ is the classical index of the elliptic operator obtained from P by twisting with ξ.

There are *Universal Coefficient Theorems* and *Künneth Theorems* for KK-theory, see for instance [138, 139, 852, 887]. The *Pimsner-Voiculescu sequences* associated to an automorphisms of a C^*-algebra are explained for KK-theory in [133, Theorem 19.6.1 on page 198].

More information about KK-theory, for instance about its construction in terms of Kasparov modules or quasi-homomorphisms, other bivariant theories such as Ext for extensions of C^*-algebras, kk-theory, E-theory, and their relation to KK-theory, generalizations of these theories to more general operator algebras than C^*-algebras, universal properties of these theories, applications to index theory, and the relevant literature can be found for instance in [133, 256, 464, 466, 474, 503], or in the papers of Kasparov [543, 544, 545, 546].

10.6 Equivariant Topological K-Theory and KK-Theory

In the sequel groups are assumed to be discrete. Given a group G, there exists an equivariant version of KK-theory. It assigns to two G-C^*-algebras A and B a $\mathbb{Z}$-graded abelian group $KK_*^G(A,B)$ and has essentially the same basic properties as non-equivariant KK-theory. Namely, it is a bi-functor, contravariant in the first and covariant in the second variable, is G-homotopy invariant, satisfies Morita equivalence and stabilization, is split exact, i.e., has long exact sequences for appropriate ideals, satisfies Bott periodicity, is compatible with finite direct products in both variables and countable direct sums in the first variable, and a homomorphism of G-C^*-algebras $f\colon A \to B$ defines an element $[f] \in KK_0^G(A,B)$. There is also an *equivariant version of Kasparov's intersection product*

$$\widehat{\otimes}_B \colon KK_j^G(A,B) \otimes KK_i^G(B,C) \to KK_{i+j}^G(A,C)$$

which has all the expected properties as in the non-equivariant case.

In particular we get on $KK_0^G(F,F)$ an interesting structure of a commutative ring with unit and it is sometimes called *representation ring* of G. If G is finite, $KK_0^G(F,F)$ is indeed isomorphic as ring to $\operatorname{Rep}_F(G)$.

There exists certain additional structures in the equivariant setting. Given a homomorphism $\alpha\colon H \to G$, there are natural *restriction homomorphisms*

$$\alpha^*\colon KK_*^G(A,B) \to KK^H(\alpha^*A, \alpha^*B) \tag{10.61}$$

where α^*A and α^*B are the H-C^*-algebras obtained from the G-C^*-algebras A and B by restring the H-action to a G-action using α. It is compatible with the equivariant Kasparov product.

Let $i\colon H \to G$ be the inclusion of groups. Given an H-C^*-algebra A, we define its *induction* i_*A, to be the G-C^*-algebra of bounded functions $f\colon G \to A$ that satisfy $f(gh) = h^{-1} \cdot f(g)$ and vanish at infinity, i.e., for every $\epsilon > 0$ there exists a finite subset $S \subseteq G/H$ such that for every $g \in G$ with $gH \notin S$ we have $||f(g)|| \leq \epsilon$. The norm is the supremums norm. Given $g \in G$ and such a function f, define $g \cdot f$ to be the function sending $g' \in G$ to $f(g^{-1}g')$.

Note that the left FG-module $FG \otimes_{FH} A$ which is the algebraic induction of A viewed as FH-module, embeds as a dense FG-submodule into i_*A by sending $g \otimes a$ to the function that maps gh to $h^{-1}a$ for $h \in H$ and $g' \in G$ with $g'H \neq gH$ to zero. In other words, we can think of $FG \otimes_{FH} A$ as the set of elements $f \in i_*A$ such that $\{gH \in G/H \mid f(g) \neq 0\}$ is finite. In contrast to modules over group rings induction i_* and restriction i^* do *not* form an adjoint pair (i_*, i^*) for equivariant C^*-algebras as the following exercise illustrates.

Exercise 10.62. Let $i\colon \{1\} \to G$ be the inclusion of the trivial group into an infinite discrete group G. Show that $\hom_G(i_*F, F)$ and $\hom_{\{1\}}(F, i^*F)$ are not isomorphic where $F = \mathbb{R}, \mathbb{C}$ denotes both the obvious $\{1\}$-C^*-algebra and the obvious G-C^*-algebra with trivial G-action.

If X is a proper H-CW-complex, then $G \times_H X$ is a proper G-CW-complex, and we obtain an isomorphism of G-C^*-algebras $i_*C_0(X) \xrightarrow{\cong} C_0(G \times_H X)$ that sends $f \in i_*C_0(X)$ to the function $G \times_H X \to F,\ (g,x) \mapsto f(g)(x)$. Given a H-C^*-algebra A and a H-C^*-algebra B, there is a natural *induction homomorphism*

$$i_*\colon KK_*^H(A,B) \to KK^G(i_*A, i_*B). \tag{10.63}$$

It is compatible with the equivariant Kasparov's intersection product respecting the units. If $j\colon G \to K$ is an inclusion, we get $(j \circ i)_* = j_* \circ i_*$.

There are *descent homomorphisms*

$$j_r^G\colon KK_*^G(A,B) \to KK_*(A \rtimes_r G, B \rtimes_r G); \tag{10.64}$$

$$j_r^G\colon KK_*^G(A,\mathbb{C}) \to KK_*(A \rtimes_r G, \mathbb{C}); \tag{10.65}$$

$$j_m^G\colon KK_*^G(A,B) \to KK_*(A \rtimes_m G, B \rtimes_m G). \tag{10.66}$$

The dual of the Green-Julg Theorem says that (10.65) is an isomorphism. The descent homomorphisms are natural and compatible with Kasparov's intersection products respecting the units.

Define the *equivariant complex K-homology* of a pair of proper G-CW-complexes (X,A) with coefficients in the complex G-C^*-algebra B by

$$K_n^G(X,A;B) := \operatorname{colim}_{C \subseteq X} KK_n^G(C_0(C, C \cap A); B) \tag{10.67}$$

where the colimit is taken over the directed system of cocompact proper G-CW-subcomplexes $C \subseteq X$, directed by inclusion, and $C_0(C, C \cap A)$ is the G-C^*-algebra of continuous functions $C \to \mathbb{C}$ that vanish on $C \cap A$ and at infinity. This group is often denoted by $RK_n(X,A;B)$ in the literature and called *equivariant K-homology with compact support* but from a topologists point of view it is better to call it equivariant K-homology in view of its description in terms of spectra, see Section 12.4. If B is $\mathbb{C}$ with the trivial G-action, we just write $K_*^G(X,A)$ for $K_*^G(X,A;\mathbb{C})$, and this is precisely the $\mathbb{Z}$-graded abelian group that we have mentioned already in Subsection 10.2.3 and will be constructed in terms of spectra in Section 12.4.

Next we explain the equivariant Chern character for equivariant complex K-homology. Denote for a proper G-CW-complex X by $\mathcal{F}(X)$ the set of all subgroups $H \subset G$ for which $X^H \neq \emptyset$, and by

$$\Lambda^G(X) := \mathbb{Z}\left[\frac{1}{\mathcal{F}(X)}\right] \tag{10.68}$$

the ring $\mathbb{Z} \subset \Lambda^G(X) \subset \Lambda^G$ obtained from $\mathbb{Z}$ by inverting the orders of all subgroups $H \in \mathcal{F}(X)$. Denote by $J^G(X)$ the set of conjugacy classes (C) of finite cyclic subgroups $C \subset G$ for which X^C is non-empty. Let $C \subset G$ be a finite cyclic subgroup. Let C_GC be the centralizer and N_GC be the normalizer of $C \subset G$. Let W_GC be the quotient N_GC/C_GC. For a specific idempotent $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$ defined in [636, Section 3] the cokernel of

$$\bigoplus_{D \subset C, D \neq C} \operatorname{ind}_D^C \colon \bigoplus_{D \subset C, D \neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$$

is isomorphic to the image of the idempotent endomorphism

$$\theta_C \colon \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

The element $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$ is uniquely determined by the property that its character sends a generator of C to 1 and all other elements in C to 0.

The next theorem is taken from [636, Theorem 0.7].

Theorem 10.69 (Equivariant Chern character for equivariant K-homology). *Let X be a proper G-CW-complex. Put $\Lambda = \Lambda^G(X)$ and $J = J^G(X)$. Let $\operatorname{im}(\theta_C) \subseteq \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$ be the image of $\theta_C \colon \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$.*

Then there is for $n \in \mathbb{Z}$ a natural isomorphism

$$\operatorname{ch}_p^G(X) \colon \bigoplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_n(C_GC \backslash X^C) \otimes_{\Lambda[W_GC]} \operatorname{im}(\theta_C) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_n^G(X).$$

Note that the isomorphism appearing in Theorem 10.69 exists already over Λ, one does not have to pass to $\mathbb{Q}$ or $\mathbb{C}$. This will be important when we will deal with the Modified Trace Conjecture in Subsection 14.8.1.

Example 10.70. In the special case where G is finite, X is the one-point-space $\{*\}$ and $n = 0$, the equivariant Chern character appearing in Theorem 10.69 reduces to an isomorphism

$$\bigoplus_{(C)\in J^G} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}[\frac{1}{|G|}][W_G C]} \operatorname{im}(\theta_C) \xrightarrow{\cong} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G)$$

where J^G is the set of conjugacy classes (C) of cyclic subgroups $C \subset G$. This is a strong version of the well-known theorem of Artin that the map induced by induction

$$\bigoplus_{(C)\in J^G} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G)$$

is surjective for any finite group G.

Exercise 10.71. Let p be an odd prime and let V be an orthogonal $\mathbb{Z}/p$-representation of dimension d such that $V^{\mathbb{Z}/p} \neq \{0\}$. Denote by SV the $\mathbb{Z}/p$-CW-complex consisting of elements $v \in V$ of norm 1. Show

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV) \cong_{\mathbb{Z}[1/p]} \begin{cases} \mathbb{Z}[1/p]^p & \text{if } d \text{ is even;} \\ \mathbb{Z}[1/p]^{2p} & \text{if } d \text{ is odd and } n \text{ is even;} \\ 0 & \text{if } d \text{ is odd and } n \text{ is even.} \end{cases}$$

Analogously to the complex case one defines *equivariant real K-homology* $KO_*^G(X, A; B)$ of a pair of proper G-CW-complexes (X, A) with coefficients in the real G-C^*-algebra B. We will abbreviate $KO_*^G(X, A) := KO_n^G(X, A; \mathbb{R})$ where $\mathbb{R}$ carries the trivial G-action, This is precisely the $\mathbb{Z}$-graded abelian group that we will be constructed in terms of spectra in Section 12.4.

For discussions of universal coefficient theorems for equivariant K-theory see [683, 851, 852].

Further information about equivariant KK-theory can be found for instance in [133, Section 20], [547], and [932].

10.7 Comparing Algebraic and Topological K-theory of C^*-Algebras

Let A be a C^*-algebra. Then $K_n(A)$ denotes in most cases topological K-theory, but it can also mean the algebraic K-theory of A considered just as a ring. To avoid this ambiguity, we will use in this Section 10.7 the superscripts TOP and ALG to make clear what we mean.

There is for any C^*-algebra over $\mathbb{R}$ or $\mathbb{C}$ a canonical map of spectra

$$\mathbf{t}(A)\colon \mathbf{K}^{\mathrm{ALG}}(A) \to \mathbf{K}^{\mathsf{TOP}}(A) \tag{10.72}$$

from the non-connective algebraic K-theory spectrum of A just considered as ring to the topological K-theory spectrum associated to the C^*-algebra A, see [846, Theorem 4 on page 851]. It induces homomorphisms of abelian groups for all $n \in \mathbb{Z}$

$$t_n(A) = K_n(\mathbf{t}(A))\colon K_n^{\mathrm{ALG}}(A) \to K_n^{\mathsf{TOP}}(A). \tag{10.73}$$

It is always an isomorphism for $n = 0$, but in general far from being a bijection as illustrated by the following

Exercise 10.74. Let X be a finite non-empty CW-complex. Prove that the comparison map $K_1(C(X)) \to K_1^{\mathsf{TOP}}(C(X))$ is never injective.

The situation is different if A is stable or if one uses coefficients in $\mathbb{Z}/k$. Namely, the following result is proved in [918, Theorem 10.9] over $\mathbb{C}$ and $n \geq 1$, but holds in the more general form below by [846, Theorem 19 on page 863], see also Higson [465].

Theorem 10.75 (Karoubi's Conjecture). *Karoubi's Conjecture is true, i.e., for any stable C^*-algebra A over $\mathbb{R}$ or $\mathbb{C}$ the canonical map $\mathbf{t}$ of* (10.72) *is weak homotopy equivalence i.e., the maps t_n of* (10.73) *are bijective for $n \in \mathbb{Z}$.*

Given an integer $k \geq 2$, we have introduced $K_n^{\mathrm{ALG}}(A; \mathbb{Z}/k)$ in Section 6.4. The analogous construction works for topological K-theory and there is the analogue of (10.73), a natural homomorphisms

$$t_n(A; \mathbb{Z}/k)\colon K_n^{\mathrm{ALG}}(A; \mathbb{Z}/k) \to K_n^{\mathsf{TOP}}(A; \mathbb{Z}/k). \tag{10.76}$$

We mention the following conjecture of Rosenberg [842, Conjecture 4.1] or [846, Conjecture 26 on page 869].

Conjecture 10.77 (Comparing algebraic and topological K-theory with coefficients for C^*-algebras). If A is a C^*-algebra and $k \geq 2$ an integer, then the comparison map

$$K_n^{\mathrm{ALG}}(A; \mathbb{Z}/k) \to K_n^{\mathsf{TOP}}(A; \mathbb{Z}/k)$$

is bijective for $n \geq 0$.

The map $K_n^{\mathrm{ALG}}(A; \mathbb{Z}/k) \to K_n^{\mathsf{TOP}}(A; \mathbb{Z}/k)$ appearing in Conjecture 10.77 is known to be bijective for $n = 1$ and to be surjective for $n \geq 1$ by [533, Theorem 2.5]. Conjecture 10.77 is true if A is stable by Theorem 10.75, or if A is commutative, see [377],[799], [842, Theorem 4.2], and [846, Theorem 27 on page 870].

A discussion about K_i-regularity and the homotopy invariance of $K_n^{\mathrm{ALG}}(A)$ for $n \leq -1$ is discussed for C^*-algebras in [846, Section 3.3.4 on page 865ff].

More information about the relation between algebraic and topological K-theory can be found in [246].

10.8 Comparing Algebraic L-Theory and Topological K-theory of C^*-Algebras

Whereas the algebraic and the topological K-theory of a C^*-algebra are very different in general, the topological K-theory of a C^* algebra is closely related to the L-theory of the C^*-algebra just considered as ring with involution. This is illustrated by the following result.

Theorem 10.78 (L-theory and topological K-theory of C^*-algebras).

(i) A generalized signature defines for any unital C^*-algebra over $\mathbb{R}$ or $\mathbb{C}$ a natural isomorphism

$$L_0^p(A) \xrightarrow{\cong} K_0(A);$$

(ii) Let A be a unital C^*-algebra over $\mathbb{C}$. Then there is for all $n \in \mathbb{Z}$ a natural isomorphism

$$K_n(A) \xrightarrow{\cong} L_n^p(A);$$

(iii) Let A be a unital C^*-algebra over $\mathbb{R}$. Then there is a natural homomorphism

$$K_1(A) \xrightarrow{\cong} L_1^h(A)$$

which is surjective and whose kernel has at most order two;

(iv) For any unital C^*-algebra over $\mathbb{R}$ or $\mathbb{C}$ there are natural isomorphisms

$$K_n(A)[1/2] \xrightarrow{\cong} L_n^p(A)[1/2] \xrightarrow{\cong} L_n^h(A)[1/2];$$

(v) Let A be a real C^*-algebra. There are natural isomorphisms

(a) $L_1^p(A) \cong \operatorname{coker}(K_0(A) \xrightarrow{\cdot\eta} K_1(A))$;
(b) $L_2^p(A) \cong \ker(K_6(A) \xrightarrow{\cdot\eta} K_7(A))$;
(c) $L_3^p(A) \cong K_7(A)$,

where η is the non-trivial element in $K_1(\mathbb{R}) \cong \mathbb{Z}/2$.

Proof. (i) See [845, Theorem 1.6].

(ii) See [707, Theorem 0.2], [722], [845, Theorem 1.8].

(iii) See [845, Theorem 1.9].

(iv) See [845, Theorem 1.11] where this result described as a consequence of Karoubi [531, 532].

(v) See [600, Theorem B]. □

10.9 Topological K-Theory for Finite Groups

Note that $\mathbb{C}G = l^1(G) = C_r^*(G) = C_{\max}^*(G)$ holds for a finite group, and analogous for the real versions.

Theorem 10.79 (Topological K-theory of the C^*-algebra of finite groups). *Let G be a finite group.*

(i) We have

$$K_n(C_r^*(G)) \cong \begin{cases} R(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \text{for } n \text{ even;} \\ 0 & \text{for } n \text{ odd,} \end{cases}$$

where $r_{\mathbb{C}}(G)$ is the number of irreducible complex G-representations;

(ii) There is an isomorphism of topological K-groups

$$KO_n(C_r^*(G,\mathbb{R})) \cong KO_n(\mathbb{R})^{r_{\mathbb{R}}(G;\mathbb{R})} \times KO_n(\mathbb{C})^{r_{\mathbb{R}}(G;\mathbb{C})} \times KO_{n+4}(\mathbb{H})^{r_{\mathbb{R}}(G;\mathbb{H})}$$

where $r_{\mathbb{R}}(G;\mathbb{R})$, $r_{\mathbb{R}}(G;\mathbb{C})$, or $r_{\mathbb{R}}(G;\mathbb{H})$ is the number of irreducible real G-representations real, complex, or quaternionic type.
Moreover $KO_n(\mathbb{C}) = K_n(\mathbb{C})$ is 2-periodic with values $\mathbb{Z}$, 0 for $n = 0, 1$, $KO_n(\mathbb{R}) = K(\mathbb{R})$ is 8-periodic with values $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 for $n = 0, 1, \ldots, 7$ and $KO_n(\mathbb{H}) = KO_{n+4}(\mathbb{R})$ for $n \in \mathbb{Z}$.

Proof. One gets isomorphisms of C^*-algebras

$$C_r^*(G) \cong \prod_{j=1}^{r_{\mathbb{C}}(G)} \mathrm{M}_{n_i}(\mathbb{C})$$

and

$$C_r^*(G,\mathbb{R}) \cong \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{R})} \mathrm{M}_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{C})} \mathrm{M}_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{H})} \mathrm{M}_{p_i}(\mathbb{H})$$

from [892, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page106]. Now the claim follows from Morita invariance and the well-known values for $K_n(\mathbb{R})$, $K_n(\mathbb{C})$, and $K_n(\mathbb{H})$, see for instance [925, page 216]. □

10.10 Notes

Bivariant algebraic K-theory is investigated in [248, 403]. More information about index theory and non-commutative geometry can be found for instance in [236, 414].

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Chapter 11
Classifying Spaces for Families

11.1 Introduction

This chapter is devoted to *classifying spaces for families of subgroups*. They are a key input in the general formulations of the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

If one wants to understand these conjectures, one only needs to know the following facts.

- A *family of subgroups* $\mathcal{F}$ is a set of subgroups of G, closed under conjugation and passing to subgroups;
- A G-CW-model for the classifying space $E_{\mathcal{F}}(G)$ is a G-CW-complex whose isotropy groups belong to $\mathcal{F}$ and whose H-fixed point set is weakly contractible for every $H \in \mathcal{F}$;
- Such a G-CW-model always exists, and two such G-CW-models are G-homotopy equivalent;
- For every G-CW-complex X whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map from X to $E_{\mathcal{F}}(G)$.

Only if one is interested in concrete computations, it is very useful to know situations where one can find small G-CW-models for specific G and $\mathcal{F}$.

We give much more information about classifying spaces for families since they are interesting in their own right and are important tools for investigating groups. After we have explained the basic G-homotopy theoretic aspects, we pass to the classifying space $\underline{E}G = E_{\mathcal{COM}}(G)$ for proper action which is the same as the classifying space for the family $\mathcal{COM}$ of compact subgroups. If G is discrete, $\underline{E}G$ reduces to $E_{\mathcal{FIN}}(G)$, where $\mathcal{FIN}$ is the family of finite subgroups. There are many prominent groups for which the are *nice geometric models* for $\underline{E}G$. The G-CW-complex $\underline{E}G$ is relevant for the Baum-Connes Conjecture. For the Farrell-Jones Conjecture we also have to deal with $\underline{\underline{E}}G = E_{\mathcal{VCY}}(G)$ for the family $\mathcal{VCY}$ of virtually cyclic subgroups, which is much harder to analyze. We systematically address the question whether there are *finite or finite dimensional G-CW-models* and what the minimal dimension of such G-CW-models for $E_{\mathcal{F}}(G)$ are for $\mathcal{F} = \mathcal{FIN}, \mathcal{VCY}$.

11.2 Definition and Basic Properties of G-CW-Complexes

Remark 11.1 (Compactly generated spaces). In the sequel we will work in the *category of compactly generated spaces*. This convenient category is explained in detail in [672, Appendix A], [909] and [987, I.4]. A reader may ignore this technical point without harm, but we nevertheless give a short explanation.

A Hausdorff space X is called *compactly generated* if a subset $A \subseteq X$ is closed if and only if $A \cap K$ is closed for every compact subset $K \subseteq X$. Given a topological space X, let $k(X)$ be the compactly generated topological space with the same underlying set as X and the topology for which a subset $A \subseteq X$ is closed if and only if for every compact subset $K \subseteq X$ the intersection $A \cap K$ is closed in the given topology on X. The identity induces a continuous map $k(X) \to X$ which is a homeomorphism if and only if X is compactly generated. The spaces X and $k(X)$ have the same compact subsets. Locally compact Hausdorff spaces and every Hausdorff space which satisfies the first axiom of countability are compactly generated. In particular metrizable spaces are compactly generated.

Working in the category of compactly generated spaces means that one only considers compactly generated spaces and whenever a topological construction such as the cartesian product or the mapping space leads out of this category, one retopologizes the result as described above to get a compactly generated space. The advantage is for example that in the category of compactly generated spaces the exponential map $\operatorname{map}(X \times Y, Z) \to \operatorname{map}(X, \operatorname{map}(Y, Z))$ is always a homeomorphism, for an identification $p\colon X \to Y$ the map $p \times \operatorname{id}_Z\colon X \times Z \to Y \times Z$ is always an identification, and, for a filtration by closed subspaces $X_1 \subset X_2 \subseteq \ldots \subseteq X$ such that X is the colimit $\operatorname{colim}_{n\to\infty} X_n$, we always get $X \times Y = \operatorname{colim}_{n\to\infty}(X_n \times Y)$.

One may also work in the category of weak Hausdorff spaces, see for instance Strickland [914].

In the sequel G is a topologically group (which is compactly generated). Subgroups are understood to be closed.

Definition 11.2 (G-CW-complex). A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subset X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n\geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration (i.e., a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed in X_n for all $n \geq 0$) and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"] & X_n.
\end{tikzcd}$$

The space X_n is called the *n-skeleton* of X. Note that only the filtration by skeletons belongs to the G-CW-structure but not the G-pushouts, only their existence is required. An *equivariant open n-dimensional cell* is a G-component of $X_n - X_{n-1}$, i.e., the preimage of a path component of $G\backslash(X_n - X_{n-1})$. The closure of an equivariant open n-dimensional cell is called an *equivariant closed n-dimensional cell*. If one has chosen the G-pushouts in Definition 11.2, then the equivariant open n-dimensional cells are the G-subspaces $Q_i(G/H_i \times (D^n - S^{n-1}))$ and the equivariant closed n-dimensional cells are the G-subspaces $Q_i(G/H_i \times D^n)$.

It is obvious what a pair of G-CW-complexes is.

Remark 11.3 (G-CW-complexes and CW-complexes with G-action). Suppose that G is discrete. A G-CW-complex X is the same as a CW-complex X with G-action such that, for each open cell $e \subseteq X$ and each $g \in G$ with $ge \cap e \neq \emptyset$, we have $gx = x$ for every $x \in e$.

The definition of a G-CW-complex appearing in Definition 11.2 has the advantage that it makes also sense for topological groups.

Example 11.4 (Simplicial actions). Let X be a simplicial complex on which the group G acts by simplicial automorphisms. Then G acts also on the barycentric subdivision X' by simplicial automorphisms. The filtration of the barycentric subdivision X' by the simplicial n-skeleton yields the structure of a G-CW-complex what is not necessarily true for X. This becomes clear if one considers the standard 2-simplex with the obvious actions of the symmetric group S_3 given by permuting the three vertices.

A map $f\colon X \to Y$ between G-CW-complexes is called *cellular* if $f(X_n) \subseteq Y_n$ holds for all $n \geq 0$.

For a subgroup $H \subseteq G$ denote by $N_G H = \{g \in G \mid gHg^{-1} = H\}$ its *normalizer* and by $W_G H = N_G H/H$ its *Weyl group.*

Lemma 11.5.

(i) Let X be a G-CW-complex and let Y be an H-CW-complex. Then $X \times Y$ with the product $G \times H$-action is a $G \times H$-CW-complex;

(ii) Let X be a G-CW-complex and let $H \subseteq G$ be a subgroup. Suppose that G is discrete or that H is open and closed in G. Then X viewed as an H-space by restriction inherits the structure of an H-CW-complex;

(iii) Consider a G-pushout

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "j_1"] \\ x_2 \arrow[r, "j_2"'] & X. \end{tikzcd}$$

Suppose that X_i for $i = 0, 1, 2$ is a G-CW-complex and that i_1 is cellular and i_2 is the inclusion of a pair of G-CW-complexes. Then X inherits the structure of a G-CW-complex;

(iv) Let X be a G-CW-complex and let $H \subseteq G$ be a subgroup. Then X^H viewed as an W_GH-space inherits the structure of a W_GH-CW-complex provided that G is discrete, or that $K \subseteq G$ is open and closed, or that G is a Lie group and $H \subseteq G$ is compact;

(v) Let X be a G-CW-complex and let $H \subseteq G$ be a normal subgroup. Then X/H viewed as an G/H-space inherits the structure of a G/H-CW-complex.

Proof. (11.5) The skeletal filtration on $X \times Y$ is given by

$$(X \times Y)_n = \bigcup_{p+q=n} X_p \times Y_q.$$

To ensure that $X \times Y$ is the colimit $\operatorname{colim}_{n\to\infty}(X \times Y)_n$, one needs to work in the category of compactly generated spaces.

(ii) Use the same filtration on X viewed as an H-space as for the G-CW-complex X.

(iii) Define the filtration on X^H given by

$$X_n = j_1\big((X_1)_n\big) \cup j_2\big((X_2)_n\big).$$

(iv) The G-action on X induces a N_GH-action on X^H, which in turn passes to a W_GH-action on X^H. Take the n-skeleton of X^H to be $(X_n)^H$. Use the fact that for every $K \subseteq G$ the space $(G/K)^H$ is a disjoint union of W_GH-orbits what is obvious if G is discrete, or if $K \subseteq G$ is open and closed, and follows for a Lie group G and compact $K \subseteq G$ for instance from [629, Theorem 1.33 on page 23].

(v) The n-skeleton of X/H is the image of X_n under the canonical projection $X \to X/H$. □

Exercise 11.6. Let $p\colon \widetilde{X} \to X$ be the universal covering of the connected CW-complex X with fundamental group π. Show that the π-space $\widetilde{X}$ inherits the structure of a π-CW-complex.

Exercise 11.7. Let p be a prime number and let X be a compact $\mathbb{Z}/p$-CW-complex. Show that X and $X^{\mathbb{Z}/p}$ are compact CW-complexes and their Euler characteristics satisfy

$$\chi(X) \equiv \chi(X^{\mathbb{Z}/p}) \mod p.$$

Definition 11.8 (Type of a G-CW-complex). A G-CW-complex is called *finite* if it is built by finitely many equivariant cells.

A G-CW-complex is called *of finite type* if each n-skeleton is a finite G-CW-complex.

A G-CW-complex is called *of dimension* $\leq n$ if $X = X_n$. It is called *n-dimensional* or of *dimension* n if $X = X_n$ and $X \neq X_{n-1}$ holds. It is called *finite dimensional* if it is of dimension $\leq n$ for some integer n.

Remark 11.9 (Slice Theorem). A *Slice Theorem* for G-CW-complexes is proved in [672, Theorem 7.1]. It says roughly, that for a G-CW-complex X we can find for any $x \in X$ an arbitrary small G_x-subspace S_x and an arbitrary small open G-invariant neighborhood U of x such that the closure of S_x is contained in U, the inclusion $\{x\} \to S_x$ is a G_x-homotopy equivalence and the canonical G-map

$$G \times_{G_x} S_x \to U, \quad (g, s) \mapsto g \cdot s$$

is a G-homeomorphism.

11.3 Proper G-Spaces

Definition 11.10 (Proper G-space). A G-space X is called *proper* if for each pair of points x and y in X there are open neighbourhoods V_x of x and W_y of y in X such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is compact.

Lemma 11.11. *A G-CW-complex X is proper if and only if all its isotropy groups are compact.*

Proof. This is shown in [629, Theorem 1.23 on page 18]. □

In particular a free G-CW-complex is always proper. However, not every free G-space is proper.

Exercise 11.12. Find a free compact $\mathbb{Z}$-space that is not proper.

Remark 11.13 (Lie Groups acting properly and smoothly on manifolds). Let G be a Lie group. If M is a proper smooth G-manifold, then an equivariant smooth triangulation induces a G-CW-structure on M. For the proof and for *equivariant smooth triangulations* we refer to [496, Theorem I and II].

Exercise 11.14. Let p be an odd prime. Show that there is no smooth free $\mathbb{Z}/p$-action on an even dimensional sphere.

11.4 Maps between G-CW-Complexes

Theorem 11.15 (Equivariant Cellular Approximation Theorem). *Let (X, A) be a pair of G-CW-complexes and let Y be a G-CW-complex. Let $f\colon X \to Y$ be a G-map such that $f|_A\colon A \to Z$ is cellular.*

Then there exists a cellular G-map $f'\colon X \to Y$ such that $f|_A = f'|_A$ and f and f' are G-homotopic relative A.

Proof. Since $X = \operatorname{colim}_{n\to\infty} X_n$ by definition, it suffices to construct inductively for $n = -1, 0, 1, 2, \ldots$ G-maps

$$h_n\colon X_n \times [0,1] \cup X \times \{0\} \to Y$$

such that $h_n(x, 0) = f(x)$ for every $x \in X_n$ and $h_n(x,t) = h_{n-1}(x,t)$ for every $x \in X_{n-1}$ and $t \in [0,1]$ hold and the map $f_n'\colon X \to Y$ sending $x \in X_n$ to $h_n(x, 1)$ is cellular. The induction beginning $n = -1$ is trivial, define $h_{-1}\colon A \times [0,1] \cup X \times \{0\} \to Y$ by sending (x,t) to $f(x)$. The induction step from $(n-1)$ to n is done as follows. Choose a G-pushout

$$\begin{array}{ccc}
\coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i} & X_{n-1} \cup A \\
\downarrow & & \downarrow \\
\coprod_{i\in I_n} G/H_i \times D^{n} & \xrightarrow{\coprod_{i\in I_n} Q_i} & X_{n} \cup A.
\end{array}$$

It yields the G-pushout

$$\begin{array}{ccc}
\coprod_{i\in I_n} G/H_i \times \left(S^{n-1} \times [0,1] \cup D^n \times \{0\}\right) & \xrightarrow{\coprod_{i\in I_n} q_i'} & X_{n-1} \times [0,1] \cup X \times \{0\} \\
\downarrow & & \downarrow \\
\coprod_{i\in I_n} G/H_i \times D^{n} \times [0,1] & \xrightarrow{\coprod_{i\in I_n} Q_i'} & X_{n} \times [0,1] \cup X \times \{0\}.
\end{array}$$

Because of the G-pushout property it suffices to explain for every $i \in I_n$ how to extend the composite

$$\phi_i\colon G/H_i \times \left(S^{n-1} \times [0,1] \cup D^n \times \{0\}\right) \xrightarrow{q_i'} X_{n-1} \times [0,1] \cup X \times \{0\} \xrightarrow{h_{n-1}} Y$$

to a G-map

$$\Phi_i\colon G/H_i \times D^n \times [0,1] \to Y$$

satisfying $\Phi_i\bigl(G/H_i \times D^n \times \{1\}\bigr) \subseteq Y_n$. This is the same as the non-equivariant problem to extend the map

$$\phi_i'\colon S^{n-1} \times [0,1] \cup D^n \times \{0\} \to Y^H$$

obtained from ϕ_i by restriction to $\{eH_i\} \times \big(S^{n-1} \times [0,1] \cup D^n \times \{0\}\big)$ to a map

$$\Phi'_i \colon D^n \times [0,1] \to Y^H$$

such that $\Phi'_i\big(D^n \times \{1\}\big) \subseteq Y_n$ since we can then define $\Phi_i(gH, x, t) := g \cdot \Phi'_i(x, t)$. It is not hard to check that this non-equivariant problem can be solved if the inclusion $Y^H_{m-1} \to Y^H_m$ is m-connected for every $m \geq 0$. Since we have the pushout of spaces

$$\begin{array}{ccc} \coprod_{i \in I_m} G/H_i^H \times S^{m-1} & \xrightarrow{\coprod_{i \in I_m} q_i^m} & Y_{m-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i^H \times D^m & \xrightarrow{\coprod_{i \in I_n} Q_i^m} & Y_m \end{array}$$

the inclusion $G/H_i^H \times S^{m-1} \to G/H_i^H \times D^m$ is m-connected and $G/H_i^H \times S^{m-1}$ is a deformation retract of an open neighborhood in $G/H_i^H \times D^m$, this follows from Blakers-Massey excision theorem, see [936, Proposition 6.4.2 on page 133]. □

A map $f \colon X \to Y$ of spaces is called a *weak homotopy equivalence* if f induces a bijection $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$ and for every $x \in X$ and $n \geq 1$ an isomorphism $\pi_n(f, x) \colon \pi_n(X, x) \to \pi_n(Y, f(x))$. A G-map $f \colon X \to Y$ of G-spaces is called a *weak G-homotopy equivalence* if $f^H \colon X^H \to Y^H$ is a weak equivalence of spaces for all subgroups $H \subseteq G$.

Theorem 11.16 (Equivariant Whitehead Theorem).

(i) Let $f \colon Y \to Z$ be a G-map between G-spaces. Then f is a weak G-homotopy equivalence if for every G-CW-complex X the map induced by f between the G-homotopy classes of G-maps

$$f_* \colon [X, Y]^G \to [X, Z]^G, \quad [h] \mapsto [f \circ h]$$

is bijective;

(ii) Let $f \colon Y \to Z$ be a G-map between G-CW-complexes. Then the following assertions are equivalent:

(a) f is a G-homotopy equivalence;
(b) f is a weak G-homotopy equivalence;
(c) For every $H \subseteq G$ that occurs as isotropy group of some point in X or Y, the map $f^H \colon X^H \to Y^H$ is a weak homotopy equivalence of spaces.

Proof. See [935, II.2.6], [629, Theorem 2.4 on page 36]. □

Exercise 11.17. Let Y be a G-space. A G-CW-approximation of Y is a G-CW-complex X together with a weak G-homotopy equivalence $f \colon X \to Y$. Show that two G-CW-approximations of Y are G-homotopy equivalent.

11.5 Definition and Basic Properties of Classifying Spaces for Families

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, to be a set of subgroups of G that is closed under conjugation with elements of G and under passing to subgroups, and listed some examples in Notation 2.63, for instance the family $\mathcal{TR}$ consisting of the trivial subgroup, the family $\mathcal{FIN}$ of finite subgroups, the family $\mathcal{VCY}$ of virtually cyclic subgroups, and the family $\mathcal{ALL}$ of all subgroups. Actually one could replace the condition that $\mathcal{F}$ is closed under taking subgroups by the weaker condition that the intersection of finitely many elements of $\mathcal{F}$ belongs to $\mathcal{F}$. Then the set of compact open subgroups is a family also.

Definition 11.18 (Classifying G-CW-complex for a family of subgroups). Let $\mathcal{F}$ be a family of subgroups of G. A model $E_{\mathcal{F}}(G)$ for the *classifying G-CW-complex for the family $\mathcal{F}$ of subgroups of G*, or sometimes also called *classifying space for the family $\mathcal{F}$ of subgroups of G*, is a G-CW-complex $E_{\mathcal{F}}(G)$ that has the following properties:

(i) All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
(ii) For any G-CW-complex Y whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{COM}}(G)$ and call it the *universal G-CW-complex for proper G-actions.*

If G is discrete, we have $\underline{E}G := E_{\mathcal{FIN}}(G)$.

In other words, $E_{\mathcal{F}}(G)$ is a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups belong to $\mathcal{F}$. In particular two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent and for two families $\mathcal{F}_0 \subseteq \mathcal{F}_1$ there is up to G-homotopy precisely one G-map $E_{\mathcal{F}_0}(G) \to E_{\mathcal{F}_1}(G)$.

Theorem 11.19 (Homotopy characterization of $E_{\mathcal{F}}(G)$). *Let $\mathcal{F}$ be a family of subgroups.*

(i) There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
(ii) A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is weakly contractible, i.e., X^H is non-empty and path connected and $\pi_n(X^H, y)$ vanishes for all $n \geq 1$ and one (and hence all) basepoints $y \in X^H$;

Proof. (i) A model can be obtained by attaching equivariant cells $G/H \times D^n$ for all $H \in \mathcal{F}$ to make the H-fixed point sets weakly contractible. See for instance [629, Proposition 2.3 on page 35]. There are also functorial constructions for discrete G generalizing the bar construction, see [265, Section 3 and Section 7].

(ii) Suppose that the G-CW-complex X is a model for $E_{\mathcal{F}}(G)$. Let Y be any

CW-complex and let $H \in \mathcal{F}$. Then there is up to G-homotopy precisely one G-map $G/H \times Y \to X$. Hence there is up to homotopy precisely one map $Y \to X^H$. This is equivalent to the condition that X^H is weakly contractible.

Suppose that X^H is weakly contractible for every $H \in \mathcal{F}$. Let (Y, B) be a G-CW-pair such that the isotropy group of any point in $Y \setminus B$ belongs to $\mathcal{F}$, and let $f_{-1}\colon B \to X$ be any G-map. We next show the existence of a G-map $f\colon Y \to X$ extending f_{-1}. Obviously this implies that X is a model for $E_{\mathcal{F}}(G)$. Since Y is the colimit over the skeletons Y_n for $n \geq -1$ and $Y_{-1} = B$, it suffices to prove for $n \geq 0$ that, for a given G-map $f_{n-1}\colon Y_{n-1} \to X$, there exists a G-map $f_n\colon Y_n \to X$ with $f_n|_{Y_{n-1}} = f_{n-1}$. Recall that by definition there exists a G-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d] & Y_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"] & Y_n
\end{tikzcd}$$

such that each H_i belong to $\mathcal{F}$. Because of the universal property of a G-pushout it remains to show for every $H \in \mathcal{F}$ that every G-map $u\colon G/H \times S^{n-1} \to X$ can be extended to a G-map $v\colon G/H \times D^n \to X$. This is equivalent to showing that every map $u'\colon S^{n-1} \to X^H$ can be extended to a map $v'\colon D^n \to X^H$. This follows from the assumption that X^H is weakly contractible. $\square$

A model for $E_{\mathcal{ALL}}(G)$ is G/G. A model for $E_{\mathcal{TR}}(G)$ is the same as a model for EG i.e, the total space of the universal G-principal bundle $EG \to BG = G\backslash EG$. In Section 11.6 we will give many interesting geometric models for classifying spaces $\underline{E}G = E_{\mathcal{FIN}}(G)$.

Exercise 11.20. Show for a discrete group G that there exists a zero-dimensional model for $E_{\mathcal{F}}(G)$ if and only if $G \in \mathcal{F}$. Is there a non-trivial connected Lie group L with a 0-dimensional model for EL?

11.6 Models for the Classifying Space for Proper Actions

In this section we present some interesting geometric models for the classifying space for proper actions $\underline{E}G$ for some discrete groups. These models will often be small in the sense that they are finite, of finite type, or finite dimensional. We will restrict ourselves to discrete groups G in this section. More information, also for non-discrete groups, can be found for instance in [103, 640].

11.6.1 Simplicial Model

Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of subsets of G of cardinality $(k+1)$. There is an obvious simplicial G-action of G on $P_\infty(G)$ coming from the group structure. We get for instance from [1, Example 2.6].

Theorem 11.21 (Simplicial model). *$P_\infty(G)$ is a model for $\underline{E}G$.*

11.6.2 Operator Theoretic Model

Let $PC_0(G)$ be the metric space of functions $f\colon G \to \{r \in \mathbb{R} \mid r \geq 0\}$ such that f is not identically zero and has finite support, where the metric comes from the supremum norm. The group G acts isometrically on $PC_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in PC_0(G)$ and $g, x \in G$. Obviously $PC_0(G)$ is a subspace of the Banach space $C_0(G)$.

Let X_G be the metric space

$$X_G = \left\{ f\colon G \to [0,1] \;\middle|\; f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

with the metric coming from the supremum norm. The group G acts isometrically on X_G by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in X_G$ and $g, x \in G$.

Theorem 11.22 (Operator theoretic model). *Both $PC_0(G)$ and X_G are G-homotopy equivalent to a G-CW-model for $\underline{E}G$.*

Proof. See [1, Theorem 2.4] and [103, page 248]. □

Remark 11.23 (Comparing $P_\infty(G)$ and X_G). The simplicial G-complex $P_\infty(G)$ of Theorem 11.21 and the G-space X_G of Theorem 11.22 have the same underlying sets but in general they have different topologies. The identity map induces a (continuous) G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism, see also [945, A.2].

11.6.3 Discrete Subgroups of Almost Connected Lie Groups

The next result is a special case of a much more general result due to Abels [1, Corollary 4.14]. Recall that a topological group L is called almost connected if $\pi_0(L)$ is finite.

Theorem 11.24 (Discrete subgroups of almost connected Lie groups). *Let L be an almost connected Lie group. Let $G \subseteq L$ be a discrete subgroup.*

Then L contains a maximal compact subgroup K, which is unique up to conjugation, and the G-space L/K is a model for $\underline{E}G$.

11.6.4 Actions on Simply Connected Non-Positively Curved Manifolds

Theorem 11.25 (Actions on simply connected non-positively curved manifolds). *Suppose that G acts properly and isometrically on the simply connected complete Riemannian manifold M with non-positive sectional curvature. Then M is a model for $\underline{E}G$.*

Proof. See [1, Theorem 4.15]. □

11.6.5 Actions on Trees and Graphs of Groups

A *tree* is a 1-dimensional CW-complex that is contractible.

Theorem 11.26 (Actions on trees). *Suppose that G acts on a tree T such that for each element $g \in G$ and each open cell e with $g \cdot e \cap e \neq \emptyset$ we have $gx = x$ for any $x \in e$. Assume that the isotropy group of each $x \in T$ is finite.*

Then T is a model for $\underline{E}G$.

Proof. Obviously T is a G-CW-complex, see Remark 11.3. Let $H \subseteq G$ be finite. If e_0 is a zero-cell in T, then $H \cdot e_0$ is finite. In the sequel we equip T with the obvious path length metric, for which each edge has length 1. Let T' be the union of all geodesics with extremities in $H \cdot e$. This is an H-invariant subtree of T of finite diameter. One shows now inductively over the diameter of T' that T' has a vertex that is fixed under the H-action, see [894, page 20] or [294, Proposition 4.7 on page 17]. Hence T^H is non-empty. If e and f are vertices in T^H, the geodesic in T from e to f must be H-invariant. Hence T^H is a connected CW-subcomplex of the tree T and hence is itself a tree. This shows that T^H is contractible. Now apply Theorem 11.19 (ii). □

11.6.6 Actions on CAT(0)-Spaces

For the notion of a CAT(0)-space we refer for instance to [155, Definition 1.1 in Chapter II.1 on page 158].

Theorem 11.27 (Actions on CAT(0)-spaces). *Let X be a proper G-CW-complex. Suppose that X has the structure of a complete* CAT(0)*-space on which G acts by isometries. Then X is a model for $\underline{E}G$.*

Proof. By [155, Corollary 2.8 in Chapter II.2 on page 179] the K-fixed point set of X is a non-empty convex subset of X and hence contractible for any compact subgroup $K \subset G$. □

This result contains as special case Theorem 11.25 and Theorem 11.26 since simply connected complete Riemannian manifolds with non-positive sectional curvature and trees are complete CAT(0)-spaces.

11.6.7 The Rips Complex of a Hyperbolic Group

A metric space $X = (X, d)$ is called *δ-hyperbolic* for a given real number $\delta \geq 0$ if for any four points x, y, z, t the following inequality holds

(11.28) $d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\} + 2\delta.$

A group G with a finite set S of generators is called *δ-hyperbolic* if the metric space (G, d_S) given by G and the word-metric d_S with respect to the finite set of generators S is δ-hyperbolic.

The *Rips complex* $P_d(G, S)$ of a group G with a finite set S of generators for a natural number d is the geometric realization of the abstract simplicial complex whose set of k-simplices consists of subsets $S' \subseteq S$ of cardinality $(k+1)$ such that $d_S(g, g') \leq d$ holds for all $g, g \in S'$. The obvious G-action by simplicial automorphisms on $P_d(G, S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G, S)'$, see Example 11.4.

Theorem 11.29 (Rips complex). *Let G be a group with a finite set S of generators. Suppose that (G, S) is δ-hyperbolic for the real number $\delta \geq 0$. Let d be a natural number with $d \geq 16\delta + 8$. Then the barycentric subdivision of the Rips complex $P_d(G, S)'$ is a finite G-CW-model for $\underline{E}G$.*

Proof. See [698], [699]. □

A metric space is called *hyperbolic* if it is δ-hyperbolic for some real number $\delta \geq 0$. A finitely generated group G is called *hyperbolic* if for one (and hence all) finite set S of generators the metric space (G, d_S) is a hyperbolic metric space. Since for metric spaces the property hyperbolic is invariant under quasiisometry and for two finite sets S_1 and S_2 of generators of G the metric spaces (G, d_{S_1}) and (G, d_{S_2}) are quasiisometric, the choice of S does not matter. Theorem 11.29 implies that for a hyperbolic group there is a finite G-CW-model for $\underline{E}G$.

The notion of a hyperbolic group is due to Gromov and has intensively been studied, see for example [155, 408, 423]. The prototype is the fundamental group of a closed hyperbolic manifold.

11.6.8 Arithmetic Groups

An arithmetic group A in a semisimple connected linear $\mathbb{Q}$-algebraic group possesses a finite A-CW-model for $\underline{E}A$. Namely, let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$, and let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup. If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left A-action is a model for $\underline{E}A$ as already explained in Theorem 11.24. The A-space $G(\mathbb{R})/K$ is not necessarily cocompact. The *Borel-Serre completion* of $G(\mathbb{R})/K$, see [144], [893], is a finite A-CW-model for $\underline{E}G$ as pointed out in [19, Remark 5.8], where a private communication with Borel and Prasad is mentioned.

11.6.9 Mapping Class Groups

Let $\Gamma^s_{g,r}$ be the *mapping class group* of an orientable compact surface $F^s_{g,r}$ of genus g with s punctures and r boundary components. This is the group of isotopy classes of orientation preserving selfdiffeomorphisms $F^s_{g,r} \to F^s_{g,r}$ that preserve the punctures individually and restrict to the identity on the boundary. We require that the isotopies leave the boundary pointwise fixed. We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface $F^s_{g,r}$ is negative. It is well-known that the associated *Teichmüller space* $\mathcal{T}^s_{g,r}$ is a contractible space on which $\Gamma^s_{g,r}$ acts properly.

Theorem 11.30 (Mapping class group). *The Teichmüler space $\mathcal{T}^s_{g,r}$ is a model for $\underline{E}\Gamma^s_{g,r}$*

Proof. This follows from [557]. □

Remark 11.31 (Finite model for $\underline{E}\Gamma^s_{g,r}$). There exist a finite $\Gamma^s_{g,r}$-CW-model for $\underline{E}\Gamma^s_{g,r}$, see [726].

11.6.10 Outer Automorphism Groups of Finitely Generated Free Groups

Let F_n be the free group of rank n. Denote by $\operatorname{Out}(F_n)$ the group of outer automorphisms of F_n, i.e., the quotient of the group of all automorphisms of F_n by the normal subgroup of inner automorphisms. Culler and Vogtmann [251, 951] have constructed a space X_n, called *outer space,* on which $\operatorname{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Fix a graph R_n with one vertex v and n-edges and identify F_n with

$\pi_1(R_n, v)$. A *marked metric graph* (g, Γ) consists of a graph Γ with all vertices of valence at least three, a homotopy equivalence $g\colon R_n \to \Gamma$ called marking, and to every edge of Γ there is assigned a positive length. This turns Γ into a metric space by the path metric. We call two marked metric graphs (g, Γ) and (g', Γ') equivalent of there is a homothety $h\colon \Gamma \to \Gamma'$ such that $g \circ h$ and h' are homotopic. Homothety means that there is a constant $\lambda > 0$ with $d(h(x), h(y)) = \lambda \cdot d(x, y)$ for all x, y. Elements in outer space X_n are equivalence classes of marked graphs. The main result in [251] is that X is contractible. Actually, for each finite subgroup $H \subseteq \operatorname{Out}(F_n)$ the H-fixed point set X_n^H is contractible [576, Proposition 3.3 and Theorem 8.1], [986, Theorem 5.1].

The space X_n contains a *spine* K_n which is an $\operatorname{Out}(F_n)$-equivariant deformation retraction. This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\operatorname{Out}(F_n)$-action is by simplicial automorphisms and cocompact. Actually the group of simplicial automorphisms of K_n is $\operatorname{Out}(F_n)$, see [156]. Hence the barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\operatorname{Out}(F_n)$.

11.6.11 Special Linear Groups of (2,2)-Matrices

In order to illustrate some of the general statements above, we consider the special example $\mathrm{SL}_2(\mathbb{Z})$.

Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. We will use either the upper half-plane model or the Poincaré disk model. The group $\mathrm{SL}_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations, i.e., a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by sending a complex number z with positive imaginary part to $\frac{az+b}{cz+d}$. This action is proper and transitive. The isotropy group of $z = i$ is $\mathrm{SO}(2)$. Since $\mathbb{H}^2$ is a simply connected Riemannian manifold whose sectional curvature is constant -1, the $\mathrm{SL}_2(\mathbb{Z})$-space $\mathbb{H}^2$ is a model for $\underline{E}\,\mathrm{SL}_2(\mathbb{Z})$ by Theorem 11.25.

One easily checks that $\mathrm{SL}_2(\mathbb{R})$ is a connected Lie group and $\mathrm{SO}(2) \subseteq \mathrm{SL}_2(\mathbb{R})$ is a maximal compact subgroup. Since the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathbb{H}^2$ is transitive and $\mathrm{SO}(2)$ is the isotropy group at $i \in \mathbb{H}^2$, we see that the $\mathrm{SL}_2(\mathbb{R})$-manifolds $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ and $\mathbb{H}^2$ are $\mathrm{SL}_2(\mathbb{R})$-diffeomorphic.

As $\mathrm{SL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, the space $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ with the obvious $\mathrm{SL}_2(\mathbb{Z})$-action is a model for $\underline{E}\,\mathrm{SL}_2(\mathbb{Z})$ by Theorem 11.24.

The group $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to the amalgamated free product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that $\mathrm{SL}_2(\mathbb{Z})$ acts cell preserving with finite stabilizers on a tree T, which has alternately two and three edges emanating from each vertex, see [894, Theorem 7 in I.4.1 on page 32 and Example 4.2 (c) in I.4.2 on page 35]. This tree is a model for $\underline{E}\,\mathrm{SL}_2(\mathbb{Z})$ by Theorem 11.26.

The other model $\mathbb{H}^2$ is a manifold. These two models must be $\mathrm{SL}_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.

Divide the Poincaré disk or the half plane model $\mathbb{H}^2$ into fundamental domains for the $\mathrm{SL}_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges whose end points lie in the interior of the Poincaré disk, is a tree T with $\mathrm{SL}_2(\mathbb{Z})$-action. This is the tree model above. The tree is a $\mathrm{SL}_2(\mathbb{Z})$-equivariant deformation retraction of $\mathbb{H}^2$. A retraction is given by moving a point p in $\mathbb{H}^2$ along a geodesic starting at the vertex at infinity that belongs to the triangle containing p, through p to the first intersection point of this geodesic with T, see for instance [894, Example 4.2 (c) in I.4.2 on page 35].

11.6.12 Groups with Appropriate Maximal Finite Subgroups

Let G be a discrete group. Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ that are maximal with respect to inclusion in $\mathcal{FIN}$. Consider the following assertions concerning G:

($\underline{\mathrm{M}}$) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

($\underline{\mathrm{NM}}$) $M \in \mathcal{MFIN}, M \neq \{1\} \implies N_G M = M$;

For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let $\{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal finite subgroups of G, i.e, each M_i is a maximal finite subgroup of G and any maximal finite subgroup of G is conjugated to M_i for precisely one element $i \in I$. By attaching free G-cells, we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$ where EG is the same as $E_{\mathcal{TR}}(G)$, i.e., a contractible free G-CW-complex.

Theorem 11.32 (Passage from EG to $\underline{E}G$). *Suppose that G satisfies ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$). Let X be the G-CW-complex define by the G-pushout*

$$\begin{tikzcd}
\coprod_{i \in I} G \times_{M_i} EM_i \arrow[r, "j_1"] \arrow[d, "u_1"'] & EG \arrow[d, "f_1"] \\
\coprod_{i \in I} G/M_i \arrow[r, "k_1"'] & X
\end{tikzcd}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

Then X is a model for $\underline{E}G$.

Proof. We have to explain why $\underline{E}G$ is a model for the classifying space for proper actions of G. Obviously it is a G-CW-complex. Its isotropy groups are all finite. We have to show for $H \subseteq G$ finite that X^H weakly contractible. We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$ and we get

$$\left(\coprod_{i \in I} G/M_i\right)^H = (G/M_{i_0})^H = \{\bullet\}.$$

Hence $X^H = \{\bullet\}$. It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence and hence $\underline{E}G$ is contractible (after forgetting the group action). □

Here are some examples of groups Q that satisfy conditions ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$):

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$.
  The conditions ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$) are satisfied by [667, Lemma 6.3];
- Fuchsian groups F.
  The conditions ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$) are satisfied by [667, Lemma 4.5]. In [667] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated;
- One-relator groups G.
  Let G be a one-relator group. Let $G = \langle (q_i)_{i \in I} \mid r \rangle$ be a presentation with one relation. We only have to consider the case where G contains torsion. Let F be the free group with basis $\{q_i \mid i \in I\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in G$ represented by s has order m, any finite subgroup of G is subconjugated to C, and for any $g \in G$ the implication $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$ holds. These claims follows from [678, Propositions 5.17, 5.18, and 5.19 in II.5 on pages 107 and 108]. Hence G satisfies conditions ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$).

Remark 11.33 (Passing to larger families). Theorem 11.32 is a special case of a general recipe to construct for two families $\mathcal{F} \subseteq \mathcal{G}$ an efficient model for $E_{\mathcal{G}}(G)$ from $E_{\mathcal{F}}(G)$ in [673, Section 2]. These models are important for concrete calculations of the left hand side appearing in the Baum-Conjecture or the Farrell-Jones Conjecture, see Chapter 17.

11.6.13 One-Relator Groups

Let G be a one-relator group. Let $G = \langle (q_i)_{i \in I} \mid r \rangle$ be a presentation with one relation. There is up to conjugacy one maximal finite subgroup C, which turns out to be cyclic. Let $p\colon \ast_{i \in I} \mathbb{Z} \to G$ be the epimorphism from the free group generated by the set I to G that sends the generator $i \in I$ to q_i. Let $Y \to \bigvee_{i \in I} S^1$ be the G-covering associated to the epimorphism p. There is a 1-dimensional unitary C-representation V and a C-map $f\colon SV \to \operatorname{res}_G^C Y$ such that the induced action on the unit sphere SV is free and the following is true: If we equip SV with the C-CW-structure with precisely one equivariant 0-cell and precisely one equivariant 1-cell and DV with the C-CW-complex structures coming from the fact that DV is the cone over SV, then the C-map f can be chosen to be cellular and we obtain a G-CW-model for $\underline{E}G$ by the G-pushout

$$\begin{tikzcd}
G \times_C SV \arrow[r, "\overline{f}"] \arrow[d] & Y \arrow[d] \\
G \times_C DV \arrow[r] & \underline{E}G
\end{tikzcd}$$

where $\overline{f}$ sends (g,x) to $gf(x)$. Thus we get a 2-dimensional G-CW-model for $\underline{E}G$ such that $\underline{E}G$ is obtained from G/C for a maximal finite cyclic subgroup $C \subseteq G$ by attaching free cells of dimensions ≤ 2. The CW-CW-complex structure on $\underline{E}G$ has precisely one 0-cell $G/C \times D^0$, one 0-cell $G \times D^0$, $(2 \cdot |I|$ many 1-cells $G \times D^1$ and $|I|$ many 2-cells $G \times D^2$. All these claims follow from [161, Exercise 2 (c) II. 5 on page 44].

If G is torsionfree, the 2-dimensional complex associated to a presentation with one relation is a model for BG, see [678, Chapter III §§9 -11].

Exercise 11.34. Let G be a one-relator group. Let $M \subseteq G$ be a maximal cyclic subgroup. Show that the inclusion induces for $n \geq 3$ an isomorphism $H_n(BM) \xrightarrow{\cong} H_n(BG)$.

Exercise 11.35. Let G be a finitely generated group. Suppose that for every integer d there is $k \geq d$ with $H_k(BG;\mathbb{Q}) \neq 0$. Show that G cannot be a hyperbolic group, an arithmetic group, a mapping class group, $\operatorname{Out}(F_n)$, or a one-relator group.

11.7 Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

In general the G-CW-models for $\underline{\underline{E}}G$ are not as nice and small than the ones for $\underline{E}G$. We illustrate this in the case $G = \mathbb{Z}^n$ for $n \geq 2$. Then a $\mathbb{Z}^n$-CW-model for $\underline{E}\mathbb{Z}^n = E\mathbb{Z}^n$ is $\mathbb{R}^n$ with the standard translation action of $\mathbb{Z}^n$.

An explicite $\mathbb{Z}^n$-CW-model for $\underline{\underline{E}}\mathbb{Z}^n$ can be constructed as follows. Choose an enumeration $\{C_i \mid i \in \mathbb{Z}\}$ of the infinite cyclic subgroups of $\mathbb{Z}^n$. Consider the space $\mathbb{R}^n \times \mathbb{R}$. For each $i \in \mathbb{Z}$ we identify in $\mathbb{R}^n \times \{i\}$ the subspace given by the $\mathbb{R}$-span of $C_i \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n$ to a point. Then we obtain a $\mathbb{Z}^n$-CW-complex X. Since the C_i-fixed point set of X consists of precisely one point, the underlying topological space X is contractible, and all isotropy groups of the $\mathbb{Z}^n$-action are infinite cyclic or trivial, X is a $\mathbb{Z}^n$-CW-model for $\underline{\underline{E}}\mathbb{Z}^n$. Note that the dimension of X is $(n+1)$. One can actually show that any $\mathbb{Z}^n$-CW-model for $\underline{\underline{E}}\mathbb{Z}^n$ has dimension greater or equal to $(n+1)$, see [673, Example 5.21].

11.7.1 Groups with Appropriate Maximal Virtually Cyclic Subgroups

Let G be a discrete group. Let $\mathcal{MVCY}$ be the subset of $\mathcal{VCY}$ consisting of elements in $\mathcal{VCY}$ that are maximal with respect to inclusion in $\mathcal{VCY}$. Consider the following assertions concerning G:

($\underline{\underline{\mathrm{M}}}$) Every infinite virtually cyclic subgroup of G is contained in a unique maximal virtually cyclic subgroup;

($\underline{\underline{\mathrm{NM}}}$) $V \in \mathcal{MVCY}, |V| = \infty \implies N_G V = V$.

For such a group there is a nice model for $\underline{\underline{E}}G$ with as few cells of type G/V with infinite virtually cyclic V as possible. Let $\{V_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal infinite virtually cyclic subgroups of G. By attaching G-cells of the type G/H for finite subgroups $H \subseteq G$, we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{V_i} \underline{E}V_i \to \underline{E}G$.

The next result is proved in [673, Corollary 2.11].

Theorem 11.36 (Passage from $\underline{E}G$ to $\underline{\underline{E}}G$). *Suppose that G satisfies ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$). Let X be the G-CW-complex define by the G-pushout*

$$\begin{tikzcd}
\coprod_{i \in I} G \times_{V_i} \underline{E}V_i \arrow[r, "j_1"] \arrow[d, "u_1"'] & \underline{E}G \arrow[d, "f_1"] \\
\coprod_{i \in I} G/V_i \arrow[r, "k_1"'] & X
\end{tikzcd}$$

where u_1 is the obvious G-map obtained by collapsing each $\underline{E}V_i$ to a point.

Then X is a model for $\underline{\underline{E}}G$.

A useful criterion for a group G to satisfy both ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$) can be found in [673, Theorem 3.1]. It implies that any hyperbolic group satisfies both ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$), see [673, Example 3.6]. On the other hand the Klein bottle group $\mathbb{Z} \rtimes \mathbb{Z}$ does not satisfy ($\underline{\underline{\mathrm{M}}}$), see [673, Example 3.7]. This is the one of the few instances where $\underline{\underline{E}}G$ behaves nicer than $\underline{E}G$ since the class of groups for which both ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$) hold is much richer than the class for which both ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$) hold.

Theorem 11.36 will be very helpful for computing the left hand side appearing in the Farrell-Jones Conjecture, see Section 17.6.

11.8 Finiteness Conditions

It has been very fruitful in group theory to investigate the question whether one can find small models for BG, for instance a finite CW-model, a CW-model of finite type or a finite dimensional CW-model, or equivalently, small G-CW-models for EG. The same question can be asked for $\underline{E}G$ and $\underline{\underline{E}}G$. For torsionfree groups there is no difference between EG and $\underline{E}G$, but for groups with torsion the space $\underline{E}G$ seems to carry much more information than EG. In this section we collect some information about finiteness conditions on EG, $\underline{E}G$, and $\underline{\underline{E}}G$. Having small models is also important for computation of the left hand sides appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture, see Chapter 17.

Throughout this section G will be a discrete group.

11.8.1 Review of Finiteness Conditions on *BG*

As an illustration we review what is known about finiteness properties of G-CW-models for EG for a discrete group G. This is equivalent to the same question about BG.

We introduce the following notation. Let R be a commutative associative ring with unit. The trivial RG-module is R viewed as RG-module by the trivial G-action. The *cohomological dimension* $\operatorname{cd}_R(M)$ of a RG-module M is ∞ if there is no finite dimensional projective RG-resolution and is equal to the integer n if there exists a projective resolution of dimension $\leq n$ for M but not of dimension $\leq n-1$. Note that M possesses a projective RG-resolution of dimension $\leq n$ if and only if for any RG-module N we have $\operatorname{Ext}^i_{RG}(M,N) = 0$ for $i \geq n+1$. The *cohomological dimension over* R of a group G, which is denoted by $\operatorname{cd}_R(G)$, is the cohomological dimension of trivial RG-module R. If $R = \mathbb{Z}$, we abbreviate $\operatorname{cd}(G) := \operatorname{cd}_{\mathbb{Z}}(G)$.

An RG-module M is of *type* FP_n, if it admits a projective RG-resolution P_* such that P_i is finitely generated for $i \leq n$, and of type FP_∞ if it admits a projective RG-resolution P_* such that P_i is finitely generated for all i. A group G is of type FP_n or FP_∞ respectively if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is of type FP_n or FP_∞ respectively.

Here is a summary of well-known statements about finiteness conditions on BG.

Theorem 11.37 (Finiteness conditions for BG**).** *Let G be a discrete group.*

(i) If there exists a finite dimensional model for BG, then G is torsionfree;

(ii)(a) There exists a CW-model for BG with finite 1-skeleton if and only if G is finitely generated;

(b) There exists a CW-model for BG with finite 2-skeleton if and only if G is finitely presented;

(c) For $n \geq 3$ there exists a CW-model for BG with finite n-skeleton if and only if G is finitely presented and of type FP_n;

(d) There exists a CW-model for BG of finite type, i.e., all skeleta are finite if and only if G is finitely presented and of type FP_∞;

(e) There exists groups G that are of type FP_2 and not finitely presented;

(iii) There is a finite CW-model for BG if and only if G is finitely presented and there is a finite free $\mathbb{Z}G$-resolution F_* for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$;

(iv) The following assertions are equivalent:

(a) The cohomological dimension over $\mathbb{Z}$ of G is ≤ 1;
(b) There is a model for BG of dimension ≤ 1;
(c) G is free;

(v) The following assertions are equivalent for $d \geq 3$:

(a) There exists a CW-model for BG of dimension d;
(b) The cohomological dimension over $\mathbb{Z}$ of G is d;

(vi) For Thompson's group F there is a CW-model of finite type for BG but no finite dimensional model for BG.

Proof. (i) Suppose we can choose a finite dimensional model for BG. Let $C \subseteq G$ be a finite cyclic subgroup. Then $C\backslash\widetilde{BG} = C\backslash EG$ is a finite dimensional model for BC. Hence there is an integer d such that we have $H_i(BC) = 0$ for $i \geq d$. This implies that C is trivial [161, (2.1) in II.3 on page 35]. Hence G is torsionfree.

(ii) See [128] and [161, Theorem 7.1 in VIII.7 on page 205].

(iii) See [161, Theorem 7.1 in VIII.7 on page 205].

(iv) See [907] and [923].

(v) See [161, Theorem 7.1 in VIII.7 on page 205].

(vi) See [162]. □

11.8.2 Cohomological Criteria for Finiteness Properties in Terms of Bredon Cohomology

We have seen that we can read off finiteness properties of BG or EG from the group cohomology of G. If one wants to investigate the same question for $E_{\mathcal{F}}(G)$ analogous statements are true if one considers modules over the $\mathcal{F}$-restricted orbit category $\mathsf{Or}_{\mathcal{F}}(G)$ in the sense of Definition 2.64. This is explained in [640, Subsection 5.2]. For instance, if $d \geq 3$ is a natural number, then there is a G-CW-model of dimension $\leq d$ for $E_{\mathcal{F}}(G)$ if and only if the trivial $\mathbb{Z}\mathsf{Or}_{\mathcal{F}}(G)$-module $\underline{\mathbb{Z}}$ has projective $\mathbb{Z}\mathsf{Or}_{\mathcal{F}}(G)$-resolution of dimension $\leq d$, see [640, Theorem 5.2 (i)]. The role of the cohomology of a group is now played by the Bredon cohomology of $E_{\mathcal{F}}(G)$. We will deal with Bredon cohomology in Example 12.2.

Other papers related to the topic of connecting geometric dimension or other finiteness properties for classifying spaces for families to algebraic analogues are [151, 380, 382, 749, 751].

11.8.3 Finite Models for the Classifying Space for Proper Actions

The specific constructions of Sections 11.6 show that there is a finite G-CW-model for the classifying space for proper actions $\underline{E}G$ if G is a cocompact discrete subgroups of an almost connected Lie group, a hyperbolic group, an arithmetic group, the outer automorphism group of a finitely generated free groups, a mapping class group, or a finitely generated one-relator group. This is also the case for an elementary amenable group of type FP_{∞}, see [574, Theorem 1.1].

If $1 \to K \to G \to Q \to 1$ is an extension of groups and there are finite models for $\underline{E}K$ and $\underline{E}Q$, one may ask whether there is a finite model for $\underline{E}G$. Some sufficient conditions for this question are given in [632, Theorem 3.2 and Theorem 3.3], for instance that K is hyperbolic or virtually poly-cyclic. However, even in the case that Q is finite and K is torsionfree with a finite model for BK, it can happen that there is no finite model for $\underline{E}G$, see [608, Example 3 on page 149 in Section 7].

11.8.4 Models of Finite Type for the Classifying Space for Proper Actions

The following result is proved in [632, Theorem 4.2].

Theorem 11.38 (Models for $\underline{E}G$ of finite type).
The following statements are equivalent for the group G.

(i) *There is a G-CW-model for $\underline{E}G$ of finite type;*
(ii) *There are only finitely many conjugacy classes of finite subgroups of G and for any finite subgroup $H \subset G$ there is a CW-model for BW_GH of finite type where $W_GH := N_GH/H$;*
(iii) *There are only finitely many conjugacy classes of finite subgroups of G and for any finite subgroup $H \subset G$ the Weyl group W_GH is finitely presented and is of type FP_{∞}.*

The comments about extensions in Subsection 11.8.3 for finite models carry over to models of finite type.

11.8.5 Finite Dimensional Models for the Classifying Space for Proper Actions

The following result follows from Dunwoody [305, Theorem 1.1].

Theorem 11.39 (A criterion for 1-dimensional models for $\underline{E}G$). *Let G be a discrete group. Then there exists a 1-dimensional model for $\underline{E}G$ if and only if the cohomological dimension of G over $\mathbb{Q}$ is less or equal to one.*

If G is finitely generated, then there is a 1-dimensional model for $\underline{E}G$ if and only if G contains a finitely generated free subgroup of finite index [537, Theorem 1]. If G is torsionfree, we rediscover the results due to Swan and Stallings stated in Theorem 11.37 (iv) from Theorem 11.39.

If G is virtually torsionfree, one defines its *virtual cohomological dimension* $\operatorname{vcd}(G)$ by the cohomological dimension $\operatorname{cd}(H)$ of any torsionfree subgroup $H \subseteq G$ of finite index. Since for any other torsionfree subgroup $K \subseteq G$ of finite index we have $\operatorname{cd}(H) = \operatorname{cd}(K)$, this notion is well-defined.

Definition 11.40 (Homotopy dimension). Given a G-space X, the *homotopy dimension* $\operatorname{hdim}^G(X) \in \{0, 1, \ldots\} \amalg \{\infty\}$ of X is defined to be the infimum over the dimensions of all G-CW-complexes Y that are G-homotopy equivalent to X.

Notation 11.41. Put for a group G

$$\underline{\operatorname{gd}}(G) := \operatorname{hdim}^G(\underline{E}G);$$
$$\underline{\underline{\operatorname{gd}}}(G) := \operatorname{hdim}^G(\underline{\underline{E}}G).$$

Lemma 11.42. *Suppose that G is virtually torsionfree. Then*

$$\operatorname{vcd}(G) \leq \underline{\operatorname{gd}}(G).$$

Proof. Choose a torsionfree subgroup $H \subseteq G$ of finite index. Then the restriction of $\underline{E}G$ to H is a model for EH. This implies $\operatorname{cd}(H) \leq \dim(\underline{E}G)$ and hence $\operatorname{vcd}(G) \leq \underline{\operatorname{gd}}(G)$. □

The next result is taken from [640, Theorem 5.24]

Theorem 11.43 (Dimension of $\underline{E}G$ for a discrete subgroup G of an almost connected Lie group). *Let L be a Lie group with finitely many path components. Then L contains a maximal compact subgroup K, which is unique up to conjugation. Let $G \subseteq L$ be a discrete subgroup of L. Then L/K with the left G-action is a model for $\underline{E}G$.*

Suppose additionally that G is virtually torsionfree. Then we have

$$\operatorname{vcd}(G) \leq \dim(L/K)$$

and equality holds if and only if $G \backslash L$ is compact.

The next result follows from [379, Theorem 1 and inequalities (1) and (2) on page 7] where also the notion of the Hirsch length for elementary amenable groups due to Hillman [478] is recalled. In the special case that there is a finite sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = \{1\}$ of subgroups such that G_{i+1} is normal in G_i and G_i/G_{i+1} is finitely generated abelian for $i = 0, 1, \ldots, (n-1)$, the Hirsch length $h(G)$ is $\sum_{i=0}^{n-1} \operatorname{rk}_{\mathbb{Z}}(G_i/G_{i+1})$.

Theorem 11.44 (Dimension of $\underline{E}G$ for countable elementary amenable groups of finite Hirsch length). *If G is an elementary amenable group, then its Hirsch length satisfies*

$$h(G) \leq \underline{\operatorname{gd}}(G).$$

If G is a countable elementary amenable group, then

$$\underline{\operatorname{gd}}(G) \leq \max\{3, h(G) + 1\}.$$

If F is a virtually poly-cyclic group G, then G is virtually torsionfree, and $\operatorname{vcd}(G)$ is finite and satisfies $\operatorname{vcd}(G) = h(G) = \underline{\operatorname{gd}}(G)$, see [640, Example 5.26].

If $H \subseteq G$ is a subgroup of finite index $[G : H]$ and there is a H-CW-model for $\underline{E}H$ of dimension $\leq d$, then there is a G-CW-model for $\underline{E}G$ of dimension $\leq d \cdot [G : H]$, see [632, Theorem 2.4]. In particular $\underline{\operatorname{gd}}(G)) \leq [G : H] \cdot \underline{\operatorname{gd}}(H)$.

Theorem 11.45 (Dimension of $\underline{E}G$ and extension). *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. Suppose that there exists a positive integer d that is an upper bound on the orders of finite subgroups of Q. Then*

$$\underline{\operatorname{gd}}(G) \leq d \cdot \underline{\operatorname{gd}}(K) + \underline{\operatorname{gd}}(Q).$$

Remark 11.46 ($\underline{\operatorname{gd}}(G)$ for locally finite groups). For a locally finite group of cardinality $\aleph_n$ the inequality $\underline{\operatorname{gd}}(G) \leq n + 1$ is proved in [295] and [673, Theorem 5.31]. The equality $\underline{\operatorname{gd}}(G) = n + 1$ is explained in [673, Example 5.32].

Exercise 11.47. Let F be a non-trivial finite group. Put $H = \bigoplus_{\mathbb{Z}} F$. Let $H \rtimes \mathbb{Z}$ be the semidirect product with respect to the shift automorphism of H. Show $\underline{\operatorname{gd}}(H) = 1$ and $\underline{\operatorname{gd}}(H \rtimes \mathbb{Z}) = 2$.

11.8.6 Brown's Problem about Virtual Cohomological Dimension and the Dimension of the Classifying Space for Proper Actions

The following problem whether the converse of Lemma 11.42 is true, is stated by Brown [160, page 32].

Problem 11.48 (Brown's problem about $\mathrm{vcd}(G) = \dim(\underline{E}G)$). For which virtually torsionfree groups G does the equality

$$\mathrm{vcd}(G) = \underline{\mathrm{gd}}(G)$$

hold?

The *length* $l(H) \in \{0, 1, \ldots\}$ of a finite group H is the supremum over all l for which there is a nested sequence $H_0 \subset H_1 \subset \ldots \subset H_l$ of subgroups H_i of H with $H_i \neq H_{i+1}$. The following result is proved in [632, Theorem 6.4] and was motivated by Brown's Problem 11.48.

Theorem 11.49 (Estimate on $\dim(\underline{E}G)$ in terms of $\mathrm{vcd}(G)$). *Let G be a group with virtual cohomological dimension $\mathrm{vcd}(G) \leq d$. Let $l \geq 0$ be an integer such that the length $l(H)$ of any finite subgroup $H \subset G$ is bounded by l.*

Then there is a G-CW-model for $\underline{E}G$ such that for any finite subgroup $H \subset G$

$$\dim(\underline{E}G^H) = \max\{3, d\} + l - l(H)$$

holds. In particular $\underline{\mathrm{gd}}(G) \leq \max\{3, d\} + l$.

However, we obtain from Leary-Petroysan [609, Corollary 1.2], see also Leary-Nucinkis [608, Example 12 on page 153 in Section 7].

Theorem 11.50 (Brown's Problem 11.48 has a negative answer in general). *Given a natural number m, there exists a group G such that there is a finite model for $\underline{E}G$ and we have $\mathrm{vcd}(G) = 2m$ and $\underline{\mathrm{gd}}(G) \geq 3m$.*

Moreover, Leary-Petroysan [609, page 732] show that the estimate in Theorem 11.49 cannot be improved, even if one considers only finite models for $\underline{E}G$.

11.8.7 Finite Dimensional Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

The following problem has triggered a lot of activities

Problem 11.51 (Relating the dimension of $\underline{E}G$ and $\underline{\underline{E}}G$). For which countable groups G do the inequalities

$$\underline{\mathrm{gd}}(G) - 1 \le \underline{\underline{\mathrm{gd}}}(G) \le \underline{\mathrm{gd}}(G) + 1$$

hold?

The inequality appearing in Problem 11.51 holds for countable elementary amenable groups, see [285, Corollary 4.4]. There are groups of type FP_∞ for which the difference $\underline{\underline{\mathrm{gd}}}(G) - \underline{\mathrm{gd}}(G)$ is arbitrary large, see [285, Example 6.5].

All possible cases of the inequality appearing in Problem 11.51 can occur, in particular there are examples of finitely presented groups G with $\underline{\underline{\mathrm{gd}}}(G) < \underline{\mathrm{gd}}(G)$, see Remark 11.55.

The next result is proved in [285, Theorem A].

Theorem 11.52 (Dimension of $\underline{\underline{E}}G$ for elementary amenable groups of finite Hirsch length). *If G is an elementary amenable group of cardinality $\aleph_n$ such that the Hirsch length $h(G)$ of G is finite, then*

$$\underline{\underline{\mathrm{gd}}}(G) \le h(G) + n + 2.$$

Theorem 11.53 (The dimension of $\underline{\underline{E}}G$).

(i) We have for any group G

$$\underline{\mathrm{gd}}(G) \le 1 + \underline{\underline{\mathrm{gd}}}(G);$$

(ii) We have

$$\underline{\mathrm{gd}}(G \times H) \le \underline{\mathrm{gd}}(G) + \underline{\mathrm{gd}}(H),$$

and

$$\underline{\underline{\mathrm{gd}}}(G \times H) \le \underline{\underline{\mathrm{gd}}}(G) + \underline{\underline{\mathrm{gd}}}(H) + 3,$$

and these inequalities cannot be improved in general;

(iii) If G satisfies condition $(\underline{\underline{M}})$ and $(\underline{\underline{NM}})$, then

$$\underline{\underline{\mathrm{gd}}}(G) \begin{cases} = \underline{\mathrm{gd}}(G) & \text{if } \underline{\mathrm{gd}}(G) \ge 2; \\ \le 2 & \text{if } \underline{\mathrm{gd}}(G) \le 1; \end{cases}$$

(iv) If $H \subseteq G$ is a subgroup of finite index $[G:H]$ then

$$\underline{\underline{\mathrm{gd}}}(G) \le [G:H] \cdot \underline{\underline{\mathrm{gd}}}(H).$$

Proof. (i) See [673, Corollary 5.4 (1)].

(ii) This is obvious for $\underline{\mathrm{gd}}(G \times H)$ and proved for $\underline{\underline{\mathrm{gd}}}(G \times H)$ in [673, Corollary 5.6 and Remark 5.7].

(iii) See [673, Theorem 5.8 (2)].

(iv) This is proved in [632, Theorem 2.4]. □

Exercise 11.54. If G is the fundamental group of a hyperbolic closed Riemannian manifold M, then

$$\mathrm{cd}(G) = \dim(N) = \underline{\mathrm{gd}}(G) = \underline{\underline{\mathrm{gd}}}(G).$$

Remark 11.55 (Virtually-poly-cyclic-groups). In [673, Theorem 5.13] a complete computation of $\underline{\underline{\mathrm{gd}}}(G)$ is presented for virtually poly-$\mathbb{Z}$ groups. The answer is much more complicated than the one for $\underline{\mathrm{gd}}(G)$, which is equal to both $\mathrm{vcd}(G)$ and the Hirsch length $h(G)$, see [640, Example 5.26]. This leads to some interesting examples in [673, Subsection 5.4]. For instance, one can construct, for $k = -1, 0, 1$, automorphisms $f_k\colon \mathrm{Hei} \to \mathrm{Hei}$ of the three-dimensional Heisenberg group Hei such that

$$\underline{\underline{\mathrm{gd}}}(\mathrm{Hei} \rtimes_{f_k} \mathbb{Z}) = 4 + k.$$

Note that $\underline{\mathrm{gd}}(\mathrm{Hei} \rtimes_f \mathbb{Z}) = \mathrm{cd}(\mathrm{Hei} \rtimes_f \mathbb{Z}) = 4$ holds for any automorphism $f\colon \mathrm{Hei} \to \mathrm{Hei}$.

The following result is taken from [643, Theorem 1.1].

Theorem 11.56 (Dimensions of $\underline{E}G$ and $\underline{\underline{E}}G$ for groups acting on $\mathrm{CAT}(0)$-spaces). *Let G be a discrete group that acts properly and isometrically on a complete proper $\mathrm{CAT}(0)$-space X. Let $\dim(X)$ be the topological dimension of X, see Definition 23.35,*

(i) We have

$$\underline{\mathrm{gd}}(G) \le \dim(X);$$

(ii) Suppose that G acts by semisimple isometries. (This is the case if we additionally assume that the G-action is cocompact.) Then

$$\underline{\underline{\mathrm{gd}}}(G) \le \dim(X) + 1.$$

Remark 11.57 ($\underline{\underline{\mathrm{gd}}}(G)$ for locally virtually cyclic groups). For a locally virtually cyclic group of cardinality $\aleph_n$ the inequality $\underline{\underline{\mathrm{gd}}}(G) \le n + 1$ is a special case of [673, Theorem 5.31].

The next result is taken from [282, Theorem A].

Theorem 11.58 (Finite dimensional models for $\underline{\underline{E}}G$ for discrete subgroups of $\mathrm{GL}_n(\mathbb{R})$). *Every discrete subgroup G of $\mathrm{GL}_n(\mathbb{R})$ admits a finite dimensional model for $\underline{\underline{E}}G$. More precisely, if m is the dimension of the Zariski closure of G in $\mathrm{GL}_n(\mathbb{R})$, then*

$$\underline{\underline{\mathrm{gd}}}(G) \le m + 1.$$

For information about $\underline{\underline{\mathrm{gd}}}(G)$ we refer for (relatively) hyperbolic groups to [515, 589], for mapping class groups of finite type surfaces to [518, 750], for mapping class groups of punctured spheres to [36], for systolic groups to [768], for braid groups to [378], for normally poly-free groups to [505], for orientable 3-manifold groups to [510], and for $\mathrm{Out}(F_n)$ to [433].

11.8.8 Low Dimensions

Besides Theorem 11.39 we have the following result proved in [673, Theorem 5.33].

Theorem 11.59 (Low-dimensional models for $\underline{E}G$ and $\underline{\underline{E}}G$).

(i) Let G be a countable group that is locally virtually cyclic. Then

$$\underline{\mathrm{gd}}(G) = \begin{cases} 0 & \text{if } G \text{ is finite;} \\ 1 & \text{if } G \text{ is infinite and either locally finite} \\ & \qquad\qquad\qquad\qquad \text{or virtually cyclic;} \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\underline{\underline{\mathrm{gd}}}(G) = \begin{cases} 0 & \text{if } G \text{ is virtually cyclic;} \\ 1 & \text{otherwise;} \end{cases}$$

(ii) Let G be a countable group satisfying $\underline{\mathrm{gd}}(G) \leq 1$. Then

$$\underline{\underline{\mathrm{gd}}}(G) = \begin{cases} 0 & \text{if } G \text{ is virtually cyclic;} \\ 1 & \text{if } G \text{ is locally virtually cyclic but} \\ & \qquad\qquad\qquad \text{not virtually cyclic;} \\ 2 & \text{otherwise.} \end{cases}$$

Exercise 11.60. Let G be a countable group. Show that G is infinite locally finite if and only if $\underline{\mathrm{gd}}(G) = \underline{\underline{\mathrm{gd}}}(G) = 1$ holds.

11.8.9 Finite Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

If G is virtually cyclic, a model for $\underline{\underline{E}}G$ is $\{\bullet\} = G/G$, which is in particular finite. There is no group known such that $\underline{\underline{E}}G$ has a finite G-CW-model and G is not virtually cyclic. This leads to the following conjecture of Juan-Pineda and Leary [515, Conjecture 1],

Conjecture 11.61 (Finite Models for $\underline{\underline{E}}G$). If a group G has a finite G-CW-model for $\underline{\underline{E}}G$, then G is virtually cyclic.

Conjecture 11.61 is known to be true in many cases since the existence of a finite G-CW-model for $\underline{\underline{E}}G$ implies that there is a finite G-CW-model for $\underline{E}G$, see [673, Corollary 5.4 (2)], and that there are only finitely many conjugacy classes of infinite virtually cyclic groups of G. Conjecture 11.61 holds for instance for hyperbolic groups, see [515, Corollary 12], elementary amenable groups, see [565, Corollary 5.8], and linear groups, see [952].

11.9 On the Homotopy Type of the Quotient Space of the Classifying Space for Proper Actions

One may think that there are more homotopy classes of CW-complexes than isomorphisms classes of groups. Namely, we can assign to any group G its classifying space BG and for two groups G and H the spaces BH and BG are homotopy equivalent if and only if G and H are isomorphic, and there are CW-complexes that are not homotopy equivalent to BG for any group G. However, here is a result due to Leary-Nucinkis [607, Theorem 1], which is in some sense the converse.

Theorem 11.62 (Every CW-complex occurs up to homotopy as quotient of a classifying space for proper group actions). *Let X be a CW-complex. Then there exists a group G such that $G\backslash\underline{E}G$ is homotopy equivalent to X. Moreover one can arrange that G contains a torsionfree subgroup of index two.*

Exercise 11.63. Let X be a CW-complex. Show that there exists a $\mathbb{Z}/2$-CW-complex Y such that Y is aspherical and X is homotopy equivalent to the $\mathbb{Z}/2$-quotient space of Y.

Remark 11.64 (Metric Kan-Thurston Theorem). Leary proves a metic Kan-Thurston Theorem in [606, Theorem A]. It yields the following variant of Theorem 11.62, see [606, Theorem 8.3]. Given a group G and proper simplicial G-complex X with connected $G\backslash X$, there exists a group $\widetilde{G}$, a cubical CAT(0)-complex E with simplicial G-action, an epimorphism of groups $p\colon \widetilde{G} \to G$, and a map $f\colon E \to X$ such that E is a model for $\underline{E}G$, the map f is $p\colon \widetilde{G} \to G$-equivariant, and for any equivariant homology theory in the sense of Definition 12.9 the pair (p,f) induces for all $n \in \mathbb{Z}$ isomorphisms $\mathcal{H}_n^{\widetilde{G}}(E) \to \mathcal{H}_n^G(X)$. An application to Isomorphism Conjectures is discussed in [606, Section 10].

The understanding of $G\backslash\underline{E}G$ and $G\backslash\underline{\underline{E}}G$ will be important for the computation of the left hand side appearing in the Baum-Conjecture or the Farrell-Jones Conjecture, see Chapter 17.

In contrast to the trivial family $\mathcal{TR}$ where EG and $BG = G\backslash EG$ carry the same information, this is not true for $\underline{E}G$ and $G\backslash\underline{E}G$. For instance, $G\backslash\underline{E}G$ is contractible if G is the infinite dihedral group $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 * \mathbb{Z}/2$, what can be seen by direct inspection, or if $G = \mathrm{SL}_3(\mathbb{Z})$, see [904, Corollary on page 8].

11.10 Notes

The notion of a classifying space for a family was introduced by tom Dieck [933].

Classifying spaces for families play a role in computations of equivariant homology and cohomology for compact Lie groups such as equivariant bordism as explained in [934, Chapter 7] and [935, Chapter III].

Classifying spaces for topological groups and appropriate families of subgroups play a key role in the construction of classifying equivariant principal bundles in [672] or the construction of the topological K-cohomology for arbitrary proper equivariant CW-complexes in [655].

More information about classifying spaces for families can be found for instance in [1, 64, 103, 242, 283, 284, 285, 381, 575, 640, 653, 673, 803, 935, 952, 953].

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Chapter 12
Equivariant Homology Theory

12.1 Introduction

This section is devoted to equivariant homology theories. They are a key input in the general formulations of the Baum-Connes Conjecture and the Farrell-Jones Conjecture. If one only wants to understand these conjectures, one only needs to browse through the Definition 12.1 of a G-homology theory, nothing more is needed from this chapter. Since G-homology theories are of general importance, we have added more material to this section. It will also be useful for concrete computations of K- and L-groups of group rings and group C^*-algebras based on the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

For a fixed group G, the notion of a *G-homology theory* $\mathcal{H}_*^G$ is the obvious generalization of the notion of a (generalized) homology theory in the non-equivariant sense. An important insight is to pass to an *equivariant homology theory* $\mathcal{H}_*^?$, see Definition 12.9. Roughly speaking, it assigns to every group G a G-homology theory $\mathcal{H}_*^G$ and links for any group homomorphisms $\alpha\colon H \to G$ the theories $\mathcal{H}_*^H$ and $\mathcal{H}_*^G$ by a so-called *induction structure*. This global point of view is the key for many applications and computations. Most of the interesting theories arise as equivariant homology theories.

Whenever one has a covariant functor from the category of small connected groupoids $\mathsf{GROUPOIDS}$ to the category of spectra $\mathsf{SPECTRA}$, one obtains an associated equivariant homology theory, see Section 12.4 Thus one can construct our main examples for equivariant homology theories, which are based on K- and L-groups of group rings and group C^*-algebras, by extending these notions from groups to groupoids, see Section 12.5.

We will provide tools for computations, namely, the *equivariant Atiyah-Hirzebruch spectral sequence*, see Subsection 12.6.1, the *p-chain spectral sequence*, see Subsection 12.6.2, and the *equivariant Chern character*, see Section 12.7. We will present some concrete examples of such computations in Sections 12.8 and 12.9.

12.2 Basics about G-Homology Theories

In this section we describe the axioms of a (proper) G-homology theory and give some basic examples. The main examples for us will be the sources of the assembly maps appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

Fix a discrete group G and an associative commutative ring Λ with unit.

Definition 12.1 (G-homology theory). A *G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of Λ-modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- G-homotopy invariance

  If f_0 and f_1 are G-homotopic G-maps of G-CW-pairs $(X,A) \to (Y,B)$, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$;
- Long exact sequence of a pair

  Given a pair (X,A) of G-CW-complexes, there is a long exact sequence

$$\ldots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X,A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X,A) \xrightarrow{\partial_n^G} \ldots$$

  where $i\colon A \to X$ and $j\colon X \to (X,A)$ are the inclusions;
- Excision

  Let (X,A) be a G-CW-pair, and let $f\colon A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F,f)\colon (X,A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(F,f)\colon \mathcal{H}_n^G(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B)$$

  for all $n \in \mathbb{Z}$;
- Disjoint union axiom

  Let $\{X_i \mid i \in I\}$ be a collection of G-CW-complexes. Denote by $j_i\colon X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i)\colon \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

  is bijective for all $n \in \mathbb{Z}$;

If $\mathcal{H}_*^G$ is defined or considered only for proper G-CW-pairs (X,A), we call it a *proper G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules.*

Example 12.2 (Bredon Homology). The most basic G-homology theory is *Bredon homology*, which was originally introduced in [152]. Recall that

$\mathsf{Or}(G)$ denotes the orbit category of G. Let X be a G-CW-complex. It defines a contravariant functor from the orbit category $\mathsf{Or}(G)$ to the category of CW-complexes by sending G/H to $\mathrm{map}_G(G/H, X) = X^H$. Composing it with the functor "cellular chain complex" yields a contravariant functor

$$C_*^{\mathsf{Or}(G)}(X)\colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{CHCOM}$$

to the category of $\mathbb{Z}$-chain complexes. Let Λ be a commutative ring and let

$$M\colon \mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MOD}$$

be a covariant functor to the abelian category of Λ-modules Λ-MOD. If $N\colon \mathsf{Or}(G) \to \mathbb{Z}$-$\mathsf{MOD}$ is a contravariant functor, one can form the tensor product over the orbit category $N \otimes_{\Lambda\mathsf{Or}(G)} M$, see for instance [629, 9.12 on page 166]. It is the quotient of the Λ-module

$$\bigoplus_{G/H \in \mathrm{ob}(\mathsf{Or}(G))} N(G/H) \otimes_{\mathbb{Z}} M(G/H)$$

by the Λ-submodule generated by

$$\{xf \otimes y - x \otimes fy \mid f\colon G/H \to G/K, x \in N(G/K), y \in M(G/H)\}$$

where xf stands for $N(f)(x)$ and fy for $M(f)(y)$. Since this is natural, we obtain a Λ-chain complex $C_*^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$. The homology of $C_*^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$ is the *Bredon homology of X with coefficients in M*

$$H_n^G(X; M) := H_n(C_*^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M). \tag{12.3}$$

This extends in the obvious way to G-CW-pairs. Thus we get a G-homology theory H_*^G with values in Λ-modules.

The description of $C_*^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$ in terms of the orbit category is conceptually the right one since it is intrinsically defined and the basic properties are easily checked following closely the non-equivariant case. For computation, the following explicit description is useful.

Fix G-pushouts

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"] & X_n
\end{tikzcd}$$

as they appear in Definition 11.2. Then the n-th Λ-chain module of the Λ-chain complex $C_*^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$ can be identified with

$$C_n^{\mathsf{Or}(G)}(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M = \bigoplus_{i \in I_n} M(G/H_i).$$

In order to define the n-th differential

$$c_n \colon \bigoplus_{i \in I_n} M(G/H_i) \to \bigoplus_{j \in I_{n-1}} M(G/H_j),$$

we specify for each pair $(i,j) \in I_n \times I_{n-1}$ a Λ-homomorphism $\alpha_{i,j} \colon M(G/H_i) \to M(G/H_j)$ such that for fixed $i \in I_n$ there are only finitely many $j \in I_{n-1}$ satisfying $\alpha_{i,j} \neq 0$.

We begin with the case $n = 1$. For $i \in I_1$, let $j(i,+)$ and $j(i,-)$ be the indices in I_0 for which $q_i^0(G/H_i \times \{\pm 1\}) \subseteq G/H_{j(i,\pm)}$ holds. Let $f(i,\pm) \colon G/H_i \to G/H_{j(i,\pm)}$ be the map induced by q_i^0. Define for $i \in I_1$ and $j \in I_0$

$$\alpha_{i,j} = \begin{cases} M(f(i,+)) - M(f(i,-)) & \text{if } j = j(i,+) \text{ and } j = j(i,-); \\ M(f(i,+)) & \text{if } j = j(i,+) \text{ and } j \neq j(i,-); \\ -M(f(i,-)) & \text{if } j \neq j(i,+) \text{ and } j = j(i,-); \\ 0 & \text{if } j \neq j(i,+) \text{ and } j \neq j(i,-). \end{cases}$$

Next we deal with the case $n \geq 2$. Let $X_{n-1,j}$ be the quotient of X_{n-1} where we collapse the $(n-2)$-skeleton and all the equivariant $(n-1)$-cells except the one for the index j to a point. The pushout above, but now for $(n-1)$ instead of n, yields a G-homeomorphism

$$\overline{Q_j^{n-1}} \colon \bigvee_{G/H_i} S^{n-1} = \left(G/H_j \times D^{n-1}\right) / \left(G/H_j \times S^{n-2}\right) \xrightarrow{\cong} X_{n-1,j}$$

where $\bigvee_{G/H_i} S^{n-1}$ is the one-point union or wedge of as many copies of S^{n-1} as there are elements in G/H_j. If $p_{gH_j} \colon \bigvee_{G/H_i} S^{n-1} \to S^{n-1}$ is the projection onto the summand belonging to $gH_j \in G/H_j$, $k \colon S^{n-1} \to G/H_i \times S^{n-1}$ is the obvious inclusion to the summand belonging to eH_i, and $\mathrm{pr}_j \colon X_{n-1} \to X_{n-1,j}$ the obvious projection, then we obtain a selfmap of S^{n-1} by the following composite

$$S^{n-1} \xrightarrow{k} G/H_i \times S^{n-1} \xrightarrow{q_i^n} X_{n-1} \xrightarrow{\mathrm{pr}_j} X_{n-1,j}$$
$$\xrightarrow{\overline{Q_j^{n-1}}^{-1}} \bigvee_{G/H_j} S^{n-1} \xrightarrow{p_{gH_j}} S^{n-1}.$$

Define $d_{i,j,gH_j} \in \mathbb{Z}$ to be the mapping degree of the map above. For $gH_j \in G/H_j^{H_i}$ we obtain a G-map

$$r_{gH_j} \colon G/H_i \to G/H_j, \quad g'H_i \mapsto g'gH_j.$$

Define

$$\alpha_{i,j} \colon M(G/H_i) \to M(G/H_j)$$

to be the sum of the maps $\sum_{gH_j \in G/H_j^{H_i}} d_{i,j,gH_j} \cdot M(r_{gH_j})$. Since because of the compactness of S^{n-1} there are for fixed $i \in I_{n-1}$ only finitely many pairs (j, gH_j) for $j \in I_{n-1}$ and $gH_j \in G/H_j$ with $d_{i,j,gH_j} \neq 0$, the definition of $\alpha_{i,j}$ makes sense and we can indeed define c_n by sending $\{x_i \mid i \in I_n\}$ to $\{\sum_{i \in I_n} \alpha_{i,j}(x_i) \mid j \in I_{n-1}\}$.

Obviously Bredon homology reduces for $G = \{1\}$ to the cellular homology of a CW-complex with coefficients in the abelian group M. It is the obvious generalization of this concept to the equivariant setting if one keeps in mind that in the equivariant situation the building blocks are equivariant cells given by G-spaces $G/H_i \times D^n$.

Exercise 12.4. Let $\mathbb{Z}/2$ act on $S^2 := \{(x_0, x_1, x_2) \mid x_i \in \mathbb{R}, x_0^2 + x_1^2 + x_2^2 = 1\}$ by the involution that sends (x_0, x_1, x_2) to $(x_0, x_1, -x_2)$. Consider the covariant functor

$$R_{\mathbb{C}} \colon \mathsf{Or}(\mathbb{Z}/2) \to \mathbb{Z}\text{-}\mathsf{MOD}$$

that sends $(\mathbb{Z}/2)/H$ to the complex representation ring $R_{\mathbb{C}}(H)$, any endomorphism in $\mathsf{Or}(\mathbb{Z}/2)$ to the identity and the morphism $\mathrm{pr} \colon (\mathbb{Z}/2)/\{1\} \to (\mathbb{Z}/2)/(\mathbb{Z}/2)$ to the homomorphism $R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(\mathbb{Z}/2)$ given by induction with the inclusion $\{1\} \to \mathbb{Z}/2$.

Show that S^2 becomes a $\mathbb{Z}/2$-CW-complex if we take $\{(1,0,0)\}$ as 0-skeleton, $\{(x_0, x_1, 0) \mid x_0^2 + x_1^2 = 1\}$ as 1-skeleton, and S^2 itself as 2-skeleton, and compute the abelian groups $H_*^{\mathbb{Z}/2}(S^2; R_{\mathbb{C}})$.

Lemma 12.5. *Let $\mathcal{H}_*^G$ be a G-homology theory. Let X be a G-CW-complex, and let $\{X_i \mid i \in I\}$ be a directed system of G-CW-subcomplexes directed by inclusion such that $X = \bigcup_{i \in I} X_i$. Then for all $n \in \mathbb{Z}$ the natural map*

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

Proof. The non-equivariant case is treated [925, Proposition 7.53 on page 121] for $I = \mathbb{N}$. The proof extends extends directly to the equivariant case, provided that $I = \mathbb{N}$. The proof of the general is left to the reader. □

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be G-homology theories. A *natural transformation of G-homology theories* $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ is a sequence of natural transformations $T_n \colon \mathcal{H}_n^G \to \mathcal{K}_n^G$ of functors from the category of G-CW-pairs to the category of Λ-modules which are compatible with the boundary homomorphisms.

Lemma 12.6. *Let $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a natural transformation of G-homology theories. Suppose that $T_n(G/H)$ is bijective for every homogeneous space G/H and $n \in \mathbb{Z}$.*

Then $T_n(X, A)$ is bijective for every G-CW-pair (X, A) and $n \in \mathbb{Z}$.

Note that one needs in Lemma 12.6 the existence of the natural transformation T. Namely, there exists two (non-equivariant) homology theories $\mathcal{H}_*$ and $\mathcal{K}_*$ such that $\mathcal{H}(\{\bullet\}) \cong \mathcal{K}_n(\{\bullet\})$ holds for $n \in \mathbb{Z}$ but there exists a CW-complex X and $m \in \mathbb{Z}$ such that $\mathcal{H}_m(X)$ and $\mathcal{K}_m(X)$ are not isomorphic. An example is topological K-homology theory K_* and the homology theory $\mathcal{H}_* = \bigoplus_{n\in\mathbb{Z}} H_{*+2n}$ for H_* singular homology.

Exercise 12.7. Give the proof of Lemma 12.6.

12.3 Basics about Equivariant Homology Theories

In this section we describe the axioms of a (proper) equivariant homology theory and give some basic examples. The point is that an equivariant homology theory assigns to every group G a G-homology theory and links them by an induction structure. It will play a key role in computations, various proofs, and the construction of the equivariant Chern character.

Let $\alpha\colon H \to G$ be a group homomorphism. Given an H-space X, define the *induction of* X *with* α to be the G-space

$$\operatorname{ind}_\alpha X = G \times_\alpha X, \tag{12.8}$$

i.e., the quotient of $G \times X$ by the right H-action $(g,x)\cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g,x) \in G \times X$. The G-actions comes from $g'\cdot(g,x) = (g'g,x)$. If $\alpha\colon H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

Definition 12.9 (Equivariant homology theory). A *(proper) equivariant homology theory with values in Λ-modules* $\mathcal{H}_*^?$ assigns to each group G a (proper) G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules (in the sense of Definition 12.1) together with the following so-called *induction structure*:

Given a group homomorphism $\alpha\colon H \to G$ and a (proper) H-CW-pair (X,A), there are for every $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^G(\operatorname{ind}_\alpha(X,A)) \tag{12.10}$$

satisfying:

- Compatibility with the boundary homomorphisms
  $\partial_n^G \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^H$;
- Functoriality
  Let $\beta\colon G \to K$ be another group homomorphism. Then we have for $n \in \mathbb{Z}$
  $$\operatorname{ind}_{\beta\circ\alpha} = \mathcal{H}_n^K(f_1)\circ\operatorname{ind}_\beta\circ\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\operatorname{ind}_{\beta\circ\alpha}(X,A))$$
  where $f_1\colon \operatorname{ind}_\beta\operatorname{ind}_\alpha(X,A) \xrightarrow{\cong} \operatorname{ind}_{\beta\circ\alpha}(X,A), \quad (k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism;

- Compatibility with conjugation

  For $n \in \mathbb{Z}$, $g \in G$, and a (proper) G-CW-pair (X, A) the homomorphism $\operatorname{ind}_{c(g)\colon G \to G}\colon \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(g)\colon G \to G}(X, A))$ agrees with $\mathcal{H}_n^G(f_2)$ for the G-homeomorphism $f_2\colon (X, A) \to \operatorname{ind}_{c(g)\colon G \to G}(X, A)$ that sends x to $(1, g^{-1}x)$ in $G \times_{c(g)} (X, A)$;
- Bijectivity

  If $\ker(\alpha)$ acts freely on $X \setminus A$, then $\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$ is bijective for all $n \in \mathbb{Z}$.

Exercise 12.11. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Show for any group G, any $g \in G$, and any $n \in \mathbb{Z}$ that induction with $c(g)\colon G \to G$ induces the identity on $\mathcal{H}_n^G(\{\bullet\})$.

Lemma 12.12. *Let $\mathcal{H}_*^?$ be a (proper) equivariant homology theory. Consider (finite) subgroups $H, K \subset G$ and an element $g \in G$ with $gHg^{-1} \subset K$. Let $R_{g^{-1}}\colon G/H \to G/K$ be the G-map sending $g'H$ to $g'g^{-1}K$ and $c(g)\colon H \to K$ be the homomorphism sending h to ghg^{-1}. Let $\mathrm{pr}\colon (\operatorname{ind}_{c(g)\colon H \to K}\{\bullet\}) \to \{\bullet\}$ be the projection. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{H}_n^H(\{\bullet\}) \arrow[r, "\mathcal{H}_n^K(\mathrm{pr})\circ \operatorname{ind}_{c(g)}"] \arrow[d, "\operatorname{ind}_H^G", "\cong"'] & \mathcal{H}_n^K(\{\bullet\}) \arrow[d, "\operatorname{ind}_K^G", "\cong"'] \\
\mathcal{H}_n^G(G/H) \arrow[r, "\mathcal{H}_n^G(R_{g^{-1}})"] & \mathcal{H}_n^G(G/K).
\end{tikzcd}$$

Proof. Let $f_1\colon \operatorname{ind}_{c(g)\colon G \to G} \operatorname{ind}_H^G\{\bullet\} \to \operatorname{ind}_K^G \operatorname{ind}_{c(g)\colon H \to K}\{\bullet\}$ be the bijective G-map sending $(g_1, g_2, \{\bullet\})$ in $G \times_{c(g)} G \times_H \{\bullet\}$ to $(g_1 g g_2 g^{-1}, 1, \{\bullet\})$ in $G \times_K K \times_{c(g)} \{\bullet\}$. The condition that induction is compatible with composition of group homomorphisms means precisely that the composite

$$\mathcal{H}_n^H(\{\bullet\}) \xrightarrow{\operatorname{ind}_H^G} \mathcal{H}_n^G(\operatorname{ind}_H^G\{\bullet\}) \xrightarrow{\operatorname{ind}_{c(g)\colon G \to G}} \mathcal{H}_n^G(\operatorname{ind}_{c(g)\colon G \to G} \operatorname{ind}_H^G\{\bullet\})$$
$$\xrightarrow{\mathcal{H}_n^G(f_1)} \mathcal{H}_n^G(\operatorname{ind}_K^G \operatorname{ind}_{c(g)\colon H \to K}\{\bullet\})$$

agrees with the composite

$$\mathcal{H}_n^H(\{\bullet\}) \xrightarrow{\operatorname{ind}_{c(g)\colon H \to K}} \mathcal{H}_n^K(\operatorname{ind}_{c(g)\colon H \to K}\{\bullet\}) \xrightarrow{\operatorname{ind}_K^G} \mathcal{H}_n^G(\operatorname{ind}_K^G \operatorname{ind}_{c(g)\colon H \to K}\{\bullet\}).$$

Naturality of induction implies $\mathcal{H}_n^G(\operatorname{ind}_K^G \mathrm{pr}) \circ \operatorname{ind}_K^G = \operatorname{ind}_K^G \circ \mathcal{H}_n^K(\mathrm{pr})$. Hence the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}_n^H(\{\bullet\}) \arrow[rrr, "\mathcal{H}_n^K(\mathrm{pr})\circ\mathrm{ind}_{c(g)H\to K}"] \arrow[d, "\cong"', "\mathrm{ind}_H^G"] & & & \mathcal{H}_n^K(\{\bullet\}) \arrow[d, "\cong"', "\mathrm{ind}_K^G"] \\
\mathcal{H}_n^G(G/H) \arrow[rrr, "\mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{pr})\circ\mathcal{H}_n^G(f_1)\circ\mathrm{ind}_{c(g)\colon G\to G}"] & & & \mathcal{H}_n^G(G/K).
\end{tikzcd}$$

By the axioms $\mathrm{ind}_{c(g)\colon G\to G}\colon \mathcal{H}_n^G(G/H) \to \mathcal{H}_n^G(\mathrm{ind}_{c(g)\colon G\to G} G/H)$ agrees with $\mathcal{H}_n^G(f_2)$ for the map $f_2\colon G/H \to \mathrm{ind}_{c(g)\colon G\to G} G/H$ that sends $g'H$ to $(g'g^{-1}, 1H)$ in $G \times_{c(g)} G/H$. Since the composite $(\mathrm{ind}_K^G \mathrm{pr}) \circ f_1 \circ f_2$ is just $R_{g^{-1}}$, Lemma 12.12 follows. □

Example 12.13 (Borel homology). Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in Λ-modules. Examples are singular homology, oriented bordism theory, or topological K-homology. Then we obtain two equivariant homology theories with values in Λ-modules in the sense of Definition 12.9 by the following constructions

$$\mathcal{H}_n^G(X,A) = \mathcal{K}_n(G\backslash X, G\backslash A);$$
$$\mathcal{H}_n^G(X,A) = \mathcal{K}_n(EG \times_G (X,A)).$$

The second one is called the *equivariant Borel homology associated to* $\mathcal{K}$.

In both cases $\mathcal{H}_*^G$ inherits the structure of a G-homology theory from the homology structure on $\mathcal{K}_*$. Induction for a group homomorphism $\alpha\colon H \to G$ is induced by the following two maps a and b. Let $a\colon H\backslash X \xrightarrow{\cong} G\backslash(G \times_\alpha X)$ be the homeomorphism sending Hx to $G(1,x)$. Define $b\colon EH \times_H X \to EG \times_G G \times_\alpha X$ by sending (e,x) to $(E\alpha(e), 1, x)$ for $e \in EH$, $x \in X$, and $E\alpha\colon EH \to EG$ the α-equivariant map induced by α. Induction for a group homomorphism $\alpha\colon H \to G$ is induced by these maps a and b. If the kernel $\ker(\alpha)$ acts freely on X, then the map b is a homotopy equivalence and hence in both cases ind_α is bijective.

Example 12.14 (Equivariant bordism). For a proper G-CW-pair (X,A), one can define the G-bordism group $\mathcal{N}_n^G(X,A)$ as the abelian group of G-bordism classes of maps $f\colon (M,\partial M) \to (X,A)$ whose sources are smooth manifolds with cocompact proper smooth G-actions. *Cocompact* means that the quotient space $G\backslash M$ is compact. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G-homology theory. There is an obvious induction structure coming from induction of equivariant spaces which is, however, only defined if the kernel of α acts freely on X. It is well-defined because of the following fact. If $\alpha\colon H \to G$ is a group homomorphism, M is an smooth H-manifold with cocompact proper smooth H-action, and $\ker(\alpha)$ acts freely, then $\mathrm{ind}_\alpha M$ is a smooth G-manifold with cocompact proper smooth G-action. The boundary of $\mathrm{ind}_\alpha M$ is $\mathrm{ind}_\alpha \partial M$.

Example 12.15 (Equivariant topological K-theory). We have explained the notion of *equivariant topological K-theory* $K_*^?$ in (10.67). If $R_{\mathbb{C}}(H)$ denotes the complex representation ring of the finite subgroup $H \subseteq G$, then

$$K_n^G(G/H) \cong K_n^H(\{\bullet\}) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Exercise 12.16. Compute $K_*^{D_\infty}(\underline{E}D_\infty)$.

In the sequel we put

$$\underline{B}G := G\backslash\underline{E}G. \tag{12.17}$$

Lemma 12.18. *Let $\mathcal{H}_*^?$ be an equivariant proper homology theory. Let G be any group. Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ.*

(i) The map $\mathcal{H}_n^{\{1\}}(BG) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^{\{1\}}(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$ is an isomorphism for all $n \in \mathbb{Z}$;
(ii) The map

$$\mathcal{H}_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^{\{1\}}(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$$

is split surjective, whereas the map

$$\mathcal{H}_n^G(EG) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda$$

is split injective.

Proof. (i) By the Atiyah-Hirzebruch spectral sequence it suffices to prove the bijectivity of the Λ-map $H_p(BG; \mathcal{H}_n^{\{1\}}(\{\bullet\})) \otimes_{\mathbb{Z}} \Lambda \to H_p(\underline{B}G; \mathcal{H}_n^{\{1\}}(\{\bullet\})) \otimes_{\mathbb{Z}} \Lambda$ for $p, q \in \mathbb{Z}$ with $p \geq 0$. The G-map $EG \to \underline{E}G$ induces a homology equivalence of projective ΛG-chain complexes $C_*(EG) \otimes_{\mathbb{Z}} \Lambda \to C_*(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda$, which is therefore a ΛG-chain homotopy equivalence. Hence it induces a Λ-chain homotopy equivalence $C_*(BG) \otimes_{\mathbb{Z}} \Lambda \to C_*(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$.

(ii) Since the following diagram commutes

$$\begin{tikzcd} \mathcal{H}_n^G(EG) \arrow[r] \arrow[d, "\mathrm{ind}_{G \to \{1\}}"', "\cong"] & \mathcal{H}_n^G(\underline{E}G) \arrow[d, "\mathrm{ind}_{G \to \{1\}}"] \\ \mathcal{H}_n^{\{1\}}(BG) \arrow[r] & \mathcal{H}_n^{\{1\}}(\underline{B}G) \end{tikzcd}$$

and has a bijection as left vertical arrow, the claim follows from assertion (i). $\square$

Example 12.19. Note that Lemma 12.18 (ii) is not true if one just considers a G-homology theory $\mathcal{H}_*^G$. Here is a counterexample. Let G be a finite group. Let M be the covariant $\mathbb{Z}\mathsf{Or}(G)$-module which sends G to $\mathbb{Z}$, G/H for $H \neq \{1\}$ to $\{0\}$, and every G-map $f\colon G \to G$ to the identity on $\mathbb{Z}$. Then the Bredon homology $H_n^G(EG;M)$ is $H_n(BG)$ and the Bredon homology $H_n^G(\underline{E}G;M) = H_n^G(G/G;M) = M(G/G)$ vanishes.

12.4 Constructing Equivariant Homology Theories Using Spectra

We briefly fix some conventions concerning spectra. Let SPACES^+ be the category of pointed compactly generated spaces. (One may also work with weakly Hausdorff spaces.) Here the objects are (compactly generated) spaces X with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. For two pointed spaces $X = (X,x)$ and $Y = (Y,y)$, define their *smash product* to be the pointed space

$$X \wedge Y = X \times Y/(\{x\} \times Y \cup X \times \{y\}), \tag{12.20}$$

and the *reduced cone* to be the pointed space

$$\operatorname{cone}(X) := X \times [0,1]/(X \times \{1\} \cup \{x\} \times [0,1]). \tag{12.21}$$

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1)$. A *map of spectra* $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ that are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$. Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category, see [13, III.2.]. The category of spectra and maps will be denoted $\mathsf{SPECTRA}$. Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) := \operatorname{colim}_{k\to\infty} \pi_{i+k}(E(k)) \tag{12.22}$$

where the ith structure map of the system $\pi_{i+k}(E(k))$ is given by the composite

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups. A spectrum $\mathbf{E}$ is called an *Ω-spectrum* if the adjoint $E_n \to \Omega E_{n+1}$ of each structure map is a weak homotopy equivalence.

Given a spectrum $\mathbf{E}$ and a pointed space X, we can define their smash product $X \wedge \mathbf{E}$ by $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ with the obvious structure maps. It is a classical result that a spectrum $\mathbf{E}$ defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n\left((X_+ \cup_{A_+} \operatorname{cone}(A_+)) \wedge \mathbf{E}\right).$$

We want to extend this to G-homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [265] where more details can be found.

In the sequel $\mathcal{C}$ is a small category. Our main example will be the orbit category $\mathsf{Or}(G)$.

Definition 12.23. A *covariant (contravariant) $\mathcal{C}$-space* X is a covariant (contravariant) functor

$$X \colon \mathcal{C} \to \mathsf{SPACES}.$$

A map between $\mathcal{C}$-spaces is a natural transformation of such functors. Analogously a *pointed $\mathcal{C}$-space* is a functor from $\mathcal{C}$ to SPACES^+ and a $\mathcal{C}$-spectrum a functor to $\mathsf{SPECTRA}$.

Example 12.24. Let Y be a left G-space. Define the associated *contravariant $\mathsf{Or}(G)$-space* $\operatorname{map}_G(-, Y)$ by

$$\operatorname{map}_G(-, Y) \colon \mathsf{Or}(G) \to \mathsf{SPACES}, \quad G/H \mapsto \operatorname{map}_G(G/H, Y) = Y^H.$$

If Y has a G-invariant base point, then $\operatorname{map}_G(-, Y)$ takes values in pointed spaces.

Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their *balanced product* to be the space

$$X \times_{\mathcal{C}} Y := \coprod_{c \in \operatorname{ob}(\mathcal{C})} X(c) \times Y(c)/\sim \tag{12.25}$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi \colon c \to d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are pointed, then one defines analogously their *balanced smash product* to be the pointed space

$$X \wedge_{\mathcal{C}} Y := \bigvee_{c \in \operatorname{ob}(\mathcal{C})} X(c) \wedge Y(c)/\sim. \tag{12.26}$$

In [265] the notation $X \otimes_{\mathcal{C}} Y$ was used for this space. Performing the same construction levelwise, one defines the *balanced smash product* $X \wedge_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$-space and a covariant $\mathcal{C}$-spectrum $\mathbf{E}$.

The proof of the next result is analogous to the non-equivariant case. Details can be found in [265, Lemma 4.4] where also cohomology theories are treated.

Theorem 12.27 (Constructing G-Homology Theories). *Let* $\mathbf{E}$ *be a covariant* $\mathsf{Or}(G)$*-spectrum. It defines a* G*-homology theory* $H_*^G(-;\mathbf{E})$ *by*

$$H_n^G(X,A;\mathbf{E}) := \pi_n\left(\operatorname{map}_G\left(-,(X_+ \cup_{A_+} \operatorname{cone}(A_+))\right) \wedge_{\mathsf{Or}(G)} \mathbf{E}\right).$$

In particular we have

$$H_n^G(G/H;\mathbf{E}) = \pi_n(\mathbf{E}(G/H)).$$

A version of the Brown representability Theorem is proved for G-homology theories and $\mathsf{Or}(G)$-spectra in [63], see also [281, Corollary 1.60 on page 36].

Example 12.28 (Bredon homology in terms of spectra). Consider a covariant functor $M\colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{MOD}$. Composing it with the functor sending a $\mathbb{Z}$-module N to its Eilenberg-MacLane spectrum $\mathbf{H}_N$, which is a spectrum such that $\pi_0(\mathbf{H}_N) \cong N$ and $\pi_n(\mathbf{H}_N) = \{0\}$ for $n \neq 0$, yields a covariant functor

$$\mathbf{H}_M\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}.$$

Then the G-homology theory $H_*^G(-;\mathbf{H}_M)$ associated to $\mathbf{H}_M$ in Theorem 12.27 agrees with Bredon homology $H_*^G(-;M)$ defined in Example 12.2.

Recall that we seek an equivariant homology theory and not only a G-homology theory. If the $\mathsf{Or}(G)$-spectrum in Theorem 12.27 is obtained from a $\mathsf{GROUPOIDS}$-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let $\mathsf{GROUPOIDS}$ be the category of small connected groupoids with covariant functors as morphisms. Recall that a *groupoid* is a category in which all morphisms are isomorphisms and that it is called *connected* if between any two objects there exists an isomorphism between them. A covariant functor $f\colon \mathcal{G}_0 \to \mathcal{G}_1$ of groupoids is called *injective* if for any two objects x, y in $\mathcal{G}_0$ the induced map $\operatorname{mor}_{\mathcal{G}_0}(x,y) \to \operatorname{mor}_{\mathcal{G}_1}(f(x),f(y))$ is injective. (We are not requiring that the induced map on the set of objects is injective.) Let $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ be the subcategory of $\mathsf{GROUPOIDS}$ with the same objects and injective functors as morphisms. For a G-set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S. The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ that satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G. It is connected if and only if G acts transitively on S. Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}^{\mathrm{inj}}, \quad G/H \mapsto \mathcal{G}^G(G/H). \tag{12.29}$$

A functor of small categories $F\colon \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that both $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects $x, y \in \mathcal{C}$ the map $\operatorname{mor}_{\mathcal{C}}(x, y) \to \operatorname{mor}_{\mathcal{D}}(F(x), F(y))$ induced by F is bijective.

Theorem 12.30 (Constructing equivariant homology theories using spectra). *Consider a covariant* $\mathsf{GROUPOIDS}$*-spectrum*

$$\mathbf{E} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}.$$

Suppose that $\mathbf{E}$ *respects equivalences, i.e., it sends an equivalence of groupoids to a weak equivalence of spectra. Then* $\mathbf{E}$ *defines an equivariant homology theory*

$$H_*^?(-; \mathbf{E})$$

whose underlying G*-homology theory for a group* G *is the* G*-homology theory associated to the covariant* $\mathsf{Or}(G)$*-spectrum* $\mathbf{E} \circ \mathcal{G}^G \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ *in the previous Theorem 12.27, i.e.,*

$$H_*^G(X, A; \mathbf{E}) = H_*^G(X, A; \mathbf{E} \circ \mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\{\bullet\}; \mathbf{E}) \cong \pi_n(\mathbf{E}(I(H)))$$

where $I(H)$ *denotes* H *considered as a groupoid with one object. The whole construction is natural in* $\mathbf{E}$*.*

Proof. We have to specify the induction structure for a homomorphism $\alpha \colon H \to G$. We only sketch the construction in the special case $A = \emptyset$.

The functor induced by α on the orbit categories is denoted in the same way

$$\alpha \colon \mathsf{Or}(H) \to \mathsf{Or}(G), \quad H/L \mapsto \operatorname{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural transformation of covariant functors $\mathsf{Or}(H) \to \mathsf{GROUPOIDS}$

$$T \colon \mathcal{G}^H \to \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the functor $\mathcal{G}^H(H/L) \to \mathcal{G}^G(G/\alpha(L))$ that sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism given by $\alpha(h) \in G$. The desired homomorphism

$$\operatorname{ind}_\alpha \colon H_n^H(X; \mathbf{E} \circ \mathcal{G}^H) \to H_n^G(\operatorname{ind}_\alpha X; \mathbf{E} \circ \mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{aligned} &\operatorname{map}_H(-, X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E} \circ \mathcal{G}^H \xrightarrow{\operatorname{id} \wedge \mathbf{E}(T)} \operatorname{map}_H(-, X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E} \circ \mathcal{G}^G \circ \alpha \\ &\xleftarrow{\cong} (\alpha_* \operatorname{map}_H(-, X_+)) \wedge_{\mathsf{Or}(G)} \mathbf{E} \circ \mathcal{G}^G \xleftarrow{\cong} \operatorname{map}_G(-, \operatorname{ind}_\alpha X_+) \wedge_{\mathsf{Or}(G)} \mathbf{E} \circ \mathcal{G}^G. \end{aligned}$$

Here $\alpha_* \operatorname{map}_H(-, X_+)$ is the pointed $\mathsf{Or}(G)$-space that is obtained from the pointed $\mathsf{Or}(H)$-space $\operatorname{map}_H(-, X_+)$ by induction, i.e., by taking the balanced product over $\mathsf{Or}(H)$ with the (discrete) $\mathsf{Or}(H)$-$\mathsf{Or}(G)$ biset $\operatorname{mor}_{\mathsf{Or}(G)}(??, \alpha(?))$, see [265, Definition 1.8]. Note that $\mathbf{E} \circ \mathcal{G}^G \circ \alpha$ is the same as the restriction of the $\mathsf{Or}(G)$-spectrum $\mathbf{E} \circ \mathcal{G}^G$ along α, which is often denoted by $\alpha^*(\mathbf{E} \circ \mathcal{G}^G)$ in the literature, see [265, Definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^*, see [265, Lemma 1.9]. The third map is the homeomorphism of $\mathsf{Or}(G)$-spaces that is the adjoint of the obvious map of $\mathsf{Or}(H)$-spaces $\operatorname{map}_H(-, X_+) \to \alpha^* \operatorname{map}_G(-, \operatorname{ind}_\alpha X_+)$ whose evaluation at H/L is given by $\operatorname{ind}_\alpha$.

It remains to show $\operatorname{ind}_\alpha$ is a weak equivalence, provided that $\ker(\alpha)$ acts freely on X. Because the second and third maps appearing in the definition above are homeomorphisms, this boils down to prove that $\operatorname{id} \wedge \mathbf{E}(T)$ is a weak equivalence, provided that $\ker(\alpha)$ acts freely on X. This follows from the fact that $T(H/L)$ is an equivalence of groupoids and hence $\mathbf{E}(T)(G/L)$ is a weak equivalence of spectra for all subgroups $L \subseteq G$ appearing as isotropy group in X since for such L the restriction of α to L induces a bijection $L \to \alpha(L)$. $\square$

Remark 12.31. In some cases the functor $\mathbf{E}$ to $\mathsf{SPECTRA}$ is only defined on $\mathsf{GROUPOIDS}^{\mathrm{inj}}$. Then one still gets an equivariant homology theory with the exception that for the induction structure one has to require that the group homomorphisms $\alpha\colon H \to G$ are injective. This does exclude the projection $G \to \{1\}$.

Example 12.32 (Bredon Homology). Let M be a covariant functor from $\mathsf{GROUPOIDS}$ to $\mathbb{Z}$-MOD. Then Bredon homology yields an equivariant homology theory if we define its value at G as the Bredon homology with coefficients in the covariant functor $M^G\colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{MOD}$ sending to G/H to $M(\mathcal{G}^G(G/H))$. This is the same as the equivariant homology theory we obtain from applying Theorem 12.30 to the functor $\mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ that sends a groupoid $\mathcal{G}$ to the Eilenberg-MacLane spectrum associated with $M(\mathcal{G})$.

Example 12.33 (Borel homology in terms of spectra). Let $\mathbf{E}$ be a spectrum. Let $H(-;\mathbf{E})$ be the (non-equivariant) homology theory associated to $\mathbf{E}$. Given a groupoid $\mathcal{G}$, denote by $E\mathcal{G}$ its classifying space. If $\mathcal{G}$ has only one object and the automorphism group of this object is G, then $E\mathcal{G}$ is a model for EG. We obtain two covariant functors

$$\begin{aligned} c_{\mathbf{E}}\colon \mathsf{GROUPOIDS} &\to \mathsf{SPECTRA}, \quad \mathcal{G} \mapsto \mathbf{E}; \\ b_{\mathbf{E}}\colon \mathsf{GROUPOIDS} &\to \mathsf{SPECTRA}, \quad \mathcal{G} \mapsto E\mathcal{G}_+ \wedge \mathbf{E}. \end{aligned}$$

Thus we obtain two equivariant homology theories $H_*^*(-;c_{\mathbf{E}})$ and $H_*^*(-;b_{\mathbf{E}})$ from Theorem 12.30. These coincide with the ones associated to $\mathcal{K}_* = H(-;\mathbf{E})$ in Example 12.13. Namely, we get for any group G and any G-CW-complex X natural isomorphisms

$$H_n^G(X; c_{\mathbf{E}}) \cong H_n(G\backslash X; \mathbf{E}); \tag{12.34}$$

$$H_n^G(X; b_{\mathbf{E}}) \cong H_n(EG \times_G X; \mathbf{E}). \tag{12.35}$$

Exercise 12.36. Let $\mathbf{E}$ and $\mathbf{F}$ be covariant functors from $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$. Let $\mathbf{t}\colon \mathbf{E} \to \mathbf{F}$ be a natural transformation such that for every $\mathcal{G} \in \operatorname{ob}(\mathsf{GROUPOIDS})$ the map $\mathbf{t}(\mathcal{G})\colon \mathbf{E}(\mathcal{G}) \to \mathbf{F}(\mathcal{G})$ is a weak equivalence of spectra.

Show that the induced transformation of equivariant homology theories $H_*^?(-;\mathbf{t})\colon H_*^?(-;\mathbf{E}) \to H_*^?(-;\mathbf{F})$ is a natural equivalence.

12.5 Equivariant Homology Theories Associated to K- and L-Theory

In this section we explain our main examples for covariant functors from $\mathsf{GROUPOIDS}$ or $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ to $\mathsf{SPECTRA}$, at least for rings as coefficients. Later we will also consider additive categories and, more generally, right exact ∞-categories.

Let RINGS be the category of associative rings with unit. Let $\mathsf{RINGS}^{\mathrm{inv}}$ be the category of rings with involution. Let C^*-$\mathsf{ALGEBRAS}$ be the category of C^*-algebras. There are classical functors for $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$

$$\mathbf{K}\colon \mathsf{RINGS} \to \mathsf{SPECTRA}; \tag{12.37}$$

$$\mathbf{L}^{\langle j \rangle}\colon \mathsf{RINGS}^{\mathrm{inv}} \to \mathsf{SPECTRA}; \tag{12.38}$$

$$\mathbf{K}^{\mathsf{TOP}}\colon C^*\text{-}\mathsf{ALGEBRAS} \to \mathsf{SPECTRA}. \tag{12.39}$$

The construction of such a non-connective algebraic K-theory functor (12.37) goes back to Gersten [406] and Wagoner [954]. The spectrum for quadratic algebraic L-theory (12.38) is constructed by Ranicki in [823]. In a more geometric formulation it goes back to Quinn [806]. In the topological K-theory case a construction for (12.39) using Bott periodicity for C^*-algebras can easily be derived from the Kuiper-Mingo Theorem, see [888, Section 2.2]. The homotopy groups of these spectra give the algebraic K-groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L-theory groups, and the topological K-groups of C^*-algebras.

We emphasize that in all three cases we need the non-connective versions of the spectra, i.e., the homotopy groups in negative dimensions are non-trivial in general, in order to ensure later that the formulations of the various Isomorphisms Conjectures do have a chance to be true.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}$, respectively $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}^{\mathrm{inv}}$ where GROUPS denotes the category of groups. Let $\mathsf{GROUPS}^{\mathrm{inj}}$ be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group C^*-algebra defines a functor $C_r^*\colon \mathsf{GROUPS}^{\mathrm{inj}} \to C^*\text{-}\mathsf{ALGEBRAS}$. The composite of these functors with the functors (12.37), (12.38) and (12.39) above yields functors

$$\mathbf{K}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{12.40}$$

$$\mathbf{L}^{\langle j\rangle}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{12.41}$$

$$\mathbf{K}^{\mathsf{TOP}}C_r^*(-,F)\colon \mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{SPECTRA}, \tag{12.42}$$

where $F = \mathbb{R}$ or $\mathbb{C}$. They satisfy

$$\begin{aligned}\pi_n(\mathbf{K}R(G)) &= K_n(RG);\\ \pi_n(\mathbf{L}^{\langle j\rangle}R(G)) &= L_n^{\langle j\rangle}(RG);\\ \pi_n(\mathbf{K}^{\mathsf{TOP}}C_r^*(G,F)) &= K_n(C_r^*(G,F)),\end{aligned}$$

for every group G and every $n \in \mathbb{Z}$. The next result essentially says that these functors can be extended to groupoids.

Theorem 12.43 (K- and L-Theory Spectra over Groupoids). *Let R be a ring (with involution). There exist covariant functors*

$$\mathbf{K}_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{12.44}$$

$$\mathbf{L}_R^{\langle j\rangle}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{12.45}$$

$$\mathbf{K}_F^{\mathsf{TOP}}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA}, \tag{12.46}$$

with the following properties:

(i) If $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ is an equivalence of (small) groupoids, then the induced maps $\mathbf{K}_R(F)$, $\mathbf{L}_R^{\langle j\rangle}(F)$, and $\mathbf{K}^{\mathsf{TOP}}(F)$ are weak equivalences of spectra;

(ii) Let $I\colon \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ be the functor sending G to G considered as a groupoid, i.e. to $\mathcal{G}^G(G/G)$. This functor restricts to a functor $\mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{GROUPOIDS}^{\mathrm{inj}}$.
There are natural transformations from $\mathbf{K}R(-)$ to $\mathbf{K}_R \circ I$, from $\mathbf{L}^{\langle j\rangle}R(-)$ to $\mathbf{L}_R^{\langle j\rangle} \circ I$ and from $\mathbf{K}C_r^*(-)$ to $\mathbf{K}^{\mathsf{TOP}} \circ I$ such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra;

(iii) For every group G and all $n \in \mathbb{Z}$ we have

$$\begin{aligned}\pi_n(\mathbf{K}_R \circ I(G)) &\cong K_n(RG);\\ \pi_n(\mathbf{L}_R^{\langle j\rangle} \circ I^{\mathrm{inv}}(G)) &\cong L_n^{\langle j\rangle}(RG);\\ \pi_n(\mathbf{K}_F^{\mathsf{TOP}} \circ I(G)) &\cong K_n(C_r^*(G,F)).\end{aligned}$$

Proof. We only sketch the strategy of the proof. More details can be found in [265, Section 2].

Let $\mathcal{G}$ be a groupoid. Similar to the group ring RG one can define an R-linear category $R\mathcal{G}$ by taking the free R-modules over the morphism sets of $\mathcal{G}$.

Composition of morphisms is extended R-linearly. By formally adding finite direct sums one obtains an additive category $R\mathcal{G}_\oplus$. Pedersen-Weibel [783], see also [197] and [668], define a non-connective algebraic K-theory functor which digests additive categories and can hence be applied to $R\mathcal{G}_\oplus$. For the comparison result one uses that for every ring R (in particular for RG) the Pedersen-Weibel functor applied to $R_\oplus$ (a small model for the category of finitely generated free R-modules) yields the non-connective K-theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L-theory case $R\mathcal{G}_\oplus$ inherits an involution and one applies the construction of Ranicki [823, Example 13.6 on page 139] to obtain the $L^{\langle 1\rangle} = L^h$-version. The versions for $j \leq 1$ can be obtained by a construction that is analogous to the Pedersen-Weibel construction for K-theory, compare Carlsson-Pedersen [202, Section 4], or by iterating the Shaneson splitting and then finally passing to a homotopy colimit, compare on the group level with [824, Section 17]. In the C^*-case one obtains from $\mathcal{G}$ a C^*-category $C^*_r(\mathcal{G})$ and assigns to it its topological K-theory spectrum. There is a construction of the topological K-theory spectrum of a C^*-category in Davis-Lück [265, Section 2]. However, the construction given there depends on two statements, which appeared in [372, Proposition 1 and Proposition 3], and those statements are incorrect, as already pointed out by Thomason in [930]. The construction in [265, Section 2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [507]. □

Exercise 12.47. Compute $H_n^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_R)$ for $n \leq 0$ and $R = \mathbb{Z}, \mathbb{C}$.

12.6 Two Spectral Sequences

In this section we state two spectral sequences which are useful for computations of equivariant homology theories.

12.6.1 The Equivariant Atiyah-Hirzebruch Spectral Sequence

Theorem 12.48 (The equivariant Atiyah-Hirzebruch spectral sequence). *Let G be a group and $\mathcal{H}_*^G$ be a G-homology theory with values in Λ-modules in the sense of Definition 12.1. Let X be a G-CW-complex.*

Then there is a spectral (homology) sequence of Λ-modules

$$(E^r_{p,q}, d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1})$$

whose E_2-term is given by the Bredon homology of Example 12.2

$$E^2_{p,q} = H_p^G(X; \mathcal{H}_q^G(-))$$

for the coefficient system given by the covariant functor

$$\mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MOD}, \quad G/H \mapsto \mathcal{H}_q^G(G/H).$$

The E^∞-term is given by

$$E_{p,q}^\infty = \operatorname{colim}_{r\to\infty} E_{p,q}^r.$$

This spectral sequence converges to $\mathcal{H}_{p+q}^G(X)$, i.e., there is an ascending filtration $F_{p,m-p}\mathcal{H}_{p+q}^G(X)$ of $\mathcal{H}_{p+q}^G(X)$ such that

$$F_{p,q}\mathcal{H}_{p+q}^G(X)/F_{p-1,q+1}\mathcal{H}_{p+q}^G(X) \cong E_{p,q}^\infty.$$

The construction of the equivariant Atiyah-Hirzebruch spectral sequence is based on the filtration of X by its skeletons. More details, actually in the more general context of spaces over a category, and a version for cohomology can be found in [265, Theorem 4.7].

Exercise 12.49. Let X be a proper G-CW-complex such that X/G with the induced CW-structure has no odd dimensional cells. Show that $K_n^G(X) = 0$ for odd $n \in \mathbb{Z}$ where K_*^G denotes the equivariant topological complex K-homology. Show that $K_n^G(X)$ for even $n \in \mathbb{Z}$ is a finitely generated free abelian group if we additionally assume that X/G is finite.

12.6.2 The *p*-Chain Spectral Sequence

Let G be a group. Recall that for a subgroup $H \subseteq G$ we denote by N_GH its normalizer and define the Weyl group $W_GH := N_GH/H$. We obtain a bijection

$$W_GH \xrightarrow{\cong} \operatorname{aut}_G(G/H), \quad gH \mapsto \big(R_{g^{-1}}\colon G/H \to G/H\big)$$

where $R_{g^{-1}}$ maps $g'H$ to $g'g^{-1}H$. Hence for any two subgroups $H, K \subseteq G$ the set $\operatorname{map}_G(G/H, G/K)$ inherits the structure of a W_GK-W_GH-biset.

A *p-chain* is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \cdots < (H_p)$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugated, but not conjugated to (H_i). For $p \geq 1$ define a W_GH_p-W_GH_0-set associated to such a p-chain by

$$\begin{aligned} &S((H_0) < \cdots < (H_p)) \\ &\quad := \operatorname{map}_G(G/H_{p-1}, G/H_p) \times_{W_GH_{p-1}} \cdots \times_{W_GH_1} \operatorname{map}_G(G/H_0, G/H_1). \end{aligned}$$

For $p = 0$ put $S(H_0) = W_GH_0$.

Let X be a G-CW-complex. Then $X^H = \mathrm{map}_G(G/H, X)$ inherits a right $W_G H$-action. In particular we get for a p-chain $(H_0) < \cdots < (H_p)$ a right $W_G H_0$-space $X(G/H_p) \times_{W_G H_p} S((H_0) < \cdots < (H_p))$.

Theorem 12.50 (The p-chain spectral sequence). *Let G be a group and $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Let X be a proper G-CW-complex.*

Then there is a spectral sequence of Λ-modules, called p-chain spectral sequence, which converges to $H^G_{p+q}(X; \mathbf{E})$ and whose E^1-term is

$$E^1_{p,q} = \bigoplus_{(H_0)<\cdots<(H_p)} \pi_q\Bigg(\big(EW_G H_0 \times (X^{H_p} \times_{W_G H_p} S((H_0) < \cdots < (H_p)))\big)_+ \wedge_{W_G H_0} \mathbf{E}(G/H_0)\Bigg)$$

where $(H_0) < \cdots < (H_p)$ runs through all p-chains consisting of finite subgroups $H_i \subseteq G$ with $X^{H_p} \neq \emptyset$.

The p-chain spectral sequence is established in [266, Theorem 2.5 (a) and Example 2.14], actually more generally for spaces over a category. There is also a more complicated version where one drops the condition that X is proper. Since then the book-keeping gets more involved and in most applications X is proper, we only deal with the proper case here.

Note that the complexity of the equivariant Atiyah-Hirzebruch spectral sequence grows with the natural number n for which one wants to compute $\mathcal{H}^G_n(X)$. The complexity of the p-chain spectral sequence growth with the maximum over all natural numbers p for which there is a p-chain $(H_0) < \cdots < (H_p)$ of finite subgroups such that X^{H_p} is non-empty.

Example 12.51 (Free G-CW-complex). Consider the situation of Theorem 12.50 and assume additionally that X is a free G-CW-complex. Then $E^1_{p,q} = 0$ for $p \geq 1$ and hence the p-chain spectral sequence predicts

$$H^G_q(X; \mathbf{E}) = \pi_q\big((EG \times X)_+ \wedge_G \mathbf{E}(G)\big).$$

But this is obviously true since the right hand side of the last equation is by definition $H^G_q(EG \times X; \mathbf{E})$ and the projection $EG \times X \to X$ is a G-homotopy equivalence and induces an isomorphism $H^G_q(EG \times X; \mathbf{E}) \xrightarrow{\cong} H^G_q(X; \mathbf{E})$.

Example 12.52 (G is finite cyclic of prime order). Let G be a finite cyclic group of prime order. Then G has only two subgroups, namely, G and $\{1\}$. Let $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum and X be a G-CW-complex. The p-chain spectral sequence of Theorem 12.50 satisfies $E^1_{p,q} = 0$ for $p \geq 2$ and hence reduces to a long exact sequence

$$\ldots \to E^1_{1,n} \xrightarrow{d^1_{1,n}} E^1_{0,n} \to H^G_n(X; \mathbf{E}) \to E^1_{1,n-1} \xrightarrow{d^1_{1,n-1}} E^1_{0,n-1} \to \ldots$$

We get

$$E^1_{0,n} = \pi_n\big((EG \times X)_+ \wedge_G \mathbf{E}(G)\big) \oplus \pi_n\big(X^G_+ \wedge \mathbf{E}(G/G)\big);$$
$$E^1_{1,n} = \pi_n\big((EG \times X^G)_+ \wedge_G \mathbf{E}(G)\big)$$

and the differential $d^1_{1,n}$ is given by the homomorphism

$$\pi_n\big((EG \times X^G)_+ \wedge_G \mathbf{E}(G)\big) \to \pi_n\big((EG \times X)_+ \wedge_G \mathbf{E}(G)\big)$$

that is induced by the inclusion $X^G \to X$, and the homomorphism (up to a sign)

$$\pi_n\big((EG \times X^G)_+ \wedge_G \mathbf{E}(G)\big) \to \pi_n\big(X^G_+ \wedge \mathbf{E}(G/G)\big)$$

coming from the projection $EG \times X^G \to X^G$.

Now suppose additionally that $\mathbf{E}$ is the constant functor $\mathsf{Or}(G) \to \mathsf{SPECTRA}$ with value the spectrum $\mathbf{F}$. Let $\mathcal{H}_*$ be the (non-equivariant) homology theory associated to $\mathbf{F}$. Then $H^G_n(X;\mathbf{E}) = \mathcal{H}_n(X/G)$ and the long exact sequence above reduces to the long exact sequence

(12.53)

$$\ldots \to \mathcal{H}_n(EG \times_G X^G) \xrightarrow{d^1_{1,n}} \mathcal{H}_n(EG \times_G X) \oplus \mathcal{H}_n(X^G) \xrightarrow{e_n} \mathcal{H}_n(X/G)$$
$$\to \mathcal{H}_{n-1}(EG \times_G X^G) \xrightarrow{d^1_{1,n-1}} \mathcal{H}_{n-1}(EG \times_G X) \oplus \mathcal{H}_{n-1}(X^G) \xrightarrow{e_{n-1}} \ldots$$

where the maps $d^1_{n,1}$ and e_n are up to sign induced by the obvious map on space level.

Exercise 12.54. Give a direct construction of the long exact sequence (12.53).

12.7 Equivariant Chern Characters

If we rationalize and have a Mackey structure on the coefficient system of an equivariant homology theory, then we can give a more direct and concrete computation via equivariant Chern characters which does avoid all the difficulties concerning spectral sequences.

12.7.1 Mackey Functors

Let Λ be an associative commutative ring with unit. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let

$$M\colon \mathsf{FGINJ} \to \Lambda\text{-}\mathsf{MOD}$$

be a bifunctor, i.e., a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from FGINJ to Λ-MOD which agree on objects. We will often denote for an injective group homomorphism $f\colon H \to G$ the map $M_*(f)\colon M(H) \to M(G)$ by ind_f and the map $M^*(f)\colon M(G) \to M(H)$ by res_f and write $\mathrm{ind}_H^G = \mathrm{ind}_f$ and $\mathrm{res}_G^H = \mathrm{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in Λ-modules if

(i) For an inner automorphism $c(g)\colon G \to G$ we have $M_*(c(g)) = \mathrm{id}\colon M(G) \to M(G)$;
(ii) For an isomorphism of groups $f\colon G \xrightarrow{\cong} H$ the composites $\mathrm{res}_f \circ \mathrm{ind}_f$ and $\mathrm{ind}_f \circ \mathrm{res}_f$ are the identity;
(iii) Double coset formula

 We have for two subgroups $H, K \subset G$

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G = \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H\cap g^{-1}Kg}$$

 where $c(g)$ is conjugation with g, i.e. $c(g)(h) = ghg^{-1}$.

Important examples of Mackey functors are $\mathrm{Rep}_F(H)$, $K_q(RH)$, $L_q(RH)$, and $K_q^{\mathsf{TOP}}(C_*^r(H,F))$ where R is an associative ring with unit and $F = \mathbb{R}, \mathbb{C}$.

Definition 12.55 (Extension to a Mackey functor). Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules. It defines a covariant functor

$$\mathcal{H}_q^?(\{\bullet\})\colon \mathsf{FGINJ} \to \Lambda\text{-}\mathsf{MOD}, \quad H \mapsto \mathcal{H}_q^G(\{\bullet\}).$$

It sends an injective homomorphism $i\colon H \to G$ to the composite $\mathcal{H}_n^H(\{\bullet\}) \xrightarrow{\mathrm{ind}_i} \mathcal{H}_n^G(G \times_H \{\bullet\}) \xrightarrow{\mathcal{H}_n^G(\mathrm{pr})} \mathcal{H}_n^G(\{\bullet\})$ where $\mathrm{pr}\colon G \times_H \{\bullet\} \to \{\bullet\}$ is the projection. We say that *the coefficients of $\mathcal{H}_*^?$ extend to a Mackey functor* if there exists a Mackey functor (M_*, M^*) such that M_* is the functor $\mathcal{H}_q^?(\{\bullet\})$ above.

Example 12.56. The functors of (12.40), (12.41), and (12.42), which send a group to the algebraic K- or L-theory of RG or to the topological K-theory of $C_r^*(G,F)$, define Mackey functors with the obvious definition of induction and restriction.

12.7.2 The Equivariant Chern Character

We can associate to a proper equivariant homology theory with values in Λ-modules $\mathcal{H}_*^?$ another Bredon type equivariant homology theory with values in Λ-modules $\mathcal{BH}_*^?$ as follows. For a group G we define

$$\mathcal{BH}_n^G(X) := \bigoplus_{p+q=n} H_p^G(X;\mathcal{H}_q^G(-))$$

where $H_p^G(X;\mathcal{H}_q^G(-))$ is the Bredon homology of X with coefficients in the covariant functor $\mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MOD}$ sending G/H to $\mathcal{H}_q^G(G/H)$. Next we show that the collection of the G-homology theories $\mathcal{BH}_*^G(X,A)$ inherits the structure of a proper equivariant homology theory. We have to specify the induction structure.

Let $\alpha\colon H \to G$ be a group homomorphism and (X,A) be a proper H-CW-pair. Induction with α yields a functor denoted in the same way

$$\alpha\colon \mathsf{Or}_{\mathcal{FIN}}(H) \to \mathsf{Or}_{\mathcal{FIN}}(G), \quad H/K \mapsto \operatorname{ind}_\alpha(H/K) = G/\alpha(K).$$

There is a natural isomorphism of $\mathsf{Or}_{\mathcal{FIN}}(G)$-chain complexes

$$\operatorname{ind}_\alpha C_*^{\mathsf{Or}_{\mathcal{FIN}}(H)}(X,A) \xrightarrow{\cong} C_*^{\mathsf{Or}_{\mathcal{FIN}}(G)}(\operatorname{ind}_\alpha(X,A))$$

and a natural adjunction isomorphism, see [634, (2.5)]

$$\begin{aligned}\left(\operatorname{ind}_\alpha C_*^{\mathsf{Or}_{\mathcal{FIN}}(H)}(X,A)\right) &\otimes_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(G)} \mathcal{H}_q^G(-) \\ &\xrightarrow{\cong} C_*^{\mathsf{Or}_{\mathcal{FIN}}(H)}(X,A) \otimes_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(H)} \left(\operatorname{res}_\alpha \mathcal{H}_q^G(-)\right).\end{aligned}$$

The induction structure on $\mathcal{H}_*^?$ yields a morphism of $R\mathsf{Or}_{\mathcal{FIN}}(H)$-modules

$$\mathcal{H}_q^H(H/?) \to \operatorname{res}_\alpha \mathcal{H}_q^G(-).$$

These maps or their inverses can be composed to a Λ-chain map

$$\begin{aligned}C_*^{\mathsf{Or}_{\mathcal{FIN}}(H)}(X,A) &\otimes_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(H)} \mathcal{H}_q^H(H/?) \\ &\xrightarrow{\cong} C_*(\operatorname{ind}_\alpha(X,A)) \otimes_{\mathbb{Z}\mathsf{Or}(G,\mathcal{FIN})} \mathcal{H}_q^G(-).\end{aligned}$$

Since X is proper and hence the Bredon homology can be defined over $\mathsf{Or}_{\mathcal{FIN}}(H)$ instead of $\mathsf{Or}(G)$, it induces a natural map

$$\operatorname{ind}_\alpha\colon H_p(X,A;\mathcal{H}_q^H(-)) \xrightarrow{\cong} H_p^G(\operatorname{ind}_\alpha(X,A);\mathcal{H}_q^G(-)).$$

Thus we obtain the required induction structure.

Define for a finite group H

(12.57)

$$S_H\left(\mathcal{H}_q^H(\{\bullet\})\right) := \operatorname{coker}\left(\bigoplus_{\substack{K\subset H\\K\neq H}} \operatorname{ind}_K^H\colon \bigoplus_{\substack{K\subset H\\K\neq H}} \mathcal{H}_q^K(\{\bullet\}) \to \mathcal{H}_q^H(\{\bullet\})\right).$$

Note that $S_H\left(\mathcal{H}_q^H(\{\bullet\}\right)$ carries a natural left $\Lambda[N_GH/H \cdot C_GH]$-module structure where $N_GH/H \cdot C_GH$ is the quotient of N_GH by the normal subgroup $H \cdot C_GH := \{h \cdot g \mid h \in H, g \in C_GH\}$. The obvious left-action of $W_GH = N_GH/H$-action on X^H yields a left $N_GH/H \cdot C_GH$-action on $C_GH\backslash X^H$ and hence a right $N_GH/H \cdot C_GH$-action by $y \cdot k := k^{-1} \cdot y$ for $y \in X^H$ and $k \in N_GH/H \cdot C_GH$.

The proof of the following result can be found in [634, Theorem 0.2 and 0.3].

Theorem 12.58 (The equivariant Chern character). *Let Λ be a commutative ring with $\mathbb{Q} \subset \Lambda$. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules in the sense of Definition 12.9. Suppose that its coefficients extend to a Mackey functor.*

(i) There is an isomorphism of proper equivariant homology theories

$$\mathrm{ch}_*^? \colon \mathcal{B}\mathcal{H}_*^? \xrightarrow{\cong} \mathcal{H}_*^?;$$

(ii) Let I be the set of conjugacy classes (H) of finite subgroups H of G. Then there is for any group G and any proper G-CW-pair (X,A) a natural isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash(X^H,A^H);\Lambda) \otimes_{\Lambda[N_GH/H \cdot C_GH]} S_H\left(\mathcal{H}_q^H(\{\bullet\})\right) \xrightarrow{\cong} \mathcal{B}\mathcal{H}_n^G(X,A).$$

Theorem 12.58 reduces the computation of $\mathcal{H}_n^G(X,A)$ to the computation of the singular or cellular homology Λ-modules $H_p(C_GH\backslash(X^H,A^H);\Lambda)$ of the CW-pairs $C_GH\backslash(X^H,A^H)$ including the obvious right W_GH-operation and of the left $\Lambda[W_GH]$-modules $S_H\left(\mathcal{H}_q^H(\{\bullet\})\right)$ which only involve the values $\mathcal{H}_q^G(G/H) = \mathcal{H}_q^H(\{\bullet\})$.

Exercise 12.59. Let Λ be a commutative ring with $\mathbb{Q} \subset \Lambda$. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules. Suppose that its coefficients extend to a Mackey functor. Consider a group G and a proper G-CW-complex X. Show that all differentials of the equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}_{p+q}^G(X)$ vanish.

Exercise 12.60. Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in $\mathbb{Q}$-modules in the sense of Definition 12.9. Suppose that its coefficients extend to a Mackey functor. Let G be a group. Consider two families of subgroups $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{FIN}$. Let $\iota_{\mathcal{F}\subseteq\mathcal{G}} \colon E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$ be the up to G-homotopy unique G-map. Show that for every n the induced map $\mathcal{H}_n^G(\iota_{\mathcal{F}\subseteq\mathcal{G}}) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$ is injective.

Remark 12.61 (Rationalizing an equivariant homology theory). Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in $\mathbb{Z}$-modules. Suppose that its coefficients extend to a Mackey functor. Then we obtain an equivariant homology theory $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_*^?$ with values in $\mathbb{Q}$-modules whose coefficients extend to a Mackey functor since $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is a flat functor and commutes with direct sums over arbitrary index sets. We can apply Theorem 12.58 to $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_*^?$ and thus obtain a rational computation of $\mathcal{H}_*^?$.

12.8 Some Rational Computations

12.8.1 Green Functors

Let $\phi\colon \Lambda \to \Lambda'$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in Λ-modules, and let N and P be Mackey functors with values in Λ'-modules. A *pairing* with respect to ϕ is a family of maps

$$m(G)\colon M(G) \times N(G) \to P(G), \quad (x, y) \mapsto m(G)(x, y) =: x \cdot y$$

where G runs through the finite groups and we require the following properties for all injective group homomorphisms $f\colon H \to G$ of finite groups:

$$\begin{aligned}
(x_1 + x_2) \cdot y &= x_1 \cdot y + x_2 \cdot y && \text{for } x_1, x_2 \in M(H), y \in N(H);\\
x \cdot (y_1 + y_2) &= x \cdot y_1 + x \cdot y_2 && \text{for } x \in M(H), y_1, y_2 \in N(H);\\
(\lambda x) \cdot y &= \phi(\lambda)(x \cdot y) && \text{for } \lambda \in \Lambda, x \in M(H), y \in N(H);\\
x \cdot \lambda' y &= \lambda'(x \cdot y) && \text{for } \lambda' \in \Lambda', x \in M(H), y \in N(H);\\
\operatorname{res}_f(x \cdot y) &= \operatorname{res}_f(x) \cdot \operatorname{res}_f(y) && \text{for } x \in M(G), y \in N(G);\\
\operatorname{ind}_f(x) \cdot y &= \operatorname{ind}_f(x \cdot \operatorname{res}_f(y)) && \text{for } x \in M(H), y \in N(G);\\
x \cdot \operatorname{ind}_f(y) &= \operatorname{ind}_f(\operatorname{res}_f(x) \cdot y) && \text{for } x \in M(G), y \in N(H).
\end{aligned}$$

A *Green functor* with values in Λ-modules is a Mackey functor U with values in Λ-modules together with a pairing with respect to $\mathrm{id}\colon \Lambda \to \Lambda$ and elements $1_G \in U(G)$ for each finite group G such that for each finite group G the pairing $U(G) \times U(G) \to U(G)$ induces the structure of an Λ-algebra on $U(G)$ with unit 1_G and for any morphism $f\colon H \to G$ in FGINJ the map $U^*(f)\colon U(G) \to U(H)$ is a homomorphism of Λ-algebras with unit. Let U be a Green functor with values in Λ-modules and M be a Mackey functor with values in Λ'-modules. A *(left) U-module structure on M* with respect to the ring homomorphism $\phi\colon \Lambda \to \Lambda'$ is a pairing such that any of the maps $U(G) \times M(G) \to M(G)$ induces the structure of a (left) module over the Λ-algebra $U(G)$ on the Λ-module $\phi^* M(G)$ that is obtained from the Λ'-module $M(G)$ by $\lambda x := \phi(\lambda) x$ for $r \in \Lambda$ and $x \in M(G)$.

The importance of the notion of a Green functor is due to the following elementary lemma which allows to deduce induction theorems for all Mackey functors that are modules over a given Green functor from the corresponding statement for the given Green functor.

Lemma 12.62. *Let $\phi\colon \Lambda \to \Lambda'$ be a homomorphism of associative commutative rings with unit. Let U be a Green functor with values in Λ-modules and let M be a Mackey functor with values in Λ'-modules such that M comes with a U-module structure with respect to ϕ. Let $\mathcal{S}$ be a set of subgroups of the finite group G. Suppose that the map*

$$\bigoplus_{H\in\mathcal{S}} \operatorname{ind}_H^G\colon \bigoplus_{H\in\mathcal{S}} U(H) \to U(G)$$

is surjective. Then the map

$$\bigoplus_{H\in\mathcal{S}} \operatorname{ind}_H^G\colon \bigoplus_{H\in\mathcal{S}} M(H) \to M(G)$$

is surjective.

Proof. By hypothesis there are elements $u_H \in U(H)$ for $H \in \mathcal{S}$ satisfying $1_G = \sum_{H\in\mathcal{S}} \operatorname{ind}_H^G u_H$ in $U(G)$. This implies for $x \in M(G)$.

$$x = 1_G \cdot x = \left(\sum_{H\in\mathcal{S}} \operatorname{ind}_H^G u_H\right) \cdot x = \sum_{H\in\mathcal{S}} \operatorname{ind}_H^G \left(u_H \cdot \operatorname{res}_G^H x\right).$$

□

Example 12.63 (Burnside ring). The *Burnside ring* $A(G)$ of a (not necessarily finite) group G is the commutative associative ring with unit $A(G)$ which is obtained by the additive Grothendieck construction applied to the commutative associative semiring with unit given by the G-isomorphism classes $[S]$ of G-sets S of finite cardinality, i.e., $|S| < \infty$, under disjoint union and cartesian product and the unit element given by $[G/G]$. For more information about the Burnside ring for not necessarily finite groups we refer to [637].

The Burnside ring defines a Mackey functor $A(?)$ by induction and restriction. The ring structure and the Mackey structure fit together to the structure of a Green functor $A(?)$ with values in $\mathbb{Z}$-modules.

Exercise 12.64. Let M be a Mackey functor with values in Λ-modules for an associative commutative ring Λ with unit. Let $\phi\colon \mathbb{Z} \to \Lambda$ be the unique ring homomorphism. Show that M inherits the structure of a module over the Green functor given by the Burnside ring with respect to ϕ.

Definition 12.65 (Swan ring). Let G be a (not necessarily finite) group. Let Λ be an associative commutative ring with unit. Denote by $\operatorname{Sw}^p(G;\Lambda)$ be the abelian group whose generators are the isomorphism classes $[M]$ of ΛG-modules M whose underlying Λ-module is finitely generated projective. For every short exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of such ΛG-modules, we require the relation $[M_0] - [M_1] + [M_2]$ in $\mathrm{Sw}^p(G;\Lambda)$. The tensor product over Λ with the diagonal G-action induces the structure of an associative commutative ring with unit $[\Lambda]$ where $[\Lambda]$ is the class of Λ equipped with the trivial G-action. We call $\mathrm{Sw}^p(G;\Lambda)$ the *Swan ring*. If $\Lambda = \mathbb{Z}$, we abbreviate $\mathrm{Sw}^p(G) := \mathrm{Sw}^p(G;\mathbb{Z})$.

If we replace finitely generated projective by finitely generated in the definition above, we denote the associated ring by $\mathrm{Sw}(G;\Lambda)$ and abbreviate $\mathrm{Sw}(G) := \mathrm{Sw}(G;\mathbb{Z})$.

Lemma 12.66. *The canoncial map $e\colon \mathrm{Sw}^p(G) \to \mathrm{Sw}(G)$ is an isomorphism.*

Proof. We only describe the definition of the inverse map $e^{-1}\colon \mathrm{Sw}(G) \to Sw^p(G)$, more details can be found in [782, Lemma 2.2]. Consider a $\mathbb{Z}G$-module M such that the underlying abelian group is finitely generated. Since $\mathrm{tors}(M)$ is a finite G-set, we can find an exact sequence of $\mathbb{Z}G$-modules $0 \to F_1 \to F_0 \to \mathrm{tors}(M) \to 0$ such that the underlying abelian groups of F_0 and F_1 are finitely generated free. One may take for F_0 the finitely generated free abelian group with the finite G-set $\mathrm{tors}(M)$ as $\mathbb{Z}$-basis. The $\mathbb{Z}G$-module $M/\mathrm{tors}(M)$ has as underlying abelian group a finitely generated free abelian group. We define $e^{-1}([M]) = [F_0] - [F_1] + [M/\mathrm{tors}(M)]$. □

Example 12.67 (Swan ring). Let R be an associative ring with unit. Let M be a $\mathbb{Z}G$-module whose underlying $\mathbb{Z}$-module is finitely generated free. It defines an exact functor $RG\text{-}\mathsf{MOD} \to RG\text{-}\mathsf{MOD}$ by taking the tensor product $M \otimes_{\mathbb{Z}} -$ with the diagonal G-action. It sends finitely generated free RG-modules to finitely generated free RG-modules by the following observations. We have the sheering RG-isomorphism

$$\mathrm{sh}\colon M \otimes_2 \mathbb{Z}G \xrightarrow{\cong} M \otimes_d \mathbb{Z}G, \quad m \otimes g \mapsto gm \otimes g \tag{12.68}$$

where $M \otimes_2 RG$ and $M \otimes_d RG$ are the RG-modules whose underlying R-module is $M \otimes_R RG$ and on which $g \in G$ acts by $g \cdot (m \otimes x) = m \otimes gx$ and $g \cdot (m \otimes x) = gm \otimes gx$. Obviously $M \otimes_2 RG$ is a finitely generated free RG-module since M is finitely generated free as an abelian group. If P is a finitely generated projective RG-module, then $M \otimes_{\mathbb{Z}} P$ is a finitely generated projective RG-module if $r \in R$ acts by $r \cdot (m \otimes p) = m \otimes rp$ and $g \in G$ acts by $g \cdot (m \otimes p) := gm \otimes gp$. We obtain a pairing

$$\mathrm{Sw}^p(G) \otimes K_n(RG) \to K_n(RG) \tag{12.69}$$

Using induction and restriction $\mathrm{Sw}^p(?)$ defines a Green functor with values in $\mathbb{Z}$-modules. There is a natural homomorphism of Green functors with values in $\mathbb{Z}$-modules

$$A(G) \to \mathrm{Sw}^p(G;\Lambda)$$

sending the class of a finite G-set S to the Λ-module with S as basis equipped with the G-action coming from the G-action on S. Thanks to the pairing above, the Mackey functor given by $K_n(R?)$ becomes a module over the Green functor given by $\mathrm{Sw}^p(?)$.

Example 12.70 (Rational representation ring). An important example of a Green functor with values in $\mathbb{Q}$-modules is the rationalized representation ring of rational representations $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$. It assigns to a finite group G the $\mathbb{Q}$-module $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(G)$ where $\mathrm{Rep}_{\mathbb{Q}}(G)$ denotes the rational representation ring of G. Note that $\mathrm{Rep}_{\mathbb{Q}}(G)$ is the same as the projective class group $K_0(\mathbb{Q}G)$ and also the same as $\mathrm{Sw}^p(G;\mathbb{Q})$. The Mackey structure comes from induction and restriction of representations. The pairing $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(G) \times \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(G)$ comes from the tensor product $P \otimes_{\mathbb{Q}} Q$ of two $\mathbb{Q}G$-modules P and Q equipped with the diagonal G-action. The unit element is the class of $\mathbb{Q}$ equipped with the trivial G-action.

Recall that $\mathrm{class}_{\mathbb{Q}}(G)$ denotes the $\mathbb{Q}$-vector space of functions $G \to \mathbb{Q}$ that are invariant under $\mathbb{Q}$-*conjugation*, i.e., we have $f(g_1) = f(g_2)$ for two elements $g_1, g_2 \in G$ if the cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ generated by g_1 and g_2 are conjugate in G. Elementwise multiplication defines the structure of a $\mathbb{Q}$-algebra on $\mathrm{class}_{\mathbb{Q}}(G)$ with the function that is constant 1 as unit element. Taking the character of a rational representation yields an isomorphism of $\mathbb{Q}$-algebras [895, Theorem 29 on page 102]

$$\chi^G \colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(G) \xrightarrow{\cong} \mathrm{class}_{\mathbb{Q}}(G). \tag{12.71}$$

We define a Mackey structure on $\mathrm{class}_{\mathbb{Q}}(?)$ as follows. Let $f \colon H \to G$ be an injective group homomorphism. For a character $\chi \in \mathrm{class}_{\mathbb{Q}}(H)$ define its induction with f to be the character $\mathrm{ind}_f(\chi) \in \mathrm{class}_{\mathbb{Q}}(G)$ given by

$$\mathrm{ind}_f(\chi)(g) = \frac{1}{|H|} \cdot \sum_{\substack{l \in G, h \in H \\ f(h) = l^{-1}gl}} \chi(h).$$

For a character $\chi \in \mathrm{class}_{\mathbb{Q}}(G)$ define its restriction with f to be the character $\mathrm{res}_f(\chi) \in \mathrm{class}_{\mathbb{Q}}(H)$ given by

$$\mathrm{res}_f(\chi)(h) := \chi(f(h)).$$

One easily checks that this yields the structure of a Green functor on $\mathrm{class}_{\mathbb{Q}}(?)$ and that the family of isomorphisms χ^G defined in (12.71) yields an isomorphism of Green functors from $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}}(?)$ to $\mathrm{class}_{\mathbb{Q}}(?)$.

12.8.2 Induction Lemmas

As already explained by Lemma 12.62, Green functors play a prominent role for induction theorems. In order to formulate two further versions, we have to introduce the following idempotents.

Let G be a finite group. There is a ring homomorphism

$$\text{card}\colon A(G) \to \prod_{H} \mathbb{Z}, \quad [S] \mapsto (|S^H|)_{(H)} \tag{12.72}$$

where the product is indexed over the conjugacy classes of subgroups of G and $|S^H|$ is the cardinality of the H-fixed point set. The ring homomorphism card is injective and has a finite cokernel. In particular it induces an isomorphism of $\mathbb{Q}$-algebras

$$\text{card}_{\mathbb{Q}}\colon \mathbb{Q} \otimes_{\mathbb{Z}} A(G) \xrightarrow{\cong} \prod_{(H)} \mathbb{Q}.$$

Now let $e_G \in \prod_{(H)} \mathbb{Q}$ be the idempotent whose value at (G) is 1 and whose value at (H) for $H \neq G$ is 0. We then define the idempotent

$$\Theta_G := \text{card}_{\mathbb{Q}}^{-1}(e_G) \quad \in \mathbb{Q} \otimes_{\mathbb{Z}} A(G). \tag{12.73}$$

For a finite cyclic group C, define the idempotent

$$\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C) \tag{12.74}$$

to be the element whose image under the isomorphism of (12.71) is the class function that sends an elements of C to 1 if it is a generator, and to 0 otherwise. The image of Θ_C under the map $\mathbb{Q} \otimes_{\mathbb{Z}} A(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(C)$ that sends a finite C-set S to the associated permutation module $\mathbb{Q}[S]$ is θ_C.

Lemma 12.75. *Let $\phi\colon \mathbb{Q} \to \Lambda$ be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in Λ-modules which is a module over the Green functor $\mathbb{Q} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{Q}}(H)$ with respect to ϕ. Then*

(i) For a finite group H the map

$$\bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} \text{ind}_C^H : \bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} M(C) \to M(H)$$

is surjective;

(ii) Let C be a finite cyclic group. Let

$$\theta_C\colon M(C) \to M(C)$$

be the map induced by the $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$*-module structure and multiplication with idempotent* θ_C *of* (12.74). *Then the inclusion of the image of* $\theta_C : M(C) \to M(C)$ *into* $M(C)$ *composed with the projection onto the cokernel of*

$$\bigoplus_{\substack{D \subset C \\ D \neq C}} \operatorname{ind}_D^C : \bigoplus_{\substack{D \subset C \\ D \neq C}} M(D) \to M(C)$$

is an isomorphism.

Proof. Let $C \subset H$ be a cyclic subgroup of the finite group H. Then we get for $h \in H$

$$\frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C(h) = \frac{1}{[H:C]} \cdot \frac{1}{|C|} \cdot \sum_{\substack{l \in H \\ l^{-1}hl \in C}} \theta_C(l^{-1}hl) = \frac{1}{|H|} \cdot \sum_{\substack{l \in H \\ \langle l^{-1}hl \rangle = C}} 1.$$

This implies in $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(H) \cong \operatorname{class}_{\mathbb{Q}}(H)$

$$1_H = \sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C \tag{12.76}$$

since for any $l \in H$ and $h \in H$ there is precisely one cyclic subgroup $C \subset H$ with $C = \langle l^{-1}hl \rangle$. Now assertion (i) follows from the following calculation for $x \in M(H)$

$$x = 1_H \cdot x = \left(\sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C \right) \cdot x = \sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H (\theta_C \cdot \operatorname{res}_H^C x).$$

It remains to prove assertion (ii). Obviously θ_C is an idempotent for any cyclic group C. We get for $x \in M(C)$ from (12.76)

$$(1_C - \theta_C) \cdot x = \left(\sum_{\substack{D \subset C \\ D \neq C}} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D \right) \cdot x = \sum_{\substack{D \subset C \\ D \neq C}} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C (\theta_D \cdot \operatorname{res}_C^D x)$$

and for $D \subset C, D \neq C$ and $y \in M(D)$

$$\theta_C \cdot \operatorname{ind}_D^C y = \operatorname{ind}_D^C (\operatorname{res}_C^D \theta_C \cdot y) = \operatorname{ind}_D^C (0 \cdot y) = 0.$$

This finishes the proof of Lemma 12.75. □

The proof of the next result is similar to the one of Lemma 12.75. Details can be found in [658, Lemma 7.2 and Lemma 7.4]. Key ingredients are Lemma 12.62, Example 12.67, and the result of Swan [919, Corollary 4.2 on page 560] which implies together with [782, page 890] that for every finite group H the cokernel of the map

$$\bigoplus_{\substack{C\subseteq H,\\ C \text{ cyclic}}} \operatorname{ind}_C^H \colon \bigoplus_{\substack{C\subseteq H\\ C \text{ cyclic}}} \operatorname{Sw}^p(C) \to \operatorname{Sw}^p(H)$$

is annihilated by $|H|^2$.

Lemma 12.77. *Let R be an associative ring with unit. Then*

(i) For a finite group H and $n \in \mathbb{Z}$ the map

$$\bigoplus_{\substack{C\subset H\\ C \text{ cyclic}}} \operatorname{ind}_C^H : \bigoplus_{\substack{C\subset H\\ C \text{ cyclic}}} \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RH)$$

is surjective;

(ii) Let C be a finite cyclic group. Let

$$\Theta_C \colon \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$$

be the map induced by the $\mathbb{Q} \otimes_{\mathbb{Z}} A(C)$-module structure and multiplication with the idempotent θ_C of (12.73). *Then the inclusion of the image of the map $\theta_C \colon \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$ into $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$ with the projection onto the cokernel of*

$$\bigoplus_{\substack{D\subset C\\ D\neq C}} \operatorname{ind}_D^C : \bigoplus_{\substack{D\subset C\\ D\neq C}} M(D) \to M(C)$$

is an isomorphism.

Remark 12.78 (L-theory analogue of Lemma 12.77). The L-theory analogue of Lemma 12.77 is also true, one has to use instead of Swan [919, Corollary 4.2 on page 560] the corresponding L-theory analogue of Dress [301, Theorem 2(a)].

For more information about Mackey and Green functors and induction theorems we refer for instance to [934, Section 6], [301] and [75].

12.8.3 Rational Computation of the Source of the Assembly Maps

Theorem 12.79 (Rational computation of the source of the assembly maps appearing in the Farrell-Jones and Baum-Connes Conjecture). *Let R be an associative ring with unit and let F be $\mathbb{R}$ or $\mathbb{C}$. Let G be a group. Denote by J be the set of conjugacy classes (C) of finite cyclic subgroups C of G.*

Then the rational Chern character of Theorem 12.58 induces isomorphisms

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \bigl(\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC)\bigr) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{K}_R)$$

and

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \bigl(\mathbb{Q} \otimes_{\mathbb{Z}} L_q^{\langle -\infty\rangle}(RC)\bigr) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}).$$

and

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \bigl(\mathbb{Q} \otimes_{\mathbb{Z}} K_q(C_r^*(C,F))\bigr) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{K}_F^{\mathsf{TOP}}).$$

Proof. This follows from Example 12.56, Theorem 12.58, Lemma 12.77, and Remark 12.78. □

Computations of $K_q(RC)$ as $\mathbb{Z}[\mathrm{aut}(C)]$-module for finite cyclic groups C and $R = \mathbb{Z}$ or R a field of characteristic zero can be found in [777].

The computations simplifies even more if we consider the case $R = \mathbb{C}$, as the following example, which is taken from [634, Example 8.11], shows.

Example 12.80 (Complex coefficients). Let T be the set of conjugacy classes (g) of elements $g \in G$. If we tensor with $\mathbb{C}$ instead of $\mathbb{Q}$ and take $R = F = \mathbb{C}$, then the isomorphism appearing in Theorem 12.79 reduce to the isomorphisms

$$\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{C}});$$

$$\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} L_q(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle});$$

$$\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathsf{TOP}}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}}),$$

where we use in the definition of $L_q(\mathbb{C})$ and $L_n(\mathbb{C}G)$ the involutions coming from complex conjugation. The targets of the maps above are isomorphic to $\mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G)$, $\mathbb{C} \otimes_{\mathbb{Z}} L_n^{\langle -\infty\rangle}(\mathbb{C}G)$, and $\mathbb{C} \otimes_{\mathbb{Z}} K_n(C_r^*(G,\mathbb{C})$ if the Farrell-Jones Conjecture and the Baum-Connes Conjecture hold for G.

12.9 Some Integral Computations

Integral computations are of course harder than rational computations. We have already provided basic tools such as the equivariant Atiyah-Hirzebruch spectral sequence and the p-chain spectral sequence in Section 12.6.

Often we are considering an equivariant homology theory and want to compute $\mathcal{H}_n^G(\underline{E}G)$ or $\mathcal{H}_n^G(\underline{\underline{E}}G)$. Sometimes one gets easy and useful computations if one has good models for $\underline{E}G$ and $\underline{\underline{E}}G$. We illustrate this in the following favorite case.

Let G be a discrete group. Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ that are maximal with respect to inclusion in $\mathcal{FIN}$. Throughout this subsection we suppose that G satisfies the conditions ($\underline{\text{M}}$) and ($\underline{\text{NM}}$) introduced in Subsection 11.6.12, where also examples of such groups G are given. Let $\{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal finite subgroups of G. Consider an equivariant homology theory $\mathcal{H}_*^?$. Recall that we put $\underline{B}G = G\backslash\underline{E}G$.

Then we obtain from Theorem 11.32 long exact sequences

$$(12.81)\qquad \cdots \to \bigoplus_{i\in I} \mathcal{H}_n^{\{1\}}(BM_i) \to \mathcal{H}_n^{\{1\}}(BG) \oplus \bigoplus_{i\in I} \mathcal{H}_n^{M_i}(\{\bullet\}) \to \mathcal{H}_n^G(\underline{E}G)$$
$$\bigoplus_{i\in I} \mathcal{H}_{n-1}^{\{1\}}(BM_i) \to \mathcal{H}_{n-1}^{\{1\}}(BG) \oplus \bigoplus_{i\in I} \mathcal{H}_{n-1}^{M_i}(\{\bullet\}) \to \cdots .$$

$$(12.82)\qquad \cdots \to \bigoplus_{i\in I} \mathcal{H}_n^{\{1\}}(BM_i) \to \mathcal{H}_n^{\{1\}}(BG) \oplus \bigoplus_{i\in I} \mathcal{H}_n^{\{1\}}(\{\bullet\}) \to \mathcal{H}_n^{\{1\}}(\underline{B}G)$$
$$\bigoplus_{i\in I} \mathcal{H}_{n-1}^{\{1\}}(BM_i) \to \mathcal{H}_{n-1}^{\{1\}}(BG) \oplus \bigoplus_{i\in I} \mathcal{H}_{n-1}^{\{1\}}(\{\bullet\}) \to \cdots .$$

We have the maps $\mathcal{H}_n^{\{1\}}(\{\bullet\}) \to \mathcal{H}_n^{M_i}(\{\bullet\})$ induced by the inclusion $\{1\} \to M_i$ and $\mathcal{H}_n^{M_i}(\{\bullet\}) \to \mathcal{H}_n^{\{1\}}(\{\bullet\})$ induced by the projection $M_i \to \{1\}$. The composite is the identity. Define

$$(12.83)\qquad \widetilde{\mathcal{H}}_n^{M_i}(\{\bullet\}) := \ker\bigl(\mathcal{H}_n^{M_i}(\{\bullet\}) \to \mathcal{H}_n^{\{1\}}(\{\bullet\})\bigr).$$

Obviously we have an isomorphism

$$\mathcal{H}_n^{M_i}(\{\bullet\}) \cong \mathcal{H}_n^{\{1\}}(\{\bullet\}) \oplus \widetilde{\mathcal{H}}_n^{M_i}(\{\bullet\}).$$

One can splice the two long exact sequences (12.81) and (12.82) together to the long exact sequence

$$\begin{aligned}(12.84)\quad \cdots \to \mathcal{H}_{n+1}^{\{1\}}(\underline{B}G) \to \bigoplus_{i\in I} \widetilde{\mathcal{H}}_n^{M_i}(\{\bullet\}) \to \mathcal{H}_n^G(\underline{E}G) \to \\ \to \mathcal{H}_n^{\{1\}}(\underline{B}G) \to \bigoplus_{i\in I} \widetilde{\mathcal{H}}_{n-1}^{M_i}(\{\bullet\}) \to \cdots .\end{aligned}$$

The long exact sequence (12.84) splits after applying $-\otimes_{\mathbb{Z}} \Lambda$, more precisely, $\mathcal{H}_n^G(\underline{E}G)\otimes_{\mathbb{Z}}\Lambda \to \mathcal{H}_n^{\{1\}}(\underline{B}G)\otimes_{\mathbb{Z}}\Lambda$ is split surjective, see Lemma 12.18 (ii).

Example 12.85 (Equivariant topological K-theory of $\underline{E}G$ for $G = \mathbb{Z}^2 \rtimes \mathbb{Z}/4$). Consider the automorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2, \quad (x,y) \mapsto (-y,x)$. It has order four. We want to show for the semidirect product $G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}/4$

$$K_n^G(\underline{E}G) \cong \begin{cases} \mathbb{Z}^9 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In this case we have the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_4 = \langle u, v, t \mid t^4 = 1,\ uv = vu,\ tut^{-1} = v,\ tvt^{-1} = u^{-1} \rangle.$$

The maximal finite subgroups are up to conjugacy given by

$$\begin{aligned} M_0 &= \langle t \rangle; \\ M_1 &= \langle ut \rangle; \\ M_2 &= \langle ut^2 \rangle. \end{aligned}$$

We have $M_0 \cong M_1 \cong \mathbb{Z}_4$ and $M_2 \cong \mathbb{Z}_2$. We obtain

$$\widetilde{K}_n^{\mathbb{Z}/m}(\{\bullet\}) \cong \begin{cases} \mathbb{Z}^{m-1} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Obviously $\underline{B}G$ is the same as $\mathbb{Z}/4\backslash T^2$ for the obvious $\mathbb{Z}/4$-action on the two-dimensional torus $T^2 = \mathbb{Z}^2\backslash \underline{E}G = \mathbb{Z}^2\backslash E\mathbb{Z}^2$. This implies because we are in dimension two, that $\underline{B}G$ has a model which is a compact 2-dimensional manifold. The rational cohomology $H^*(\underline{B}G)$ agrees with $H^*(T^2;\mathbb{Q})^{\mathbb{Z}/4}$. Since $\mathbb{Z}/4$ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, its action on T^2 is orientation preserving. This implies that $\mathbb{Z}/4$ acts trivial on $H^p(T^2;\mathbb{Q})$ for $p = 0, 2$. Since $\mathbb{Z}/4$ acts freely on $\mathbb{Z}^2 = H_1(T^2;\mathbb{Z})$ outside $\{0\}$, we conclude $H^1(T^2;\mathbb{Q})^{\mathbb{Z}/4} \cong \hom_{\mathbb{Z}}(H_1(T^2;\mathbb{Z})^{\mathbb{Z}/4}, \mathbb{Q}) \cong \{0\}$. We conclude that $\underline{B}G = \mathbb{Z}/4\backslash T^2$ has the rational cohomology of S^2 and hence is homeomorphic to S^2. This implies that $K_0(\underline{B}G) \cong \mathbb{Z}^2$ and $K_1(\underline{B}G) = 0$.

The group G satisfies conditions ($\underline{\text{M}}$) and ($\underline{\text{NM}}$) by a direct check or because of Subsection 11.6.12 since the $\mathbb{Z}/4$ action on $\mathbb{Z}^2$ given by α is free outside 0. Now the claim follows from the long exact sequence (12.84) applied in the case $\mathcal{H}_*^? = K_*^?$.

Since G satisfies the Baum-Connes Conjecture, we have $K_n(C_r^*(G)) \cong K_n^G(\underline{E}G)$.

Exercise 12.86. Determine all finite subgroups $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ and compute for any of these $K_n^G(\underline{E}G)$ for $n \in \mathbb{Z}$ and $G = \mathbb{Z}^2 \rtimes F$.

The long exact sequence (12.84) will be a key ingredient in computations of $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$, and $K_n(C_r^*(G))$, provided that G satisfies the Farrell-Jones Conjecture and the Baum-Connes Conjecture, see Theorem 17.12.

Already for group homology the long exact sequence (12.84) contains valuable information as we explain next.

Example 12.87 (Group homology). Suppose that G satisfies ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$). Let $\mathcal{H}_*$ be given by the Borel homology, i.e., $\mathcal{H}^G(X) := H_n(EG \times X)$ for H_n singular homology with coefficients in $\mathbb{Z}$, see Example 12.13. Then (12.84) reduces to the long exact sequence where $H_n(G) := H_n(BG)$ is the group homology and $\widetilde{H}_n(G) := \ker(H_n(G) \to H_n(\{1\})$

$$\cdots \to H_{n+1}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{H}_n(M_i) \to H_n(G)$$
$$\to H_n(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{H}_{n-1}(M_i) \to \cdots .$$

In particular we get for $n \geq \dim(\underline{B}G) + 2$ an isomorphism

$$\bigoplus_{i \in I} H_n(M_i) \xrightarrow{\cong} H_n(G).$$

Example 12.88 (The group homology of certain extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F). Consider an extension $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$. Then the conditions ($\underline{\mathrm{M}}$) and ($\underline{\mathrm{NM}}$) are satisfied by [667, Lemma 6.3], and there is an n-dimensional model for $\underline{E}G$ whose underlying space is $\mathbb{R}^2$.

Even in the case where F is a finite cyclic group, the computation of the homology of G is not at all easy. It is carried in [267, Theorem 2.1] provided that $|F|$ is a prime. More information in the case where there are no restriction on $|F|$, can be found in [603].

Based on the material of this section, we will compute the group homology of one-relators groups in Lemma 17.21 (iii) and Lemma 17.27.

12.10 Equivariant Homology Theory over a Group and Twisting with Coefficients

Next we present a slight variation of the notion of an equivariant homology theory introduced in Section 12.3. We have to treat this variation since we later want to study coefficients over a fixed group Γ which we will then pullback via group homomorphisms with Γ as target. For instance, we may be interested in the algebraic K-theory of a twisted groups ring $R_\alpha G$ for some homomorphism $\alpha\colon G \to \operatorname{aut}(R)$. More generally, we will later consider additive G-categories as coefficients.

Fix a group Γ. A group (G,ξ) over Γ is a group G together with a group homomorphism $\xi\colon G \to \Gamma$. A map $\alpha\colon (G_1,\xi_1) \to (G_2,\xi_2)$ of groups over Γ is a group homomorphisms $\alpha\colon G_1 \to G_2$ satisfying $\xi_2 \circ \alpha = \xi_1$. Let Λ be an associative commutative ring with unit.

Definition 12.89 (Equivariant homology theory over a group Γ). An *equivariant homology theory $\mathcal{H}_*^{?\downarrow\Gamma}$ with values in Λ-modules over a group Γ* assigns to every group (G,ξ) over Γ a G-homology theory $\mathcal{H}_*^{G,\xi}$ with values in Λ-modules and comes with the following so-called *induction structure*: given a homomorphism $\alpha\colon (H,\mu) \to (G,\xi)$ of groups over Γ and an H-CW-pair (X,A), there are for each $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^{H,\mu}(X,A) \to \mathcal{H}_n^{G,\xi}(\alpha_*(X,A)) \tag{12.90}$$

satisfying

- Compatibility with the boundary homomorphisms
  $\partial_n^{G,\xi} \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^{H,\mu}$;
- Functoriality
  Let $\beta\colon (G,\xi) \to (K,\nu)$ be another morphism of groups over Γ. Then we have for $n \in \mathbb{Z}$
  $$\operatorname{ind}_{\beta\circ\alpha} = \mathcal{H}_n^{K,\nu}(f_1) \circ \operatorname{ind}_\beta \circ \operatorname{ind}_\alpha\colon \mathcal{H}^{H,\mu}H_n(X,A) \to \mathcal{H}_n^{K,\nu}((\beta\circ\alpha)_*(X,A))$$
  where $f_1\colon \beta_*\alpha_*(X,A) \xrightarrow{\cong} (\beta\circ\alpha)_*(X,A), \quad (k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism;
- Compatibility with conjugation
  Let (G,ξ) be a group over Γ. Fix $g \in G$ such that $\xi \circ c(g) = \xi$. Then the conjugation homomorphisms $c(g)\colon G \to G$ defines a morphism $c(g)\colon (G,\xi) \to (G,\xi)$ of groups over Γ. Let $f_2\colon (X,A) \to c(g)_*(X,A)$ be the G-homeomorphism that sends x to $(1,g^{-1}x)$ in $G \times_{c(g)} (X,A)$.
  Then for every $n \in \mathbb{Z}$ and every G-CW-pair (X,A) the homomorphism $\operatorname{ind}_{c(g)}\colon \mathcal{H}_n^{G,\xi}(X,A) \to \mathcal{H}_n^{G,\xi}(c(g)_*(X,A))$ agrees with $\mathcal{H}_n^G(f_2)$;
- Bijectivity

If $\ker(\alpha)$ acts freely on $X \setminus A$, then $\operatorname{ind}_\alpha \colon \mathcal{H}_n^{H,\mu}(X,A) \to \mathcal{H}_n^{G,\xi}(\operatorname{ind}_\alpha(X,A))$ is bijective for all $n \in \mathbb{Z}$.

Definition 12.89 reduces to Definition 12.9 if one puts $\Gamma = \{1\}$.
The analog of Lemma 12.12 in this setting is obvious and easily checked.
The proof of Theorem 12.30 to this setting as explained in [70, Lemma 7.1].

Theorem 12.91 (Constructing equivariant homology theories over a group using spectra). *Let Γ be a group. Denote by $\mathsf{GROUPOIDS} \downarrow \Gamma$ the category of small connected groupoids over Γ considered as a groupoid with one object. Consider a covariant functor*

$$\mathbf{E} \colon \mathsf{GROUPOIDS} \downarrow \Gamma \to \mathsf{SPECTRA}$$

that sends equivalences of groupoids to weak equivalences of spectra.

Then we can associate to it an equivariant homology theory $\mathcal{H}_*^{?\downarrow\Gamma}(-;\mathbf{E})$ with values in $\mathbb{Z}$-modules over Γ such that for every group (G,μ) over Γ and subgroup $H \subseteq G$ we have a natural identification

$$\mathcal{H}_n^{H,\xi|_H}(\{\bullet\};\mathbf{E}) = \mathcal{H}_n^{G,\xi}(G/H,\mathbf{E}) = \pi_n(\mathbf{E}(H,\xi|_H)).$$

There are obvious twisted analogues of the functors mentioned in Section 12.5, see (13.10) together with Remark 13.12 and (13.15) together with Remark 13.17, and also [70, Theorem 6.1].

Remark 12.92. Equivariant Chern characters have only be constructed for equivariant homology theories but not for the more general notion of an equivariant homology theory over a group Γ. It is conceivable they exist, provided that the coefficients of the homology theory $\mathcal{H}_*^?$ over Γ extend to a Mackey functor over Γ, where we leave it to the reader to figure out what the latter condition means. For this claim there are many details to be checked and we have not done this. It seems also to be plausible that the equivariant homology theories over a group Γ given by the algebraic K and L-theory for a ring (with involution) coming with a homomorphism $\Gamma \to \operatorname{aut}(R)$ do have the property that the coefficients of the homology theories $\mathcal{H}_*^?(-;\mathbf{K}_R)$ and $\mathcal{H}_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle})$ over Γ extend to a Mackey functor over Γ and hence that there exists equivariant Chern characters for them.

Remark 12.93. Note that the proof of Lemma 12.18 (ii) does not extend to an equivariant homology theory over a non-trivial group Γ because we cannot pass to the quotient by G anymore. However, if the coefficients of the homology theory $\mathcal{H}_*^?$ over Γ extend to a Mackey functor over Γ and we do have an equivariant Chern character, then it is still true that the map $\mathcal{H}_n^{G,\xi}(EG) \to \mathcal{H}_n^{G,\xi}(\underline{E}G)$ is rationally injective for every $n \in \mathbb{Z}$ and every group $\xi \colon G \to \Gamma$ over Γ. So this would yield the rational injectivity of the maps

$$H_n^{G,\xi}(EG;\mathbf{K}_R) \to H_n^{G,\xi}(\underline{E}G;\mathbf{K}_R);$$
$$H_n^{G,\xi}(EG;\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^{G,\xi}(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}),$$

for every $n \in \mathbb{Z}$ and every $\xi\colon G \to \Gamma$. The reader should note that we have proved this only in the case $\Gamma = \{1\}$, see Lemma 12.18 (ii).

Exercise 12.94. Let Γ be a group. Let R be a ring with a homomorphism $\alpha\colon \Gamma \to \operatorname{aut}(R)$. Let $\xi\colon G \to \Gamma$ be a group over Γ such that G is finite.

Show that the map $H_n^{G,\xi}(EG;\mathbf{K}_R) \to H_n^{G,\xi}(\underline{E}G;\mathbf{K}_R) = K_n(R_{\alpha\circ\xi}G)$ is rationally injective for every $n \in \mathbb{Z}$.

12.11 Notes

Equivariant stable cohomotopy has been introduced in [637] for arbitrary groups G and proper finite G-CW-complexes and extended to proper G-CW-complexes in [281, Example 3.43 on page 107]. A version of the Segal Conjecture in this setting is proved in [649]. A systematic study of the equivariant homotopy category for proper G-CW-complex can be found in [281]. There it is explained in [281, Remark 3.44 on page 107] that the classical notion of an $RO(G)$-grading is taken over by a kind of $K_G^0(\underline{E}G)$-grading.

If one is dealing with equivariant topological K-theory, then there exists a Chern character where one does not have to fully rationalize, it suffices to invert the orders of all the isotropy groups of the proper G-CW-complex under consideration, see [636].

There are also equivariant cohomology theories and a cohomological version of the equivariant Chern character, see [638]. It can be used to extend the Atiyah-Segal Completion Theorem for finite groups to infinite groups and proper G-CW-complexes, see [655, 656]. It leads to rational computations of $K^*(BG)$ also for not necessarily finite groups, see [508, 641].

An equivariant Chern character for equivariant topological K-theory after complexification has been introduced in [101].

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Chapter 13
The Farrell-Jones Conjecture

13.1 Introduction

In this chapter we discuss the Farrell-Jones Conjecture for K- and L-theory for arbitrary groups and rings. It predicts that certain assembly maps

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \to K_n(RG);$$
$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to L_n^{\langle -\infty\rangle}(RG),$$

are bijective for all $n \in \mathbb{Z}$. The targets are the algebraic K- or L-groups of the group ring RG, which one wants to understand. The source is an expression that depends only on the values of these K- and L-groups on virtually cyclic subgroups of G and is therefore much more accessible. The version above is often the one which is relevant in concrete applications, but nevertheless we will consider generalizations, for instance to twisted group rings and twisted involutions. The both most general and most important version will be the Full Farrell-Jones Conjecture 13.27. It implies all other variants of the Farrell-Jones Conjecture which appear in this book, see Section 13.11, It has very nice inheritance properties, see Section 13.7, which are in general not shared by the other variants.

A status report of the Full Farrell-Jones Conjecture 13.27 will be given in Theorem 16.1. It is known for a large class of groups.

The main point about the Full Farrell-Jones Conjecture 13.27 is that it implies a great variety of other prominent conjectures such as the ones due to Bass, Borel, Kaplanski, and Novikov, and leads to very deep and interesting results about manifolds and groups, as we will record and explain in Section 13.12. Often these applications are much more appealing and easier to comprehend than the rather technical Full Farrell-Jones Conjecture 13.27. The author's favorite is the Borel Conjecture, which predicts that two aspherical closed topological manifolds are homeomorphic if and only if their fundamental groups are isomorphic and any homotopy equivalence between them is homotopic to a homeomorphism.

Section 13.10 deals with the question whether one can reduce the family of virtually cyclic subgroups to a smaller family of subgroups, for instance to all finite subgroups or just to the family consisting of the trivial subgroup. Section 13.13 presents of a short discussion of G-theory.

We have tried to keep this chapter as much as possible independent of the other chapters, so that one may start reading directly here.

13.2 The Farrell-Jones Conjecture with Coefficients in Rings

Let G be a (discrete) group. Recall that a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules for some commutative associative unital ring Λ assigns to every G-CW-pair (X,A) and integer $n \in \mathbb{Z}$ a Λ-module $\mathcal{H}_n^G(X,A)$ such that the obvious generalization to G-CW-pairs of the axioms of a (non-equivariant generalized) homology theory for CW-complexes holds, i.e., G-homotopy invariance, the long exact sequence of a G-CW-pair, excision, and the disjoint union axiom are satisfied. The precise definition of a G-homology theory can be found in Definition 12.1 and of a G-CW-complex in Definition 11.2, see also Remark 11.3.

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, to be a set of subgroups of G that is closed under conjugation with elements of G and passing to subgroups. Denote by $E_{\mathcal{F}}(G)$ a model for the classifying space for the family $\mathcal{F}$ of subgroups of G, i.e., a G-CW-complex $E_{\mathcal{F}}(G)$ whose isotropy groups belong to $\mathcal{F}$ and for which for each $H \in \mathcal{F}$ the H-fixed point set $E_{\mathcal{F}}(G)^H$ is weakly contractible. Such a model always exists and is unique up to G-homotopy, see Definition 11.18 and Theorem 11.19. Recall that $\underline{E}G$ and $\underline{\underline{E}}G$ are abbreviations for $E_{\mathcal{FIN}}(G)$ and $E_{\mathcal{VCY}}(G)$ where $\mathcal{FIN}$ is the family of finite subgroups and $\mathcal{VCY}$ is the family of virtually cyclic subgroups, i.e., subgroups that are either finite or contain $\mathbb{Z}$ as a subgroup of finite index.

13.2.1 The *K*-Theoretic Farrell-Jones Conjecture with Coefficients in Rings

Given a ring R, there is a specific G-homology theory $H_n^G(-;\mathbf{K}_R)$ with values in $\mathbb{Z}$-modules which has the property that $H_n^G(G/H;\mathbf{K}_R) \cong K_n(RH)$ holds for all $n \in \mathbb{Z}$ and subgroups $H \subseteq G$, where $K_n(RH)$ is the nth algebraic K-group of the group ring RH. Its construction can be used in the sequel as a black box. We have already given some details, namely, it is given by the equivariant homology theory $H_*^?(-;\mathbf{K}_R)$ evaluated at G that is associated to the covariant functor $\mathbf{K}_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ of (12.44) in Theorem 12.30.

Conjecture 13.1 (K-theoretic Farrell-Jones Conjecture with coefficients in the ring R). Given a group G and a ring R, we say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in the ring R* if the assembly map induced by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \to H_n^G(G/G;\mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

In many of the proofs the coefficients rings do not play a role, and therefore it is reasonable to consider the following stronger variant that is now a statement about the group G itself.

Conjecture 13.2 (K-theoretic Farrell-Jones Conjecture with coefficients in rings). We say that the group G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in rings* if the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in R holds for every ring R.

Exercise 13.3. Show that Conjecture 13.2 does not hold for $G = \mathbb{Z}$ if one replaces $\mathcal{VCY}$ by $\mathcal{FIN}$ in Conjecture 13.1.

Conjecture 13.2 makes also sense for twisted group rings $R_\alpha G$, see Remark 13.12.

13.2.2 The *L*-Theoretic Farrell-Jones Conjecture with Coefficients in Rings

The situation for L-theory is similar. Namely, given a ring with involution R, there is a specific G-homology theory $H_n^G(-;\mathbf{L}_R^{\langle -\infty \rangle})$ with values in $\mathbb{Z}$-modules that has the property that $H_n^G(G/H;\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RH)$ holds for all $n \in \mathbb{Z}$ and subgroups $H \subseteq G$, where $L_n^{\langle -\infty \rangle}(RH)$ is the nth algebraic L-groups of the group ring with involution RH with decoration $\langle -\infty \rangle$. Its construction can be used in the sequel as a black box. We have already given some details, namely, it is given by the equivariant homology theory $H_*^?(-;\mathbf{L}_R^{\langle -\infty \rangle})$ evaluated at G that is associated to the covariant functor $\mathbf{L}_R^{\langle -\infty \rangle} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ of (12.45) in Theorem 12.30.

Conjecture 13.4 (L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R). Given a group G and ring with involution R, we say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R* if the assembly map induced by the projection $E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(\mathrm{pr}) \colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Exercise 13.5. Show that Conjecture 13.4 holds for $G = \mathbb{Z}$ if one replaces $\mathcal{VCY}$ by $\mathcal{FIN}$.

If we invert 2, it is expected that one can replace $\mathcal{VCY}$ by $\mathcal{FIN}$ in general.

Conjecture 13.6 (L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R after inverting 2). Given a group G and ring with involution R, we say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R after inverting* 2 if the assembly map induced by the projection $E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$ after inverting 2.

Conjecture 13.7 (L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution). A group G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution* if the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R holds for every ring with involution R.

Conjecture 13.8 (L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting 2). We say that a group G satisfies *the L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting* 2 if the L-theoretic Farrell-Jones Conjecture 13.6 with coefficients in the ring with involution R after inverting 2 holds for every ring with involution R.

Remark 13.9 (The decoration $\langle -\infty \rangle$ is necessary). One can define for any decoration $j \in \{n \in \mathbb{Z} \mid n \leq 1\} \amalg \{-\infty\}$ an assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle j \rangle}) \to H_n^G(G/G; \mathbf{L}_R^{\langle j \rangle}) = L_n^{\langle j \rangle}(RG).$$

But in general one can only hope that it is bijective if one chooses $j = -\infty$. Counterexamples for $G = \mathbb{Z}^2 \times F$ for a finite group F, $R = \mathbb{Z}$ and $j = -1, 0, 1$, which is also sometimes denoted by $j = p, h, s$, are constructed in [356].

If we invert 2, the decorations do not play a role because of the Rothenberg sequences, see Subsection 9.10.4.

Conjectures 13.7 and 13.8 makes also sense for twisted group rings $R_\alpha G$, see Remark 13.17.

13.3 The Farrell-Jones Conjecture with Coefficients in Additive Categories

There are situations where one wants to consider *twisted groups rings* $R_\alpha G$, sometimes also denoted by $R_\alpha[G]$, for some group homomorphism $\alpha\colon G \to \mathrm{aut}(R)$ to the group of ring automorphisms of R. Elements in $R_\alpha G$ are given by formal finite sums $\sum_{g \in G} r_g \cdot g$, and addition and multiplication is given by

$$\left(\sum_{g\in G} r_g \cdot g\right) + \left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g\in G} (r_g + s_g) \cdot g;$$
$$\left(\sum_{g\in G} r_g \cdot g\right) \cdot \left(\sum_{g\in G} s_g \cdot g\right) := \sum_{g\in G} \left(\sum_{\substack{h,k\in G,\\ g=hk}} r_h \cdot \alpha(h)(s_k)\right) \cdot g.$$

So the decisive relation for the multiplication is $(r\cdot h)\cdot(s\cdot k) = (r\cdot\alpha(h)(s))\cdot hk$. Or even, more generally, one may want to consider crossed product rings, see for instance [76, Section 4].

When considering L-theory, one considers a ring with involution R and wants to allow to twist the involution on RG by an orientation homomorphism $w\colon G \to \operatorname{center}(R)$ satisfying $\overline{w(g)} = w(g)$ resulting in the *w-twisted involution on RG* is given by

$$\overline{\sum_{g\in G} r_g \cdot g} := \sum_{g\in G} w(g) \cdot \overline{r_g} \cdot g^{-1}.$$

The situation becomes even more involved if one wants to consider crossed product rings with involution. Details are explained in [76, Section 4].

It turns out that one can nicely treat these generalization of group rings and involutions by looking at additive G-categories (with involution).

There is another crucial reason why it is useful to look at coefficients in additive G-categories (with involution). These versions of the Farrell-Jones Conjecture with coefficients in additive G-categories (with involution) have much better inheritance properties than the one with coefficients in rings (with involution) as we will explain below in Section 13.7, for instance they pass to subgroups.

The details are given for additive G-categories and K-theory in [90], the case of additive G-categories with involution is treated for the K-theory taking the involution into account and for L-theory in [76]. Since we can use this general approach essentially as a black box, we give only a brief summary here, following the notation of [76].

13.3.1 The K-theoretic Farrell-Jones Conjecture with Coefficients in Additive G-Categories

Let $\mathcal{A}$ be an additive G-category in the sense of [76, Definition 2.1], i.e., an additive categories with G-action by functors of additive categories. Note that we use left actions here, whereas in [76] right actions are considered. Let $\mathsf{GROUPOIDS} \downarrow G$ be the category of connected groupoids over $I(G)$. Recall that for a group G we denote by $I(G)$ the groupoid with one object and G as its automorphism group. We obtain from [76, Section 5] a contravariant functor to the category ADDCAT of small additive categories

$$\mathsf{GROUPOIDS} \downarrow G \to \mathsf{ADDCAT}, \quad \mathrm{pr}\colon \mathcal{G} \to I(G) \ \mapsto \ \int_{\mathcal{G}} \mathcal{A} \circ \mathrm{pr}\,.$$

Composing it with the functor sending an additive category to its non-connective K-theory spectrum, see for instance [197, 668, 783], yields a functor

$$\mathbf{K}_{\mathcal{A}}\colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA}. \tag{13.10}$$

By Theorem 12.91 we obtain an equivariant homology theory over G in the sense of Definition 12.89. In particular its evaluation at G yields a G-homology theory $H_*^G(-;\mathbf{K}_{\mathcal{A}})$.

Conjecture 13.11 (K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories). We say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories* if for every additive G-category $\mathcal{A}$ and every $n \in \mathbb{Z}$ the assembly map induced by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^g(\mathrm{pr};\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr)$$

is bijective.

Remark 13.12 (The setting with additive G-categories as coefficients encompasses the setting with rings as coefficients). Let $\alpha\colon G \to \mathrm{aut}(R)$ be a group homomorphism. Then we have already introduced the twisted group ring $R_\alpha(G)$ above. For a suitable choice of an additive G-category $\mathcal{A}$, the assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr)$$

appearing in Conjecture 13.11 reduces to the assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}_{R,\alpha})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{R,\alpha}) \to H_n^G(G/G;\mathbf{K}_{R,\alpha}) = K_n(R_\alpha G)$$

where for any subgroup $H \subseteq G$ and integer $n \in \mathbb{Z}$ we have $\pi_n\bigl(\mathbf{K}_{R,\alpha}(I(H))\bigr) = K_n(R_{\alpha|_H}H)$. If α is trivial, this is precisely the assembly map appearing in Conjecture 13.1. More details, even for crossed product rings, can be found in [76, Section 4 and 6].

In particular we get that the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings holds for G if the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories holds for G.

Exercise 13.13. Let R be a ring. Define a category $\underline{R}_\oplus$ as follows. For each integer $m \in \mathbb{Z}$ with $m \geq 0$ we have one object $[m]$. For $m,n \geq 1$ the set of morphisms from $[m]$ to $[n]$ is the set $M_{m,n}(R)$ of (m,n)-matrices with entries in R. The set of morphisms from $[0]$ to $[m]$ and from $[m]$ to $[0]$ consist of precisely one element. Composition is given by matrix multiplication.

Show that $\underline{R}_{\oplus}$ can be equipped with the structure of a small additive category and that it is equivalent as an additive category to the category of finitely generated free R-modules.

Remark 13.14 (Involutions and K-theory). Let $\mathcal{A}$ be an additive G-category with involution in the sense of [76, Definition 4.22], i.e., an additive category with involution coming with G-action by functors of additive categories with involution.

Then the involution induces involutions on the source and target of the K-theoretic assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

of Conjecture 13.11 and the assembly map is compatible with them.

13.3.2 The L-theoretic Farrell-Jones Conjecture with Coefficients in Additive G-Categories with Involution

Let $\mathcal{A}$ be an additive G-category with involution in the sense of [76, Definition 4.22]. We obtain from [76, Section 7] a contravariant functor to the category $\mathsf{ADDCAT}_{\mathrm{inv}}$ of small additive categories with involution

$$\mathsf{GROUPOIDS} \downarrow G \to \mathsf{ADDCAT}_{\mathrm{inv}}, \quad \mathrm{pr} \colon \mathcal{G} \to I(G) \ \mapsto \ \int_{\mathcal{G}} \mathcal{A} \circ \mathrm{pr}\,.$$

Composing it with the functor sending an additive category with involution $\mathcal{A}$ to its L-theory spectrum $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$, yields a functor

$$\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle} \colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA} \tag{13.15}$$

where for the construction of the spectrum L-theory $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$ associated to an additive category with involution $\mathcal{A}$ we refer to Ranicki [823, Chapter 13]. By Theorem 12.91 we obtain an equivariant homology theory over G in the sense of Definition 12.89. In particular its evaluation at G yields a G-homology theory $H_n^G(-; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle})$.

Conjecture 13.16 (L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution). We say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution* if for every additive G-category with involution $\mathcal{A}$ and every $n \in \mathbb{Z}$ the assembly map given by the projection $E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \\ \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\bigl(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\bigr)$$

is bijective.

Remark 13.17 (The setting of additive G-categories with involution as coefficients encompasses the setting with rings with involution as coefficients). Let R be a ring with involution. Consider a group homomorphism $\alpha\colon G \to \mathrm{aut}(R)$ satisfying $\overline{\alpha(g)(r)} = \alpha(g)(\overline{r})$, and a group homomorphism $w\colon G \to \mathrm{center}(R)$ satisfying $\overline{w(g)} = w(g)$. Then we have already introduced the twisted group ring $R_\alpha(G)$ above. It inherits an involution by

$$\overline{\sum_{g\in G} r_g \cdot g} := \sum_{g\in G} w(g)\cdot \alpha(g^{-1})(\overline{r_g})\cdot g^{-1},$$

and we denote this ring with involution by $R_{\alpha,w}G$. For a suitable choice of an additive G-category with involution $\mathcal{A}$, the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})$$

appearing in Conjecture 13.16 reduces to the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{R,\alpha,w}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{R,\alpha,w}^{\langle -\infty\rangle}) \to H_n^G(G/G; \mathbf{L}_{R,\alpha,w}^{\langle -\infty\rangle})$$

where for any subgroup $H \subseteq G$ and integer $n \in \mathbb{Z}$ we have $\mathbf{L}_{R,\alpha,w}^{\langle -\infty\rangle}(I(H)) = L_n^{\langle -\infty\rangle}(R_{\alpha|_H}H, w|_H)$. If α and w are trivial, this is precisely the assembly map appearing in Conjecture 13.4. More details, even for crossed product rings, can be found in [76, Theorem 0.4, Section 4 and 8].

In particular we get that the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution holds for G if the L-theoretic Farrell-Jones Conjecture 13.16 with coefficients in additive G-categories with involution holds for G.

Exercise 13.18. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of additive categories. Show that it is an equivalence of additive categories if and only if for every two objects A and B in $\mathcal{A}$ the induced map $\mathrm{mor}_{\mathcal{A}}(A_0, A_1) \to \mathrm{mor}_{\mathcal{B}}(F(A_0), F(A_1))$ sending f to $F(f)$ is bijective and for each object B in $\mathcal{B}$ there exists an object A in $\mathcal{A}$ such that $F(A)$ and B are isomorphic in $\mathcal{B}$.

Remark 13.19 (Eilenberg swindle for L-theory). There is an obvious version of Theorem 6.37 (iii) for the algebraic L-theory $\mathbf{L}^{\langle -\infty\rangle}(\mathcal{A})$ of an additive category $\mathcal{A}$ with involution.

13.4 The K-theoretic Farrell-Jones Conjecture with Coefficients in Higher Categories

Comment 9 (by W.): The precise formulation of this subsection has to be discussed with Christoph.

Let $\mathcal{C}$ be an right exact G-∞-category in the sense of **Comment 10 (by W.)**: Add reference, probably to Chapter 8. We obtain from (8.1) a covariant functor

$$\mathbf{K}_{\mathcal{C}} \colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA}.$$

By Theorem 12.91 we obtain an equivariant homology theory over G in the sense of Definition 12.89. In particular its evaluation at G yields a G-homology theory $H_*^G(-;\mathbf{K}_{\mathcal{C}})$.

Conjecture 13.20 (K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories). We say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories* if for every right exact G-∞-category $\mathcal{C}$ and every $n \in \mathbb{Z}$ the assembly map given by the projection

$$H_n^G(\mathrm{pr};\mathbf{K}_{\mathcal{C}}) \colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{C}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{C}}(I(G))\bigr)$$

is bijective.

Comment 11 (by W.): The next remark is under construction.

Remark 13.21 (For K-theory the setting of higher G-categories encompasses all other settings for K-theory.). The assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}_{\mathcal{C}}) \colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{C}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{C}}(I(G))\bigr)$$

appearing in K-theoretic Farrell-Jones Conjecture 13.20 with coefficients in higher G-categories reduces to the the assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr)$$

appearing in Conjecture 13.11 if we take for $\mathcal{C}$ the higher G-category $\mathcal{C}(\mathcal{A})$ associated to an additive G-category. **Comment 12 (by W.)**: We have to add an explanation probably referring to Chapter 8.

Moreover, the K-theoretic Farrell-Jones Conjecture 13.20 with coefficients in higher G-categories implies the Farrell-Jones Conjecture 15.59 for A-theory (with coefficients). **Comment 13 (by W.)**: We have to add an explanation probably referring to Chapter 8. This is also mentioned and references for the proof are discussed in Theorem 15.63 (iii).

Recall from Remark 13.12 that Conjecture 13.11 and hence also Conjecture 13.20 imply the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings. One may ask whether there is a version of the Farrell-Jones

Conjecture for ring spectra as coefficients. Such a version is contained in Farrell-Jones Conjecture 13.20. **Comment 14 (by W.)**: Add a discussion. There seem to be no place in the literature where a K-theoretic version of the Farrell-Jones Conjecture is formulated for ring spectra. Note that one can think of the Farrell-Jones Conjecture 15.59 for A-theory (with coefficients) as the Farrell-Jones Conjecture with coefficients in the sphere spectrum. **Comment 15 (by W.)**: Add a discussion.

Remark 13.22 (L-theory version of Conjecture 13.20). It has not been worked out in detail how to construct the assembly map for the L-theory version of the K-theoretic Farrell-Jones Conjecture 13.20 with coefficients in higher G-categories with Poincaré structure and to prove the conjecture for the same class of groups as it has be done for the other versions. Such a version should of course imply the L-theoretic Farrell-Jones Conjecture 13.16 with coefficients in additive G-categories with involution.

13.5 Finite Wreath Products

The versions of the Farrell-Jones Conjecture discussed above do not carry over to overgroups of finite index. To handle this difficulty, we consider finite wreath products.

Let G and F be groups. Their *wreath product* $G \wr F$ is defined as the semidirect product $(\prod_F G) \rtimes F$ where F acts on $\prod_F G$ by permuting the factors. For our purpose the following elementary lemma is crucial.

Lemma 13.23.

(i) There is an embedding $(H \wr F_1) \wr F_2 \to H \wr (F_1 \wr F_2)$;
(ii) If F_1 and F_2 are finite, then $F_1 \wr F_2$ is finite;
(iii) Let G be an overgroup of H of finite index. Then there is subgroup $N \subseteq H$ of H that satisfies $[G : N] < \infty$ and is normal in G, and a finite group F such that G embeds into $N \wr F$.

Proof. (i) See [578, Lemma 1.21].

(ii) This is obvious.

(iii) Let S denote a system of representatives of the cosets G/H. Since G/H is by assumption finite, $N := \bigcap_{s \in S} sHs^{-1}$ is a finite index normal subgroup of G and is contained in H. Now G can be embedded in $N \wr G/N$, see [297, Section 2.6]. □

Conjecture 13.24 (K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products). We say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products* if for any finite group F

the group $G \wr F$ satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive $G \wr F$-categories.

Conjecture 13.25 (L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution with finite wreath products). We say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution with finite wreath products* if for any finite group F the group $G \wr F$ satisfies the L-theoretic Farrell-Jones Conjecture 13.16 with coefficients in additive $G \wr F$-categories with involution.

Conjecture 13.26 (K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products). We say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products* if for any finite group F the group $G \wr F$ satisfies the K-theoretic Farrell-Jones Conjecture 13.20 with coefficients in higher $G \wr F$-categories.

13.6 The Full Farrell-Jones Conjecture

Next we can formulate the version of the Farrell-Jones Conjecture which is the most general one, implies all other ones, and has the best inheritance properties.

Conjecture 13.27 (Full Farrell-Jones Conjecture). We say that a group satisfies the *Full Farrell-Jones Conjecture* if G satisfies the following three conjectures:

- the K-theoretic Farrell-Jones Conjecture 13.24 with coefficients in additive G-categories with finite wreath products;
- the L-theoretic Farrell-Jones Conjecture 13.25 with coefficients in additive G-categories with involution with finite wreath products;
- the K-theoretic Farrell-Jones Conjecture 13.26 with coefficients in higher G-categories with finite wreath products.

Despite the fact that Conjecture 13.26 implies Conjecture 13.24, see Remark 13.21, we list Conjecture 13.24 above in Conjecture 13.27 for the reader's convenience. Recall that the version with rings as coefficients do follow from the versions with additive categories as coefficients, see Remarks 13.12 and 13.17.

13.7 Inheritance Properties of the Farrell-Jones Conjecture

In this section we discuss the inheritance properties of the various versions of the Farrell-Jones Conjectures above. Both the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution do not have good inheritance properties. The reason why we have introduced the other variants is that they do have some remarkable inheritance properties.

Definition 13.28 (Farrell-Jones groups). Let $\mathcal{FJ}$ be the class of groups that satisfy the Full Farrell-Jones Conjecture 13.27. We call a (discrete) group G a *Farrell-Jones group* if G belongs to $\mathcal{FJ}$.

Theorem 13.29 (Inheritance properties of the Full Farrell-Jones Conjecture).

(i) Passing to subgroups
Let $H \subseteq G$ be an inclusion of groups and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
(ii) Passing to overgroups of finite index
Let G be an overgroup of H with finite index $[G : H]$. If H belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;
(iii) Passing to finite wreath products
If G belongs to $\mathcal{FJ}$, then $G \wr F$ belongs to $\mathcal{FJ}$ for any finite group F;
(iv) Passing to finite direct products
If the groups G_0 and G_1 belong to $\mathcal{FJ}$; then $G_0 \times G_1$ belongs to $\mathcal{FJ}$,
(v) Group extensions
Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that the groups K and Q belongs to $\mathcal{FJ}$ and that for any infinite cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{FJ}$.
Then G belongs to $\mathcal{FJ}$;
(vi) Colimits over directed systems
Let $\{G_i \mid i \in I\}$ be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to $\mathcal{FJ}$.
Then the colimit $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;
(vii) Passing to free products
Consider a collection of groups $\{G_i \mid i \in I\}$ such that G_i belongs to $\mathcal{FJ}$ for each $i \in I$. Then $*_{i \in I} G_i$ belongs to $\mathcal{FJ}$.

Proof. (i) We begin with the case of additive G-categories as coefficients.

Assertion (i) is proved in [90, Theorem 4.5] for Conjecture 13.11, and in [76, Theorem 0.10] for Conjecture 13.16. Now assertion (i) follows for the version of Full Farrell-Jones Conjecture 13.27 for additive G-categories as coefficients since $H \wr F$ is a subgroup of $G \wr F$ for every subgroup $H \subseteq G$.

The proof of assertion (i) for the version with higher G-categories as coefficients is analogous and can be found in [172, Theorem 1.6 (1)].

(ii) This follows from Lemma 13.23 and assertion (i).

(iii) This follows from Lemma 13.23 and assertion (i).

(iv) We begin with the case of additive G-categories as coefficients.

The versions of the Farrell-Jones Conjecture 13.11 and 13.16 are true for virtually finitely generated abelian groups by [71, Theorem 3.1]. Hence they hold in particular for the product of two virtually cyclic subgroups. By inspecting the proof of [578, Lemma 3.15], we see that the assertion (iv) holds for the Farrell-Jones Conjectures 13.11 and 13.16.

Next we prove assertion (iv) for the version of Full Farrell-Jones Conjecture 13.27 with additive G-categories as coefficients. Suppose it holds for G_1 and G_2. Let F be any finite subgroup. We have to show that versions of the Farrell-Jones Conjecture 13.11 and 13.16 holds for $(G_1 \times G_2) \wr F$. By assumption they both hold for $G_1 \wr F$ and $G_2 \wr F$. Since $(G_1 \times G_2) \wr F$ is a subgroup of $(G_1 \wr F) \times (G_2 \wr F)$ by [578, Lemma 1.197] and Conjecture 13.11 and 13.16 pass to subgroups by the argument given in assertion (i), assertion (iv) holds for the Full Farrell-Jones Conjecture 13.27 with additive G-categories as coefficients.

The proofs of assertions (iv) for the version of the Full Farrell-Jones Conjecture 13.27 for higher G-categories are analogous and can be found in in [172, Theorem 1.7 (11)].

(v) We begin with the case of additive G-categories as coefficients.

The following version of assertion (v) is proved in [76, Theorem 0.9] for Conjecture 13.16.

Property (E)

Let $1 \to K \to G \to Q \to 1$ be an extension of groups. If for any virtually cyclic subgroup $V \subseteq Q$ the group $p^{-1}(V)$ and the group Q satisfy Conjecture 13.16, then G satisfies Conjecture 13.16.

The proof of property (E) for Conjecture 13.11 is analogous. Finally we conclude from [578, Lemma 3.16] and assertion (iv) that property (E) holds also for the Full Farrell-Jones Conjecture 13.27 for additive G-categories.

Because of assertion (ii), we can replace in property (E) the assumption that V is virtually cyclic by the assumption that V is trivial or infinite cyclic. This finishes the proof of assertions (v) for additive G-categories as coefficients.

The proofs of assertions (v) for the version of the Full Farrell-Jones Conjecture 13.27 for higher G-categories is analogous and can be found in in [172, Theorem 1.7 (13)].

(vi) We begin with the case of additive G-categories as coefficients.

Assertion (vi) is proved in [76, Theorem 0.8] for Conjecture 13.16, the proof for Conjecture 13.11 is completely analogous. Now assertion (vi) follows for the version of Full Farrell-Jones Conjecture 13.27 with additive G-

categories since there is an obvious isomorphism for a finite group F, see [578, Lemma 1.20],

$$\operatorname{colim}_{i\to\infty}(G_i \wr F) \xrightarrow{\cong} \big(\operatorname{colim}_{i\to\infty} G_i\big) \wr F.$$

The proof of assertion (vi) for the version of the Full Farrell-Jones Conjecture 13.27 for higher G-categories is analogous and can be found in in [172, Theorem 1.6 (12)].

(vii) Because of assertion (vi) it suffices to consider the case where I is finite. An obvious induction argument over the cardinality of the finite set I reduces the claim to the case $I = \{1,2\}$.

Let G_1 and G_2 be groups. Let $\mathrm{pr}\colon G_1 * G_2 \to G_1 \times G_2$ be the canonical projection. Let $C \subseteq G_1 \times G_2$ be a cyclic subgroup. Then there exists a free group F and a finite group H such that $\mathrm{pr}^{-1}(C)$ is a subgroup of $F \wr H$, see [578, Lemma 3.21]. (In the statement of [578, Lemma 3.21] the assumption countable appears but the proof goes through in the general case without modifications.) A finitely generated free group satisfies the Full Farrell-Jones Conjecture 13.27 by [88, Remark 6.4] and [172, Theorem 1.6 (3)] since it is a hyperbolic group. Hence F satisfies the Full Farrell-Jones Conjecture 13.27 by assertion (vi). We conclude from assertion (iii) that $F \wr H$ satisfies the Full Farrell-Jones Conjecture 13.27. Hence $\mathrm{pr}^{-1}(C)$ satisfies the Full Farrell-Jones Conjecture 13.27 for every cyclic subgroup $C \subseteq G_1 \times G_2$ by assertion (i). The product $G_1 \times G_2$ satisfies the Full Farrell-Jones Conjecture 13.27 by assertion (iv). Now assertion (v) implies that $G_1 * G_2$ satisfies the Full Farrell-Jones Conjecture 13.27. $\square$

Exercise 13.30. Consider an epimorphism of groups $G \to Q$ whose kernel is finite. Suppose that Q satisfies the Full Farrell-Jones Conjecture 13.27.

Show that G satisfies the Full Farrell-Jones Conjecture 13.27.

Exercise 13.31. Suppose that the Full Farrell-Jones Conjecture 13.27 holds for all groups that occur as fundamental groups of a connected orientable closed 4-manifold.

Show that then the Full Farrell-Jones Conjecture 13.27 holds for all groups.

13.8 Splitting the Assembly Map from $\mathcal{FIN}$ to $\mathcal{VCY}$

In the sequel we denote for two families $\mathcal{F} \subseteq \mathcal{G}$ by

$$\iota_{\mathcal{F}\subseteq\mathcal{G}}\colon E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G) \tag{13.32}$$

the up to G-homotopy unique G-map. Note that $\iota_{\mathcal{F}\subseteq\mathcal{ALL}}\colon E_{\mathcal{F}}(G) \to E_{\mathcal{ALL}}(G) = G/G$ is the projection.

Theorem 13.33 (Splitting the K-theoretic assembly map from $\mathcal{FIN}$ to $\mathcal{VCY}$). *Let G be a group*

(i) *Let $\mathcal{A}$ be an additive G-category. Let n be any integer. Then*

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{A}}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}}\big)\to H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}\big)$$

is split injective. In particular we obtain a natural splitting

$$H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}\big)\xrightarrow{\cong}H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}}\big)\oplus H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{A}}\big).$$

Moreover, there exists specific $\mathsf{Or}(G)$-spectrum $\mathbf{NK}_{\mathcal{A}}$ and a natural isomorphism

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}_I};\mathbf{NK}_{\mathcal{A}}\big)\xrightarrow{\cong}H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{A}}\big)$$

where $\mathcal{VCY}_I$ is the family of virtually cyclic subgroups of type I;

(ii) *Let $\mathcal{C}$ be a right exact G-∞-category. Let n be any integer. Then*

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{C}}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{C}}\big)\to H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{C}}\big)$$

is split injective.

Proof. (i) See [92, Theorem 1.3] or[670, Theorem 0.1].
(ii) This follows from [170, Corollary 1.13] and [169, Theorem 1.1.5]. □

Whereas in [92, Theorem 1.3] just a splitting is established, in [670, Theorem 0.1], an explicit $\mathsf{Or}(G)$-spectra $\mathbf{NK}_{\mathcal{A}}^G$ is constructed and the relative terms including the involution on the K-groups are further analyzed, in particular they are identified with K-groups of Nil-categories. **Comment 16 (by W.)**: There may be gap in [670, Theorem 0.1] which we have to fix. For rings see also Lafont-Ortiz [591].

For L-theory one has at least the following version that is mentioned after Theorem 1.3 in [92] for rings. The argument carries over to additive G-categories with involution.

Theorem 13.34 (Splitting the L-theoretic assembly map from $\mathcal{FIN}$ to $\mathcal{VCY}$). *Let $\mathcal{A}$ be an additive G-category with involution such that there exists an integer N with the property that $\pi_n\big(\mathbf{K}_{\mathcal{A}}(I(V))\big)=0$ for all virtually cyclic subgroups V of G and all $n\le N$. Then*

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\to H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)$$

is split injective.

It is not clear whether the condition about $\pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(V))\bigr)$ appearing in Theorem 13.34, which is needed for the proposed proof, is necessary. If we consider group rings RG, this condition is automatically satisfied if R is regular and the order of every finite subgroup of G is invertible in R, e.g., R is a field of characteristic zero.

13.9 Splitting Rationally the Assembly Map from $\mathcal{TR}$ to $\mathcal{FIN}$

Lemma 13.35. *Let G be a group and let R be a ring (with involution). Then the relative assembly maps*

$$\begin{aligned} H_n(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}};\mathbf{K}_R)\colon H_n(E_{\mathcal{TR}}(G);\mathbf{K}_R) &\to H_n(E_{\mathcal{FIN}}(G);\mathbf{K}_R);\\ H_n(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}};\mathbf{L}_R^{\langle-\infty\rangle})\colon H_n(E_{\mathcal{TR}}(G);\mathbf{L}_R^{\langle-\infty\rangle}) &\to H_n(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle-\infty\rangle});\\ K_n^G(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}})\colon K_n^G(E_{\mathcal{TR}}(G)) &\to K_n^G(E_{\mathcal{FIN}}(G));\\ KO_n^G(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}})\colon KO_n^G(E_{\mathcal{TR}}(G)) &\to KO_n^G(E_{\mathcal{FIN}}(G)), \end{aligned}$$

are split injective after applying $-\otimes_{\mathbb{Z}}\mathbb{Q}$ for $n\in\mathbb{Z}$.

Proof. This follows Lemma 12.18 (ii). □

Remark 13.36. Note that Lemma 13.35 is only stated in the case that we consider the untwisted coefficients rings R. It is conceivable that it holds also in the case where we allow a twisting $\alpha\colon G\to \operatorname{aut}(R)$, but the details of a proof of this statement has not been worked out in detail and are definitely more complicated as in the untwisted case, see Remark 12.93.

The proof of Lemma 13.35 carries over to also for additive categories and right-exact ∞-categories as coefficients provided that the G-actions on these are trivial.

Example 13.37 (The L-theory assembly map for the trivial family is not injective in general). Consider the group $\mathbb{Z}/3$. Then

$$H_1(B\mathbb{Z}/3;\mathbf{L}(\mathbb{Z}))\to L_1(\mathbb{Z}[\mathbb{Z}/3])$$

is not injective. Namely, the target is known to be trivial, but the source is non-trivial. This can be seen by inspecting the Atiyah-Hirzebruch spectral sequence converging to $H_{p+q}(B\mathbb{Z}/3;\mathbf{L}(\mathbb{Z}))$ with E^2-term

$$H_p(B\mathbb{Z}/3,L_q(\mathbb{Z}))=\begin{cases}\mathbb{Z}/3 & p\ge 1, p\text{ odd}, q\equiv 0 \mod 4;\\ L_q(\mathbb{Z}) & p=0;\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\mathrm{Wh}(\mathbb{Z}/3)$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$, and $K_n(\mathbb{Z}[\mathbb{Z}/3])$ for $n \leq -1$ vanish by Theorem 3.115, Theorem 3.116 (iva), Theorem 4.22 (i) and (v), Example 2.106 so that the decorations for the L-groups do not play a role by Theorem 9.106.

Example 13.38 (The K-theory assembly map for the trivial family is not injective in general). An easy calculation using the Atyiah-Hirzebruch spectral sequence shows that the K-theoretic assembly map $H_n(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}};\mathbf{K}_R)\colon H_n(E_{\mathcal{TR}}(G);\mathbf{K}_R) \to H_n(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$ is not injective if $n = 2$, $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $R = \mathbb{F}_p$ for an odd prime p, see [943]. No such example is known to the author for $R = \mathbb{Z}$.

13.10 Reducing the Family of Subgroups for the Farrell-Jones Conjecture

Next we explain that one sometimes can reduce the family of virtually cyclic subgroups $\mathcal{VCY}$ to a smaller family.

A virtually cyclic group V is called *of type I* if it admits an epimorphism to the infinite cyclic group, and *of type II* if it admits an epimorphism onto the infinite dihedral group. The elementary proof of the following result can be found in [670, Lemma 1.1].

Lemma 13.39. *Let V be an infinite virtually cyclic group.*

(i) V is either of type I or of type II;
(ii) The following assertions are equivalent:
  (a) V is of type I;
  (b) $H_1(V)$ is infinite;
  (c) $H_1(V)/\operatorname{tors}(V)$ is infinite cyclic;
  (d) The center of V is infinite;
(iii) There exists a unique maximal normal finite subgroup $K_V \subseteq V$, i.e., K_V is a finite normal subgroup and every normal finite subgroup of V is contained in K_V;
(iv) Let $Q_V := V/K_V$. Then we obtain a canonical exact sequence

$$1 \to K_V \xrightarrow{i_V} V \xrightarrow{p_V} Q_V \to 1.$$

Moreover, Q_V is infinite cyclic if and only if V is of type I and Q_V is isomorphic to the infinite dihedral group if and only if V is of type II;
(v) Let $f\colon V \to Q$ be any epimorphism onto the infinite cyclic group or onto the infinite dihedral group. Then the kernel of f agrees with K_V;

Exercise 13.40. Let $\phi\colon V \to W$ be a homomorphism of infinite virtually cyclic groups with infinite image. Then ϕ maps K_V to K_W and we obtain the following canonical commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & K_V \arrow[r, "i_V"] \arrow[d, "\phi_K"] & V \arrow[r, "p_V"] \arrow[d, "\phi"] & Q_V \arrow[r] \arrow[d, "\phi_Q"] & 1 \\
1 \arrow[r] & K_W \arrow[r, "i_W"] & W \arrow[r, "p_W"] & Q_W \arrow[r] & 1
\end{tikzcd}$$

with injective ϕ_Q.

Exercise 13.41. Show that a group G is infinite virtually cyclic if and only if it admits a proper cocompact isometric action on $\mathbb{R}$.

In the sequel we denote by $\mathcal{VCY}_I$ the family of subgroups that are either finite or infinite virtually cyclic of type I.

Definition 13.42 (Hyperelementary group). Let p be a prime. A (possibly infinite) group G is called *p-hyperelementary* if it can be written as an extension $1 \to C \to G \to P \to 1$ for a cyclic group C and a finite group P whose order is a power of p.

We call G *hyperelementary* if G is p-hyperelementary for some prime p.

If G is finite, this reduces to the usual definition. Note that for a finite p-hyperelementary group G one can arrange that the order of the finite cyclic group C appearing in the extension $1 \to C \to G \to P \to 1$ for a finite p-group P is prime to p. Subgroups and quotient groups of p-hyperelementary groups are p-hyperelementary again. For a group G and a prime p let $\mathcal{HE}_p$ and $\mathcal{HE}$ respectively be the class of (possibly infinite) subgroups that are p-hyperelementary or hyperelementary respectively.

The following result is taken from [71, Theorem 8.2].

Theorem 13.43 (Hyperelementary induction). *Let G be a group and let $\mathcal{A}$ be an additive G-category (with involution). Then both relative assembly maps*

$$H_n(\iota_{\mathcal{HE},\mathcal{VCY}};\mathbf{K}_{\mathcal{A}})\colon H_n^G\bigl(E_{\mathcal{HE}}(G);\mathbf{K}_{\mathcal{A}}\bigr) \to H_n^G\bigl(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}\bigr)$$

and

$$H_n(\iota_{\mathcal{HE},\mathcal{VCY}};\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G\bigl(E_{\mathcal{HE}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\bigr) \to H_n^G\bigl(E_{\mathcal{VCY}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}\bigr)$$

induced by the up to G-homotopy unique G-map $\iota_{\mathcal{HE},\mathcal{VCY}}; E_{\mathcal{HE}}(G) \to E_{\mathcal{VCY}}(G)$ are bijective for all $n \in \mathbb{Z}$.

Comment 17 (by W.): Does Theorem 13.43 has an analogue for ∞-categories? Here Ullmann-Winges [942, Theorem 8.7] is relevant. (I doubt it since one has to pass to Dress groups in the setting of higher categories).

13.10.1 Reducing the Family of Subgroups for the Farrell-Jones Conjecture for K-Theory

Theorem 13.44 (Passage from $\mathcal{VCY}_I$ to $\mathcal{VCY}$ for K-theory). *Let G be a group.*

(i) Let $\mathcal{A}$ be an additive G-category. Then the relative assembly map

$$H_n^G\big(\iota_{\mathcal{VCY}_I \subseteq \mathcal{VCY}}; \mathbf{K}_{\mathcal{A}}\big)\colon H_n^G\big(E_{\mathcal{VCY}_I}(G); \mathbf{K}_{\mathcal{A}}\big) \to H_n\big(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{A}}\big)$$

is bijective for all $n \in \mathbb{Z}$;

(ii) Let $\mathcal{C}$ be a right exact G-∞-category. Then the relative assembly map

$$H_n^G\big(\iota_{\mathcal{VCY}_I \subseteq \mathcal{VCY}}; \mathbf{K}_{\mathcal{C}}\big)\colon H_n^G\big(E_{\mathcal{VCY}_I}(G); \mathbf{K}_{\mathcal{C}}\big) \to H_n\big(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{C}}\big)$$

is bijective for all $n \in \mathbb{Z}$;

Proof. (i) See [271, Remark 1.6].

(ii) The argument for assertion (i) goes through, since the K-theoretic Farrell-Jones Conjecture 13.20 with coefficients in higher G-categories holds for finitely $\mathcal{F}$-amenable groups, actually for finitely homotopy $\mathcal{F}$-amenable groups, see [172, Theorem 5.1]. □

Theorem 13.45 (Passage from $\mathcal{HE}_I$ to $\mathcal{VCY}$ for K-theory and additive G-categories as coefficients). *Let G be a group and $\mathcal{A}$ be an additive G-category. Let $\mathcal{HE}_I$ be the family of subgroups of G given by the intersection $\mathcal{VCY}_I \cap \mathcal{HE}$.*

Then the relative assembly map

$$H_n^G\big(\iota_{\mathcal{HE}_I \subseteq \mathcal{VCY}}; \mathbf{K}_{\mathcal{A}}\big)\colon H_n^G\big(E_{\mathcal{HE}_I}(G); \mathbf{K}_{\mathcal{A}}\big) \to H_n\big(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{A}}\big)$$

is bijective for all $n \in \mathbb{Z}$.

Proof. This follows from Theorem 13.43, Theorem 13.44 (ii), Theorem 15.9 (ii), and Lemma 15.14. □

Theorem 13.45 implies that we get equivalent conjectures if we replace in Conjectures 13.1, 13.2, and 13.11 the family $\mathcal{VCY}$ by the smaller family $\mathcal{HE}_I$.

Exercise 13.46. Fix a prime p. Show that an infinite subgroup $H \subset G$ belongs to $\mathcal{HE}_p \cap \mathcal{VCY}_I$ if and only if H is isomorphic to $P \rtimes_\phi \mathbb{Z}$ for some finite p-group P and an automorphism $\phi\colon P \to P$ whose order is a power of p.

Exercise 13.47. Let p be a prime. Let G be an infinite virtually cyclic group of type I that is p-hyperelementary. Let R be a regular ring.

Show that the map induced by the projection $\mathrm{pr}\colon E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to H_n^G(G/G;\mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$ after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.

Theorem 13.48 (Reduction to the family $\mathcal{FIN}$ for algebraic K-theory with regular rings as coefficients).

Let G be a group and let R be a regular ring coming with a homomorphism $G \to \operatorname{aut}(R)$. Let $\mathcal{P}(G,R)$ be the set of primes which are not invertible in R and for which G contains an element of order p.

Then for all $m \in \mathbb{Z}$ the assembly map

$$H_m^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to H_m^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$$

is an $\mathcal{P}(G,R)$-isomorphism, i.e., it becomes an isomorphism after inverting all primes in $\mathcal{P}(G,R)$.

Proof. See [651, Theorem 1.2]. Actually, additive categories with coefficient are treated in [651, Theorem 9.1]. □

Exercise 13.49. Let G be a group and let R be a regular ring. Suppose that $\mathbb{Q} \subseteq R$ or that G is torsionfree.

Then for all $m \in \mathbb{Z}$ the assembly map

$$H_m^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to H_m^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$$

is an isomorphism.

One can reduce the families by extending the classical induction theorems for finite groups due to Dress to our setting. This is carried out in detail in [75]. There only rings as coefficients are treated but the proofs carry over to the setting of additive G-categories. For instance for K-theory one has to extend the relevant pairing of the Swan group for group rings to additive categories. We leave the details to the reader and just record some results. Recall that $\mathcal{FCY}$ is the family of finite cyclic subgroups.

Theorem 13.50 (Reductions to families contained in $\mathcal{FIN}$ for algebraic K-theory with rings as coefficients). *Let G be a group and R be a ring.*

(i) Then the relative assembly map

$$H_n^G\big(\iota_{(\mathcal{HE}\cap\mathcal{FIN})\subseteq\mathcal{FIN}};\mathbf{K}_R\big)\colon H_n^G\big(E_{\mathcal{HE}\cap\mathcal{FIN}}(G);\mathbf{K}_R\big) \xrightarrow{\cong} H_n\big(E_{\mathcal{FIN}}(G);\mathbf{K}_R\big)$$

is bijective for all $n \in \mathbb{Z}$;

(ii) Let p be a prime. Then the relative assembly map

$$H_n^G\big(\iota_{(\mathcal{HE}_p\cap\mathcal{FIN})\subseteq\mathcal{FIN}};\mathbf{K}_R\big)\colon H_n^G\big(E_{\mathcal{HE}_p\cap\mathcal{FIN}}(G);\mathbf{K}_R\big) \to H_n\big(E_{\mathcal{FIN}}(G);\mathbf{K}_R\big)$$

is bijective for all $n \in \mathbb{Z}$ *after applying* $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$;

(iii) The relative assembly map

$$H_n^G\big(\iota_{\mathcal{FCY} \subseteq \mathcal{FIN}}; \mathbf{K}_R\big) \colon H_n^G\big(E_{\mathcal{FCY}}(G); \mathbf{K}_R\big) \to H_n\big(E_{\mathcal{FIN}}(G); \mathbf{K}_R\big)$$

is bijective for all $n \in \mathbb{Z}$ *after applying* $\mathbb{Q} \otimes_{\mathbb{Z}} -$;

Proof. By the Transitivity Principle, see Theorem 15.13, it suffices to prove the assertions only in the special case where G is finite and in particular $H_n\big(E_{\mathcal{FIN}}(G); \mathbf{K}_R\big)$ reduces to $K_n(RG)$. Then the claim follows from [75, Theorem 2.9 and Lemma 4.1]. □

Note that in Theorem 13.50 we consider only rings with trivial G-action. It is conceivable that it carries over twisted group rings and, more generally, to additive G-categories, but we have not checked the details of a proof of this claim.

Next we state and prove the following results, which will be needed for the proof of Theorem 13.62 (v).

Lemma 13.51. *Consider a ring* R*, a group* G*, and* $m \in \mathbb{Z}$*. Suppose that, for every finite group* H *and every group automorphism* $\phi \colon H \xrightarrow{\cong} H$ *with the property that the semidirect product* $H \rtimes_{\Phi} \mathbb{Z}$ *is isomorphic to a subgroup of* G*, and every* $n \in \mathbb{Z}, n \geq 0$*, the assembly map*

$$H_m^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_m^{H \rtimes_\phi \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]}) = K_m((R[\mathbb{Z}^n])[H \rtimes_\phi \mathbb{Z}])$$

is an isomorphism where we consider the $\mathbb{Z}$*-CW-complex* $E\mathbb{Z}$ *as a* $H \rtimes_\phi \mathbb{Z}$*-CW-complex by restriction with the projection* $H \rtimes_\phi \mathbb{Z} \to \mathbb{Z}$.

Then the canonical map

$$H_i^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \xrightarrow{\cong} H_i^G(E_{\mathcal{VCY}}(G), \mathbf{K}_R)$$

is bijective for $i \leq m$.

Proof. Theorem 13.44 implies that for $i \in \mathbb{Z}$ the map

$$H_i^G(E_{\mathcal{VCY}_I}(H \rtimes_\phi \mathbb{Z}), \mathbf{K}_R) \to H_i^G(E_{\mathcal{VCY}}(H \rtimes_\phi \mathbb{Z}), \mathbf{K}_R)$$

is bijective. Hence it suffices to show that, for $i \in \mathbb{Z}$ with $i \leq m$, the canonical map

$$H_i^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \xrightarrow{\cong} H_i^G(E_{\mathcal{VCY}_I}(G), \mathbf{K}_R)$$

is bijective. Thanks to the Transitivity Principle appearing in Theorem 15.12, this has only to be done in the special case where G is a virtually cyclic group of type I.

Consider any finite group H and any group automorphism $\phi \colon H \xrightarrow{\cong} H$. Since $E\mathbb{Z}$ with the $H \rtimes_\phi \mathbb{Z}$ action coming from restriction with the projection

$H \rtimes_\phi \mathbb{Z} \to \mathbb{Z}$ is a model for $E_{\mathcal{FIN}}(H \times_\phi \mathbb{Z})$ and $\{\bullet\}$ is a model for $E_{\mathcal{VCY}_I}(H \times_\phi \mathbb{Z})$, it remains to show that the assembly map

$$H_i^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to H_i^{H \rtimes_\phi \mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) = K_i(R[H \rtimes_\phi \mathbb{Z}])$$

is bijective for $i \leq m$. This will be achieved by proving inductively for $n = 0, 1, 2, \dots$ that this map is bijective for $m - n \leq i \leq m$ provided that $H_m^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_m^{H \rtimes_\phi \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]})$ is bijective.

The induction beginning $n = 0$ is trivial. The induction step from $(n-1)$ to n is done as follows. The Bass-Heller-Swan decomposition for the ring $R[\mathbb{Z}^{n-1}]$ can be implemented on the spectrum level, see for instance [668, Theorem 4.2], and yields because of the identity $(R[\mathbb{Z}^{n-1}])[\mathbb{Z}] = R[\mathbb{Z}^n]$ for every $H \rtimes_\phi \mathbb{Z}$-$CW$-complex X and every $i \in \mathbb{Z}$ an isomorphism, natural in X,

$$\begin{aligned} H_m^{H \rtimes_\phi \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \oplus H_{m-1}^{H \rtimes_\phi \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \oplus H_m^{H \rtimes_\phi \mathbb{Z}}(X; \mathbf{NK}_{R[\mathbb{Z}^{n-1}]}) \\ \oplus H_m^{H \rtimes_\phi \mathbb{Z}}(X; \mathbf{NK}_{R[\mathbb{Z}^{n-1}]}) \xrightarrow{\cong} H_m^{H \rtimes_\phi \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^n]}). \end{aligned}$$

Since a direct sum of an isomorphism is again an isomorphism and we can apply the latter isomorphism to $X = E\mathbb{Z}$ and $X = \{\bullet\}$, the map

$$H_k^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \to H_k^{H \rtimes_\phi \mathbb{Z}}(E\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^{n-1}]})$$

is bijective for $k = m-1, m$. Now the induction hypothesis implies that

$$H_i^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to H_i^{H \rtimes_\phi \mathbb{Z}}(\{\bullet\}; \mathbf{K}_R)$$

is bijective for $m - n \leq i \leq m$. This finishes the proof of Lemma 4.14. $\square$

Consider a ring R together with a ring automorphism $\Psi\colon R \xrightarrow{\cong} R$. We can think of Ψ as a group homomorphism $\Psi\colon \mathbb{Z} \to \operatorname{aut}(R)$. For a subgroup $L \subseteq \mathbb{Z}$, let $\mathbf{K}(R_{\psi|_L}[L])$ be the non-connective algebraic K-theory spectrum of the $\Psi|_L$-twisted group ring of L with coefficient in R for the group homomorphism $\Psi|_L\colon L \to \operatorname{aut}(R)$. We obtain a covariant $\mathsf{Or}(\mathbb{Z})$-spectrum $\mathbf{K}_{R,\Psi}$ by sending $\mathbb{Z}/L$ to $\mathbf{K}(R_{\Psi|_L}[L])$. Note that for two subgroups $L, L' \subseteq \mathbb{Z}$ the set $\operatorname{mor}_{\mathsf{Or}(\mathbb{Z})}(\mathbb{Z}/L, \mathbb{Z}/L')$ is empty if $L \not\subseteq L'$, and consists of precisely one element, the canonical projection $\mathbb{Z}/L \to \mathbb{Z}/L'$ if $L \subseteq L'$. In the case $L \subseteq L'$ the functor $\mathbf{K}_{R,\Psi}$ sends this morphism to the map of spectra induced by the inclusion of rings $R_{\Psi|_L}[L] \to R_{\Psi|_{L'}}[L']$.

Lemma 13.52. *Let R be a regular ring and $\Psi\colon R \to R$ be a ring automorphism. Then the map*

$$H_m^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R,\Psi}) \to H_m^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R,\Psi}) = K_m(R_\Psi[\mathbb{Z}])$$

is an isomorphisms for all $m \in \mathbb{Z}$.

Proof. This is a special case of Theorem 13.48 but we describe as an illustration a more elementary proof.

There is a twisted Bass-Heller-Swan decomposition for non-negative K-theory, see [671, Theorem 0.1], which reduces to the desired isomorphism if the twisted Nil terms $NK_m(R,\Psi)$ vanish for $m \in \mathbb{Z}$. By inspecting the definition of the non-connective K-theory spectrum of [668] one sees that it suffices to show the bijectivity

$$H_m^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{R[\mathbb{Z}^n],\Psi[\mathbb{Z}^n]}) \to H_m^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_{R[\mathbb{Z}^n],\Psi[\mathbb{Z}^n]}) = K_m((R[\mathbb{Z}^n])_{\Psi[\mathbb{Z}^n]}[\mathbb{Z}])$$

for all $n,m \in \mathbb{Z}$ with $m \geq 1$ and $n \geq 0$. Since R is regular, Theorem 3.80 (ii) shows that $R[\mathbb{Z}^n]$ is regular for every $n \geq 0$. Hence it suffices to prove Lemma 13.52 only for $m \geq 1$. This has already be done by Waldhausen [955, Theorem 4 on page 138 and the Remark on page 216]. One may also refer to [419, Remark on page 362].

One may also refer for the proof of Lemma 13.52 to [80, Theorem 7.8 and Theorem 10.1], where more generally additive categories are treated. □

Consider a group H together with an automorphism $\phi\colon H \to H$. Let $p\colon H \rtimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ be the projection. Then we get from the adjunction between p_* and p^*, see [265, Lemma 1.9], for every $\mathbb{Z}$-CW-complex X and all $m,n \in \mathbb{Z}, n \geq 0$ an isomorphism, natural in X

$$H_m^{H\rtimes_\phi \mathbb{Z}}(p^*X;\mathbf{K}_{R[\mathbb{Z}^n]}) \xrightarrow{\cong} H_m^{\mathbb{Z}}(X;p_*\mathbf{K}_{R[\mathbb{Z}^n]}). \tag{13.53}$$

From the definitions we get

$$p_*\mathbf{K}_{R[\mathbb{Z}^n]}(\mathbb{Z}/L) = \mathbf{K}_{R[\mathbb{Z}^n]}((H \rtimes_\phi \mathbb{Z})/p^{-1}(L)) = \mathbf{K}(R[\mathbb{Z}^n][H \rtimes_{\phi|_L} L])$$

for any object $\mathbb{Z}/L$ in $\mathsf{Or}(\mathbb{Z})$. Let $\Phi\colon RH \to RH$ be the ring automorphism induced by ϕ. It yields a ring automorphism $\Phi[\mathbb{Z}^n]\colon RH[\mathbb{Z}^n] \to RH[\mathbb{Z}^n]$. We have defined a covariant $\mathsf{Or}(\mathbb{Z})$-spectrum $\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}$ before Lemma 13.52, just take $\Psi = \Phi[\mathbb{Z}^n]$. There is a weak equivalence of covariant $\mathsf{Or}(\mathbb{Z})$-spectra

$$\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]} \xrightarrow{\cong} p_*\mathbf{K}_{R[\mathbb{Z}^n]}$$

coming from the identification $R[H][\mathbb{Z}^n]_{\Phi|_L[\mathbb{Z}^n]}[L] = R[\mathbb{Z}^n][H \rtimes_{\phi|_L} L]$. This implies using [265, Theorem 3.11] that the next lemma is true.

Lemma 13.54. *We get for every $\mathbb{Z}$-CW-complex X and all $m,n \in \mathbb{Z}, n \geq 0$ an isomorphism, natural in X*

$$H_m^{\mathbb{Z}}(X;\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) \xrightarrow{\cong} H_m^{H\times_\phi \mathbb{Z}}(p^*X;\mathbf{K}_{R[\mathbb{Z}^n]}).$$

Lemma 13.55. *Let H be a finite group and let $\phi\colon H \xrightarrow{\cong} H$ be an automorphism. Let R be a Artinian ring. Then the map*

$$H_0^{H \rtimes_\phi \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_0^{H \rtimes_\phi \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]}) = K_0((R[\mathbb{Z}^n])[H \rtimes_\phi \mathbb{Z}])$$

is an isomorphisms for all $n \in \mathbb{Z}, n \geq 0$.

Proof. We conclude from Lemma 13.54 that it remains to show that the map

$$H_0^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) \to H_0^{\mathbb{Z}}(\{\bullet\}; \mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) = K_0(RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}])$$

is bijective for all $n \geq 1$.

Denote by $J \subseteq RH$ the Jacobson radical of RH. Since RH is Artinian, J is nilpotent, i.e., there exists a natural number m with $J^m = \{0\}$, see [594, Theorem 4.12 on page 56]. The ring RH/J is a semisimple Artinian ring, see [594, Definition 20.1 on page 311 and (20.3) on page 312], and in particular regular.

The ring automorphism $\Phi\colon RH \to RH$ induced by ϕ obviously satisfies $\Phi(J) = J$ and hence induces a ring automorphism $\overline{\Phi}\colon RH/J \to RH/J$. Hence we get a commutative diagram induced by the projection $RH \to RH/J$.

$$\begin{array}{ccc} H_0^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) & \longrightarrow & K_0(RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}]) \\ \downarrow & & \downarrow \\ H_0^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{(RH/J)[\mathbb{Z}^n],\overline{\Phi}[\mathbb{Z}^n]}) & \longrightarrow & K_0((RH/J)[\mathbb{Z}^n]_{\overline{\Phi}[\mathbb{Z}^n]}[\mathbb{Z}]). \end{array} \tag{13.56}$$

We have the short exact sequence of abelian groups $0 \to J \to RH \to RH/J \to 0$. It induces a short exact sequence of abelian groups

$$0 \to J[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]|_{J[\mathbb{Z}^n]}}[\mathbb{Z}] \to RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}] \to (RH/J)[\mathbb{Z}^n]_{\overline{\Phi}[\mathbb{Z}^n]}[\mathbb{Z}] \to 0.$$

Hence we can identify the ring $(RH/J)[\mathbb{Z}^n]_{\overline{\Phi}[\mathbb{Z}^n]}[\mathbb{Z}]$ with the quotient of the ring $RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}]$ by the ideal $J[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]|_{J[\mathbb{Z}^n]}}[\mathbb{Z}]$. Recall that an ideal I in a ring is nilpotent if and only if there is a natural number l such that for any collection of l elements $i_1, i_2, \ldots i_l$ in I the product $i_1 i_2 \cdots i_l$ vanishes. Since J is nilpotent, we conclude that the ideal $J[\mathbb{Z}^n] \rtimes_{\Phi[\mathbb{Z}^n]|_{J[\mathbb{Z}^n]}}[\mathbb{Z}]$ is nilpotent. Hence the right vertical arrow in the diagram (13.56) is bijective by Lemma 2.124.

Next we show that the left vertical arrow in the diagram (13.56) is bijective. Since $E\mathbb{Z}$ is a free $\mathbb{Z}$-CW-complex, we conclude from the equivariant Atiyah-Hirzebruch spectral sequence described in Theorem 12.48 that it suffices to show for every i that the map $K_i(RH[\mathbb{Z}^n]) \to K_i((RH/J)[\mathbb{Z}^n])$ is bijective for all $i \leq 0$.

Since J is a nilpotent two-sided ideal of RH, $J[\mathbb{Z}^n]$ is a nilpotent two-sided ideal of $RH[\mathbb{Z}^n]$. We can identify $(RH/J)[\mathbb{Z}^n]$ with $(RH[\mathbb{Z}^n])/(J[\mathbb{Z}^n])$. Hence $K_0(RH[\mathbb{Z}^n]) \to K_0((RH/J)[\mathbb{Z}^n])$ is bijective by Lemma 2.124. We conclude $K_i(RH[\mathbb{Z}^n]) = 0$ for $i \leq -1$ from Theorem 4.16 (ii). Since RH/J is regular and hence $RH/J[\mathbb{Z}^n]$ is regular by Theorem 3.80 (ii), we conclude from Theorem 4.7 that $K_i((RH/J)[\mathbb{Z}^n]) = 0$ for $i \leq -1$. Hence the left vertical arrow in the diagram (13.56) is bijective. The lower vertical arrow in the diagram (13.56) is bijective because of Lemma 13.52 applied to the automorphism $\overline{\Phi}[\mathbb{Z}^n]$. We conclude that the upper vertical arrow in the diagram (13.56) is bijective. This finishes the proof of Lemma 13.55 □

13.10.2 Reducing the Family of Subgroups for the Farrell-Jones Conjecture for *L*-Theory

Theorem 13.57 (Passage from $\mathcal{FIN}$ to $\mathcal{VCY}_I$ for L-theory). *Let G be a group and let $\mathcal{A}$ be an additive G-category with involution. Let n be any integer. Then*

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}_I};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\to H_n^G\big(E_{\mathcal{VCY}_I}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)$$

is bijective.

Proof. The argument given in [639, Lemma 4.2] goes through since it is based on the Wang sequence for a semidirect product $G\rtimes\mathbb{Z}$ which can be generalized for additive G-categories with involutions as coefficients. □

The last result is very useful when G does not contain virtually cyclic subgroups of type II since then one can replace in Conjectures 13.4, 13.7 and 13.16 the family $\mathcal{VCY}$ by the family $\mathcal{FIN}$. (This is not true for Conjecture 13.25 since $G\wr F$ for a finite group F may contain a virtually cyclic subgroup of type II even in the case that G does not contain a virtually cyclic subgroup of type II.)

Exercise 13.58. Consider the group extension $1\to F\to G\xrightarrow{f}\mathbb{Z}^d\to 1$ for a finite group F. Show that there exists a spectral sequence converging to $L_{p+q}^{\langle-\infty\rangle}(\mathbb{Z}G)$ whose E^2-term is given by $H_p(C_*(E\mathbb{Z}^d)\otimes_{\mathbb{Z}[\mathbb{Z}^d]}L_q^{\langle-\infty\rangle}(\mathbb{Z}F))$ where the $\mathbb{Z}^d$-action on $L_q^{\langle-\infty\rangle}(\mathbb{Z}F)$ is induced by the conjugation action of G on F.

Let p be a prime. A finite group G is called *p-elementary* if it is isomorphic to $C\times P$ for a cyclic group C and a p-group P such that the order $|C|$ is prime to p. Let $\mathcal{E}_p$ be the class of of finite subgroups that are p-elementary.

Theorem 13.59 (Bijectivity of the L-theoretic assembly map from $\mathcal{FIN}$ to $\mathcal{VCY}$ after inverting 2). *Let G be a group and let R be a ring with involution.*

(i) The relative assembly map

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{L}_R^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)\to H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)$$

is bijective for all $n\in\mathbb{Z}$ after applying $\mathbb{Z}[1/2]\otimes_{\mathbb{Z}}-$;

(ii) Put

$$\mathcal{F} = \bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p$$

Then the relative assembly map

$$H_n^G\big(\iota_{\mathcal{F}\subseteq\mathcal{FIN}};\mathbf{L}_R^{\langle -\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{F}}(G);\mathbf{L}_R^{\langle -\infty\rangle}\big) \to H_n\big(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}\big)$$

is bijective for all $n \in \mathbb{Z}$

Proof. (i) See [657, Proposition 74 on page 747].

(ii) This is a variation of the proof of assertion (ii) of Theorem 13.50 taking [75, Section 15] into account. ◻

Theorem 13.59 shows that Conjecture 13.4 implies Conjecture 13.6 and hence Conjecture 13.7 implies Conjecture 13.8.

Note that in Theorem 13.59 we consider only rings with involution with trivial G-action. It is conceivable that it carries over twisted group rings, but we have not checked the details of a proof of this claim. It is unclear whether it carries over to additive categories with involutions since UNil-terms have not been defined and investigated for additive categories.

13.11 The Full Farrell-Jones Conjecture Implies All Its Variants

Recall that the Full Farrell-Jones Conjecture 13.27 implies the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, see Remarks 13.12 and 13.17. In this section we give the proofs that Conjectures 13.2 and 13.7 imply all the variants we have stated before at various places for rings as coefficients. So the Full Farrell-Jones Conjecture is the "master" conjecture that implies all variants stated in this book for rings as coefficients.

For the reader's convenience we recall all these variants below before we show how they follow from Full Farrell-Jones Conjecture 13.27.

13.11.1 List of Variants of the Farrell-Jones Conjecture

We begin with the K-theoretic variants.

Conjecture 2.60 (Farrell-Jones Conjecture for $K_0(R)$ for torsionfree G and regular R). Let G be a torsionfree group and let R be a regular ring. Then the map induced by the inclusion of the trivial group into G

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective.

In particular we get for any principal ideal domain R and torsionfree G

$$\widetilde{K}_0(RG) = 0.$$

Conjecture 2.67 (Farrell-Jones Conjecture for $K_0(RG)$ for regular R with $\mathbb{Q} \subseteq R$). Let R be a regular ring with $\mathbb{Q} \subseteq R$ and G be a group.

Then the homomorphism

$$I_{\mathcal{FIN}}(G,F) \colon \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG).$$

coming from the various inclusions of finite subgroups of G into G is a bijection.

Here is a stronger version of Conjecture 2.67.

Conjecture 2.69. [Farrell-Jones Conjecture for $K_0(RG)$ for regular R] Let R be a regular ring and let G be a group. Let $\mathcal{P}(G,R)$ be the set of primes which are not invertible in R and for which G contains an element of order p.

Then the homomorphism

$$I_{\mathcal{FIN}}(G,F) \colon \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

coming from the various inclusions of finite subgroups of G into G is an $\mathcal{P}(G,R)$-isomorphism, i.e., an isomorphism after inverting all prime in $\mathcal{P}(G,R)$.

Conjecture 2.72 (Farrell-Jones Conjecture for $K_0(RG)$ for an Artinian ring R) Let G be a group and R be an Artinian ring. Then the canonical map

$$I_{\mathcal{FIN}}(G,R) \colon \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

is an isomorphism.

Conjecture 2.102 (The rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture) The change of ring maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Z}G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Q}G)$$

is trivial.

Conjecture 3.109 (Farrell-Jones Conjecture for $K_0(RG)$ and $K_1(RG)$ for regular R and torsionfree G). Let G be a torsionfree group, and let R be a regular ring. Then the maps defined in (3.26) and (3.27)

$$A_0\colon K_0(R) \xrightarrow{\cong} K_0(RG);$$
$$A_1\colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(RG),$$

are both isomorphisms. In particular the groups $\mathrm{Wh}_0^R(G)$ and $\mathrm{Wh}_1^R(G)$, see Definition 3.28, vanish.

Conjecture 3.110 (Farrell-Jones Conjecture for $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ for torsionfree G). Let G be a torsionfree group. Then $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Conjecture 4.18 (The Farrell-Jones Conjecture for negative K-theory and regular coefficient rings). Let R be a regular ring and G be a group such that for every finite subgroup $H \subseteq G$ the element $|H| \cdot 1_R$ of R is invertible in R. Then

$$K_n(RG) = 0 \quad \text{for} \quad n \le -1.$$

Conjecture 4.20 (The Farrell-Jones Conjecture for negative K-theory of the ring of integers in an algebraic number field). Let R be the ring of integers in an algebraic number field. Then, for every group G, we have

$$K_n(RG) = 0 \quad \text{for } n \le -2,$$

and the map

$$\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_{-1}(RH) \xrightarrow{\cong} K_{-1}(RG)$$

is an isomorphism.

Conjecture 4.21 (The Farrell-Jones Conjecture for negative K-theory and Artinian rings as coefficient rings) Let G be a group and let R be a Artinian ring. Then

$$K_n(RG) = 0 \quad \text{for} \quad n \le -1.$$

Conjecture 5.22 (Farrell-Jones Conjecture for $\mathrm{Wh}_2(G)$ for torsionfree G). Let G be a torsionfree group. Then $\mathrm{Wh}_2(G)$ vanishes.

Conjecture 6.53 (Farrell-Jones Conjecture for torsionfree groups and regular rings for K-theory). Let G be a torsionfree group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Conjecture 6.59 (Nil-groups for regular rings and torsionfree groups). Let G be a torsionfree group and let R be a regular ring. Then

$$NK_n(RG) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Conjecture 6.74 (Farrell-Jones Conjecture for torsionfree groups for homotopy K-theory). Let G be a torsionfree group. Then the assembly map

$$H_n(BG; \mathbf{KH}(R)) \to KH_n(RG)$$

is an isomorphism for every $n \in \mathbb{Z}$ and every ring R.

Conjecture 6.76 (Comparison of algebraic K-theory and homotopy K-theory for torsionfree groups). Let R be a regular ring and let G be a torsionfree group. Then the canonical map

$$K_n(RG) \to KH_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture 13.1 (K-theoretic Farrell-Jones Conjecture with coefficients in the ring R) Given a group G and a ring R, we say that G satisfies the *K-theoretic Farrell-Jones Conjecture with coefficients in the ring R* if the assembly map induced by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Next we list the L-theoretic variants.

Conjecture 9.114 (Farrell-Jones Conjecture for torsionfree groups for L-theory) Let G be a torsionfree group. Let R be any ring with involution.

Then the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Conjecture 13.6 (L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R after inverting 2) Given a group G and ring with involution R, we say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R after inverting 2* if the assembly map given by the projection $E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$ after inverting 2.

Conjecture 13.4 (L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R) Given a group G and ring with involution R, we say that G satisfies the *L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R* if the assembly map given by the projection $E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Finally we mention the following Novikov type conjectures.

Conjecture 13.60 (K-theoretic Novikov Conjecture). A group G satisfies the *K-theoretic Novikov Conjecture* if the assembly map

$$H_n(BG;\mathbf{K}(\mathbb{Z})) = H_n^G(EG;\mathbf{K}(\mathbb{Z})) \to H_n^G(G/G;\mathbf{K}(\mathbb{Z})) = K_n(\mathbb{Z}G)$$

is rationally injective for all $n \in \mathbb{Z}$.

Conjecture 13.61 (L-theoretic Novikov Conjecture). A group G satisfies the *L-theoretic Novikov Conjecture* if the assembly map

$$\begin{aligned} H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) = H_n^G(EG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \\ \to H_n^G(G/G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G) \end{aligned}$$

is rationally injective for all $n \in \mathbb{Z}$.

13.11.2 Proof of the Variants of the Farrell-Jones Conjecture

Theorem 13.62 (The Full Farrell-Jones Conjecture implies all other variants).

(i) The Full Farrell-Jones Conjecture 13.27 implies the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution;
(ii) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjecture 6.53, whereas the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involutions implies Conjecture 13.6 and Conjecture 9.114;
(iii) Conjecture 6.53 implies Conjectures 2.60, 3.109, and 3.110;
(iv) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 2.67, 2.69, 2.102, and 4.18;
(v) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 2.72 and 4.21;
(vi) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjecture 4.20;
(vii) Conjecture 6.53 implies Conjecture 5.22;

(viii) Conjecture 6.53 implies Conjecture 6.59;
(ix) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 6.74 and 6.76;
(x) The K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$ implies the K-theoretic Novikov Conjecture 13.60;
(xi) The L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring $\mathbb{Z}$ implies the L-theoretic Novikov Conjecture 13.61. The L-theoretic Novikov Conjecture 13.61 implies the Novikov Conjecture 9.137;
(xii) The Full Farrell-Jones Conjecture 13.27 implies all other variants of the Farrell-Jones Conjecture.

Proof. (i) See Remarks 13.12 and 13.17.

(ii) Conjecture 13.4 implies Conjecture 13.6 by Theorem 13.59.

Next we show why Conjecture 13.2 implies Conjecture 6.53 and why Conjecture 13.7 implies Conjecture 9.114. Every torsionfree virtually cyclic group is isomorphic to $\mathbb{Z}$ by Lemma 13.39. By the Transitivity Principle 15.13 applied to $\mathcal{TR} \subseteq \mathcal{VCY}$ it suffices to show that the assembly maps

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to K_n(R\mathbb{Z});$$
$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R\mathbb{Z}),$$

are bijective for $n \in \mathbb{Z}$. This follows for K-theory from the Bass-Heller-Swan decomposition, see Theorem 6.16, and for L-theory from the Shaneson splitting, see (9.109).

(iii) Since R is regular, the negative K-groups of R vanish by Theorem 4.7. Hence the Atiyah-Hirzebruch spectral sequence, which has E^2-term $E^2_{p,q} = H_p(BG; K_q(R))$ and converges to $H_{p+q}(BG; \mathbf{K}(R))$, is a first quadrant spectral sequence. The edge homomorphism $H_0(BG; K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}(R))$ at $(0,0)$ is bijective. There is an obvious identification $H_0(BG; K_0(R)) \cong K_0(R)$. Under this identification the edge homomorphism composed with the assembly map appearing in Conjecture 6.53 turns out to be the change of rings map $K_0(R) \to K_0(RG)$. Hence we conclude from Conjecture 6.53 that $K_0(R) \to K_0(RG)$ is bijective as predicted by Conjecture 2.60. Inspecting the Atiyah-Hirzebruch spectral yields an exact sequence $0 \to H_0(BG; K_1(R)) \to H_1(BG; \mathbf{K}(R)) \to H_1(BG; K_0(R)) \to 0$. Under the obvious identification $H_0(BG; K_1(R)) = K_1(R)$ the composite of $H_0(BG; K_1(R)) \to H_1(BG; \mathbf{K}(R))$ with the assembly map appearing in Conjecture 6.53 turns out to be the change of rings map $K_1(R) \to K_1(RG)$. Since $H_1(BG; K_0(R)) = G/[G,G] \otimes K_0(R)$, we obtain an exact sequence

$$0 \to K_1(R) \to K_1(RG) \to G/[G,G] \otimes K_0(R) \to 0.$$

Next one checks that the composite of the map $K_1(RG) \to G/[G,G] \otimes K_0(R)$ appearing in the sequence above with the map A_1 appearing in Conjecture 3.109 is the obvious projection. This implies Conjecture 3.109, and hence also Conjecture 3.110.

(iv) See [87, Theorem 1.5], [657, Proposition 87 on page 754] and [657, paragraph before Conjecture 79 on page 750] for the proof for Conjectures 2.67, 2.102, and 4.18. For the proof of Conjecture 2.69 is analogous if one uses Theorem 13.48.

(v) We conclude from Lemma 13.51 and Lemma 13.55 that the assembly map $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \to K_n(RG)$ is an isomorphism for $n \leq 0$. We have $K_i(RH) = 0$ for every finite group H and every $i \leq -1$ by Theorem 4.16 (ii). We conclude from the equivariant Atiyah-Hirzebruch spectral sequence described in Theorem 12.48 that $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) = 0$ holds for $n \leq -1$ and that $H_0^{H \rtimes_\phi \mathbb{Z}}(E_{\mathcal{FIN}}(H \times_\phi \mathbb{Z}), \mathbf{K}_R)$ is the 0-th Bredon homology of $E_{\mathcal{FIN}}(H \times_\phi \mathbb{Z})$ with coefficients in the covariant functor $\mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{MOD}$ sending G/K to $K_n(RK)$. This 0-th Bredon homology can be identified with $\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{FIN}}(G)} K_0(RH)$. Under this identification the bijective assembly map $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \to K_n(RG)$ becomes the canonical map $\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$.

(vi) See [657, page 749]. The proof goes through if we replace $\mathbb{Z}$ by the ring R of integers in an algebraic number field since the results appearing in [353] for $\mathbb{Z}$ have been extended to R by Juan-Pineda [513].

(vii) We conclude from [619] that the second Whitehead group can be identified with the cokernel of the assembly map

$$H_2(\mathrm{pr}; \mathbf{K}_R) \colon H_2^G(EG; \mathbf{K}_R) = H_2(BG; \mathbf{K}(\mathbb{Z})) \to H_2^G(EG; \mathbf{K}_R) = K_2(\mathbb{Z}G).$$

(viii) We conclude from Theorem 3.80 that $R[t]$ is regular. We have the obvious commutative diagram

$$\begin{tikzcd}
H_n(BG; \mathbf{K}(R[t])) \arrow[r, "\cong"] \arrow[d, "\cong"'] & K_n(R[t]G) = K_n(RG[t]) \arrow[d] \\
H_n(BG; \mathbf{K}(R)) \arrow[r, "\cong"'] & K_n(RG)
\end{tikzcd}$$

whose horizontal arrows are bijective by the assumption that Conjecture 13.2 holds and whose left vertical arrow is bijective since $K_n(R[t]) \to K_n(R)$ is bijective for all $n \in \mathbb{Z}$ by Theorem 6.16 (ii). Hence the right vertical arrow is bijective which implies by definition $NK_n(RG) = 0$.

(ix) This follows from [74, Theorem 8.4 and Remark 8.6].

(x) This follows from Theorem 13.33 and Lemma 13.35.

(xi) The L-theoretic Novikov Conjecture 13.61 follows from the L-theoretic Farrell-Jones Conjecture 13.4 because of Theorem 13.34 and Lemma 13.35.

For the proof that the L-theoretic Novikov Conjecture 13.61 for G implies the Novikov Conjecture 9.137 for G, we refer to [570, Lemma 23.2 on page 192] and [825, Proposition 6 on page 300]. Or just take a look at Remark 9.143

and use the fact that under the Chern character the assembly map

$$\operatorname{asmb}_n^G\colon \bigoplus_{k\in\mathbb{Z}} H_{n+4k}(BG;\mathbb{Q}) \to L_n^h(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q},$$

appearing in Remark 9.143 can be identified with the assembly map appearing in L-theoretic Novikov Conjecture 13.61 for G.

(xii) The Full Farrell-Jones Conjecture 13.27 implies Conjectures 13.2 and 13.7, see Remarks 13.12 and 13.17. Now the claim follows from all the other assertions which we have already proved. □

13.12 Summary of the Applications of the Farrell-Jones Conjecture

We have discussed at various places applications and consequences of the various versions of the Farrell-Jones Conjecture. In Theorem 13.62 we have explained that the Full Farrell-Jones Conjecture 13.27 implies all of these variants of the Farrell-Jones Conjecture and hence all these applications and consequences. For the reader's convenience we list now all these applications and where they are treated in this book or in the literature.

- *Wall's Finiteness Obstruction*
  Wall's finiteness obstruction of a connected finitely dominated CW-complex X takes values in $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ and vanishes if and only if X is homotopy equivalent to a finite CW-complex, see Section 2.5. For torsionfree $\pi_1(X)$ Conjecture 2.60 predicts that $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes and hence X is always homotopy equivalent to a finite CW-complex, see Remark 2.61.
- *Kaplansky's Idempotent Conjecture*
  Kaplansky's Idempotent Conjecture 2.73 predicts for an integral domain R and a torsionfree group G that all idempotents of RG are trivial. See Section 2.9.
- *The Bass Conjectures*
  The Bass Conjecture 2.91 for fields of characteristic zero as coefficients says for a field F of characteristic zero and a group G that the Hattori-Stallings homomorphism of (2.87) induces an isomorphism

  $$\mathrm{HS}_{FG}\colon K_0(FG)\otimes_{\mathbb{Z}} F \to \operatorname{class}_F(G)_f.$$

  This essentially generalizes character theory for finite dimensional representations over finite groups to finitely generated projective modules over infinite groups.
  The Bass Conjecture 2.98 for integral domains as coefficients predicts for a commutative integral domain R, a group G, and $g\in G$ that for every

finitely generated projective RG-module the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial provided that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
For more information about the Bass Conjectures, we refer to Section 2.10.

- *Whitehead torsion*
One can assign to a homotopy equivalence $f\colon X \to Y$ of connected finite CW-complexes its Whitehead torsion $\tau(f)$, which takes values in the Whitehead group $\mathrm{Wh}(\pi_1(Y))$, see Sections 3.3. It vanishes if and only f is a simple homotopy equivalence, see Section 3.4
An h-cobordism of dimension ≥ 6 is trivial if and only if its Whitehead torsion vanishes, see Theorem 3.47.
If the group G is torsionfree, then Conjecture 3.110 predicts that $\mathrm{Wh}(G)$ vanishes. Hence Conjecture 3.110 implies that a homotopy equivalence of connected finite CW-complexes is simple if $\pi_1(Y)$ is torsionfree, and that every connected h-cobordism W of dimension ≥ 6 with torsionfree $\pi_1(W)$ is trivial, see Remark 3.112.
- *Bounded h-cobordisms*
There are so-called bounded h-cobordisms, controlled over $\mathbb{R}^k$, for $k \geq 1$. They are trivial (for dimension ≥ 6) if and only if certain elements in negative K-groups $\widetilde{K}_{1-k}(\mathbb{Z}G)$ vanish, see Section 4.3. Conjecture 4.18 predicts for a torsionfree group G the vanishing of $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$.
- *Pseudoisotopy and the second Whitehead group*
There is a certain obstruction for pseudoisotopies to be trivial, which take values in $\mathrm{Wh}_2(G)$, see Section 5.6. Conjecture 5.22 predicts for a torsionfree group G the vanishing of $\mathrm{Wh}_2(G)$.
- *Whitehead spaces and pseudoisotopy spaces*
One can assign to a compact manifold M its pseudoisotopy spaces $\mathcal{P}(M)$ and $\mathcal{P}^{\mathsf{DIFF}}(M)$, Whitehead spaces $\mathrm{Wh}^{\mathsf{PL}}(X)$ and $\mathrm{Wh}^{\mathsf{DIFF}}(X)$, and its A-theory $A(X)$ in the sense of Waldhausen, see Section 7.2 and 7.3. There also exist non-connective versions. There are various relations between these spaces. The homotopy groups of $A(M)$ are related to the K-groups $K_n(\mathbb{Z}[\pi_1(M)])$.
Conjecture 6.53 predicts for a torsionfree group G and a regular ring R that the assembly map
$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$
is an isomorphism for $n \in \mathbb{Z}$. It implies for an aspherical closed manifold M for all $n \geq 0$, see Theorem 7.27 and Theorem 7.32,

$$\begin{aligned}
\pi_n(\mathrm{Wh}^{\mathsf{PL}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong 0;\\
\pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong 0;\\
\pi_n(\mathrm{Wh}^{\mathsf{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(M;\mathbb{Q});\\
\pi_n(\mathcal{P}^{\mathsf{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M;\mathbb{Q}).
\end{aligned}$$

- *Automorphisms of manifolds*
  If Conjecture 6.53 and Conjecture 9.114 hold for the torsionfree group G and the ring $R = \mathbb{Z}$, then some rational computations of the homotopy groups of the automorphism group of an aspherical closed manifold M with $G = \pi_1(M)$ can be found in Theorems 9.192 and 9.193.
- *Novikov Conjecture*
  The Novikov Conjecture 9.137 for a group G predicts the homotopy invariants of the higher signatures

  (13.63) $$\operatorname{sign}_x(M,u) := \langle \mathcal{L}(M) \cup u^*x, [M]_{\mathbb{Q}} \rangle \qquad \in \mathbb{Q}$$

  of a closed oriented manifold M coming with a reference map $f\colon M \to BG$ for an element $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$, see Subsection 9.14.1.
  We conclude from Theorem 13.62 (xi) that the L-theoretic Farrell Jones Conjecture 13.4 for the group G and the ring $\mathbb{Z}$ implies the Novikov Conjecture 9.137 for G.
- *Borel Conjecture*
  The Borel Conjecture 9.160 predicts that any aspherical closed topological manifold M is topologically rigid, i.e, if N is another aspherical closed topological manifold with $\pi_1(M) \cong \pi_1(N)$, then M and N are homeomorphic and any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.
  Let G be a finitely presented group. Suppose that it satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$. Then Theorem 9.168 shows that every aspherical closed topological manifold of dimension ≥ 5 with G as fundamental group is topologically rigid.
- *Poincaré duality groups*
  Conjecture 9.180 predicts that a finitely presented group is an n-dimensional Poincaré duality group if and only if it is the fundamental group of an aspherical closed n-dimensional topological manifold.
  Suppose that the torsionfree group G is a finitely presented Poincaré duality group of dimension $n \geq 6$ and satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$.

Let X be a Poincaré complex of dimension ≥ 6 with $\pi_1(X) \cong G$. Suppose that its Spivak normal fibration has a TOP-reduction.

Then X is homotopy equivalent to a compact homology ANR-manifold satisfying the disjoint disk property, see Theorem 9.181.

- *Boundaries of hyperbolic groups*
  As a consequence of the Farrell-Jones Conjecture, we get Theorem 9.185 which says for a torsionfree hyperbolic group G and $n \geq 6$ that the following statements are equivalent:
  - The boundary ∂G is homeomorphic to S^{n-1};
  - There is an aspherical closed topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$, and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n;

  Moreover the aspherical closed topological manifold M appearing above is unique up to homeomorphism.
- *Stable Cannon Conjecture*
  A stable version of the Cannon Conjecture is known to be true, see Theorem 9.189.
- *Product decompositions of aspherical closed manifolds*
  Theorem 9.191 deals with the question when for an aspherical closed topological manifold M a given algebraic decomposition $\pi_1(M) = G_1 \times G_2$ comes from the topological decomposition $M = M_1 \times M_2$. Theorem 9.191 is a consequence of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring $R = \mathbb{Z}$.
- *Classification of manifolds homotopy equivalent to certain torus bundles over lens spaces.*
  The K-theoretic Farrell-Jones Conjecture 13.1 and the L-theoretic Farrell-Jones Conjecture 13.4 play a key role in the paper [268] where a classification of manifolds homotopy equivalent to certain torus bundles over lens spaces is presented. See also [969].
- *Fibering manifolds*
  The K-theoretic Farrell-Jones Conjecture 13.1 and the L-theoretic Farrell-Jones Conjecture 13.4 play a key role in the paper [360] where the question is treated when for an aspherical closed manifold B and a map $p\colon M \to B$ from some closed connected manifold M the map p is homotopic to Manifold Approximate Fibration.
- *The Atiyah Conjecture*
  Conjecture 2.67 is related to the Atiyah Conjecture which makes predictions about the possibly values of the L^2-Betti numbers of coverings of closed Riemannian manifolds, see Remark 2.71.
- *Homotopy invariance of $\tau^{(2)}(M)$*
  Suppose that the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring R with involution is rationally true for $R = \mathbb{Z}$, i.e., the

rationalized assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for $n \in \mathbb{Z}$.
Then the Hirzebruch-type invariant $\tau^{(2)}(M)$ is a homotopy invariant, see Remark 14.60.

- *Homotopy invariance of the (twisted) L^2-torsion*
  The K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $R = \mathbb{Z}$ implies the homotopy invariance of L^2-torsion and of the L^2-torsion function, see [647, Theorem 7.5 (4)]. The twisted L^2-torsion function is related to the Thurston norm for appropriate 3-manifolds in [390].
- *Vanishing of κ-classes for aspherical closed manifolds*
  The vanishing of κ-classes for aspherical closed manifolds is analyzed in [458] using as one input the Full Farrell-Jones Conjecture 13.27.
- *Classification of 4-manifolds*
  Sometimes the Farrell-Jones Conjecture is needed as input in the (stable) classification of certain 4-manifolds, see for instance [438, 439, 552].
- *Group actions on manifolds*
  Applications of the Farrell-Jones Conjecture to manifolds with group actions are given for instance in [186, 237, 238, 243, 269, 650].

13.13 *G*-Theory

Instead of considering finitely generated projective modules, one may apply the standard K-theory machinery to the category of finitely generated modules. This leads to the definition of the groups $G_n(R)$ for $n \geq 0$. One can define them also for negative n using [884]. We have described $G_0(R)$ and $G_1(R)$ already in Definitions 2.1 and 3.1. One may ask whether versions of the Farrell-Jones Conjectures for G-theory instead of K-theory might be true. The answer is negative as the following discussion explains.

For a finite group H the ring $\mathbb{C}H$ is semisimple. Hence any finitely generated $\mathbb{C}H$-module is automatically projective and $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$. Recall that a group G is called *virtually poly-cyclic* if there exists a subgroup of finite index $H \subseteq G$ together with a filtration $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_r = H$ such that H_{i-1} is normal in H_i and the quotient H_i/H_{i-1} is cyclic. More generally for all $n \in \mathbb{Z}$ the forgetful map

$$f\colon K_n(\mathbb{C}G) \to G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic since then $\mathbb{C}G$ is regular [864, Theorem 8.2.2 and Theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [844, Corollary 53.26 on page 293]. In particular this applies to virtually cyclic groups and so the left hand side of the Farrell-Jones assembly map does not see the difference between K- and G-theory if we work with complex coefficients. We obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \arrow[r] \arrow[d, "\cong"'] & K_0(\mathbb{C}G) \arrow[d, "f"] \\
\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} G_0(\mathbb{C}H) \arrow[r] & G_0(\mathbb{C}G)
\end{tikzcd}$$

where, as indicated, the left hand vertical map is an isomorphism. Conjecture 2.67 which follows from the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{C}$ predicts that the upper horizontal arrow is an isomorphism. A G-theoretic analogue of Conjecture 2.67 would say that the lower horizontal map is an isomorphism. There are however cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map f on the right is not injective or not surjective, and hence the lower vertical arrow cannot be injective or surjective.

If G contains a non-abelian free subgroup, then the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [635, Theorem 9.66 on page 364] and hence the map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ has an infinite kernel since $[\mathbb{C}G]$ generates an infinite cyclic subgroup in $K_0(\mathbb{C}G)$. Note that Conjecture 13.1 is known for non-abelian free groups.

Conjecture 13.1 is also known for $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ and hence $K_0(\mathbb{C}A)$ is countable, whereas $G_0(\mathbb{C}A)$ is not countable [635, Example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know the answer to the following questions:

Question 13.64. If G is an amenable group for which there is an upper bound on the orders of its finite subgroups, is then the forgetful map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ an isomorphism?

Question 13.65. If the group G is not amenable, is then $G_0(\mathbb{C}G) = \{0\}$?

To our knowledge the answer to Question 13.65 is not even known in the special case $G = \mathbb{Z} * \mathbb{Z}$.

For more information about $G_0(\mathbb{C}G)$, we refer for instance to [635, Subsection 9.5.3].

Exercise 13.66. Let $H \subseteq G$ be a subgroup of G possessing an epimorphism $f\colon H \to \mathbb{Z}$. Show that the class of $\mathbb{C}[G/H]$ in $G_0(\mathbb{C}G)$ is trivial.

13.14 Notes

The original formulation of the Farrell-Jones Conjecture with rings as coefficients appears in [351, 1.6 on page 257]. Our formulation differs from the original one, but is equivalent, see Remark 15.42.

Proofs of some of the inheritance properties above are also given in [447, 862].

The inheritance properties of the Farrell-Jones Conjecture under actions of trees is discussed in [74], see also Section 6.9 and Section 15.7. The situation is much more complicated than for the Baum-Connes Conjecture 14.11 with coefficients where the optimal result holds, see Theorem 14.31 (v) and Remark 14.35.

In the sequel we consider classes $\mathcal{C}$ of groups that are closed under taking subgroups and passing to isomorphic groups. Examples are the classes of virtually cyclic or of finite groups. Given a group G, let $\mathcal{C}(G)$ be the family of subgroups of G that belong to $\mathcal{C}$. The relevant family of subgroups appearing in Conjectures 13.1, 13.2, 13.4, 13.7, 13.11, 13.16, 13.24, 13.25, and 13.27 is always given by $\mathcal{C}(G)$ where $\mathcal{C}$ is the class of virtually cyclic subgroups. We have proved various theorems where $\mathcal{C}$ could be chosen to be smaller, for instance to be the class of virtually cyclic groups of type I or of hyperelementary groups, see Theorems 13.43, 13.44, and 13.45. One may ask whether there is always a class $\mathcal{C}_{\min}$ for which such a conjecture holds for all groups G and which is minimal. Of course for the class of all groups such a conjecture will hold for trivial reasons. In the worst case $\mathcal{C}_{\min}$ may be just the class of all groups. A candidate for $\mathcal{C}_{\min}$ may be the intersection of all the classes $\mathcal{C}$ of groups for which the conjecture is true for all groups, but we do not know whether this intersection does satisfies the conjecture for all groups, see also Section 15.15 and in particular Lemma 15.101. At least we know that the intersection of two classes of groups $\mathcal{C}_0$ and $\mathcal{C}_1$ for which one of the Conjectures 13.11, 13.16, 13.24, 13.25 and 13.27 holds for all groups, satisfies this conjecture for all groups as well. We also know for two classes of subgroups $\mathcal{C} \subseteq \mathcal{D}$ such that $\mathcal{D}$ satisfies one of the Conjectures 13.11, 13.16, 13.24, 13.25 and 13.27 for all groups if $\mathcal{C}$ does. These claims follow from Theorem 15.9 (ii) and (iv), Theorem 15.13 (ii) and Lemma 15.14.

Further variants of the Farrell-Jones Conjecture for other theories such as A-theory, topological cyclic homology and Hochschild homology, homotopy K-theory, and the K-theory of Hecke algebras of totally disconnected groups will be discussed in Sections 15.10, 15.11, 15.12, and 15.13.

A coarse version of the Farrell-Jones Conjecture is treated in [1013].

A version of the Farrell-Jones Conjecture for polyhedra is proved in[140].

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Chapter 14
The Baum-Connes Conjecture

14.1 Introduction

In this chapter we discuss the Baum-Connes Conjecture 14.9 for the topological K-theory of the reduced group C^*-algebra $C_r^*(G, F)$ for $F = \mathbb{R}, \mathbb{C}$. It predicts that certain assembly maps

$$\begin{aligned} K_n^G(E_{\mathcal{FIN}}(G)) &\to K_n(C_r^*(G, \mathbb{C})); \\ KO_n^G(E_{\mathcal{FIN}}(G)) &\to KO_n(C_r^*(G, \mathbb{R})), \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$. The target is the topological K-theory of $C_r^*(G, F)$, which one wants to understand. The source is an expression that depends only on the values of these topological K-groups on finite subgroups of G and is therefore much more accessible. The version above is often the one which is relevant in concrete applications, but there is also a more general version, the Baum-Connes Conjecture 14.11 with coefficients, where one allows coefficients in a G-C^*-algebra. Note that in contrast to the Full Farrell-Jones Conjecture 13.27 it suffices to consider finite subgroups instead of virtually cyclic subgroups.

A status report of the Baum-Connes Conjecture 14.9 and its version 14.11 with coefficients will be given in Section 16.4.

The main point about the Baum-Connes Conjecture 14.9 is that it implies a great variety of other prominent conjectures such as the ones due to Kadison and Novikov, and leads to very deep and interesting results about manifolds and C^*-algebras, as we will record and explain in Section 14.8.

Variants of the Baum-Connes Conjecture 14.9 and its versions 14.11 with coefficient are presented in Section 14.5.

We will discuss the inheritance properties of the Baum-Connes Conjecture 14.11 with coefficients in Section 14.6.

We have tried to keep this chapter as much as possible independent of the other chapters, so that one may start reading directly here.

14.2 The Analytic Version of the Baum-Connes Assembly Map

Let A be a G-C^*-algebra over $F = \mathbb{R}, \mathbb{C}$. Denote by $A \rtimes_r G$ the C^*-algebra over F given by the reduced crossed product, see [786, 7.7.4 on page 262]. If A is $\mathbb{R}$ or $\mathbb{C}$ with the trivial G-action, this is the reduced real or complex reduced group C^*-algebra $C_r^*(G, \mathbb{R})$ or $C_r^*(G, \mathbb{C})$, see Subsection 10.3.1. Denote by $K_n(A \rtimes_r G)$ and $KO(A \rtimes_r G)$ their topological K-theory, as introduced in Subsection 10.3.2.

Let X be a proper G-CW-complex. Denote by $K_*^G(X; A)$ and $KO_*^G(X; A)$ the complex and real equivariant topological K-theory of X with coefficients in A, see Section 10.6. Note that $K_*^G(-; A)$ and $KO_*^G(-; A)$ are G-homology theories in the sense of Definition 12.1 such that $K_n^G(G/H; A) = K_n(A \rtimes H)$ and $KO_n^G(G/H; A) = KO_n(A \rtimes_r H)$ hold for any finite subgroup $H \subseteq G$ and $n \in \mathbb{Z}$, provided that we consider proper G-CW-complexes only.

We want to explain the *analytic Baum-Connes assembly map*

$$\operatorname{asmb}_A^{G,\mathbb{C}}(X)_n \colon K_n^G(X; A) \to K_n(A \rtimes_r G); \tag{14.1}$$

$$\operatorname{asmb}_A^{G,\mathbb{R}}(X)_n \colon KO_n^G(X; A) \to KO_n(A \rtimes_r G). \tag{14.2}$$

We will only treat the case $F = \mathbb{C}$, the case $F = \mathbb{R}$ is analogous.

We first consider the special case where X is proper and cocompact and then explain how the map extends by a colimit argument to arbitrary proper G-CW-complexes. Note that for a proper and cocompact G-CW-complex X we can identify $K_n^G(X; A)$ with the equivariant KK-groups $KK_n^G(C_0(X), A)$, see Section 10.6.

One description is in terms of indices with values in C^*-algebras. Namely, one assigns to a Kasparov cycle representing an element in $KK_n^G(C_0(X), A)$ its C^*-valued index in $K_n(A \rtimes G)$ in the sense of Mishchenko-Fomenko [723], thus defining a map $KK_n^G(C_0(X), A) \to K_n(A \rtimes G)$, provided that X is proper and cocompact. This is the approach appearing in [103].

The other equivalent approach is based on the Kasparov product. Given a proper cocompact G-CW-complex X, one can assign to it an element $[p_X] \in KK_0^G(\mathbb{C}, C_0(X) \rtimes_r G)$. Now define the map (14.1) by the composite of a descent map and a map coming from the Kasparov product

$$\begin{aligned} KK_n^G(C_0(X), A) &\xrightarrow{j_r^G} KK_n(C_0(X) \rtimes_r G, A \rtimes_r G) \\ &\xrightarrow{[p_X] \widehat{\otimes}_{C_0(X) \rtimes_r G} -} KK_n(\mathbb{C}, A \rtimes_r G) = K_n(A \rtimes G). \end{aligned}$$

For some information about these two approaches and their identification, we refer to [596] in the torsionfree case and to [171, 568] in the general case.

This extends to arbitrary proper G-CW-complexes X by the following argument. If $f \colon X \to Y$ is a G-map of proper cocompact G-CW-complexes, then f is a proper map (after forgetting the group action). Hence composition with f defines a homomorphism of G-C^*-algebras $C_0(f)\colon C_0(Y) \to C_0(X)$. Denote by $KK_n^G(C_0(f), \mathrm{id}_A)\colon KK_n^G(C_0(X), A) \to KK_n^G(C_0(Y), A)$ the induced map on equivariant KK-groups. One easily checks $\mathrm{asmb}^{G,\mathbb{C}}(Y)_n \circ KK_n^G(C_0(f), \mathrm{id}_A) = \mathrm{asmb}^{G,\mathbb{C}}(X)_n$. We conclude by inspecting definition (10.67) that for any proper G-CW-complex X the canonical map

$$\operatorname{colim}_{C \subseteq X} K_n^G(C) \xrightarrow{\cong} K_n^G(X)$$

is an isomorphism where C runs through the finite G-CW-subcomplexes of X directed by inclusion. Hence by a colimit argument over the directed systems of proper cocompact G-CW-subcomplexes the definition above for proper compact G-CW-complexes extends to the desired assembly maps (14.1) for any proper G-CW-complex X. Moreover, for any G-map of proper G-CW-complexes $f\colon X \to Y$, we obtain again by passing to the colimit a homomorphism $K_n^G(f)\colon K_n^G(X; A) \to K_n^G(Y; A)$ satisfying

$$\mathrm{asmb}^{G,\mathbb{C}}(Y)_n \circ K_n^G(f; A) = \mathrm{asmb}^{G,\mathbb{C}}(X)_n; \tag{14.3}$$

$$\mathrm{asmb}^{G,\mathbb{R}}(Y)_n \circ KO_n^G(f; A) = \mathrm{asmb}^{G,\mathbb{R}}(X)_n. \tag{14.4}$$

14.3 The Version of the Baum-Connes Assembly Map in Terms of Spectra

There is also a version of the Baum-Connes assembly map, which is very close to the construction of the one for the Farrell-Jones Conjecture. Namely, if we apply Theorem 12.30, taking Remark 12.31 into account, to the functor

$$\mathbf{K}_F^{\mathsf{TOP}}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA},$$

of (12.46) for $F = \mathbb{R}, \mathbb{C}$, then we obtain an equivariant homology theory $H_*^?(-; \mathbf{K}_F^{\mathsf{TOP}})$ in the sense of Definition 12.9 such that we get for every inclusion $H \subseteq G$ of groups natural identifications

$$H_n^G(G/H; \mathbf{K}_F^{\mathsf{TOP}}) \cong H_n^H(H/H; \mathbf{K}_F^{\mathsf{TOP}}) \cong \pi_n(\mathbf{K}_F^{\mathsf{TOP}} \circ I(H)) = K_n(C_r^*(H, F)).$$

Note that $H_n^?(X; \mathbf{K}_F^{\mathsf{TOP}})$ is defined for any G-CW-complex X, whereas the definition of $K_n^G(X)$ and $KO_n(X)$ in terms of KK-theory only makes sense for proper G-CW-complexes.

We get assembly maps induced by the projection

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}})\colon H_n^G(X; \mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}}) \to H_n^G(G/G; \mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}}) = K_n(C_r^*(G, \mathbb{C})); \tag{14.5}$$

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}})\colon H_n^G(X; \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}) \to H_n^G(G/G; \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}) = K_n(C_r^*(G, \mathbb{R})). \tag{14.6}$$

The assembly maps (14.1) and (14.5) are identified in [265, Section 6]. Unfortunately, the proof is based on an unpublished preprint by Carlsson-Pedersen-Roe. Another proof of the identification is given in [444, Corollary 8.4] and [730].

The identification above in the general case where one allows coefficients in a G-C^*-algebra A, is carried out in [171, 568].

Consider a proper G-CW-complex X. One sometimes finds in the literatur the notation

$$RK_n^G(X) := \operatorname{colim}_{C \subseteq X} KK_n^G(C_0(X), \mathbb{C}) \tag{14.7}$$

where C runs through the finite G-CW-subcomplexes of X directed by inclusion. By definition and by the discussion above we get for every proper G-CW-complex X identifications, natural in X,

$$RK_n^G(X) = K_n^G(X) = H_n^G(X; \mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}}) \tag{14.8}$$

and analogous in the real case.

14.4 The Baum-Connes Conjecture

Recall that a model for the *classifying space for proper G-actions* is a G-CW-complex $\underline{E}G = E_{\mathcal{FIN}}(G)$ such that $\underline{E}G^H$ is non-empty and contractible for each finite subgroup $H \subseteq G$ and empty for each infinite subgroup $H \subseteq G$. Two such models are G-homotopy equivalent. See Definition 11.18 and Theorem 11.19.

Conjecture 14.9 (Baum-Connes Conjecture). A group G satisfies the *Baum-Connes Conjecture* if the assembly maps

$$\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_r^*(G,\mathbb{C}));$$
$$\operatorname{asmb}^{G,\mathbb{R}}(\underline{E}G)_n \colon KO_n^G(\underline{E}G) \to KO_n(C_r^*(G,\mathbb{R})),$$

defined in (14.1) and (14.2) are bijective for all $n \in \mathbb{Z}$ in the special case that A is $\mathbb{C}$ or $\mathbb{R}$ respectively with the trivial G-action.

Exercise 14.10. Show $K_n^G(\underline{E}G) \cong \mathbb{Z}^k$ for $k, n \in \mathbb{Z}$, $k \geq 1$ and $G = \mathbb{Z} \times \mathbb{Z}/k$.

Conjecture 14.11 (Baum-Connes Conjecture with coefficients). A group G satisfies the *Baum-Connes Conjecture with coefficients* if the assembly maps

$$\operatorname{asmb}_A^{G,\mathbb{C}}(\underline{E}G)_n \colon K_n^G(\underline{E}G; A) \to K_n(A \rtimes_r G);$$
$$\operatorname{asmb}_A^{G,\mathbb{R}}(\underline{E}G)_n \colon KO_n^G(\underline{E}G; A) \to KO_n(A \rtimes_r G),$$

defined in (14.1) and (14.2) are bijective for all $n \in \mathbb{Z}$ and all G-C^*-algebras A over $F = \mathbb{R}, \mathbb{C}$.

Remark 14.12 (Counterexample to the Baum-Connes Conjecture 14.11 with coefficients and a modified version). We will discuss the status and further applications of the Baum-Connes Conjecture 14.11 with coefficients in Section 14.8 and 16.4, but immediately want to point out that there exists counterexamples to the version 14.11 with coefficients, see [470], but no counterexample to the Baum-Connes Conjecture 14.9 is known.

In [105] a new formulation of the Baum-Connes Conjecture with coefficients is given by considering a different crossed product for which the counterexamples mentioned above are not counterexamples anymore, see also [180], and no counterexample is known to the author's knowledge. The new version takes care of the problem that there exists groups G together with short exact sequences of G-C^*-algebras $0 \to I \to A \to B \to 0$ for which the induced sequence $0 \to I \rtimes G \to A \rtimes G \to B \rtimes G \to 0$ is not exact anymore and it is hence not clear that there exists a long exact sequence

$$\begin{aligned} \cdots \to K_n(I \rtimes G) \to K_n(A \rtimes G) \to K_n(B \rtimes G) \\ \to K_{n-1}(I \rtimes G) \to K_{n-1}(A \rtimes G) \to K_{n-1}(B \rtimes G) \to \cdots \end{aligned}$$

whose existence is a consequence of the Baum-Connes Conjecture 14.11 with coefficients. The new version still has the flaw that the left hand side of the assembly map is functorial under arbitrary group homomorphism, whereas this is unknown for the right hand side, compare Remark 14.20.

The original source of the Baum-Connes Conjecture (with coefficients) is [103, Conjecture 3.15 on page 254].

Remark 14.13 (The complex case implies the real case). The complex version of the Baum-Connes Conjecture 14.9 and 14.11 implies automatically the real version, see [110, 880].

Remark 14.14 (The torsionfree case). There are canonical isomorphisms $K_*^G(EG) \xrightarrow{\cong} K_*(BG)$ and $KO_*^G(EG) \xrightarrow{\cong} KO_*(BG)$. Suppose that G is torsionfree. Then EG is a model for $\underline{E}G$ and under the identification above the assembly map appearing in the Baum-Connes Conjecture 14.9 agrees with the one appearing in the Baum-Connes Conjecture for torsionfree groups 10.44. Hence the Baum-Connes Conjecture for torsionfree groups 10.44 is a special case of the Baum-Connes Conjecture 14.9.

Exercise 14.15. Let $f\colon H \to G$ be a group homomorphism of torsionfree groups. Suppose that H and G satisfy the Baum-Connes Conjecture 14.9 and the induced map on group homology $H_n(f)\colon H_n(H) \to H_n(G)$ is bijective for $n \in \mathbb{Z}$. Show that then $K_n(C_r^*(G,\mathbb{C})) \cong K_n(C_r^*(H,\mathbb{C}))$ and $KO_n(C_r^*(G,\mathbb{R})) \cong KO_n(C_r^*(H,\mathbb{R}))$ holds for all $n \in \mathbb{Z}$.

14.5 Variants of the Baum-Connes Conjecture

In this section we discuss some variants of the Baum-Connes Conjecture.

14.5.1 The Baum-Connes Conjecture for Maximal Group C^*-Algebras

There are also versions of the Baum-Connes assembly map using the maximal crossed product $A \rtimes_m G$, see [786, 7.6.5 on page 257] for a G-C^*-algebra A over F or the maximal group C^*-algebra $C^*_m(G,F)$ for $F = \mathbb{R}, \mathbb{C}$. Namely, there are assembly maps

$$\operatorname{asmb}^{G,\mathbb{C},m}_A(X)_*\colon K^G_*(X;A) \to K_*(A \rtimes_m G); \tag{14.16}$$

$$\operatorname{asmb}^{G,\mathbb{R},m}_A(X)_*\colon KO^G_*(X;A) \to KO_*(A \rtimes_m G), \tag{14.17}$$

which reduce for $A = \mathbb{R}, \mathbb{C}$ equipped with the trivial G-action to assembly maps

$$\operatorname{asmb}^{G,\mathbb{C},m}(\underline{E}G)_n\colon K^G_n(\underline{E}G) \to K_n(C^*_m(G,\mathbb{C})); \tag{14.18}$$

$$\operatorname{asmb}^{G,\mathbb{R},m}(\underline{E}G)_n\colon KO^G_n(\underline{E}G) \to KO_n(C^*_m(G,\mathbb{R})). \tag{14.19}$$

In the sequel we only consider the complex case, the corresponding statements are true over $\mathbb{R}$ as well.

There is always a C^*-homomorphism $p\colon A \rtimes_m G \to A \rtimes_r G$, and we obtain the following factorization of the Baum-Connes assembly map of (14.1)

$$\operatorname{asmb}^{G,\mathbb{C}}_A(X)_*\colon K^G_n(X;A) \xrightarrow{\operatorname{asmb}^{G,\mathbb{C},m}_A(X)_*} K_*(A \rtimes_m G) \xrightarrow{K_*(p)} K_*(A \rtimes_r G).$$

The Baum-Connes Conjecture 14.11 implies that the map $\operatorname{asmb}^{G,\mathbb{C},m}_A(\underline{E}G)_*$ is always injective, and that it is surjective if and only if the map $K_*(p)$ is bijective.

Remark 14.20 (Functoriality of the Baum-Connes assembly map). Note that the source of the assembly maps $\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n\colon K^G_n(\underline{E}G) \to K_n(C^*_r(G))$ and $\operatorname{asmb}^{G,\mathbb{C}.m}(\underline{E}G)_n\colon K^G_n(\underline{E}G) \to K_n(C^*_m(G))$ are functorial in G. The target $K_n(C^*_m(G))$ is also functorial in G since $C^*_m(G)$ is functorial in G, and the assembly map $\operatorname{asmb}^{G,\mathbb{C}.m}(\underline{E}G)_n\colon K^G_n(\underline{E}G) \to K_n(C^*_m(G))$ is natural in G.

However, it is not known whether the target $K_n(C^*_r(G))$ is functorial in G and we have already explained in Subsection 10.3.1 that not every group homomorphism $\alpha\colon G \to H$ induces a homomorphism of C^*-algebras $C^*_r(G) \to C^*_r(H)$. This is irritating since the Baum-Connes Conjecture 14.9 implies that $K_n(C^*_r(G))$ is also functorial in G.

The same problem is still present in the new formulation of the Baum-Connes Conjecture with coefficients in [105].

Remark 14.21 (The Baum-Connes Conjecture does not hold in general for the maximal group C^*-algebra). It is known that the assembly map $\operatorname{asmb}_A^{G,\mathbb{C},m}(\underline{E}G)_*$ of (14.16) is in general *not* surjective. Namely, $K_0(p)$ is not injective if G is any infinite group with property (T), compare for instance the discussion in [519]. There are infinite groups with property (T) for which the Baum-Connes Conjecture is known, see [582] and also [901]. Hence there are counterexamples to the conjecture that $\operatorname{asmb}^{G,\mathbb{C},m}(\underline{E}G)_n$ is surjective.

Remark 14.22 (The Baum-Connes Conjecture for the maximal group C^*-algebra holds for A-T-menable groups). A countable group G is called *K-amenable* if the map $p\colon C^*_{\max}(G) \to C^*_r(G)$ induces a KK-equivalence, see [253]. This implies in particular that the map $K_n(p)$ above is an isomorphism for all $n \in \mathbb{Z}$. A-T-menable groups are K-amenable, see [469] and they satisfy the Baum-Connes Conjecture 14.9, see Theorem 16.7 (ia). Hence for A-T-menable groups the assembly map $\operatorname{asmb}_A^{G,\mathbb{C},m}(\underline{E}G)_*$ of (14.16) is bijective for all $n \in \mathbb{Z}$. This is also true for the real version of the assembly map (14.19).

14.5.2 The Bost Conjecture

Some of the strongest results about the Baum-Connes Conjecture are proven using the so-called Bost Conjecture, see [584, page 798]. The Bost Conjecture is the version of the Baum-Connes Conjecture where one replaces the reduced group C^*-algebra $C^*_r(G,F)$ by the Banach algebra $L^1(G,F)$. One still can define the topological K-theory of $L^1(G,F)$ and the assembly map in this context.

Conjecture 14.23 (Bost Conjecture). The assembly maps

$$\begin{aligned}\operatorname{asmb}^{G,\mathbb{C},L^1}(\underline{E}G)_n\colon K_n^G(\underline{E}G) &\to K_n(L^1(G,\mathbb{C}));\\ \operatorname{asmb}^{G,\mathbb{R},L^1}(\underline{E}G)_n\colon KO_n^G(\underline{E}G) &\to KO_n(L^1(G,\mathbb{R})),\end{aligned}$$

are isomorphism for all $n \in \mathbb{Z}$.

In the sequel we only consider the complex case, the corresponding statements are true over $\mathbb{R}$ as well.

Again the left hand side coincides with the left hand side of the Baum-Connes assembly map. There is a canonical map of Banach $*$-algebras $q\colon L^1(G) \to C^*_r(G)$. We obtain a factorization of the Baum-Connes assembly map appearing in the Baum-Connes Conjecture 14.9

$$\operatorname{asmb}_A^{G,\mathbb{C}}(\underline{E}G)_*\colon K_n^G(\underline{E}G) \xrightarrow{\operatorname{asmb}^{G,\mathbb{C},L^1}(\underline{E}G)_*} K_*(L^1(G)) \xrightarrow{K_*(q)} K_*(C_r^*(G)) \tag{14.24}$$

Every group homomorphism $G \to H$ induces a homomorphism of Banach algebras $L^1(G) \to L^1(H)$ and the assembly map appearing in Conjecture 14.23 is natural in G.

The disadvantage of $L^1(G)$ is however that indices of operators tend to take values in the topological K-theory of the group C^*-algebras, not in $K_n(L^1(G))$. Moreover the representation theory of G is closely related to the group C^*-algebra, whereas the relation to $L^1(G)$ is not well understood.

There is also a version of the Bost Conjecture with coefficients in a C^*-algebra:

$$\operatorname{asmb}_A^{G,\mathbb{C},L^1}(\underline{E}G)_*\colon K_*^G(\underline{E}G;A) \to K_*(A \rtimes_{L^1} G) \tag{14.25}$$

For more information about the Bost Conjecture 14.23, we refer for instance to [70, 584, 586, 773, 774, 901].

14.5.3 The Strong and the Integral Novikov Conjecture

We mention the following conjectures, which actually follow from the Baum-Connes Conjecture 14.9.

Conjecture 14.26 (Strong Novikov Conjecture). A group G satisfies the *Strong Novikov Conjecture* if the assembly maps appearing in (10.42) or (10.43)

$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_*\colon K_n(BG) \to K_n(C_r^*(G,\mathbb{C}));$$
$$\operatorname{asmb}^{G,\mathbb{R}}(BG)_*\colon KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})),$$

are rationally injective for all $n \in \mathbb{Z}$.

Conjecture 14.27 (Integral Novikov Conjecture).
A torsionfree group G satisfies the *Integral Novikov Conjecture* if the assembly map appearing in (10.42) or (10.43) are injective for all $n \in \mathbb{Z}$.

The assembly maps appearing in the Integral Novikov Conjecture 14.26 agree with the assembly maps appearing in the Baum-Connes Conjecture for torsionfree groups.

The Integral Novikov Conjecture makes only sense for torsionfree groups.

Exercise 14.28. Find a finite group G for which there cannot be an injective map from $K_1(BG)$ to $K_1(C_r^*(G))$.

Theorem 14.29 (The Baum-Connes Conjecture implies the Novikov Conjecture). *Given a group G, the Baum-Connes Conjecture 14.9 for G implies the Strong Novikov Conjecture 14.26 for G and the Strong Novikov Conjecture 14.26 for G implies the Novikov Conjecture 9.137 for G.*

Proof. The implication that the Baum-Connes Conjecture 14.9 implies the Strong Novikov Conjecture 14.26 follows from Lemma 13.35. For proofs that the Strong Novikov Conjecture 14.26 implies the Novikov Conjecture 9.137 we refer to Kasparov [547, § 9], [539] or Kaminker-Miller [525]. □

14.5.4 The Coarse Baum Connes Conjecture

We briefly explain the Coarse Baum-Connes Conjecture, a variant of the Baum-Connes Conjecture that applies to metric spaces and not only to groups. Its importance lies in the fact that isomorphism results about the Coarse Baum-Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological K-theory, see Theorem 16.15.

Let X be a metric space that is proper, i.e., closed balls are compact. Let H_X a separable Hilbert space with a faithful nondegenerate $*$-representation of $C_0(X)$. Let $T\colon H_X \to H_X$ be a bounded linear operator. Its *support* $\operatorname{supp} T \subset X \times X$ is defined as the complement of the set of all pairs (x, x') for which there exist functions ϕ and $\phi' \in C_0(X)$ such that $\phi(x) \neq 0$, $\phi'(x') \neq 0$ and $\phi' T \phi = 0$. The operator T is said to be a *finite propagation operator,* if there exists a constant α such that $d(x, x') \leq \alpha$ holds for all pairs in the support of T. The operator is said to be *locally compact* if ϕT and $T\phi$ are compact for every $\phi \in C_0(X)$. An operator is called *pseudolocal* if $\phi T \psi$ is a compact operator for all pairs of continuous functions ϕ and ψ with compact and disjoint supports.

The *Roe-algebra* $C^*(X)$ is the operator-norm closure of the $*$-algebra of all locally compact finite propagation operators on H_X. The algebra $D^*(X)$ is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological K-theory of the quotient $K_*(D^*(X)/C^*(X))$ agrees with K-homology $K_{*-1}(X)$. A metric space is called *uniformly contractible* if for every $R > 0$ there exists $S > R$ such that for every $x \in X$ the inclusion of open balls $B_R(x) \to B_S(x)$ is nullhomotopic. For a uniformly contractible proper metric space the coarse assembly map $K_n(X) \to K_n(C^*(X))$ is the boundary map in the long exact sequence associated to the short exact sequence of C^*-algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0.$$

For general metric spaces one first approximates the metric space by spaces with nice local behavior, compare [838].

For simplicity we only explain the case where X is a discrete metric space. Let $P_d(X)$ be *the Rips complex for a fixed distance* d, i.e., the geometric realization of the abstract simplicial complex with vertex set X where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip $P_d(X)$ with the spherical metric, compare [1006].

A discrete metric space has *bounded geometry* if for each $r > 0$ there exists a natural number $N(r)$ such that for all x the open ball of radius r centered at $x \in X$ contains at most $N(r)$ elements.

Conjecture 14.30 (Coarse Baum-Connes Conjecture). Let X be a discrete metric space of bounded geometry. Then for $n \in \mathbb{Z}$ the coarse assembly map

$$\operatorname{colim}_{d\to\infty} K_n(P_d(X)) \to \operatorname{colim}_{d\to\infty} K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

A counterexample to the surjectivity part is constructed in [470, Section 6]. The injectivity part of this conjecture is false if one drops the bounded geometry hypothesis, see [299, 1007].

The Coarse Baum-Connes Conjecture for a finitely generated discrete group G (considered as a metric space) can be interpreted as a case of the Baum-Connes Conjecture 14.11 with coefficients for the group G with a certain specific choice of coefficients, see [1011].

Further information about the coarse Baum-Connes Conjecture can be found for instance in [227, 374, 395, 396, 471, 472, 474, 752, 838, 994, 995, 1006, 1007, 1008, 1005].

14.6 Inheritance Properties of the Baum-Connes Conjecture

Similar to the Farrell-Jones Conjecture, the Baum-Connes Conjecture 14.11 with coefficients has much better inheritance properties than the Baum-Connes Conjecture 14.9. Namely, we have

Theorem 14.31 (Inheritance properties of the Baum-Connes Conjecture with coefficients).

(i) Passing to subgroups
Let $H \subseteq G$ be an inclusion of groups. If G satisfies the Baum-Connes Conjecture 14.11 with coefficients, then H satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(ii) Group extensions
Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that for

any finite subgroup $H \subseteq Q$ the group $p^{-1}(H)$ satisfies the Baum-Connes Conjecture 14.11 with coefficients and that the group Q satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Then G satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(iii) Passing to finite direct products

If the groups G_0 and G_1 satisfy the Baum-Connes Conjecture 14.11 with coefficients, then $G_0 \times G_1$ satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(iv) Directed unions

Let G be a union of the directed system of subgroups $\{G_i \mid i \in I\}$.

If each group G_i satisfies the Baum-Connes Conjecture 14.11 with coefficients, then G satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(v) Actions on trees

Suppose that G acts on a tree without inversion. Assume that the Baum-Connes Conjecture 14.11 with coefficients holds for the stabilizers of any of the vertices.

Then the Baum-Connes Conjecture 14.11 with coefficients holds for G;

(vi) Amalgamated free products

Let G_0 be a subgroup of G_1 and G_2 and G be the amalgamated free product $G = G_1 *_{G_0} G_2$. Suppose G_i satisfies the Baum-Connes Conjecture 14.11 with coefficients for $i = 0, 1, 2$.

Then G satisfies Baum-Connes Conjecture 14.11 with coefficients;

(vii) HNN extension

Let G be an HN extension of the group H. Suppose that G satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Then G satisfies the Baum-Connes Conjecture 14.11 with coefficients;

Proof. (i) This has been stated in [103], a proof can be found for instance in [209, Theorem 2.5].

(ii) See [770, Theorem 3.1].

(iii) This follows from assertion (ii).

(iv) See [70, Theorem 1.8 (ii)].

(v) This is proved by Oyono-Oyono [771, Theorem 1.1].

(vi) and (vii) These are special case of assertion (v). □

Exercise 14.32. Show that the Baum-Connes Conjecture 14.11 with coefficients holds for any abelian group and any free group.

Exercise 14.33. Let G be the fundamental group of the orientable closed surface of genus $g \geq 1$. Show

$$K_n(C_r^*(G,\mathbb{C})) = \begin{cases} \mathbb{Z}^2 & n \text{ is even;} \\ \mathbb{Z}^g & n \text{ is odd.} \end{cases}$$

Remark 14.34 (The Baum-Connes Conjecture with coefficients is is not compatible with colimits in general). The Baum-Connes Conjecture with coefficients is not compatible with colimits in general. This is in contrast to the Full Farrell-Jones Conjecture 13.27, see Theorem 13.29 (vi) and to the Bost Conjecture (14.25) with coefficients, see [70, Theorem 1.8 (i)]. The Baum-Connes Conjecture 14.11 with coefficients is known for hyperbolic groups, see [582, 901]. Now let G be a colimit of a directed system of hyperbolic groups $\{G_i \mid i \in I\}$ (whose structure maps $G_i \to G_j$ are not injective). Suppose that the Baum-Connes Conjecture 14.11 with coefficients passes to colimits of directed systems of groups. Then the Baum-Connes Conjecture 14.11 with coefficients holds for G as well. However, there exists a group G which is a colimit of hyperbolic groups and contains appropriate expanders so that that [470] applies and hence the Baum-Connes Conjecture 14.11 with coefficients does not hold for G. The construction of such a group is described in [39, 767].

Remark 14.35 (The Farrell-Jones Conjecture and actions on trees). The inheritance properties of the Baum-Connes Conjecture 14.11 with coefficients for actions on trees, see Theorem 14.31 (v), is very useful. It does not hold for the Full Farrell-Jones Conjecture 13.27. The main reason is that in the Baum-Connes setting the family $\mathcal{FIN}$ suffices, whereas in the Farrell-Jones setting we have to use the family $\mathcal{VCY}$ since in the Farrell-Jones setting Nil-phenomenons occur which are not present in the Baum-Connes setting. Nevertheless, some partial results about this question in the Farrell-Jones setting can be found in [74]. Alternatively, one uses actions on trees to compute $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$, see Section 15.7, and treats the relative group $H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G);\mathbf{K}_R)$ separately, for which the results of Section 13.10 are very useful. Thanks to the splitting results of Section 13.8 one can put these two computations together to get a full description of $H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$. The analogous remark applies to L-theory.

Remark 14.36 (Passing to overgroups of finite index). It is not known whether the Baum-Connes Conjecture 14.11 with coefficients passes to overgroups of finite index. The same is true for the K- and L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories (with involution), see Conjecture 13.11 and Conjecture 13.16. This was the reason why we have introduced in Section 13.5 the versions "with finite wreath products". One can do the same in the Baum-Connes setting.

14.7 Reducing the Family of Subgroups for the Baum-Connes Conjecture

The following result is proved in [75, Theorem 0.5] based on a Completion Theorem, see [655, Theorem 6.5] and a Universal Coefficient Theorem, see [137, 508]. An argument for the complex case using equivariant Euler classes is given by Mislin and Matthey [694] for the complex case. It is not clear to us whether it is possible to extend the methods of [694] to the real case.

Theorem 14.37 (Reducing the family of subgroups for the Baum-Connes Conjecture). *For any group G the relative assembly maps*

$$K_n^G(E_{\mathcal{FCY}}(G)) \to K_n^G(E_{\mathcal{FIN}}(G));$$
$$KO_n^G(E_{\mathcal{FCY}}(G)) \to KO_n^G(E_{\mathcal{FIN}}(G)),$$

are bijective for all $n \in \mathbb{Z}$ where $\mathcal{FCY}$ is the family of finite cyclic subgroups.

Remark 14.38 ($\mathcal{FCY}$ is the smallest family for the Baum-Connes Conjecture). Let C be a finite cyclic group and $\mathcal{F}$ be a family of subgroups of C. Then the assembly map

$$K_0^C(E_{\mathcal{F}}(C)) \to K_0^C(C/C) = R_{\mathbb{C}}(C)$$

is surjective if and only if $\mathcal{F}$ consists of all subgroups. This follows from [636, Theorem 0.7 and Lemma 3.4] since they predict that the homomorphism induced by the various inclusions

$$\bigoplus_{D \in \mathcal{F}} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C)$$

is rationally surjective and hence C must be contained in $\mathcal{F}$.

Let $\mathcal{C}$ be a class of groups that is closed under taking subgroups and passing to isomorphic groups. Examples are the classes of finite cyclic groups or of finite groups. Given a group G, let $\mathcal{C}(G)$ be the family of subgroups of G that belong to G. Suppose that for any group G the assembly map

$$K_n^G(E_{\mathcal{C}(G)}(G)) \to K_n^G(G/G)$$

is bijective. The considerations above imply that $\mathcal{C}$ has to contain all finite cyclic subgroups. So, roughly speaking, $\mathcal{FCY}$ is the smallest family for which one can hope that the Baum-Connes Conjecture 14.9 is true for all groups.

14.8 Applications of the Baum-Connes Conjecture

The Baum-Connes Conjecture for torsionfree groups 10.44 follows from the Baum-Connes Conjecture 14.9, see Remark 14.14, and implies, see Subsections 10.4.1 and 10.4.2,

- *Trace Conjecture 10.50 for torsionfree groups*
  For a torsionfree group G the image of

$$\mathrm{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

- *Kadison Conjecture 10.52*
  If G is a torsionfree group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.

The Baum-Connes Conjecture 14.9 implies by Theorem 14.29

- *Strong Novikov Conjecture 14.26*
  The assembly maps

$$\mathrm{asmb}^{G,\mathbb{C}}(BG)_* : K_n(BG) \to K_n(C_r^*(G,\mathbb{C}));$$
$$\mathrm{asmb}^{G,\mathbb{R}}(BG)_* : KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})).$$

  of (10.42) and (10.43) are rationally injective for all $n \in \mathbb{Z}$.

The strong Novikov Conjecture 14.26 (and hence also the Baum-Connes Conjecture 14.9) implies, see Subsection 10.4.3,

- *Zero-in-the-spectrum Conjecture 10.55*
  If $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M, then zero is in the spectrum of the minimal closure of the pth Laplacian on $\widetilde{M}$ for some $p \in \{0, 1, \ldots, \dim M\}$.

Moreover, we have already shown in Theorem 14.29 that the Baum-Connes Conjecture 14.9 implies

- *Novikov Conjecture* 9.137
  Higher signatures are homotopy invariant.

Next we deal with some other conjectures which follows from the Baum-Connes Conjecture.

14.8.1 The Modified Trace Conjecture

Denote by Λ^G the subring of $\mathbb{Q}$ that is obtained from $\mathbb{Z}$ by inverting all orders $|H|$ of finite subgroups H of G, i.e.,

$$\Lambda^G = \mathbb{Z}\left[|H|^{-1} \mid H \subset G,\ |H| < \infty\right]. \tag{14.39}$$

The following conjecture generalizes Conjecture 10.50 to the case where the group need no longer be torsionfree. For the standard trace see (10.48).

Conjecture 14.40 (Trace Conjecture, modified). Let G be a group. Then the image of the homomorphism induced by the standard trace

$$\text{(14.41)} \qquad \operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

is contained in Λ^G.

The following result is proved in [636, Theorem 0.3].

Theorem 14.42. *Let G be a group. Then the image of the composite*

$$K_0^G(E_{\mathcal{FIN}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n \otimes_{\mathbb{Z}} \operatorname{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\operatorname{tr}_{C_r^*(G)}} \mathbb{R}$$

is Λ^G. Here $\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n$ is the map appearing in the Baum-Connes Conjecture 14.9. In particular the Baum-Connes Conjecture 14.9 implies the Modified Trace Conjecture 14.40.

The original version of the Trace Conjecture due to Baum and Connes [102, page 21] makes the stronger statement that the image of $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$ is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$ where $H \subset G$ runs though all finite subgroups of G. Roy has constructed a counterexample to this version in [865] based on her article [866]. The examples of Roy do *not* contradict the Modified Trace Conjecture 14.40 or the Baum-Connes Conjecture 14.9.

Exercise 14.43. The G be a finite group. Show that the image of the trace map $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$ is $\{n \cdot |G|^{-1} \mid n \in \mathbb{Z}\}$.

14.8.2 The Stable Gromov-Lawson-Rosenberg Conjecture

The Stable Gromov-Lawson-Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. It is related to the real version of the Baum-Connes Conjecture 14.9.

Let $\Omega_n^{\mathrm{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG. Given an element $[u \colon M \to BG] \in \Omega_n^{\mathrm{Spin}}(BG)$, we can take the $C_r^*(G,\mathbb{R})$-valued index of the equivariant Dirac operator associated to the G-covering $\overline{M} \to M$ determined by u. Thus we get a homomorphism

$$\text{(14.44)} \qquad \operatorname{ind}_{C_r^*(G,\mathbb{R})} \colon \Omega_n^{\mathrm{Spin}}(BG) \to KO_n(C_r^*(G,\mathbb{R})).$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 1. We fix such a choice, the particular choice does not matter for the sequel. Note that $\operatorname{ind}_{C_r^*(\{1\},\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G,\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G,\mathbb{R}))$. In particular

$$\operatorname{ind}_{C_r^*(G,\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(G,\mathbb{R})}(M \times B), \tag{14.45}$$

if we identify $KO_n(C_r^*(G,\mathbb{R})) = KO_{n+8}(C_r^*(G,\mathbb{R}))$ via Bott periodicity.

Conjecture 14.46 (Stable Gromov-Lawson-Rosenberg Conjecture). Let M be a connected closed Spin-manifold of dimension $n \geq 5$. Let $u_M \colon M \to B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\operatorname{ind}_{C_r^*(\pi_1(M),\mathbb{R})}([M, u_M]) = 0 \quad \in KO_n(C_r^*(\pi_1(M),\mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner-Lichnerowicz formula [841]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [913, Section 3].

Theorem 14.47 (The Baum-Connes Conjecture implies the Stable Gromov-Lawson-Rosenberg Conjecture). *If the assembly map for the real version of the Baum-Connes Conjecture 14.9 is injective for the group G, then the Stable Gromov-Lawson-Rosenberg Conjecture 14.46 is true for all closed* Spin-*manifolds of dimension* ≥ 5 *with* $\pi_1(M) \cong G$.

The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture since in dimension four new obstructions, the Seiberg-Witten invariants, occur. The unstable version of this conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\operatorname{ind}_{C_r^*(\pi_1(M),\mathbb{R})}([M, u_M]) = 0$. Schick [878] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau, see also [306]. There are counterexamples with $\pi \cong \mathbb{Z}^4 \times \mathbb{Z}/3$. However for appropriate $\rho \colon \mathbb{Z}/3 \to \operatorname{aut}(\mathbb{Z}^4)$ the unstable version does hold for $\pi \cong \mathbb{Z}^4 \rtimes_\rho \mathbb{Z}/3$ and $\dim(M) \geq 5$, see [267, Theorem 0.7 and Remark 0.9]. More infinite groups for which the unstable version holds are presented in [490, Theorem 6.3].

Since the Baum-Connes Conjecture 14.9 is true for finite groups (for the trivial reason that $E_{\mathcal{FIN}}(G) = \{\bullet\}$ for finite groups G), Theorem 14.47 implies that the Stable Gromov-Lawson Conjecture 14.46 holds for finite fundamental groups, see also [853]. It is not known at the time of writing whether the unstable version is true for finite fundamental groups.

The index map appearing in (14.44) can be factorized as a composite

$$\operatorname{ind}_{C_r^*(G,\mathbb{R})} \colon \Omega_n^{\mathrm{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{\operatorname{asmb}^{G,\mathbb{C}}(BG)_n} KO_n(C_r^*(G,\mathbb{R})) \tag{14.48}$$

where D sends $[M, u]$ to the class of the G-equivariant Dirac operator of the G-manifold $\overline{M}$ given by u and $\operatorname{asmb}^{G,\mathbb{C}}(BG)_n$ is the real version of the classical assembly map. The homological Chern character defines an isomorphism

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG;\mathbb{Q}).$$

Recall that associated to M there is the *$\widehat{A}$-class*

$$\widehat{\mathcal{A}}(M) \in \prod_{p \geq 0} H^p(M;\mathbb{Q}) \tag{14.49}$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (14.48) sends the class of $u\colon M \to BG$ to $u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}})$, i.e., the image of the Poincaré dual of $\widehat{\mathcal{A}}(M)$ under the map induced by u in rational homology. Hence $D([M,u]) = 0$ if and only if $u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}})$ vanishes. For $x \in \prod_{k \geq 0} H^k(BG;\mathbb{Q})$ define the *higher $\widehat{A}$-genus of* (M,u) *associated to* x to be

$$\widehat{A}_x(M,u) = \langle \widehat{\mathcal{A}}(M) \cup u^*x, [M]_{\mathbb{Q}} \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}}) \rangle \ \in \mathbb{Q}. \tag{14.50}$$

The vanishing of $\widehat{\mathcal{A}}(M)$ is equivalent to the vanishing of all higher $\widehat{A}$-genera $\widehat{A}_x(M,u)$. The following conjecture is a weak version of the Stable Gromov-Lawson-Rosenberg Conjecture.

Conjecture 14.51 (Homological Gromov-Lawson-Rosenberg Conjecture). Let G be a group. Then for any closed Spin-manifold M, which admits a Riemannian metric with positive scalar curvature, the $\widehat{A}$-genus $\widehat{A}_x(M,u)$ vanishes for all maps $u\colon M \to BG$ and elements $x \in \prod_{k \geq 0} H^k(BG;\mathbb{Q})$.

From the discussion above we obtain the following result.

Lemma 14.52. *If the assembly map*

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to KO_n(C_r^*(G,\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$, then the Homological Gromov-Lawson-Rosenberg Conjecture 14.51 holds for G.

The following conjecture is due to Gromov-Lawson [426, page 313].

Conjecture 14.53 (Aspherical closed manifolds carry no Riemannian metric with positive scalar curvature). An aspherical closed manifold carries no Riemannian metric with positive scalar curvature.

Conjecture 14.53 is true in dimensions 4 and 5 by Chodosh-Li-Liokumovich [229] and Gromov [425].

Lemma 14.54. *Let M be an aspherical closed* Spin*-manifold whose fundamental group satisfies the Homological Gromov-Lawson-Rosenberg Conjecture 14.51.*

Then M satisfies Conjecture 14.53, i.e., M carries no Riemannian metric with positive scalar curvature.

Proof. Suppose M carries a Riemannian metric of positive scalar curvature. Since M is aspherical, we can take $M = BG$ for $G = \pi_1(M)$ and $f = \mathrm{id}_G$ in Conjecture 14.51. Since $\widehat{\mathcal{A}}(M)_0 = 1$, we get for all $x \in H^{\dim(M)}(M;\mathbb{Q})$ that $\langle x, [M]\rangle = 0$ holds, a contradiction. □

Exercise 14.55. Let $F \to M \to S^1$ be a fiber bundle such that F is an orientable closed surface and M is a closed spin-manifold. Show that M carries a Riemannian metric with positive scalar curvature if and only if F is S^2.

The (moduli) space of metrics of positive scalar curvature of closed spin manifolds is studied in [147, 148, 250, 308, 449, 882].

14.8.3 L^2-Rho-Invariants and L^2-Signatures

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Let M be an orientable connected closed Riemannian manifold. Denote by $\eta(M) \in \mathbb{R}$ the *eta-invariant* of M and by $\eta^{(2)}(\widetilde{M}) \in \mathbb{R}$ the L^2-*eta-invariant* of the $\pi_1(M)$-covering given by the universal covering $\widetilde{M} \to M$. Let $\rho^{(2)}(M) \in \mathbb{R}$ be the L^2-*rho-invariant* that is defined to be the difference $\eta^{(2)}(\widetilde{M}) - \eta(M)$. These invariants were studied by Cheeger and Gromov [225, 226]. They show that $\rho^{(2)}(M)$ depends only on the diffeomorphism type of M and is in contrast to $\eta(M)$ and $\eta^{(2)}(\widetilde{M})$ independent of the choice of Riemannian metric on M. The following conjecture is taken from Mathai [693].

Conjecture 14.56 (Homotopy Invariance of the L^2-Rho-Invariant for Torsionfree Groups). If $\pi_1(M)$ is torsionfree, then $\rho^{(2)}(M)$ is a homotopy invariant.

Theorem 14.57 (Homotopy Invariance of $\rho^{(2)}(M)$). *Let M be an oriented connected closed manifold M of odd dimension such that $G = \pi_1(M)$ is torsionfree. Suppose that the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra, see Subsection 14.5.1, is surjective.*

Then $\rho^{(2)}(M)$ is a homotopy invariant.

Proof. This is proved by Keswani [560, 561]. □

Remark 14.58 (L^2-signature Theorem). Let X be a $4n$-dimensional Poincaré space over $\mathbb{Q}$. Let $\overline{X} \to X$ be a normal covering with torsionfree covering group G. Suppose that the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra is surjective see Subsection 14.5.1, or suppose that the rationalized assembly map for L-theory

$$H_{4n}(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q}\to L_{4n}^{\langle -\infty\rangle}(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is an isomorphism. Then the following L^2-signature theorem is proved in Lück-Schick [665, Theorem 0.3]

$$\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X). \tag{14.59}$$

If one drops the condition that G is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space X with a finite G covering $\overline{X}\to X$ for which $\operatorname{sign}(\overline{X})\neq |G|\cdot\operatorname{sign}(X)$ holds, see [823, Example 22.28], [966, Corollary 5.4.1]. If X is a closed topological manifold, then (14.59) is true for all groups G, see [665, Theorem 0.2].

Remark 14.60. Chang-Weinberger [213] assign to an oriented connected closed $(4k-1)$-dimensional manifold M a Hirzebruch-type invariant $\tau^{(2)}(M)\in\mathbb{R}$ as follows. By a result of Hausmann [457] there is an oriented connected closed $4k$-dimensional smooth manifold W with $M=\partial W$ such that the inclusion $\partial W\to W$ induces an injection on the fundamental groups. Define $\tau^{(2)}(M)$ as the difference $\operatorname{sign}^{(2)}(\widetilde{W})-\operatorname{sign}(W)$ of the L^2-signature of the $\pi_1(W)$-covering given by the universal covering $\widetilde{W}\to W$ and the signature of W. This is indeed independent of the choice of W. We conjecture that $\rho^{(2)}(M)=\tau^{(2)}(M)$ is always true. Chang-Weinberger [213] use $\tau^{(2)}$ to prove that, if $\pi_1(M)$ is not torsionfree, there are infinitely many diffeomorphically distinct smooth manifolds of dimension $4k+3$ with $k\geq 1$ which are tangentially simple homotopy equivalent to M.

Suppose that the L-theoretic Farrell-Jones Conjecture13.4 with coefficients in the ring with involution is rationally true for $R=\mathbb{Z}$, i.e., the rationalized assembly map

$$H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q}\to L_n^{\langle -\infty\rangle}(\mathbb{Z}G)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is an isomorphism for $n\in\mathbb{Z}$. We mention without proof that then $\tau^{(2)}(M)$ is a homotopy invariant.

Remark 14.61 (Obstructions for knots to be slice). Cochran-Orr-Teichner give in [232] new obstructions for a knot to be slice, which are sharper than the Casson-Gordon invariants. They use L^2-signatures and the Baum-Connes Conjecture 14.9. We also refer to the survey article [231] about non-commutative geometry and knot theory.

14.9 Notes

The Baum-Connes Conjecture has also been formulated and proved for (not necessarily discrete) topological groups, see for instance [103, 106, 210, 586]. It is interesting for representation theory, see for instance [107].

The Baum-Connes assembly maps in terms of localizations of triangulated categories are considered in [497, 498, 499, 701, 702, 703].

Certain so-called Cuntz-Lie C^*-algebras, see [254, 255], were classified in [613, Corollary 1.3]. The main difficulty is to compute the topological K-theory of these C^*-algebras, which boils down to the computation of the topological C^*-algebra of certain crystallographic groups. This in turn leads via the Baum-Connes Conjecture to an open conjecture about group homology which was solved in the case needed for this application, see [603, 604].

Other classification results whose proof uses the Baum-Connes Conjecture 14.9, can be found in [310, Theorem 0.1].

We propose that one should also construct a Baum-Connes assembly map for the Fréchet algebra $\mathcal{R}(G)$ associated to a group G. This will lead to the intriguing factorization of the Baum-Connes assembly map

$$K_n^G(\underline{E}G) \to K_n(\mathcal{R}(G)) \to K_n(L^1(G)) \to K_n(C_r^*(G)).$$

There is some hope that the methods of proof for the K-theoretic Farrell-Jones Conjecture carry over to group Fréchet algebras. This would lead for instance to the proof of the bijectivity of $K_n^G(\underline{E}G) \to K_n(\mathcal{R}(G))$ for (not necessarily cocompact) lattices in second countable locally compact Hausdorff groups with finitely many path components. Note that the Baum-Connes Conjecture 14.9 is open for $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.

For more information about the Baum-Connes Conjecture and its applications we refer for instance to [103, 409, 467, 475, 476, 477, 584, 585, 657, 727, 791, 850, 879, 945].

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Chapter 15
The (Fibered) Meta- and Other Isomorphism Conjectures

15.1 Introduction

In this section we deal with Isomorphism Conjectures in their most general form. Namely, given a G-homology theory $\mathcal{H}_*^G$, the Meta-Isomorphism Conjecture 15.2 predicts that, for a group G and a family $\mathcal{F}$ of subgroups of G, the map induced by the projection $E_{\mathcal{F}}(G) \to G/G$

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

is bijective for all $n \in \mathbb{Z}$.

If we take special examples for $\mathcal{H}_*^G$ and $\mathcal{F}$, then we obtain the Farrell-Jones Conjecture for a ring R (with involution), see Conjectures 13.1 and 13.4, and the Baum-Connes Conjecture 14.9. We will also introduce a Fibered Meta-Isomorphism Conjecture 15.8 which is more general and has much better inheritance properties, see Section 15.6. The versions of the Farrell-Jones Conjecture with coefficients in additive categories, see Conjectures 13.11 and 13.16, and the Baum-Connes Conjecture 14.11 with coefficients are automatically fibered, see Theorem 15.9, and hence have good inheritance properties.

The main tool to reduce the family of subgroups is the Transitivity Principle, which we discuss in Section 15.5.

Section 15.7 is devoted to actions on trees and their implications, such as the existence of Mayer-Vietoris sequences associated to amalgamated free products and Wang sequences associated to semidirect products with $\mathbb{Z}$, or more generally to HNN-extensions.

In Section 15.8 we pass to the special case where the homology theory comes from a functor from spaces to spectra which respects weak homotopy equivalences and disjoint unions, and discuss inheritance properties in this framework.

By specifying the functor from spaces to spectra, we obtain the Farrell-Jones Conjecture for Waldhausen's A-theory for pseudoisotopy and Whitehead spaces in Section 15.10 We also deal with topological Hochschild homology and cyclic homology in Section 15.11. We explain the Farrell-Jones Conjecture for homotopy K-theory in Section 15.12. The only instance where we will consider not necessarily discrete groups is the Farrell-Jones Conjecture 15.78 for the algebraic K-theory of the Hecke algebra of a totally disconnected locally compact second countable Hausdorff group.

In Section 15.14 interesting relations between these conjectures are discussed, namely, between the Farrell-Jones Conjecture for the K-theory of groups rings, for A-theory, and for pseudoisotopy, between the L-theoretic Farrell-Jones Conjecture and the Baum-Connes Conjecture, and between the Farrell-Jones Conjecture for K-theory and homotopy K-theory. We will briefly also relate the geometric surgery sequence in the topological category to an analytic Surgery Exact Sequence.

15.2 The Meta-Isomorphism Conjecture

Let G be a (discrete) group. Let $\mathcal{H}_*^G$ be a G-homology theory with values in Λ-modules for some commutative associative ring with unit Λ. Recall that it assigns to every G-CW-pair (X, A) and integer $n \in \mathbb{Z}$ a Λ-module $\mathcal{H}_n^G(X, A)$ such that the obvious generalization to G-CW-pairs of the axioms of a (non-equivariant generalized) homology theory for CW-complexes holds, i.e., G-homotopy invariance, the long exact sequence of a G-CW-pair, excision, and the disjoint union axiom are satisfied. The precise definition of a G-homology theory can be found in Definition 12.1 and of a G-CW-complex in Definition 11.2, see also Remark 11.3.

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, to be a set of subgroups of G that is closed under conjugation with elements of G and passing to subgroups. Let $\mathcal{F}$ be a family of subgroups of G. Denote by $E_{\mathcal{F}}(G)$ a model for the classifying G-CW-complex for the family $\mathcal{F}$ of subgroups of G, i.e., a G-CW-complex $E_{\mathcal{F}}(G)$ whose isotropy groups belong to $\mathcal{F}$ and for which for each $H \in \mathcal{F}$ the H-fixed point set $E_{\mathcal{F}}(G)^H$ is weakly contractible. Such a model always exists and is unique up to G-homotopy, see Definition 11.18 and Theorem 11.19.

The projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ induces for all integers $n \in \mathbb{Z}$ a homomorphism of Λ-modules

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G), \tag{15.1}$$

which is sometimes called *assembly map*.

Conjecture 15.2 (Meta-Isomorphism Conjecture). The group G satisfies the *Meta-Isomorphism Conjecture* with respect to the G-homology theory $\mathcal{H}_*^G$ and the family $\mathcal{F}$ of subgroups of G if the assembly map

$$\mathcal{H}_n(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

of (15.1) is bijective for all $n \in \mathbb{Z}$.

If we choose $\mathcal{F}$ to be the family $\mathcal{ALL}$ of all subgroups, then G/G is a model for $E_{\mathcal{ALL}}(G)$ and the Meta-Isomorphism Conjecture 15.2 is obviously true. The point is to find an as small as possible family $\mathcal{F}$. The idea of the

Meta-Isomorphism Conjecture 15.2 is that one wants to compute $\mathcal{H}_n^G(G/G)$, which is the unknown and the interesting object, by assembling it from the values $\mathcal{H}_n^G(G/H)$ for $H \in \mathcal{F}$, which are usually much more accessibly since the structure of the groups H is easy. For instance $\mathcal{F}$ could be the family $\mathcal{FIN}$ of finite subgroups or the family $\mathcal{VCY}$ of virtually cyclic subgroups.

The various Isomorphism Conjectures are now obtained by specifying the G-homology theory $\mathcal{H}_*^G$ and the family $\mathcal{F}$. For instance, the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R and the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R are equivalent to the Meta-Isomorphism Conjecture 15.2 if we choose $\mathcal{F}$ to be $\mathcal{VCY}$ and $\mathcal{H}_n^G$ to be $H_n^G(-;\mathbf{K}_R)$ and $H_n^G(-;\mathbf{L}_R^{\langle -\infty \rangle})$. The Baum-Connes Conjecture 14.9 is equivalent the Meta-Isomorphism Conjecture 15.2 if we choose $\mathcal{F}$ to be $\mathcal{FIN}$ and $\mathcal{H}_n^G$ to be $K_n^G(-) = H_n^G(-;\mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}})$ or $KO_n^G(-) = H_n^G(-;\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}})$. The analogous statement holds for the versions with coefficients in additive G-categories (with involutions), Conjectures 13.11, 13.16, and 14.11.

Exercise 15.3. Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ-modules in the sense of Definition 12.9. Fix a class of groups $\mathcal{C}$ that is closed under isomorphisms, taking subgroups and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G that belong to $\mathcal{C}$. Then we obtain for each group G an assembly map induced by the projection $E_{\mathcal{C}(G)}(G) \to G/G$.

$$\mathcal{H}_n^G(E_{\mathcal{C}(G)}(G)) \to \mathcal{H}_n^G(G/G).$$

Explain that using the induction structure of $\mathcal{H}_*^?$ we can turn the source and target to be functors from the category of groups to the category of Λ-modules such that the assembly maps yield a natural transformation of such functors.

15.3 The Fibered Meta-Isomorphism Conjecture

Given a group homomorphism $\phi\colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family of subgroups of K by

$$\phi^*\mathcal{F} := \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}. \tag{15.4}$$

If ϕ is an inclusion of subgroups, we also write

$$\mathcal{F}|_K := \phi^*\mathcal{F} = \{H \subseteq K \mid H \in \mathcal{F}\}. \tag{15.5}$$

If $\psi\colon H \to K$ is another group homomorphism, then

$$\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}. \tag{15.6}$$

Exercise 15.7. Let $\phi\colon K \to G$ be a group homomorphism. Consider a family $\mathcal{F}$ of subgroups of G and a G-CW-model $E_{\mathcal{F}}(G)$. Show that its restriction to K by $\phi\colon K \to G$ is a K-CW-complex which is a model for $E_{\phi^*\mathcal{F}}(K)$.

Consider an equivariant homology theory $\mathcal{H}_*^?$ over the group Γ with values in Λ-modules in the sense of Definition 12.89.

Conjecture 15.8 (Fibered Meta-Isomorphism Conjecture). A group (G,ξ) over Γ satisfies the *Fibered Meta-Isomorphism Conjecture with respect to $\mathcal{H}_*^?$ and the family $\mathcal{F}$ of subgroups of G* if for each group homomorphism $\phi\colon K \to G$ the group K satisfies the Meta-Isomorphism Conjecture 15.2 with respect to the K-homology theory $\mathcal{H}_*^{K,\xi\circ\phi}$ and the family $\phi^*\mathcal{F}$ of subgroups of K.

15.4 The Farrell-Jones Conjecture with Coefficients in Additive or Higher Categories is Fibered

We will see that it is important for inheritance properties to pass to the fibered version. It turns out that the fibered version is automatically built into the versions of the Farrell-Jones Conjecture with coefficients in additive G-categories (with involution).

Theorem 15.9 (The Farrell-Jones Conjecture with coefficients in additive G-categories (with involutions) is automatically fibered).

(i) Let $\phi\colon K \to G$ be a group homomorphism. Let $\mathcal{F}$ be a family of subgroups of G. Suppose that the assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr)$$

is bijective for every $n \in \mathbb{Z}$ and every additive G-category $\mathcal{A}$.
Then the assembly map

$$H_n^K(\mathrm{pr})\colon H_n^K(E_{\phi^*\mathcal{F}}(K);\mathbf{K}_{\mathcal{B}}) \to H_n^K(K/K;\mathbf{K}_{\mathcal{B}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{B}}(I(K))\bigr)$$

is bijective for every $n \in \mathbb{Z}$ and every additive K-category $\mathcal{B}$.
The anlogous statement holds for higher categories as coefficients;

(ii) Suppose that G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories.
Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for the group (G,id_G) over G, the family $\mathcal{VCY}$ and the equivariant homology theory $H_*^?(-;\mathbf{K}_{\mathcal{A}})$ over G for every additive G-category $\mathcal{A}$.
The anlogous statement holds for higher categories as coefficients;

(iii) Let $\phi\colon K \to G$ be a group homomorphism. Let $\mathcal{F}$ be a family of subgroups of G. Suppose that the assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\big)$$

is bijective for every $n \in \mathbb{Z}$ and every additive G-category with involution $\mathcal{A}$.
Then the assembly map

$$H_n^K(\mathrm{pr})\colon H_n^K(E_{\phi^*\mathcal{F}}(G);\mathbf{L}_{\mathcal{B}}^{\langle -\infty\rangle}) \to H_n^K(K/K;\mathbf{L}_{\mathcal{B}}^{\langle -\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{B}}^{\langle -\infty\rangle}(I(K))\big)$$

is bijective for every $n \in \mathbb{Z}$ and every additive K-category with involution $\mathcal{B}$;

(iv) Suppose that G satisfies the L-theoretic Farrell-Jones Conjecture 13.16 with coefficients in additive G-categories with involution.
Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for the group (G,id_G) over G, the family $\mathcal{VCY}$ and the equivariant homology theory $H_*^?(-;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})$ over G for every additive G-category with involution $\mathcal{A}$;

Proof. (i) See [90, Corollary 4.3]. **Comment 19 (by W.)**: Add a reference for higher additive categories. See for instance [330, Remark 2.-10].

(ii) This follows from assertion (i) by taking $\mathcal{B} = \phi^*\mathcal{A}$ since a direct inspection of the definitions in [76, Section 9] shows that the K-homology theory obtained by taking in $H_*^?(-;\mathbf{K}_{\mathcal{A}})$ the variable ? to be ϕ is the same as the K-homology theory $\mathbf{H}_*^K(-;\mathbf{K}_{\phi^*\mathcal{A}})$ associated to the additive K-category $\phi^*\mathcal{A}$. **Comment 20 (by W.)**: Check that this argument passes to higher categories.

(iii) See [76, Theorem 11.3].

(iv) This follows from (iii) by the same proof as it appears in assertion (ii). □

It is useful to have the Fibered Meta Conjecture 15.8 available since there are other situations where it is not known to formulate it with adequate coefficients, as it is possible in the Farrell-Jones setting for K- and L-theory.

15.5 Transitivity Principles

In this subsection we treat only equivariant homology theories $\mathcal{H}_*^?$ to keep the notation and exposition simple. The generalizations to an equivariant homology theory over a group Γ are obvious, just equip each group occurring below with the appropriate reference map to Γ.

Lemma 15.10. *Let G be a group, and let $\mathcal{F}$ be a family of subgroups of G. Let m be an integer. Let Z be a G-CW-complex. For $H \subseteq G$ let $\mathcal{F}|_H$ be the family of subgroups of H given by $\{L \subseteq H \mid L \in \mathcal{F}\}$. Suppose for* *each $H \subseteq G$ occurring as isotropy group in Z that the maps induced by the projection* $\mathrm{pr}_H \colon E_{\mathcal{F}|_H}(H) \to H/H$

$$\mathcal{H}_n^H(\mathrm{pr}_H) \colon \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

satisfy one of the following conditions

(i) They are bijective for $n \in \mathbb{Z}$ with $n \leq m$;
(ii) They are bijective for $n \in \mathbb{Z}$ with $n \leq m-1$ and surjective for $n = m$.

Then the maps induced by the projection $\mathrm{pr}_2 \colon E_{\mathcal{F}}(G) \times Z \to Z$

$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \to \mathcal{H}_n^G(Z)$$

satisfies the same condition.

Proof. We first prove the claim for finite dimensional G-CW-complexes by induction over $d = \dim(Z)$. The induction beginning $\dim(Z) = -1$, i.e. $Z = \emptyset$, is trivial. In the induction step from $(d-1)$ to d we choose a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d & \longrightarrow & Z_d \end{array}$$

If we cross it with $E_{\mathcal{F}}(G)$, we obtain another G-pushout of G-CW-complexes. The various projections induce a map from the Mayer-Vietoris sequence of the latter G-pushout to the Mayer-Vietoris sequence of the first G-pushout. By the Five Lemma (or its obvious variant if we consider assumption (ii)) it suffices to prove that the following maps

$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times S^{d-1}\right) \to \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times S^{d-1}\right);$$
$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) \to \mathcal{H}_n^G(Z_{d-1});$$
$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times D^d\right) \to \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times D^d\right),$$

satisfy condition (i) or(ii). This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and G-homotopy invariance of $\mathcal{H}_*^?$ the claim follows for the third map if we can show for any $H \subseteq G$ which occurs as isotropy group in Z that the maps

$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \to \mathcal{H}^G(G/H) \tag{15.11}$$

satisfy condition (i) or(ii). The G-map

$$G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G) \to G/H \times E_{\mathcal{F}}(G) \quad (g,x) \mapsto (gH, gx)$$

is a G-homeomorphism where res_G^H denotes the restriction of the G-action to an H-action. Since $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$, the H-space $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$. We conclude from the induction structure that the map (15.11) can be identified with the map

$$\mathcal{H}_n^H(\mathrm{pr}_H)\colon \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}^H(H/H)$$

which satisfies condition (i) or (ii) by assumption. This finishes the proof in the case that Z is finite dimensional. The general case follows by a colimit argument using Lemma 12.5. ☐

Theorem 15.12 (Transitivity Principle for equivariant homology). *Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of the group G. Suppose for every $H \in \mathcal{G}$ that the maps induced by the projection*

$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

satisfy one of the following conditions

(i) They are bijective for $n \in \mathbb{Z}$ with $n \leq m$;
(ii) They are bijective for $n \in \mathbb{Z}$ with $n \leq m-1$ and surjective for $n = m$.

Then the maps induced by the up to G-homotopy unique G-map $\iota_{\mathcal{F}\subseteq\mathcal{G}}\colon E_{\mathcal{F}}(G) \to E_G(\mathcal{G})$

$$\mathcal{H}_n^G(\iota_{\mathcal{F}\subseteq\mathcal{G}})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

satisfy the same condition.

Proof. If we equip $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G)$ with the diagonal G-action, it is a model for $E_{\mathcal{F}}(G)$. Now apply Lemma 15.10 in the special case $Z = E_{\mathcal{G}}(G)$. ☐

This implies the following transitivity principle for the Fibered Isomorphism Conjecture. At the level of spectra this transitivity principle is due to Farrell and Jones [351, Theorem A.10].

Theorem 15.13 (Transitivity Principle). *Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G.*

(i) Assume that for every element $H \in \mathcal{G}$ the group H satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively for $\mathcal{F}|_H$.
Then the group G satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to $\mathcal{G}$ if and only if G satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to $\mathcal{F}$;
(ii) The group G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to $\mathcal{G}$ if G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to $\mathcal{F}$;

Proof. (i) We first treat the (slightly harder) case of the Fibered Meta-Isomorphism Conjecture 15.8

Consider a group homomorphism $\phi\colon K \to G$. Then for every subgroup H of K we conclude

$$(\phi|_H)^*(\mathcal{F}|_{\phi(H)}) = (\phi^*\mathcal{F})|_H$$

from (15.6) where $\phi|_H\colon H \to \phi(H)$ is the group homomorphism induced by ϕ. For every element $H \in \phi^*\mathcal{G}$ the map

$$\mathcal{H}_n^H(E_{(\phi|_H)^*(\mathcal{F}|_{\phi(H)})}(H)) = \mathcal{H}_n^H(E_{\phi^*\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

is bijective for all $n \in \mathbb{Z}$ by the assumption that the element $\phi(H) \in \mathcal{G}$ satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}|_{\phi(H)}$. Hence by Theorem 15.12 applied to the inclusion $\phi^*\mathcal{F} \subseteq \phi^*\mathcal{G}$ of families of subgroups of K we get an isomorphism

$$\mathcal{H}_n^K(\iota_{\phi^*\mathcal{F}\subseteq\phi^*\mathcal{G}})\colon \mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \xrightarrow{\cong} \mathcal{H}_n^K(E_{\phi^*\mathcal{G}}(K)).$$

Therefore the map $\mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \to \mathcal{H}_n^K(K/K)$ is bijective for all $n \in \mathbb{Z}$ if and only if the map $\mathcal{H}_n^K(E_{\phi^*\mathcal{G}}(K)) \to \mathcal{H}_n^K(K/K)$ is bijective for all $n \in \mathbb{Z}$.

The argument for the Meta-Isomorphism Conjecture 15.8 is analogous, just specialize the argument above to the case $\phi = \mathrm{id}_G$.

(ii) We want to apply assertion (i). We have to show that for every element $H \in \mathcal{G}$ the group H satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for $\mathcal{F}|_H$, provided that G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to $\mathcal{F}$. This follows from the elementary Lemma 15.16 below since $\mathcal{F}|_H = i^*\mathcal{F}$ for the inclusion $i\colon H \to G$. □

Note that assertion (ii) of Theorem 15.13 is only formulated for the fibered version.

The Fibered Isomorphism Conjecture is also well behaved with respect to finite intersections of families of subgroups.

Lemma 15.14. *Let G be a group, and let $\mathcal{F}$ and $\mathcal{G}$ be families of subgroups. Suppose that G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for both $\mathcal{F}$ and $\mathcal{G}$.*

Then G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the family $\mathcal{F} \cap \mathcal{G} := \{H \subseteq G \mid H \in \mathcal{F} \text{ and } H \in \mathcal{G}\}$.

Proof. Obviously $\mathcal{F} \cup \mathcal{G} := \{H \subseteq G \mid H \in \mathcal{F} \text{ or } H \in \mathcal{G}\}$ is a family of subgroups of G.

Consider a group homomorphism $\phi\colon K \to G$. We have to show that the Meta-Isomorphism Conjecture 15.2 holds for G with respect to $\phi^*(\mathcal{F} \cap \mathcal{G})$.

Choose G-CW-models $E_{\mathcal{F}\cap\mathcal{G}}(G)$, $E_{\mathcal{F}}(G)$ and $E_{\mathcal{G}}(G)$ such that $E_{\mathcal{F}\cap\mathcal{G}}(G)$ is a G-CW-subcomplex of both $E_{\mathcal{F}}(G)$ and $E_{\mathcal{G}}(G)$. This can be arranged by a mapping cylinder construction. Define a G-CW-complex

$$X = E_{\mathcal{F}}(G) \cup_{E_G(\mathcal{F}\cap\mathcal{G})} E_{\mathcal{G}}(G).$$

For any subgroup $H \subseteq G$ we get

$$X^H = E_{\mathcal{F}}(G)^H \cup_{E_G(\mathcal{F}\cap\mathcal{G})^H} E_{\mathcal{G}}(G)^H.$$

If $E_{\mathcal{F}}(G)^H$ and $E_{\mathcal{G}}(G)^H$ are empty, the same is true for X^H. If $E_{\mathcal{F}}(G)^H$ is empty, then $E_{\mathcal{G}}(G)^H = X^H$. If $E_{\mathcal{G}}(G)^H$ is empty, then $E_{\mathcal{F}}(G)^H = X^H$. If $E_{\mathcal{F}}(G)^H$, $E_{\mathcal{G}}(G)^H$, and $E_{\mathcal{F}\cap\mathcal{G}}(G)^H$ are weakly contractible, the same is true for X^H. Hence X is a model for $E_{\mathcal{F}\cup\mathcal{G}}(G)$. If we apply restriction with ϕ, we get a decomposition of $E_{\phi^*(\mathcal{F}\cup\mathcal{G})}(K) = \phi^* E_{\mathcal{F}\cup\mathcal{G}}(G)$ as the union of $E_{\phi^*\mathcal{F}}(K) = \phi^* E_{\mathcal{F}}(G)$ and $E_{\phi^*\mathcal{G}}(K) = \phi^* E_{\mathcal{G}}(G)$ such that the intersection of $E_{\phi^*\mathcal{F}}(K)$ and $E_{\phi^*\mathcal{G}}(K)$ is $E_{\phi^*(\mathcal{F}\cap\mathcal{G})}(K) = \phi^* E_{\mathcal{F}\cap\mathcal{G}}(G)$. By assumption and by Theorem 15.13 (ii) the Fibered Meta-Isomorphism Conjecture 15.8 holds for G with respect to $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{F} \cup \mathcal{G}$. Hence the Meta-Isomorphism Conjecture 15.2 holds for G with respect to $\phi^*(\mathcal{F}\cup\mathcal{G})$, $\phi^*\mathcal{F}$, and $\phi^*\mathcal{G}$. Using the Mayer-Vietoris sequence for the decomposition of $E_{\phi^*\mathcal{F}\cup\phi^*\mathcal{G}}(K)$ above and the Five Lemma, we conclude that Meta-Isomorphism Conjecture 15.2 holds for G with respect to $\phi^*(\mathcal{F} \cap \mathcal{G})$. Since $\phi\colon K \to G$ is an arbitrary group homomorphism with target G, the group G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the family $\mathcal{F} \cap \mathcal{G}$. □

Exercise 15.15. Assume that the Fibered Meta-Isomorphism Conjecture 15.8 holds for $G = \mathbb{Z}$, the family $\mathcal{F} = \mathcal{FIN}$, and the equivariant homology theory $H_*^?(-;\mathbf{K}_R)$ for a given ring R.

Show that then we have $NK_n(RG) = 0$ for every group G and $n \in \mathbb{Z}$.

15.6 Inheritance Properties of the Fibered Meta-Isomorphism Conjecture

The Fibered Meta-Isomorphism Conjecture 15.8 has better inheritance properties than the Meta-Isomorphism Conjecture 15.2.

In this subsection we treat only equivariant homology theories $\mathcal{H}_*^?$ for simplicity. The generalizations to an equivariant homology theory over a group Γ are obvious.

Lemma 15.16. *Let $\phi\colon K \to G$ be a group homomorphism and $\mathcal{F}$ be a family of subgroups. If $(G, \mathcal{F})$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8 then $(K, \phi^*\mathcal{F})$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8.*

Proof. If $\psi\colon L \to K$ is a group homomorphism, then $\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}$ by (15.6). □

Exercise 15.17. Fix a class of groups $\mathcal{C}$ that is closed under isomorphisms and taking subgroups, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G that belong to $\mathcal{C}$. Suppose that the Fibered Meta-Isomorphism Conjecture 15.8 holds for $(G, \mathcal{C}(G))$. Let $H \subseteq G$ be a subgroup.

Show that $(H, \mathcal{C}(H))$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8.

Lemma 15.18. *Fix a class of groups $\mathcal{C}$ that is closed under isomorphisms, taking subgroups, and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G that belong to $\mathcal{C}$. Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension of groups. Suppose that $(Q, \mathcal{C}(Q))$ and $(p^{-1}(H), \mathcal{C}(p^{-1}(H))$ for every $H \in \mathcal{C}(Q)$ satisfy the Fibered Meta-Isomorphism Conjecture 15.8.*

Then $(G, \mathcal{C}(G))$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8.

Proof. By Lemma 15.16 the pair $(G, p^*\mathcal{C}(Q))$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8. Obviously $\mathcal{C}(G) \subseteq p^*\mathcal{C}(Q)$. Because of the Transitivity Principle 15.13 (i) it remains to show for each $L \in p^*\mathcal{C}(Q)$ that the pair $(L, \mathcal{C}(L))$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8. Since $L \subseteq p^{-1}(p(L))$ and $p(L) \in \mathcal{C}(Q)$ holds, we conclude from Exercise 15.17 that this follows from the assumption that $(p^{-1}(H), \mathcal{C}(p^{-1}(H))$ satisfy the Fibered Meta-Isomorphism Conjecture 15.8 for every $H \in \mathcal{C}(Q)$. $\square$

Fix an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules. Let X be a G-CW-complex. Let $\alpha\colon H \to G$ be a group homomorphism. Denote by α^*X the H-CW-complex obtained from X by *restriction with* α. Recall that α_*Y denotes the induction of an H-CW-complex Y and is a G-CW-complex. The functors α_* and α^* are adjoint to one another. In particular the adjoint of the identity on α^*X is a natural G-map

$$f(X, \alpha)\colon \alpha_*\alpha^*X \to X. \tag{15.19}$$

It sends an element in $G \times_\alpha \alpha^*X$ given by (g, x) to gx. Define the Λ-map

$$a_n = a_n(X, \alpha)\colon \mathcal{H}_n^H(\alpha^*X) \xrightarrow{\operatorname{ind}_\alpha} \mathcal{H}_n^G(\alpha_*\alpha^*X) \xrightarrow{\mathcal{H}_n^G(f(X,\alpha))} \mathcal{H}_n^G(X).$$

If $\beta\colon G \to K$ is another group homomorphism, then by the axioms of an induction structure the composite $\mathcal{H}_n^H(\alpha^*\beta^*X) \xrightarrow{a_n(\beta^*X,\alpha)} \mathcal{H}_n^G(\beta^*X) \xrightarrow{a_n(X,\beta)} \mathcal{H}_n^K(X)$ agrees with $a_n(X, \beta \circ \alpha)\colon \mathcal{H}_n^H(\alpha^*\beta^*X) = \mathcal{H}_n^H((\beta \circ \alpha)^*X) \to \mathcal{H}_n^G(X)$ for a K-CW-complex X.

Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$ and $\phi_{i,j}\colon G_i \to G_j$ for $i, j \in I, i \leq j$. We obtain for every G-CW-complex X a system of Λ-modules $\{\mathcal{H}_n^{G_i}(\psi_i^*X) \mid i \in I\}$ with structure maps $a_n(\psi_j^*X, \phi_{i,j})\colon \mathcal{H}_n^{G_i}(\psi_i^*X) \to \mathcal{H}_n^{G_j}(\psi_j^*X)$. We get a map of Λ-modules

(15.20)
$$t_n^G(X,A) := \operatorname{colim}_{i \in I} a_n(X,\psi_i)\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*(X,A)) \;\to\; \mathcal{H}_n^G(X,A).$$

Definition 15.21 ((Strongly) continuous equivariant homology theory). An equivariant homology theory $\mathcal{H}_*^?$ is called *continuous* if for every group G and every directed system of subgroups $\{G_i \mid i \in I\}$ of G with $G = \bigcup_{i \in I} G_i$ the Λ-map, see (15.20),

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

An equivariant homology theory $\mathcal{H}_*^?$ over Γ is called *strongly continuous* if for every group G and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$ the Λ-map

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

The next result is taken from [70, Lemma 3.4].

Lemma 15.22. *Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$. Let (X,A) be a G-CW-pair. Suppose that $\mathcal{H}_*^?$ is strongly continuous.*

Then the Λ-homomorphism, see (15.20)

$$t_n^G(X,A)\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*(X,A)) \;\xrightarrow{\cong}\; \mathcal{H}_n^G(X,A)$$

is bijective for every $n \in \mathbb{Z}$.

The proof of the next result is based on Lemma 15.22.

Lemma 15.23. *Fix a class of groups $\mathcal{C}$ that is closed under isomorphisms, taking subgroups, and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G that belong to $\mathcal{C}$. Let G be a group.*

(i) Let G be the directed union of subgroups $\{G_i \mid i \in I\}$. Suppose that $\mathcal{H}_*^?$ is continuous and for every $i \in I$ the Meta-Isomorphism Conjecture 15.2 holds for G_i and $\mathcal{C}(G_i)$.
Then the Meta-Isomorphism Conjecture 15.2 holds for G and $\mathcal{C}(G)$;

(ii) Let G be the directed union of subgroups $\{G_i \mid i \in I\}$. Suppose that $\mathcal{H}_*^?$ is continuous and for every $i \in I$ the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for G_i and $\mathcal{C}(G_i)$ is injective for all $n \in \mathbb{Z}$.
Then the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for G and $\mathcal{C}(G)$ is injective for all $n \in \mathbb{Z}$;

(iii) Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i\colon G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and for every $i \in I$ the Fibered Meta-Isomorphism Conjecture 15.8 holds for G_i and $\mathcal{C}(G_i)$.
Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for G and $\mathcal{C}(G)$.

Proof. (i) is proved in [74, Proposition 3.4].

(ii) The proof of [74, Proposition 3.4] for isomorphism yields also a proof for the injectivity version since the colimit over a directed system is an exact functor and hence preserves injectivity.

(iii) See [70, Theorem 5.6]. $\square$

Remark 15.24 (Injectivity and the Transitivity principle). For colimits over a directed system of subgroups, we did get a statement about injectivity in Lemma 15.23 (ii), essentially since the colimit over a directed system is an exact functor. We cannot prove such injectivity statement for assertion (iii) since its proof uses the Transitivity Principle 15.13 for which the injectivity version is not true in general, essentially, because the Five Lemma does not has a version for injectivity.

15.7 Actions on Trees

In this subsection we treat only equivariant homology theories $\mathcal{H}^?_*$ for simplicity. The generalizations to an equivariant homology theory over a group Γ are obvious.

Given a subgroup $H \subseteq G$, we obtain a G-homeomorphism $G \times_H \underline{E}G|_H \xrightarrow{\cong} G/H \times \underline{E}G$ sending (g,z) to (gH, gz) where G acts diagonally on the target. The inverse sends (gH,z) to $(g, g^{-1}z)$. Since $\underline{E}G|_H$ is a model for $\underline{E}H$, we obtain a G-homotopy equivalence

$$\mu(H)\colon G \times_H \underline{E}H \xrightarrow{\simeq} G/H \times \underline{E}G. \tag{15.25}$$

Recall that we obtain for any subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ from the induction structure an isomorphism

$$\operatorname{ind}_H^G\colon \mathcal{H}^H_n(\underline{E}H) \xrightarrow{\cong} \mathcal{H}^G_n(G \times_H \underline{E}H). \tag{15.26}$$

In the sequel we denote by pr the obvious projection and by ι the obvious inclusion.

Lemma 15.27. *Suppose that G acts on the tree T by automorphisms of trees without inversion. Let $\mathcal{H}^?_*$ be an equivariant homology theory.*

(i) We can write T as a G-pushout

$$\begin{tikzcd}
\coprod_{j\in J} G/K_j \times S^0 \arrow[r, "q"] \arrow[d, "k"'] & \coprod_{i\in I} G/H_i \arrow[d, "\overline{k}"] \\
\coprod_{j\in J} G/K_j \times D^1 \arrow[r, "\overline{q}"'] & T
\end{tikzcd}$$

where there are for every $j \in J$ two elements $i(j,+)$ and $i(j,-)$ in I such that the restriction of q to G/K_j considered as G-subspace of $\coprod_{j\in J} G/K_j \times S^0$ by

$$G/K_j = G/K_j \times \{\pm 1\} \subseteq G/K_j \times S^0 \subseteq \coprod_{j\in J} G/K_j \times S^0$$

is given by the composite of a G-map $\widehat{q}_{j,\pm 1}\colon G/K_j \to G/H_{i(j,\pm)}$ with the canonical inclusion $G/H_{i(j,\pm)} \to \coprod_{i\in I} G/H_i$;

(ii) We obtain a long exact sequence

$$\cdots \to \bigoplus_{j\in J} \mathcal{H}_n^{K_j}(\underline{E}K_j) \xrightarrow{t_n(j,+)-t_n(j,-)} \bigoplus_{i\in I} \mathcal{H}_n^{H_i}(\underline{E}H_i) \xrightarrow{s_n} \mathcal{H}_n^G(\underline{E}G)$$
$$\to \bigoplus_{j\in J} \mathcal{H}_{n-1}^{K_j}(\underline{E}K_j) \xrightarrow{t_{n-1}(j,+)-t_{n-1}(j,-)} \bigoplus_{i\in I} \mathcal{H}_{n-1}^{H_i}(\underline{E}H_i) \xrightarrow{s_{n-1}} \cdots$$

where $t_n(j,\pm)$ is given by the composite

$$\mathcal{H}_n^{K_j}(\underline{E}K_j) \xrightarrow{\operatorname{ind}_{K_j}^G} \mathcal{H}_n^G(G \times_{K_j} \underline{E}K_j) \xrightarrow{\mathcal{H}_n^G(\mu(K_j))} \mathcal{H}_n^G(G/K_j \times \underline{E}G)$$
$$\xrightarrow{\mathcal{H}_n^G(\widehat{q}_{j,\pm 1}\times \mathrm{id}_{\underline{E}G})} \mathcal{H}_n^G(G/H_{i(j,\pm)} \times \underline{E}G) \xrightarrow{\mathcal{H}_n^G(\mu(H_{i(j,\pm)}))^{-1}} \mathcal{H}_n^G(G \times_{H_{i(j,\pm)}} \underline{E}H_{i(j,\pm)})$$
$$\xrightarrow{(\operatorname{ind}_{H_{i(j,\pm)}}^G)^{-1}} \mathcal{H}_n^{H_{i(j,\pm)}}(\underline{E}H_{i(j,\pm)}) \xrightarrow{\iota} \bigoplus_{i\in I} \mathcal{H}_n^{H_i}(\underline{E}H_i)$$

and s_n is the direct sum of the maps for $i \in I$

$$\mathcal{H}_n^{H_i}(\underline{E}(H_i)) \xrightarrow{\operatorname{ind}_{H_i}^G} \mathcal{H}_n^G(G \times_{H_i} \underline{E}(H_i))$$
$$\xrightarrow{\mathcal{H}_n^G(\mu(H_i))} \mathcal{H}_n^G(G/H \times \underline{E}(G)) \xrightarrow{\mathcal{H}_n^G(\mathrm{pr})} \mathcal{H}_n^G(\underline{E}G).$$

Proof. (i) Since G acts on T by automorphisms of trees without inversion, T is a 1-dimensional G-CW-complex and the G-pushout just describes how the 1-skeleton is obtained from the 0-skeleton $\coprod_{i\in I} G/H_i$.

(ii) If we cross the G-pushout of assertion (i) with $\underline{E}G$ using the diagonal

G-action, we obtain the G-pushout

(15.28)
$$\begin{tikzcd} \coprod_{j\in J} G/K_j \times \underline{E}G \times S^0 \arrow[r, "q\times \mathrm{id}_{\underline{E}G}"] \arrow[d, "k\times \mathrm{id}_{\underline{E}G}"'] & \coprod_{i\in I} G/H_i \times \underline{E}G \arrow[d, "\overline{k}\times \mathrm{id}_{\underline{E}G}"] \\ \coprod_{j\in J} G/K_j \times \underline{E}G \times D^1 \arrow[r, "\overline{q}\times \mathrm{id}_{\underline{E}G}"'] & T \times \underline{E}G \end{tikzcd}$$

The H-fixed point set T^H is a non-empty subtree and therefore contractible for every finite subgroup $H \subseteq G$, see [894, Theorem 15 in 6.1 on page 58 and 6.3.1 on page 60]. We conclude that the projection $\underline{E}G \times T \to \underline{E}G$ is a G-homotopy equivalence from the Equivariant Whitehead Theorem, see for instance [629, Theorem 2.4 on page 36]. The desired long exact sequence can be derived from the Mayer-Vietoris sequence associated to the G-pushout (15.28) using the identifications (15.25) and (15.26). $\square$

Lemma 15.29. *Suppose that G acts on the tree T by automorphisms of trees without inversion. Let $\mathcal{H}_*^?$ be an equivariant homology theory, Suppose that the Meta-Isomorphism Conjecture 15.2 holds for G with respect to $\mathcal{FIN}$. Assume that for any isotropy group H of the G-action on T the Meta-Isomorphism Conjecture 15.2 holds for H with respect to $\mathcal{FIN}$.*

(i) The projection $T \to \{\bullet\}$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(T) \xrightarrow{\cong} \mathcal{H}_n^G(\{\bullet\});$$

(ii) Write T as a G-pushout as described in Lemma 15.27 (i). *Let $g(j,\pm)$ be an element in G such that $g(j,\pm)K_j g(j,\pm))^{-1} \subseteq H_{i(j,\pm)}$ and the G-map $\widehat{q}_{j,\pm 1}\colon G/K_j \to G/H_{i_\pm(j)}$ is given by $gK_j \mapsto gg(j,\pm)^{-1}H_{i(j,\pm)}$. Let $c(g(j,\pm))\colon K_j \to H_{i(j,\pm)}$ be the group homomorphism sending k to $g(j,\pm))kg(j,\pm))^{-1}$.*
We get a long exact sequence

$$\cdots \to \bigoplus_{j\in J} \mathcal{H}_n^{K_j}(\{\bullet\}) \xrightarrow{t'_n(j,+)-t'_n(j,-)} \bigoplus_{i\in I} \mathcal{H}_n^{H_i}(\{\bullet\}) \xrightarrow{s'} \mathcal{H}_n^G(\{\bullet\})$$
$$\to \bigoplus_{j\in J} \mathcal{H}_{n-1}^{K_j}(\{\bullet\}) \xrightarrow{t'_{n-1}(j,+)-t'_{n-1}(j,-)} \bigoplus_{i\in I} \mathcal{H}_{n-1}^{H_i}(\{\bullet\}) \xrightarrow{s'_{n-1}} \cdots$$

where $t'_n(j,\pm)$ is given by the composite

$$\mathcal{H}_n^{K_j}(\{\bullet\}) \xrightarrow{\mathrm{ind}_{c(g(j,\pm))}} \mathcal{H}_n^{H_{i(j,\pm)}}(\mathrm{ind}_{c(g(j,\pm))}\{\bullet\})$$
$$\xrightarrow{\mathrm{pr}} \mathcal{H}_n^{H_{i(j,\pm)}}(\{\bullet\}) \xrightarrow{\iota} \bigoplus_{i\in I} \mathcal{H}_n^{H_i}(\{\bullet\})$$

and s'_n is the direct sum of the maps for $i \in I$

$$\mathcal{H}_n^{H_i}(\{\bullet\}) \xrightarrow{\operatorname{ind}_{H_i}^G} \mathcal{H}_n^G(G \times_{H_i} \{\bullet\}) \xrightarrow{\mathcal{H}_n^G(\mathrm{pr})} \mathcal{H}_n^G(\{\bullet\}).$$

Proof. (i) We have already explained in the proof of Lemma 15.27 (ii) that the projection $\underline{E}G \times T \to \underline{E}G$ is a G-homotopy equivalence. By assumption the projection $\underline{E}G \to \{\bullet\}$ induces for all $n \in \mathbb{Z}$ isomorphisms $\mathcal{H}_n^G(\underline{E}G) \to \mathcal{H}_n^G(\{\bullet\})$. Hence the projection $\underline{E}G \times T \to \{\bullet\}$ induces for all $n \in \mathbb{Z}$ isomorphisms $\mathcal{H}_n^G(\underline{E}G \times T) \to \mathcal{H}_n^G(\{\bullet\})$. By Lemma 15.10 and the assumptions on T the projection $\underline{E}G \times T \to T$ induces for all $n \in \mathbb{Z}$ isomorphisms $\mathcal{H}_n^G(\underline{E}G \times T) \to \mathcal{H}_n^G(T)$. Hence the projection $T \to \{\bullet\}$ induces for all $n \in \mathbb{Z}$ isomorphisms $\mathcal{H}_n^G(T) \to \mathcal{H}_n^G(\{\bullet\})$.

(ii) This follows from Lemma 12.12 and Lemma 15.27 (ii). □

Example 15.30 (Amalgamated free products). Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ-modules. Let G be the amalgamated free product $G_1 *_{G_0} G_2$ for a common subgroup G_0 of the groups G_1 and G_2. Suppose that G_i for $i = 0, 1, 2$ and G satisfy the Meta-Isomorphism Conjecture 15.2 with respect to the family $\mathcal{FIN}$. Then there is a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{H}_n^{G_0}(\{\bullet\}) \to \mathcal{H}_n^{G_1}(\{\bullet\}) \oplus \mathcal{H}_n^{G_1}(\{\bullet\}) \to \mathcal{H}_n^{G}(\{\bullet\}) \\ \to \mathcal{H}_{n-1}^{G_0}(\{\bullet\}) \to \mathcal{H}_{n-1}^{G_1}(\{\bullet\}) \oplus \mathcal{H}_{n-1}^{G_1}(\{\bullet\}) \to \cdots \end{aligned}$$

Namely, there is a 1-dimensional G-CW-complex T whose underlying space is a tree such that the 1-skeleton is obtained from the 0-skeleton by the G-pushout

$$\begin{array}{ccc} G/G_0 \times S^0 & \xrightarrow{q} & G/G_1 \coprod G/G_2 \\ \downarrow & & \downarrow \\ G/G_0 \times D^1 & \longrightarrow & T \end{array}$$

where q is the disjoint union of the canonical projection $G/G_0 \to G/G_1$ and $G/G_0 \to G/G_2$, see [894, Theorem 7 in §4.1 on page 32]. Now the desired long exact sequence is the one appearing in Lemma 15.29 (ii).

Suppose that G_0, G_1, G_2 and G satisfy the Baum-Connes Conjecture 14.9, which is equivalent the Meta-Isomorphism Conjecture 15.2 if we choose $\mathcal{F}$ to be $\mathcal{FIN}$ and $\mathcal{H}_n^G$ to be $H_n^G(-; \mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}})$. Then we obtain a long exact sequence

(15.31)

$$\begin{aligned}
\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) &\xrightarrow{K_n(C_r^*(i_1)) \oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \\
&\xrightarrow{K_n(C_r^*(j_1)) - K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \\
&\xrightarrow{K_{n-1}(C_r^*(i_1)) \oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1)) \\
&\xrightarrow{K_{n-1}(C_r^*(j_1)) - K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots
\end{aligned}$$

where i_1,i_2, j_1 and j_2 are the obvious inclusions. Actually, such long exact Mayer-Vietoris sequence exists always for an amalgamated free product $G = G_1 *_{G_0} G_2$, see Pimsner [796, Theorem 18 on page 632].

Suppose that G_0, G_1, G_2 and G satisfy the K-theoretic Farrell Conjecture Conjecture 13.1 with coefficients in the regular ring R with $\mathbb{Q} \subseteq R$. Then we obtain using Theorem 13.48 a long exact sequence

$$\begin{aligned}
(15.32) \quad \cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) &\xrightarrow{K_n(Ri_1) \oplus K_n(Ri_2)} K_n(RG_1) \oplus K_n(RG_2) \\
&\xrightarrow{K_n(Rj_1) - K_n(Rj_2)} K_n(RG) \xrightarrow{\partial_n} K_{n-1}(RG_0) \\
&\xrightarrow{K_{n-1}(Ri_1) \oplus K_{n-1}(Ri_2)} K_{n-1}(RG_2) \oplus K_{n-1}(RG_1) \\
&\xrightarrow{K_{n-1}(Rj_1) - K_{n-1}(Rj_2)} K_{n-1}(RG) \xrightarrow{\partial_{n-1}} \cdots .
\end{aligned}$$

Without extra assumptions on R the long exact sequence above does not exist, certain Nil-terms enter, see Theorem 6.62.

Suppose that G_0, G_1, G_2 and G satisfy the L-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R with involution. Then we obtain using Theorem 13.59 (i) a long exact sequence

$$\begin{aligned}
(15.33) \quad \cdots \xrightarrow{\partial_{n+1}} & L_n(RG_0)[1/2] \\
&\xrightarrow{L_n(Ri_1)[1/2] \oplus L_n(Ri_2)[1/2]} L_n(RG_1)[1/2] \oplus L_n(RG_2)[1/2] \\
&\xrightarrow{L_n(Rj_1)[1/2] - L_n(Rj_2)[1/2]} L_n(RG)[1/2] \xrightarrow{\partial_n} L_{n-1}(RG_0)[1/2] \\
&\xrightarrow{L_{n-1}(Ri_1)[1/2] \oplus L_{n-1}(Ri_2)[1/2]} L_{n-1}(RG_2)[1/2] \oplus L_{n-1}(RG_1)[1/2] \\
&\xrightarrow{L_{n-1}(Rj_1)[1/2] - L_{n-1}(Rj_2)[1/2]} L_{n-1}(RG)[1/2] \xrightarrow{\partial_{n-1}} . \cdots
\end{aligned}$$

Note that the decoration of the L-groups does not play a role since we invert 2. Actually, such long exact Mayer-Vietoris sequence exists always for an amalgamated free product $G = G_1 *_{G_0} G_2$, see Cappell [192]. Without inverting 2 the long exact sequence above does not exist, certain UNil-terms enter.

Exercise 15.34. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Let $\phi\colon G \to G$ be a group automorphism. Let $G \times_\phi \mathbb{Z}$ be the associated semidirect product.

Denote by $i\colon G \to G \rtimes_\phi \mathbb{Z}$ the obvious inclusion. Suppose that G and $G \times_\phi \mathbb{Z}$ satisfy the Meta-Isomorphism Conjecture 15.2 with respect to the family $\mathcal{FIN}$.

Prove the existence of a long exact sequence

$$\cdots \to \mathcal{H}_n^G(\{\bullet\}) \xrightarrow{\phi_*-\mathrm{id}} \mathcal{H}_n^G(\{\bullet\}) \xrightarrow{k_*} \mathcal{H}_n^{G\rtimes_\phi \mathbb{Z}}(\{\bullet\})$$
$$\to \mathcal{H}_{n-1}^G(\{\bullet\}) \xrightarrow{\phi_*-\mathrm{id}} \mathcal{H}_{n-1}^G(\{\bullet\}) \xrightarrow{k_*} \cdots$$

where $\phi_*\colon \mathcal{H}_n^G(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$ and k_* come from the induction structure and the identification $\mathrm{ind}_\phi\{\bullet\} = \{\bullet\}$ and the projection $\mathrm{ind}_i\{\bullet\} \to \{\bullet\}$.

Explain that this reduces in the case of the Baum-Connes Conjecture to the long exact sequence

$$\cdots \to K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(\phi))-\mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_\phi \mathbb{Z}))$$
$$\to K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(\phi))-\mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots,$$

and similar for the K-theoretic Farrell-Jones Conjecture for a regular ring R with $\mathbb{Q} \subseteq R$ and the L-theoretic Farrell-Jones Conjecture after inverting 2.

15.8 The Meta-Isomorphism Conjecture for Functors from Spaces to Spectra

Let $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor. Throughout this section we will assume that it respects weak equivalences and disjoint unions, i.e., a weak homotopy equivalence of spaces $f\colon X \to Y$ is sent to a weak homotopy equivalence of spectra $\mathbf{S}(f)\colon \mathbf{S}(X) \to \mathbf{S}(Y)$ and for a collection of spaces $\{X_i \mid i \in I\}$ for an arbitrary index set I the canonical map

$$\bigvee_{i\in I} \mathbf{S}(X_i) \to \mathbf{S}\Bigl(\coprod_{i\in I} X_i\Bigr)$$

is weak homotopy equivalence of spectra. We obtain a covariant functor

$$\mathbf{S}^B\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}, \quad \mathcal{G} \mapsto \mathbf{S}(B\mathcal{G}) \tag{15.35}$$

where $B\mathcal{G}$ is the classifying space of the category $\mathcal{G}$ which is the geometric realization of the simplicial set given by its nerve and denoted by $B^{\mathrm{bar}}\mathcal{G}$ in [265, page 227]. Denote by $H_n^?(-;\mathbf{S}^B)$ be the equivariant homology theory in the sense of Definition 12.9 which is associated to $\mathbf{S}_B$ by the construction of Theorem 12.30. It has the property that for any group G and subgroup $H \subseteq G$ we have canonical identifications

$$H_n^G(G/H;\mathbf{S}^B) \cong H_n^H(H/H;\mathbf{S}^B) \cong \pi_n(\mathbf{S}(BH)).$$

Conjecture 15.36 (Meta-Isomorphism Conjecture for functors from spaces to spectra). Let $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor that respects weak equivalences and disjoint unions. The group G satisfies the *Meta-Isomorphism Conjecture for* $\mathbf{S}$ with respect to the family $\mathcal{F}$ of subgroups of G if it satisfies the Meta-Isomorphism Conjecture 15.2 for the G-homology theory $H_*^G(-;\mathbf{S}^B)$, i.e., the assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{S}^B) \to H_n^G(G/G;\mathbf{S}^B)$$

is bijective for all $n \in \mathbb{Z}$.

Example 15.37 (The Farrell-Jones Conjecture in the setting of functors from spaces to spectra). In the sequel $\Pi(X)$ denotes the fundamental groupoid of a space X. If we take the covariant functor to be the one that sends a space X to $\mathbf{K}_R(\Pi(X))$, $\mathbf{L}_R^{\langle -\infty\rangle}(\Pi(X))$, or $\mathbf{K}_F^{\mathsf{TOP}}(\Pi(X))$ respectively, see Theorem 12.43, then the Meta-Isomorphism Conjecture 15.36 for $\mathbf{S}$ for a group G and the family $\mathcal{VCY}$, $\mathcal{VCY}$, or $\mathcal{FIN}$ respectively is equivalent to the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R, the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R, or the Baum-Connes Conjecture 14.9 respectively. This follows from the obvious natural weak equivalence of groupoids $\mathcal{G} \xrightarrow{\simeq} \Pi(B\mathcal{G})$.

Let G be a group and Z be a G-CW-complex. Define a covariant $\mathsf{Or}(G)$-spectrum

$$\mathbf{S}_Z^G\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \mathbf{S}(G/H \times_G Z) \tag{15.38}$$

where $G/H \times_G Z$ is the orbit space of the diagonal left G-action on $G/H \times S$. Note that there is an obvious homeomorphism $G/H \times_G Z \xrightarrow{\cong} H\backslash Z$.

Conjecture 15.39 (Meta-Isomorphism Conjecture for functors from spaces to spectra with coefficients). Let $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor that respects weak equivalences and disjoint unions. The group G satisfies the *Meta-Isomorphism Conjecture for* $\mathbf{S}$ *with coefficients* with respect to the family $\mathcal{F}$ of subgroups of G if for any free G-CW-complex Z the pair $(G,\mathcal{F})$ satisfies the Meta-Isomorphism Conjecture 15.2 for $H_*^G(-;\mathbf{S}_Z^G)$, i.e., the assembly map

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{S}_Z^G) \to H_n^G(G/G;\mathbf{S}_Z^G)$$

is bijective for all $n \in \mathbb{Z}$.

Exercise 15.40. Let $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor that respects weak equivalences and disjoint unions. Suppose that it satisfies the Meta-Isomorphism Conjecture 15.39 for every group G and the trivial family

$\mathcal{TR}$ consisting of one element, the trivial subgroup. Show that then we obtain for every connected CW-complex X a weak homotopy equivalence

$$E\pi_1(X)_+ \wedge_{\pi_1(X)} \mathbf{S}(\widetilde{X}) \to \mathbf{S}(X).$$

Show that $\pi_n\big(E\pi_1(X)_+ \wedge_{\pi_1(X)} \mathbf{S}(\widetilde{X})\big)$ and $\pi_n\big(B\pi_1(X)_+ \wedge \mathbf{S}(\{\bullet\})\big)$ are not isomorphic in general, but that they are isomorphic if $\widetilde{X}$ is contractible or $\mathbf{S}$ is of the shape $Y \mapsto \mathbf{T}(\Pi(Y))$ for some covariant functor $\mathbf{T}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$.

Example 15.41 ($Z = EG$). If we take $Z = EG$ in Conjecture 15.39, then Conjecture 15.39 reduces to Conjecture 15.36 since there is a natural homotopy equivalence $G/H \times_G EG \xrightarrow{\simeq} B\mathcal{G}^G(G/H)$ and hence we get a weak homotopy equivalence of $\mathsf{Or}(G)$-spectra $\mathbf{S}^G_{EG} \xrightarrow{\simeq} \mathbf{S}^B(\mathcal{G}^G(G/?))$.

Remark 15.42 (Relation to the original formulation). In [351, Section 1.7 on page 262] Farrell and Jones formulate a fibered version of their conjectures for a covariant functor $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ for every (Serre) fibration $\xi\colon Y \to X$ over a connected CW-complex X. In our setup this corresponds to choosing Z to be the total space of the fibration obtained from $Y \to X$ by pulling back along the universal covering $\widetilde{X} \to X$. This space Z is a free G-CW for $G = \pi_1(B)$. Note that an arbitrary free G-CW-complex Z can always be obtained in this fashion from the fiber bundle $EG \times_G Z \to BG$ up to G-homotopy, compare [351, Corollary 2.2.1 on page 263].

We sketch the proof of this identification. Let A be a G-CW-complex. Let $\mathcal{E}(X)$ be the G-quotient of the diagonal $G = \pi_1(X)$-action on $A \times \widetilde{X}$ and let $f\colon \mathcal{E}(X) \to X$ be the obvious projection. Denote by $\widehat{p}\colon \mathcal{E}(\xi) \to \mathcal{E}(X)$ the pullback of ξ with f. Let $q\colon \mathcal{E}(\xi) \to A/G$ be the composite of $\widehat{p}$ with the map $\mathcal{E}(X) \to A/G$ induced by the projection $A \times X \to A$. This is a stratified fibration, and one can consider the spectrum $\mathbb{H}(A/G; \mathcal{S}(q))$ in the sense of Quinn [808, Section 8]. Put

$$\mathcal{H}^G_n(A;\xi) := \pi_n(\mathbb{H}(A/G;\mathcal{S}(q)).$$

The projection $\mathrm{pr}\colon A \to G/G$ induces a map

$$a_n(A)\colon \mathcal{H}^G_n(A;\xi) \to \mathcal{H}^G_n(G/G;\xi) = \pi_n(\mathbf{S}(Y)) \tag{15.43}$$

which is the assembly map in [351, Section 1.7 on page 262] if we take $A = E_{\mathcal{VCY}}(G)$. The construction of $\mathcal{H}^G_n(A;\xi) := \mathbb{H}(A/G;\mathcal{S}(q))$ is very complicated, but, fortunately, for us only two facts are relevant. We obtain by $\mathcal{H}^G_*(-;\xi)$ a G-homology theory in the sense of Definition 12.1, and for every $H \subseteq G$ we get a natural identification $\mathcal{H}^G_n(G/H;\xi) = \mathbf{S}^G_Z(G/H)$. Hence the functor G-CW-$\mathsf{COM} \to \mathsf{SPECTRA}$ sending $A \to \mathbb{H}(A/G;\mathcal{S}(q))$ is weakly excisive and its restriction to $\mathsf{Or}(G)$ is the functor $\mathbf{S}^G_Z$. Corollary 18.16 implies that the map (15.43) can be identified with the map induced by the projection $A \to G/G$

$$H_n^G(A; \mathbf{S}_Z^G) \to H_n^G(G/G; \mathbf{S}_Z^G) = \pi_n(\mathbf{S}(Z/G)) = \pi_n(\mathbf{S}(Y))$$

which appears in Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients.

Remark 15.44 (The condition free is necessary in Conjecture 15.39). In general Conjecture 15.39 is not true if we drop the condition that Z is free. Take for instance $Z = G/G$. Then Conjecture 15.39 predicts that the projection $E_{\mathcal{F}}(G)/G \to G/G$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$H_n(\mathrm{pr}; \mathbf{S}(\{\bullet\}))\colon H_n(E_{\mathcal{F}}(G)/G; \mathbf{S}(\{\bullet\})) \to H_n(\{\bullet\}, \mathbf{S}(\{\bullet\}))$$

where $H_*(-; \mathbf{S}(\{\bullet\}))$ is the (non-equivariant) homology theory associated to the spectrum $\mathbf{S}(\{\bullet\})$. This statement is in general wrong, except in extreme cases such as $\mathcal{F} = \mathcal{ALL}$.

The proof of the next theorem will be given at the end of Section 15.9.

Theorem 15.45 (Inheritance properties of the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients). *Let* $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ *be a covariant functor that respects weak equivalences and disjoint unions. Fix a class of groups* $\mathcal{C}$ *that is closed under isomorphisms, taking subgroups, and taking quotients.*

(i) Suppose that the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients holds for the group G *and the family of subgroups* $\mathcal{C}(G) := \{K \subseteq G, K \in \mathcal{C}\}$ *of* G. *Let* $H \subseteq G$ *be a subgroup.*
Then Conjecture 15.39 holds for $(H, \mathcal{C}(H))$*;*

(ii) Let $1 \to K \to G \xrightarrow{p} Q \to 1$ *be an extension of groups. Suppose that* $(Q, \mathcal{C}(Q))$ *and* $(p^{-1}(H), \mathcal{C}(p^{-1}(H))$ *for every* $H \in \mathcal{C}(Q)$ *satisfy Conjecture 15.39.*
Then $(G, \mathcal{C}(G))$ *satisfies Conjecture 15.39;*

(iii) Suppose that Conjecture 15.39 is true for $(H_1 \times H_2, \mathcal{C}(H_1 \times H_2))$ *for every* $H_1, H_2 \in \mathcal{C}$.
Then for two groups G_1 *and* G_2 *Conjecture 15.39 is true for the direct product* $G_1 \times G_2$ *and the family* $\mathcal{C}(G_1 \times G_2)$*, if and only it is true for* $(G_k, \mathcal{C}(G_k))$ *for* $k = 1, 2$*;*

(iv) Suppose that, for any directed systems of spaces $\{X_i \mid i \in I\}$ *indexed over an arbitrary directed set* I*, the canonical map*

$$\mathrm{hocolim}_{i \in I} \mathbf{S}(X_i) \to \mathbf{S}\big(\mathrm{hocolim}_{i \in I} X_i\big)$$

is a weak homotopy equivalence. Let $\{G_i \mid i \in I\}$ *be a directed system of groups over a directed set* I *(with arbitrary structure maps). Put* $G = \mathrm{colim}_{i \in I} G_i$. *Suppose that Conjecture 15.39 holds for* $(G_i, \mathcal{C}(G_i))$ *for every* $i \in I$*;*
Then Conjecture 15.39 holds for $(G, \mathcal{C}(G))$.

Exercise 15.46. Let $\mathcal{C}$ be the class of finite groups or let $\mathcal{C}$ be the class of virtually cyclic subgroups. Suppose that Conjecture 15.39 holds for $(H, \mathcal{C}(H))$ if H contains a subgroup K of finite index such that K is a finite product of finitely generated free groups.

Show that for a collection of groups $\{G_i \mid i \in I\}$ Conjecture 15.39 is true for the free product $*_{i\in I} G_i$ and the family $\mathcal{C}(*_{i\in I} G_i)$, if and only it is true for $(G_i, \mathcal{C}(G_i))$ for every $i \in I$.

Lemma 15.47. *Suppose that there is a spectrum* $\mathbf{E}$ *such that* $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ *is given by* $Y \mapsto Y_+ \wedge \mathbf{E}$.

(i) Then, for any group G*, any* G*-*CW*-complex* X *that is contractible (after forgetting the* G*-action), and any free* G*-*CW*-complex* Z*, the projection* $X \to G/G$ *induces for all* $n \in \mathbb{Z}$ *an isomorphism*

$$H_n^G(X; \mathbf{S}_Z^G) \xrightarrow{\cong} H_n^G(G/G; \mathbf{S}_Z^G);$$

(ii) Both Conjecture 15.36 and Conjecture 15.39 for $\mathbf{S}$ *hold for every group* G *and every family* $\mathcal{F}$ *of subgroups of* G.

Proof. (i) There are natural isomorphisms of spectra

$$\begin{aligned}
&\operatorname{map}_G(G/?, X)_+ \wedge_{\mathsf{Or}(G)} \big((G/? \times_G Z)_+ \wedge \mathbf{E}\big)\\
&\xrightarrow{\cong} \big((\operatorname{map}_G(G/?), X) \times_{\mathsf{Or}(G)} G/?) \times_G Z\big)_+ \wedge \mathbf{E}\\
&\xrightarrow{\cong} (X \times_G Z)_+ \wedge \mathbf{E}
\end{aligned}$$

where the second isomorphism comes from the G-homeomorphism

$$\operatorname{map}_G(G/?), X) \times_{\mathsf{Or}(G)} G/? \xrightarrow{\cong} X$$

of [265, Theorem 7.4 (1)]. Since Z is a free G-CW-complex and X is contractible (after forgetting the group action), the projection $X \times_G Z \to G/G \times_G Z$ is a homotopy equivalence and hence induces a weak homotopy equivalence

$$(X \times_G Z)_+ \wedge \mathbf{E} \xrightarrow{\simeq} (G/G \times_G Z)_+ \wedge \mathbf{E},$$

Thus we get a weak homotopy equivalence

$$\operatorname{map}_G(G/?), X)_+ \wedge_{\mathsf{Or}(G)} \big((G/? \times_G Z)_+ \wedge \mathbf{E}\big) \to (G/G \times_G Z)_+ \wedge \mathbf{E}.$$

Under the identifications coming from the definitions

$$\begin{aligned}
H_n^G(X; \mathbf{S}_Z^G) &:= \pi_n \left(\operatorname{map}_G(G/?), X)_+ \wedge_{\mathsf{Or}(G)} \big((G/? \times_G Z)_+ \wedge \mathbf{E}\big)\right),\\
H_n^G(G/G; \mathbf{S}_Z^G) &= \pi_n \left((G/G \times_G Z)_+ \wedge \mathbf{E}\right),
\end{aligned}$$

this weak homotopy equivalence induces on homotopy groups the isomorphism $H_n^G(X; \mathbf{S}_Z^G) \to H_n^G(G/G; \mathbf{S}_Z^G)$.

(ii) This follows from assertion (i). □

Lemma 15.48. *Let* $\mathbf{S}, \mathbf{T}, \mathbf{U} \colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ *be covariant functors that respects weak equivalences and disjoint unions. Let* $\mathbf{i} \colon \mathbf{S} \to \mathbf{T}$ *and* $\mathbf{p} \colon \mathbf{T} \to \mathbf{U}$ *be natural transformations such that for any space* Y *the map of spectra* $\mathbf{S}(Y) \xrightarrow{\mathbf{i}(Y)} \mathbf{T}(Y) \xrightarrow{\mathbf{p}(Y)} \mathbf{U}(Y)$ *is up to weak homotopy equivalence a cofibration of spectra.*

(i) Then we obtain for every group G *and all* G*-CW-complexes* X *and* Z *a natural long exact sequence*

$$\begin{aligned} \cdots \to H_n^G(X;\mathbf{S}_Z^G) \to H_n^G(X;\mathbf{T}_Z^G) \to H_n^G(X;\mathbf{U}_Z^G) \\ \to H_{n-1}^G(X;\mathbf{S}_Z^G) \to H_{n-1}^G(X;\mathbf{T}_Z^G) \to H_{n-1}^G(X;\mathbf{U}_Z^G) \to \cdots; \end{aligned}$$

(ii) Let G *be a group and* $\mathcal{F}$ *be a family of subgroups of* G*. Then Conjecture 15.36 or Conjecture 15.39 respectively holds for all three functors* $\mathbf{S}$*,* $\mathbf{T}$ *and* $\mathbf{U}$ *for* $(G,\mathcal{F})$ *if Conjecture 15.36 or Conjecture 15.39 respectively holds for two of the functors* $\mathbf{S}$*,* $\mathbf{T}$ *and* $\mathbf{U}$ *for* $(G,\mathcal{F})$*.*

Proof. (i) This is a consequence of the fact following from the version for spectra of [265, Theorem 3.11] that we obtain an up to weak homotopy equivalence cofiber sequence of spectra

$$\begin{aligned} \operatorname{map}_G(G/?,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{S}(G/? \times_G Z) \to \operatorname{map}_G(G/?,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{T}(G/? \times_G Z) \\ \to \operatorname{map}_G(G/?,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{U}(G/? \times_G Z). \end{aligned}$$

(ii) This follows from assertion (i) and the Five Lemma. □

15.9 Proof of the Inheritance Properties

This section is entirely devoted to the proof of Theorem 15.45.

Let $\mathbf{S} \colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be a covariant functor. Throughout this section we will assume that it respects weak equivalences and disjoint unions.

Lemma 15.49. *Let* $\psi \colon K_1 \to K_2$ *be a group homomorphism.*

(i) If Z *is a* K_1*-CW-complex and* X *is a* K_2*-CW-complex, then there is a natural isomorphism*

$$H_n^{K_1}(\psi^*X;\mathbf{S}_Z^{K_1}) \xrightarrow{\cong} H_n^{K_2}(X;\mathbf{S}_{\psi_*Z}^{K_2});$$

(ii) If Z *is a* K_2*-CW-complex and* X *is a* K_1*-CW-complex, then there is a natural isomorphism*

$$H_n^{K_1}(X;\mathbf{S}_{\psi^*Z}^{K_1}) \xrightarrow{\cong} H_n^{K_2}(\psi_*X;\mathbf{S}_Z^{K_2}).$$

Proof. (i) The fourth isomorphisms appearing in [265, Lemma 1.9] implies that it suffices to construct a natural weak homotopy equivalence of $\mathsf{Or}(K_2)$-spectra

$$u(\psi, Z)\colon \psi_* \mathbf{S}_Z^{K_1} \xrightarrow{\cong} \mathbf{S}_{\psi_* Z}^{K_2}$$

where $\psi_* \mathbf{S}_Z^{K_1}$ is the $\mathsf{Or}(K_2)$-spectrum obtained by induction in the sense of [265, Definition 1.8] with the functor $\mathsf{Or}(\psi)\colon \mathsf{Or}(K_1) \to \mathsf{Or}(K_2),\ K_1/H \mapsto \psi_*(K_1/H)$ applied to the $\mathsf{Or}(K_1)$-spectrum $\mathbf{S}_Z^{K_1}$. For a homogeneous space K_2/H, we define $u(\psi, Z)(K_2/H)$ to be the composite

$$\begin{aligned}
\psi_* \mathbf{S}_Z^{K_1}(K_2/H) &:= \operatorname{map}_{K_2}(\psi_*(K_1/?), K_2/H) \times_{\mathsf{Or}(K_1)} \mathbf{S}\,(K_1/? \times_{K_1} Z) \\
&\xrightarrow{\cong} \operatorname{map}_{K_1}(K_1/?, \psi^*(K_2/H)) \times_{\mathsf{Or}(K_1)} \mathbf{S}(K_1/? \times_{K_1} Z) \\
&\xrightarrow{\cong} \mathbf{S}(\psi^*(K_2/H) \times_{K_1} Z) \\
&\xrightarrow{\cong} \mathbf{S}(K_2/H \times_{K_2} \psi_* Z) \\
&=: \mathbf{S}_{\psi_* Z}^{K_2}(K_2/H).
\end{aligned}$$

Here the first map comes from the adjunction isomorphism

$$\operatorname{map}_{K_2}(\psi_*(K_1/?), K_2/H) \xrightarrow{\cong} \operatorname{map}_{K_1}(K_1/?, \psi^*(K_2/H)),$$

and the third map comes from the canonical homeomorphism

$$\psi^*(K_2/H) \times_{K_1} Z \xrightarrow{\cong} K_2/H \times_{K_2} \psi_* Z.$$

The second map is the special case $T = \psi^* K_2/?$ of the natural weak homotopy equivalence defined for any K_1-set T

$$\kappa(T)\colon\ \operatorname{map}_{K_1}(K_1/?, T) \times_{\mathsf{Or}(K_1)} \mathbf{S}\,(K_1/? \times_{K_1} Z) \xrightarrow{\cong} \mathbf{S}(T \times_{K_1} Z)$$

that is given by $(u\colon K_1/? \to T) \times s \mapsto \mathbf{S}(u \times_{K_1} \mathrm{id}_Z)(s)$. If T is a homogeneous K_1-set, then $\kappa(T)$ is an isomorphism by the Yoneda Lemma. Since ψ is compatible with disjoint unions, $\mathbf{S}$ is compatible with disjoint unions up to weak homotopy equivalence by assumption, and every K_1-set is the disjoint union of homogeneous K_1-set, $\kappa(T)$ is a weak homotopy equivalence for all K_1-sets T.

(ii) The third isomorphisms appearing in [265, Lemma 1.9] implies that it suffices to construct a natural weak homotopy equivalence of $\mathsf{Or}(K_1)$-spectra

$$v(\psi, Z)\colon \psi^* \mathbf{S}_Z^{K_2} \xrightarrow{\cong} \mathbf{S}_{\psi^* Z}^{K_1}$$

where $\psi^* \mathbf{S}_Z^{K_2}$ is the $\mathsf{Or}(K_1)$-spectrum obtained by restriction in the sense of [265, Definition 1.8] with the functor $\mathsf{Or}(\psi)\colon \mathsf{Or}(K_1) \to \mathsf{Or}(K_2),\ K_1/H \mapsto \psi_*(K_1/H)$ applied to the $\mathsf{Or}(K_2)$-spectrum $\mathbf{S}_Z^{K_2}$. Actually, we obtain even an isomorphism $v(\psi, Z)$ using the adjunction

$$\psi_*(K_1/H) \times_{K_2} Z \cong K_1/H \times_{K_1} \psi^* Z$$

for any subgroup $H \subseteq K_1$. $\square$

Note that for a homomorphism $\phi\colon H \to G$ the restriction $\phi^* Z$ of a free G-CW-complex Z is free again if and only if ϕ is injective. We have already explained in Remark 15.44 that the assumption that Z is free is needed in Conjecture 15.39. In the Fibered Meta-Isomorphism Conjecture 15.8 it is crucial not to require that $\phi\colon H \to G$ is injective since we want to have good inheritance properties such as the one appearing in assertion (iii) of Lemma 15.23, which will be crucial for the proof of assertion (iv) of Theorem 15.45. Therefore we are forced to introduce the following construction.

Consider a group G and a G-CW-complex Z. We want to define an equivariant homology theory $H_*^?(-;\mathbf{S}_Z^{\downarrow G})$ over G in the sense of Definition 12.89. Given a group homomorphism $\phi\colon K \to G$, define the associated K-homology theory

$$H_*^{K,\phi}(-;\mathbf{S}_Z^{\downarrow G}) := H_*^K(-;\mathbf{S}_{EK \times \phi^* Z}^K).$$

Given group homomorphisms $\psi\colon K_1 \to K_2$, $\phi_1\colon K_1 \to G$, and $\phi_2\colon K_2 \to G$ with $\phi_2 \circ \psi = \phi_1$, a K_1-CW-complex X, and $n \in \mathbb{Z}$, we have to define a natural map

$$H_n^{K_1}(X;\mathbf{S}_{EK_1 \times \phi_1^* Z}^{K_1}) \to H_n^{K_2}(\psi_* X;\mathbf{S}_{EK_2 \times \phi_2^* Z}^{K_2}).$$

We get the isomorphism $H_n^{K_2}(\psi_* X;\mathbf{S}_{EK_2 \times \phi_2^* Z}^{K_2}) = H_n^{K_1}(X;\mathbf{S}_{\psi^*(EK_2 \times \phi_2^* Z)}^{K_2})$ from Lemma 15.49 (ii). Hence it suffices to specify a K_1-map

$$EK_1 \times \phi_1^* Z \to \psi^*(EK_2 \times \phi_2^* Z) = \psi^*(EK_2) \times \phi_1^* Z.$$

The homomorphism $\psi\colon K_1 \to K_2$ induces a K_1-map $EK_1 \to \psi^*(EK_2)$ and we can take its product with $\mathrm{id}_{\phi_1^* Z}$.

The proof of the next Lemma is left to the reader.

Lemma 15.50. *Given a group G and a G-CW-complex Z, all the axioms of an equivariant homology theory over G, see Definition 12.89, are satisfied by $H_*^?(-;\mathbf{S}_Z^{\downarrow G})$.*

Exercise 15.51. Let G be a group and Z be a G-CW-complex. Consider the functor

$$\mathbf{E}\colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA}, \quad p\colon \mathcal{G} \to I(G) \;\mapsto\; \mathbf{S}\big(E\mathcal{G} \times_{\mathcal{G}} p^* Z\big).$$

Here $E\mathcal{G}$ is the classifying $\mathcal{G}$-CW-complex associated to $\mathcal{G}$, see [265, Definition 3.8] for which we use the functorial model $E^{\mathrm{bar}}\mathcal{G}$ of [265, page 230], we consider Z as a $I(G)$-CW-complex and hence get a $\mathcal{G}$-CW-complex $p^* Z$ by restriction with $p\colon \mathcal{G} \to I(G)$, and the space $E\mathcal{G} \times_{\mathcal{G}} p^* Z$ is defined in [265, Definition 1.4].

Show that the equivariant homology theory $H_*^?(-;\mathbf{E})$ over G associated to $\mathbf{E}$ in Theorem 12.91 is isomorphic to $H_*^?(-;\mathbf{S}_Z^{\downarrow G})$.

Lemma 15.52. *Let $\phi\colon H \to K$ and $\psi\colon K \to G$ be a group homomorphisms.*

(i) Let X be a G-CW-complex and let Z be a K-CW-complex. Then we obtain a natural isomorphism

$$H_n^{H,\phi}(\phi^*\psi^*X;\mathbf{S}_Z^{\downarrow K}) \xrightarrow{\cong} H_n^G(X;\mathbf{S}^G_{(\psi\circ\phi)_*(EH\times\phi^*Z)});$$

(ii) Let X be a H-CW-complex and let Z be a G-CW-complex. Then we obtain a natural isomorphism

$$H_n^{H,\phi}(X;\mathbf{S}_{\psi^*Z}^{\downarrow K}) \xrightarrow{\cong} H_n^{H,\psi\circ\phi}\bigl(X;\mathbf{S}_Z^{\downarrow G}\bigr).$$

Proof. (i) We have by definition

$$H_n^{H,\phi}(\phi^*\psi^*X;\mathbf{S}_Z^{\downarrow K}) := H_n^H(\phi^*\psi^*X;\mathbf{S}^H_{EH\times\phi^*Z}).$$

Now apply Lemma 15.49 (i).

(ii) We get by definition

$$\begin{aligned} H_n^{H,\phi}(X;\mathbf{S}_{\psi^*Z}^{\downarrow K}) &:= H_n^H(X;\mathbf{S}^H_{EH\times\phi^*\psi^*Z}) \\ &= H_n^H(X;\mathbf{S}^H_{EH\times(\psi\circ\phi)^*Z}) =: H_n^{H,\psi\circ\phi}\bigl(X;\mathbf{S}_Z^{\downarrow G}\bigr). \end{aligned}$$

□

Conjecture 15.53 (Fibered Meta-Isomorphism Conjecture for a functor from spaces to spectra with coefficients). We say that $\mathbf{S}$ satisfies the Fibered Meta-Isomorphism Conjecture for a functor from spaces to spectra with coefficients for the group G and the family of subgroups $\mathcal{F}$ of G if for any G-CW-complex Z the equivariant homology theory $H_*^?(-;\mathbf{S}_Z^{\downarrow G})$ over G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the group (G,id_G) over G and the family $\mathcal{F}$.

Note that Conjecture 15.39 is a statement about $H_*^G(-\mathbf{S}_Z^G)$, whereas Conjecture 15.53 is a statement about $H_*^G(-;\mathbf{S}_Z^{\downarrow G})$.

Lemma 15.54. *Let $\psi\colon K \to G$ be a group homomorphism.*

(i) Suppose that the Meta Conjecture 15.39 with coefficients holds for the group G and the family $\mathcal{F}$. Then the Fibered Meta Conjecture 15.53 with coefficients holds for the group K and the family $\psi^*\mathcal{F}$;

(ii) If the Fibered Meta Conjecture 15.53 with coefficients holds for the group G and the family $\mathcal{F}$, then the Meta Conjecture 15.39 with coefficients holds for the group G and the family $\mathcal{F}$;

(iii) Suppose that the Fibered Meta Conjecture 15.53 with coefficients holds for K and the family $\mathcal{F}$. Then for every G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_n(-;\mathbf{S}_Z^{\downarrow G})$ over G for the group (K,ψ) over G and the family $\mathcal{F}$ of subgroups of K.

Proof. (i) This follows from Lemma 15.52 (i) since in the notation used there we have $\phi^*\psi^*E_{\mathcal{F}}(G) = \phi^*E_{\psi^*\mathcal{F}}(K)$ and $\phi^*\psi^*G/G = H/H$, and $(\psi \circ \phi)_*(EH \times \phi^*Z)$ is a free G-CW-complex.

(ii) This follows from the fact that for a free G-CW-complex Z the projection $EG \times Z \to Z$ is a G-homotopy equivalence and hence we get a natural isomorphism

$$H_n^{G,\mathrm{id}_G}(X;\mathbf{S}_Z^{\downarrow G}) := H_n^G(X;\mathbf{S}^G_{EG\times Z}) \xrightarrow{\cong} H_n^G(X;\mathbf{S}_Z^G)$$

for every G-CW-complex X and $n \in \mathbb{Z}$.

(iii) This follows from Lemma 15.52 (ii). □

Lemma 15.55. *Suppose that, for any directed systems of spaces $\{X_i \mid i \in I\}$ indexed over an arbitrary directed set I, the canonical map*

$$\mathrm{hocolim}_{i\in I}\,\mathbf{S}(X_i) \to \mathbf{S}\big(\mathrm{hocolim}_{i\in I}\,X_i\big)$$

is a weak homotopy equivalence.

Then for every group G and G-CW-complex Z the equivariant homology theory over G given by $H_*^?(\_\mathbf{S}_Z^{\downarrow G})$ is strongly continuous.

Proof. We only treat the case $\mathrm{id}_G\colon G \to G$, the case of a group $\psi\colon K \to G$ over G is completely analogous. Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \mathrm{colim}_{i\in I}\,G_i$. Let $\psi_i\colon G_i \to G$ be the structure map for $i \in I$.

The canonical map

$$\mathrm{hocolim}_{i\in I}\,\mathbf{S}(EG_i \times_{G_i} \psi_i^*Z) \to \mathbf{S}\big(\mathrm{hocolim}_{i\in I}(EG_i \times_{G_i} \psi_i^*Z)\big) \tag{15.56}$$

is by assumption a weak homotopy equivalence. We have the homeomorphisms

$$EG_i \times_{G_i} \psi_i^*Z \xrightarrow{\cong} (\psi_i)_*EG_i \times_G Z;$$
$$\big(\mathrm{hocolim}_{i\in I}(\psi_i)_*EG_i\big) \times_G Z \xrightarrow{\cong} \mathrm{hocolim}_{i\in I}\big((\psi_i)_*EG_i \times_G Z\big).$$

They induce a homeomorphism

$$\mathbf{S}(\mathrm{hocolim}_{i\in I}(EG_i \times_{G_i} \psi_i^*Z)\big) \xrightarrow{\cong} \mathbf{S}\big((\mathrm{hocolim}_{i\in I}(\psi_i)_*EG_i) \times_G Z\big) \tag{15.57}$$

The canonical map

$$\mathrm{hocolim}_{i\in I}(\psi_i)_*EG_i \to EG$$

is a G-homotopy equivalence. The proof of this fact is a special case of the argument appearing in the proof of [673, Theorem 4.3 on page 516]. It induces a weak homotopy equivalence

$$\mathbf{S}\big((\operatorname{hocolim}_{i\in I}(\psi_i)_*EG_i)\times_G Z\big)\to\mathbf{S}(EG\times_G Z). \tag{15.58}$$

Hence we get by taking the composite of the maps (15.56), (15.57) and (15.58) a weak homotopy equivalence

$$\operatorname{hocolim}_{i\in I}\mathbf{S}(EG_i\times_{G_i}\psi_i^*Z)\to\mathbf{S}(EG\times_G Z).$$

It induces after taking homotopy groups for every $n\in\mathbb{Z}$ an isomorphism

$$\operatorname{colim}_{i\in I}\pi_n\big(\mathbf{S}(EG_i\times_{G_i}\psi_i^*Z)\big)\to\pi_n\big(\mathbf{S}(EG\times_G Z)\big)$$

which is by definition the same as the canoncial map

$$\operatorname{colim}_{i\in I}H_n^{G_i,\psi_i}(G_i/G_i;\mathbf{S}_Z^{\downarrow G})\to H_n^{G,\mathrm{id}_G}(G/G;\mathbf{S}_Z^{\downarrow G}).$$

This finishes the proof of Lemma 15.55. □

Proof of Theorem 15.45. (i) Consider a free H-CW-complex Z. Let $i\colon H\to G$ be the inclusion. Then i_*Z is a free G-CW-complex, $i^*E_{\mathcal{C}(G)}(G)$ is a model for $E_{\mathcal{C}(H)}(H)$,and $i^*G/G=K/K$. From Lemma 15.49 (i), we obtain a commutative diagram with isomorphisms as vertical maps

$$\begin{tikzcd}
H_n^H(E_{\mathcal{C}(H)}(H);\mathbf{S}_Z^H) \arrow[r] \arrow[d, "\cong"'] & H_n^H(H/H;\mathbf{S}_Z^G) \arrow[d, "\cong"] \\
H_n^G(E_{\mathcal{C}(G)}(G);\mathbf{S}_{i_*Z}^G) \arrow[r] & H_n^G(G/G;\mathbf{S}_{i_*Z}^G)
\end{tikzcd}$$

where the horizontal maps are induced by the projections. The lower map is bijective by assumption. Hence the upper map is bijective as well.

(ii) Since $(Q,\mathcal{C}(Q))$ and $(p^{-1}(H),\mathcal{C}(p^{-1}(H)))$ for every $H\in\mathcal{C}(Q)$ satisfy the Meta-Isomorphism Conjecture Conjecture 15.39 with coefficients by assumption, we conclude from Lemma 15.54 (i) that the Fibered Meta-Isomorphism Conjecture 15.53 with coefficients holds for the group G and the family $p^*\mathcal{C}(Q)$ and that for every $H\in\mathcal{C}(Q)$ the Fibered Meta-Isomorphism Conjecture 15.53 with coefficients holds for $p^{-1}(H)$ and the family $\mathcal{C}(p^{-1}(H))=\mathcal{C}(G)|_{p^{-1}(H)}$. Lemma 15.54 (iii) implies that for every $H\in\mathcal{C}(Q)$ and G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_n^?(-;\mathbf{S}_Z^{\downarrow G})$ over G for the group $(p^{-1}(H)\subseteq G)$ over G and the family $\mathcal{C}(G)|_{p^{-1}(H)}$. Since for every $L\in p^*\mathcal{C}(Q)$ we have $p(L)\in\mathcal{C}(Q)$ and hence $L\subseteq p^{-1}(p(L))$, we conclude from Lemma 15.16 that for every $L\in p^*\mathcal{C}(Q)$ and G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_n^?(-;\mathbf{S}_Z^{\downarrow G})$ over

G for the group $(L \subseteq G)$ over G and the family $\mathcal{C}(G)|_L$. The Transitivity Principle 15.13 (i) implies that for every G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_n^?(-;\mathbf{S}_Z^{\downarrow G})$ over G for the group (G, id_G) over G and the family $\mathcal{C}(G)$, in other words, the Fibered Meta-Isomorphism Conjecture 15.53 with coefficients holds for G and the family $\mathcal{C}(G)$. We conclude from Lemma 15.54 (ii) that the Meta-Isomorphism Conjecture 15.39 holds for the group G and the family $\mathcal{C}(G)$.

(iii) If the Meta-Isomorphism Conjecture 15.39 with coefficients holds for $(G_1 \times G_1, \mathcal{C}(G_1 \times G_2))$, it holds for G_k and the family $\mathcal{C}(G_k) = \mathcal{C}(G_1 \times G_2)|_{G_k}$ for $k = 1, 2$ by assertion (i).

Suppose that the Meta-Isomorphism Conjecture 15.39 with coefficients holds for $(G_k, \mathcal{C}(G_k))$ for $k = 1, 2$. By assertion (ii) applied to the obvious exact sequence $1 \to H_2 \to G_1 \times H_2 \to G_1 \to 1$, Conjecture 15.39 holds for $(G_1 \times H_2, \mathcal{C}(G_1 \times H_2))$ for every $H_2 \in \mathcal{C}(G_2)$. By assertion (ii) applied to the obvious exact sequence $1 \to G_1 \to G_1 \times G_2 \to G_2 \to 1$ Conjecture 15.39 with coefficients holds for $(G_1 \times G_2, \mathcal{C}(G_1 \times G_2))$.

(iv) Since the Meta-Isomorphisms Conjecture Conjecture 15.39 holds for G_i and $\mathcal{C}(G_i)$ for every $i \in I$ by assumption, we conclude from Lemma 15.54 (i) that the Fibered Meta-Isomorphism Conjecture 15.53 with coefficients holds for the group G_i and the family $\mathcal{C}(G_i)$ for every $i \in I$. Lemma 15.54 (iii) implies that for every $i \in I$ and G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_n(-;\mathbf{S}_Z^{\downarrow G})$ over G for the group $\psi_i \colon G_i \to G$ over G and the family $\mathcal{C}(G_i)$. We conclude from Lemma 15.23 (iii) and Lemma 15.55 that for every G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory $H_*^?(-;\mathbf{S}_Z^{\downarrow G})$ over G for the group (G, id_G) over G and the family $\mathcal{C}(G)$, in other words, the Fibered Meta-Isomorphism Conjecture 15.53 with coefficients holds for the group G and the family $\mathcal{C}(G)$. We conclude from Lemma 15.54 (ii) that the Meta-Isomorphism Conjecture Conjecture 15.39 with coefficients holds for the group G and the family $\mathcal{C}(G)$. This finishes the proof of Theorem 15.45. $\square$

15.10 The Farrell-Jones Conjecture for A-Theory, Pseudoisotopy, and Whitehead Spaces

Conjecture 15.59 (Farrell-Jones Conjecture for A-theory (with coefficients)). A group G satisfies the *Farrell-Jones Conjecture for A-theory* if the Meta-Isomorphism Conjecture 15.36 for functors from spaces to spectra applied to the case $\mathbf{S} = \mathbf{A}$ for the functor non-connective A-theory $\mathbf{A}$ introduced in (7.12) holds for $(G, \mathcal{VCY})$.

A group G satisfies the *Farrell-Jones Conjecture for A-theory with coefficients* if the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients applied to the case $\mathbf{S} = \mathbf{A}$ for the functor non-connective A-theory $\mathbf{A}$ introduced in (7.12) holds for $(G, \mathcal{VCY})$.

Note that $\mathbf{A}$ respects weak equivalences and disjoint unions, see Theorem 7.16.

Exercise 15.60. Suppose that G is torsionfree and satisfies the Farrell-Jones Conjecture 15.59 for A-theory. Show that $\pi_n(\mathbf{A}(BG)) = 0$ for $n \leq -1$ and $\pi_0(\mathbf{A}(BG)) \cong \mathbb{Z}$.

Conjecture 15.61 (Farrell-Jones Conjecture for (smooth) pseudoisotopy (with coefficients)). A group G satisfies the *Farrell-Jones Conjecture for (smooth) pseudoisotopy* if the Meta-Isomorphism Conjecture 15.36 for functors from spaces to spectra applied to the case $\mathbf{S} = \mathbf{P}$ or $\mathbf{S} = \mathbf{P}^{\mathsf{DIFF}}$ for the functor non-connective (smooth) pseudoisotopy $\mathbf{P}$ and $\mathbf{P}^{\mathsf{DIFF}}$ of Definition 7.1 holds for $(G, \mathcal{VCY})$.

A group G satisfies the *Farrell-Jones Conjecture for (smooth) pseudoisotopy with coefficients* if the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients applied to the case $\mathbf{S} = \mathbf{P}$ or $\mathbf{S} = \mathbf{P}^{\mathsf{DIFF}}(X)$ for the functor non-connective (smooth) pseudoisotopy $\mathbf{P}$ and $\mathbf{P}^{\mathsf{DIFF}}$ of Definition 7.1 holds for $(G, \mathcal{VCY})$.

Conjecture 15.62 (Farrell-Jones Conjecture for (smooth) Whitehead spectra (with coefficients)). A group G satisfies the *Farrell-Jones Conjecture for (smooth) Whitehead spectra* if the Meta-Isomorphism Conjecture 15.36 for functors from spaces to spectra applied to the case $\mathbf{S} = \mathbf{Wh}$ or $\mathbf{S} = \mathbf{Wh}^{\mathsf{DIFF}}$ for the functor non-connective (smooth) Whitehead spectra $\mathbf{Wh}$ and $\mathbf{Wh}^{\mathsf{DIFF}}$ of Remark 7.34 holds for $(G, \mathcal{VCY})$.

A group G satisfies the *Farrell-Jones Conjecture for (smooth) Whitehead spectra with coefficients* if the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients applied to the case $\mathbf{S} = \mathbf{Wh}$ or $\mathbf{S} = \mathbf{Wh}^{\mathsf{DIFF}}$ for the functor non-connective (smooth) Whitehead spectra $\mathbf{Wh}$ and $\mathbf{Wh}^{\mathsf{DIFF}}$ of Remark 7.34 holds for $(G, \mathcal{VCY})$.

Note that $\mathbf{P}$ and $\mathbf{P}^{\mathsf{DIFF}}$ respect weak equivalences and disjoint unions, see Theorem 7.3.

Theorem 15.63.

(i) *The following assertions are equivalent for a group G:*

  (a) *The Farrell-Jones Conjecture 15.59 for A-theory holds for G;*
  (b) *The Farrell-Jones Conjecture 15.61 for pseudoisotopy holds for G;*
  (c) *The Farrell-Jones Conjecture 15.61 for smooth pseudoisotopy holds for G;*
  (d) *The Farrell-Jones Conjecture 15.62 for Whitehead spectra holds for G;*

(e) *The Farrell-Jones Conjecture 15.62 for smooth Whitehead spectra holds for G;*

(ii) *Assertion* (i) *holds also for the versions of the conjectures with coefficients;*

(iii) *Suppose that the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories, see Conjecture 13.20 holds for G. Then the versions with coefficients of the Conjectures 15.59, 15.61 and 15.62*

Proof. Assertions (i) and (ii) are proved in [330, Lemma 3.3].

(iii) is proved in [169, Example 1.9 and Corollary 7.71] **Comment 21 (by W.)**: This reference seems correspond to the version v1. What is the correct reference in v3? Maybe Example 1.1.11 and Corollary 7.5.6 **Comment 22 (by W.)**: The proof of this assertion may also be discussed in Chapter 8 and we have to add an appropriate reference. It has also to be synchronized with Remark 13.21. □

15.11 The Farrell-Jones Conjecture for Topological Hochschild and Cyclic Homology

There are the notions of Hochschild homology and cyclic homology of algebras, which are defined in the algebraic setting, see for instance Connes [236] or Loday [621]. One of the important insights of Waldhausen was that one can define an analogue of algebraic K-theory for rings where one "spacifies" the constructions. This led to A-theory which we have described in Chapter 7. These circle of ideas motivated also the definition of topological Hochschild homology by Bökstedt and then of topological cyclic homology by Bökstedt-Hsiang-Madsen [141], which are better approximations of the algebraic K-theory than their original algebraic counterparts. A systematic study how much algebraic cyclic homology detects from algebraic K-theory of group rings is presented in [658] showing that the topological versions are much more effective. Roughly speaking, in the topological versions one replaces rings by ring spectra and tensor products by (highly structured and strictly commutative) smash products. The role of the ring $\mathbb{Z}$ of integers, which is initial in the category of rings, is now played by the sphere spectrum $\mathbb{S}$, which is initial in the category of ring spectra. We refer for further information to the book by Dundas-Goodwillie-McCarthy [302] and the survey article by Madsen [681].

Given a symmetric ring spectrum $\mathbb{A}$ and a prime p, one can define functors see [659, (14.1) and Example 14.3]

$$\mathbf{THH}_{\mathbb{A}} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{15.64}$$

$$\mathbf{TC}_{\mathbb{A};p} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}, \tag{15.65}$$

such that for a group G considered as groupoid $I(G)$ the value of these functors is the topological Hochschild homology and the topological cyclic homology with respect to the prime p of the group ring spectrum $\mathbb{A}[G] := \mathbb{A} \wedge G_+$. From Theorem 12.30 we obtain equivariant homology theories $\mathcal{H}_*^?(-;\mathbf{THH}_{\mathbb{A}})$ and $\mathcal{H}_*^?(-;\mathbf{TC}_{\mathbb{A};p})$ satisfying for any group G and subgroup $H \subseteq G$

$$\begin{aligned}\mathcal{H}_n^G(G/H;\mathbf{THH}_{\mathbb{A}}) &= \mathcal{H}_n^H(H/H;\mathbf{THH}_{\mathbb{A}}) = \pi_n\big(\mathbf{THH}(\mathbb{A}[H])\big);\\ \mathcal{H}_n^G(G/H;\mathbf{TC}_{\mathbb{A};p}) &= \mathcal{H}_n^H(H/H;\mathbf{TC}_{\mathbb{A};p}) = \pi_n\big(\mathbf{TC}(\mathbb{A}[H];p)\big).\end{aligned}$$

15.11.1 Topological Hochschild Homology

The following theorem is taken from [659, Theorem 1.19]. The notion of a very well pointed spectrum and of a connective$^+$-spectrum are introduced in [659, Subsection 4J]. These are mild condition that are satisfied by the sphere spectrum $\mathbb{S}$ and the Eilenberg-MacLane spectrum of a discrete ring.

Theorem 15.66 (The Farrell-Jones Conjecture holds for topological Hochschild homology). *Let G be a group and $\mathcal{F}$ be a family of subgroups. Let $\mathbb{A}$ be a very well pointed symmetric ring spectrum. Then the map induced by the projection* $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{THH}_{\mathbb{A}}) \to H_n^G(G/G;\mathbf{THH}_{\mathbb{A}}) = \pi_n\big(\mathbf{THH}(\mathbb{A}[G])\big)$$

is split injective for all $n \in \mathbb{Z}$. If $\mathcal{F}$ contains all cyclic subgroups, then it is bijective for all $n \in \mathbb{Z}$.

Topological Hochschild homology is one of the rare instances where an Isomorphism Conjecture is known for all groups and an interesting family of subgroups, namely the family of all cyclic subgroups, and the reasons are not completely elementary.

15.11.2 Topological Cyclic Homology

For the rest of this subsection we assume that $\mathbf{A}$ is connective$^+$.

The assembly map for topological cyclic homology

$$H_n^G(E_{\mathcal{VCY}}(G);\mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G;\mathbf{TC}_{\mathbb{A}.p}) = \pi_n\big(\mathbf{TC}_p(\mathbb{A}[G])\big)$$

for the family $\mathcal{VCY}$ of virtually cyclic subgroups is not bijective in general. For instance, it is not surjective for $n = -1$ if $\mathbb{A} = \mathbb{Z}_{(p)}$ and G is either finitely generated free abelian or torsionfree hyperbolic, but not cyclic, see [660, Theorem 1.5]. More counterexamples against surjectivity are presented in [660, Remark 6.7]. Counterexamples against rational injectivity are described in [660, Remark 1.9] based on [659, Remark 3.7].

There are also some positive results.

Theorem 15.67 (Bijectivity of the assembly map for topological cyclic homology for finite groups and the family of cyclic subgroups). *If G is finite, then the assembly map for the family of cyclic subgroups*

$$H_n^G(E_{\mathcal{CYC}}(G);\mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G;\mathbf{TC}_{\mathbb{A}.p}) = \pi_n l(\mathbf{TC}_p(\mathbb{A}[G]))$$

is bijective for all $n \in \mathbb{Z}$.

Proof. See [660, Theorem 1.1]. □

Exercise 15.68. Let S_3 be the symmetric group on the set $\{1,2,3\}$. Let C_2 and C_3 be any cyclic subgroups of S_3 of order 2 and 3.

Show that for any prime p there is a weak equivalence

$$\mathbf{TC}(\mathbb{A}[C_2];p) \vee \big((EC_2)_+ \wedge_{C_2} \widetilde{\mathbf{TC}}(\mathbb{A}[C_3];p)\big) \xrightarrow{\simeq} \mathbf{TC}(\mathbb{A}[S_3];p)$$

where C_2 acts on C_3 by sending the generator to its inverse, and $\widetilde{\mathbf{TC}}(\mathbb{A}[G];p)$ is the homotopy cofiber of the map $\mathbf{TC}(\mathbb{A};p) \to \mathbf{TC}(\mathbb{A}[G];p)$ induced by the inclusion.

Theorem 15.69. *Let G be a group and p be a prime.*

(i) *Assume that there is a G-CW-model for $E_{\mathcal{FIN}}(G)$ of finite type. Then the map induced by the projection* $\mathrm{pr}\colon E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G;\mathbf{TC}_{\mathbb{A};p}) = \pi_n\big(\mathbf{TC}(\mathbb{A}[G];p)\big)$$

is split injective for all $n \in \mathbb{Z}$;

(ii) *Assume that G is hyperbolic or virtually finitely generated abelian. Then the map induced by the projection* $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to G/G$

$$H_n^G(E_{\mathcal{VCY}}(G);\mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G;\mathbf{TC}_{\mathbb{A};p}) = \pi_n\big(\mathbf{TC}(\mathbb{A}[G];p)\big)$$

is injective for all $n \in \mathbb{Z}$;

Proof. See [660, Theorem 1.4], □

A more general result about rational injectivity of the assembly map for topological cyclic homology can be found in [660, Theorem 1.8].

One of the reasons why topological cyclic homology is much harder than topological Hochschild homology is that in the construction of topological cyclic homology a homotopy inverse limits occurs and taking smash product does not commute with homotopy inverse limits in general, see [661]. This is the main reason for the existence of the counterexamples above.

Remark 15.70 (Pro-systems). If one does not pass to the assembly maps but argues on the level of pro-systems, then there is a kind of assembly map for pro-systems for any group G and the family $\mathcal{CYC}$ of cyclic subgroups which is indeed a pro-isomorphism, see [660, Theorem 1.3]. In other words, a pro-system version of the Farrell-Jones Conjecture for topologically cyclic homology holds for any group G and any connective$^+$ spectrum $\mathbb{A}$ for the family $\mathcal{CYC}$ of cyclic subgroups.

More information about topological cyclic homology and its applications to algebraic K-theory via the cyclotomic trace can be found for instance in [302, 462, 742].

15.12 The Farrell-Jones Conjecture for Homotopy K-Theory

Let $\mathbf{E}\colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$ be a (covariant) functor from the category ADDCAT of small additive categories. In [668, Definition 8.1] its *homotopy stabilization* is constructed that consists of a covariant functor

$$\mathbf{EH}\colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$$

together with a natural transformation

$$\mathbf{h}\colon \mathbf{E} \to \mathbf{EH}$$

We call $\mathbf{E}$ homotopy stable if $\mathbf{h}(\mathcal{A})$ is an equivalence for any object $\mathcal{A}$ in ADDCAT.

This construction has the following basic properties. Given an automorphism $\Phi\colon \mathcal{A} \to \mathcal{A}$, let $\mathcal{A}_\Phi[t]$ be the additive category of twisted polynomials with coefficients in $\mathcal{A}$, see [671, Definition 1.2]. Let $\mathrm{ev}_0^+\colon \mathcal{A}_\Phi[t] \to \mathcal{A}$ be the functor of additive categories given by taking $t = 0$ and let $\mathbf{i}^+\colon \mathcal{A} \to \mathcal{A}_\Phi[t]$ be the obvious inclusion see [671, (1.10) and (1.12)].

Lemma 15.71. *Let* $\mathbf{E}\colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$ *be a covariant functor.*

(i) $\mathbf{EH}$ *is homotopy stable;*
(ii) Suppose that $\mathbf{E}$ *is homotopy stable. Let* $\mathcal{A}$ *be any additive category with an automorphism* $\Phi\colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$*. Then the maps*

$$\mathbf{E}(\mathrm{ev}_0^+)\colon \mathbf{E}(\mathcal{A}_\Phi[t]) \xrightarrow{\simeq} \mathbf{E}(\mathcal{A});$$
$$\mathbf{E}(\mathbf{i}^+)\colon \mathbf{E}(\mathcal{A}) \xrightarrow{\simeq} \mathbf{E}(\mathcal{A}_\Phi[t]),$$

are weak homotopy equivalences;

(iii) The functor $\mathbf{E}$ *is homotopy stable if and only if for every additive category* $\mathcal{A}$ *the inclusion* $\mathcal{A} \to \mathcal{A}[t]$ *induces a weak homotopy equivalence* $\mathbf{E}(\mathcal{A}) \to \mathbf{E}(\mathcal{A}[t])$.

Proof. (i) and (ii) See [668, Lemma 8.2].

(iii) The only if statement follows from assertion (ii). The if-statement is a direct consequence of the definition of $\mathbf{EH}$, see [668, Definition 8.1]. □

Lemma 15.71 (ii) essentially says that homotopy stable automatically implies homotopy stable in the twisted sense.

Remark 15.72 (Universal property of EH). Note that Lemma 15.71 (i) says that up to weak homotopy equivalence the transformation $\mathbf{h} \colon \mathbf{E} \to \mathbf{EH}$ is universal (from the left) among transformations $\mathbf{f} \colon \mathbf{E} \to \mathbf{F}$ to homotopy stable functors $\mathbf{F} \colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$ since we obtain a commutative square whose lower vertical arrow is a weak homotopy equivalence

$$\begin{tikzcd} \mathbf{E} \arrow[r, "\mathbf{h}"] \arrow[d, "\mathbf{f}"] & \mathbf{EH} \arrow[d, "\mathbf{Hf}"] \\ \mathbf{F} \arrow[r, "\simeq", "\mathbf{h}"'] & \mathbf{FH} \end{tikzcd}$$

Definition 15.73 (Homotopy K-theory). Let $\mathbf{K} \colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$ be the covariant functor that sends an additive category to its non-connective K-theory spectrum, see for instance [197, 668, 783]. Define the *homotopy K-theory functor*

$$\mathbf{KH} \colon \mathsf{ADDCAT} \to \mathsf{SPECTRA}$$

to be the homotopy stabilization of $\mathbf{K}$.

The next result is taken from [668, Lemma 8.6].

Theorem 15.74 (Bass-Heller-Swan decomposition for homotopy K-theory). *Let $\mathcal{A}$ be an additive category with an automorphism $\Phi \colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$. Then we get for all $n \in \mathbb{Z}$ a weak homotopy equivalence*

$$\mathbf{a} \colon \mathbf{T}_{\mathbf{K}(\Phi^{-1})} \xrightarrow{\simeq} \mathbf{KH}(\mathcal{A}_{\Phi}[t, t^{-1}])$$

where $\mathbf{T}_{\mathbf{KH}(\Phi^{-1})}$ is the mapping torus of the selfmap $\mathbf{KH}(\Phi^{-1}) \colon \mathbf{KH}(\mathcal{A}) \to \mathbf{KH}(\mathcal{A})$.

Remark 15.75 (Identification with Weibel's definition). Weibel has defined a version of homotopy K-theory for a ring R by a simplicial construction in [977]. It is not hard to check using Remark 15.72, which applies also to the constructions of [977] instead of $\mathbf{H}$, that $\pi_i(\mathbf{KH}(\mathcal{R}))$ can be identified with the one in [977] if $\mathcal{R}$ is a skeleton of the category of finitely generated free R-modules.

Conjecture 15.76 (Farrell-Jones Conjecture for homotopy K-theory with coefficients in additive G-categories). We say that G satisfies the *Farrell-Jones Conjecture with coefficients for homotopy K-theory in additive G-categories* if for every additive G-category $\mathcal{A}$ and every $n \in \mathbb{Z}$ the assembly map given by the projection $\mathrm{pr}\colon E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{KH}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{KH}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{KH}_{\mathcal{A}}(I(G))\bigr)$$

is bijective where $\mathbf{KH}_{\mathcal{A}}\colon \mathsf{GROUPOIDS} \downarrow G \to \mathsf{SPECTRA}$ is analogously defined as the functor appearing in (13.10) but with $\mathbf{K}$ replaced by $\mathbf{KH}$.

The version of Conjecture 15.76 has been treated for rings in [74].

Conjecture 15.77 (Farrell-Jones Conjecture for homotopy K-theory with coefficients in additive G-categories with finite wreath products). We say that G satisfies the *Farrell-Jones Conjecture with coefficients for homotopy K-theory in additive G-categories with finite wreath products* if for any finite group F the group $G \wr F$ satisfies the Farrell-Jones Conjecture with coefficients for homotopy K-theory in additive $G \wr F$-categories 15.76.

15.13 The Farrell-Jones Conjecture for Hecke Algebras

There is one instance where one can formulate the Farrell-Jones Conjecture for non-discrete groups, namely, for the algebraic K-theory of a Hecke algebra $\mathcal{H}(G)$ of a totally disconnected locally compact second countable Hausdorff group G.

Denote by $\mathcal{H}(G)$ the *Hecke algebra* of G that consists of locally constant functions $G \to \mathbb{C}$ with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra $\mathcal{H}(G)$ plays the same role for G as the complex group ring $\mathbb{C}G$ for a discrete group G and reduces to this notion if G happens to be discrete. There is a G-homology theory $\mathcal{H}_*^G$ with the property that for any open and closed subgroup $H \subseteq G$ and all $n \in \mathbb{Z}$ we have $\mathcal{H}_n^G(G/H) = K_n(\mathcal{H}(H))$ where $K_n(\mathcal{H}(H))$ is the algebraic K-group of the Hecke algebra $\mathcal{H}(H)$. There is also the notion of a classifying space $E_{\mathcal{KO}}(G)$ for the family of compact-open subgroups of G. Note that $\mathcal{KO}$ is not closed under passing to subgroups but at least under finite intersections, which suffices to our purposes. The space $E_{\mathcal{KO}}(G)$ is characterized by the property that for any G-CW-complex X whose isotropy groups are compact-open, there is up to G-homotopy precisely one G-map from X to $E_{\mathcal{KO}}(G)$. More information about this space and the comparison with the classifying space for numerable G-spaces $J_{\mathcal{KO}}(G)$ can be found in [640]. The following conjecture has appeared already in [657, Conjecture 119 on page 773].

Conjecture 15.78 (The Farrell-Jones Conjecture for the algebraic K-theory of Hecke-Algebras). For a totally disconnected locally compact second countable Hausdorff group G the assembly map

$$\mathcal{H}_n^G(E_{\mathcal{KO}}(G)) \to \mathcal{H}^G(\{\bullet\}) = K_n(\mathcal{H}(G)) \tag{15.79}$$

induced by the projection $\mathrm{pr}\colon E_{\mathcal{KO}}(T) \to \{\bullet\}$ is an isomorphism for all $n \in \mathbb{Z}$.

In the case $n = 0$ this reduces to the statement that

$$\operatorname{colim}_{G/H \in \mathsf{Or}_{\mathcal{KO}}(G)} K_0(\mathcal{H}(H)) \to K_0(\mathcal{H}(G)) \tag{15.80}$$

is an isomorphism. Some evidence for this comes for instance from [260], where the bijectivity of (15.80) has been proved rationally for a reductive p-adic group G. For $n \le -1$ one obtains the statement that $K_n(\mathcal{H}(G)) = 0$. The Hecke algebra $\mathcal{H}(G)$ and its projective class group $K_0(\mathcal{H}(G))$ are closely related to the theory of smooth representations of G, see for instance [122, 885, 886]. The G-homology theory can be constructed using an appropriate functor $\mathbf{K}_{\mathcal{H}}\colon \mathsf{Or}_{\mathcal{KO}}(G) \to \mathsf{SPECTRA}$ and the recipe explained in Theorem 12.27. The desired functor $\mathbf{K}_{\mathcal{H}}$ is constructed in [873].

All this is explained and carried out in the preprints [83, 81, 84] actually also for twisted Hecke algebras with respect to a central character and more general coefficient than $\mathbb{C}$. Moreover, the following result is proved in [81, Corollaries 1.8 and 1.18] and [84, Theorem 1.1].

Theorem 15.81. *Suppose that G is a modulo a compact subgroup isomorphic to a closed subgroup of a reductive p-adic group. Then Conjecture 15.78 is true, the map* (15.80) *is bijective, and $K_n(\mathcal{H}(G))$ vanishes for $n \le -1$.*

15.14 Relations among the Isomorphisms Conjectures

15.14.1 The Farrell-Jones Conjecture for *K*-Theory and for *A*-Theory

Let G be a group and let X be a G-CW-complex. We get from the linearization map of (7.17) a natural map

$$L_n^G(X)\colon H_n^G(X; \mathbf{A}^B) \to H_n^G(X; \mathbf{K}_{\mathbb{Z}}). \tag{15.82}$$

if we take Example 15.37 into account and $\mathbf{A}^B$ is defined by (15.35) for $\mathbf{S} = \mathbf{A}$ for the functor $\mathbf{A}$ of (7.12). We conclude from Theorem 7.18 and the equivariant Atiyah Hirzebruch spectral sequence, see Theorem 12.48, that $L_n^G(X)$ is bijective for $n \le 1$, surjective for $n = 2$ and rationally bijective for all $n \in \mathbb{Z}$. If we take $X = E_{\mathcal{VCY}}(G)$ and $X = G/G$ we obtain a commutative diagram where the horizontal maps are assembly maps and the vertical maps are given by the maps (15.82).

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{A}^B) \arrow[r] \arrow[d] & H_n^G(G/G;\mathbf{A}^B) = \pi_n(\mathbf{A}(BG)) \arrow[d] \\
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \arrow[r] & H_n^G(G/G;\mathbf{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}G).
\end{tikzcd}$$

We conclude that for $n \in \mathbb{Z}$ with $n \leq 1$ the upper arrow is bijective if and only if the lower arrow is bijective We also conclude for every for $n \in \mathbb{Z}$ and that the lower arrow is rationally bijective if and only if the lower arrow is rationally bijective for $n \in \mathbb{Z}$. This gives some interesting relations between the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$ and the Farrell-Jones Conjecture 15.59 for A-theory (without coefficients). For instance, they are equivalent in degrees $n \leq 1$, and they are rationally equivalent.

The case where we allow in the Farrell-Jones Conjecture 15.59 for A-theory coefficients is more complicated since in Theorem 7.18 (ii) the assumption occurs that the space under consideration has to be aspherical. Consider a free G-CW-complex Z that is simply connected (but not necessarily contractible). Then $\pi_1(G/H \times_G Z) \cong H$. We still get a commutative diagram

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{A}_Z^G) \arrow[r] \arrow[d] & H_n^G(G/G;\mathbf{A}_Z^G) = \pi_n(\mathbf{A}(G\backslash Z)) \arrow[d] \\
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \arrow[r] & H_n^G(G/G;\mathbf{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}G)
\end{tikzcd}$$

but we only know that the vertical arrows are bijective for $n \leq 1$ and surjective for $n = 2$, but *not* anymore that they are rationally bijective for all $n \in \mathbb{Z}$.

15.14.2 The Farrell-Jones Conjecture for *A*-Theory, Pseudoisotopy, and Whitehead Spaces

The Farrell Jones Conjecture 15.59 for A-theory (with coefficients) and the Farrell-Jones Conjecture 15.61 for (smooth) pseudoisotopy (with coefficients) andf the Farrell-Jones Conjecture 15.62 for Whitehead spaces are equivalent. This follows from the non-connective analogues of (7.24) and (7.26) for pseudoisotopy, and from the non-connective analogues of (7.30), and (7.31) for smooth pseudoisotopy, see [794], using Lemma 15.47 (ii) and Lemma 15.48 (ii).

15.14.3 The Farrell-Jones Conjecture for K-Theory and for Topological Cyclic Homology

The basic reason why topological cyclic homology is a powerful approximation of algebraic K-theory is the cyclotomic trace due to Bökstedt-Hsiang-Madsen [141]. It can extended to the equivariant setting and thus be used together with the linearization map (7.17) to construct the following commutative diagram, which is closely related to the main diagram in [659, (3.1)] for $n \geq 0$,

(15.83)

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}^{\geq 0}) \arrow[r] & H_n^G(G/G;\mathbf{K}_{\mathbb{Z}}^{\geq 0}) = K_n(\mathbb{Z}G) \\
H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}_{\mathbb{Z}}^{\geq 0}) \arrow[r] \arrow[u, "H_n(\iota_{\mathcal{FCY}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathbb{Z}}^{\geq 0})", "\cong_{\mathbb{Q}}"'] & H_n^G(G/G;\mathbf{K}_{\mathbb{Z}}^{\geq 0}) = K_n(\mathbb{Z}G) \arrow[u, "\cong_{\mathbb{Q}}", "\mathrm{id}"'] \\
H_n^G(E_{\mathcal{FCY}}(G);\mathbf{A}^{\geq 0}) \arrow[r] \arrow[u, "H_n(E_{\mathcal{FCY}}(G);\mathbf{L}^{\geq 0})", "\cong_{\mathbb{Q}}"'] \arrow[d, "H_n(E_{\mathcal{FCY}}(G);\mathbf{ct}^{\geq 0})"'] & H_n^G(G/G;\mathbf{A}^{\geq 0}) = A_n(BG) \arrow[u, "\cong_{\mathbb{Q}}", "L_n"'] \arrow[d, "\mathrm{ct}_n"] \\
H_n^G(E_{\mathcal{FCY}}(G);\mathbf{TC}_{\mathbb{S}}) \arrow[r] & H_n^G(G/G;\mathbf{TC}_{\mathbb{S},p}) = TC_n(BG,p)
\end{tikzcd}$$

where $\mathcal{FCY}$ is the family of finite cyclic subgroups of G, the superscript ≥ 0 indicates that we consider the 0-connective covers, the vertical arrows from the third row to the second row come from the linearization map, and the vertical arrows from the third row to the fourth row come from the cyclotomic trace. All arrows marked with $\cong_{\mathbb{Q}}$ are known to be rationally bijective. This follows from the maps induced by the linearization from Theorem 7.18. For the map $H_n(\iota_{\mathcal{FCY}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathbb{Z}}^{\geq 0})$, this follows from Theorem 13.48 and further computations based equivariant Chern characters using Theorem 12.79 and [659, Example 12.12]. Rationally the natural map $H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}^{\geq 0}) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}})$ is split injective and has a cokernel that is given by an expression involving the groups $K_{-1}(\mathbb{Z}C)$ for finite cyclic subgroups $C \subseteq G$. Thus the diagram (15.83) implies that the K-theoretic Farrell-Jones assembly map is rationally injective, ignoring certain contributions from the collection of the groups $K_{-1}(\mathbb{Z}C)$ for finite cyclic subgroups $C \subseteq G$, provided that the lowermost horizontal arrow is rationally injective. This is the basic idea in the proof of rational injectivity results for the K-theoretic Farrell-Jones assembly map presented in [659, Theorem 1.13], where the actual argument is more involved and uses the C-functors as well.

A rational computation of $K_n(\mathbb{Z}G)$ is given in Theorem 17.4, provided that if G satisfies the satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$. With the methods mentioned above, one can detect under certain conditions the source of the map appearing in Theorem 17.4 if one ignores the summand for $q = -1$

15.14.4 The *L*-Theoretic Farrell-Jones Conjecture and the Baum-Connes Conjecture

In the sequel [1/2] stands for inverting 2 at the level of spectra or abelian groups. Note that for a spectrum $\mathbf{E}$ we have a natural isomorphism $\pi_n(\mathbf{E})[1/2] \xrightarrow{\cong} \pi_n(\mathbf{E}[1/2])$.

One can construct the following commutative diagram

(15.84)
$$\begin{tikzcd}
H_n^G(\underline{E}G;\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle})[1/2] \arrow[r] \arrow[d, "l"', "\cong"] & L_n^{\langle -\infty\rangle}(\mathbb{Z}G)[1/2] \arrow[d, "\mathrm{id}"', "\cong"] \\
H_n^G(\underline{E}G;\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle}[1/2]) \arrow[r] & L_n^{\langle -\infty\rangle}(\mathbb{Z}G)[1/2] \\
H_n^G(\underline{E}G;\mathbf{L}_{\mathbb{Z}}^{p}[1/2]) \arrow[r] \arrow[u, "i_0", "\cong"'] \arrow[d, "i_1"', "\cong"] & L_n^{p}(\mathbb{Z}G)[1/2] \arrow[u, "\cong", "j_0"'] \arrow[d, "\cong"', "j_1"] \\
H_n^G(\underline{E}G;\mathbf{L}_{\mathbb{Q}}^{p}[1/2]) \arrow[r] \arrow[d, "i_2"', "\cong"] & L_n^{p}(\mathbb{Q}G)[1/2] \arrow[d, "j_2"] \\
H_n^G(\underline{E}G;\mathbf{L}_{\mathbb{R}}^{p}[1/2]) \arrow[r] \arrow[d, "i_3"', "\cong"] & L_n^{p}(\mathbb{R}G)[1/2] \arrow[d, "j_3"] \\
H_n^G(\underline{E}G;\mathbf{L}_{C_r^*(?,\mathbb{R})}^{p}[1/2]) \arrow[r] & L_n^{p}(C_r^*(G,\mathbb{R}))[1/2] \\
H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}[1/2]) \arrow[r] \arrow[u, "i_4", "\cong"'] & K_n^{\mathsf{TOP}}(C_r^*(G,\mathbb{R}))[1/2] \arrow[u, "\cong", "j_4"'] \\
H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}})[1/2] \arrow[u, "\cong", "l"'] \arrow[d, "i_5"'] & K_n^{\mathsf{TOP}}(C_r^*(G,\mathbb{R}))[1/2] \arrow[u, "\mathrm{id}", "\cong"'] \arrow[d, "j_5"] \\
H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{C}}^{\mathsf{TOP}})[1/2] \arrow[r] & K_n(C_r^*(G))[1/2]
\end{tikzcd}$$

where all horizontal maps are assembly maps and the vertical arrows are induced by transformations of functors $\mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$. These transformations are induced by change of rings maps except the one from $\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}[1/2]$ to $\mathbf{L}^p_{C_r^*(?,\mathbb{R})}[1/2]$, which is much more complicated and carried out in [599, 600]. Actually, it does not exists without inverting two on the spectrum level. Since it is a weak equivalence, the maps i_4 and j_4 are bijections.

On the level of homotopy groups the comparison between the algebra L-theory and the topological K-theory of a real and of a complex C^*-algebra have already been explained in Theorem 10.78, namely we obtain isomorphisms

(15.85) $\quad KO_n(A)[1/2] \xrightarrow{\cong} L_n^p(A)[1/2], \quad$ if A is a real C^*-algebra;

(15.86) $\quad K_n(A) \xrightarrow{\cong} L_n^p(A), \quad$ if A is a complex C^*-algebra.

Since for any finite group H each of the following maps is known to be a bijection because of [823, Proposition 22.34 on page 252] and $\mathbb{R}H = C_r^*(H,\mathbb{R})$

$$L_n^p(\mathbb{Z}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{Q}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{R}H)[1/2] \xrightarrow{\cong} L_n^p(C_r^*(H,\mathbb{R})),$$

we conclude from the equivariant Atiyah Hirzebruch spectral sequence, see Theorem 12.48, that the vertical arrows i_1, i_2, and i_3 are isomorphisms. The arrow j_1 is bijective by [821, page 376]. The maps l are isomorphisms for general results about localizations.

The lowermost vertical arrows i_5 and j_5 are known to be split injective, a splitting comes by restriction with the inclusions $C_r^*(G,\mathbb{R}) \to C_r^*(G,\mathbb{C})$. The following conjecture is already raised as question in [570, Remark 23.14 on page 197], see also [599, Conjecture 1 in Subsection 5.2].

Conjecture 15.87 (Passage for L-theory from $\mathbb{Q}G$ to $\mathbb{R}G$ to $C_r^*(G,\mathbb{R})$). The maps j_2 and j_3 appearing in diagram (15.84) are bijective.

One easily checks

Lemma 15.88. *Let G be a group.*

(i) Suppose that G satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring R for $R = \mathbb{Q}$ and $R = \mathbb{R}$ and the complex version of the Baum-Connes Conjecture 14.9. Then G satisfies Conjecture 15.87;

(ii) Suppose that G satisfies Conjecture 15.87. Then G satisfies the L-theoretic Farrell-Jones Conjecture 13.4 for the ring $\mathbb{Z}$ after inverting 2 if and only if G satisfies the real version of the Baum-Connes Conjecture 14.9 after inverting 2;

(iii) Suppose that the assembly map appearing in the complex version of the Baum-Connes Conjecture 14.9 is (split) injective after inverting 2. Then the assembly map appearing in L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring for $R = \mathbb{Z}$ is (split) injective after inverting 2.

Proof. This follows from Theorem 13.59 (i), Remark 14.13 and the diagram (15.84). □

15.14.5 Mapping Surgery to Analysis

Let X be a connected CW-complex with fundamental group π. Let $\widetilde{X} \to X$ be its universal covering. Denote by ϵ one of the decorations s, h or p. We have constructed functors $\mathbf{L}_{\mathbb{Z}}^{\epsilon}$ and $\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ in Theorem 12.43. We obtain maps of spectra

$$X_+ \wedge \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\{*\}) \xleftarrow{\simeq} \widetilde{X}_+ \wedge_{\pi} \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\mathcal{G}^{\pi}(\pi)) \longrightarrow \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\mathcal{G}^{\pi}(\pi/\pi));$$

$$X_+ \wedge \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}(\{*\}) \xleftarrow{\simeq} \widetilde{X} \wedge_{\pi} \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}(\mathcal{G}^{\pi}(\pi)) \longrightarrow \mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}(\mathcal{G}^{\pi}(\pi/\pi)).$$

Here $\{*\}$ denotes the trivial groupoid with one object, the horizontal arrows pointing to the left are defined in the obvious way and are weak homotopy equivalences since $\widetilde{X}$ is a free π-CW-complex with $\pi\backslash\widetilde{X} = X$ and $\mathcal{G}^{\pi}(\pi) \to \{*\}$ is an equivalence of groupoids, and the horizontal arrows to the right are assembly maps composed with maps induced by a fixed π-map $\widetilde{X} \to E\pi$. (If one wants to get rid of the dependency of a choice of π-map $\widetilde{X} \to E\pi$, one can consider $\Pi(\pi/H \times_{\pi} \widetilde{X})$ instead of $\mathcal{G}^{\pi}(\pi/H)$ for objects π/H in $\mathsf{Or}(\pi)$.)

Denote by $\mathbf{S}^{\epsilon}(X)$ and $\mathbf{D}(X)$ respectively the homotopy fiber of the arrow pointing to the right in the first and second row above.

After taking homotopy groups we obtain long exact sequences

$$(15.89)\quad \cdots \to H_{n+1}(X; \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\{*\})) \to L_{n+1}^{\epsilon}(\mathbb{Z}\pi) \to \pi_n(\mathbf{S}^{\epsilon}(X)) \\ \to H_n(X; \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\{*\})) \to L_n^{\epsilon}(\mathbb{Z}\pi) \to \cdots,$$

and

$$(15.90)\quad \cdots \to KO_{n+1}(X) \to KO_{n+1}(C_r^*(\pi, \mathbb{R})) \to \pi_n(\mathbf{D}(X)) \\ \to KO_n(X) \to KO_n(C_r^*(\pi, \mathbb{R})) \to \cdots.$$

After inverting 2 there is a zigzag of natural transformation from $\mathbf{K}_{\mathbb{R}}^{\mathsf{TOP}}[1/2]$ $\mathbf{L}_{\mathbb{Z}}^{\epsilon}[1/2]$ as explained in Subsection 15.14.4. It yields a map between long exact sequence

(15.91)
$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
KO_{n+1}(X)[1/2] \arrow[r] \arrow[d] & H_{n+1}(X;\mathbf{L}^{\epsilon}_{\mathbb{Z}}(\{*\}))[1/2] \arrow[d] \\
KO_{n+1}(C^*_r(\pi,\mathbb{R})) \arrow[r] \arrow[d] & L^{\epsilon}_{n+1}(\mathbb{Z}\pi)[1/2][1/2] \arrow[d] \\
\pi_n(\mathbf{D}(X))[1/2] \arrow[r] \arrow[d] & \pi_n(\mathbf{S}^{\epsilon}(X))[1/2] \arrow[d] \\
KO_n(X)[1/2] \arrow[r] \arrow[d] & H_n(X;\mathbf{L}^{\epsilon}_{\mathbb{Z}}(\{*\}))[1/2] \arrow[d] \\
KO_n(C^*_r(\pi,\mathbb{R}))[1/2] \arrow[r] \arrow[d] & L^{\epsilon}_n(\mathbb{Z}\pi)[1/2] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Lemma 15.92. *Suppose that π satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution $\mathbb{Z}$ and the Baum-Connes Conjecture 14.9 for the real group C^*-algebra.*

Then the map

$$\pi_n(\mathbf{D}(M))[1/2] \xrightarrow{\cong} \pi_n(\mathbf{S}^{\epsilon}(M))[1/2]$$

is bijective for $n \in \mathbb{Z}$.

Proof. The first and fourth horizontal arrow in the diagram 15.91 are bijective since there are given by transformation of homology theories and their evaluation at $\{\bullet\}$ is known to be bijective. The Rothenberg sequences of Subsection 9.10.4, Theorem 13.59 (i), and the diagram (15.84) together with the assumption that π satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution $\mathbb{Z}$ and the Baum-Connes Conjecture 14.9 for the real group C^*-algebra imply that the second and fifth horizontal arrow in the diagram 15.91 are bijective. Now apply the Five Lemma to the diagram (15.91). $\square$

Now consider the case $X = M$ for a closed orientable topological manifold M of dimension d. Then the part of the sequence (15.89) for $n \geq d$ can identified with the long exact surgery sequence in the topological category, see Theorem 9.130 see for instance [823, Theorem 18.5 on page 198] or [579].

Some extra care is necessary at the end in degree d since one has to pass to the 1-connective cover of the L-theory spectrum. In particular we get an identification of $\pi_d(\mathbf{S}^s(M))$ with the topological structure set $\mathcal{S}_d^{\mathrm{TOP},s}(M)$, see Subsection 9.12.1, which is the central object of study in the classification of topological manifolds. Note that in view of Lemma 15.92 one can hope for an identification of $\mathcal{S}_d^{\mathrm{TOP},s}(M)$ after inverting 2 with $\pi_d(\mathcal{D}(M))$, which is an object related to topological K-theory of spaces and C^*-algebras. An analytic Surgery Exact Sequence in terms of the topological K-theory of C^*-algebra associated to M is constructed in [477, Section 1]

Problem 15.93 (Identification of analytic Surgery Exact Sequences). Identify the real version of the analytic Surgery Exact Sequence appearing in [477, Section 1] with the exact sequence (15.90) for a closed orientable manifold of dimension d.

Note that Higson-Roe have to work with smooth manifolds since they want to apply index theory. So they have to consider the surgery sequence in the smooth category. They construct a diagram relating the Surgery Exact Sequence in the smooth category to their analytic Surgery Exact Sequence.

A more direct approach to the map comparing the surgery sequence in the smooth category to the analytic Surgery Exact Sequence is given in Piazza-Schick [792].

A comparison map starting with the Surgery Exact Sequence in the topological category is constructed in Zenobi [1014] using the approach of [792] and Lipschitz structures.

Recall that the Surgery Exact Sequence in the topological category is an exact sequence of abelian groups, what is not true for the smooth category. It is not clear whether the construction in Zenobi [1014] is compatible with the structures of an abelian groups on the topological and analytic structure sets.

Note that that the comparison maps appearing in [477, 792, 1014] go in the opposite direction, namely from L-theory to KO-theory, in comparison with the transformations appearing in [599, 600].

So one can state the following problem after Problem 15.93 has been solved:

Problem 15.94 (Identification of transformations from the Surgery Exact Sequence to its analytic counterpart). Identify the comparison map (15.91) from the Surgery Exact Sequence in the topological category to the analytic surgery sequence appearing in [477, Section 5] with the comparison map appearing in Zenobi [1014].

15.14.6 The Baum-Connes Conjecture and the Bost Conjecture

We have the a factorization of the Baum-Connes assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients

$$\operatorname{asmb}_A^{G,\mathbb{C}}(\underline{E}G)_*\colon K_n^G(\underline{E}G;A) \xrightarrow{\operatorname{asmb}_A^{G,\mathbb{C},L^1}(\underline{E}G)_*} K_*(A \rtimes_{L^1} G) \xrightarrow{K_*(q)} K_*(A \rtimes_r G).$$

Recall that the Bost Conjecture with coefficients predicts the bijectivity of the first map. We have also mentioned that there are counterexamples to the Baum-Connes Conjecture Conjecture 14.11 with coefficients. The group G involved in these counterexamples can be constructed as colimits of hyperbolic groups. For such colimits the Bost Conjecture with coefficients is known to be true. Hence for such a group G the map $K_*(q)\colon K_*(A \rtimes_{L^1} G) \to K_*(A \rtimes_r G)$ fails to be bijective. More details about this discussion can be found in [70, Section 1.5].

15.14.7 The Farrell-Jones Conjecture for K-Theory and for Homotopy K-theory

Theorem 15.95 (The K-theoretic Farrell-Jones Conjecture implies the Farrell-Jones Conjecture for homotopy K-theory). *If G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories, then G satisfies also the Farrell-Jones Conjecture 15.76 for homotopy K-theory with coefficients in additive G-categories.*

Proof. See [668, Theorem 9.1 (iii)]. □

Remark 15.96 (Implications of the homotopy K-theory version to the K-theory version). Next we discuss some cases where the Farrell-Jones Conjecture 15.76 for homotopy K-theory with coefficients in additive G-categories gives implications for the injectivity part of the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R. These all follow by inspecting for a ring R the following commutative diagram

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \arrow[r] & H_n^G(\{\bullet\};\mathbf{K}_R) = K_n(RG) \arrow[dd, "KH(\mathbf{h})"] \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \arrow[u, "\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}}", "\cong"'] \arrow[d, "h"', "\cong"] & \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{KH}_R) \arrow[r] & H_n^G(\{\bullet\};\mathbf{KH}_R) = KH_n(RG)
\end{tikzcd}$$

where the two vertical arrows pointing downwords are induced by the transformation $\mathbf{h}\colon \mathbf{K} \to \mathbf{KH}$, the map $\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}}$ is induced by the inclusion of families $\mathcal{FIN} \subseteq \mathcal{VCY}$ and the two horizontal arrows are the assembly maps for K-theory and homotopy K-theory.

Suppose that R is regular and the order of any finite subgroup of G is invertible in R. Then the two left vertical arrows are known to be bijections. This follows for $\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}}$ from [657, Proposition 70 on page 744] and for h from [265, Lemma 4.6] and the fact that RH is regular for all finite subgroups H of G and hence $K_n(RH) \to KH_n(RH)$ is bijective for all $n \in \mathbb{Z}$ by Theorem 6.16. Hence the (split) injectivity of the lower horizontal arrow implies the (split) injectivity of the upper horizontal arrow.

Suppose that R is regular. Then the two left vertical arrows are rational bijections. This follows for $\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}}$ from [670, Theorem 0.3]. To show it for h it suffices because of [265, Lemma 4.6] to show that $K_n(RH) \to KH_n(RH)$ is rationally bijective for each finite group H and $n \in \mathbb{Z}$. By the version of the spectral sequence appearing in [977, 1.3] for non-connective K-theory, it remains to show that $N^pK_n(RH)$ vanishes rationally for all $n \in \mathbb{Z}$. Since $R[t]$ is regular if R is, this boils down to show that $NK_p(RH)$ is rationally trivial for any regular ring R and any finite group H. The proof that $NK_p(RH)$ is rationally trivial for any regular ring R and any finite group H can be found for instance in [670, Theorem 9.4]. Hence the upper horizontal arrow is rationally injective if the lower horizontal arrow is rationally injective.

The next conjecture generalizes Conjecture 6.76 from torsionfree groups to arbitrary groups.

Conjecture 15.97 (K-theory versus homotopy K-theory for regular rings). Let G be a group. Suppose that R is regular and the order of any finite subgroup of G is invertible in R.

Then the natural map

$$K_n(RG) \to KH_n(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Exercise 15.98. Suppose that G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories. Then G satisfies Conjecture 15.97.

Exercise 15.99. Let G be a group. Suppose that R is regular and the order of any finite subgroup of G is invertible in R. Suppose that Conjecture 15.97 is true for G. Show that then $NK_n(RG) = 0$ holds for all $n \in \mathbb{Z}$.

15.15 Notes

One can also define a version of the Meta-Isomorphism Conjecture 15.2 or of the Fibered Meta-Isomorphism Conjecture 15.8 with finite wreath products, compare Section 13.5. Let $\mathcal{C}$ be a class of groups closed under isomorphisms and taking subgroups and quotients. Let $\mathcal{H}_*^?$ be an equivariant homology theory.

Definition 15.100 (Fibered Meta-Isomorphism Conjecture with finite wreath products).
A group G satisfies the *Fibered Isomorphism Conjecture with finite wreath products* with respect to $\mathcal{H}_*^?$ and $\mathcal{C}$ if for any finite group F the wreath product $G \wr F$ satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to $\mathcal{H}_*^?$ and the family $\mathcal{C}(G \wr F)$ consisting of subgroups of $G \wr F$ that belong to $\mathcal{C}$.

The inheritance properties for the Fibered Meta-Isomorphism Conjecture 15.8 plus the passage to overgroups of finite index do also hold for the Fibered Meta-Isomorphism Conjecture 15.100 with finite wreath products , see [578, Section 3].

Proofs of some of the inheritance properties above are also given in [447, 862].

One may ask whether one can find abstractly for the Fibered Meta-Isomorphism Conjecture 15.8 a smallest family for which it is true. For instance what happens if one takes the intersection of all families for which the Fibered Meta-Isomorphism Conjecture 15.8 is true. This questions turns out to be equivalent to the difficult and unsolved question whether the Fibered Meta-Isomorphism Conjecture 15.8 holds for an infinite product of groups, provided that for each of these groups the Fibered Meta-Isomorphism Conjecture 15.8 is true,

The following observation is taken from [803, Section 7]. Fix an equivariant homology theory $\mathcal{H}^?$. Take for simplicity Γ to be the trivial group when considering the Fibered Meta Isomorphism Conjecture 15.8.

We consider the following properties:

- (P)
  For any set $\{(G_i, \mathcal{F}_i) \mid i \in I\}$ for G_i a group and $\mathcal{F}_i$ a family of subgroups of G_i such that $(G_i, \mathcal{F}_i)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every $i \in I$, the group $\prod_{i \in I} G_i$ with respect to the family

  $$\prod_{i \in I} \mathcal{F}_i := \left\{ H \subseteq \prod_{i \in I} G_i \;\middle|\; \exists H_i \in \mathcal{F}_i \text{ for every } \; i \in I \text{ with } H \subseteq \prod_{i \in I} H_i \right\}$$

  satisfies the Fibered Meta Isomorphism Conjecture 15.8.

- (I)
  For any group G and families of subgroups $\{\mathcal{F}_i \mid i \in I\}$ of G such that $(G, \mathcal{F}_i)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every $i \in I$, the pair $(G, \bigcap_{i \in I} \mathcal{F}_i)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8.

Lemma 15.101. *The properties (I) and (P) are equivalent.*

Exercise 15.102. Prove Lemma 15.101 using Lemma 15.16.

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Chapter 16
Status

16.1 Introduction

In this chapter we give a status report about the class of groups for which the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1, the Baum-Connes 14.11 with coefficients, see Theorem 16.7, the Baum-Connes Conjecture 14.9, see Theorem 16.12, and the Novikov Conjecture 9.137, see Section 16.7 have been proved. We discuss also injectivity results in Sections 16.5 and 16.6. In order to restrict the length of the exposition, we do not present the long history of these results and concentrate only on the current state of art, although this unfortunately means that certain papers, which were spectacular breakthroughs at the time of their writing and had a big impact on the following papers, may not appear here.

A review of and a status report for some classes of groups is given in Section 16.8. This may be helpful for a reader who is interested in a certain class of groups, although this means that there are some repetitions of statements of results.

At the time of writing no counterexamples to the Full Farrell-Jones Conjecture 13.27, the Baum-Connes Conjecture 14.9 without coefficients, and the Novikov Conjecture 9.137 are known to the author. These conjectures are open in general. In Section 16.10 we explain that the search for counterexamples is not easy at all. In Subsection 16.10.5 we mention a few results that are consequences of the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R and for which there exist independent proofs for all groups.

16.2 Status of the Full Farrell-Jones Conjecture

The most general form of the Farrell-Jones Conjecture is the Full Farrell-Jones Conjecture 13.27. It has the best inheritance properties and all variants of the Farrell-Jones Conjecture presented in this book are special cases of it.

Theorem 16.1 (Status of the Full Farrell-Jones Conjecture 13.27). *Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture 13.27 is true. Then*

(i) The following classes of discrete groups belong to $\mathcal{FJ}$:

(a) *Hyperbolic groups;*
(b) *Finite dimensional* CAT(0)*-groups;*
(c) *Virtually solvable groups;*
(d) *(Not necessarily cocompact) lattices in path connected second countable locally compact Hausdorff groups.*
More generally, if L is a (not necessarily cocompact) lattice in a second countable locally compact Hausdorff group G such that $\pi_0(G)$ is discrete and belongs to $\mathcal{FJ}$, then L belongs to $\mathcal{FJ}$;
(e) *Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;*
(f) *The groups $\mathrm{GL}_n(\mathbb{Q})$ and $\mathrm{GL}_n(F(t))$ for $F(t)$ the function field over a finite field F;*
(g) *S-arithmetic groups;*
(h) *The mapping class group $\Gamma^s_{g,r}$ group of a closed orientable surface of genus g with r boundary components and s punctures for $g, r, s \ge 0$;*
(i) *Fundamental groups of graphs of abelian groups;*
(j) *Fundamental groups of graphs of virtually cyclic groups;*
(k) *Artin's full braid groups B_n;*
(l) *Coxeter groups;*
(m) *Groups in the class $\mathrm{AC}(\mathcal{VSOLV})$ defined in* (20.46) *for the class $\mathcal{VSOLV}$ of virtually solvable groups;*
(n) *Groups which acts properly and cocompactly on a finite product of hyperbolic graphs.*

(ii) The class $\mathcal{FJ}$ has the following inheritance properties:

(a) Passing to subgroups
Let $H \subseteq G$ be an inclusion of groups. If G belongs to $\mathcal{FJ}$, then H belongs to $\mathcal{FJ}$;
(b) Passing to finite direct products
If the groups G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 \times G_1$ belongs to $\mathcal{FJ}$;
(c) Group extensions
Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any infinite cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{FJ}$ and that the groups K and Q belong to $\mathcal{FJ}$.
Then G belongs to $\mathcal{FJ}$;
(d) Group extensions with virtually torsionfree hyperbolic groups as kernel
Let $1 \to K \to G \to Q \to 1$ be an extension of groups such that K is virtually torsionfree hyperbolic and Q belongs to $\mathcal{FJ}$. Then G belongs to $\mathcal{FJ}$;
(e) Colimits over directed systems
Let $\{G_i \mid i \in I\}$ be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to $\mathcal{FJ}$.
Then the colimit $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;

(f) Passing to free products
Consider a collection of groups $\{G_i \mid i \in I\}$ *such that* G_i *belongs* $\mathcal{FJ}$ *for each* $i \in I$*. Then* $*_{i \in I} G_i$ *belongs to* $\mathcal{FJ}$*;*

(g) Passing to overgroups of finite index
Let G *be an overgroup of* H *with finite index* $[G : H]$*. If* H *belongs to* $\mathcal{FJ}$*, then* G *belongs to* $\mathcal{FJ}$*;*

(h) Graph products
A graph product of groups, each of which belongs to $\mathcal{FJ}$*, belongs to* $\mathcal{FJ}$ *again.*

Proof. We begin with assertion (i) about classes of groups belong to $\mathcal{FJ}$.

(ia) This is proved for K-theory with coefficients in additive G-categories in [86, Main Theorem] and for L-theory with coefficients in additive G-categories in [77, Theorem B], but not including the "with finite wreath product" property. How this can be included, is explained in [88, Remark 6.4]. The proof for K-theory with coefficients in higher G-categories can be found in [172, Theorem 1.7 (3)].

(ib) This is proved for K-theory with coefficients in additive G-categories in degree ≤ 1 and for L-theory with coefficients in additive G-categories in all degrees in [77, Theorem B]. The argument why the K-theory case with coefficients in additive G-categories holds in all degrees can be found in [973, Theorem 1.1 and Theorem 3.4]. Note that for a finite dimensional CAT(0)-group G and a finite group F the wreath product $G \wr F$ is a finite dimensional CAT(0)-group again so that the passage to the version with finite wreath products is automatically true. The proof for K-theory with coefficients in higher G-categories can be found in [172, Theorem 1.71.7 (2)].

(ic) See [974, Theorem 1.1]. (The special case of nearly crystallographic groups is treated by Farrell-Wu [359].) and [172, Theorem 1.7 (4)].

(id) See [526, Theorem 8] whose proof is based on the case of a cocompact lattices in an almost connected Lie groups handled in [71, Theorem 1.2 and Remark 1.4] and [172, Theorem 1.7 (6)].

(ie) In dimension 3 this is proved in [71, Corollary 1.3 and Remark 1.4], where Roushon's papers [862, 863] are used, and in [172, Theorem 1.7 (7)]. The dimensions 1 and 2 can be handled directly or reduced to dimension 3 by crossing with D^1.

(if) See [868, Theorem 8.13] and [172, Theorem 1.7 (5)].

(ig) This follows from assertion (if) and the inheritance property passing to subgroups, see assertion (iia) since any S-arithmetic group is a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ or of $\mathrm{GL}_n(F(t))$ for $F(t)$ the function field over a finite field F.

(ih) See Bartels-Bestvina [69, Theorem A and Remark 9.4] and [172, Theorem 1.7 (9)].

(ii) See Gandini-Meinert-Rüping [399] and [172, Theorem 1.7 (8)].

(ij) See Wu [1000] and [172, Theorem 1.7 (8)].

(ik) The pure Artin braid group P_n is a strongly poly-surface group in the sense of Definition 16.23 by [37, Theorem 2.1]. Hence it satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.24. Since the full braid group B_n contains P_n as a subgroup of finite index, B_n satisfies the Full Farrell-Jones Conjecture 13.27 by assertion (iig).

(il) The argument in Bartels-Lück [77, page 636] for the version without "finite wreath products" extends directly to the case with "finite wreath products".

(im) See Theorem 20.47.

in Such a group G is strongly transfer reducible in the sense of Definition 20.38 by inspecting the proof of [554, Theorem 6.1 and Example 2.9] and Theorem 23.45. Now apply Theorem 20.39.

Finally we deal with the assertion (ii) about inheritance properties. Here we can refer to Theorem 13.29 except for assertions (iid) and (iih).

Assertion (iih) is proved by Gandini-Rüping [400] for additive G-categories as coefficients. The argument carries over to the setting of higher G-categories as coefficients since it is based only on inheritance properties which do also hold for higher G-categories as coefficients.

Assertion (iid) follows from assertion (im) and from Bestvina-Fujiwara-Wigglesworth [129, Theorem 2.3].

This finishes the proof of Theorem 16.1. □

Exercise 16.2. Let G be a cocompact torsionfree lattice in an almost connected Lie group L with $\dim(L) \geq 5$. Let M be an aspherical closed manifold with fundamental group G. Let $K \subseteq L$ be a maximal compact subgroup. Show that then M is homeomorphic to $G \backslash L/K$.

Exercise 16.3. Let U be a group that is universal finitely presented, i.e., any finitely presented group is isomorphic to a subgroup of G. (Such a group exists by Higman [463, page 456], and there is even a universal finitely presented groups which is the complement of an embedded S^2 in S^4, see [412, Corollary 3.4].) Show that the Full Farrell-Jones Conjecture 13.27 holds for all groups if and only if it holds for U.

Exercise 16.4. Let $S \subseteq R$ be a subring of R such that R as right S-module is finitely generated free. Suppose that for every natural number m the group $GL_m(S)$ belongs to $\mathcal{FJ}$. Show that $GL_n(R)$ belongs to $\mathcal{FJ}$ for every natural number n.

16.3 Status of the Farrell-Jones Conjecture for Homotopy *K*-Theory

Theorem 16.5 (Status of the Farrell-Jones Conjecture for homotopy K-theory). *Let $\mathcal{FJKH}$ be the class of groups for which the Farrell-*

Jones Conjecture 15.77 for homotopy K-theory with coefficients in additive G-categories with finite wreath products is true.

(i) The class $\mathcal{FJKH}$ contains the class $\mathcal{FJ}$ of groups for which the Full Farrell-Jones Conjecture 13.27 holds. (The class $\mathcal{FJ}$ is analyzed in Theorem 16.1.) Moreover, $\mathcal{FJKH}$ contains all elementary amenable groups and all one-relator groups;

(ii) The class $\mathcal{FJKH}$ has the following inheritance properties:

(a) Passing to subgroups
Let $H \subseteq G$ be an inclusion of groups. If G belongs to $\mathcal{FJKH}$, then also H belongs to $\mathcal{FJKH}$;

(b) Passing to finite direct products
If the groups G_0 and G_1 belong to $\mathcal{FJKH}$, then $G_0 \times G_1$ belong to $\mathcal{FJKH}$;

(c) Group extensions
Let $1 \to K \to G \to Q \to 1$ be an extensions of groups. If K and Q belong to $\mathcal{FJKH}$, then G belongs to $\mathcal{FJKH}$;

(d) Directed colimits
Let $\{G_i \mid i \in I\}$ be a direct system of subgroups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to $\mathcal{FJKH}$, then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJKH}$;

(e) Passing to free products
Consider a collection of groups $\{G_i \mid i \in I\}$ such that G_i belongs $\mathcal{FJKH}$ for each $i \in I$. Then $*_{i \in I} G_i$ belongs to $\mathcal{FJKH}$;

(f) Passing to overgroups of finite index
Let G be an overgroup of H with finite index $[G : H]$. If H belongs to $\mathcal{FJKH}$, then G belongs to $\mathcal{FJKH}$;

(g) Graph products
A graph product of groups each of which belongs to $\mathcal{FJKH}$ belongs to $\mathcal{FJKH}$ again;

(h) Actions on trees
If G acts on a tree T without inversion such that every stabilizer group G_x of any vertex x in T belongs to $\mathcal{FJKH}$. Then G belongs to $\mathcal{FJKH}$.

Proof. This follows from Theorem 15.95 and [668, Remark 9.3] except for assertion (iig). Here the arguments of [400] apply also directly to homotopy K-theory, the situation is actually easier because of assertion (iic). $\square$

The class of groups $\mathcal{FJKH}$ is larger and has better inheritance properties than the class $\mathcal{FJ}$. The decisive difference is that we can use for the homotopy K-theory the family $\mathcal{FIN}$ instead of the family $\mathcal{VCY}$. This is essentially a consequence of and reflected by Theorem 15.74.

Exercise 16.6. Let G be a torsionfree elementary amenable group and let R be regular.

Show that then the assembly map $H_n(BG; \mathbf{K}(R)) \to K_n(RG)$ is split injective.

Comment 23 (by W.): Is there a version of homotopy K-theory for higher categories? Compare with the construction in [668, Section 8].

16.4 Status of the Baum-Conjecture (with coefficients)

We have introduced the Baum-Connes Conjecture 14.11 with coefficients in Section 14.4.

Theorem 16.7 (Status of the Baum-Connes 14.11 with coefficients). *Let $\mathcal{BC}$ be the class of groups for which the Baum-Connes Conjecture 14.11 with coefficients holds.*

(i) The following classes of groups belong to $\mathcal{BC}$.

(a) A-T-menable groups;
(b) $\mathrm{CAT}(0)$-cubical groups in the sense of [158], i.e., groups which act properly and cocompactly on a finite-dimensional CAT(0)-cubical complex with bounded geometry.
(c) G is a countable subgroup of $\mathrm{GL}_2(F)$ for a field F;
(d) Hyperbolic groups;
(e) One-relator groups;
(f) Fundamental groups of compact 3-manifolds (possibly with boundary);
(g) Artin's full braid groups B_n;
(h) Thompson's groups F, T, and V;
(i) Coxeter groups;

(ii) The class $\mathcal{BC}$ has the following inheritance properties:

(a) Passing to subgroups
Let $H \subseteq G$ be an inclusion of groups. If G belongs to $\mathcal{BC}$, then H belongs to $\mathcal{BC}$;
(b) Passing to finite direct products
If the groups G_0 and G_1 belong to $\mathcal{BC}$, the also $G_0 \times G_1$ belongs to $\mathcal{BC}$;
(c) Group extensions
Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any finite subgroup $F \subseteq Q$ the group $p^{-1}(F)$ belongs to $\mathcal{BC}$ and that the group Q belongs to $\mathcal{BC}$.
Then G belongs to $\mathcal{BC}$;
(d) Directed unions
Let $\{G_i \mid i \in I\}$ be a direct system of subgroups of G indexed by the directed set I such that $G = \bigcup_{i \in I} G_i$. Suppose that G_i belongs to $\mathcal{BC}$ for every $i \in I$.
Then G belongs to $\mathcal{BC}$;

(e) Actions on trees
Let G be a countable group acting without inversion on a tree T. Then G belongs to $\mathcal{BC}$ if and only if the stabilizers of each of the vertices of T belong to $\mathcal{BC}$.
In particular $\mathcal{BC}$ is closed under amalgamated products and HNN-extensions.

Proof. We begin with assertion (i) about classes of groups belong to $\mathcal{BC}$.

(ia). This is proved by Higson-Kasparov [469, Theorem 1.1].

(ib) See [158]. This follows also from assertion (ia).

(ic) Such groups are a-T-menable by [429, Theorem 4]. Now apply assertion (ia).

(id) This is proved by Lafforgue [587, Théorème 0.4], see also [802]. The proof without coefficients can be found in Mineyev-Yu [717].

(ie) See Oyono-Oyono [771, Corollary 1.3].

(if) Let M be a closed Seifert manifold. Then there is an extension $1 \to \mathbb{Z} \to \pi_1(M) \to Q \to 1$ such that Q contains a subgroup H of finite index that is isomorphic to the fundamental group of a closed surface S, see [460, Theorem 12.2 on page 118]. If S carries the structure of a hyperbolic manifold, $\pi_1(S)$ and hence Q are hyperbolic and belongs to $\mathcal{BC}$ by assertion (id). If S does not carry the structure of a hyperbolic manifold, its fundamental group and hence Q are virtually finitely generated abelian and hence belong to $\mathcal{BC}$ by assertion (ia). Now assertions (ia) and (iic) imply that $\pi_1(M)$ belongs to $\mathcal{BC}$.

Let M be a closed hyperbolic 3-manifold. Then its fundamental group is hyperbolic and hence belongs to $\mathcal{BC}$ by assertion (id).

Let M be a compact irreducible manifold with infinite fundamental group such that its boundary is non-trivial or is Haken. Then $\pi_1(M)$ can be obtained from the trivial group by a finite number of HNN extensions and free amalgamated products. See [957, proof of Proposition 19.5 (6) on page 253] where the condition orientable is only assumed for simplicity, or see [460, Theorem 13.3 on page 141]. Hence $\pi_1(M)$ belongs to $\mathcal{BC}$ by assertion (iie). Let M be an irreducible closed 3-manifold. If it does not contain an incompressible torus, it is either Seifert or hyperbolic by the proof of Thurston's Geometrization Conjecture due to Perelman, see for instance Morgan-Tian [734] and hence belongs to $\mathcal{BC}$. If it contains an incompressible torus, it is Haken and hence belongs to $\mathcal{BC}$ by the argument above. We conclude that $\pi_1(M)$ belongs to $\mathcal{BC}$ for any compact irreducible 3-manifold. Since any prime 3-manifold that is not irreducible is an S^1-bundle over S^2, see [460, Lemma 3.13 on page 28], and hence belongs to $\mathcal{BC}$ by assertion (ia), any compact prime 3-manifold M belongs to $\mathcal{BC}$. Since any compact 3-manifold is a connected sum of prime compact 3-manifolds, see [460, Theorem 3.15 on page 31], assertion (if) follows from assertion (iie).

(ig) See Schick [881, Theorem 20].

(ih) These groups are a-T-menable by Farley [333], and hence we can apply assertion (ia).

(ii) Since a finitely generated Coxeter group is a-T-menable, it satisfies the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ia). By a colimit argument based on Theorem 16.7 (iid) every Coxeter group satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Finally we deal with the assertion (ii) about inheritance properties.

(iia) See Chabert-Echterhoff [209, Theorem 2.5].

(iib) See Chabert-Echterhoff [209, Theorem 3.17], or Oyono-Oyono [770, Corollary 7.12].

(iic) See Oyono-Oyono [770, Theorem 3.1].

(iid) This follows from Bartels-Echterhoff-Lück [70, Theorem 5.6 (i) and Lemma 6.2].

(iie) This is proved by Oyono-Oyono [771, Theorem 1.1]. □

Exercise 16.8. Let $1 \to K \to G \to Q \to 1$ be an extension of groups such that K and Q satisfy the Baum-Connes Conjecture 14.11 with coefficients and Q is torsionfree. Show that then G satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Exercise 16.9. Let G be a torsionfree group. Suppose that $\mathbb{C}G$ has an idempotent different from 0, 1. Show that then G cannot be a subgroup of a hyperbolic group, a finite dimensional CAT(0)-group, a lattice in an almost connected Lie group, the fundamental group of a manifold of dimension ≤ 3, an amenable group, a mapping class group, or a one-relator group.

Remark 16.10 (Passing to overgroups of finite index). It is not known in general whether a group G belongs to $\mathcal{BC}$, i.e., G satisfies the Baum-Connes Conjecture 14.11 with coefficients if a subgroup of finite index does. Partial answers to this question are given by Schick [881, Theorem 20].

This suggests to systematically implement the with "finite wreath product version" in the Baum-Connes setting, as we did in the Farrell-Jones setting, see Section 13.5.

Remark 16.11 (The Status of the Baum-Connes Conjecture for topological groups). We have only dealt with the Baum-Connes Conjecture for discrete groups. The Baum Connes Conjecture (with coefficients) makes also sense for second countable locally compact Hausdorff groups. Here some results in this setting.

Higson-Kasparov [469] treat the Baum-Connes Conjecture with coefficients for second countable locally compact Hausdorff groups which are a-T-menable

Julg-Kasparov [521, Theorem 5.4 (i)] prove the Baum-Connes Conjecture with coefficients for connected Lie groups L whose Levi-Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \cdots \times SO(n_k, 1) \times SU(m_1, 1) \times \cdots \times SU(m_l, 1)$$

for a compact group K. The Baum-Connes Conjecture with coefficients for $Sp(n, 1)$ is proved by Julg [520].

The Baum-Connes Conjecture without coefficients has been proven by Chabert-Echterhoff-Nest [210] for second countable almost connected Hausdorff groups, based on the work of Higson-Kasparov [469] and Lafforgue [586].

Next we deal with the Baum-Connes Conjecture 14.9 without coefficients for (discrete) groups. Recall that all groups which satisfy the Baum-Connes 14.11 with coefficients do in particular satisfy the Baum-Connes Conjecture 14.9. Below are some case some of which are not covered by this implication.

A *length function* on G is a function $L\colon G \to \mathbb{R}_{\geq 0}$ such that $L(1) = 0$, $L(g) = L(g^{-1})$ for $g \in G$ and $L(g_1 g_2) \leq L(g_1) + L(g_2)$ for $g_1, g_2 \in G$ holds. The word length metric L_S associated to a finite set S of generators is an example. A length function L on G has *property (RD)* ("rapid decay") if there exist $C, s > 0$ such that for any $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have

$$||\rho_G(u)||_\infty \leq C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s} \right)^{1/2}$$

where $||\rho_G(u)||_\infty$ is the operator norm of the bounded G-equivariant operator $l^2(G) \to l^2(G)$ coming from right multiplication with u. A group G has *property (RD)* if there is a length function which has property (RD). This notion is due to Jolissaint [512]. More information about property (RD) can be found for instance in [221, 223], [583], and [945, Chapter 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. A simply connected complete Riemannian manifold with non-positive sectional curvature is bolic. We refer to [540, Section 2] for a precise definition.

Theorem 16.12 (Status of the Baum-Connes Conjecture (without coefficients)). *A group G satisfies the Baum-Connes Conjecture 14.9 (without coefficients) if it satisfies one of the following conditions.*

(i) The group G is a discrete subgroup of a connected Lie groups L whose Levi-Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \cdots \times SO(n_k, 1) \times SU(m_1, 1) \times \cdots \times SU(m_l, 1)$$

for a compact group K;

(ii) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space;

(iii) The group G is a discrete finite covolume subgroup of the isometry groups of a simply connected complete Riemannian manifold with pinched negative sectional curvature;
(iv) The group G is a discrete subgroup of $Sp(n,1)$.

Proof. (i) See Julg-Kasparov [521].
(ii) See Lafforgue [582] or [901].
(iii) See [223, Corollary 0.3].
(iv) See Julg [520]. □

16.5 Injectivity Results in the Baum-Connes Setting

There are cases where one can show that the assembly maps appearing in the Farrell-Jones setting or Baum-Connes setting are injective without knowing that they are bijective. There is no case where one can prove surjectivity but does not know bijectivity as well. This is a common phenomenon in algebraic topology where surjectivity arguments often contain an injectivity argument, essentially one applies the surjectivity argument to a cycle whose boundary is the image of a cycle representing an element in the kernel of the assembly map. Moreover, this shows that in general surjectivity results are harder than injectivity results.

The main value of surjectivity statements is that they allow to interprete elements in the K- or L-groups homologically and thus to obtain valuable information. The injectivity statements are interesting since they imply the Novikov Conjecture or give some idea how large the K- and L-groups are.

Theorem 16.13 (Split injectivity of the assembly map appearing in the Baum-Connes Conjecture 14.9 (without coefficients) for fundamental groups of complete Riemannian manifolds with non-positive sectional curvature). *The assembly map appearing in the Baum-Connes Conjecture 14.9 is split injective if G is the fundamental group of complete Riemannian manifold with non-positive sectional curvature.*

Proof. See Kasparov [546, Theorem 6.7]. □

More general results for bolic spaces are proved in Kasparov-Skandalis [541].

A metric space (X,d) admits a *uniform embedding into Hilbert space* or sometimes also called *coarse embedding into Hilbert space* if there exist a separable Hilbert space H, a map $f\colon X \to H$, and non-decreasing functions ρ_1 and ρ_2 from $[0,\infty) \to \mathbb{R}$ such that $\rho_1(d(x,y)) \le ||f(x)-f(y)|| \le \rho_2(d(x,y))$ for $x,y \in X$ and $\lim_{r\to\infty} \rho_i(r) = \infty$ for $i=1,2$. A metric is *proper* if for each $r>0$ and $x \in X$ the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G-invariant length metric d admits a uniform embedding into Hilbert space is independent of the choice of d since the induced coarse structure does not depend on d, see [902, page 808]. We mention that for a finitely generated group any left invariant word length metric is an example of a proper left G-invariant length metric.

For more information about groups admitting a uniform embedding into Hilbert space we refer to [300, 429].

The next result is due to Yu [1008, Theorem 2.2 and Proposition 2.6].

Theorem 16.14 (Status of the Coarse Baum-Connes Conjecture). *The Coarse Baum-Connes Conjecture 14.30 is true for a discrete metric space X of bounded geometry if X admits a uniform embedding into Hilbert space. In particular a countable group G satisfies the Coarse Baum-Connes Conjecture 14.30 if G equipped with a proper left G-invariant length metric admits a uniform embedding into Hilbert space.*

Theorem 16.15 (Split injectivity of the assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients). *Let G be a countable group. Then for any C^*-algebra A the assembly map appearing in the Baum-Connes Conjecture 14.11*

$$K_n^G(\underline{E}G; A) \to K_n(A \rtimes_r G);$$

is split injective if the group G has one of the following properties:

(i) The group G admits a proper left G-invariant length metric for which G admits a uniform embedding into Hilbert space;
(ii) The group G admits a proper left G-invariant length metric for which G admits a uniform embedding into a Banach space with property (H);
(iii) The group G is a subgroup of $\mathrm{GL}_n(F)$ for some field F and natural number n;
(iv) The group G is a subgroup of an almost connected Lie group.

Proof. (i) This is proved by Skandalis-Tu-Yu [902, Theorem 6.1] using ideas of Higson [468] and Theorem 16.14.

(ii) See Kasparov-Yu [542, Theorem 1.3].

(iii) Assertion (i) applies to G by Guentner-Higson-Weinberger [429, Theorem 2 and 3].

(iv) Assertion (i) applies to G by Guentner-Higson-Weinberger [429, Theorem 7]. □

Exercise 16.16. Let G be a group such that for any finitely generated subgroup $H \subseteq G$ and every H-C^*-algebra A the assembly map $K_n^H(\underline{E}H; A) \to K_n(A \rtimes_r H)$ injective.

Show that then the assembly map $K_n^G(\underline{E}G; A) \to K_n(A \rtimes_r G)$ is injective for every G-C^*-algebra A. Prove the analogous statement for the K-theoretic and L-theoretic assembly maps with coefficients in additive categories (with involution) and the family of virtually cyclic subgroups.

Split injectivity of the Baum-Connes assembly map (for trivial coefficients) is proved under certain conditions about the compactifications of the model for the space for proper G-actions by Rosenthal [859] based on techniques developed by Carlsson-Pedersen [202].

Remark 16.17 (Groups Acting Amenably on a Compact Space). A continuous action of a discrete group G on a compact space X is called *topologically amenable* if there exists a sequence

$$p_n \colon X \to M^1(G) = \{f \colon G \to [0,1] \mid \sum_{g \in G} f(g) = 1\}$$

of weak-$*$-continuous maps such that for each $g \in G$ one has

$$\lim_{n \to \infty} \sup_{x \in X} ||g * (p_n(x) - p_n(g \cdot x))||_1 = 0.$$

More information about this notion can be found for instance in [25, 26]. It should not be confused with the notion of an *amenable action* of a group G on a set X, where amenable in this context means that there exists a a G-invariant mean on X. Note that the following statements are equivalent:

- the group G is amenable;
- The action of G on G by multiplication is amenable;
- The obvious action on G on the one-point-space is topologically amenable.

A group G is called *boundary amenable*, if admits a topologically amenable action on a compact metric space in the sense above.

Higson-Roe [473, Theorem 1.1 and Proposition 2.3] show that a finitely generated group is boundary amenable if and only if it belongs to the class A defined in [1008, Definition 2.1], and hence admits a uniform embedding into Hilbert space. Hence Theorem 16.15 (i) implies the result of Higson [468, Theorem 1.1] that the assembly map $K_n^G(\underline{E}G; A) \to K_n(A \rtimes_r G)$ appearing in the Baum-Connes Conjecture 14.11 with coefficients is split injective if G is boundary amenable.

Finally we mention that a finitely generated group G is boundary amenable if and only if the reduced group C^*-algebra $C_r^*(G)$ is exact, i.e., the minimal tensor product with it preserves short exact sequences of C^*-algebras, see for instance [409, Proposition 9.9].

16.6 Injectivity Results in the Farrell-Jones Setting

Theorem 16.18 (Split injectivity of the assembly map appearing in the L-theoretic Farrell Jones Conjecture with coefficients in the

ring $\mathbb{Z}$ for fundamental groups of complete Riemannian manifolds with non-positive sectional curvature). *The assembly map appearing in the L-theoretic Farrell Jones Conjecture 13.4 with coefficients in the ring $\mathbb{Z}$ is split injective if G is the fundamental group of complete Riemannian manifold with non-positive sectional curvature.*

Proof. See [371, Theorem 2.3]. □

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any $R > 0$ there exists a cover $\mathcal{U}$ of X with the property that the diameter of the members of $\mathcal{U}$ is uniformly bounded and every open ball of radius R intersects at most $(n+1)$ elements of $\mathcal{U}$, see [424, page 29]. The asymptotic dimension of a finitely generated group is the asymptotic dimension of its Cayley graph (and is independent of the choice of set of finite generators.)

For a torsionfree group G with finite asymptotic dimension and a finite model for BG and any ring R the split injectivity of $H_n(BG;\mathbf{K}(R)) \to K_n(RG)$ is proved by Bartels [93, Theorem 1.1] and by Carlsson-Goldfarb [201, Main Theorem on page 406]. The L-theory version is proved in Bartels [93, Section 7] as well, provided that there exists a natural number N with $K_{-i}(R) = 0$ for $i \geq N$.

The notion of finite decomposition complexity was introduced and studied by Guentner-Tessera-Yu [430, 431]. It is a weaker notion than finite asymptotic dimension. The split injectivity of the assembly maps $H_n(BG;\mathbf{K}(R)) \to K_n(RG)$ and of $H_n(BG;\mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$ for a torsionfree group G with finite model for BG and finite decomposition complexity is proved by Ramras-Tessera-Yu [815, Theorem 1.1] and Guentner-Tessera-Yu [430, page 334] for any ring R (with involution), provided that in the L-theory case there exists a natural number N with $K_{-i}(R) = 0$ for $i \geq N$.

Kasprowski [549, Theorem 8.1] proved for a group G with finite dimensional model for $E_{\mathcal{FIN}}(G)$ and finite quotient finite decomposition complexity, a strengthening of the notion of finite decomposition complexity, and a global upper bound on the orders of the finite subgroups that the assembly map $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to K_n(RG)$ is split injective for all $n \in \mathbb{Z}$. An L-theory version is proved in [549, Theorem 9.1].

The paper [549] uses ideas of [91]. Kasprowski [549, page 566] points out a gap in the proof of [91] which has the consequence that the results in [91] are only proved under the additional assumption that there is a finite model for $E_{\mathcal{FIN}}(G)$.

The papers by Kasprowski [550, 551] are based on [549] and lead to the following two results.

Theorem 16.19 (Injectivity of the Farrell-Jones assembly map for $\mathcal{FIN}$ for subgroups of almost connected Lie groups). *Let G be a subgroup of an almost connected Lie group. Suppose that G admits a finite dimensional model for the classifying space $E_{\mathcal{FIN}}(G)$.*

(i) Let $\mathcal{A}$ be an additive G-category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$;

(ii) Let $\mathcal{A}$ be an additive G-category with involution. Suppose that there exists $N \geq 0$ such that $\pi_{-i}\big(\mathbf{K}_{\mathcal{A}}(I(A))\big) = 0$ holds for all $i \geq N$ and all virtually abelian subgroups $A \subseteq G$.
Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$;

(iii) Let $\mathcal{C}$ be a right exact G-∞-category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{C}}) = \pi_n\big(\mathbf{K}_{\mathcal{C}}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$, provided that G is finitely generated; ***Comment 24 (by W.)****: Christoph says that one get rid of the assumption "finitely generated" using Kasprowski's argument in [550, Section 7]. Shall we mention or explain this?*

(iv) A subgroup G of an almost connected Lie group admits a finite dimensional model for $E_{\mathcal{FIN}}(G)$ if and only if there exists $N \in \mathbb{N}$ such that every finitely generated abelian subgroup of G has rank at most N.

Proof. (i) and (ii) If G is finitely generated, this is proved in [550, Theorem 1.1 and Theorem 6.1]. Since every group is the union of its finitely generated subgroups, the general case for injectivity follows from Lemma 15.23 (ii). One obtains even split injectivity since also the retraction is natural, see [550, Section 7].

(iii) This is proved in [169, Theorem 1.1].

(iv) See [550, Proposition 1.3]. □

Theorem 16.20 (Injectivity of the Farrell-Jones assembly map for $\mathcal{FIN}$ for linear groups). *Let R be a commutative ring with unit and let $G \subseteq \mathrm{GL}_n(R)$ be a subgroup. Suppose that G admits a finite dimensional model for the classifying space $E_{\mathcal{FIN}}(G)$.*

(i) Let $\mathcal{A}$ be any additive G-category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$;

(ii) Let $\mathcal{A}$ be any additive G-category with involution. Suppose that there exists $N \geq 0$ such that $\pi_{-i}(\mathbf{K}_{\mathcal{A}}(I(H)) = 0$ holds for all $i \geq N$ and all virtually nilpotent subgroups $H \subseteq G$.

Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$;

(iii) Let $\mathcal{C}$ be a right exact G-∞-category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{C}}(I(G))\big)$$

is split injective for all $n \in \mathbb{Z}$;

Proof. If G is finitely generated, this is proved in [551, Theorem 1.1]. Since every group is the union of its finitely generated subgroups, the general case for injectivity follows from Lemma 15.23 (ii). One obtains even split injectivity since also the retraction is natural, as explained in [550, Section 7]. The case of higher G-categories is proved in [169, Theorem 1.1]. □

Split injectivity of the K- and L-theoretic Farrell-Jones assembly map (for trivial coefficients) is proved under certain conditions about the compactifications of the model for the space for proper G-actions by Rosenthal [855, 856, 857], based on techniques developed by Carlsson-Pedersen [202].

16.7 Status of the Novikov Conjecture

Recall that the Novikov Conjecture 9.137 holds for a group G if one of the following conditions is satisfied:

- The assembly map

$$\begin{aligned} H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) = H_n^G(EG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \\ \to H_n^G(G/G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G) \end{aligned}$$

  is rationally injective for all $n \in \mathbb{Z}$, see Theorem 13.62 (xi);
- The assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G)$$

  is rationally injective, see Lemma 13.35 and Theorem 13.62 (xi);
- The L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring $\mathbb{Z}$ holds, see Theorem 13.62 (xi);
- The assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

  is rationally injective for all $n \in \mathbb{Z}$, see Theorem 14.29;

- The assembly map

$$K_n^G(E_{\mathcal{FIN}}(G))) \to K_n(C_r^*(G))$$

is rationally injective for all $n \in \mathbb{Z}$, see Lemma 13.35 and Theorem 14.29;
- The Baum-Connes Conjecture 14.9 holds for G, see Theorem 14.29.

Hence all groups appearing in Theorems 16.1, 16.7, 16.12, 16.13, 16.15, 16.18, or 16.19 satisfy the Novikov Conjecture 9.137. In particular a group G satisfies Novikov Conjecture 9.137 if G is a countable discrete subgroup of one of the following type of groups:

- Hyperbolic groups (or more generally directed colimits of hyperbolic groups);
- Finite dimensional CAT(0)-groups;
- Almost connected Lie groups;
- (Not necessarily cocompact) lattices in second countable locally compact Hausdorff groups G for which $\pi_0(G)$ is discrete and belongs to $\mathcal{FJ}$;
- $\mathrm{GL}_n(F)$ for a field F and some natural number n;
- S-arithmetic groups;
- mapping class groups;
- Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;
- A-T-menable groups and hence also amenable and elementary amenable groups;
- One-relator groups;
- Coxeter groups;
- Thompson's groups F, T and V;
- Artin's full braid groups B_n;
- $\mathrm{Out}(F_n)$ or more generally, $\mathrm{Out}(\Gamma)$ for a torsionfree hyperbolic group or a right-angled Artin group Γ, see [130].

Furthermore, the Novikov Conjecture 9.137 is satisfied for a countable group G if one of the following conditions are satisfied:

- G is the fundamental group of a complete Riemannian manifold with non-positive sectional curvature;
- The group G admits a proper left G-invariant length metric for which G admits a uniform embedding into Hilbert space;
- The group G admits a proper left G-invariant length metric for which G admits a uniform embedding into a Banach space with property (H);
- G has a finite model for BG and finite asymptotic dimension, see [1007], or, more generally, has a finite model for BG and finite decomposition complexity, Guentner-Tessera-Yu [430, page 334];
- G is a geometrically discrete subgroup of a volume preserving diffeomorphism of any smooth compact manifold, see [411]. See also [410] where the condition volume preserving does not occur anymore.

A Banach version of the strong Novikov conjecture is proved in [327] for groups having polynomially bounded higher-order combinatorial functions. This includes all automatic groups. If the group G is of type F_∞, is polynomially contractible, and has property (RD), it satisfies the strong Novikov Conjecture 14.26.

More information about the Novikov Conjecture and its status can be found for instance in [1010].

16.8 Review of and Status Report for Some Classes of Groups

16.8.1 Hyperbolic Groups

The definition and the basic properties of the notion of a *hyperbolic group* can be found for instance in [155, 278, 408, 642]. Examples are free groups and fundamental groups of closed Riemannian manifolds with negative sectional curvature.

Almost all conjectures in this book about groups are satisfied for hyperbolic groups since they satisfy both the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia), and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (id).

16.8.2 Lacunary Hyperbolic Groups

A finitely generated group is a *lacunary hyperbolic group* if one of its asymptotic cones is an $\mathbb{R}$-tree, see Olshanskii-Osin-Sapir [764]. Since they are directed colimits of hyperbolic groups, see [764, Theorem 1.1], they satisfy the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia) and (iie). It is not known whether lacunary hyperbolic groups satisfy the Baum-Connes Conjecture 14.9.

A lacunary hyperbolic group is finitely presented if and only if it is hyperbolic. This is due to Kapovich-Kleiner, see [764, Theorem 8.1].

There are rather exotic examples of lacunary hyperbolic group. For instance a finitely generated torsionfree non-cyclic group all whose proper subgroups are cyclic is constructed by Ol'shanskii [762]. It is a lacunary hyperbolic group. This follows from [764, Theorem 1.1].

Other examples of lacunary hyperbolic groups are constructed in [39]. These finitely generated groups do contain (in a weak sense) an infinite expander. Hence they admit no uniform embedding into a Hilbert space (or into any l^p with $1 \leq p < \infty$) and any infinite dimensional linear representation of these groups has infinite image. Note that for these group a counterexample to the Baum-Connes Conjecture 14.11 with coefficients is constructed by Higson-Lafforgue-Skandalis [470]. (This lead Baum-Guentner-Willet [105] to reformulate the Baum-Connes Conjecture 14.11 with coefficients by introducing a new crossed product, see also [180] for which no counterexamples are known so far.)

The class of lacunary groups contains some non–virtually cyclic elementary amenable groups and and some infinite torsion groups. More examples of exotic lacunary hyperbolic groups are discussed in [764] and [872, Section 4].

16.8.3 Relative Hyperbolic Groups

For the definition and basic information about relative hyperbolic groups we refer for instance to [150, 167, 331, 423, 769, 926, 927].

The following result is taken from Bartels [67, Remark 4.7] where the notion of a relative hyperbolic groups following Bowditch [150] is used.

Theorem 16.21 (The Full Farrell-Jones Conjecture and relatively hyperbolic groups). *Let G be a countable group which is relatively hyperbolic to the subgroups $P_1, P_2, \dots, P_n$. If $P_1, P_2, \dots, P_n$ satisfy the Full Farrell-Jones Conjecture 13.27, then G satisfies the Full Farrell-Jones Conjecture 13.27.*

The analogue of assertion (iid) of Theorem 16.1, which is due to Bestvina-Fujiwara-Wigglesworth [129, Theorem 2.3], has been studied for certain relative hyperbolic groups by Andrew-Guerch-Hughes [31]. The strategy of the proof of Bartels [67] is used by Knopf [564, Corollary 4.2] to study the Farrell-Jones Conjecture for groups acting acylindrically on a simplicial tree.

16.8.4 Systolic Groups

Let G be a group which acts cocompactly and properly on a systolic complex with the Isolated Flats Property by simplicial automorphisms. Then G is relatively hyperbolic to the family of virtually abelian groups by Elsner [321, Theorem B]. Hence Theorem 16.21 implies that G satisfies the Full Farrell-Jones Conjecture 13.27.

16.8.5 Finite Dimensional CAT(0)-Groups

A CAT(0)-*group* is a group admitting a cocompact proper isometric action on a CAT(0)-space X. We call it a *finite dimensional* CAT(0)-*group* if we can additionally arrange that X has finite topological dimension. Basic properties of this notion of a can be found for instance in [155, 642]. Examples for finite dimensional CAT(0)-groups are fundamental groups of closed Riemannian manifolds with non-positive sectional curvature.

A finite dimensional CAT(0)-group satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ib).

It is not known whether every finite dimensional CAT(0)-group satisfies the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9. If G admits a cocompact proper isometric action on a CAT(0)-space with the Isolated Flats Property in the sense of [223, Definition 3.1], then the Baum-Connes Conjecture 14.9 holds for G, see [223, Corollary 0.3 b]. If G is a CAT(0)-cubical groups in the sense of [158], then the Baum-Connes Conjecture 14.9 holds for G, see [158].

16.8.6 Limit Groups

Limit groups as they appear for instance in [891] have been in the focus of geometric group theory for the last years. Expositions about limit groups are for instance [212, 778]. Alibegović-Bestvina [22] have shown that limit groups are CAT(0)-groups. It is not hard to check that their proof shows that a limit group is even a finite dimensional CAT(0)-group. Hence every limit group satisfies the Full Farrell-Jones Conjecture 13.27.

16.8.7 Fundamental Groups of Complete Riemannian Manifolds with Non-Positive Sectional Curvature

Let π be the fundamental group of a complete Riemannian manifold M. Let sec denote its sectional curvature.

- M closed and $\sec(M) < 0$
  If M is closed and has negative sectional curvature, then π is hyperbolic and hence satisfies both the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia), and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (id).
- M closed and $\sec(M) \leq 0$
  If M is closed and has non-positive sectional curvature, then π is a finite dimensional CAT(0)-group and satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ib). It is not known whether all such π satisfy the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9.
- $C_1 \leq \sec(M) \leq C_2 < 0$ and finite volume
  Let M be a complete Riemannian manifold which is pinched negatively

curved and has finite volume. Then π satisfies the Full Farrell-Jones Conjecture 13.27 since π is relatively hyperbolic with respect to the family of virtually finitely generated nilpotent groups, see [150], or [331, Theorem 4.11], and we can apply Theorem 16.1 (ic) and Theorem 16.21.
If we additionally assume that the curvature tensor has bounded derivatives, then also the Baum-Connes Conjecture 14.9 holds for G by Chatterji-Ruan [223, Corollary 0.3 a]. Lattices in rank one Lie groups are examples for π.

- M A-regular and $\sec(M) \leq 0$
A complete Riemannian manifold M is called *A-regular* if there exists a sequence of positive real numbers A_0, A_1, A_2, ... such that $||\nabla^n K|| \leq A_n$ where $||\nabla^n K||$ is the supremum-norm of the n-th covariant derivative of the curvature tensor K. Every locally symmetric space is A-regular since ∇K is identically zero.
Let M be a complete Riemannian manifold with non-positive sectional curvature that is A-regular. Then $\pi = \pi_1(M)$ satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$ in degree $n \leq 1$ and the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution $\mathbb{Z}$, see Farrell-Jones [355, Proposition 0.10 and Lemma 0.12]. Since π is torsionfree, this implies that $\mathrm{Wh}(\pi)$, $\widetilde{K}_0(\mathbb{Z}\pi)$ and $K_n(\mathbb{Z}\pi)$ for $n \leq -1$ all vanish and Conjecture 9.114 holds for $R = \mathbb{Z}$.
- $C_1 \leq \sec(M) \leq C_2 < 0$
Let M be a complete Riemannian manifold with pinched negative curvature. Then there is another Riemannian metric for which M is negatively curved complete and A-regular. This fact is mentioned in Farrell-Jones [355, page 216] and attributed there to Abresch [2] and Shi [897]. Hence the conclusions above for complete Riemannian manifold M with non-positive sectional curvature that is A-regular do also hold for pinched negatively curved complete Riemannian manifolds.
- $\sec(M) \leq 0$
If M is a complete Riemannian manifold with non-positive sectional curvature, we have already stated some injectivity results for π in Theorem 16.13 and Theorem 16.18.
In particular π satisfies the Novikov Conjecture 9.137 by Theorem 13.62 (xi) or Theorem 14.29.

16.8.8 Lattices

A discrete subgroup G of a locally compact second countable Hausdorff group Γ is called a *lattice* if the quotient space Γ/G has finite covolume with respect to the Haar measure of Γ.

Every lattice G in Γ satisfies the Full Farrell-Jones Conjecture 13.27 if $\pi_0(\Gamma)$ is discrete and belongs to the class $\mathcal{FJ}$ introduced and analyzed in

Theorem 16.1, for instance if Γ is path connected or an almost connected Lie group. This follows from Theorem 16.1 (id).

It is a prominent open problem to decided whether lattices satisfy the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9. This is not even known for lattices in almost connected Lie groups. The case $\mathrm{SL}_n(\mathbb{Z})$ is still open for $n \geq 3$. By [223, Corollary 0.3 a] lattices G in rank one Lie groups satisfy the Baum-Connes Conjecture 14.9. Some other lattices satisfying the Baum-Connes Conjecture 14.9 come from Theorem 16.12.

16.8.9 S-Arithmetic Groups

Every S-arithmetic group satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ig). This is not known for the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9, the group $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ is still an open problem.

16.8.10 Linear Groups

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are in general open for linear groups, i.e., $\mathrm{GL}_n(F)$ for some field F. The same statement holds for the Baum-Connes Conjecture.

The Novikov-Conjecture holds by Theorem 14.29 and Theorem 16.15 (iii) and Exercise 16.16 for any countable subgroup of $\mathrm{GL}_n(F)$ for a field F.

16.8.11 Subgroups of Almost Connected Lie Groups

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for discrete subgroups of almost connected Lie groups in general. The same statement holds for the Baum-Connes Conjecture 14.9.

The Novikov-Conjecture holds by Theorem 14.29 and Theorem 16.15 (iv) and Exercise 16.16 for any countable subgroup of an almost connected Lie group.

16.8.12 Virtually Solvable Groups

Virtually solvable groups satisfy both the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ic) and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (ia).

16.8.13 A-T-menable, Amenable and Elementary Amenable Groups

A group G is called *amenable* if there is a (left) G-invariant linear operator $\mu\colon L^\infty(G,\mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ that satisfies for all $f \in l^\infty(G,\mathbb{R})$

$$\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\}.$$

The latter condition is equivalent to the condition that μ is bounded and $\mu(f) \geq 0$ if $f(g) \geq 0$ for all $g \in G$.

The *class of elementary amenable* groups is defined as the smallest class of groups that has the following properties:

(i) It contains all finite and all abelian groups;
(ii) It is closed under taking subgroups;
(iii) It is closed under taking quotient groups;
(iv) It is closed under extensions, i.e., if $1 \to H \to G \to K \to 1$ is an exact sequence of groups and H and K belong to the class, then also G;
(v) It is closed under *directed unions*, i.e., if $\{G_i \mid i \in I\}$ is a directed system of subgroups such that $G = \bigcup_{i \in I} G_i$ and each G_i belongs to the class, then G belongs to the class.

Since the class of amenable groups has all the properties mentioned above, every elementary amenable group is amenable. The converse is not true. For more information about amenable and elementary amenable groups, we refer for instance to [635, Section 6.4.1] or [776].

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property*, if G admits a metrically proper isometric action on some affine Hilbert space. *Metrically proper* means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite.

An extensive treatment of such groups is presented in [228, 946]. Any a-T-menable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients, and under finite products. It is not closed under semidirect products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of $SO(n,1)$ and $SU(n,1)$, Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes. A group G has Kazhdan's *property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. For more information about this property we refer for instance to [113]. An infinite a-T-menable group does not have property (T). Since $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Every a-T-menable, every amenable, and every elementary-amenable group satisfies the Baum-Connes Conjecture 14.11 with coefficients. This follows from Theorem 16.7 (ia) in the a-T-menable case. Since every group is the directed union of its finitely generated subgroups, every finitely generated group is countable, and every countable amenable group is a-T-menable, the claim follows for amenable groups and hence also for elementary amenable groups from Theorem 16.7 (iid).

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for elementary amenable groups. The main problem in the Farrell-Jones setting is that one has to deal with virtually cyclic subgroups in its formulation and for the inheritance property under extensions, see Theorem 16.7 (iic), whereas in the Baum-Connes setting finite subgroups suffice. This also explains that elementary amenable groups satisfy the Farrell-Jones Conjecture 15.77 for homotopy K-theory with coefficients in additive G-categories with finite wreath products, see Theorem 16.5 (i).

The L-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2 holds for elementary amenable groups by [458, Theorem 5.2.1].

16.8.14 Three-Manifold Groups

Let M be a (not necessarily compact) manifold (possibly with boundary) of dimension ≤ 3. Then $\pi_1(M)$ satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ie).

If we additionally assume that M is compact, $\pi_1(M)$ satisfies the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (if).

The reason why in the Farrell-Jones setting we do not need compact, is the inheritance property under directed colimits of directed systems of subgroups, see Theorem 16.1 (iie), which is not available in the Baum-Connes setting where we need that all structure maps are injective, see Theorem 16.7 (iid).

Exercise 16.22. Let G be the fundamental group of a knot complement.

Show for any regular ring R that the projection $\mathrm{pr}\colon G \to G/[G,G] \cong \mathbb{Z}$ induces for every ring R an isomorphism $K_n(RG) \to K_n(R[G/[G,G]])$ and we get an isomorphism $K_n(RG) \cong K_n(R) \oplus K_{n-1}(R)$.

Show for any ring R with involution $L_n^{\langle -\infty \rangle}(RG) \cong L_n^{\langle -\infty \rangle}(R) \oplus L_n^{\langle -\infty \rangle}(R)$.

16.8.15 One-Relator Groups

The Baum-Connes Conjecture 14.11 with coefficients holds for one-relator groups by Theorem 16.7 (ie).

A consequence of Newman's spelling theorem, see [740], is that a one-relator groups which is not torsionfree is hyperbolic and hence satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia).

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for torsionfree one-relator groups. Note that not all one-relator groups are solvable, hyperbolic, or finite dimensional CAT(0)-groups, so that we cannot apply Theorem 16.1 in general.

Nevertheless the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R is known if R is regular and G is a subgroup of a torsionfree one-relator group by Waldhausen [957, Theorem 19.4 on page 249] in the connective case and by Bartels-Lück [74, Theorem 0.11] for the non-connective version. Recall that in this special case Conjecture 13.1 boils down to Conjecture 6.53.

The L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in any ring with involution R holds after inverting two for torsionfree one-relator groups by Cappell [192, Corollary 8].

All Baumslag-Solitar groups satisfy the Full Farrell-Jones Conjecture 13.27, see Farrell-Wu [361] for the version without “finite wreath products” and Gandini-Meinert-Rüping [399, Corollary 1.1].

16.8.16 Selfsimilar Groups

We use the notion of selfsimilar group as presented in [94, Section 3] which is slightly more general than the classical notion defined for instance in [95, 739]. Selfsimilar groups are groups acting in a recursive manner on a regular rooted tree RT_d. If the recursion of every element involves only a linearly growing subtree of T_d, the group is said to be bounded.

The Full Farrell-Jones Conjecture 13.27 is proved by Bartholdi [94, Theorem A] for bounded selfsimilar groups since these are subgroups of finite dimensional CAT(0)-groups and hence Theorem 16.1 (ib) and (iia) applies. Using Theorem 16.1 (ib) and (iib) Bartholdi [94, Theorem C] proves the Full Farrell-Jones Conjecture 13.27 for Aleshin-Grigorchuk groups, Gupta-Sidki groups, and generalized Grigorchuk groups whose definition and intriguing properties are reviewed in [94, Section 4].

16.8.17 Strongly Poly-Surface Groups

Definition 16.23 (Strongly poly-surface group). Let G be a group with a finite filtration $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = G$.

We call G *strongly poly-surface* if the filtration satisfies the following conditions:

(i) G_i is normal in G for $i = 0, 1, 2, \ldots, d$;
(ii) For every $i \in \{1, 2, \ldots, d\}$ and $g \in G$, there is a (not necessarily compact) surface S (possibly with boundary) with torsionfree $\pi_1(S)$, a diffeomorphism $f\colon S \to S$, and an isomorphism $\alpha\colon G_i/G_{i-1} \xrightarrow{\cong} \pi_1(S)$ such that the following diagram commutes

$$\begin{tikzcd} G_i/G_{i-1} \arrow[r, "c_g"] \arrow[d, "\alpha"'] & G_i/G_{i-1} \arrow[d, "\alpha"] \\ \pi_1(S) \arrow[r, "\pi_1(f)"] & \pi_1(S) \end{tikzcd}$$

where c_g is induced by conjugation with $g \in G$.

Note that condition (ii) is automatically satisfied if S is a closed surface.

Theorem 16.24 (The Full Farrell-Jones Conjecture for strongly poly-surface groups). *A strongly poly surface group G satisfies the Full Farrell-Jones Conjecture 13.27.*

Proof. Fix a filtration $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = G$ as it occurs in Definition 16.23. We show by induction over $i = 0, 1, 2, \ldots, d$ that G/G_{d-i} satisfies the Full Farrell-Jones Conjecture 13.27. The induction beginning $i = 0$ is trivial, the induction step from $(i-1)$ to i done as follows.

Consider the exact sequence $1 \to G_{d-i+1}/G_{d-i} \to G/G_{d-i} \xrightarrow{p} G/G_{d-i+1} \to 1$. By induction hypothesis G/G_{d-i+1} satisfies the Full Farrell-Jones Conjecture 13.27. Since $G_{d-i+1}/G_{d-i} \cong \pi_1(S)$, the group G_{d-i+1}/G_{d-i} satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ie). Consider any infinite cyclic subgroup $C \subseteq G/G_{d-i+1}$. Choose $g \in G$ such that the image of g under $p\colon G/G_{d-i} \to G/G_{d-i+1}$ sends g to a generator of C. Hence $p^{-1}(C)$ is isomorphic to $G_{d-i+1}/G_{d-i} \rtimes_{c_g} \mathbb{Z}$. From the assumptions about G, we get a diffeomorphism $f\colon S \to S$ of a surface S such that $p^{-1}(C)$ is isomorphic to $\pi_1(T_f)$. Since T_f is a 3-manifold, $\pi_1(T_f)$ satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ie). We conclude from Theorem 16.1 (iic) that G/G_{d-i} satisfies the Full Farrell-Jones Conjecture 13.27. ∎

Exercise 16.25. Let G be a group with a filtration $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = G$ such that G_{i-1} is normal in G_i and G_i/G_{i-1} is torsionfree and isomorphic to the fundamental group of a compact manifold of dimension ≤ 3 (possibly with boundary) for all i. Show that the Baum-Connes Conjecture 14.11 with coefficients holds for G.

16.8.18 Normally Poly-Free Groups

A group G is called *poly-free* if there is a finite filtration $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = G$ such that $G_{i-1} \subseteq G_i$ is normal and G_i/G_{i-1} is free (of possibly infinite rank) for $i = 1, 2, \ldots, d$. The Baum-Connes Conjecture 14.11 with coefficients holds for poly-free groups G by Theorem 16.7 (iic), (iid), and (iie).

The Full Farrell-Jones Conjecture 13.27 is not known for all poly-free groups.

We call a group a *normally poly-free group* if there is a finite filtration $\{1\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = G$ such that $G_{i-1} \subseteq G$ is normal and G_i/G_{i-1} is free (of possibly infinite rank) for $i = 1, 2, \ldots, d$.

Theorem 16.26 (The Full Farrell-Jones Conjecture for normally poly-free groups). *A normally poly-free group satisfies the Full Farrell-Jones Conjecture 13.27.*

Proof. This is proved by Brück-Kielak-Wu [163] using the proof for the case of a finitely generated free group extended by $\mathbb{Z}$ due to Bestvina-Fujiwara-Wigglesworth [129]. □

Exercise 16.27. Let $1 \to K \to G \to Q \to 1$ be an extension of groups such that K is the fundamental group of a (possibly non-compact) connected manifold (possibly with boundary) of dimension ≤ 2.

Show that G satisfies the Full Farrell-Jones Conjecture 13.27 if Q does.

16.8.19 Virtually Torsionfree Hyperbolic by Infinite Cyclic Groups

If H is a virtually torsionfree hyperbolic group and $\phi\colon H \to H$ is an automorphism, then $G = H \rtimes_\phi \mathbb{Z}$ satisfies the Full Farrell-Jones Conjecture 13.27. This follows from [129, Proposition 2.2 and Theorem 2.3] using [69, Remark 9.4]. Note that this implies the more general assertion (iid) appearing in Theorem 16.1.

There is no counterexample to the conjecture that every hyperbolic group is virtually torsionfree.

16.8.20 Coxeter Groups

For the definition of and information about Coxeter groups we refer to [274]. Every Coxeter group satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (il) and the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ii).

16.8.21 Right-Angled Artin groups

Every right-angled Artin group can be embedded into a right-angled Coxeter groups as a subgroup of finite index, see [277]. Hence every right-angled Artin group satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (il) and (iia) Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ii) and (iia).

For more information about Right-Angled Artin groups for refer for instance to [219].

16.8.22 Artin groups

The Full Farrell-Jones Conjecture 13.27 and the Baum-Connes Conjecture 14.11 are open for Artin groups, only some partial results are known.

It is an open problem whether every Artin groups admits a cocompact proper isometric action on a complete CAT(0)-space. This is known in some cases, see for instance Haettel [435, 436]. It seems to be also an open question whether Artin groups are A-T-menable.

Even Artin groups of type FC satisfy the Full Farrell-Jones Conjecture 13.27 by Huang-Osajda [489, Corollary], see also [163, Corollary B] and [1001].

The Baum-Connes Conjecture 14.11 with coefficients is proved for some Artin groups by Haettel [436, Corollary C].

16.8.23 Braid Groups

Artin's full braid groups P_n satisfies satisfy both the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ik) and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (ig).

16.8.24 Mapping Class Groups

Let $F^s_{g,r}$ be the orientable compact surface of genus g with r boundary components and s punctures where s punctures means the choice of s pairwise distinct points. Let $\operatorname{Diff}(F^s_{g,r}, \mathrm{rel})$ be the group of orientation preserving diffeomorphisms $F^s_{g,r} \to F^s_{g,r}$ that leave the boundary and the punctures pointwise fixed. Then the *mapping class group* $\Gamma^s_{g,r}$ is defined to be $\pi_0(\operatorname{Diff}(S^s_{g,r}, \mathrm{rel}))$, the group of isotopy classes of such diffeomorphisms. All mapping class groups satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 ih.

16.8.25 Out(F_n)

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for $\mathrm{Out}(F_n)$ for $n \geq 3$. The same statement holds for the Baum-Connes Conjecture.

The group $\mathrm{Out}(F_n)$ is boundary amenable by a result of Bestvina-Guiardel-Horbez [130]. Hence the assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients is rationally injective, see Remark 16.17, and therefore also the Novikov Conjecture holds for any subgroup of $\mathrm{Out}(F_n)$, see Section 14.8. Actually, in [130] also other groups than F_n, for instance torsionfree hyperbolic groups, and right-angled Artin groups, are treated.

At least the rational injectivity of the K-theoretic Farrell-Jones assembly map with coefficients in $\mathbb{Z}$ (disregarding some K_{-1}-term contribution) follows from [659] for $\mathrm{Out}(F_n)$.

16.8.26 Thompson's Groups

Thompson defined the groups F, T, and V in some handwritten notes from 1965. Thompson's group V is the group of right-continuous automorphisms f of $[0,1]$ that map dyadic rational numbers to dyadic rational numbers, that are differentiable except at finitely many dyadic rational numbers, and such that, on each interval on which f is differentiable, f is affine with derivative a power of 2. The group F is the subgroup of V consisting of homeomorphisms. The group T is the subgroup of V consisting of those elements that induce homeomorphisms of the circle where the circle is regarded as $[0,1]$ with 0 and 1 identified. These groups have some unusual properties. It is an open question whether F is amenable. It is known that F is not elementary amenable.

Farley [333] has shown that F, T, and V are a-T-menable and hence satisfy the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (ia).

The Full Farrell-Jones Conjecture 13.27, and actually even the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for F, T, and V.

At least the rational injectivity of the K-theoretic Farrell-Jones assembly map with coefficients in $\mathbb{Z}$ (disregarding some K_{-1}-term contribution) follows from [659] for T using [404].

16.8.27 Helly Groups

The Full Farrell-Jones Conjecture 13.27 is proved for Helly groups by Chalopin-Chepoi-Genevois-Osajda [211, Section 7.5] using [554]. This implies that the Full Farrell-Jones Conjecture 13.27 holds also for weak Garside groups of finite type, see Huang-Osajda [489, Theorem and Corollary].

16.8.28 Groups Satisfying Homological Finiteness Conditions

So far the groups for which we were able to prove the Farrell-Jones Conjecture or the Baum-Connes Conjecture satisfy some geometric conditions, often a reminiscence of non-positive sectional curvature. At least for the K-theoretic Farrell-Jones Conjecture there are results where no geometric conditions but some finiteness conditions are required. The celebrated prototype of such a result is the following theorem due to Boekstedt-Hsiang-Madsen [141].

Theorem 16.28 (Bökstedt-Hsiang-Madsen Theorem). *Let G be a group such that $H_i(G;\mathbb{Z})$ is finitely generated for all $i \geq 0$. Then G satisfies the K-theoretic Novikov Conjecture 13.60, i.e., the assembly map*

$$H_n(BG;\mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is rationally injective for all $n \in \mathbb{Z}$.

This raises the question under which finiteness conditions one can show that the assembly map appearing in the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$ is rationally injective. Recall from Theorem 13.48 that for a group G and a regular ring R the map

$$H_n^G(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_R)\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \xrightarrow{\cong} H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \tag{16.29}$$

is bijective for all $n \in \mathbb{Z}$ after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$.

The source of the map (16.29) has already been computed rationally using equivariant Chern characters in Theorem 12.79

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \big(\mathbb{Q} \otimes_R K_q(RC)\big) \xrightarrow{\cong} \mathbb{Q} \otimes_R H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R). \tag{16.30}$$

By the isomorphisms (16.29) and (16.30), the assembly map appearing in the Farrell-Jones Conjecture 13.1 with coefficients in the regular ring R becomes rationally a map

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \big(\mathbb{Q} \otimes_R K_q(RC)\big) \to \mathbb{Q} \otimes_R K_n(RG). \tag{16.31}$$

So the question above is equivalent to the question whether the map (16.31) is rationally injective.

From now on we consider the special case $R = \mathbb{Z}$. The restriction of the map (16.31) to the summand corresponding to $C = \{1\}$ is rationally the same as the map appearing in in Theorem 16.28. Hence a positive answer to the question above implies Theorem 16.28.

The main result of [659] says that under certain finiteness assumptions, which are for instance satisfied if there is a model for $E_{\mathcal{FIN}}(G)$ of finite type, and certain number theoretic conditions, which are implied by the Leopoldt-Schneider Conjecture, the assembly map (16.31) is rational injective if we ignore the summands for $q = -1$. This summand cannot be detected since topological cyclic homology does not see K_{-1}. Note that Theorem 16.30 just detects the summand for $C = \{1\}$ and does not see the ones for non-trivial C. Nevertheless, the methods and proofs of [659] are based on the ideas of [141].

As an illustration we mention two easy to formulate consequences of the results of [659, Main Theorem 1.13] where the necessary input from number theory is known to be true and therefore does not appear in the assumptions, similar to the situation in Theorem 16.28.

Theorem 16.32 (Rationally injectivity of the colimit map for finite subgroups for the Whitehead group). *Let G be a group. Assume that for every finite cyclic subgroup C of G the abelian groups $H_1(BC_GC;\mathbb{Z})$ and $H_2(BC_GC;\mathbb{Z})$ associated to their centralizers C_GC are finitely generated.*

Then the canonical map

$$\operatorname{colim}_{H\in \mathtt{Sub}G(\mathcal{FIN})} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Wh}(H) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Wh}(G)$$

is injective.

Theorem 16.33 (Eventual injectivity of the rational K-theoretic assembly map for $R = \mathbb{Z}$). *Let G be a group. Assume that there is a finite G-CW-model for $E_{\mathcal{FIN}}(G)$.*

Then there exists an integer $L > 0$ such that the rationalized Farrell-Jones assembly map (16.31) is injective for all $n \geq L$. The bound L only depends on the dimension of $E_{\mathcal{FIN}}(G)$ and on the orders of the finite cyclic subgroups of G.

16.9 Open Cases

Here is a list of interesting groups for which the Full Farrell-Jones Conjecture 13.27 is open in general:

- elementary amenable, amenable, a-T-menable groups;
- $\mathrm{Out}(F_n)$;
- Artin groups;
- Thompson's groups F, V, and T;
- Torsionfree one-relator groups;
- Linear groups;
- Subgroups of almost connected Lie groups;
- Residual finite groups;
- (Bi-)Automatic groups

Here is a list of interesting groups for which the Baum-Connes Conjecture 14.11 with coefficients is open in general:

- Finite dimensional CAT(0)-groups.
- Fundamental groups of closed Riemannian manifolds with non-positive sectional curvature;
- Lattices in almost connected Lie groups, for instance $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$;
- S-arithmetic groups;
- $\mathrm{Out}(F_n)$ for $n \geq 3$;
- Mapping class groups (of higher genus);
- Linear groups;
- Subgroups of almost connected Lie groups;
- Residual finite groups;
- (Bi-)Automatic groups

16.10 How Can We Find Counterexamples?

We are not aware of any group for which the Full Farrell-Jones Conjecture 13.27 is known to be false. The same statement holds for the Baum-Connes Conjecture 14.9 without coefficients and the Novikov Conjecture 9.137.

16.10.1 Is the Full Farrell-Jones Conjecture True for All Groups?

It is hard to believe that the Full Farrell-Jones Conjecture 13.27 is true for all groups since there have been so many prominent conjectures about groups which were open for some time and for which finally counterexamples were found. On the other hand the conjecture is known for so many groups so that we currently have no strategy to find counterexamples, as we will illustrate below.

We have already mentioned that the groups that come from the construction of Arzhantseva-Delzant [39] and yield counterexamples to the Baum-Connes Conjecture 14.11 with coefficients by Higson-Lafforgue-Skandalis [470] are colimits of hyperbolic groups and hence satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia) and (iie).

Baum-Guentner-Willet [105] give a reformulation of the Baum-Connes Conjecture 14.11 with coefficients by introducing a new crossed product, see also [180], for which no counterexamples are known so far.

We have already discussed the problem about the Baum-Connes Conjecture 14.9, which does not occur for the Full Farrell-Jones Conjecture 13.27, that the left hand side of the Baum-Connes Conjecture is functorial under group homomorphism and there is no reason why the right hand side should have this property, see Remark 14.12. The new version of Baum-Guentner-Willet [105] still faces this problem. This sheds additional doubts on the Baum-Connes Conjecture.

16.10.2 Exotic Groups

One does not know of a property of a group for which one may expect that groups with this property are automatically counterexamples to the Full Farrell-Jones Conjecture 13.27 or to the Baum-Connes Conjecture 14.9. Next we list some groups with an exotic property for which the Full Farrell-Jones Conjecture 13.27 is known to be true at least for some groups with this exotic property.

- Finitely generated infinite torsion p-groups
  Given a large enough prime p, there exists an infinite finitely generated group all of whose proper subgroups are finite cyclic groups of order p, see [763]. These groups are lacunary hyperbolic groups and hence satisfy the Full Farrell-Jones Conjecture 13.27, see Subsection 16.8.2.
  Other examples of finitely generated infinite torsion p-groups are mentioned in Subsection 16.8.16.
- Groups with expanders
  There exists a group G that is a colimit of hyperbolic groups and contains appropriate expanders, see [39]. It satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia) and (iie).
- Selfsimilar groups
  See Subsection 16.8.16.
- Infinite torsionfree simple groups
  There exists finitely presented torsionfree simple CAT(0)-groups, see [175,

Corollary 5.4 and Theorem 5.5]. They satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ib).

- Groups which do not possess a finite dimensional model or a model of finite type for BG or $\underline{B}G$
  Examples of such groups satisfying the Full Farrell-Jones Conjecture 13.27 can easily be constructed using Theorem 16.1 (iif).
- Groups with property (T)
  There are hyperbolic groups that have property (T). They satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia).
- Groups for which certain decision problems are unsolvable.
  A lot of groups for which the Full Farrell-Jones Conjecture 13.27 is known and some decision problems such as the isomorphism problem, conjugacy problem and membership problem are unsolvable can be found in Bridson [154].

Also the results about groups with some homological finiteness conditions of Subsection 16.8.28 indicate that the search for counterexamples for the Farrell-Jones Conjecture is not easy.

In order to find counterexamples one seems to need completely new ideas, maybe from random groups or logic. It is unlikely that the counterexample is a concrete group, but rather a group with certain strange properties, for which existence can be shown by abstract methods but not by a concrete construction.

It is probably easier to find counterexamples to surjectivity than to injectivity.

16.10.3 Infinite Direct Products

Nothing is known about infinite products. It would be very interesting if one can show that for family of groups $\{G_i \mid i \in I\}$ (with infinite I) the Full Farrell-Jones Conjecture 13.27 is true for the direct product $\prod_{i\in I} G_i$ if it holds for each G_i. (Note that the corresponding statement is true for the direct sum $\bigoplus_{i\in I} G_i$ by Theorem 16.1 (iib) and (iie).) In view of Theorem 16.1 (iia) this would imply that the Full Farrell-Jones Conjecture 13.27 is stable under inverse limits over directed systems of groups. This would have the immediate consequence that the Full Farrell-Jones Conjecture 13.27 is true for all residually finite groups. On the other hand it may be worthwhile to look at infinite direct products in order to find a counterexample.

For this discussion see also Lemma 15.101.

16.10.4 Exotic Aspherical Closed Manifolds

One may look also for counterexamples to one of the conjecture which follow from the Full Farrell-Jones Conjecture 13.27, for instance to the Borel Conjecture 9.160. There are indeed aspherical closed manifolds with unusual properties, but the fundamental groups of some of them do satisfy the Full Farrell-Jones Conjecture 13.27 and hence the Borel Conjecture. Note that we have already discussed aspherical closed manifolds with exotic properties in Subsection 9.15.1.

Davis constructed for every $n \geq 4$ aspherical closed manifolds of dimension n whose universal covering is not homeomorphic to Euclidean space [273, Corollary 15.8]. In particular, these manifolds do not support metrics of non-positive sectional curvature. The fundamental groups of these examples are finite index subgroups of Coxeter groups W. Thus they satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (il) and (iia). In particular these manifolds are indeed topologically rigid, provided that $n \geq 5$.

Davis and Januszkiewicz [276, Theorem 5b.1] used Gromov's hyperbolization technique to construct for every $n \geq 5$ an aspherical closed n-dimensional manifold M such that the universal covering $\widetilde{M}$ is a finite dimensional CAT(0)-space whose fundamental group at infinite is non-trivial. In particular, these universal covers are not homeomorphic to Euclidean space. Because these examples are in addition non-positively curved polyhedron, their fundamental groups are finite dimensional CAT(0)-groups. There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [220] and produces an aspherical closed manifold M whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic. The fundamental groups of these manifolds M satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia) and (ib). In particular these manifolds M are topologically rigid.

Davis-Januszkiewicz [276, Theorem 5a.1 and Corollary 5a.4] constructed a 4-manifold N such that $\pi_1(N)$ is a finite dimensional CAT(0)-group and $N \times T^k$ for $k \geq 1$ is not homotopy equivalent to a PL-manifold. Since $\pi_1(N \times T^k)$ is a finite dimensional CAT(0)-group and $\dim(N \times T^k) \geq 5$ for $k \geq 1$, the manifolds $N \times T^k$ for $k \geq 1$ are topologically rigid by Theorem 16.1 (ib).

Davis-Fowler-Lafont [275] constructed using the work of Manolescu [691, 690] non-triangulable aspherical closed manifolds with hyperbolic fundamental group in all dimensions ≥ 6. In particular these manifolds M are topologically rigid since hyperbolic groups satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ia).

Belegradek [114, Corollary 5.1], and Weinberger, see [272, Section 13], proved that for every $n \geq 4$ there is an aspherical closed manifold of dimension n whose fundamental group has an unsolvable word problem. Note that a finitely presented group with unsolvable word problem is not a CAT(0)-group, not hyperbolic, not automatic, not asynchronously automatic, not residually finite, and not linear over any commutative ring, see [114, Remark 5.2]. So we do not know whether it satisfies the Full Farrell-Jones Conjecture 13.27 or the Borel Conjecture 9.160.

The proofs of the results above are based on the *reflection group trick* as it appears for instance in [272, Sections 8,10 and 13]. It can be summarized as follows.

Theorem 16.34 (Reflection group trick). *Let G be a group that possesses a finite model for BG. Then there is an aspherical closed manifold M and a map $i\colon BG \to M$ and $r\colon M \to BG$ such that $r \circ i = \mathrm{id}_{BG}$.*

An interesting immediate consequence of the reflection group trick is that many well-known conjectures about groups hold for every group that possesses a finite model for BG if and only if it holds for the fundamental group of every aspherical closed manifold, see also [272, Sections 11].

Exercise 16.35. Suppose that Farrell-Jones Conjecture 6.53 for torsionfree groups and regular rings holds for the fundamental group of any aspherical closed manifold. Show that it then holds for all groups G with a finite model for BG.

Prove the analogous statement for the L-theoretic Farrell-Jones Conjecture 9.114 for torsionfree groups.

The upshot of the discussion is that one does not know of a property of aspherical closed manifolds, such as being not triangulable, for which one may expect that the Borel Conjecture 9.160 automatically fails if this property is satisfied.

16.10.5 Some Results Which Hold for All Groups

Here is a result which holds for all (discrete) groups, is non-trivial and related to the Farrell-Jones Conjecture. Let $i\colon H \to G$ be the inclusion of a normal subgroup $H \subset G$. It induces a homomorphism $i_0\colon \mathrm{Wh}(H) \to \mathrm{Wh}(G)$. The conjugation actions of G on H and on G induce G-actions on $\mathrm{Wh}(H)$ and on $\mathrm{Wh}(G)$ which turns out to be trivial on $\mathrm{Wh}(G)$. Hence i_0 induces homomorphisms

$$i_1\colon \mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) \to \mathrm{Wh}(G); \tag{16.36}$$

$$i_2\colon \mathrm{Wh}(H)^G \to \mathrm{Wh}(G). \tag{16.37}$$

Theorem 16.38 (Rational injectivity of $\mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) \to \mathrm{Wh}(G)$ for normal finite $H \subseteq G$). *Let $i\colon H \to G$ be the inclusion of a normal finite subgroup H into an arbitrary group G. Then the maps i_1 and i_2 defined in* (16.36) *and* (16.37) *have finite kernel.*

Proof. See [635, Theorem 9.38 on page 354]. □

We omit the details of the proof that the result of Theorem 16.38 can be also deduced from the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$.

Exercise 16.39. Let G be a group with vanishing Whitehead group. Show that each element in the center has order order 1, 2, 3, 4, or 6.

We have already stated a more advanced detection result for $\mathbb{Q}$ and $\mathbb{C}$ as coefficients, see Theorem 6.78, which also holds for all groups. Recall also Theorem 16.32, which requires only very mild conditions on the group G, and Theorem 16.33, which holds for all groups.

Another non-trivial consequence of the Farrell-Jones Conjecture which holds for all groups has been discussed in Remark 2.97.

Furthermore, Yu [1009, Theorem 1.1], see also Cortinas-Tartaglia [247], proved that the K-theoretic assembly map $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{S}}) \to K_n(\mathcal{S}G)$ appearing in the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R is rationally injective for every group G, provided that R is the ring $\mathcal{S}$ of Schatten class operators of an infinite dimensional separable Hilbert space.

16.11 Notes

There are groups for which the Full Farrell-Jones Conjecture 13.27 is not known to be true but weaker versions of it have been proved. For example, the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R is known if R is regular and G belongs to the class $\mathcal{CL}'$ described in [74, Definition 0.10]. The class $\mathcal{CL}'$ contains for instance all torsionfree 1-relator groups.

The class of groups for which the L-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2 is analyzed in [458, Proposition 5.2.2 and Lemma 5.2.3]; actually the more general fibered version is treated. It contains for instance all elementary amenable groups. The result and its proof is analogous to Theorem 16.5.

A proof of the Full Farrell-Jones Conjecture 13.27 for finite-dimensional CAT(0)-groups has been extended to a larger class of group which also contains all hyperbolic groups by Kasprowski-Rüping [554, Theorem 6.1]. In particular they prove it for all groups acting properly and cocompactly on a finite products of hyperbolic graphs, see [554, Theorem 1.1] as already mentioned in Theorem 16.1 (in).

The bijectivity of the algebraic K-theoretic assembly map for certain coefficients coming from C^*-algebras is proved by Cortinas-Tartaglia [245, Corollary 1.5] for a-T-menable groups G by reducing it to the Baum-Connes Conjecture.

Gonzalez-Acuna-Gordon-Simon [412, Theorem 5.6, Corollary 5.7, Theorem 5.8] show that the problem whether the projective class group, the Whitehead group, or the L-groups of a group is trivial, cannot be decided. So it is possible that the problem, whether a group G satisfies the Farrell-Jones Conjecture holds, cannot be decided.

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Chapter 17
Guide for Computations

17.1 Introduction

One major goal is to compute K- and L-groups such as $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$, and $K_n(C_r^*(G))$. Assuming that the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R, the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R, or the Baum-Connes Conjecture 14.9 hold for G, this reduces to the computation of the left hand side of the corresponding assembly maps, namely, of $H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$, $H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty \rangle})$, or $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathsf{TOP}}) = K_n^G(E_{\mathcal{FIN}}(G))$. This is much easier since here we can use standard methods from algebraic topology. The main general tools are the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, the p-chain spectral sequence, see Theorem 12.50, and equivariant Chern characters, see Theorem 12.58. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a general reasonable answer after rationalization, but integral computations have only been done case by case, and there seems to be no general pattern for a general answer. Often the key is a good understanding of how one can built $E_{\mathcal{FIN}}(G)$ from EG and how one can built $E_{\mathcal{VCY}}(G)$ from $E_{\mathcal{FIN}}(G)$. These passages have already been studied in Theorems 11.32 and 11.36.

17.2 *K*- and *L*-Groups for Finite Groups

For the computations of $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$, $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty \rangle})$, and $H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathsf{TOP}}) = K_n^G(E_{\mathcal{FIN}}(G))$, one needs to understand $K_n(RH)$, $L_n^{\langle -\infty \rangle}(RH)$, and $K_n(C_r^*(H))$ for finite groups H since these are the values of $H_n^G(G/H;\mathbf{K}_R)$, $H_n^G(G/H;\mathbf{L}_R^{\langle -\infty \rangle})$, and $H_n^G(G/H;\mathbf{K}^{\mathsf{TOP}}) = K_n^G(G/H)$ for homogeneous spaces G/H for finite subgroups $H \subseteq G$.

For a finite group G we have given information about $K_0(\mathbb{Z}G)$ in Section 2.12, about $K_1(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ in Section 3.12, about $K_n(\mathbb{Z}G)$ for $n \le -1$ in Example 4.12, Section 4.5 and Example 5.15, about $K_2(\mathbb{Z}G)$ and $\mathrm{Wh}_2(G)$ in Section 5.8, about $L_n^{\langle j \rangle}(\mathbb{Z}G)$ in Section 9.22 and about $K_n(C_r^*G)$ and $KO_n(C_r^*(G;\mathbb{R}))$ in Section 10.9.

Let us summarize. There is a complete formula for $K_{-1}(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G) = 0$ for $n \le -2$. One has a good understanding of $\mathrm{Wh}(G)$. A complete computation of $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ for arbitrary primes p is out of reach. A complete computation of $K_n(\mathbb{Z})$ is not known for all $n \in \mathbb{Z}$. We have already mentioned Borel's formula for $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$ in Theorem 6.24. The L-groups of $\mathbb{Z}G$ are pretty well understood for finite groups G. The values of $K_n(C_r^*(G))$ and $K_n(C_r^*(G;\mathbb{R}))$ are explicitly known for finite groups G and are in the complex case in contrast to the real case always torsionfree.

17.3 The Passage from $\mathcal{FIN}$ to $\mathcal{VCY}$

In the Baum-Connes setting it is enough to consider the family $\mathcal{FIN}$. In the Farrell-Jones Conjecture we have to pass from $\mathcal{FIN}$ to $\mathcal{VCY}$. This passage has been discussed on detail already in Section 13.8. We get splittings

$$\begin{aligned} &H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{K}_R\big) \\ &\qquad\cong H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{K}_R\big) \oplus H_n^G\big(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}}\big), \end{aligned}$$

and under mild K-theoretic assumptions

$$\begin{aligned} &H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}\big) \\ &\qquad\cong H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}\big) \oplus H_n^G\big(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}\big). \end{aligned}$$

We have also explained in Theorem 13.44 that in K-theory it suffices to replace $\mathcal{VCY}$ by $\mathcal{VCY}_I$ and in Theorem 13.57 that in L-theory there is no difference between $\mathcal{FIN}$ and $\mathcal{VCY}_I$.

If we are only interested in rational information, then there is no difference between $\mathcal{FIN}$ and $\mathcal{VCY}$ when we are dealing with the algebraic K-theory of groups rings RG for regular rings R, see Theorem 13.48, and when we are dealing with L-theory, see Theorem 13.59 (i).

For L-theory the Tate cohomology of the K-theory is important when one is comparing different decoration, see Subsection 9.10.4. Therefore the next result is sometimes useful.

In [670, Definition 8.5] the condition is formulated that the infinite virtually cyclic subgroups of type I of G are orientable. This condition is automatically satisfied if one of the following conditions is satisfied, see [670, Lemma 8.7 and Lemma and 8.8], [360, Theorem 9.1].

- Let G is hyperbolic and all infinite virtually cyclic subgroups of G are of type I;
- G is a torsionfree hyperbolic group;
- G is a CAT(0)-group which does not contain the Klein bottle group $\mathbb{Z} \rtimes \mathbb{Z}$ as subgroup and all of whose infinite virtually cyclic subgroups of G are of type I;
- G is a torsionfree CAT(0)-group that does not contain the Klein bottle group $\mathbb{Z} \rtimes \mathbb{Z}$ as subgroup.

Lemma 17.1. *Let R be a ring with involution. Suppose that the infinite virtually cyclic subgroups of type I of G are orientable.*

Then for all $j, n \in \mathbb{Z}$ the obvious map between the Tate cohomology groups

$$\widehat{H}^n(\mathbb{Z}/2; H_j^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R)) \xrightarrow{\cong} \widehat{H}^n(\mathbb{Z}/2; H_j^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R))$$

is an isomorphism.

Proof. It suffices to show

$$\widehat{H}^n(\mathbb{Z}/2; H_j^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R)) = 0.$$

This is a direct consequence of [670, Theorem 0.1 and Theorem 0.2]. □

17.4 Mayer-Vietoris Sequences and Wang Sequences

We have explained in Section 15.7 how an action of G on a tree T yields a long exact sequence involving the isotropy groups. In particular we get for an amalgamated free product a Mayer-Vietoris sequence and for a semidirect product with $\mathbb{Z}$, or, more generally, for HNN-extension, a long exact Wang sequence.

We want to illustrate this in the case $G = \mathrm{SL}_2(\mathbb{Z})$. We have already explained in Subsection 11.6.11 that $\mathrm{SL}_2(\mathbb{Z})$ is the free amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. Since the inclusion $\mathbb{Z}/2 \to \mathbb{Z}/6$ is split injective, we obtain from the long exact sequence appearing in Theorem 15.27 (ii) for every equivariant homology theory $\mathcal{H}_*^?$ an isomorphism

$$\mathcal{H}_n^{\mathbb{Z}/4}(\{\bullet\}) \oplus \operatorname{coker}\bigl(\mathcal{H}_n^{\mathbb{Z}/2}(\{\bullet\}) \to \mathcal{H}_n^{\mathbb{Z}/6}(\{\bullet\})\bigr) \xrightarrow{\cong} \mathcal{H}_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\,\mathrm{SL}_2(\mathbb{Z})).$$

This yields isomorphisms

$$\begin{aligned} K_n(C_r^*(\mathbb{Z}/4;\mathbb{C})) \oplus \operatorname{coker}\bigl(K_n(C_r^*(\mathbb{Z}/2;\mathbb{C})) \to K_n(C_r^*(\mathbb{Z}/6;\mathbb{C}))\bigr)& \\ \cong K_n(C_r^*(\mathrm{SL}_2(\mathbb{Z});\mathbb{C}));& \end{aligned}$$

$$\begin{aligned} KO_n(C_r^*(\mathbb{Z}/4;\mathbb{R})) \oplus \operatorname{coker}\bigl(KO_n(C_r^*(\mathbb{Z}/2;\mathbb{R})) \to KO_n(C_r^*(\mathbb{Z}/6;\mathbb{R}))\bigr)& \\ \cong KO_n(C_r^*(\mathrm{SL}_2(\mathbb{Z});\mathbb{R}));& \end{aligned}$$

$$\begin{aligned} L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/4])[1/2] \oplus \operatorname{coker}\bigl(L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/2])[1/2] \to L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/6])[1/2]\bigr)& \\ \cong L_n^{\langle -\infty\rangle}(R[\mathrm{SL}_2(\mathbb{Z})])[1/2];& \end{aligned}$$

$$L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/4]) \oplus \operatorname{coker}\bigl(L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/2]) \to L_n^{\langle -\infty\rangle}(R[\mathbb{Z}/6])\bigr) \\ \cong H_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\,\mathrm{SL}_2(\mathbb{Z}); \mathbf{L}_R^{\langle -\infty\rangle});$$

$$K_n(R[\mathbb{Z}/4])\oplus\operatorname{coker}\bigl(K_n(R[\mathbb{Z}/2]) \to K_n(R[\mathbb{Z}/6])\bigr) \cong H_n^{\mathrm{SL}_2}(\mathbb{Z})(\underline{E}\,\mathrm{SL}_2(\mathbb{Z}); \mathbf{K}_R).$$

Since $\mathrm{SL}_2(\mathbb{Z})$ is hyperbolic, we get from Theorem 11.36, Theorem 13.44, and Theorem 13.57 isomorphisms

$$H_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\,\mathrm{SL}_2(\mathbb{Z}); \mathbf{L}_R^{\langle -\infty\rangle}) \oplus \bigoplus_V H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(R[\mathrm{SL}_2(\mathbb{Z})]);$$

$$H_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\,\mathrm{SL}_2(\mathbb{Z}); \mathbf{K}_R) \oplus \bigoplus_V H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{K}_R), \cong K_n(R[\mathrm{SL}_2(\mathbb{Z})])$$

where V runs through a complete system of representatives of infinite virtually cyclic subgroups of type II in the L-theory case and through a complete system of representatives of infinite virtually cyclic subgroups of type I in the K-theory case.

Since every infinite cyclic subgroup of type I of $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}/2$, we conclude from Theorem 4.3, and Theorem 6.21 that $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{K}_{\mathbb{Z}})$ vanishes for $n \le 1$ for any infinite virtually cyclic subgroup of type I of $\mathrm{SL}_2(\mathbb{Z})$. Hence we get for $n \le 1$ an isomorphism

$$K_n(\mathbb{Z}[\mathbb{Z}/4]) \oplus \operatorname{coker}\bigl(K_n(\mathbb{Z}[\mathbb{Z}/2]) \to K_n(\mathbb{Z}[\mathbb{Z}/6])\bigr) \cong K_n(\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]).$$

We conclude from Theorem 2.112 (i), Theorem 3.115, Theorem 3.116 (iv), Example 4.10, and Theorem 4.22 (i) and (v) that $\mathrm{Wh}(\mathrm{SL}_2(\mathbb{Z}))$, $\widetilde{K}_0(\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})])$, and $K_n(\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})])$ for $n \le -2$ vanish and the inclusion $\mathbb{Z}/6 \to \mathrm{SL}_2(\mathbb{Z})$ induces an isomorphism $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z} \to K_{-1}(\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})])$.

Exercise 17.2. Prove

$$K_n(C_r^*(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C})) \cong \begin{cases} \mathbb{Z}^8 & n \text{ even;} \\ 0 & n \text{ odd,} \end{cases}$$

and

$$KO_n(C_r^*(\mathrm{SL}_2(\mathbb{Z}); \mathbb{R})) \cong \begin{cases} \mathbb{Z}^5 & n \equiv 0 \mod (8); \\ (\mathbb{Z}/2)^2 & n \equiv 1 \mod (8); \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}^3 & n \equiv 2 \mod (8); \\ \{0\} & n \equiv 3 \mod (8); \\ \mathbb{Z}^5 & n \equiv 4 \mod (8); \\ \{0\} & n \equiv 5 \mod (8); \\ \mathbb{Z}^3 & n \equiv 6 \mod (8); \\ \{0\} & n \equiv 7 \mod (8). \end{cases}$$

Exercise 17.3. Let D_8 be the dihedral group of order eight and C be its center, which is a group of order two. Let G be the group $D_8 *_C D_8$. Prove

$$K_0(\mathbb{C}G) \cong \mathbb{Z}^8 \oplus \mathbb{Z}/2;$$
$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^8 \oplus \mathbb{Z}/2 & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

17.5 Rational Computations for Infinite Groups

Next we state what is known rationally about the K- and L-groups of an infinite (discrete) group, provided the Farrell-Jones Conjectures 13.1 or 13.4 or the Baum-Connes Conjecture 14.9 holds.

17.5.1 Rationalized Algebraic K-Theory

The next result follows from Theorem 12.79 and Theorem 13.48. For $R = \mathbb{Z}$ see also Grunewald [427, Corollary on page 165].

Theorem 17.4 (Rational computations of $K_n(RG)$ for regular R). *Let R be regular ring, e.g., R is $\mathbb{Z}$. Suppose that the group G satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R.*

Then we have for all $n \in \mathbb{Z}$ a natural isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in J} H_p(C_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \cdot \big(\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC)\big) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG)$$

where we use the notation from Theorem 12.79.

Computations of $K_q(RC)$ as $\mathbb{Z}[\mathrm{aut}(C)]$-module for finite cyclic groups C and $R = \mathbb{Z}$ or R a field of characteristic zero can be found in [777].

Exercise 17.5. If in Theorem 17.4 we drop the condition that R is regular, show that then we still know that the map appearing there is split injective.

Example 17.6 (A Formula for $K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$ for R the ring of integers in an algebraic number field). Let R be the ring of integers in an algebraic number field, e.g., $R = \mathbb{Z}$. Note that then R is regular by Theorems 2.21 and 2.23. Suppose that the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R is true for G. Then Conjecture 4.20 is true by Theorem 13.62. Hence we obtain from Theorem 2.104, Theorem 4.22 (i), and Theorem 17.4 an isomorphism

$$\widetilde{K}_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{(C) \in (\mathcal{FCY})} H_1(BC_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that $\widetilde{K}_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains only contributions from $K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ for finite cyclic subgroups $C \subseteq G$.

Exercise 17.7. Suppose that the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{Z}$ is true for G and that any element of finite order has prime power order. Show that then $\widetilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes.

17.5.2 Rationalized Algebraic *L*-Theory

The next result follows from Theorem 9.106, Theorem 12.79, and Theorem 13.59 (i).

Theorem 17.8 (Rational computation of algebraic L-theory). *Suppose that the group G satisfies the L-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2.*

Then we get for all $j \in \mathbb{Z}, j \leq 2$ and $n \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in J} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot \big(\mathbb{Q} \otimes_{\mathbb{Z}} L_q^{\langle j \rangle}(RC)\big) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(RG)$$

where we use the L-theoretic version of the notation of Theorem 12.79.

Exercise 17.9. Let F be a finite group of odd order. Put $G = F \wr \mathbb{Z}$. Show for all decorations $j \in \mathbb{Z}, j \leq 2$

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Q} & n \equiv 1 \mod 4; \\ \{0\} & n \equiv 3 \mod 4. \end{cases}$$

17.5.3 Rationalized Topological *K*-Theory

The next result is taken from [636, Theorem 0.7]. Let Λ^G be the ring $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ that is obtained from $\mathbb{Z}$ by inverting the orders of the finite subgroups of G.

Theorem 17.10 (Rational computation of topological K-theory). *Suppose that the group G satisfies the Baum-Connes Conjecture 14.9.*

Then there is an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} K_p(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \Theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G$$

where we use the notation of Theorem 12.79.

If we tensor with $\mathbb{Q}$, we get an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BC_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}. \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

17.5.4 The Complexified Comparison Map from Algebraic to Topological *K*-theory

If we consider $R = \mathbb{C}$ as coefficient ring and apply $- \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $- \otimes_{\mathbb{Z}} \mathbb{Q}$, the formulas simplify. Suppose that G satisfies the Baum-Connes Conjecture 14.9 and K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{C}$. Recall that $\operatorname{con}(G)_f$ is the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $\langle g \rangle$ the cyclic subgroup generated by g.

Then we get the following commutative square whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homomorphisms, see [634, Theorem 0.5],

$$\begin{tikzcd}
\bigoplus_{p+q=n} \bigoplus_{(g)\in\operatorname{con}(G)_f} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) \arrow[r, "\cong"] \arrow[d] & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d] \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in\operatorname{con}(G)_f} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathsf{TOP}}(\mathbb{C}) \arrow[r, "\cong"] & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}.
\end{tikzcd}$$

Suslin [917, Theorem 4.9] has proved that the algebraic K-theory of $\mathbb{C}$ in dimensions $2n$ for $n \geq 1$ a unique divisible group and hence admits no nontrivial map to $\mathbb{Z}$. This implies that the canonical map from the algebraic K-theory of $\mathbb{C}$ to the topological K-theory of $\mathbb{C}$ is trivial in all dimensions except dimension zero where it is a bijection. Thus rationally we understand by the diagram above the comparison map from algebraic K-theory of the complex group ring to the topological K-theory of the group C^*-algebra provided that G satisfies the Baum-Connes Conjecture 14.9 and K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring $\mathbb{C}$.

Remark 17.11 (Separation of Variables). In Theorems 17.4, 17.8, and 17.10 and in Subsection 17.5.4 we see a principle which we call *separation of variables*: There is a group homology part that is independent of the coefficient ring R and the K- or L-theory under consideration and a part depending only on the values of the theory under consideration on RC or $C_r^*(C)$ for all finite cyclic subgroups $C \subseteq G$.

17.6 Integral Computations for Infinite Groups

As mentioned above, no general pattern for integral calculations is known or expected. We give some examples where computations are possible and which shall illustrate the techniques.

17.6.1 Groups Satisfying Conditions ($\underline{\text{M}}$) and ($\underline{\text{NM}}$)

We mention at least one situation where a certain class of groups can be treated systematically. Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ that are maximal in $\mathcal{FIN}$.

Consider the following assertions concerning G:

($\underline{\text{M}}$) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

($\underline{\text{NM}}$) $M \in \mathcal{MFIN}, M \neq \{1\} \implies N_G M = M$.

Denote by $\widetilde{K}_n(C_r^*(G))$ the cokernel of $K_n(C_r^*(\{1\})) \to K_n(C_r^*(G))$, by $\widetilde{KO}_n(C_r^*(G;\mathbb{R}))$ the cokernel of $KO_n(C_r^*(\{1\};\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R}))$, and by $\overline{L}_n^{\langle j\rangle}(RG)$ the cokernel of $L_n^{\langle j\rangle}(R) \to L_n^{\langle j\rangle}(RG)$. This coincides with $\widetilde{L}_n^{\langle j\rangle}(R)$, that is defined for any ring R with involution to be the cokernel of $L_n^{\langle j\rangle}(\mathbb{Z}) \to L_n^{\langle j\rangle}(R)$ if $R = \mathbb{Z}G$, but not in general if we replace $\mathbb{Z}$ by other coefficients. Recall that $\mathrm{Wh}_n^R(G)$ is the $(n-1)$-th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \to \mathbf{K}(RG)$. The next result is taken from [266, Theorem 5.1], except for assertion (ii). It is a direct consequence of the existence of a nice model for $E_{\mathcal{FIN}}(G)$, see Theorem 11.32, the long exact sequence (12.84) and Lemma 12.18 (ii). Recall that we abbreviate $\underline{E}G = E_{\mathcal{FIN}}(G)$ and $\underline{B}G = G\backslash E_{\mathcal{FIN}}(G)$.

Theorem 17.12 (Fundamental exact sequences for groups satisfying conditions ($\underline{\text{M}}$) and ($\underline{\text{NM}}$)). *Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ. Suppose that the group G satisfies conditions ($\underline{\text{M}}$) and ($\underline{\text{NM}}$). Let $\{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal finite subgroups of G. Then:*

(i) If G satisfies the Baum-Connes Conjecture 14.9, then there is a short exact sequence of topological K-groups

$$0 \to \bigoplus_{i \in I} \widetilde{K}_n(C_r^*(M_i)) \to K_n(C_r^*(G)) \to K_n(\underline{B}G) \to 0$$

where the maps $\widetilde{K}_n(C_r^*(M_i)) \to K_n(C_r^*(G))$ *are induced by the inclusions* $H \to G$.

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$;

(ii) *If* G *satisfies the Baum-Connes Conjecture 14.9, then there is a long exact sequence of topological* K*-groups*

$$\cdots \to KO_{n+1}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{KO}_n(C_r^*(M_i;\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R})) \\ \to KO_n(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{KO}_{n-1}(C_r^*(M_i;\mathbb{R})) \to \cdots$$

where the maps $\widetilde{KO}_n(C_r^*(H;\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R}))$ *are induced by the inclusions* $H \to G$.

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$, *more precisely the* Λ*-homomorphism* $KO_n(C_r^*(G;\mathbb{R})) \otimes_{\mathbb{Z}} \Lambda \to KO_n(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$ *is split surjective;*

(iii) *Suppose that every infinite virtually cyclic subgroup of* G *is of type* I, *and* G *satisfies the* L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution* R.

Then for all $n \in \mathbb{Z}$ *there is an exact sequence*

$$\ldots \to H_{n+1}(\underline{B}G;\mathbf{L}^{\langle -\infty \rangle}(R)) \to \bigoplus_{i \in I} \overline{L}_n^{\langle -\infty \rangle}(RM_i) \\ \to L_n^{\langle -\infty \rangle}(RG) \to H_n(\underline{B}G;\mathbf{L}^{\langle -\infty \rangle}(R)) \to \ldots$$

where the maps $\overline{L}_n^{\langle -\infty \rangle}(RH) \to L_n^{\langle -\infty \rangle}(RG)$ *are induced by the inclusions* $H \to G$.

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$, *more precisely*

$$L_n^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \Lambda \to H_n(\underline{B}G;\mathbf{L}^{\langle -\infty \rangle}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of Λ*-modules;*

(iv) *If* G *satisfies the* K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring* R, *then there is for* $n \in \mathbb{Z}$ *an isomorphism*

$$H_n(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G);\mathbf{K}_R) \oplus \bigoplus_{i \in I} \mathrm{Wh}_n^R(M_i) \xrightarrow{\cong} \mathrm{Wh}_n^R(G)$$

where the maps $\mathrm{Wh}_n^R(H) \to \mathrm{Wh}_n^R(G)$ *are induced by the inclusions* $H \to G$.

Remark 17.13 (Role of $\underline{B}G$). Theorem 17.12 illustrates that for such computations a good understanding of the geometry of the orbit space $\underline{B}G$ is necessary. This can be hard to figure out, even for at the first glance nice groups with pleasant geometric properties such as crystallographic groups. In general $\underline{B}G$ can be very complicated, see Theorem 11.62.

Remark 17.14. In [266] it is explained that the following classes of groups do satisfy the assumption appearing in Theorem 17.12 and what the conclusions are in the case $R = \mathbb{Z}$.

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
- Fuchsian groups F;
- One-relator groups G.

Theorem 17.12 is generalized in [639], in order to treat for instance the semidirect product of the discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. For this group $\underline{B}G$ is S^3.

Exercise 17.15. Let G be a group satisfying conditions ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$) appearing in Subsection 11.7.1. Show that then we obtain for any ring R an isomorphism

$$\bigoplus_{(V)} H_n^V(E_{\mathcal{FIN}}(V) \to \{\bullet\}; \mathbf{K}_R) \xrightarrow{\cong} H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

where (V) runs through the conjugacy classes of maximal infinite virtual cyclic subgroups.

In general the L-theoretic Farrell-Jones assembly map is not an isomorphism if one replace the decoration $\langle -\infty \rangle$ by the decoration p, h, or s, see Remark 13.9. This can be very unpleasant since for applications one needs the decorations s or h. The situation is better when G is torsionfree, as explained in Theorem 9.106. Here is a situation where the situation is still optimal although G is not torsionfree.

Theorem 17.16. *Consider a group G with an orientation character $w \colon G \to \{\pm 1\}$ satisfying the following conditions:*

- *Conditions (M) and (NM) are satisfied;*
- *All virtually cyclic subgroups are of type I;*
- *The infinite virtually cyclic subgroups of type I of G are orientable in the sense of [670, Definition 8.5];*
- *G satisfies the K-theoretic Farrell-Jones Conjectures 13.1 and the L-theoretic Farrell-Jones Conjectures 13.1 with coefficients in $\mathbb{Z}$.*

Then:

(i) The assembly maps

$$H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \to L_n^{\langle j \rangle}(\mathbb{Z}G, w)$$

are bijective for all $n \in \mathbb{Z}$ and $j \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$;

(ii) Let $\mathcal{M} = \{M_i \mid i \in I\}$ be a complete set of representative of the conjugacy classes of maximal finite subgroups of G. There are canonical isomorphisms for $n \in \mathbb{Z}$ and $j \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$

$$\bigoplus_{i \in I} H_n^{M_i}(EM_i \to M_i/M_i; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}) \xrightarrow{\cong} H_n^G(EG \to G/G; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle})$$

Proof. (i) In the sequel of this proof we omit the orientation characters from the notaion.

We conclude from the L-theoretic Farrell-Jones Conjecture 13.4 and Theorem 13.57 that the assembly map

$$H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{-\infty}\big) \to L_n^{-\infty}(\mathbb{Z}G)$$

is bijective.

Consider $j \in \{-2, -3, \ldots\}$. There is a commutative diagram

$$\begin{tikzcd}
H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big) \arrow[r] \arrow[d, "\cong"'] & L_n^{\langle j+1 \rangle}(\mathbb{Z}G) \arrow[d, "\cong"] \\
H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] & L_n^{\langle j \rangle}(\mathbb{Z}G)
\end{tikzcd}$$

where the vertical maps are change of decoration maps, which can be implemented on the level of spectra, and the horizontal arrows are the assembly maps. Since $\widetilde{K}_j(\mathbb{Z}H) = 0$ holds for finite groups and $j \leq -2$ by Theorem 4.22 (i), we conclude from the Rothenberg sequences of Subsection 9.10.4 that the map $L_n^{\langle j+1 \rangle}(\mathbb{Z}H) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}H)$ is bijective for $j \leq -2$ and $n \in \mathbb{Z}$. By the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, the left vertical arrow is bijective. We conclude from the K-theoretic Farrell-Jones Conjectures 13.1 that $\widetilde{K}_j(\mathbb{Z}G) = 0$ holds for $j \leq -2$, see Conjecture 4.20 and Theorem 13.62 (vi). The Rothenberg sequence (9.102) implies that the right vertical arrow is bijective.

There is by construction a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{j \to -\infty} H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] \arrow[d, "\cong"'] & \operatorname{colim}_{j \to -\infty} L_n^{\langle j \rangle}(\mathbb{Z}G) \arrow[d, "\cong"] \\
H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\big) \arrow[r] & L_n^{\langle -\infty \rangle}(\mathbb{Z}G)
\end{tikzcd}$$

with bijective vertical arrows. This implies that

$$H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle j \rangle}\big) \to L_n^{\langle j \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$ and $j \in \{-1, -2, \ldots\} \amalg \{-\infty\}$.

It remains to show for $j \in \{1, 0, -1\}$ that the map above is bijective for j and all $n \in \mathbb{Z}$ if it is bijective for $j-1$ and all $n \in \mathbb{Z}$.

The Tate spectra constructions of [984] imply that there is a covariant $\mathsf{GROUPOIDS}$-spectrum $\mathbf{T}_{\mathbb{Z}}^{\langle j \rangle}$ such that there is a cofibration sequence of covariant $\mathsf{GROUPOIDS}$-spectra $\mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle} \to \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle} \to \mathbf{T}_{\mathbb{Z}}^{\langle j \rangle}$ such that for every group G and subgroup $H \subseteq G$ the associated long exact sequence

$$\begin{aligned} \cdots \to H_n^G(G/H, \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}) \to H_n^G(G/H, \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}) \to H_n^G(G/H; \mathbf{T}_{\mathbb{Z}}^{\langle j \rangle}) \\ H_{n-1}^G(G/H, \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}) \to H_{n-1}^G(G/H, \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}) \to \cdots \end{aligned}$$

can be identified with the Rothenberg sequence of (9.102)

$$\begin{aligned} \ldots \to L_n^{\langle j+1 \rangle}(\mathbb{Z}H) \to L_n^{\langle j \rangle}(\mathbb{Z}H) \to \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j^{\mathbb{Z}}(H)) \\ \to L_{n-1}^{\langle j+1 \rangle}(\mathbb{Z}H) \to L_{n-1}^{\langle j \rangle}(\mathbb{Z}H) \to \ldots. \end{aligned}$$

Next we consider the commutative diagram

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] \arrow[d] & L_n^{\langle j \rangle}(\mathbb{Z}G) \arrow[d] \\ H_n^G\big(E_{\mathcal{FIN}}(G); \mathbf{T}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] \arrow[d] & \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j^{\mathbb{Z}}(G)) \arrow[d] \\ H_{n-1}^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j+1 \rangle}\big) \arrow[r] \arrow[d] & L_{n-1}^{\langle j+1 \rangle}(\mathbb{Z}G) \arrow[d] \\ H_{n-1}^G\big(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] \arrow[d] & L_{n-1}^{\langle j \rangle}(\mathbb{Z}G) \arrow[d] \\ H_{n-1}^G\big(E_{\mathcal{FIN}}(G); \mathbf{T}_{\mathbb{Z}}^{\langle j \rangle}\big) \arrow[r] \arrow[d] & \widehat{H}^{n-1}(\mathbb{Z}/2; \mathrm{Wh}_j^{\mathbb{Z}}(G)) \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

where the horizontal maps are the assembly maps, the left long exact column comes from the cofibration sequence of $\mathsf{GROUPOIDS}$-spectra above and the right column is the long exact Rothenberg sequence of (9.102). By the Five Lemma it suffices to show that the map induced by $E_{\mathcal{FIN}}(G) \to G/G$

$$H_n^G\bigl(E_{\mathcal{FIN}}(G);\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) \to H_n^G\bigl(G/G;\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) = \widehat{H}^n(\mathbb{Z}/2;\mathrm{Wh}_j^{\mathbb{Z}}(G))$$

is bijective for $j \le 1$ and $n \in \mathbb{Z}$. This will follow from the following commutative diagram

$$\begin{tikzcd}
H_n^G\bigl(E_{\mathcal{FIN}}(G);\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) \arrow[r] \arrow[d, "\cong"] & \widehat{H}^n(\mathbb{Z}/2;\mathrm{Wh}_j^{\mathbb{Z}}(G)) \\
H_n^G\bigl(EG \to E_{\mathcal{FIN}}(G);\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) & \\
\bigoplus_{i\in I} H_n^F\bigl(EF \to F/F;\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) \arrow[u, "\cong"] & \widehat{H}^n\left(\mathbb{Z}/2, H_j^G(EG \to E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}})\right) \arrow[uu, "\cong"] \\
\bigoplus_{i\in I} H_n^F\bigl(F/F;\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr) \arrow[u, "\cong"] & \widehat{H}^n\left(\mathbb{Z}/2, H_j^G(EG \to E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}})\right) \arrow[u, "\cong"] \\
\bigoplus_{i\in I} \pi_n(\mathbf{T}^{\langle j\rangle}(\mathbb{Z}M_i)) \arrow[u, "\cong"] & \widehat{H}^n\left(\mathbb{Z}/2, \bigoplus_{i\in I} H_j^{M_i}(EM_i \to M_i/M_i;\mathbf{K}_{\mathbb{Z}})\right) \arrow[u, "\cong"] \\
\bigoplus_{i\in I} \widehat{H}^n(\mathbb{Z}/2,\mathrm{Wh}_j(\mathbb{Z}M_i)) \arrow[u, "\cong"] \arrow[r, "\cong"'] & \widehat{H}^n\left(\mathbb{Z}/2, \bigoplus_{i\in I} \mathrm{Wh}_j(\mathbb{Z}M_i)\right) \arrow[u, "\cong"]
\end{tikzcd}$$

as soon as we have shown that all arrows except the upper horizontal one are bijective. The bijectivity of these arrows come from the following observations. Since $\mathrm{Wh}_j^{\mathbb{Z}}(\{1\})$ and hence $\pi_n(\mathbf{T}_{\mathbb{Z}}^{j})(G/\{1\}) = H^2(\mathbb{Z}/2;\mathrm{Wh}_j^{\mathbb{Z}}(\{1\}))$ vanishes for $j \le 1$ and $n \in \mathbb{Z}$, the groups $H_n^G\bigl(EG;\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr)$ and $H_n^{M_i}\bigl(EM_i;\mathbf{T}_{\mathbb{Z}}^{\langle j\rangle}\bigr)$ vanish and one can consider the long exact sequences associated to $EG \to E_{\mathcal{FIN}}(G)$ and $EM_i \to M_i/M_i$. One can apply excision to the G-pushout appearing in Theorem 11.32. The map

$$H_j^G(EG \to E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \xrightarrow{\cong} H_j^G(EG \to G/G;\mathbf{K}_{\mathbb{Z}}) = \mathrm{Wh}_j^{\mathbb{Z}}(G)$$

is bijective since G satisfies the K-theoretic Farrell-Jones-Conjecture. The map

$$\widehat{H}^n\left(\mathbb{Z}/2, H_j^G(EG \to E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}})\right) \xrightarrow{\cong} \widehat{H}^n\left(\mathbb{Z}/2, H_j^G(EG \to E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}})\right)$$

is bijective by Lemma 17.1.

(ii) This follows from assertion (i) and excision applied to G-pushout appearing and excision applied to the Theorem 11.32. This finishes the proof of Theorem 17.16. ◻

More information about $H_n^G(E_{\mathcal{FIN}}(V) \to \{\bullet\}; \mathbf{K}_R)$ can be found in Theorem 13.33 or in [670] where also identifications with twisted Nil-categories are discussed.

Many of the following results are based on Theorem 17.12.

Exercise 17.17. Let G be a one-relator group that is not torsionfree. Let C be a maximal finite subgroup of G. Suppose that $m := |C|$ is odd. Show that the canonical map

$$H_n^C(EC \to C/C; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle}) \to H_n^G(EG \to G/G; \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle})$$

is an isomorphism for $j \in \{2, 1, 0, \ldots\} \amalg \{-\infty\}$ and $n \in \mathbb{Z}$.

17.6.2 Torsionfree One-Relator Groups

Let $G = \langle s_i, i \in I \mid r \rangle$ be the presentation of a one-relator group G. Denote by F the free group on the set of generators $\{s_i \mid i \in I\}$. Note that r is an element in F. The group G is torsionfree if and only for any element $s \in F$ and natural number m satisfying $r = s^m$ we get $m = 1$, see [536] or [678, Proposition 5.17 on page 107]. Throughout this section we will assume that G is torsionfree.

We begin with the following lemma.

Lemma 17.18. *Let X be the 2-dimensional CW-complex given by the pushout*

(17.19)
$$\begin{tikzcd} S^1 \arrow[r, "q"] \arrow[d, "i"'] & \bigvee_{i \in I} S^1 \arrow[d, "\overline{i}"] \\ D^2 \arrow[r, "Q"'] & X. \end{tikzcd}$$

Let $d_i \in \mathbb{Z}$ be the degree of the composition $S^1 \xrightarrow{q} \bigvee_{i \in I} S^1 \xrightarrow{\mathrm{pr}_i} S^1$ where pr_i is the projection onto the i-th summand. Let $\mathcal{H}_*$ be any (non-equivariant) generalized homology theory satisfying the disjoint union axiom.

(i) Suppose that $d_i = 0$ holds for all $i \in I$. Then we get for $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n(X) \cong \mathcal{H}_n(\{\bullet\}) \oplus \bigoplus_{i \in I} \mathcal{H}_{n-1}(\{\bullet\}) \oplus \mathcal{H}_{n-2}(\{\bullet\});$$

(ii) Suppose that there is one $i \in I$ with $d_i \neq 0$. Then we have an isomorphism

$$\mathcal{H}_n(X) \xrightarrow{\cong} \mathcal{H}_n(X,\{\bullet\}) \oplus \mathcal{H}_n(\{\bullet\}),$$

and a short exact sequence

$$0 \to H_1(X) \otimes_{\mathbb{Z}} \mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_n(X,\{\bullet\}) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(X), \mathcal{H}_{n-2}(\{\bullet\})) \to 0;$$

(iii) Let d be the common greatest divisor of the finite set $\{|d_i| \mid i \in I, d_i \neq 0\}$, provided that $\{|d_i| \mid i \in I, d_i \neq 0\}$ is non-empty.
Then $H_1(X) \cong \bigoplus_{i \in I} \mathbb{Z}$ if $\{|d_i| \mid i \in I, d_i \neq 0\}$ is empty or if $d = 1$. If $d \geq 2$, then $H_1(X) \cong \mathbb{Z}/d \bigoplus_{i \in J} \mathbb{Z}$ where the set $|J|$ has cardinality $|I| - 1$ if $|I|$ is finite, and the same cardinality as $|I|$ if I is infinite.

Proof. We can assume without loss of generality that the pushout (17.19) above consists of base point preserving maps, otherwise change q up to homotopy to be base point preserving. From the Mayer-Vietoris sequence of the pair $(X, \{\bullet\})$ and the projection $X \to \{\bullet\}$, we obtain an isomorphism

$$\mathcal{H}_n(X) \xrightarrow{\cong} \mathcal{H}_n(X,\{\bullet\}) \oplus \mathcal{H}_n(\{\bullet\}).$$

(i) If we apply $S^1 \wedge -$ to (17.19), we obtain a pushout of pointed spaces

$$\begin{tikzcd} S^2 \arrow[r, "q"] \arrow[d, "i"'] & \bigvee_{i \in I} S^2 \arrow[d, "\bar{i}"] \\ D^3 \arrow[r, "Q"'] & S^1 \wedge X \end{tikzcd}$$

Since S^2 is simply connected, one gets using the Hurewicz Theorem an isomorphism $\bigoplus_{i \in I} H_2(S^2) \xrightarrow{\cong} \pi_2\left(\bigvee_{i \in I} S^2\right)$. We conclude that $\mathrm{id}_{S^1} \wedge q$ is pointed nullhomotopic. Hence we obtain a pointed homotopy equivalence $S^1 \wedge X \xrightarrow{\simeq} S^3 \vee \bigvee_{i \in I} S^2$. Now assertion (i) from the suspension isomorphism.
(ii) Since S^1 is compact, only finitely many of the numbers d_i are different from zero. We get for any abelian group A a group homomorphism

$$D(A)\colon A \to \bigoplus_{i \in I} A, \quad a \mapsto (d_i \cdot a)_{i \in I}.$$

The long exact sequence

$$(17.20) \quad \cdots \mathcal{H}_{n-1}(\{\bullet\}) \xrightarrow{D(\mathcal{H}_{n-1}(\{\bullet\}))} \bigoplus_{i \in I} \mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_n(X,\{\bullet\}) \\ \to \mathcal{H}_{n-2}(\{\bullet\}) \xrightarrow{D(\mathcal{H}_{n-2}(\{\bullet\}))} \bigoplus_{i \in I} \mathcal{H}_{n-2}(\{\bullet\}) \to \cdots$$

comes from the long Mayer-Vietoris sequence of the pushout of pointed spaces (17.19) above and with the identification derived from the disjoint union axiom and the suspension isomorphism

$$\bigoplus_{i\in I}\mathcal{H}_{n-1}(\{\bullet\}) \xrightarrow{\cong} \mathcal{H}_n\left(\bigvee_{i\in I} S^1, \{\bullet\}\right).$$

If we take $\mathcal{H}_*$ to be singular homology with integer coefficients, we see that $D(A)$ is obtained from $D(\mathbb{Z})$ by $D(A) = D(\mathbb{Z}) \otimes_A \mathrm{id}_A$ and there is a short exact sequence $0 \to \mathbb{Z} \xrightarrow{D(\mathbb{Z})} \bigoplus_{i\in I}\mathbb{Z} \to H_1(X) \to 0$. This implies

$$\begin{aligned}\operatorname{coker}(D(A)) &= H_1(X)\otimes_{\mathbb{Z}} A;\\ \ker(D(A)) &= \operatorname{Tor}_1^{\mathbb{Z}}(H_1(X), A).\end{aligned}$$

(iii) This follows from the short exact sequence $\mathbb{Z} \xrightarrow{D(\mathbb{Z})} \bigoplus_{i\in\mathbb{Z}}\mathbb{Z} \to H_1(G) \to 0$. $\square$

We denote by $H_n(Y;A)$ the singular homology of a space Y with coefficients in the abelian group A and abbreviate $H_n(Y) := H_n(Y;\mathbb{Z})$. Note that the group homology $H_n(G)$ is $H_n(BG)$ and $H_1(G) = G/[G,G]$.

Lemma 17.21. *Suppose that the one-relator-group G is torsionfree. Let $\mathcal{H}_*$ be any (non-equivariant) generalized homology theory.*

(i) If r lies in $[F,F]$, we get isomorphisms

$$\mathcal{H}_n(BG) \cong \mathcal{H}_n(\{\bullet\}) \oplus \bigoplus_{i\in I}\mathcal{H}_{n-1}(\{\bullet\}) \oplus \mathcal{H}_{n-2}(\{\bullet\});$$

(ii) If r does not lie in $[F,F]$, then we get isomorphisms

$$\mathcal{H}_n(BG) \cong \mathcal{H}_n(\{\bullet\}) \oplus \mathcal{H}_n(BG,\{\bullet\}),$$

and a short exact sequence

$$\begin{aligned}0 \to H_1(BG)\otimes_{\mathbb{Z}}\mathcal{H}_{n-1}(\{\bullet\}) &\to \mathcal{H}_n(BG,\{\bullet\})\\ &\to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(BG);\mathcal{H}_{n-2}(\{\bullet\})) \to 0;\end{aligned}$$

(iii)

$$H_n(BG;A) \cong \begin{cases} A & n=0;\\ \bigoplus_{i\in I} A & n=1 \text{ and } r\in[F,F];\\ H_1(X)\otimes_{\mathbb{Z}} A & n=1 \text{ and } r\notin[F,F];\\ A & n=2 \text{ and } r\in[F,F];\\ \operatorname{Tor}_1^{\mathbb{Z}}(H_1(X);A) & n=2 \text{ and } r\notin[F,F];\\ 0 & n\geq 3.\end{cases}$$

Proof. Consider the pushout

$$\begin{tikzcd} S^1 \arrow[r, "q"] \arrow[d, "i"'] & \bigvee_{i \in I} S^1 \arrow[d, "\overline{i}"] \\ D^2 \arrow[r, "Q"'] & Z \end{tikzcd}$$

where the upper vertical arrow is given by the word $r \in *_{i \in I}\mathbb{Z} = \pi_1\left(\bigvee_{i \in I} S^1\right)$. Then Z is a model for BG, see [678, Chapter III §§9 -11].

(i) This follows from Lemma 17.18 (i).

(ii) This follows from Lemma 17.18 (ii).

(iii) This follows from assertions (i) and (ii) applied to the special case that $\mathcal{H}_*$ is singular homology with coefficients in the abelian group A. □

Recall that the Baum-Connes-Conjecture 10.44 for torsionfree groups holds for every torsionfree one-relator group G predicting isomorphisms

$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_n \colon K_n(BG) \to K_n(C_r^*(G;\mathbb{C}));$$
$$\operatorname{asmb}^{G,\mathbb{R}}(BG)_n \colon KO_n(BG) \to KO_n(C_r^*(G;\mathbb{R})).$$

Hence we get from Lemma 17.21 (i) in the case that r belongs to $[F, F]$

$$K_n(C_r^*(G;\mathbb{C})) \cong K_n(\{\bullet\}) \oplus \bigoplus_{i \in I} K_{n-1}(\{\bullet\}) \oplus K_{n-2}(\{\bullet\}) \cong \begin{cases} \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} & n \text{ even;} \\ \mathbb{Z}^2 & n \text{ odd,} \end{cases}$$

and

$$KO_n(C_r^*(G;\mathbb{R})) \cong KO_n(\{\bullet\}) \oplus \bigoplus_{i \in I} KO_{n-1}(\{\bullet\}) \oplus KO_{n-2}(\{\bullet\}).$$

If r does not belong to $[F, F]$, then get from Lemma 17.21 (ii)

$$K_n(C_r^*(G;\mathbb{C})) \cong \begin{cases} \mathbb{Z} & n \text{ even;} \\ H_1(G) & n \text{ odd,} \end{cases}$$

$$KO_n(C_r^*(G;\mathbb{R})) \cong KO_n(\{\bullet\}) \oplus KO_n(BG, \{\bullet\}),$$

and a short exact sequence

$$0 \to H_1(G) \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}) \to KO_n(BG, \{\bullet\}) \\ \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(G), KO_{n-2}(\{\bullet\})) \to 0.$$

The computation for $K_*(C_r^*(G))$ agrees with the one in [111].

Recall that Farrell-Jones Conjecture 6.53 for torsionfree groups and regular rings for K-theory holds for torsionfree one-relator groups predicting for a regular ring R an isomorphism $H_n(BG;\mathbf{K}(R)) \to K_n(RG)$ for $n \in \mathbb{Z}$, and one can apply Lemma 17.21 to $H_n(BG;\mathbf{K}(R))$. Moreover, the Farrell-Jones Conjecture 9.114 for torsionfree groups for L-theory predicts that the assembly map $H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) \to L_n^{\langle -\infty\rangle}(RG)$ is bijective for $n \in \mathbb{Z}$, and is known for torsionfree one-relator groups to be true after inverting 2. So Lemma 17.21 can also be used to compute $K_n(RG)$ and $L_n^{\langle -\infty\rangle}(RG)[1/2]$ if one understands $K_n(R)$ and $L_n^{\langle -\infty\rangle}(R)[1/2]$.

Exercise 17.22. Let $G = \langle s_1, s_2, \ldots, s_n \mid r\rangle$ be a finitely generated (not necessarily torsionfree) one-relator group where r is given by the word $s_{i_1}^{m_1} s_{i_2}^{m_2} \cdots s_{i_l}^{m_l}$ for $i_j \in \{1,2,\ldots,n\}$ and $m_j \in \mathbb{Z}$. Define for $j = 1,2,\ldots,n$

$$d_j = \sum_{\substack{k\in\{1,2,\ldots,n\}\\ i_k = j}} m_k.$$

Show that $H_1(G) \cong \mathbb{Z}^n$ if all the numbers d_j are trivial, and $H_1(G) \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}/d$ if not all the numbers d_j are zero and d is the common greatest divisor of $\{|d_j| \mid j = 1,2\ldots,l, d_j \neq 0\}$.

Exercise 17.23. Consider the 1-relator group $G = \langle s_1, s_2 \mid s_1 s_2 s_1 s_2^{-1} s_1^{-2}\rangle$. Compute the algebraic K-groups $K_n(\mathbb{C}[\mathbb{Z}/m \times G])$ for $n \leq 1$ and any natural number m.

Exercise 17.24. Let G be the non-trivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$. Compute $L_n^s(\mathbb{Z}[G])$ for $n \in \mathbb{Z}$.

17.6.3 One-Relator Groups with Torsion

Let $G = \langle s_i, i \in I \mid r\rangle$ be the presentation of a one-relator group G. For the remainder of this subsection we assume that G is not torsionfree.

Then there exists a maximal non-trivial finite subgroup $C \subseteq G$, unique up to conjugation. It is cyclic. Let $m \geq 2$ be its order. Denote by F the free group on the set of generators $\{s_i \mid i \in I\}$. Note that then r is an element in F. The natural number m can be characterized as the largest natural number for which there exists a word $s \in F$ with $r = s^m$. Note that for such s the cyclic group C of order m is generated by the class $\overline{s}$ in G represented by s and every torsion element in G is conjugated to some power of $\overline{s}$. This was proved by Karras-Magnus-Solitar, see [536] or [678, Proposition 5.17 on page 107].

Let $p\colon BG \to \underline{B}G$ be the up to homotopy unique canonical map and let $i\colon C \to A$ be the inclusion. The Mayer-Vietoris sequence of the G-quotient of the G-pushout appearing in Theorem 11.32 yields the long exact sequence

$$(17.25) \quad \cdots \to \mathcal{H}_n(BC,\{\bullet\}) \xrightarrow{\mathcal{H}_n(Bi)} \mathcal{H}_n(BG,\{\bullet\}) \xrightarrow{\mathcal{H}_n(p)} \mathcal{H}_n(\underline{B}G,\{\bullet\})$$
$$\to \mathcal{H}_{n-1}(BC,\{\bullet\}) \xrightarrow{\mathcal{H}_{n-1}(Bi)} \mathcal{H}_{n-1}(BG,\{\bullet\}) \xrightarrow{\mathcal{H}_{n-1}(p)} \mathcal{H}_{n-1}(\underline{B}G,\{\bullet\}) \to \cdots$$

for any (non-equivariant) generalized homology theory $\mathcal{H}_*$. Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ. Then sequence (17.25) splits into short split exact sequences after applying $- \otimes_{\mathbb{Z}} \Lambda$, more precisely, the Λ-map $\mathcal{H}_n(BG,\{\bullet\}) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n(\underline{B}G,\{\bullet\}) \otimes_{\mathbb{Z}} \Lambda$ is split surjective for every $n \in \mathbb{Z}$, see Lemma 12.18 (ii).

By inspecting the model for $\underline{E}G$ of Subsection 11.6.13 and dividing out the G-action, we obtain a pushout

$$(17.26) \qquad \begin{tikzcd} S^1 \arrow[r, "\underline{q}"] \arrow[d, "i"'] & \bigvee_{i\in I} S^1 \arrow[d, "\bar{i}"] \\ D^2 \arrow[r, "\underline{Q}"'] & \underline{B}G \end{tikzcd}$$

where the attaching map $\underline{q}$ is given by the element s. Note that we can apply Lemma 17.21 and get information about $\mathcal{H}_n(\underline{B}G)$ for any (non-equivariant) generalized homology theory $\mathcal{H}_*$. As an illustration we compute the singular homology $H_n(BG;A)$ with coefficients in the abelian group A

Lemma 17.27. *(i) The inclusion $C \to G$ induces isomorphisms*

$$H_n(Bi;A)\colon H_n(BC;A) \xrightarrow{\cong} H_n(BG;A)$$

for $n \geq 3$;

(ii) We obtain an exact sequence

$$0 \to H_2(BC;A) \xrightarrow{H_1(Bi;A)} H_2(BG;A) \xrightarrow{H_2(p;A)} H_2(\underline{B}G;A) \to H_1(BC;A)$$
$$\xrightarrow{H_1(Bi;A)} H_1(BG;A) \xrightarrow{H_1(p;A)} H_1(\underline{B}G;A) \to 0.$$

We have

$$H_n(BC,A) \cong \begin{cases} \ker(m \cdot \mathrm{id}\colon A \to A) & n \geq 2, n \text{ even}; \\ \operatorname{coker}(m \cdot \mathrm{id}\colon A \to A) & n \geq 1, n \text{ odd}, \end{cases}$$

and

$$H_n(\underline{B}G;A) \cong \begin{cases} A & n = 2 \text{ and } s \in [F,F]; \\ \operatorname{Tor}_1^{\mathbb{Z}}(C,A) & n = 2 \text{ and } s \notin [F,F]; \\ \{0\} & n \geq 3; \end{cases}$$

(iii) If $A = \mathbb{Z}$ and $s \in [F,F]$, we get a short exact sequence

$$0 \to H_2(BG) \xrightarrow{H_2(p;A)} H_2(\underline{B}G) \to C \to 0,$$

the groups $H_2(BG)$ *and* $H_2(\underline{B}G)$ *are infinite cyclic, and the homomorphisms* $H_1(p)\colon H_1(BG) \xrightarrow{\cong} H_1(\underline{B}G)$ *is bijective.*
If $A = \mathbb{Z}$ *and* $s \notin [F,F]$*, we get a short exact sequence*

$$0 \to C = H_1(BC) \xrightarrow{H_1(Bi)} H_1(BG) \xrightarrow{H_2(p)} H_1(\underline{B}G) \to 0,$$

and the groups $H_2(BG)$ *and* $H_2(\underline{B}G)$ *are trivial.*

Proof. (i) and (ii) These follow from the long exact sequence (17.25), and the fact that $\dim(\underline{B}G)$ is two, and Lemma 17.21 applied to the pushout (17.26).
(iii) This follows from assertions (i) and (ii) using the fact that the class $\underline{s} \in G$ represented by s is a generator of the finite cyclic group C. □

Recall that the Baum-Connes Conjecture 14.11 with coefficients holds for one-relator groups. Hence the assembly maps

$$K_n^G(\underline{E}G) \to K_n(C_r^*(G;\mathbb{C}));$$
$$KO_n^G(\underline{E}G) \to KO_n(C_r^*(G;\mathbb{R})),$$

are bijective for all $n \in \mathbb{Z}$.

Recall that the Full Farrell-Jones Conjecture 13.27 holds for one-relator groups with torsion. If R is a regular ring with $\mathbb{Q} \subseteq R$ then we obtain an isomorphism for every $n \in \mathbb{Z}$, see Theorem 13.48

$$H_n^G(\underline{E}G;\mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

If m is odd, any virtually cyclic subgroup of G is of type I, and we obtain for any ring with involution and $n \in \mathbb{Z}$ an isomorphism, see Theorem 13.57,

$$H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_R^{\langle -\infty \rangle}(RG).$$

If m is even, we know at least that this map is bijective after inverting two.

In any cases we want to compute the source of the assembly maps. A far reaching strategy is to use Theorem 17.12 after one has computed $K^G(\underline{B}G)$, $KO^G(\underline{B}G)$, $H_n(\underline{B}G;\mathbf{K}(R))$, or $H_n(\underline{B}G;\mathbf{L}^{\langle -\infty \rangle})$ by applying Lemma 17.18 to (17.26)

Example 17.28 (Topological K-theory in the complex case). We carry this out for $K_n(C_r^*(G))$. Since $K_n(\{\bullet\})$ is $\mathbb{Z}$ for even n and trivial for odd n, we get from Lemma 17.18 applied to (17.26) and Lemma 17.27 (iii)

$$K_n(\underline{B}G) \cong \begin{cases} \mathbb{Z}^2 & s \in [F,F] \text{ and } n \text{ even;} \\ \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} & s \in [F,F] \text{ and } n \text{ odd} \\ \mathbb{Z} & s \notin [F,F] \text{ and } n \text{ even;} \\ H_1(\underline{B}G) \cong \operatorname{coker}\bigl(H_1(C) \to H_1(G)\bigr) & s \notin [F,F] \text{ and } n \text{ odd.} \end{cases}$$

We get from Theorem 17.12 (i) the short exact sequence

$$0 \to \widetilde{K}_n(C_r^*(C)) \to K_n(C_r^*(G)) \to K_n(\underline{B}G) \to 0$$

which splits after inverting m. Since $\widetilde{K}_n(C_r^*(\mathbb{Z}/m)) \cong \mathbb{Z}^{m-1}$ for even n and is $\{0\}$ for odd n, we get

$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^{m+1} & s \in [F,F] \text{ and } n \text{ even;} \\ \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} & s \in [F,F] \text{ and } n \text{ odd;} \\ \mathbb{Z}^m & s \notin [F,F] \text{ and } n \text{ even;} \\ \operatorname{coker}\bigl(H_1(i)\colon H_1(C) \to H_1(G)\bigr) & s \notin [F,F] \text{ and } n \text{ odd.} \end{cases}$$

This computation for $K_*(C_r^*(G))$ agrees with the one in [111] since $F/[F,F]$ is torsionfree and hence $r \in [F,F] \iff s \in [F,F]$.

The following example is illuminating since it combines a lot of the material and methods we have presented so far in this book.

Example 17.29. Consider the finitely generated one-relator group

$$G = \langle s_1, s_2, s_3 \mid r\rangle \text{ for } r = s_1^6 s_2^9 s_1^{21} s_2^9 s_1^{21} s_2^9 s_1^{15}.$$

Put $s = s_1^6 s_2^9 s_1^{15}$. Then $r = s^3$. If m is a natural number for which $r = s'^m$ for some word s', then $m = 1, 3$. Hence G has a maximal finite subgroup C generated by the element $\overline{s} \in G$ represented by s and C is cyclic of order 3. We can compute $H_1(G)$ using the recipe stated in Exercise 17.22 and obtain $H_1(G) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/9$. Since s does not belong to $[F,F]$, we get from Lemma 17.27

$$H_n(G) \cong \begin{cases} \mathbb{Z}/3 & n \geq 3 \text{ and } n \text{ odd;} \\ 0 & n \geq 2 \text{ and } n \text{ even;} \\ \mathbb{Z}^2 \oplus \mathbb{Z}/9 & n = 1; \\ \mathbb{Z} & n = 0. \end{cases}$$

We get from Example 17.28

$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^3 & n \text{ even;} \\ \mathbb{Z}/3 & n \text{ odd.} \end{cases}$$

We conclude from Theorem 10.79 (ii) that $\widetilde{KO}_n(C_r^*(C;\mathbb{R}))$ is $\mathbb{Z}$ for n even and $\{0\}$ for n odd.

We conclude from Lemma 17.18 (ii) in the case $\mathcal{H}_* = KO_*$ applied to the pushout (17.26) an isomorphism

$$KO_n(\underline{B}G) \xrightarrow{\cong} KO_n(\underline{B}G, \{\bullet\}) \oplus KO_n(\{\bullet\}),$$

and a short exact sequence

$$\begin{aligned} 0 \to H_1(\underline{B}G) \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}) \to KO_n(\underline{B}G, \{\bullet\}) \\ \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(\underline{B}G), KO_{n-2}(\{\bullet\})) \to 0. \end{aligned}$$

Since $H_1(\underline{B}G) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3$ by Lemma 17.27 (iii), this implies

$$KO_n(\underline{B}G) \cong KO_n(\{\bullet\}) \oplus KO_{n-1}(\{\bullet\}) \oplus KO_{n-1}(\{\bullet\}) \oplus \mathbb{Z}/3 \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}).$$

Since $\widetilde{KO}_n(C_r^*(C;\mathbb{R}))$ is $\mathbb{Z}$ or trivial, we obtain from Theorem 17.12 (ii) for every $n \in \mathbb{Z}$ a short exact sequence

$$0 \to \widetilde{KO}_n(C_r^*(C;\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R})) \to KO_n(\underline{B}G) \to 0$$

which splits after inverting 3.

Next we consider the case where n is odd. Then $\widetilde{KO}_n(C_r^*(C;\mathbb{R}))$ vanishes, and we get an isomorphism $KO_n(C_r^*(G;\mathbb{R})) \cong KO_n(\underline{B}G)$. Thus we get

$$KO_n(C_r^*(G;\mathbb{R})) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 & n \equiv 1 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 & n \equiv 5 \mod 8; \\ \{0\} & n \equiv 7 \mod 8. \end{cases}$$

Next we consider the case where n is even. Then $KO_n(\underline{B}G)$ contains no 3-torsion and we get

$$\begin{aligned} KO_n(C_r^*(G;\mathbb{R})) &\cong \widetilde{KO}_n(C_r^*(C;\mathbb{R})) \oplus KO_n(\underline{B}G) \\ &\cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \equiv 0 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 2 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z} & n \equiv 4 \mod 8; \\ \mathbb{Z} & n \equiv 6 \mod 8. \end{cases} \end{aligned}$$

Let $V \subseteq G$ be an infinite virtually cyclic subgroup of type I. Then we can find an exact sequence $1 \to H \to V \to \mathbb{Z} \to 0$. Any finite subgroup of G is subconjugated to C and hence we can find $g \in G$ with $gHg^{-1} \subseteq C$. Since $gVg^{-1} \subseteq N_G(gHg^{-1})$ and $N_GC = C$, we get $H = \{1\}$ and hence $V \cong \mathbb{Z}$.

Suppose that there exists an infinite virtually cyclic subgroup $V \subseteq G$ be of type II. It contains an infinite cyclic subgroup V' of type I satisfying $[V : V'] = 2$. Since we have already proved that V' is infinite cyclic. V' must be $\mathbb{Z}/2 \times \mathbb{Z}/2$. This contradicts the fact that an finite subgroup of G is subconjugated to $C \cong \mathbb{Z}/3$. Thus we have shown that any infinite virtually cyclic subgroup of G is infinite cyclic.

We conclude from Theorem 6.16, and the Transitivity Principle 15.12 that the assembly map $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}})$ is bijective for all $n \in \mathbb{Z}$. Theorem 13.57 implies that the assembly map $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ is bijective for all $n \in \mathbb{Z}$.

We conclude from Theorem 17.12 (iv) that the inclusion $C \to M$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$\mathrm{Wh}_n^{\mathbb{Z}}(C) \xrightarrow{\cong} \mathrm{Wh}_n^{\mathbb{Z}}(G).$$

Since $\mathrm{Wh}(\mathbb{Z}/3)$ by Theorem 3.115 and Theorem 3.116 (iii), $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$ by Theorem 2.112 (i), and $K_n(\mathbb{Z}[\mathbb{Z}/3])$ for $n \leq -1$ by Theorem 4.10 all vanish, also the groups $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$, and $K_n(\mathbb{Z}G)$ for $n \leq -1$ vanish.

We conclude from Theorem 9.106 that the L-groups of $\mathbb{Z}G$ are independent of the decoration, namely, for every $j \in \mathbb{Z}, j \leq -1$ and every $n \in \mathbb{Z}$ the forgetful maps induce isomorphisms

$$L_n^s(\mathbb{Z}G) \xrightarrow{\cong} L_n^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^p(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

The same statement is true for the L-groups of $\mathbb{Z}C$. We conclude from Theorem 9.196

$$\overline{L}_n^{\langle -\infty \rangle}(\mathbb{Z}C) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (4); \\ 0 & n \equiv 1 \mod (4); \\ \mathbb{Z} & n \equiv 2 \mod (4); \\ 0 & n \equiv 3 \mod (4). \end{cases}$$

Hence we get from Theorem 17.12 (iii) for $n \in \mathbb{Z}$ a short exact sequence

$$0 \to \overline{L}_n^{\langle -\infty \rangle}(\mathbb{Z}C) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \to H_n(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0$$

which splits after inverting 3.

We obtain from Theorem 17.18 (ii) an isomorphism

$$H_n(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \cong H_n(\underline{B}G, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \oplus H_n(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))$$

and the short exact sequence

$$0 \to H_1(\underline{B}G) \otimes_{\mathbb{Z}} L_{n-1}^{\langle -\infty \rangle}(\mathbb{Z}) \to H_n(\underline{B}G, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \\ \to \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\underline{B}G), L_{n-2}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0.$$

We get from Lemma 17.27 (ii) and (iii)

$$H_n(\underline{B}G) \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 & n = 1; \\ 0 & \text{otherwise} \end{cases}$$

Hence we get for every decoration j

$$L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (4); \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 & n \equiv 1 \mod (4); \\ \mathbb{Z}/2 & n \equiv 2 \mod (4); \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod (4). \end{cases}$$

17.6.4 Fuchsian Groups

Let F be a cocompact Fuchsian group with presentation

$$F = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_t \mid \\ c_1^{\gamma_1} = \cdots = c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1}[a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

for integers $g, t \geq 0$ and $\gamma_i > 1$. Then $\underline{B}F$ is a closed orientable surface of genus g. The following is a consequence of Theorem 17.12 and Remark 17.14, see [667] for details. Lower algebraic K-theory has also been computed in [120].

Theorem 17.30 (K-and L-groups of Fuchsian groups).

(i) There are isomorphisms

$$K_n(C_r^*(F)) \cong \begin{cases} \left(2 + \sum_{i=1}^t (\gamma_i - 1)\right) \cdot \mathbb{Z} & n = 0; \\ (2g) \cdot \mathbb{Z} & n = 1; \end{cases}$$

(ii) The inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism

$$\bigoplus_{i=1}^t \mathrm{Wh}_n(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_n(F)$$

for $n \leq 1$;

(iii) There are isomorphisms

$$L_n(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1+\sum_{i=1}^t \left\lfloor \frac{\gamma_i}{2} \right\rfloor\right) \cdot \mathbb{Z}[1/2] & n \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \ (4); \\ \left(1+\sum_{i=1}^t \left\lfloor \frac{\gamma_i - 1}{2} \right\rfloor\right) \cdot \mathbb{Z}[1/2] & n \equiv 2 \ (4); \\ 0 & n \equiv 3 \ (4), \end{cases}$$

where $\lfloor r \rfloor$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r. From now on suppose that each γ_i is odd. Then we get for $\epsilon = p$ and s

$$L_n^\epsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \bigoplus \left(1+\sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & n \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \ (4); \\ \mathbb{Z}/2 \bigoplus \left(1+\sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 2 \ (4); \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \ (4). \end{cases}$$

For $\epsilon = h$ we do not know an explicit formula for $L_n^\epsilon(\mathbb{Z}F)$. The problem is that no general formula is known for the 2-torsion contained in $\widetilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$, for m odd since it is given by the term $\widehat{H}^2(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [60, Theorem 2].

Exercise 17.31. Let F be a Fuchsian group as above. Show that its Whitehead group $\mathrm{Wh}(F)$ is a free abelian group of rank $\bigoplus_{i=1}^t \lfloor \gamma_i/2 \rfloor + 1 - \delta(\gamma_i)$ where $\delta(\gamma_i)$ is the number of divisors of γ_i.

17.6.5 Torsionfree Hyperbolic Groups

Theorem 17.32 (Farrell-Jones Conjecture for torsionfree hyperbolic groups for K-theory). *Let G be a non-trivial torsionfree hyperbolic group.*

(i) We obtain for all $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_C \bigl(NK_n(R) \oplus NK_n(R)\bigr) \xrightarrow{\cong} K_n(RG)$$

where C runs through a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups;

(ii) The abelian groups $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$, and $\mathrm{Wh}(G)$ vanish;

(iii) We get for every ring R with involution and $n \in \mathbb{Z}$ an isomorphism

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

For every $j \in \mathbb{Z}, j \leq 2$, and $n \in \mathbb{Z}$, the natural map

$$L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is bijective;

(iv) We get for any $n \in \mathbb{Z}$ isomorphisms

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G));$$
$$KO_n(BG) \xrightarrow{\cong} KO_n(C_r^*(G;\mathbb{R})).$$

Proof. (i) By [673, Corollary 2.11, Theorem 3.1 and Example 3.6], see also Theorem 11.36, there is a G-pushout

$$\begin{tikzcd} \coprod_C G \times_C EC \arrow[r, "i"] \arrow[d, "p"] & E_{\mathcal{FIN}}(G) \arrow[d] \\ \coprod_C G/C \arrow[r] & E_{\mathcal{VCY}}(G) \end{tikzcd}$$

where i is an inclusion of G-CW-complexes, p is the obvious projection. Hence we obtain using Theorem 6.16 an isomorphism

$$\begin{aligned} H_n^G\bigl(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R\bigr) &\cong \bigoplus_C H_n^G\bigl(G \times_C EC \to G/C; \mathbf{K}_R\bigr) \\ &\cong \bigoplus_C H_n^C\bigl(EC \to \{\bullet\}; \mathbf{K}_R\bigr) \\ &\cong \bigoplus_C \bigl(NK_n(R) \oplus NK_n(R)\bigr). \end{aligned}$$

We obtain from Theorem 13.33 an isomorphism

$$\begin{aligned} H_n^G\bigl(E_{\mathcal{VCY}}(G); \mathbf{K}_R\bigr) &\cong H_n^G\bigl(EG; \mathbf{K}_{\mathcal{A}}\bigr) \oplus H_n^G\bigl(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R\bigr) \\ &\cong H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_C \bigl(NK_n(R) \oplus NK_n(R)\bigr). \end{aligned}$$

Since G satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia) Theorem 13.62 implies that G satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R.

(ii) Since G satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia) and hence by Theorem 13.62 Conjectures 3.110 and 4.18, assertion (ii) follows.

(iii) Since G satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (ia), Theorem 13.62 implies that G satisfies Conjecture 9.114. Now assertion (iii) follows from assertion (ii) and Theorem 9.106.

(iv) This follows from the fact that G satisfies the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (id) and from Remark 14.14. $\square$

17.6.6 Hyperbolic Groups

Not necessarily torsionfree hyperbolic groups are treated in [663, Theorem 1.1] which says the following.

Theorem 17.33 (Hyperbolic groups). *Let G be a hyperbolic group in the sense of Gromov [423], and let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of G.*

(i) For each $n \in \mathbb{Z}$ there is an isomorphism

$$H_n^G\big(\underline{E}G; \mathbf{K}_R\big) \oplus \bigoplus_{V \in \mathcal{M}} H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{K}_R\big) \xrightarrow{\cong} K_n(RG);$$

(ii) For each $n \in \mathbb{Z}$ there is an isomorphism

$$H_n^G\big(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}\big) \oplus \bigoplus_{V \in \mathcal{M}} H_n^V\big(\underline{E}V \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}\big) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG),$$

provided that there exists $n_0 \le -2$ such that $K_n(RV) = 0$ holds for all $n \le n_0$ and all virtually cyclic subgroups $V \subseteq G$. (The latter condition is satisfied if $R = \mathbb{Z}$ or if R is regular with $\mathbb{Q} \subseteq R$.)

17.6.7 L-Theory of Torsionfree Groups

Throughout this subsection, let G be a torsionfree group satisfying Conjecture 9.114, i.e., we have the isomorphism

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

Thus we obtain from Subsection 15.14.4 an isomorphism

$$KO(BG)[1/2] \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)[1/2]. \tag{17.34}$$

Example 17.35 (p-torsion in L-groups). Let $n \ge 3$ be an odd natural number. Consider the group automorphism

$$\alpha\colon \mathbb{Z}^2 \to \mathbb{Z}^2, (a,b) \mapsto (a + nb, b).$$

Let G be the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Obviously there is an orientable aspherical closed smooth 3-manifold M that is the total space in a locally trivial fiber bundle $T^2 \to M \to S^1$ whose fundamental group is G, namely, the mapping torus of the selfdiffeomorphism $S^1 \times S^1 \to S^1 \times S^1$ sending (z_1, z_2) to $(z_1 z_2^n, z_2)$. The group G satisfies the Full Farrell-Jones Conjecture and hence Conjecture 9.114. One easily computes

$$H_k(M;\mathbb{Z}) \cong H_1(G) \cong \begin{cases} \mathbb{Z} & k = 0,2,3; \\ \mathbb{Z} \oplus \mathbb{Z}/n & k = 1; \\ 0 & \text{otherwise.} \end{cases}$$

An elementary spectral sequence argument shows

$$L_n^{\langle -\infty \rangle}(\mathbb{Z}G)[1/2] \cong KO_k(M;\mathbb{Z})[1/2] \cong \begin{cases} \mathbb{Z}[1/2] & k = 0,2,3 \mod 4; \\ \mathbb{Z}[1/2] \oplus \mathbb{Z}/n & k = 1 \mod 4; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$ can contain p-torsion for any odd prime p. Recall that for finite groups G only 2-torsion occurs in $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$ by Theorem 9.196 (ii).

Exercise 17.36. Let p be a prime. Show for any $n \geq 6$ and any decoration $j \in \{2,1,0,-1,\ldots\} \amalg \{-\infty\}$ that there is an orientable aspherical closed smooth manifold M of dimension n such that $L_k^{\langle j \rangle}(\mathbb{Z}\pi_1(M))$ contains non-trivial p-torsion for every $k \in \mathbb{Z}$.

Since we have the decomposition of spectra after localization at 2

$$\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})_{(2)} = \prod_{k \in \mathbb{Z}} \mathbf{K}(\mathbb{Z}_{(2)}, 4k) \times \prod_{k \in \mathbb{Z}} \mathbf{K}(\mathbb{Z}/2, 4k-2),$$

see Remark 9.133 in the connective case and [928, Theorem $\mathrm{A}_{(2)}$ on page 178] in the periodic case, we obtain for any torsionfree group G satisfying Conjecture 9.114

$$L_n^{\langle -\infty \rangle}(\mathbb{Z}G)_{(2)} \cong \prod_{k \in \mathbb{Z}} H_{n+4k}(BG;\mathbb{Z}_{(2)}) \times \prod_{k \in \mathbb{Z}} H_{n+4k-2}(BG;\mathbb{Z}/2). \tag{17.37}$$

17.6.8 Cocompact NEC-Groups

A calculation of $\mathrm{Wh}_n(G)$, $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$, and $K_n(C_*^r(G))$ for 2-dimensional crystallographic groups G and more general cocompact NEC-groups G is presented in [667], see also [779]. For these groups the orbit spaces $\underline{B}G$ are compact surfaces possibly with boundary.

17.6.9 Crystallographic Groups

A *crystallographic group of dimension* n is a discrete group that acts cocompactly, properly, and isometrically on the Euclidean space $\mathbb{R}^n$ for some $n \geq 0$. One does not have a complete calculation of K-and L-groups of integral group rings or reduced group C^*-algebras of crystallographic groups except in dimension two as mentioned above in Subsection 17.6.8. Computations of the lower and middle algebraic K-theory of the integral group ring of split three-dimensional crystallographic groups are carried out by Farley-Ortiz [334], see also [24].

As an illustration we mention the following result taken from [604, Theorem 0.1].

Theorem 17.38 (Computation of the topological K-theory of $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ for free conjugation action). *Consider the extension of groups $1 \to \mathbb{Z}^n \to G \to \mathbb{Z}/m \to 1$ such that the conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$. Let $\mathcal{M}$ be the set of conjugacy classes of maximal finite subgroups of G.*

(i) We obtain an isomorphism

$$\omega_1 \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(\underline{B}G).$$

Restriction with the inclusion $k \colon \mathbb{Z}^n \to G$ induces an isomorphism

$$k^* \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}.$$

Induction with the inclusion k yields a homomorphism

$$\overline{k_*} \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(G)).$$

It fits into an exact sequence

$$0 \to \widehat{H}^{-1}(\mathbb{Z}/m, K_1(C_r^*(\mathbb{Z}^n))) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\overline{k_*}} K_1(C_r^*(G)) \to 0$$

where $\widehat{H}^*(\mathbb{Z}/m; M)$ denotes the Tate cohomology *of $\mathbb{Z}/m$ with coefficients in a $\mathbb{Z}[\mathbb{Z}/m]$-module M. In particular $\overline{k_*}$ is surjective and its kernel is annihilated by multiplication with m;*

(ii) There is an exact sequence

$$0 \to \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M)\in\mathcal{M}} i_M} K_0(C_r^*(G)) \xrightarrow{\omega_0} K_0(\underline{B}G) \to 0$$

where $\widetilde{R}_{\mathbb{C}}(M)$ is the kernel of the map $R_{\mathbb{C}}(M) \to \mathbb{Z}$ sending the class $[V]$ of a complex M-representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$ and the map i_M comes from the inclusion $M \to G$ and the identification $R_{\mathbb{C}}(M) = K_0(C_r^*(M))$. We obtain a homomorphism

$$\overline{k_*} \oplus \bigoplus_{(M) \in \mathcal{M}} i_M \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \to K_0(C_r^*(G)).$$

It is injective. It is bijective after inverting m;

(iii) We have

$$K_i(C_r^*(G)) \cong \mathbb{Z}^{s_i}$$

where

$$s_i = \begin{cases} \left(\sum_{(M)\in\mathcal{M}}(|M|-1)\right) + \sum_{l\in\mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l}\mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ even;} \\ \sum_{l\in\mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l+1}\mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ odd;} \end{cases}$$

(iv) If m is even, then $s_1 = 0$ and

$$K_1(C_r^*(G)) \cong \{0\}.$$

The numbers s_i can be made more explicite, see [604]. For instance, if $m = p$ for a prime number p, then there exists a natural number k that is determined by the property $n = (p-1)\cdot k$, and we get:

$$(17.39)\qquad s_i = \begin{cases} p^k\cdot(p-1) + \frac{2^n+p-1}{2p} + \frac{p^{k-1}\cdot(p-1)}{2} & p \neq 2 \text{ and } i \text{ even;} \\ \frac{2^n+p-1}{2p} - \frac{p^{k-1}\cdot(p-1)}{2} & p \neq 2 \text{ and } i \text{ odd;} \\ 3\cdot 2^{k-1} & p = 2 \text{ and } i \text{ even;} \\ 0 & p = 2 \text{ and } i \text{ odd.} \end{cases}$$

Exercise 17.40. Consider the automorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2, (a,b) \mapsto (b, -a-b)$. Then $\phi^3 = \mathrm{id}$. Show

$$K_i\left(C_r^*(\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}/3)\right) \cong \begin{cases} \mathbb{Z}^8 & i \text{ even;} \\ \{0\} & i \text{ odd.} \end{cases}$$

Theorem 17.38 in the special case where m is a prime number, is treated in [267].

The groups appearing in Theorem 17.38 are crystallographic groups, see [604, Lemma 3.1].

The proof of Theorem 17.38 is surprisingly complicated. It is based on computations of the group homology of $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ by Langer-Lück [603, Theorem 0.5]. They prove a conjecture of Adem-Ge-Pan-Petrosyan [17, Conjecture 5.2] which says that the associated Lyndon-Hochschild-Serre spectral sequence collapses in the strongest sense, in the special case that the conjugation action of $\mathbb{Z}/m$ of $\mathbb{Z}^n$ is free outside the origin $0 \in \mathbb{Z}^n$. Moreover, it uses generalizations of the Atiyah-Segal Completion Theorem for finite groups to infinite groups, see Lück-Oliver [655, 656]. Interestingly the conjecture of Adem-Ge-Pan-Petrosyan is disproved in general by Langer-Lück [603, Theorem 0.6].

17.6.10 Virtually Finitely Generated Free Abelian Groups

One does not have a complete calculation of the K-groups and L-groups of integral group rings or group C^*-algebras of crystallographic groups and hence not of virtually finitely generated abelian groups. The favorite situation is the one occurring in Remark 17.14 when one considers groups G occurring in an extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$. Then the computation can be found in [663, Theorem 1.7].

17.6.11 $\mathrm{SL}_3(\mathbb{Z})$

Since $\mathrm{SL}_3(\mathbb{Z})$ satisfies the Full Farrell-Jones Conjecture 13.27, see Theorem 16.1 (id), Theorem 13.62 implies that G satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in $\mathbb{Z}$. Using this fact the following result is proved in [908] and [944].

Theorem 17.41 (Lower and middle K-theory of the integral group ring of $\mathrm{SL}_3(\mathbb{Z})$). *The groups $K_n(\mathbb{Z}[\mathrm{SL}_3(\mathbb{Z})])$ for $n \leq -2$, $\widetilde{K}_0(\mathbb{Z}[\mathrm{SL}_3(\mathbb{Z})])$, and $\mathrm{Wh}(\mathrm{SL}_3(\mathbb{Z}))$ are trivial. For an appropriate subgroup $C_6 \subseteq \mathrm{SL}_3(\mathbb{Z})$, which is cyclic of order six, the inclusion $C_6 \to \mathrm{SL}_3(\mathbb{Z})$ induces an isomorphism*

$$\mathbb{Z} \cong K_{-1}(\mathbb{Z}[C_6]) \xrightarrow{\cong} K_{-1}(\mathbb{Z}[\mathrm{SL}_3(\mathbb{Z})]).$$

The following result is taken from [869, Corollary 2] in the complex case and from [490, Theorem 4.2] in the real case.

Theorem 17.42 (Topological equivariant K-theory of $E_{\mathcal{FIN}}(\mathrm{SL}_3(\mathbb{Z}))$).

(i) The abelian group $K_n^{\mathrm{SL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{SL}_3(\mathbb{Z})))$ is $\mathbb{Z}^8$ for even n and vanishes for odd n;
(ii) We have for $n = 0, 1, 2, \ldots, 7$

$$KO_n^{\mathrm{SL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{SL}_3(\mathbb{Z}))) = \mathbb{Z}^8, \mathbb{Z}/2^8, \mathbb{Z}/2^8, \{0\}, \mathbb{Z}^8, \{0\}, \{0\}, \{0\}$$

and the remaining groups are given by 8-fold Bott periodicity.

The groups $K_n^{\mathrm{GL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{GL}_3(\mathbb{Z})))$ are determined in [869, Corollary 4], and the groups $KO_n^{\mathrm{GL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{GL}_3(\mathbb{Z})))$ are determined in [490, Corollary 3.3].

Recall that the Baum-Connes Conjecture is not known to be true for $\mathrm{SL}_3(\mathbb{Z})$. So it would be interesting to compute $K_n(C_r^*(\mathrm{SL}_3(\mathbb{Z}); \mathbb{C}))$ and $KO_n(C_r^*(\mathrm{SL}_3(\mathbb{Z}); \mathbb{R}))$ directly and to compare the result with the computations of Theorem 17.42.

17.6.12 Right Angled Artin Groups

The group homology, the algebraic K- and L-groups, and the topological K-groups of right-angled Artin groups, and, more generally, of graph products is computed in [553, Section 6] where more generally graph products are handled.

Let X be a finite simplicial graph on the vertex set V and suppose that we are given a collection of groups $\mathcal{W} = \{W_v \mid v \in V\}$. Then the *graph product* $W(X, \mathcal{W})$ is defined as the quotient of the free product $*_{v \in V} W_v$ of the collection of groups $\mathcal{W}$ by introducing the relations

$$\{[g, g'] = 1 \mid v, v' \in V, \text{ there is an edge joining } v \text{ and } v', g \in W_v, g' \in W_{v'}\}.$$

In other words, elements of subgroups W_v and $W_{v'}$ commute if there is an edge joining v and v'. This notion is due to Green [420].

A *right-angled Artin group* is a graph product $W = W(X, \mathcal{W})$ for which each of the groups W_v is infinite cyclic. For general information about right-angled Artin groups we refer for instance to Charney [219]. Denote by Σ be flag complex associated to the finite simplicial graph X. Let $\mathcal{P}$ be the poset of simplices of Σ, both ordered by inclusion where the empty subcomplex and the empty simplex are allowed and the dimension of the empty simplex is defined to be -1. Note that W is torsionfree. In the sequel we denote by r_k the number of k-simplices in $\mathcal{P}$.

Let $\mathcal{K}_*$ be any generalized non-equivariant homology theory with values in Λ-modules. Then

$$\bigoplus_{\sigma \in \mathcal{P}} \mathcal{K}_{n-\dim(\sigma)-1}(\{\bullet\}) \xrightarrow{\cong} \mathcal{K}_n(BW).$$

If we take for $\mathcal{K}_*$ singular homology $H_*(-;\Lambda)$ with coefficients in Λ, this boils down to the well-known isomorphism of Λ-modules

$$\Lambda^{r_{n-1}} \xrightarrow{\cong} H_n(BW;\Lambda). \tag{17.43}$$

In particular we get the following relation for the Euler characteristics

$$\chi(BW) = 1 - \chi(\Sigma).$$

Theorem 17.44 (The algebraic K-theory and L-theory of right-angled Artin groups).

(i) Let R be a regular ring. Then there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(R) \xrightarrow{\cong} K_n(RW).$$

In particular we get $K_n(RW) = 0$ for $n \leq -1$.

If we take $R = \mathbb{Z}$, we conclude that $K_n(\mathbb{Z}W)$ for $n \le -1$, $\widetilde{K}_0(\mathbb{Z}W)$, and $\mathrm{Wh}(W)$ *vanish.*

(ii) Let R be a ring with involution. Then there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} L^{\langle -\infty \rangle}_{n-\dim(\sigma)-1}(R) \xrightarrow{\cong} L^{\langle -\infty \rangle}_n(RW).$$

Theorem 17.45 (The topological K-theory of right-angled Artin groups). *There are explicit isomorphisms of abelian groups*

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(\mathbb{C}) \xrightarrow{\cong} K_n(C^*_m(W)) \cong K_n(C^*_r(W));$$

$$\bigoplus_{\sigma \in \mathcal{P}} KO_{n-\dim(\sigma)-1}(\mathbb{R}) \xrightarrow{\cong} KO_n(C^*_m(W;\mathbb{R})) \cong KO_n(C^*_r(W;\mathbb{R})).$$

In particular we get an isomorphism of abelian groups

$$K_n(C^*_m(W)) \cong K_n(C^*_r(W)) \cong \mathbb{Z}^{t_n},$$

if we put $t_n = \sum_{\substack{k \in \{-1,0,1,2,\dots,\dim(\Sigma)\} \\ (n-k) \text{ odd}}} r_k$.

Exercise 17.46. Let G be $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$ where we consider $\mathbb{Z}$ as a subgroup of $\mathbb{Z}^2$ by sending n to $(n, 0)$. Compute $H_*(G)$, $K_*(C^*_r(G;\mathbb{C}))$, and $KO_*(C^*_r(G;\mathbb{R}))$

17.6.13 Right Angled Coxeter Groups

Recall that a *right-angled Coxeter group* is a graph product $W = W(X, \mathcal{W})$ for which each of the groups W_v is cyclic of order two. The group homology, the algebraic K- and L-groups, and the topological K-groups of right-angled Coxeter groups, and, more generally, of graph products is computed in [553, Section 7]. The result are nearly as explicite as in the case of right-angled Artin groups which we have presented in Subsection 17.6.12.

For instance, the integral group homology $H_n(W;\mathbb{Z})$ is in degree $n \ge 1$ an explicite $\mathbb{F}_2$-vector space, $K_n(\mathbb{Z}W) = 0$ for $n \le -1$, $\widetilde{K}_0(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = 0$, and $K_1(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = 0$. Next we state the result for the topological K-theory.

Theorem 17.47 (The topological K-theory of right-angled Coxeter groups). *There are for every $n \in \mathbb{Z}$ isomorphisms*

$$\bigoplus_{\sigma \in \mathcal{P}} K_n(\mathbb{C}) \xrightarrow{\cong} K_n(C_m^*(W)) \cong K_n(C_r^*(W));$$
$$\bigoplus_{\sigma \in \mathcal{P}} KO_n(\mathbb{R}) \xrightarrow{\cong} KO_n(C_m^*(W;\mathbb{R})) \cong KO_n(C_r^*(W;\mathbb{R})).$$

In particular there are isomorphisms of abelian groups

$$K_n(C_m^*(W)) \cong K_n(C_r^*(W)) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \text{ is even;} \\ \{0\} & \text{otherwise;} \end{cases}$$
$$KO_n(C_m^*(W;\mathbb{R})) \cong KO_n(C_r^*(W;\mathbb{R})) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \equiv 0 \mod 4; \\ (\mathbb{Z}/2)^r & \text{if } n \equiv 1,2 \mod 8; \\ \{0\} & \text{otherwise,} \end{cases}$$

where r is the number of simplices (including the empty simplex) in $\mathcal{P}$.

The computation of the topological K-theory of the complex reduced group C^*-algebra of a right-angled Coxeter group is also done by Sanchez-Garcia [870] using the Davis complex as a model for $\underline{E}W$. The real case is treated by Fuentes [394].

Exercise 17.48. Let G be a group that is isomorphic to some amalgamated free product of the form $(\mathbb{Z}/2)^3 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$. Compute $K_n(C_r^*(G;\mathbb{C}))$ and $KO_n(C_r^*(G;\mathbb{R}))$ for $n \in \mathbb{Z}$.

17.6.14 Fundamental Groups of 3-Manifolds

The algebraic K-theory $K_n(R[\pi_1(M)])$ has been computed for a compact connected 3-manifold M in [517] based on Theorem 16.1 (ie) and [510] modulo Nil-terms of the ring R. We at least present the computation for an already interesting special case, also including the algebraic L-theory.

Theorem 17.49 (K-and L-groups of 3-manifolds). *Let M be a compact connected orientable 3-manifold with fundamental group π and prime decomposition $M \cong M_1 \sharp M_2 \sharp \cdots \sharp M_r$.*

(i) Suppose that R is a regular ring. Then we get for $n \in \mathbb{Z}$

$$\overline{K}_n(R\pi) \cong \bigoplus_{i=1}^{n} \overline{K}_n(R[\pi_1(M_i)]);$$
$$K_n(R\pi) \cong 0 \quad \textit{if } n \leq -1,$$

where $\overline{K}_n(RG)$ is the cokernel of the split injective map $K_n(R) \to K_n(RG)$. If π is torsionfree, then there is an isomorphism

$$H_n(B\pi;\mathbf{K}_R) \xrightarrow{\cong} K_n(R\pi);$$

(ii) Let R be a ring with involution. Suppose that π contains no 2-torsion. We get for $n \in \mathbb{Z}$

$$\overline{L}_n^{\langle -\infty\rangle}(R\pi) \cong \bigoplus_{i=1}^{n} \overline{L}_n^{\langle -\infty\rangle}(R[\pi_1(M_i)])$$

where $\overline{L}_n^{\langle -\infty\rangle}(RG)$ is the cokernel of the split injective map $L_n^{\langle -\infty\rangle}(R) \to L_n^{\langle -\infty\rangle}(RG)$.
If π is torsionfree, then there is an isomorphism

$$H_n(B\pi;\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(R\pi).$$

Proof. We conclude from Theorem 16.1 (ie) that π satisfies the Full Farrell-Jones Conjecture 13.27.

Note that $\pi \cong *_{i=1}^r \pi_1(M_i)$. The Kurosh Subgroup Theorem, see [678, Theorem 1.10 on page 178], says for a subgroup $H \subseteq \pi$ that $H \cong (*_{j\in J} H_j) * F$ where each H_j is the intersection of H with some conjugate of $\pi_1(M_i)$ and F is a free group. Note that $\pi_1(M_i)$ is either finite or torsionfree since every irreducible 3-manifold with infinite fundamental group is aspherical by the Sphere Theorem, see [460, 4.3 on page 40], and a prime 3-manifold that is not irreducible is a S^2 bundle over S^1, see [460, Lemma 3.13 on page 28]. Every torsionfree virtually cyclic group is isomorphic to $\mathbb{Z}$. A virtually cyclic group V is isomorphic to a non-trivial free product $L_1 * L_2$ if and only if V is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$. Hence any virtually cyclic subgroup V of π is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$.

(i) Since R is regular, we conclude from Lemma 13.51 and Lemma 13.52 that the assembly map

$$H_n^\pi(\underline{E}\pi;\mathbf{K}_R) \to K_n(R\pi)$$

is an isomorphism for $n \in \mathbb{Z}$. We conclude from Example 15.30 that the obvious map $\bigoplus_{i=1}^n \overline{K}_n(R[\pi_1(M_i)]) \to \overline{K}_n(R\pi)$ is bijective. The claim in the special case that π is torsionfree follows from Conjecture 6.53, which holds for π by Theorem 13.62 (xii).

(ii) The assembly map

$$H_n^\pi(\underline{E}\pi;\mathbf{L}_R^{\langle -\infty\rangle}) \to L_n^{\langle -\infty\rangle}(R\pi)$$

is an isomorphism by Theorem 13.57 since every virtually cyclic subgroup of π is isomorphic to $\mathbb{Z}$. We conclude from Example 15.30 that the obvious map $\bigoplus_{i=1}^n \overline{L}_n^{\langle -\infty\rangle}(R[\pi_1(M_i)]) \to \overline{L}_n^{\langle -\infty\rangle}(R\pi)$ is bijective. The claim in the special case that π is torsionfree follows from Conjecture 9.114, which holds for π by Theorem 13.62 (xii). $\square$

Exercise 17.50. Let M be a connected orientable irreducible closed 3-manifold with infinite fundamental group π. Show that $L_n^{\langle i \rangle}(\mathbb{Z}\pi)$ is independent of the decoration and that we have isomorphisms

$$\begin{aligned} L_0(\mathbb{Z}\pi) &\cong \mathbb{Z} \oplus \hom_{\mathbb{Z}}(\pi, \mathbb{Z}/2); \\ L_1(\mathbb{Z}\pi) &\cong \pi/[\pi,\pi] \oplus \mathbb{Z}/2; \\ L_2(\mathbb{Z}\pi) &\cong \mathbb{Z}/2 \oplus \hom_{\mathbb{Z}}(\pi, \mathbb{Z}); \\ L_3(\mathbb{Z}\pi) &\cong \mathbb{Z} \oplus (\pi/[\pi,\pi] \otimes_{\mathbb{Z}} \mathbb{Z}/2). \end{aligned}$$

17.7 Applications of Some Computations

17.7.1 Classification of Some C^*-algebras

Theorem 17.38 is an important input in the classification of certain C^*-algebras associated to number fields by Li-Lück [613]. Here the key point is the rather surprising result that the topological K-groups are all torsion-free what is not the case for the group homology. Actually, it is intriguing that the topological complex K-groups are finitely generated free abelian groups in many of the examples presented in Subsection 17.6.

Another application of the computation of the topological K-theory of group C^*-algebras can be found in [310], namely, to the structure of crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

17.7.2 Unstable Gromov-Lawson Rosenberg Conjecture

We have already discussed in Subsection 14.8.2 that Schick [878] constructed counterexamples to the unstable version of the Gromov-Lawson-Rosenberg Conjecture with fundamental group $\pi \cong \mathbb{Z}^4 \times \mathbb{Z}/3$. However for appropriate $\rho\colon \mathbb{Z}/3 \to \mathrm{aut}(\mathbb{Z}^4)$ the unstable version does hold for $\pi \cong \mathbb{Z}^4 \rtimes_\rho \mathbb{Z}/3$ and $\dim(M) \geq 5$. This is proved by Davis-Lück [267, Theorem 0.7 and Remark 0.9] based on explicite calculations of the topological K-theory of the reduced real group C^*-algebra of $\mathbb{Z}^4 \rtimes_\rho \mathbb{Z}/3$. More infinite groups for which the unstable version holds are presented in [490, Theorem 6.3].

17.7.3 Classification of Certain Manifolds with Infinite Not Torsionfree Fundamental Groups

Manifolds homotopy equivalent to the total space of certain fiber bundles over lens spaces with tori as fiber are classified by Davis-Lück [268]; see also [969]. Here the key input is the calculation of algebraic K-and L-groups of integral group rings of groups of the shape $\pi = \mathbb{Z} \rtimes_\rho \mathbb{Z}/p$ for odd primes p where the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free outside the origine. Note that π is infinite and not torsionfree. This is one of the few classification result about a class of closed manifolds whose fundamental group is not obtained from torsionfree and finite groups using amalgamated free products and HNN-extensions.

17.8 Notes

The lower and middle algebraic K-theory of integral group rings of certain reflection groups has been computed by Lafont-Ortiz [590] and by Lafont-Margurn-Ortiz [588], of $\varGamma_3 := O^+(3,1) \cap \mathrm{GL}_4(\mathbb{Z})$ by Ortiz [765, 766], of Bianchi groups by Berkove-Farrell-Pineda-Pearson [118], and of pure braid groups by Aravinda-Farrell-Roushon [37]. The lower and middle algebraic K-theory of integral group rings or mapping class group of genus 1 is computed in [119]. The topological K-theory of the complex group C^*-algebra of cocompact 3-dimensional hyperbolic reflection group is computed by by Lafont-Ortiz-Rahm-Sanchez-Gracia [592]. Computations of the algebraic K-groups $K_n(RG)$ for Artin groups G of dihedral type can be found in [33].

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Chapter 18
Assembly Maps

18.1 Introduction

In this chapter we discuss assembly maps and the assembly principle im general.

We recall the homological approach in Section 18.2 which we have used in this book.

We give the version in terms of spectra in Section 18.3. Actually, in all concrete situations, such as in the Farrell-Jones Conjecture for K-and L-theory and pseudoisotopy or the Baum-Connes Conjecture, the assembly map can be implemented in terms of spectra. This can easily be identified with the elementary approach in terms of homotopy colimits, which nicely illustrates the name assembly, but works only, if we confine ourselves to classifying spaces of families of subgroups, see Section 18.4. The approach in terms of homotopy colimits is the quickest and most natural approach for a homotopy theorist.

The universal property of assembly is explained in Section 18.5. Roughly speaking, it says that the assembly map is the best approximation of a weakly homotopy invariant functor $\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$ from the left by a weakly excisive functor $G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$, where weakly excisive essentially means that after taking homotopy groups the functor yields a G-homology theory. This is very helpful to identify the various versions of the assembly maps appearing in the literature with our homological approach since the constructions of the assembly maps can be very complicated and is much easier to use the universal property to establish the desired identifications than to go through the actual definitions. The universal property will be exploited to identify the various assembly maps in Section 18.6.

This universal approach explains the philosophical background of assembly and presents a uniform approach to the assembly map in all cases, such as the Farrell-Jones Conjecture or the Baum-Conjecture. It is important to have also the other more geometric or operator-theoretic definitions of assembly maps in terms of surgery theory or index theory at hand in order to apply the Farrell-Jones Conjecture and the Baum-Connes Conjecture to geometric problems, such as the topological rigidity of closed aspherical manifolds or the existence of a Riemannian metric with positive sectional curvature.

The homological or homotopy theoretic approach to assembly maps is best suited for computations based on the Isomorphism Conjectures, but not necessarily for their proofs, where the approach using index theory or controlled topology come into play.

18.2 Homological Approach

The homological version of assembly is manifested in the Meta-Isomorphism Conjecture 15.2. Recall that it predicts for a group G, a family $\mathcal{F}$ of subgroups of G, and a G-homology theory $\mathcal{H}_*^G$ in the sense of Definition 12.1 that the map induced by the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ for $E_{\mathcal{F}}(G)$ the classifying space of the family $\mathcal{F}$ in the sense of Definition 11.18,

$$\mathcal{H}_n(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G) \tag{18.1}$$

is bijective for all $n \in \mathbb{Z}$. The various conjectures due to Baum-Connes and Farrell-Jones are special cases where one specifies $\mathcal{F}$ and $\mathcal{H}_*^G$.

18.3 Extension from Homogenous Spaces to G-CW-Complexes

Let $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum, i.e., a covariant functor $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$. We get an extension of $\mathbf{E}$ to the category G-CW-COM of G-CW-complexes by

$$\mathbf{E}_{\%}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}, \quad X \mapsto \mathrm{map}_G(-,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}, \tag{18.2}$$

where $\mathrm{map}_G(-,X)$ and $\wedge_{\mathsf{Or}(G)}$ have been defined in Example 12.24 and in (12.25). The projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ for $E_{\mathcal{F}}(G)$ induces a map of spectra

$$\mathbf{E}_{\%}(\mathrm{pr})\colon \mathbf{E}_{\%}(E_{\mathcal{F}}(G)) \to \mathbf{E}_{\%}(G/G). \tag{18.3}$$

After taking homotopy groups we get for all $n \in \mathbb{Z}$ a homomorphism

$$\pi_n\big(\mathbf{E}_{\%}(\mathrm{pr})\big)\colon \pi_n\big(\mathbf{E}_{\%}(E_{\mathcal{F}}(G))\big) \to \pi_n\big(\mathbf{E}_{\%}(G/G)\big). \tag{18.4}$$

We have constructed a G-homology theory $H_*^G(-;\mathbf{E})$ with the property that $H_n^G(G/H;\mathbf{E}) \cong \pi_n(\mathbf{E}(G/H))$ holds for all $n \in \mathbb{Z}$ and subgroups $H \subseteq G$ in Theorem 12.27. The G-homology theories relevant for the Baum-Connes and the Farrell-Jones Conjecture are given by specifying such covariant functors $\mathbf{E}$. It follows essentially from the definitions that the map (18.1) for $\mathcal{H}_*^G = H_*^G(-;\mathbf{E})$ agrees with the map (18.4).

18.4 Homotopy Colimit Approach

Consider a covariant functor $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$. Recall that $\mathsf{Or}_{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-restricted orbit category, see Definition 2.64. If the G-homology theory $\mathcal{H}_*^G$ is given by $H_*^G(-;\mathbf{E})$, one can identify the assembly map (18.4) with the map

$$\pi_n(\mathbf{p})\colon \pi_n\left(\operatorname{hocolim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E}\right) \to \pi_n(\mathbf{E}(G/G)) \tag{18.5}$$

where the map of spectra

$$\mathbf{p}\colon \operatorname{hocolim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E} \to \operatorname{hocolim}_{\mathsf{Or}(G)} \mathbf{E} = \mathbf{E}(G/G)$$

comes from the inclusion of categories $\mathsf{Or}_{\mathcal{F}}(G) \to \mathsf{Or}(G)$ and the fact that G/G is a terminal object in $\mathsf{Or}(G)$. For more information about homotopy colimits and the identification of the maps (18.1), (18.4), and (18.5) we refer to [265, Sections 3 and 5].

This interpretation is one explanation for the name *assembly*. If the assembly map (18.5) is bijective for all $n \in \mathbb{Z}$, or, equivalently, the map $\mathbf{p}$ above is is a weak homotopy equivalence, we have a recipe to assemble $\mathbf{E}(G/G)$ from its values $\mathbf{E}(G/H)$, where H runs through $\mathcal{F}$. The idea is tha $\mathcal{F}$ consists of well-understood subgroups, for which one knows the values $\mathbf{E}(G/H)$ for $H \subseteq G$ and hence $\operatorname{hocolim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E}$, whereas $\mathbf{E}(G/G)$ is the object, which one wants to understand and is very hard to access.

18.5 Universal Property

In this section we characterize assembly maps by a universal property. This is useful for identifying different constructions of assembly maps.

Lemma 18.6. *Let* $\mathbf{E}$ *be a covariant* $\mathsf{Or}(G)$*-spectrum. Then:*

(i) The canonical map

$$\mathbf{E}_{\%}(X) \cup_{\mathbf{E}_{\%}(f)} \mathbf{E}_{\%}(Y) \to \mathbf{E}_{\%}(X \cup_f Y)$$

is an isomorphism of spectra where (X, A) *is a* G*-CW-pair and* $f\colon A \to Y$ *is a cellular* G*-map;*

(ii) The canonical map

$$\operatorname{colim}_{i \in I} \mathbf{E}_{\%}(X_i) \to \mathbf{E}_{\%}(X)$$

is an isomorphism of spectra where $\{X_i \mid i \in I\}$ *is a directed system of* G*-CW-subcomplexes of the* G*-CW-complex* X *directed by inclusion and satisfying* $X = \bigcup_{i \in I} X_i$*;*

(iii) The canonical map

$$Z_+ \wedge \mathbf{E}_\%(X) \to \mathbf{E}_\%(Z \times X)$$

is an isomorphism of spectra where Z is a CW-complex (with trivial G-action) and X is a G-CW-complex;

(iv) The canonical map

$$\mathbf{E}_\%(G/H) \to \mathbf{E}(G/H)$$

is an isomorphism of spectra for all $H \in \mathcal{F}$.

Proof. One easily checks that the H-fixed point set functor $\operatorname{map}_G(G/H, -)$ commutes with attaching a G-space to a G-space along a G-map and with directed unions of G-CW-subcomplexes. Assertions (i) and (ii) follow from the fact that $- \wedge_{\mathsf{Or}(G)} \mathbf{E}$ commutes with colimits since it has an right adjoint, see [265, Lemma 1.5]. Assertions (iii) and (iv) follow by inspecting the definition of $\mathbf{E}_\%$. □

Lemma 18.7. *Let $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Then the extension $\mathbf{E} \mapsto \mathbf{E}_\%$ is uniquely determined up to isomorphism of G-CW-COM-spectra by the properties of Lemma 18.6.*

Proof. Let $\mathbf{E} \mapsto \mathbf{E}_\$$ be another such extension. There is a (a priori not necessarily continuous) set-theoretic natural transformation

$$\mathbf{T}(X) : \mathbf{E}_\%(X) = X_+ \wedge_{\mathsf{Or}(G)} \mathbf{E} \longrightarrow \mathbf{E}_\$(X)$$

which sends an element represented by $(x\colon G/H \longrightarrow X, e)$ in $\operatorname{map}_G(G/H, X) \times \mathbf{E}(G/H)$ to $\mathbf{E}_\$(x)(e)$. Since any G-CW-complex is constructed from orbits G/H with $H \in \mathcal{F}$ via products with disks and disjoint unions, attaching a G-space to a G-space along a G-map, and is the directed union over its skeletons, and $\mathbf{T}(G/H)$ is an isomorphism of spectra for $H \subseteq G$, $\mathbf{T}(X)$ is an isomorphism of spectra for all G-CW-complexes X. □

Lemma 18.7 is a characterization of $\mathbf{E} \mapsto \mathbf{E}_\%$ up to isomorphism. Next we give a homotopy theoretic characterization.

Definition 18.8 ((Weakly) excisive). We call a covariant functor

$$\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$$

(weakly) homotopy invariant if it sends G-homotopy equivalences to (weak) homotopy equivalences of spectra.

The functor $\mathbf{E}$ is *(weakly) excisive* if it has the following four properties:

- It is (weakly) homotopy invariant;
- The spectrum $\mathbf{E}(\emptyset)$ is (weakly) contractible;

- It respects homotopy pushouts up to (weak) homotopy equivalence, i.e., if the G-CW-complex X is the union of G-CW-subcomplexes X_1 and X_2 with intersection X_0, then the canonical map from the homotopy pushout of $\mathbf{E}(X_2) \longleftarrow \mathbf{E}(X_0) \longrightarrow \mathbf{E}(X_2)$ to $\mathbf{E}(X)$ is a (weak) homotopy equivalence of spectra;
- It respects disjoint unions up to (weak) homotopy, i.e., the natural map $\bigvee_{i\in I} \mathbf{E}(X_i) \to \mathbf{E}(\coprod_{i\in I} X_i)$ is a (weak) homotopy equivalence for all index sets I.

Exercise 18.9. Let $\mathbf{E}\colon \mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ be an excisive functor for the trivial group. Show that the functor $G\text{-}\mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ sending X to $\mathbf{E}(X/G)$ is excisive.

Notation 18.10. If $\mathbf{E}\colon G\text{-}\mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ is a covariant functor, we denote $(\mathbf{E}|_{\mathsf{Or}(G)})_\%$ by $\mathbf{E}_\%$ again where $\mathbf{E}|_{\mathsf{Or}(G)}$ is the composite of $\mathbf{E}$ with the obvious inclusion $\mathsf{Or}(G) \to G\text{-}\mathsf{CW\text{-}COM}$.

The following result has been proved for $G = \{1\}$ in Weiss-Williams [982].

Theorem 18.11 (Universal Property of assembly).

(i) Suppose $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ *is a covariant functor. Then* $\mathbf{E}_\%$ *is excisive;*

(ii) Suppose $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ *is a covariant functor. Then we obtain a G-homology theory* $H_n^G(-;\mathbf{E})$ *in the sense of Definition 12.1 from Theorem 12.27, and we get for every pair (X,A) of G-CW-complexes (X,A) a natural isomorphism*

$$H_n^G(X,A;\mathbf{E}) \cong \operatorname{coker}\bigl(\pi_n(\mathbf{E}_\%(\emptyset_+)) \to \pi_n(\mathbf{E}_\%(X/A))\bigr).$$

If $A = \emptyset$, this becomes an isomorphism

$$H_n^G(X;\mathbf{E}) \cong \pi_n(\mathbf{E}_\%(X));$$

(iii) Let $\mathbf{T}\colon \mathbf{E} \to \mathbf{F}$ *be a transformation of (weakly) excisive functors* $\mathbf{E}$ *and* $\mathbf{F}$ *from* $G\text{-}\mathsf{CW\text{-}COM}$ *to* $\mathsf{SPECTRA}$ *so that* $\mathbf{T}(G/H)$ *is a (weak) homotopy equivalence of spectra for all $H \subseteq G$.*
Then $\mathbf{T}(X)$ *is a (weak) homotopy equivalence of spectra for all G-CW-complexes X;*

(iv) For any (weakly) homotopy invariant functor $\mathbf{E}$ *from* $G\text{-}\mathsf{CW\text{-}COM}$ *to* $\mathsf{SPECTRA}$*, there is a (weakly) excisive functor*

$$\mathbf{E}^\%\colon G\text{-}\mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$$

and natural transformations

$$\mathbf{A}_\mathbf{E}\colon \mathbf{E}^\% \to \mathbf{E};$$
$$\mathbf{B}_\mathbf{E}\colon \mathbf{E}^\% \to \mathbf{E}_\%,$$

which induce (weak) homotopy equivalences of spectra $\mathbf{A_E}(G/H)$ for all $H \subseteq G$ and (weak) homotopy equivalences of spectra $\mathbf{B_E}(X)$ for all G-CW-complexes X.

The constructions $\mathbf{E}_\%$, $\mathbf{E}^\%$, $\mathbf{A_E}$ and $\mathbf{B_E}$ are natural in $\mathbf{E}$.

Moreover, $\mathbf{E}$ is (weakly) excisive if and only if $\mathbf{A_E}(X)$ is a (weak) homotopy equivalence of spectra for all G-CW-complexes X.

Proof. (i) follows from Lemma 18.6 after one has shown that in the situation of Lemma 18.6 (i) the canoncial map from the homotopy pushout of spectra to the pushout of spectra is a weak homotopy equivalence. This follows from the fact that the inclusion of $\mathbf{E}_\%(A) \to \mathbf{E}_\%(X)$ is on each level a cofibration of spaces.

(ii) There is an obvious G-homotopy equivalence of pointed G-CW-complexes $X_+ \cup_{A_+} \operatorname{cone}(A_+) \to X/A$. Hence we get from the definitions

$$H_n^G(X, A; \mathbf{E}) = \pi_n\bigl(\operatorname{map}_G(-, X/A) \wedge_{\mathsf{Or}(G)} \mathbf{E}\bigr).$$

Now the assertion follows from the cofibration sequence of spectra

$$\begin{aligned}
&\mathbf{E}_\%(\emptyset_+) = \operatorname{map}_G(-, \emptyset_+)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E} \\
\to\ &\mathbf{E}_\%(X/A) = \operatorname{map}_G(-, X/A)_+ \wedge_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E} \to \operatorname{map}_G(-, X/A) \wedge_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E}.
\end{aligned}$$

(iii) Use the fact that a (weak) homotopy colimit of homotopy equivalences of spectra is again a (weak) homotopy equivalence of spectra.

(iv) See [265, Theorem 6.3]. $\square$

Exercise 18.12. Show that a covariant functor $\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$ is weakly excisive if and only if the assignment sending a pair (X, A) of G-CW-complexes to $\operatorname{coker}\bigl(\pi_n(\mathbf{E}(\emptyset_+)) \to \pi_n(\mathbf{E}(X/A))\bigr)$ defines a G-homology in the sense of Definition 12.1.

Exercise 18.13. Let $\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$ be a weakly excisive functor such that $\pi_n(\mathbf{E}(G/H))$ is finitely generated for every $H \subseteq G$ and $n \in \mathbb{Z}$. Then $\pi_n(\mathbf{E}(X))$ is finitely generated for every finite G-CW-complex X and $n \in \mathbb{Z}$.

Definition 18.14 (Homotopy theoretic assembly transformation). Given a covariant functor $\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$, we call the transformation appearing in Theorem 18.11 (iv)

$$\mathbf{A_E}\colon \mathbf{E}^\% \to \mathbf{E}$$

the *homotopy theoretic assembly transformation.*

Remark 18.15 (No continuity condition E). One may be tempted to define a natural transformation $\mathbf{S}\colon \mathbf{E}_\% \to \mathbf{E}$ as indicated in the proof of

Lemma 18.7. Then $\mathbf{S}(X)$ is a well-defined bijection of sets but is not necessarily continuous because we do not want to assume that $\mathbf{E}$ is continuous, i.e., that the induced map from $\hom_{\mathcal{C}}(X,Y)$ to $\hom_{\mathcal{C}}(\mathbf{E}(X),\mathbf{E}(Y))$ is continuous for all G-CW-complexes X and Y. This is the reason why we have to pass to the more complicated construction of $\mathbf{E}^{\%}$ and only obtain a zigzag

$$\mathbf{E}_{\%} \xleftarrow{\mathbf{B_E}} \mathbf{E}^{\%} \xrightarrow{\mathbf{A_E}} \mathbf{E},$$

which suffices for all our purposes. The construction of this zigzag uses the (weak) homotopy invariance of $\mathbf{E}$ and does not require any continuity condition for $\mathbf{E}$.

Theorem 18.11 implies

Corollary 18.16. Let $\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$ be a weakly excisive functor. Denote by $\mathbf{E}|_{\mathsf{Or}(G)}$ its restriction to a covariant functor $\mathsf{Or}(G) \to \mathsf{SPECTRA}$.

Then we obtain for all $n \in \mathbb{Z}$ and G-CW-complex X an isomorphism, natural in X,

$$\pi_n(\mathbf{E}(X)) \xrightarrow{\cong} H_n^G(X;\mathbf{E}|_{\mathsf{Or}(G)}).$$

In particular we get for every family of subgroups $\mathcal{F}$ and $n \in \mathbb{Z}$ a commutative diagram with isomorphisms as vertical arrows

$$\begin{tikzcd}
\pi_n(\mathbf{E}(E_{\mathcal{F}}(G))) \arrow[r, "\pi_n(\mathbf{E}(\mathrm{pr}))"] \arrow[d, "\cong"'] & \pi_n(\mathbf{E}(G/G)) \arrow[d, "\cong"] \\
H_n^G(E_{\mathcal{F}}(G);\mathbf{E}|_{\mathsf{Or}(G)}) \arrow[r, "H_n^G(\mathrm{pr};\mathbf{E}|_{\mathsf{Or}(G)})"] & H_n^G(G/G;\mathbf{E}|_{\mathsf{Or}(G)}).
\end{tikzcd}$$

Exercise 18.17. Consider the covariant functor

$$\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}, \quad X \mapsto \mathbf{K}_R(\Pi(EG \times_G X))$$

where $\Pi(EG \times_G X)$ is the fundamental groupoid of the space $EG \times_G X$ and $\mathbf{K}_R\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ has been defined in (12.44). Suppose that $\mathbf{E}$ is weakly excisive.

Show that then for every family $\mathcal{F}$ of subgroups the assembly map induced by the projection $E_{\mathcal{F}}(G) \to G/G$

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_R) \to H_n^G(G/G;\mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Remark 18.18 (Universal property of the homotopy theoretic assembly transformation). Next we explain why Theorem 18.11 characterizes the homotopy theoretic assembly map

$$\mathbf{A_E} \colon \mathbf{E}^\% \longrightarrow \mathbf{E}$$

in the sense that it is the universal approximation from the left by a (weakly) excisive functor of a (weakly) homotopy invariant functor $\mathbf{E}$ from G-CW-COM to $\mathsf{SPECTRA}$ up to (weak) homotopy equivalence. Namely, let $\mathbf{T} \colon \mathbf{F} \to \mathbf{E}$ be a transformation of covariant functors from G-CW-COM to $\mathsf{SPECTRA}$ such that $\mathbf{F}$ is (weakly) excisive and $\mathbf{T}(G/H)$ is a (weak) homotopy equivalence for all $H \subseteq G$. Then for any G-CW-complex X the following diagram commutes

$$\begin{tikzcd}
\mathbf{F}^\%(X) \arrow[r, "\mathbf{A_F}(X)"] \arrow[d, "\mathbf{T}^\%(X)"'] & \mathbf{F}(X) \arrow[d, "\mathbf{T}(X)"] \\
\mathbf{E}^\%(X) \arrow[r, "\mathbf{A_E}(X)"'] & \mathbf{E}(X)
\end{tikzcd}$$

and $\mathbf{A_F}(X)$ and $\mathbf{T}^\%(X)$ are (weak) homotopy equivalences. Hence one may say that $\mathbf{T}(X)$ factorizes over $\mathbf{A_E}(\mathbf{X})$ up to (weak) homotopy equivalence.

In particular we obtain for every G-CW-complex X a commutative diagram with an isomorphism as vertical arrow

$$\begin{tikzcd}
\pi_n(\mathbf{F}(X)) \arrow[dd, "\pi_n(\mathbf{T}^\%(X))\circ\pi_n(\mathbf{A_F}(X))^{-1}"', "\cong"] \arrow[dr, "\pi_n(\mathbf{T}(X))"] & \\
& \pi_n(\mathbf{E}(X)). \\
\pi_n(\mathbf{E}^\%(X)) \arrow[ur, "\pi_n(\mathbf{A_E}(X))"'] &
\end{tikzcd}$$

18.6 Identifying Assembly Maps

In this section we explain and summarize that we can identify all the various assembly maps we have studied so far.

We recall that we have the following versions of assembly maps.

- The Meta-Isomorphism Conjecture 15.2 with respect to the G-homology theory $\mathcal{H}_*^G$ and the family $\mathcal{F}$ of subgroups of G, where the assembly map

$$\mathcal{H}_n(\mathrm{pr}) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

  comes from the projection $\mathrm{pr} \colon E_{\mathcal{F}}(G) \to G/G$;

- The Meta-Isomorphism Conjecture 15.2, where the equivariant homology theory comes from a functor $\mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ respecting equivalences, see Theorem 12.30 and Section 12.5;
- The Meta-Isomorphism Conjecture 15.36 for functors from spaces to spectra;
- The homotopy theoretic assembly transformation in the sense of Definition 18.14;
- For the L-theoretic Farrell-Jones Conjecture and G the fundamental group of an aspherical closed manifold, the assembly map given by taking surgery obstructions, see the sketch of the proof of Theorem 9.168 in Subsection 9.15.3;
- For the Baum-Connes Conjecture in terms of index theory, see Section 14.2.

Remark 18.19 (The homotopy theoretic assembly transformation and the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients). Consider a functor $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ which respects weak equivalences and disjoint unions. Given a group G and a free G-CW-complex Z, we get a functor functor

$$\mathbf{S}_Z^G\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}, \quad X \mapsto \mathbf{S}(X \times_G Z)$$

whose restriction to $\mathsf{Or}(G)$ is denoted in the same way and has already been introduced in (15.38). The Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients predicts for a family $\mathcal{F}$ of subgroups of G that the map

$$H_n^G(\mathrm{pr}; \mathbf{S}_Z^G)\colon H_n^G(E_{\mathcal{F}}(G); \mathbf{S}_Z^G) \to \mathcal{H}_n^G(G/G; \mathbf{S}_Z^G) \tag{18.20}$$

induced by the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ is bijective for all $n \in \mathbb{Z}$. This map can be identified with the corresponding map for the homotopy theoretic assembly map

$$\pi_n\big(\mathbf{A}_{\mathbf{S}_Z^G}(E_{\mathcal{F}}(G))\big)\colon \pi_n\big((\mathbf{S}_Z^G)^{\%}(E_{\mathcal{F}}(G))\big) \to \pi_n\big(\mathbf{S}_Z^G(E_{\mathcal{F}}(G))\big) \tag{18.21}$$

by the following argument. Because of Theorem 18.11 (ii) the map (18.20) can be identified with the map induced by the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$

$$\pi_n\big((\mathbf{S}_Z^G)_{\%}(\mathrm{pr})\big)\colon \pi_n\big((\mathbf{S}_Z^G)_{\%}(E_{\mathcal{F}}(G))\big) \to \pi_n\big((\mathbf{S}_Z^G)_{\%}(G/G)\big),$$

and hence by Theorem 18.11 (iv) with the map

$$\pi_n\big((\mathbf{S}_Z^G)^{\%}(\mathrm{pr})\big)\colon \pi_n\big((\mathbf{S}_Z^G)^{\%}(E_{\mathcal{F}}(G))\big) \to \pi_n\big((\mathbf{S}_Z^G)^{\%}(G/G)\big). \tag{18.22}$$

We have the following commutative diagram

$$\begin{tikzcd}
(\mathbf{S}_Z^G)^{\%}(E_{\mathcal{F}}(G)) \arrow[r, "(\mathbf{S}_Z^G)^{\%}(\mathrm{pr})"] \arrow[d, "\mathbf{A}_{\mathbf{S}_Z^G}(E_{\mathcal{F}}(G))"'] & (\mathbf{S}_Z^G)^{\%}(G/G) \arrow[d, "\mathbf{A}_{\mathbf{S}_Z^G}(G/G)"] \\
\mathbf{S}_Z^G(E_{\mathcal{F}}(G)) \arrow[r, "(\mathbf{S}_Z^G)^{\%}(\mathrm{pr})"'] & \mathbf{S}_Z^G(G/G).
\end{tikzcd}$$

The right vertical arrow is a weak homotopy equivalence by Theorem 18.11 (iv). Since Z is a free G-CW-complex and $E_{\mathcal{F}}(G)$ is contractible (after forgetting the group action), the map $\mathrm{id} \times_G \mathrm{pr}\colon Z \times_G E_{\mathcal{F}}(G) \to Z \times_G G/G$ is a homotopy equivalence and hence the lower horizontal arrow is a weak homotopy equivalence. Hence we get an identification of the maps (18.21) and (18.22). Thus we have identified the maps (18.20) and (18.21).

Example 18.23 (The Farrell-Jones Conjecture and the Baum-Connes Conjecture in the setting of the homotopy theoretic assembly transformation). In the sequel $\Pi(X)$ denotes the fundamental groupoid of a space X. If we take in Remark 18.19 the covariant functor $\mathbf{S}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ to be the one, which sends a space X to $\mathbf{K}_R(\Pi(X))$ or $\mathbf{L}_R^{\langle -\infty\rangle}(\Pi(X))$ respectively, see Theorem 12.43, then we conclude from Example 15.37 and Remark 18.19 that the assembly map appearing in the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \to H_n^G(G/G;\mathbf{K}_R) = K_n(RG)$$

or the assembly map appearing in the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG)$$

respectively can be identified with the map induced on homotopy groups by the homotopy theoretic assembly map

$$\pi_n\bigl(\mathbf{S}_{EG}^G(E_{\mathcal{VCY}}(G))^{\%}(\mathrm{pr})\bigr)\colon \pi_n\bigl((\mathbf{S}_{EG}^G)^{\%}(E_{\mathcal{VCY}}(G))\bigr) \to \pi_n\bigl((\mathbf{S}_{EG}^G)^{\%}(E_{\mathcal{VCY}}(G))\bigr).$$

In the Baum-Connes setting we get an identification of the assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}^{\mathsf{TOP}})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathsf{TOP}}) \to H_n^G(G/G;\mathbf{K}^{\mathsf{TOP}}) = K_n(C_r^*(G))$$

with the map

$$\begin{aligned}\pi_n\bigl(\mathbf{S}_{EG}^G(E_{\mathcal{FIN}}(G))^{\%}(\mathrm{pr})\bigr)\colon \pi_n\bigl((\mathbf{S}_{EG}^G)^{\%}(E_{\mathcal{FIN}}(G))\bigr)& \\ \to \pi_n\bigl((\mathbf{S}_{EG}^G)^{\%}(E_{\mathcal{FIN}}(G))\bigr),&\end{aligned}$$

if we take $\mathbf{S} = \mathbf{K}^{\mathsf{TOP}}(\Pi(X))$, see Theorem 12.43, and analogously in the real case.

We have explained in Remark 15.42 the identification of the original formulation of the fibered Farrell-Jones Conjecture for covariant functors from SPACES to $\mathsf{SPECTRA}$, e.g., for pseudoisotopy, K-theory and L-theory, due to Farrell-Jones [351, Section 1.7 on page 262] with the setting we are using in the Meta-Isomorphism Conjecture 15.39 for functors from spaces to spectra with coefficients.

We have discussed the various Baum-Connes assembly maps and their relations already in Sections 14.2 and 14.3.

We have explained the relationship between the L-theoretic assembly map in terms of spectra, which we are using here, and the surgery obstruction map appearing in the geometric Surgery Exact Sequence the sketch of the proof of Theorem 9.168 in Subsection 9.15.3.

18.7 Notes

The Baum-Connes assembly maps in terms of localizations of triangulated categories are considered in [497, 498, 499, 701, 702, 703]. A categorial approach in terms of codescent is presented in [61].

Chain complex versions of the L-theoretic assembly map for additive categories are intensively studied by Ranicki [823] and Kühl-Macko-Mole [579, Section 11] emphasizing the aspect of comparing local Poincaré duality and global Poincaré duality.

The idea of the geometric assembly map is due to Quinn [807, 812] and its algebraic counterpart was introduced by Ranicki [823]. See also Loday [620]. The basic and uniform approach to assembly as presented in this chapter is sometimes called the Davis-Lück approach and was developed in [265].

For more information about assembly maps we refer for instance to the survey article [648].

Comment 25 (by W.): Are there other places where we should say something about the name assembly, for instance in the introduction or in part III?

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Chapter 19
Motivation, Summary, and History of the Proofs of the Farrell-Jones Conjecture

19.1 Introduction

The purpose of this chapter is to present basic ideas and motivations for the proofs of the Farrell-Jones Conjecture and some information about their long history without getting lost in technical details. So it will be a soft introduction to the methods of proofs conveying only ideas. Moreover, we also want to provide some insight why some input such as controlled topology, transfers, and flows occurs, which one might not expect at the first glance since so far the assembly maps are purely homotopy theoretic notions. We refer the interested reader, who wants to see more details, to Chapters 21, 22, 23, 24, and 25.

We also want to explain why it is rather difficult to say something about all the proofs in full detail since the proofs and their methods have been moving targets; many new ideas and technical modifications have entered during the last decades until today so that sometimes the original ideas cannot be recognized anymore and the overwhelming variety of the different proofs cannot be presented in detail in this book. The most advanced presentation of a framework of a proof will be given in Chapter 25 where we will work in the setting of higher categories as coefficients, which is more general than considering additive categories or rings as coefficients. We will not deal with the Farrell-Jones Conjecture for reductive p-adic groups, see Bartels-Lück [81, 83], which is the next level of complexity since we refine ourselves in this book to discrete groups and do not consider topological groups.

The original formulation of the Farrell-Jones Conjecture appears in [351, 1.6 on page 257]. Of course it had many previous versions, some of them can be found in Subsection 13.11.1.

19.2 Homological Aspects

We have already explained in the Introduction of this book, see Chapter 1, how homological aspects concerning the topological K-theory $K_*(C_r^*(G))$ of the reduced group C^*-algebra $C_r^*(G)$ of G and the algebraic K-theory $K_*(RG)$ and algebraic L-theory $L_*^{\langle -\infty \rangle}(RG)$ of the group ring RG lead to the assembly maps

$$K_n^G(E_{\mathcal{VCY}}(G)) \xrightarrow{\cong} K_n(C_r^*(G));$$
$$H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \xrightarrow{\cong} K_n(RG);$$
$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG).$$

They appear in the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3 which predict that these assembly maps are bijections for all $n \in \mathbb{Z}$. Moreover we have explained in Chapter 18 that, after passing to the spectrum version of $K_*(C_r^*(G))$, $K_*(RG)$, and $L_*^{\langle -\infty\rangle}(RG)$ as functors from the category of G-CW-complexes to the category of spectra, these assembly maps are characterized by the universal property that they are the best approximation from the left by an excisive functor and do have interpretations in terms of homotopy colimits over the orbit category. So the first attempt to prove these conjectures is to show that these functors are excisive. However, this direct strategy has never really worked out, at least not with further sophisticated input. The problem is to isolate the reason why these functors are excisive in general. It is unclear which basic properties of the K- and L-theory of group rings or reduced group C^*-algebras guarantee excisiveness.

19.3 Constructing Detection maps

The next idea is just to construct an inverse to these assembly maps. In the Baum-Connes setting this is a successful strategy relying on the equivariant Kasparov product and the Dirac-Dual Dirac Method, see Section 26.2. In the Farrell-Jones setting this has nearly never worked out. The main reason is that it is hard to construct detecting maps with the algebraic K- or L-theory of group rings as source. There are interesting attempts to do this, most prominently the cyclotomic trace for the algebraic K-theory of groups rings, or Chern characters for the topological K-theory of C^*-algebras with values in cyclic homology, but these give inverses to the assembly maps only in a very few instances. However, they can be used to show injectivity results, as explained, see Sections 16.5 and 16.6. Note that surjectivity results are more valuable than injectivity results since they give some insight about elements in the K- or L-groups under consideration and imply many other conjectures, whereas injectivity only describes some portion of the K- or L-groups under consideration and do not have so many consequences with the exception of the Novikov Conjecture which is essentially an injectivity claim about assembly maps. Moreover, surjectivity result can often be easily turned into bijectivity results by considering relative versions.

In the Farrell-Jones setting the most successful method for proving bijectivity results is controlled topology, as motivated and explained next.

19.4 Controlled Topology

19.4.1 Two Classical Results

Let α be an open cover of a space Y. Two maps $f, g\colon X \to Y$ are called *α-close* if for every $x \in X$ there is $U_x \in \alpha$ satisfying $f(x), g(x) \in U_x$. They are called *α-homotopic* if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$ and $h_1 = g$ hold and for every $x \in X$ there is $U_x \in \alpha$ satisfying $h(\{x\} \times [0,1]) \subseteq U_x$. A map $f\colon X \to Y$ is an *α-domination* if there is a map $g\colon Y \to X$ such that $f \circ g$ is α-homotopic to the identity id_Y. In such a situation, g is called a *right α-homotopy inverse* for f. We call $f\colon X \to Y$ an *α-homotopy equivalence* if f is an α-domination and, for some right α-homotopy inverse g, the composite $g \circ f$ is $f^{-1}(\alpha)$-homotopic to the identity id_X where $f^{-1}(\alpha)$ denotes the cover $\{f^{-1}(U) \mid U \in \alpha\}$ of X. We call g an *α-homotopy inverse of f*.

Recall that a map $f\colon X \to Y$ is *proper* if $f^{-1}(C)$ is compact for every compact subset $C \subseteq Y$.

Obviously a homeomorphism $f\colon X \to Y$ is an α-homotopy equivalence for every α and a proper map.

The next result is due to Chapman and Ferry, see [218].

Theorem 19.1 (α-Approximation Theorem). *Let N be a topological manifold of dimension n and α be an open cover of N. Then there is an open cover β of N with the following property: If M is a topological manifold and $f\colon (M, \partial M) \to (N, \partial N)$ is a proper β-homotopy equivalence of pairs such that either $n \geq 6$ or ($n \geq 5$ and ∂f is a homeomorphism) hold, then f is α-close to a homeomorphism.*

The following result is a special case of a theorem due to Ferry [364, Theorem 1]. Its proof relies on the α-Approximation Theorem 19.1.

Theorem 19.2. *Let M be a closed topological manifold of dimension $n \geq 5$. Equip M with a metric generating the given topology. Then there is $\epsilon > 0$ with the following property: Every surjective map $f\colon M \to N$ to some closed manifold N of dimension n for which the diameter of $f^{-1}(y)$ for every $y \in N$ is less than ϵ is homotopic to a homeomorphism.*

The next result follows from Quinn [808, Theorem 2.7] which is closely related to the work of Chapman and Ferry [217, 218, 363].

Let M_0 be a closed topological manifold of dimension $n \geq 5$. Equip M with a metric generating the given topology. An h-cobordism $(W; M_0, M_1, f_0, f_1)$ is called *ϵ-controlled* if for $i = 0, 1$ the composite $M_i \xrightarrow{f_i} \partial_i W \xrightarrow{j_i} W$ for j_i the inclusion possesses a retraction $r_i\colon W \to M_i$ coming with a homotopy $H_i\colon j_i \circ r_i \simeq \mathrm{id}_W$ such that for every $w \in W$ the subset of M_0 given by $r_0 \circ H_i(\{w\} \times [0,1])$ has a diameter less than ϵ, in other words, the images of all the tracks of the two homotopies H_0 and H_1 under r_0 have diameter less than ϵ.

An ϵ-controlled h-cobordism $(W; M_0, M_1, f_0, f_1)$ has an *ϵ-product structure* if there is additionally a homeomorphism $F\colon W \xrightarrow{\cong} M_0 \times [0,1]$ such that $F \circ j_0 \circ f_0$ sends $x \in M_0$ to $(x, 0)$ and r_0 and $\mathrm{pr}_{M_0} \circ F$ for the projection $\mathrm{pr}_{M_0}\colon M_0 \times [0,1] \to M_0$ are ϵ-homotopic, in the sense that there exists a homotopy $L\colon W \times [0,1] \to M_0$ between them such that the diameter of the subset $L(\{x\} \times [0,1])$ of M_0 is less than ϵ for every $w \in W$. In particular every ϵ-controlled h-cobordism $(W; M_0, M_1, f_0, f_1)$ with an ϵ-product structure is trivial and hence has vanishing Whitehead torsion.

Theorem 19.3 (Thin h-Cobordism Theorem). *Let M_0 be a closed topological manifold of dimension $n \geq 5$. Equip M_0 with a metric generating the given topology.*

Then for every $\epsilon > 0$ there exists δ with $0 < \delta < \epsilon$ such that every topological h-cobordism $(W; M_0, M_1, f_0, f_1)$ over M_0 which is δ-controlled has an ϵ-product structure. In particular there exists a $\delta > 0$ such that every every topological h-cobordism $(W; M_0, M_1, f_0, f_1)$ over M_0 which is δ-controlled is trivial.

19.4.2 The Strategy of Gaining Control

Let N and M_0 be closed topological manifold of dimension $n \geq 5$ equipped with a metric generating the given topology. Then there exists $\epsilon > 0$ with the following properties:

- Let M be a closed manifold and $f\colon M \to N$ be a homotopy equivalence which is *ϵ-controlled* in the sense that it is α-homotopy equivalence for the open covering α of N consisting of all open balls of radius $\epsilon/2$. Then by the α-Approximation Theorem 19.1 f is homotopic to a homeomorphism and in particular has trivial Whitehead torsion. So in order to prove that N is topological rigid in the sense of Definition 9.159, it suffices to show that a given homotopy equivalence $g\colon M \to N$ is homotopic to an ϵ-controlled homotopy equivalence. Roughly speaking, to achieve up to homotopy a homeomorphism, it suffices to gain ϵ-control.
- An h-cobordism $(W; M_0, M_1, f_0, f_1)$ over M_0 is trivial and hence has vanishing Whitehead torsion if we can show that it is ϵ-controlled. This follows from the Thin h-Cobordism Theorem 19.3. In particular in order to show that $\mathrm{Wh}(\pi_1(N))$ vanishes, it suffices to show because of the s-Cobordism Theorem 3.47 that, for any h-cobordism $(W; M_0, M_1, f_0, f_1)$ over M_0, we can find another an h-cobordism $(W'; M_0, M_1', f_0', f_1')$ over M_0 such that $(W; M_0, M_1, f_0, f_1)$ and $(W'; M_0, M_1', f_0', f_1')$ have the same Whitehead torsion and the new h-cobordism $(W'; M_0, M_1', f_0', f_1')$ is ϵ-controlled.

Hence to prove the Farrell-Jones Conjecture 3.110 for $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ for torsionfree G or the Borel Conjecture 9.160, a promising strategy is to gain control, i.e., turning an h-cobordism or a homotopy equivalence to an ϵ-controlled one without changing the associated class in the Whitehead group. (One can also achieve the $\widetilde{K}_0(\mathbb{Z}G)$ part of Conjecture 3.110 using the Bass-Heller-Swan decomposition 3.73 and replacing N by $N \times T^n$.)

This turns to be out a major breakthrough since it allows to bring in completely new methods, namely, geometric methods, into the play. This was pioneered by Farrell and Jones, in particular in their seminal papers [344, 345]. They used the Thin h-Cobordism Theorem 19.3, which did not play a role anymore in more recent proofs.

19.4.3 Controlled Algebra

Fix an infinite cardinal κ. Let $\mathcal{F}^{\kappa}(R)$ be a small model for the category of all free R-modules which admit a basis B with $\mathrm{card}(B) \le \kappa$ and possesses direct sums over index sets of cardinality $\le \kappa$.

We have to consider this cardinal κ and $\mathcal{F}^{\kappa}(R)$ and consider only countable groups and spaces whose cardinality is less or equal to κ for set theoretic reasons which the reader may ignore in the sequel. Denote by $\mathcal{F}^{f}(R) \subseteq \mathcal{F}^{\kappa}(R)$ the full subcategory consisting of all free R-modules which admit a finite basis B. For more information about these issues and $\mathcal{F}^{\kappa}(R)$ see for instance [90, Lemma 9.2].

Definition 19.4 (Geometric modules). Let G be a group, R be a ring, and X be a free G-space with $\mathrm{card}(X) \le \kappa$. We define the *additive category* $\mathsf{GM}^G(X)$ *of geometric modules over* X as follows.

An object M is a collection $\{M_x \mid x \in X\}$ of objects in $\mathcal{F}^{\kappa}(R)$ such that $M_{gx} = M_x$ holds for every $x \in X$ and $g \in G$. Define the *support* of an object

$$\mathrm{supp}(M) = \{x \in X \mid M_x \neq \{0\}\} \subseteq X.$$

Given two objects $M = \{M_x \mid x \in X\}$ and $N = \{N_y \mid y \in X\}$, a morphism $f\colon M \to N$ consists of a collection of R-homomorphisms $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in X\}$ such that $f_{gx,gy} = f_{x,y}$ holds for $x, y \in X$ and $g \in G$ and for every $x \in x$ the set $\{y \in Y \mid f_{x,y} \neq 0\}$ is finite and for every $y \in X$ the set $\{x \in x \mid f_{x,y} \neq 0\}$ is finite. Define the *support* of a morphism

$$\mathrm{supp}(f) = \{(x, y) \in X \times X \mid f_{x,y} \neq \{0\}\} \subseteq X \times X.$$

If $P = \{P_z \mid z \in X\}$ is an object and $g\colon N \to P$ is a morphism, define the composite

$$g \circ f = \{(g \circ f)_{x,z}\colon M_x \to P_z \mid x, z \in X\}\colon M \to P$$

by $(g \circ f)_{x,z} = \sum_{y \in Y} g_{y,z} \circ f_{x,y}$. Define the identity

$$\mathrm{id}_M = (\mathrm{id}_M)_{x,y} \mid x, y \in X\}\colon M_x \to M_y$$

of the object M by $(\mathrm{id}_M)_{x,y} = \mathrm{id}_{M_x}$ for $x = y$ and by $(\mathrm{id}_M)_{x,y} = 0$ for $x \neq y$.

Given two morphisms $f, g\colon M \to N$ and $m, n \in \mathbb{Z}$, define the morphism $m \cdot f + n \cdot g\colon M \to N$ by $(m \cdot f + n \cdot g)_{x,y} = m \cdot f_{x,y} + n \cdot g_{x,y}$ for $x, y \in X$. The direct sum of two objects M and N is defined by $(M \oplus N)_x = M_x \oplus N_x$ for $x \in X$.

Denote by $\mathsf{GM}^G(X)^f$ the full additive subcategory of $\mathsf{GM}^G(X)$ consisting of those objects $M = \{M_x \mid x \in X\}$ such that M_x belongs to $\mathcal{F}^f(R)$ for all $x \in X$ and the support $\mathrm{supp}(M) = \{x \in X \mid M_x \neq \{0\}\}$ is G-cofinite, i.e., there is a finite subset S of X with $\mathrm{supp}(M) = G \cdot S$, or, equivalently, $G\backslash \mathrm{supp}(M)$ is finite.

The additive category $\mathsf{GM}^G(X)$ is equivalent to the additive category $\mathcal{F}^\kappa(RG)$. Namely, there is an equivalence of additive categories

$$F\colon \mathsf{GM}^G(X) \to \mathcal{F}^\kappa(RG) \tag{19.5}$$

defined as follows. Given an object $M = \{M_x \mid x \in X\}$ in $\mathsf{GM}^G(X)$, we obtain an RG-module whose underlying R-module is $\bigoplus_{x \in X} M_x$ and $g \in G$ acts by sending $\{m_x \mid x \in X\}$ to $\{m_{g^{-1}x} \mid x \in X\}$. The G-action is well-defined since $M_x = M_{gx}$ holds for $x \in X$ and $g \in G$ by assumption. Since G acts freely on X, each M_x is free, $F(M)$ is isomorphic to the free RG-module $RG \otimes_R \left(\bigoplus_{y \in S} M_y\right)$ for a set S with $\mathrm{supp}(M) = G \cdot S$. Hence we can choose an object V_M in $\mathcal{F}^\kappa(RG)$ and an RG-isomorphism $\xi_M\colon \bigoplus_{x \in X} M_x \xrightarrow{\cong} V_M$ and define $F(M) = V_M$.

Given a morphism $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in X\}\colon M \to N$ we get an RG-homomorphism $\eta_f\colon \bigoplus_{x \in X} M_x \to \bigoplus_{y \in X} N_y$ by sending $(u_x \mid x \in X\}$ to $\{v_y \mid y \in X\}$ with $v_y = \sum_{x \in X} f_{x,y}(u_x)$. Now define $F(f)$ to be the composite $\xi_N \circ \eta_f \circ \xi_M^{-1}$.

The functor F induces an equivalence of additive categories

$$F^f\colon \mathsf{GM}^G(X)^f \to \mathcal{F}^f(RG) \tag{19.6}$$

Exercise 19.7. Show that the functors F and F^f are equivalences of additive categories.

The additive categories $\mathsf{GM}^G(X)$ and $\mathsf{GM}^G(X)^f$ becomes much more interesting than $\mathcal{F}^\kappa(RG)$ and $\mathcal{F}^f(RG)$ if we bring the notion of control into play. Namely, suppose that we have a metric space $Z = (Z, d)$ with free isometric G-action together with a G-map $p\colon X \to Z$. Given $\epsilon \geq 0$, we call a morphism $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in X\}\colon M \to N$ *ϵ-controlled* if the implication $x, y \in X, f_{x,y} \neq 0 \implies d(p(x), p(y)) \leq \epsilon$ holds. An automorphism $f\colon M \xrightarrow{\cong} M$ is called an *ϵ-controlled automorphism* if both f and f^{-1} are ϵ-controlled.

Geometric modules were introduced by Connell-Hollowingsworth [234]. Their theory was developed further by, among others, Pedersen and Quinn and is sometimes referred to as *controlled algebra*. More information can be found in the survey article [785]. One can find an algebraic proof of the topological invariance of Whitehead torsion in [785, Section 5].

Next we give a kind of algebraic version of the Thin h-Cobordism Theorem 19.3 taken from [66, Theorem 1.2.8].

An *abstract simplicial complex* $\Sigma = (\Sigma, V)$ consists of a set V and a family Σ of non-empty finite subsets of V such that for every element σ in Σ, and every non-empty subset $\tau \subseteq \sigma$, the subset τ also belongs to Σ and for each $v \in V$ the subset $\{v\}$ belongs to Σ. In the sequel we will often identify $v \in V$ with $\{v\} \in \Sigma$. The *dimension* $\dim(\sigma)$ of a simplex is defined to be $|\sigma| - 1$. The dimension $\dim(\Sigma)$ is the supremum of the dimension of all simplices of Σ. A *map of simplicial complexes* $f\colon (\Sigma, V) \to (\Sigma', V')$ is a map $f\colon V \to V'$ such that for any element $\sigma \in \Sigma$ the subset $f(\sigma) \subseteq V'$ belongs to Σ'. The *barycentric subdivision* Σ' of an abstract simplicial complex Σ is the abstract simplicial complex whose set of vertices is Σ and whose set of simplices consists of non-empty finite subsets of Σ which are totally ordered. Note that a d-simplex in Σ' is the same as a flag $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_d$ of elements $\sigma_i \in \Sigma$.

We equip the *geometric realization* $|\Sigma|$ of Σ, which consists of functions $b\colon V \to [0,1]$ whose support $\operatorname{supp}(b) = \{v \in V \mid f(v) \neq 0\}$ is finite and belongs to Σ and satisfies $\sum_{v \in V} b(v) = 1$, with the L^1-metric given by $d_{L^1}(b, b') = \sum_{v \in V} |b(v) - b'(v)|$.

An *abstract simplicial G-complex* is an abstract simplicial complex Σ together with G-action by simplicial automorphisms. The G-action on Σ induces an isometric G-action on $|\Sigma|$ equipped with its L^1-metric. Let $\mathcal{F}$ be a family of subgroups. We call Σ an *abstract simplicial $(G, \mathcal{F})$-complex* if the isotropy group $G_b = \{g \in G \mid gb = b\}$ for every $b \in |\Sigma|$ belongs to $\mathcal{F}$. Note that $|\Sigma|$ is not necessarily a G-CW-complex, but $|\Sigma'|$ for the barycentric subdivision Σ' of Σ is. If the isotropy group of each vertex $v \in V$ belongs to $\mathcal{F}$ and $\mathcal{F}'$ is the family of subgroups of G consisting of those subgroups which contain a subgroup of finite index belonging to $\mathcal{F}$, then Σ and Σ' are abstract simplicial $(G, \mathcal{F}')$-complexes and $|\Sigma'|$ is a G-CW-complex whose isotropy groups belong to $\mathcal{F}'$.

Theorem 19.8 (Algebraic Thin h-Cobordism Theorem). *Given a natural number N, there exists $\epsilon_N > 0$ with the following property. Consider*

(i) A family $\mathcal{F}$ of subgroups of G;
(ii) An abstract simplicial $(G, \mathcal{F})$-complex Z of dimension $\leq N$;
(iii) A free G-space X together with a map $p\colon X \to |Z|$;
(iv) An automorphism $a\colon M \to M$ in $\mathsf{GM}^G(X)^f$ which is ϵ_N-controlled with respect to p and the L^1-metric on $|Z|$.

Then the class $[F^f(a)] \in K_1(\mathbb{Z}G)$ of the RG-automorphism $F^f(a)\colon F(M) \xrightarrow{\cong} F(M)$ of the finitely generated free RG-module $F^f(M)$ for the functor F^f

of (19.6) *is contained in the image of the assembly map* $H_1(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}) \to K_1(\mathbb{Z}G)$.

The Algebraic Thin h-Cobordism Theorem 19.8 follows from [77, Theorem 5.3] and implies the Thin h-Cobordism Theorem 19.3 as explained in [66, Remark 1.2.11 and Remark 1.2.9]. There is also a converse to the Algebraic Thin h-Cobordism Theorem 19.8 as discussed in [66, Remark 1.2.11 and Remark 1.2.15]. It says, roughly speaking, that any element appearing in the image of the assembly map can be realized as $[F^f(a)]$ for appropriate Z, X, p, and a.

Remark 19.9 (Control-Strategy). The considerations above lead to the following *Control-Strategy* for proving the Farrell-Jones Conjecture.

(i) Interprete elements in the target group $K_n(\mathbb{Z}G)$ of the assembly map as a kind of cycles and the source of the assembly map $H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)$ as *controlled cycles*, i.e., cycles satisfying certain control conditions related to the family $\mathcal{F}$;
(ii) Identify the assembly map as a kind of *forget control map*;
(iii) For a specific group G and a specific family $\mathcal{F}$, develop a strategy how to change a cocycle without changing its class in $K_n(\mathbb{Z}G)$ such that the new representative satisfies the necessary control conditions to ensure that the it defines an element in $H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)$. This proves surjectivity of the assembly map. One may call this process *gaining control*;
(iv) Use a relative version of part (iii) to prove injectivity of the assembly map. One may call this process *gaining relative control*;

The strategy for L-theory is completely analogous.

Example 19.10 (Singular homology). Next we illustrate this strategy in a much easier and classical instance, namely, singular homology, by repeating how one proves excision for it.

Let X be a topological space, and let $C_*^{\mathrm{sing}}(X; R)$ be the singular chain complex of X with coefficients in the ring R. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be a cover of X, i.e., a collection of subsets U_i such that the union of their interiors U_i° is X. Denote by $S_n^{\mathcal{U}}(X)$ the subset of the set $S_n(X)$ of those singular n-simplices $\sigma \colon \Delta_n \to X$ for which there exists $i \in I$ satisfying $\mathrm{im}(\sigma) \subseteq U_i$. Let $C_*^{\mathrm{sing},\mathcal{U}}(X; R)$ be the R subchain complex of $C_*^{\mathrm{sing}}(X; R)$ whose nth chain module consists of elements of the shape $\sum_{\sigma \in S_n^{\mathcal{U}}(X)} r_\sigma \cdot \sigma$. Let $i_*^{\mathcal{U}} \colon C_*^{\mathrm{sing},\mathcal{U}}(X; R) \to C_*^{\mathrm{sing}}(X; R)$ be the inclusion. The main ingredient in the proof of excision is to show that i_* is a homology equivalence. Then excision follows by applying the result above to $\mathcal{U} = \{X \setminus A, B\}$ for $A \subseteq B \subseteq X$ with $\overline{A} \subseteq B^\circ$.

The proof that $i_*^{\mathcal{U}} \colon C_*^{\mathrm{sing},\mathcal{U}}(X; R) \to C_*^{\mathrm{sing}}(X; R)$ is a homology equivalence is based on the construction of the subdivision operator which subdivides Δ_n into a bunch of smaller copies of Δ_n and replaces the singular simplex $\sigma \colon \Delta_n \to X$ by the sum of the singular simplices obtained by restricting to these smaller pieces. This process does not change the homology class but can be used to arrange that the representing cycle lies in $C_*^{\mathrm{sing},\mathcal{U}}(X;R)$. This implies surjectivity of $H_n(i_*^{\mathcal{U}})\colon H_n(C_*^{\mathrm{sing},\mathcal{U}}(X;R)) \to H_n(C_*^{\mathrm{sing}}(X;R))$. One obtains injectivity by applying these construction to an $(n+1)$-simplex $\tau\colon \Delta_{n+1} \to X$, provided that the restriction of τ to faces of Δ_{n+1} does already lie in $S_n^{\mathcal{U}}(X)$.

Roughly speaking, the process of gaining control is realized by subdivision.

19.4.4 Controlled Algebra Defined Using the Open Cone

In order to carry out the Control Strategy discussed in Remark 19.9, one needs to find the equivalent setup of the homotopy theoretic construction of $H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_R)$, but now in the controlled setting. The basic idea is to construct additive categories (with involution) which encode $\mathcal{F}$ and the relevant control conditions and to consider their K- or L-groups.

An obvious drawback of the notion of ϵ-controlled morphisms between geometric modules, see Subsection 19.4.3, is that they do not form a subcategory of the additive category of geometric modules. The composite of two ϵ-controlled morphism is 2ϵ-controlled but not necessary ϵ-controlled. The same applies to ϵ-controlled automorphisms. In order to fix this problems, Pedersen-Weibel [784] considered for a finite PL-subcomplex X of S^n (for large n) the open cone $O(X) = \{sx \mid s \in \mathbb{R}, s > 0, x \in X\} \subseteq \mathbb{R}^{n+1}$ with the metric induced from maximums metric on $\mathbb{R}^{n+1}$ and introduced a quotient category in which every morphism has for any $\epsilon > 0$ a representative that is ϵ-controlled. They used this construction to produce a geometric homology theory digesting these finite PL-complexes $X \subseteq S^n$ with coefficients in the K-theory spectrum $\mathbf{K}_R$ of a ring R, which is a delooping of the homology theory associated to the algebraic K-theory spectrum $\mathbf{K}_R$ sending X to the homotopy groups of the spectrum $X_+ \wedge \mathbf{K}_R$. This construction can easily be extended to additive categories $\mathcal{A}$ as coefficients instead of a ring R as coefficient.

The idea of the open cone $O(X)$ is that, given a constant $C > 0$, for two points x and y in X the implication $d(sx,sy) \le C \implies d(x,y) \le \frac{C}{s}$ holds for $s > 0$. Hence $d(x,y)$ becomes arbitrary small if $d(sx,sy) \le C$ holds for large enough s. More generally, given constants $C > 0$ and $R > 0$, we can find for every $\epsilon > 0$ a real number $T > 0$ such that for $x, y \in X$ and $s, t > 0$ the implication

$$d(sx,ty) \le C, |t-s| \le R, t \ge T \implies d(x,y) \le \epsilon$$

holds. These points sx and ty will be points contained in the support $\operatorname{supp}(f) = \{(sx,ty) \in O(X) \times O(X) \mid f_{sx,ty} \ne 0\}$ of a morphism $f = \{f_{sx,ty}\}$ in $\mathsf{GM}^{\{1\}}(O(X))$. Our setup ensures the we get an additive subcategory of

$\mathsf{GM}^{\{1\}}(O(X))$ if we consider only those morphisms $f = \{f_{sx,ty}\}$ for which there exists constants $C > 0$ and $R \geq 0$ satisfying $d(tx, sy) \leq C$ and $|t - s| \leq R$ for every $(sx, ty) \in \operatorname{supp}(f)$. One may think of the inclusion of this subcategory to $\mathsf{GM}^{\{1\}}(O(X))$ as a forget control functor.

19.4.5 Continuous Control

Roughly speaking, the idea is to introduce a new non-compact coordinate, for instance the distance from the origin in $\mathbb{R}^{n+1}$ in the open cone $O(X)$ appearing in Subsection 19.4.4, so that bounded control for objects or morphisms over the given space X correspond to ϵ-controlled morphisms in the new extended space for which the ϵ can be chosen to be smaller and smaller the farer out the objects and morphisms are with respect to this new coordinate. In principle one uses the observation that bounded plus bounded is bounded (in contrasts to the wrong statement ϵ plus ϵ is ϵ) so that bounded controlled morphisms form a subcategory. One has to consider germs of morphisms where it is allowed to ignore everything which is only bounded in this new coordinate, or, equivalently, where only the asymptotic behaviour at ∞ matters. Therefore one takes the quotient by the category of those objects and morphisms that live in a bounded region with the respect to the new coordinate, in other word, do not get arbitrary close to ∞, This quotient has the desired property that for every morphism and $\epsilon > 0$, we can find a representative that is ϵ-controlled. Taking this quotient has the side effect that one deals with a delooping of the desired homology theory.

The constructions of Pedersen-Weibel [784] have undergone a long lasting mutation through various steps, in order to get a better and better setting. For instance, one needs to design equivariant versions, and the theory should just digest G-CW-complexes without any choice of embeddings into an open cone or so.

For these development we refer to the papers by Bartels-Farrell-Jones-Reich [72, 73], Bartels-Lück-Reich [86], and Bartels-Lück [77]. The most advanced setup is presented in Bartels-Lück [81] where for the first time the Farrell-Jones Conjecture is considered for topological groups, namely, for totally disconnected groups such as reductive p-adic groups. We will not discuss this long process but we will give details about the constructions in [81] in the discrete case in Chapter 22, where we also give the full proof that we indeed get a G-homology theory digesting arbitrary G-CW-complexes. The construction of the $\mathcal{TOD}$-sequence in Section 22.5 is the detailed and mathematically complete manifest of the discussion above.

As an illustration we want to describe the notion of continuous control (in the non-equivariant setting) which will replace the open cone construction, can digest any CW-complex X without any embedding into S^n, and does not need a choice of a metric.

We define an additive subcategory $\mathcal{O}(X)$ of $\mathsf{GM}^{\{1\}}(X \times \mathbb{N})$ as follows, where $\mathbb{N}$ denotes the natural numbers. The suport of an object $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$ is defined to be $\operatorname{supp}(M) = \{(x,s) \in X \times \mathbb{N} \mid M_{x,s} \neq \{0\}\}$. We require for an object M in $\mathcal{O}(X)$:

- *Compact support over* X
  The set $\{x \in X \mid \exists s \in \mathbb{N} \text{ with } (x,s) \in \operatorname{supp}(M)\}$ is contained in a compact subset of X;
- *Locally finiteness over* $\mathbb{N}$
  For every $n \in \mathbb{N}$ the set $\{x \in X \mid (x,n) \in \operatorname{supp}(M)\}$ is finite.

We require for the support

$$\operatorname{supp}(f) = \{((x,s),(y,t)) \in (X \times \mathbb{N}) \times (X \times \mathbb{N}) \mid \{f_{(x,s),(y,t)} \neq 0\}$$

of a morphism $f = \{f_{(x,s),(y,t)}\}$ in $\mathcal{O}(X)$:

- *Bounded control in the* $\mathbb{N}$ *direction*
  There is $N \in \mathbb{N}$ such that $|t-s| \leq N$ holds for $((x,s),(y,t)) \in \operatorname{supp}(f)$;
- *Continuous control*
  For every $z \in X$, open neighborhood V of z, and $r \in \mathbb{N}$, there exists an open neighborhood U of z with $U \subseteq V$ and $R \in \mathbb{N}$ with $r \leq R$ such that the implication

  $$((x,s),(y,t)) \in \operatorname{supp}(f), x \in U, s \geq R \implies y \in V, t \geq r$$

  holds.

The condition above ensures that the morphisms become more and more controlled in the X direction as farer we go out in the $\mathbb{N}$-direction. The other conditions will be needed to construct the transfer or certain quotient categories. One may also consider the full additive subcategory $\tau(X)$ of $\mathcal{O}(X)$ where we additionally require for an object M that there exists a natural number $m \in \mathbb{N}$ for which the implication $(x,s) \in \operatorname{supp}(M) \implies s \leq m$ holds. Then the quotient category $\mathcal{D}(X) = \mathcal{O}(X)/\mathcal{T}(X)$ can be thought of equivalence classes of objects and morphism in $\mathcal{O}(X)$ where we identify two of them if they agree outside of a bounded region in the $\mathbb{N}$-direction. In this category $\mathcal{D}(X)$ we can always find representatives in $\mathcal{O}(X)$ which are with respect to the X direction arbitrary good controlled since we can put all modules and morphism to be zero in any region bounded in the $\mathbb{N}$-direction. The precise definition of the quotient category $\mathcal{D}(X)$ will be given in Chapter 21, where also a weak homotopy fibration sequence of non-connective spectra,

$$\mathbf{K}(\mathcal{T}(X)) \to \mathbf{K}(\mathcal{O}(X)) \to \mathbf{K}(\mathcal{D}(X)) \tag{19.11}$$

is established. It will be the key ingredient to show that the functor sending X to $\mathbf{K}(\mathcal{D}(X))$ is weakly excisive in the sense of Definition 18.8 for $G = \{1\}$, or, equivalently, that we get a homology theory with values in abelian groups by sending a CW-complex X to $K_{n+1}(\mathcal{D}(X))$ for $n \in \mathbb{Z}$. An Eilenberg swindle towards infinity in the $\mathbb{N}$-direction will show that $K_n(\mathcal{O}(\{\bullet\}))$ vanishes for all $n \in \mathbb{Z}$. It is not hard to see that $\mathcal{T}(X)$ is equivalent to $\mathsf{GM}^{\{1\}}(X)^f$ and hence we get from the equivalence (19.6) an identification $K_n(\tau(X)) = K_n(R)$. Thus we obtain an identification $K_n(R) = K_{n+1}(\mathcal{D}(\{\bullet\}))$. We conclude from the universal property of assembly maps, see Theorem 18.11 and Remark 18.18 that we get natural identifications $H_n(X; \mathbf{K}(R)) \cong \pi_{n+1}(\mathcal{D}(X))$. Furthermore, if we take $X = BG$, the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

appearing in Conjecture 6.53 can be identified with a map

$$\pi_{n+1}(\mathcal{D}(BG)) \to K_n(RG)$$

which can be thought of as a forget control map.

All this will be fully explained in Chapter 22, also in the equivariant setting. In particular there is for any G-CW-complex X an equivariant version of (19.11)

$$\mathbf{K}(\mathcal{T}^G(X)) \to \mathbf{K}(\mathcal{O}^G(X)) \to \mathbf{K}(\mathcal{D}^G(X))$$

such that the assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

appearing in the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R can be identified with the homomorphism

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{VCY}}(G))) \to K_{n+1}(\mathcal{D}^G(G/G))$$

induced by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to G/G$ which can be thought of as a forget control map.

Exercise 19.12. Show that the inclusion $I\colon \mathsf{GM}^{\{1\}}(X) \to \mathcal{T}(X)$ coming from the inclusion $X \to X \times \mathbb{N}$ sending x to $(x, 0)$ is an equivalence of additive categories

19.5 Gaining Control by Using Flows and Transfers

In this Section we briefly sketch the basic ideas appearing in the seminal papers by Farrell-Jones [344, 345]. These papers do of course rely on earlier work by Farrell and Jones and other mathematicians, which we will not explain here. For us it is important to explain briefly the main idea in these two papers to prove the vanishing of the Whitehead group $\mathrm{Wh}(G)$ for torsionfree groups G which occur as fundamental groups of certain closed manifolds. For simplicity we only consider the case $G = \pi_1(M)$ for an orientable hyperbolic closed smooth Riemannian manifold of dimension $d \geq 5$.

The key ingredient is to lift an element x in the Whitehead group $\mathrm{Wh}(\pi_1(M))$ to the Whitehead group $\mathrm{Wh}(\pi_1(STM))$ of the fundamental group of the total space STM of the sphere tangent bundle $p\colon STM \to M$ by a transfer map $p^*\colon \mathrm{Wh}(\pi_1(M)) \to \mathrm{Wh}(\pi_1(STM))$ and to use the geometric flow on STM and the hyperbolic structure on M to show that this element $p^*(x)$ has a representative with good enough control ensuring that $p^*(x)$ vanishes. The composite of the transfer p^* with the obvious map $p_*\colon \mathrm{Wh}(\pi_1(STM)) \to \mathrm{Wh}(\pi_1(M))$ induced by the isomorphism $\pi_1(p)\colon \pi_1(STM) \to \pi_1(M)$ satisfies $p_* \circ p^* = 2 \cdot \mathrm{id}_{\mathrm{Wh}(\pi_1(M))}$ if d is odd, since the fiber of p is an even dimensional sphere S^{d-1} and hence has Euler characteristic 2. This implies $2x = 0$, if d is odd, To get rid of the factor 2, Farrell and Jones replaced the sphere bundle $p\colon STM \to M$ by a kind of upper hemisphere bundle $p_+\colon S_+TM \to M$ whose fiber is the upper hemisphere S^{d-1}_+ and hence contractible and therefore has Euler characteristic 1. Then the composite $\mathrm{Wh}(\pi_1(M)) \xrightarrow{(p_+)^*} \mathrm{Wh}(\pi_1(S_+TM)) \xrightarrow{(p_+)_*} \mathrm{Wh}(\pi_1(M))$ is the identity for all $d \geq 5$, and one can still show using the geometric flow on S_+TM and the hyperbolic structure on M that $(p_+)^*(x)$ vanishes if $d \geq 5$. (All this claims about the transfers will be explained in Example 24.14, which is a consequence of Theorem 24.13.)

We will give more information about the transfer in Chapter 24 and will refine ourself for the remainder of this section to explain why every element in $\mathrm{Wh}(STM)$ vanishes if M is a hyperbolic closed smooth Riemannian manifold. Farrell and Jones used the Algebraic Thin h-Cobordism Theorem 19.8 and the fact that every element in the Whitehead group $\mathrm{Wh}(\pi_1(STM))$ can be realized by the Whitehead torsion of an h-cobordism over STM, see Theorem 3.47 (i). The main ingredients in proof of Farrell and Jones was to use the geodesic flow and its specific properties due to the hyperbolic structure to convert an arbitrary h-cobordism into a thin one without changing its Whitehead torsion in $\mathrm{Wh}(\pi_1(M))$. Having the Algebraic Thin h-Cobordism Theorem 19.8 in mind, we just will explain how the geodesic flow can be used to turn automorphism of geometric modules into ϵ-controlled ones without changing their Whitehead torsion in the Whitehead group. For this we look at the very specific case, namely, the geodesic flow on the half plane model $\mathbb{H}^2$ for the two-dimensional hyperbolic space.

Consider two points with coordinates (x_1, y_1) and (x_2, y_2) in $\mathbb{H}^2$. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the y-direction. However, there is a fundamental problem: if $y_1 \neq y_2$, then the distance of these points will be bounded from below by a constant $C > 0$, regardless how long we let them flow to infinity. Therefore we make the following prearrangement. Suppose that $y_1 < y_2$. Then we first let the point (x_1, y_1) flow so that it reaches a position where $y_1 = y_2$ and do nothing to the point (x_2, y_2), and then we let both points flow simultaneously. Inspecting the hyperbolic metric, one sees that the distance between the two points (x_1, τ) and (x_2, τ) goes to zero if τ goes to infinity. This is the basic idea to gain control in the negatively curved case. In some sense we will see this *wait and then flow together* principle in the more general theorems about flows which we will present in Chapter 23. Note that moving along a flow is a continuous process and therefore should not change the associated homology class or element in the Whitehead group. It should also be clear what it means for instance to move an object or a morphism in $\mathsf{GM}^{\{1\}}(X \times \mathbb{N})$ a long flow, just move the positions of the modules $M_{(x,s)}$ and morphisms $f_{((x,s),(y,t))}$ accordingly. All of this works also in the case, where M is a closed Riemannian manifold with strictly negative sectional curvature.

Exercise 19.13. Consider two points (x_1, y_1) and (x_2, y_2) in the half plane model $\mathbb{H}^2$. Denote by $\gamma_{(x_k,y_k)}(t)$ the point obtained by flowing upwards starting with (x_k, x_k) along the geodesic given by the vertical line though (x_k, y_k) for $k = 1, 2$. Show for the hyperbolic metric d_{hyp}

$$\lim_{t\to\infty} d_{\mathrm{hyp}}(\gamma_{(x_1,y_1)}(t), \gamma_{x_2,y_2}(t)) = |\ln(y_2) - \ln(y_1)|.$$

Later Farrell-Jones could also deal with the case where M is a closed Riemannian manifold with non-positive sectional curvature, see for instance [350]. This case is significantly harder as illustrated next. Again, consider the half plane model, but this time equip it with the flat Riemannian metric coming from Euclidean inner product on $\mathbb{R}^2$. Then the same construction makes sense, but the distance between two points (x_1, τ) and (x_2, τ) is unchanged if we change τ. The basic first idea is to choose a so-called focus point far away, say $f := \big((x_1 + x_2)/2, \tau + 169356991\big)$, and then let (x_1, τ) and (x_2, τ) flow along the rays emanating from them and passing through the focus point f. In the beginning the effect is indeed that the distance becomes smaller, but as soon as we have passed the focus point the distance grows again. Either one uses the idea of simultaneously moving the focus point towards infinity while the points x_1 and x_2 flow towards it, as Farrell and Jones did, or stops flowing when has reached the focus point. We will use the second solution. In particular we want to fix a base point x_0 and want to carry out all the construction inside the closed ball $\overline{B}_R(x_0)$ for large $R > 0$.

The problem with this ideas is obvious, we must describe this process in a functorial way and carefully check all the estimates to guarantee the desired effects.

Another problem is that we later need to make everything equivariant. So if the group G acts isometrically (and does not necessarily leave the origin x_0 fixed), there are points $x \in \overline{B}_R(x_0)$ and $g \in G$ such that gx lands outside $\overline{B}_R(x)$. Then we have to use the radial projection to pull back gx to $\overline{B}_R(x_0)$. With this modification we of course do not get a strict G-action on $\overline{B}_R(x_0)$

but an up to homotopy (and actually up to higher homotopies) well-defined G-action. This is the reason why in the CAT(0)-setting one has to deal with these kind of non-strict G-actions. Moreover, we also have to deal with the problem that the focus point f may also not be fixed under the G-action.

We give a quantitative version of the sketch of ideas above for $\mathbb{R}^n$ with the Euclidian metric d. For two distinct points $a, b \in \mathbb{R}$, define

$$c_{a,b} \colon \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto \begin{cases} a & t \leq 0; \\ a + \frac{t}{d(a,b)} \cdot (b-a) & 0 \leq t \leq d(a,b); \\ b & t \geq d(a,b). \end{cases}$$

Note that the restriction of c to $[0, d(a,b)]$ is the geodesic line starting at a and ending at b and is constant for $t \leq 0$ and $t \geq d(a,b)$.

Lemma 19.14. *Fix $x_0 \in \mathbb{R}^n$ and real numbers r', r'', β, and L satisfying $r', L, \beta > 0$ and $r'' > 2\beta$. Put $T := r'' + r'$. Fix $x_1, x_2 \in \overline{B}_\beta(x_0)$. Let x be any point in $B_{r'+r''+L}(x_0)$. Put $\tau := d(x_2, x) - d(x_1, x)$.*

Then we get for all $t \in [T - r', T + r']$

$$\begin{aligned} d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) &\leq \frac{4 \cdot \beta \cdot (r' + \beta + L)}{r''}; \\ c_{x_1,x}(t) &\in \overline{B}_{2r'+r''+2\beta}(x_1); \\ c_{x_2,x}(t+\tau) &\in \overline{B}_{2r'+r''+2\beta}(x_2). \end{aligned}$$

Note that the larger we take r'' (without changing r', β, and L), the smaller $d(c_{x_1,x}(t), c_{x_2,x}(t+\tau))$ becomes for $t \in [T - r', T + r']$ and that the geodesic triangle with x,x_1, and x_2 as vertices lies in $\overline{B}_{r'+r''+\beta+L}(x_0)$. Actually, the obvious analogon of Lemma 19.14 holds in any CAT(0)-space. The contents of Lemma 19.14 will be stated in more generality in Proposition 23.30 and Theorem 23.34.

In the situation of Lemma 19.14 the points x_1 and x_2 flow towards the focal point x and everything takes place in a fixed ball around a fixed base point x_0. The wait and then flow together principle is reflected in Lemma 19.14 by the appearance of τ.

Exercise 19.15. Give the proof of Lemma 19.14.

More details about the discussion of this subsection will be given in Chapter 23.

19.6 Notes

Farrell-Hsiang used in [340] a beautiful combination of controlled topology and induction theory a la Dress to prove that the Whitehead group of fundamental groups of compact flat Riemannian manifolds is trivial. This general method, often called Farrell-Hsiang method, has been refined and used further, see for example [78, 342, 343, 346, 813, 942, 996]. This will be explained in some more detail in Chapter 20, notably in Sections 20.2 and 20.9.

There is a survey articles about continuously controlled algebra by Rosenthal [858] and about controlled K-theory by Quinn [814],

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Chapter 20
Conditions about a Group Implying the Farrell-Jones Conjecture

20.1 Introduction

In this chapter we want to isolate geometric properties about a group G which guarantee that the strategy of proofs discussed in Chapter 19 works out. So we want to describe a bunch of geometric conditions which imply the Farrell-Jones Conjecture but do not contain any K-theoretic or homotopy theoretic data. This may be useful for someone who wants to show the Farrell-Jones Conjecture for a new class of groups since she or he needs only to check that this class satisfies one of the properties (or some appropriate variation or generalization) appearing below without having to deal with the proofs relying on homotopy theory and K-theory that these properties do imply the Farrell-Jones Conjecture.

We do this in chronological order taking into account that these conditions have been reformulated and generalized over the last decades. Here is a list of the different notions which will treat:

- *Farrell-Hsiang groups* in Section 20.2;
- *Strictly transfer reducible groups – almost equivariant version* in Section 20.3;
- *Strictly transfer reducible groups – cover version* in Section 20.4;
- *Transfer reducible groups* in Section 20.5;
- *Strongly transfer reducible groups* in Section 20.6;
- *Finitely $\mathcal{F}$-amenable groups* in Section 20.7;
- *Finitely homotopy $\mathcal{F}$-amenable groups* in Section 20.8;
- *Dress-Farrell-Hsiang groups* in Section 20.9;
- *Dress-Farrell-Hsiang-Jones groups* in Section 20.10.

Remark 20.1. These various notions come in two flavours, in terms of covers or in terms of almost equivariant maps, where in general the first version implies the second. This is essentially a consequence of results such as Proposition 20.22 or Lemma 20.42.

Some of the notions above imply one another as the next result shows.

Lemma 20.2.

(i) Strictly transfer reducible groups – cover version $\implies$ strictly transfer reducible groups – almost equivariant version*;*
(ii) Strictly transfer $\mathcal{F}$-reducible – almost equivariant version *and* finitely presented $\implies$ transfer $\mathcal{F}$-reducible*;*

(iii) Strictly transfer reducible groups – cover version $\implies$ finitely $\mathcal{F}$-amenable*;*
(iv) Strongly transfer $\mathcal{F}$-reducible $\implies$ finitely homotopy $\mathcal{F}$-amenable*;*
(v) Strongly transfer $\mathcal{F}$-reducible $\implies$ transfer $\mathcal{F}$-reducible*;*
(vi) Finitely $\mathcal{F}$-amenable $\implies$ finitely homotopy $\mathcal{F}$-amenable*;*
(vii) Farrell-Hsiang over $\mathcal{F}$ $\implies$ Dress-Farrell-Hsiang over $\mathcal{F}$*;*
(viii) Dress-Farrell-Hsiang over $\mathcal{F}$ $\implies$ Dress-Farrell-Hsiang-Jones over $\mathcal{F}$*;*
(ix) Finitely homotopy $\mathcal{F}$-amenable $\implies$ Dress-Farrell-Hsiang-Jones over $\mathcal{F}$*.*

Proof. (i) see Lemma 20.25.

(ii) This follows directly from the definitions.

(iii) An N-transfer space is a compact metrizable finite-dimensional contractible ANR by Lemma 20.15, A compact metrizable topological space X is an ER if and only if it is a finite-dimensional contractible ANR. **Comment 26 (by W.)**: Is there a reference for this claim? Hence any N-transfer space is a compact ER. Now the assertion follows from Lemma 20.42.

(iv) This follows from the argument appearing in the proof of assertion iv using a variation of Lemma 20.42. and the fact that an ANR is an AR if and only if it is contractible, see [488, Theorem 7.1 and Proposition 7.2 in Chapter III on page 96].

(v) This follows directly from the definitions.

(vi) This follows from Lemma 20.42.

(vii) This follows directly from the definitions.

(viii) see [172, Remark 7.2 (2)].

(ix) see [172, Remark 7.2 (1)]. □

Remark 20.3. Note that by Lemma 20.2 the notion of a *Dress-Farrell-Hsiang-Jones group* is the most general one if we ignore *transfer reducible groups*. Namely every *Farrell-Hsiang group*, *strictly transfer reducible group – almost equivariant version*, *strictly transfer reducible group – cover version*, *strongly transfer reducible group*, *finitely $\mathcal{F}$-amenable group*, *finitely homotopy $\mathcal{F}$-amenable group*, or *Dress-Farrell-Hsiang group* is a *Dress-Farrell-Hsiang-Jones group.*

The notion of *transfer reducible groups* deals only with homotopy G-actions and not with strong homotopy G-actions or strict G-actions and does therefore not imply *Dress-Farrell-Hsiang-Jones group*. Note that also the conclusions for *transfer reducible groups* predicts only that the K-theoretic assembly map is 1-connected and not that it is a weak equivalence, cf. Theorem 20.31 and Theorem 20.61.

Proofs of the Farrell-Jones Conjecture for prominent classes such as hyperbolic groups or finite-dimensional CAT(0)-groups are based on showing that they fall into one of the classes above. We will explain for the various classes which versions of the Farrell-Jones Conjecture is known for them.

20.2 Farrell-Hsiang Groups

The next definition is equivalent to [78, Definition 1.1].

Definition 20.4 (Farrell-Hsiang group). Let G be a finitely generated group G and $\mathcal{F}$ be a family of subgroups. We call G a *Farrell-Hsiang group with respect to* $\mathcal{F}$, if there exists a natural number N such that for one (and hence all) finite set S of generators we can find for every $\epsilon > 0$:

(i) A finite group F together with a group homomorphism $p\colon G \to F$;
(ii) For every $H \in \mathcal{H}(F)$ an abstract simplicial $(G, \mathcal{F})$-complex Σ_H of dimension $\leq N$ where $\mathcal{H}(F)$ denotes the set of hyperelementary subgroups of F;
(iii) For every $H \in \mathcal{H}(F)$ a map $f_H\colon p^{-1}(H) \to |\Sigma_H|$ that is (ϵ, S)-*almost* G-*equivariant*, i.e., we have $d_{L^1}(f_H(sx), sf_H(x)) \leq \epsilon$ for all $x \in p^{-1}(H)$ and all $s \in S$.

The appearance of the hyperelementary subgroups in Definition 20.4 is due to the result of Swan [919, Corollary 4.2] that for a finite group F and the family $\mathcal{H}(F)$ of hyperelementary subgroup there are elements $\tau_H \in \mathrm{Sw}^p(H)$ for $H \in \mathcal{H}(F)$ satisfying

$$1_{\mathrm{Sw}^p(F)} = \sum_{H \in \mathcal{H}} \mathrm{ind}_H^F(\tau_H) \in \mathrm{Sw}^p(F), \tag{20.5}$$

where $\mathrm{Sw}^p(F)$ denotes the Swan ring defined in Definition 12.65 and the homomorphisms $\mathrm{ind}_H^F\colon \mathrm{Sw}^p(H) \to \mathrm{Sw}^p(F)$ are induced by induction. This is the key ingredient in induction theorems a la Dress, see for instance [75, Section 2], and leads for instance to Theorem 13.43. There is also an L-theoretic version due to Dress [301, Theorem 2]

$$1_{\mathrm{GW}(F)} = \sum_{H \in \mathcal{H}} \mathrm{ind}_H^F(\sigma_H) \in \mathrm{GW}(F) \tag{20.6}$$

for Dress' equivariant Witt ring $\mathrm{GW}(F)$ and elements $\sigma_H \in \mathrm{GW}(H)$ for $H \in \mathcal{H}(F)$.

It is often not so easy to check that a finitely generated group G is a Farrell-Hsiang group. The proof for $\mathbb{Z}^2 \rtimes \mathbb{Z}/2$ can be found in [71, Lemma 3.8].

The proof of the next theorem can be found in [78, Theorem 1.2]. It combines methods from controlled geometry and induction theory.

Theorem 20.7 (Hsiang-Farrell groups and the Farrell-Jones Conjecture). *Let G be a finitely generated group G and $\mathcal{F}$ be a family of subgroups such that G is a Hsiang-Farrell group with respect to the family $\mathcal{F}$ in the sense of Definition 20.4.*

Then the assembly maps

$$H_n^G(\mathrm{pr};;\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr)$$

and

$$H_n^G(\mathrm{pr};\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\bigl(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\bigr)$$

are bijective for every additive G-category (with involution) and $n \in \mathbb{Z}$.

Remark 20.8. Definition 20.4 can be weakened if one is only interested in the L-theoretic Farrell–Jones conjecture. Then it suffices to consider all subgroups H of F that are either 2-hyperelementary or p-elementary for some odd prime p. In other words p-hyperelementary subgroups that are not p-elementary can be ignored for all odd primes p.

In the setting of higher categories one has to enlarge the class of hyperelementary groups as explained in Section 20.9.

20.3 Strictly Transfer Reducible Groups – Almost Equivariant version

Definition 20.9 (N-transfer space X). Let N be a natural number. An *N-transfer space* is a compact metric space X possessing the following property:

For any $\delta > 0$ there exists an abstract simplicial complex K of dimension at most N, maps $i\colon X \to |K|$ and $r\colon |K| \to X$, and a homotopy $h\colon X \times [0,1] \to X$ from $r \circ i$ to id_X which is δ-controlled, i.e., for every $x \in X$ the diameter of the subset $h(\{x\} \times [0,1])$ of X is smaller than δ.

Remark 20.10 (No uniform bound on the dimensions). In Definition 20.9 and also in [66, Definition 1.3.1] it is required that there is a natural number N such that the dimensions of the simplicial complexes K appearing in Definition 20.9 is uniformly bounded by N. It turns out that this condition is not needed, cf., Remark 20.49. However, it is satisfied in all the applications, e.g., to hyperbolic groups, finite-dimensional CAT(0)-groups, or mapping class groups. **Comment 27 (by W.)**: Check this Remark 20.10 later when Chapter 25 is finished.

Definition 20.11 (Strictly $\mathcal{F}$-transfer reducible group– almost equivariant version). Let G be a finitely generated group, and let $\mathcal{F}$ be a family of subgroups. We call G *strictly $\mathcal{F}$-transfer reducible* if there exists a natural number N such that for one (and hence all) finite set S of generators there exists for any given $\epsilon > 0$

(i) an N-transfer space X in the sense of Definition 20.9 equipped with a G-action;

(ii) an abstract simplicial $(G, \mathcal{F})$-complex Σ of dimension $\leq N$;
(iii) a map $f\colon X \to |\Sigma|$ that is (ϵ, S)-*almost* G-*equivariant*, i.e., we have $d_{L^1}(f(sx), sf(x)) \leq \epsilon$ for every $s \in S$ and every $x \in X$.

Note that [85, Theorem 1.2] implies that hyperbolic groups are strictly $\mathcal{VCY}$-transfer reducible. If there exists a group G which is strictly $\mathcal{F}$-transfer reducible, then $\mathcal{F}$ must contain all cyclic subgroups of G, see [66, Remark 1.3.9].

In the sequel we denote for a family of subgroups $\mathcal{F}$ of G by $\mathcal{F}_2$ the family of subgroups of G consisting of those group $H \subseteq G$ for which H or a subgroup $H' \subseteq H$ of index $[H : H'] = 2$ belong to $\mathcal{F}$. For instance, $(\mathcal{VCY}_I)_2 = \mathcal{VCY}$ and $\mathcal{FIN}_2 = \mathcal{FIN}$.

Theorem 20.12 (Strictly transfer $\mathcal{F}$-reducible groups and the Farrell-Jones Conjecture). *Let G be a finitely generated group, and let $\mathcal{F}$ be a family of subgroups such that G is strictly $\mathcal{F}$-transfer reducible in the sense of Definition 20.11.*

Then the assembly maps

$$\begin{aligned} H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) &\to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big); \\ H_n^G(\mathrm{pr}; \mathbf{H}_{\mathcal{C}})\colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_{\mathcal{C}}) &\to H_n^G(G/G; \mathbf{H}_{\mathcal{C}}) = \pi_n\big(\mathbf{K}_{\mathcal{C}}(I(G))\big), \end{aligned}$$

are bijective for every additive G-category $\mathcal{A}$, every right exact G-∞-category $\mathcal{C}$, and every $n \in \mathbb{Z}$, and the assembly map

$$\begin{aligned} H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle})\colon & H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \\ & \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G))\big), \end{aligned}$$

is bijective for every additive G-category with involution $\mathcal{A}$ and every $n \in \mathbb{Z}$.

Later we will give as an illustration a proof of a special case of Theorem 20.12 in Proposition 24.24.

Remark 20.13 (Meaning and proof of Theorem 20.12). Theorem 20.12 implies that every strictly $\mathcal{VCY}$-transfer reducible group satisfies both the K-theoretic Farrell-Jones Conjecture 13.11 and the L-theoretic Farrell-Jones Conjecture 13.16 with coefficients in additive G-categories with involution.

The K-theoretic part of Theorem 20.12 for additive categories is a minor reformulation of [86, Theorem 1.1], as explained in [66, Theorem A, Remark 1.3.7 and Remark 1.3.8]. So the K-theoretic part of the proof of Theorem 20.12 follows from [86, Theorem 1.1].

The same ideas apply also to the L-theoretic part, see [77, Theorem B]. Note that the passage from $\mathcal{F}$ to $\mathcal{F}_2$ for L-theory is due to [77, Lemma 9.2 and Remark 9.3]. This is consistent with the fact that Theorem 13.44 holds for the K-theoretic version but not for the L-theoretic version.

The K-theory part for higher categories (and hence also the one for additive categories) follows from Theorem 20.61 using Remark 20.3.

The passage to almost equivariant maps as pursued by Bartels [66] is illuminating, since it better isolates what is needed for proofs of the Farrell-Jones Conjecture, see also Remark 20.26.

Exercise 20.14. Let Σ be a finite simplicial complex such that $|\Sigma|$ is contractible. Let G be a group which acts simplicially on Σ. Denote by $\mathcal{F}(\Sigma)$ the family of subgroups of Σ which occur as subgroups of isotropy groups of $|\Sigma|$. Show:

(i) The assembly maps appearing in Theorem 20.12 are isomorphisms for the family $\mathcal{F}(\Sigma)$;
(ii) Each isotropy group of $|\Sigma|$ has finite index in G.

In connection with Exercise 20.14 Theorem 20.53 is interesting.

Next we prove the following lemma which we have already used in the proof of Lemma 20.2.

Lemma 20.15. *Let X be an N-transfer space in the sense of Definition 20.9. Then X is a compact metrizable* ANR *with* $\dim(X) \leq N$.

Proof. By definition X is a compact metric space.

Next we show that X satisfies $\dim(X) \leq N$. Consider an open cover $\mathcal{U}$ of X. Since X is compact, there exists a finite subcover $\mathcal{V} = \{V_1, \dots, V_l\}$ of $\mathcal{U}$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{V}$. Define $V_i' = \{x \in V_i \mid \overline{B}_{\delta/2}(x) \subseteq V_i\}$ for $i = 1, 2, \dots, l$ where $\overline{B}_{\delta/2}(x)$ is the closed ball of radius $\delta/2$ around x. Then $\mathcal{V}' = \{V_1', \dots, V_l'\}$ is an open cover of X. Choose an abstract simplicial complex K of dimension at most N, maps $i\colon X \to |K|$ and $r\colon |K| \to X$, and a homotopy $h\colon X \times [0,1] \to X$ from $r \circ i$ to id_X which is $\delta/2$-controlled, i.e., for every $x \in X$ the diameter of the subset $h(\{x\} \times [0,1])$ of X is smaller than $\delta/2$. Note that this implies that for every $x \in X$ we have $d_X(x, r \circ i(x)) < \delta/2$. Since the image of i is compact, it is contained in K_0 for a finite subcomplex K_0 of K. Hence we can assume without loss of generality that K itself is a finite abstract simplicial complex of dimension $\leq N$. This implies $\dim(|K|) \leq N$. Consider the open covering $r^{-1}(\mathcal{V}') = \{r^{-1}(V_1'), \dots, r^{-1}(V_l')\}$. Choose an open cover $\mathcal{W}$ of $|K|$ which is a refinement of $r^{-1}(\mathcal{V}')$ and satisfies $\dim(\mathcal{W}) \leq N$. Consider the open cover $i^{-1}(\mathcal{W}) = \{i^{-1}(W) \mid W \in \mathcal{W}\}$ of X. Obviously we have $\dim(i^{-1}(\mathcal{W})) \leq N$ and $i^{-1}(\mathcal{W})$ is a refinement of the open cover $i^{-1}(r^{-1}(\mathcal{V}')) = \{(r \circ i)^{-1}(U_1'), \dots, (r \circ i)^{-1}(U_l')\}$. We have $(r \circ i)^{-1}(V_i') \subseteq V_i$ for $i = 1, \dots, l$. Hence $i^{-1}(\mathcal{W})$ is a refinement of $\mathcal{V}$ and hence of $\mathcal{U}$.

An N-transfer space is an ANR by [488, Theorem 6.3 in Chapter IV on page 139] since for every $\delta > 0$ there exists a finite simplicial complex Σ, maps $i\colon X \to |K|$ and $r\colon |K| \to X$, and a homotopy $h\colon X \times [0,1] \to X$ from $r \circ i$ to id_X which is δ-controlled. $\square$

20.4 Strictly Transfer Reducible Groups – Cover Version

Next we state the version of strictly transfer reducible as it appears [86, Theorem 1.1]. and give more details about some of the claims appearing in Remark 20.13.

We begin with recalling the criterion of [86, Theorem 1.1], where only the K-theoretic version is treated. Its extension to L-theory follows from the proof of [77, Theorem B].

Definition 20.16 (Strictly $\mathcal{F}$-transfer reducible group – cover version). Let G be a finitely generated group. Let $\mathcal{F}$ be a family of subgroups of G. Suppose:

(i) There exists a G-space X such that the underlying space X is the realization of a finite-dimensional abstract simplicial complex K;
(ii) There exists a G-space $\overline{X}$ which contains X as an open G-subspace such that the underlying space of $\overline{X}$ is compact, metrizable, and contractible;
(iii) Assumption 20.17 holds;
(iv) Assumption 20.19 holds for $\mathcal{F}$.

Next we give some explanations about the conditions appearing in Definition 20.16.

Assumption 20.17 (Weak Z-set condition).
There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$.

In order to state the second assumption we introduce the notion of an open $\mathcal{F}$-cover.

Definition 20.18 ((Open) $\mathcal{F}$-cover). Let Y be a G-space. Let $\mathcal{F}$ be a family of subgroups of G. An *$\mathcal{F}$-cover* of Y is a collection $\mathcal{U}$ of subsets of Y such that the following conditions are satisfied:

(i) $Y = \bigcup_{U \in \mathcal{U}} U$;
(ii) For $g \in G$, $U \in \mathcal{U}$ the set $g(U) := \{gx \mid x \in U\}$ belongs to $\mathcal{U}$;
(iii) For $g \in G$ and $U \in \mathcal{U}$ we have $g(U) = U$ or $U \cap g(U) = \emptyset$;
(iv) For every $U \in \mathcal{U}$, the subgroup $\{g \in G \mid g(U) = U\}$ lies in $\mathcal{F}$.

We call an $\mathcal{F}$-cover $\mathcal{U}$ of Y *open* if each $U \in \mathcal{U}$ is open.

Consider an open $\mathcal{F}$-cover $\mathcal{U}$. Then its nerve $\mathrm{Nerv}(\mathcal{U})$ is a simplicial complex with cell preserving simplicial G-action and hence a G-CW-complex. (A G-action on a simplicial complex is called *cell preserving*, if for every simplex σ and element $g \in G$ such that the intersection of the interior σ° of σ with $g\sigma^\circ$ is non-empty we have $gx = x$ for every $x \in \sigma$. Note that a simplicial action is not necessarily cell preserving, but the induced simplicial action on the barycentric subdivision is cell preserving.) Moreover all isotropy groups of its geometric realization $|\operatorname{Nerv}(\mathcal{U})|$ lie in $\mathcal{F}$, in other words, $\operatorname{Nerv}(\mathcal{U})$ is a simplicial $(G, \mathcal{F})$-complex. Recall that by definition the dimension $\dim(\mathcal{U})$ of an open cover is the dimension of the CW-complex $|\operatorname{Nerv}(\mathcal{U})|$.

If G is a finitely generated discrete group, then d_G denotes the word metric with respect to some chosen finite set of generators. Recall that d_G depends on the choice of the set of generators but its quasi-isometry class is independent of it.

Assumption 20.19 (Wide open $\mathcal{F}$-covers)**.** *There exists $N \in \mathbb{N}$, which only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an open $\mathcal{F}$-cover $\mathcal{U}(\beta)$ of $G \times \overline{X}$ equipped with the diagonal G-action with the following two properties:*

(i) For every $g \in G$ and $x \in \overline{X}$ there exists $U \in \mathcal{U}(\beta)$ such that

$$B_\delta(g) \times \{x\} \subset U.$$

Here $B_\delta(g)$ denotes the open β-ball around g in G with respect to the word metric d_G, i.e., the set $\{h \in G \mid d_G(g,h) < \beta\}$;

(ii) The dimension of the open cover $\mathcal{U}(\beta)$ is smaller than or equal to N.

Exercise 20.20. Let X be a G-space. Let $G \times_1 X$ be the topological space $G \times X$ with the G-action given by $g' \cdot (g,x) = (g'g, x)$ and let $G \times_d X$ be the topological space $G \times X$ with the diagonal G-action given by $g' \cdot (g,x) = (g'g, g'x)$. Show that $G \times_1 X$ and $G \times_d X$ are G-homeomorphic.

Next we describe some of the geometric constructions in [86].

Let (Z,d) be a metric space. Let $\mathcal{U}$ be a finite dimensional cover of Z by open sets. Recall that points in the geometric realization of the nerve $|\operatorname{Nerv}(\mathcal{U})|$ are formal sums $x = \sum_{U \in \mathcal{U}} x_U U$, with $x_U \in [0,1]$ such that $\sum_{U \in \mathcal{U}} x_U = 1$ and such that the intersection of all the U-s with $x_U \neq 0$ is non-empty, i.e., $\{U \mid x_U \neq 0\}$ is a simplex in the nerve of $\mathcal{U}$. There is a well-defined map

$$f = f^{\mathcal{U}} \colon Z \to |\operatorname{Nerv}(\mathcal{U})|, \quad x \mapsto \sum_{U \in \mathcal{U}} f_U(x) U \tag{20.21}$$

where

$$f_U(x) = \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)} \quad \text{with} \quad a_U(x) = d(x, Z - U) = \inf\{d(x,u) \mid u \notin U\}.$$

If Z is a G-space, d is G-invariant, and $\mathcal{U}$ is an open $\mathcal{F}$-cover, then the map $f = f^{\mathcal{U}}$ is G-equivariant.

The proof of the following proposition can be found in [86, Proposition 5.3].

Proposition 20.22. *Let $Z = (Z,d)$ be a metric space and let $\beta \geq 1$. Suppose $\mathcal{U}$ is an open cover of Z of dimension less than or equal to N with the property* *that for every $z \in Z$ there exists $U \in \mathcal{U}$ such that the open ball $B_\delta(z)$ of radius δ around z lies in U.*

Then the map $f^{\mathcal{U}} \colon Z \to |\operatorname{Nerv}(\mathcal{U})|$ of (20.21) *has the contracting property that for $z, z' \in X$ satisfying $d(z,z') \leq \frac{\beta}{4N}$ we get*

$$d_{L^1}(f^{\mathcal{U}}(z), f^{\mathcal{U}}(z')) \leq \frac{16N^2}{\beta} \cdot d(z,z').$$

Note that if β gets bigger, the estimate applies more often and $f^{\mathcal{U}}$ contracts stronger.

Let $\overline{X}$ be as in Definition 20.16. Next we define a G-invariant metric d_C depending on a constant $C > 0$ on the G-space $G \times \overline{X}$. Recall that $\overline{X}$ is assumed to be metrizable. We choose some (not necessarily G-invariant) metric $d_{\overline{X}}$ on $\overline{X}$ which generates the topology. Recall that we have already fixed some choice of a word-metric d_G on G.

Definition 20.23. Let $C > 0$. For (g,x), $(h,y) \in G \times \overline{X}$ set

$$d_C((g,x),(h,y)) = \inf \sum_{i=1}^{n} C d_{\overline{X}}(g_i^{-1}x_{i-1}, g_i^{-1}x_i) + d_G(g_{i-1}, g_i)$$

where the infimum is taken over all finite sequences $(g_0,x_0),\dots,(g_n,x_n)$ with $(g_0,x_0) = (g,x)$ and $(g_n,x_n) = (h,y)$.

The elementary proof of the next proposition can be found in [86, Proposition 4.3].

Proposition 20.24.

(i) We obtain a G-invariant metric d_C on $G \times \overline{X}$ equipped with the diagonal action by d_C;
(ii) We get $d_G(g,h) \leq d_C((g,x),(h,y))$ for all $g,h \in G$ and $x,y \in \overline{X}$;
(iii) We get $d_G(g,h) = d_C((g,x),(h,x))$ for all $g,h \in G$ and $x \in \overline{X}$.

The next lemma illustrates Remark 20.1.

Lemma 20.25. *Let G be a finitely generated group and $\mathcal{F}$ be a family of subgroups. Suppose that G is strictly transfer $\mathcal{F}$-reducible in the sense of Definition 20.16*

Then G is strictly $\mathcal{F}$-transfer reducible in the sense of Definition 20.11.

Proof. Let N be the number appearing in Assumption 20.19. By possibly enlarging N we can arrange that the dimension of the finite dimensional abstract simplicial complex K whose geometric realization is X is less or equal to N. Consider any $\epsilon > 0$. As N-transfer space, as it is required in Definition 20.11, we take $\overline{X}$. Note that $\overline{X}$ is indeed an N-transfer space by Assumption 20.17 since any compact subset of X is a contained in $|L|$ for a finite simplicial subcomplex L of K for which obviously $\dim(L) \leq N$ holds.

Let $\mathcal{U}$ be the open covering appearing in Assumption 20.19. Then we take Σ to be the simplicial complex given by the nerve of $\mathcal{U}$ and we have by definition $|\Sigma| = |\operatorname{Nerv}(\mathcal{U})|$.

Fix a finite set of generators S and let d_G be the corresponding word metric on G. Fix $C > 0$. The function $\overline{X} \to \mathbb{R}$ sending x to $d_C((e,x),(s,sx))$ is continuous. Since $\overline{X}$ is compact, we can find a constant D such that $d_C((e,x),(s,sx)) \leq D$ holds for every $x \in \overline{X}$ and $s \in S$. Choose $\beta > 0$ satisfying $4ND \leq \beta$ and $\frac{16N^2}{\epsilon} < \beta$. Then we get $d_C((e,x),(s,sx)) \leq \frac{\beta}{4N}$ for every $x \in \overline{X}$ and $s \in S$. Let $f_{\mathcal{U}} \colon G \times \overline{X} \to |\operatorname{Nerv}(\mathcal{U})|$ be the G-map defined in (20.21). Proposition 20.22 implies that $d_{L^1}(f_{\mathcal{U}}(e,x), f_{\mathcal{U}}(s,sx)) < \epsilon$ holds for every $x \in \overline{X}$ and $s \in S$.

Define the desired map $f \colon \overline{X} \to |\operatorname{Nerv}(\mathcal{U})|$ by sending x to $f_{\mathcal{U}}(e,x)$. Since we have

$$d_{L^1}(f(sx), sf(x)) = d_{L^1}(f_{\mathcal{U}}(e), sf_{\mathcal{U}}(e,x)) = d_{L^1}(f_{\mathcal{U}}(e), f_{\mathcal{U}}(s,sx)) < \epsilon,$$

the group G is strictly $\mathcal{F}$-transfer reducible group in the sense of Definition 20.11. $\square$

Because of Lemma 20.25 Theorem 20.12 applies also to groups which are strictly transfer $\mathcal{F}$-reducible in the sense of Definition 20.16

In some sense one can also get the other direction of the implication appearing in Lemma 20.25 since maps from a topological space to the geometric realization of a finite dimensional simplicial space translate to finite dimensional covers of the source, as we can pull back standard coverings of the simplicial complex.

Remark 20.26 (Role of the compactification). Note that in Definition 20.11 the compactification $\overline{X}$ appearing in Definition 20.16 does not occur anymore and hence the criterion may be easier to verify. Moreover, this formulation isolates in a nice fashion what is really needed for the proof of the Farrell-Jones Conjecture. On the other hand, in many cases where the Farrell-Jones Conjecture has been proved, such as hyperbolic groups, finite dimensional CAT(0)-groups, or mapping class groups, these compactifications $\overline{X}$ and in particular their boundary $\partial X = \overline{X} \setminus X$ were well-known and did play a role and did lead to the necessary constructions, often, since the elements on the boundary corresponds to geodesic rays emanating in the space X and going to infinity. It is conceivable that for future proofs for new groups Definition 20.11 may be more appropriate, but we also expect that some shadow of the notion of a compactification and its boundary and of non-positive curvature will occur in future proofs which follow and generalize the actual ones.

20.5 Transfer Reducible Groups

The strict versions of transfer reducible of the previous Sections 20.3 and 20.4 were sufficient to treat hyperbolic groups. In order to handle CAT(0) groups, one has to pass to the following generalizations of this notion where homotopy coherent group actions come in and one has to drop *strict*.

Definition 20.27 (Homotopy action of a (finitely presented) group on a space). A *homotopy action* of a group G on a space X is a group homomorphism $\rho\colon G \to [X,X]$ to the monoid of homotopy classes of self homotopy equivalences of X.

Let G be finitely presented group with a finite presentation $\langle S \mid R\rangle$. A *homotopy action of the finitely presented group* (φ, h) of G on X is given by the following data:

(i) For every $s \in S \cup S^{-1} = \{s \in G, s \text{ or } s^{-1} \text{ belongs to } S\}$, we have a map

$$\varphi_s\colon X \to X;$$

(ii) For every word $r = s_1 s_2 \cdots s_n \in R$ for $s_i \in S \cup S^{-1}$, we have a homotopy

$$h_r\colon \varphi_{s_1} \circ \varphi_{s_2} \circ \cdots \circ \varphi_{s_n} \simeq \mathrm{id}_X .$$

Note that a homotopy action of the finitely presented group G yields a homotopy G-action, but is a stronger notion since the choice of the homotopies h for the relations is part of the structure.

The next definition is just the condition appearing in [66, Theorem B] and motivated by [77, Definition 1.8].

Definition 20.28 (Transfer $\mathcal{F}$-reducible group). Let G be a finitely generated presented group and let $\mathcal{F}$ be a family of subgroups. We call G *transfer $\mathcal{F}$-reducible* if for one (and hence all) finite presentation $\langle S \mid R\rangle$ there exists a natural number N such that there is for any given $\epsilon > 0$

(i) an N-transfer space X in the sense of Definition 20.9 equipped with a homotopy G-action (φ, h) in the sense of Definition 20.27;
(ii) an abstract simplicial $(G, \mathcal{F})$-complex Σ of dimension $\leq N$;
(iii) a map $f\colon X \to |\Sigma|$ that is $(\epsilon, \langle S \mid R\rangle)$-*almost G-equivariant*, i.e., it satisfies:
   (a) We have $d_{L^1}(f(sx), sf(x)) < \epsilon$ for every $s \in S$ and every $x \in X$;
   (b) For every $x \in X$ and $r \in R$, the diameter of the subset $h_r(\{x\} \times [0,1])$ of X is $\leq \epsilon$.

Remark 20.29. Definition 20.28 is just the condition appearing in [66, Theorem B] and motivated by [77, Definition 1.8], one replaces the formulation in terms of open coverings by the formulation in terms of almost equivariant maps, in the spirit of Remark 20.1 or of Lemma 20.25 and its proof. In particular a group which satisfies the notion of transfer reducible over $\mathcal{F}$ in the sense of [77, Definition 1.8] is transfer $\mathcal{F}$-reducible group in the sense of Definition 20.28.

Remark 20.30. Note that [79, Main Theorem] implies that finite-dimensional CAT(0)-groups are finitely presented and transfer $\mathcal{VCY}$-reducible. A sketch of this proof can also be found in [66, Section 1.5]. We recall that a finite-dimensional CAT(0)-group is a group admitting a cocompact proper isometric action on a CAT(0)-space which has finite topological dimension.

Theorem 20.31 (Transfer reducible groups and the Farrell-Jones Conjecture). *Let $\mathcal{F}$ be a family of subgroups. Let G be a finitely presented group coning with a presentation $\langle S \mid R\rangle$ such that G is transfer $\mathcal{F}$-reducible in the sense of Definition 20.28.*

Then the assembly map

$$H_n^G(\mathrm{pr};\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

is bijective for $n \le 0$ and surjective for $n = 1$ for every additive G-category, and the assembly map

$$H_n^G(\mathrm{pr};\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{F}_2}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \\ \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}(I(G))\big)$$

is bijective for every $n \in \mathbb{Z}$ and every additive G-category with involution.

Theorem 20.31 is a reformulation of [86, Theorem 1.1] as pointed out in [66, Remarks 1.3.15 and 1.3.18] for the K-theory version. Its extension to L-theory follows from the proof of [77, Theorem B].

20.6 Strongly Transfer Reducible Groups

In Theorem 20.31 we deal only with lower and middle K-theory. In order to treat also higher algebraic K-theory, one needs to take higher homotopies into account.

The next definition is taken from [172, Definition 5.2], see also Wegner [973, Definition 2.1].

Definition 20.32 (Strong homotopy action). A *strong homotopy action*, sometimes also called *homotopy coherent G-action*indexhomotopy action!homotopy coherent G-action of a group on a space, (Γ, Z) of a group G on a topological space Z consists of a map

$$\Gamma\colon \coprod_{k=0}^{\infty}\left(\left(\prod_{j=1}^{k} G \times [0,1]\right) \times G \times Z\right) \to Z$$

satisfying

$$\begin{aligned}&\Gamma(g_k, t_k, \ldots, g_1, t_1, g_0, z)\\ &= \begin{cases} \Gamma(g_k, t_k, \ldots, g_j, \Gamma(g_{j-1}, t_{j-1}, \ldots, g_0, z)) & t_j = 0, 1 \leq j \leq k;\\ \Gamma(g_k, t_k, \ldots, t_{j+1}, g_j g_{j-1}, t_{j-1}, \ldots, g_0, z) & t_j = 1, 1 \leq j \leq k;\\ \Gamma(g_k, t_k, \ldots, g_2, t_2, g_1, z) & g_0 = e;\\ \Gamma(g_k, t_k, \ldots, g_{j+1}, t_{j+1} t_j, g_{j-1}, \ldots, g_0, z) & g_j = e, 1 \leq j \leq k-2;\\ \Gamma(g_{k-1}, t_{k-1}, \ldots, g_0, z) & g_k = e;\\ x & g_0 = e, k = 0. \end{cases}\end{aligned}$$

Here we use the convention that non-existing entries are dropped, e.g., g_k, t_k in the first line if $j = k$ or the entry t_{j-1} in the second line if $j = 1$.

Next we present the notion of *strongly transfers reducible over $\mathcal{F}$*, which we prefer to call *strongly $\mathcal{F}$-transfer reducible*, due to Wegner [973, Definition 3.1], where all the higher homotopies are taken into account.

Given a strong homotopy action Γ in the sense of Definition 20.32, we need to introduce the following notions. For $k \in \mathbb{N}$, $g \in G$, and a subset $S \subseteq G$ containing e and g we define a subset of $\operatorname{map}(X, X)$

(20.33)

$$F_g(\Gamma, S, k) := \{\Gamma(g_k, t_k, \ldots g_0, ?)\colon X \to X \mid g_i \in S, t_j \in [0,1], g_k \cdots g_0 = g\}$$

For $(g, x) \in G \times X$ we define

$$S^0_{\Gamma,S,k}(g, x) = \{(g, x)\} \subseteq G \times X \tag{20.34}$$

and

$$S^1_{\Gamma,S,k}(g, x) \subseteq G \times X \tag{20.35}$$

as the subset of all $(h, y) \in G \times X$ with the property that here are $a, b \in S$, $f \in F_a(\Gamma, S, k)$, and $f' \in F_b(\Gamma, S, k)$ satisfying both $f(x) = f'(y)$ and $h = ga^{-1}b$. For $n \geq 2$ define inductively

$$S^n_{\Gamma,S,k}(g, x) \subseteq G \times X \tag{20.36}$$

by

$$S^n_{\Gamma,S,k}(g, x) = \bigcup_{\{(h,y) \in S^{n-1}_{\Gamma,S,k}(g,x)\}} S^1_{\Gamma,S,k}(h, y).$$

Exercise 20.37. Let X be a G-space X. Consider the G-action as a homotopy G-action in the sense of Definition 20.32 in the obvious way. Define the subsets of G by

$$\begin{aligned} S[k] &:= \{g_0 g_1 \cdots g_k \mid g_i \in S\};\\ S[k, n] &:= \{a_1^{-1} b_1 \cdots a_n^{-1} b_n \mid a_1, \ldots a_n, b_1, \ldots b_n \in S[k]\}. \end{aligned}$$

Show that then the sets $F_g(\Gamma, S, k)$ of (20.33), $S^1_{\Gamma,S,k}(g,x)$ of (20.35), and $S^n_{\Gamma,S,k}(g,x)$ of (20.36) reduce to

$$\begin{aligned} F_g(\Gamma, S, k) &= \{l_g \colon X \to X \mid g \in S[k]\} \subseteq \operatorname{map}(X, X); \\ S^1_{\Gamma,S,k}(g,x) &= \{(gu, u^{-1}x) \mid u \in S[k,1]\}; \\ S^n_{\Gamma,S,k}(g,x) &= \{(gv, v^{-1}x) \mid v \in S[k,n]\}. \end{aligned}$$

Definition 20.38 (Strongly transfer $\mathcal{F}$- reducible). A group G is *strongly $\mathcal{F}$-transfer reducible*, if there exists a natural number N with the following property: For all subsets $S \subseteq G$, which satisfy $S = \{g^{-1} \mid g \in S\}$ and contain the unit $e \in G$, and all natural numbers n, k there are

- a N-transfer space X in the sense of Definition 20.9;
- a strong homotopy action Γ in the sense of Definition 20.32;
- An open $\mathcal{F}$-cover $\mathcal{U}$ of $G \times X$, where the G-action on $G \times X$ is given by $g' \cdot (g,x) = (g'g, x)$, of dimension at most N such that for every $(g,x) \in G \times X$ there exists $U \in \mathcal{U}$ with $S^n_{\Gamma,S,k}(g,x) \subseteq U$.

Hyperbolic groups are strongly transfer reducible over $\mathcal{VCY}$ by the proof of [77, Proposition 2.1], as explained in [973, Example 3.2]. Wegner [973, Theorem 3.4] explains that finite-dimensional CAT(0)-groups are strongly transfer reducible over $\mathcal{VCY}$ by [79, Main Theorem].

Theorem 20.39 (Strongly transfer $\mathcal{F}$-reducible groups and the Farrell-Jones Conjecture). *Let G be a group and $\mathcal{F}$ be a family of subgroups such that G is strongly $\mathcal{F}$-transfer reducible.*

Then the assembly maps

$$\begin{aligned} H_n(\mathrm{pr}; \mathbf{K}_\mathcal{A}) \colon H^G_n(E_\mathcal{F}(G); \mathbf{K}_\mathcal{A}) &\to H^G_n(G/G; \mathbf{K}_\mathcal{A}) = \pi_n\big(\mathbf{K}_\mathcal{A}(I(G))\big); \\ H^G_n(\mathrm{pr}; \mathbf{H}_\mathcal{C}) \colon H^G_n(E_\mathcal{F}(G); \mathbf{H}_\mathcal{C}) &\to H^G_n(G/G; \mathbf{H}_\mathcal{C}) = \pi_n\big(\mathbf{K}_\mathcal{C}(I(G))\big), \end{aligned}$$

are bijective for every additive G-category $\mathcal{A}$, every right exact G-∞-category $\mathcal{C}$, and every $n \in \mathbb{Z}$, and the assembly map

$$\begin{aligned} H^G_n(\mathrm{pr}; \mathbf{L}^{\langle -\infty \rangle}_\mathcal{A}) \colon H^G_n(E_{\mathcal{F}_2}(G); \mathbf{L}^{\langle -\infty \rangle}_\mathcal{A}) & \\ \to H^G_n(G/G; \mathbf{L}^{\langle -\infty \rangle}_\mathcal{A}) &= \pi_n\big(\mathbf{L}^{\langle -\infty \rangle}_\mathcal{A}(I(G))\big), \end{aligned}$$

is bijective for every additive G-category with involution $\mathcal{A}$ and every $n \in \mathbb{Z}$.

Proof. For the K-theoretic part in the setting of additive categories we refer to [973, Theorem 1.1], whose prove is based on [77, Theorem B]. The more general proof of the K-theory version for higher categories follows from Remark 20.3 and Theorem 20.61.

The L-theory part follows already from Theorem 20.31 since strongly $\mathcal{F}$-transfer reducible implies transfer $\mathcal{F}$-reducible by Lemma 20.2 (v). □

We have explained in Section 19.5 why in the CAT(0)-setting one needs to consider strong homotopy G-actions instead of strict G-actions.

Theorem 20.40 (Strongly transfer $\mathcal{VCY}$-reducible groups and the Full Farrell-Jones Conjecture). *Let G be a group such that G is strongly $\mathcal{VCY}$-transfer reducible.*

Then G is a Farrell-Jones group, i,e., it satisfies the Full Farrell-Jones Conjecture 13.27.

Proof. Let F be a finite subgroup. Let $\mathcal{VCY}^l$ be the family of subgroups H of $G \wr F$ such that there is subgroup $H' \subseteq H$ of finite index such that H' is isomorphic to a finite product $V_1 \times V_2 \cdots \times \times V_k$ for virtually cyclic groups V_i. Then $G \wr F$ satisfies the Farrell-Jones Conjecture for K and L-theory with additive G-categories as coefficients, see Conjecture 13.11 and Conjecture 13.16, with respect to the family $\mathcal{VCY}^l$ by [88, Theorem 5.1 (ii)]. Every element in $\mathcal{VCY}^l$ is virtually abelian and hence satisfies the Farrell-Jones Conjecture for K and L-theory with additive G-categories as coefficients, see Theorem 16.1 (ic). We conclude from the Transitivity principle 15.13 that $G \wr F$ satisfies the Farrell-Jones Conjecture for K and L-theory with additive G-categories as coefficients. Hence G satisfies the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products, see Conjecture 13.24 and Conjecture 13.25. It remains to show that G satisfies the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products, see Conjecture 13.26. This follows from Remark 20.3 and Theorem 20.62. $\square$

20.7 Finitely $\mathcal{F}$-Amenable Groups

Let G be a group and let $\mathcal{F}$ be a family of subgroups. The next definition is taken from [69, Introduction] which is motivated by [86, Theorem 1.1].

Definition 20.41 (Finitely $\mathcal{F}$-amenable group action). For a natural number N, a G-action on a space X is called *N-$\mathcal{F}$-amenable*, if for all finite subsets S of G there exists an open $\mathcal{F}$-cover $\mathcal{U}$ in the sense of Definition 20.18 of $G \times X$ equipped with the diagonal G-action $g \cdot (h, x) = (gh, gx)$ satisfying:

- The dimension of $\mathcal{U}$ is at most N;
- The open $\mathcal{F}$-cover cover $\mathcal{U}$ is S-long (in the group coordinate), i.e., for every $(g, x) \in G \times X$ there is $U \in \mathcal{U}$ with $gS \times \{x\} \subseteq U$.

A G-action on a space X is called *finitely $\mathcal{F}$-amenable* if it is N-$\mathcal{F}$-amenable for some natural number N.

The proof of the next lemma can be found in [69, Lemma 4.2], whose proof is based on [432, Proposition 4.2]. It is useful for studying how N-$\mathcal{F}$-amenability behaves under finite extensions, see [69, Section 4.1].

Lemma 20.42. *Let G be a group G and $\mathcal{F}$ be a family of subgroups. Then the following statements are equivalent for a compact metric space X and a G-action on it:*

(i) The G-action on X is N-$\mathcal{F}$-amenable in the sense of Definition 20.41;
(ii) For every finite subset $S \subseteq G$ and every $\epsilon > 0$, there exists an abstract simplicial $(G, \mathcal{F})$-complex Σ of dimension at most N together with a map $f\colon X \to |\Sigma|$ that is (ϵ, S)-almost G-equivariant, i.e., we have $d_{L^1}(f(sx), sf(x)) \le \epsilon$ for every $s \in S$ and every $x \in |\Sigma|$.

Exercise 20.43. Suppose that G is finitely generated. Let S_1 and S_2 be two finite sets of generators. Then the second condition appearing in Lemma 20.42 holds for S_1 if and only if holds for S_2.

Recall that a metric space X is an ER (= Euclidean retract) if it can be embedded in some $\mathbb{R}^n$ as a retract. A compact metric space X is an ER if and only if it is a finite-dimensional contractible ANR, **Comment 28 (by W.)**: Do we have a reference for this claim? or, equivalently, if it is finite-dimensional, locally contractible, and contractible.

Definition 20.44 (Finitely $\mathcal{F}$-amenable group). We call a group G *finitely $\mathcal{F}$-amenable* if G admits a finitely $\mathcal{F}$-amenable action on a compact ER.

Theorem 20.45 (Finitely $\mathcal{F}$-amenable actions and the Farrell-Jones Conjecture). *Let G be a group and $\mathcal{F}$ be a family of subgroups. Suppose that G is finitely $\mathcal{F}$-amenable.*

Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{A}}(I(G))\bigr);$$
$$H_n^G(\mathrm{pr}; \mathbf{H}_{\mathcal{C}})\colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_{\mathcal{C}}) \to H_n^G(G/G; \mathbf{H}_{\mathcal{C}}) = \pi_n\bigl(\mathbf{K}_{\mathcal{C}}(I(G))\bigr),$$

are bijective for every additive G-category $\mathcal{A}$, every right exact G-∞-category $\mathcal{C}$, and every $n \in \mathbb{Z}$, and the assembly map

$$\begin{aligned} H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle})\colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) & \\ \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) &= \pi_n\bigl(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G))\bigr), \end{aligned}$$

is bijective for every additive G-category with involution $\mathcal{A}$ and every $n \in \mathbb{Z}$.

Proof. This follows from the axiomatic results in [77, Theorem 1.1] and [86, Theorem 1.1], as explained in [69, Theorem 4.8] for additive G-categories (with involution) as coefficients. (In [69, Theorem 4.8] it is required that $\mathcal{F}$ is closed under passage to over groups of finite index but this is not necessary, see also [67, Theorem 4.3]). The K-theoretic version for higher G-categories as coefficients follows from Remark 20.3 and Theorem 20.62. ◻

Let $\mathcal{C}$ be a class of groups that is closed under isomorphisms and taking subgroups. Let $\mathfrak{ac}(\mathcal{C})$ be the class of all groups G that admit a finitely generated $\mathcal{C}(G)$-amenable action on an ER where $\mathcal{C}(G)$ is the family of subgroups of G which belong to $\mathcal{C}$. Because of the action on a one-point-space we get $\mathcal{C} \subseteq \mathfrak{ac}(\mathcal{C})$. Starting with $\mathfrak{ac}^0(\mathcal{C}) = \mathcal{C}$, we can define inductively $\mathfrak{ac}^{n+1}(\mathcal{C}) = \mathfrak{ac}(\mathfrak{ac}^n(\mathcal{C}))$. We set

$$\mathsf{AC}(\mathcal{C}) = \bigcup_{n=0}^{\infty} \mathfrak{ac}^n(\mathcal{C}). \tag{20.46}$$

Let $\mathcal{VNIL}$ be the class of virtually nilpotent groups and $\mathcal{VSOLV}$ be the class of virtually solvable groups.

Theorem 20.47 (Groups in $\mathsf{AC}(\mathcal{VSOLV})$ satisfy the Full Farrell-Jones Conjecture). *Every group in $\mathsf{AC}(\mathcal{VSOLV})$ satisfies the Full Farrell-Jones Conjecture 13.27.*

Proof. This follows from the Transitivity Principle, see Theorem 15.13, Theorem 16.1 (ic), and Theorem 20.45 for additive G-categories as coefficients as explained in [69, Corollary 4.10 and Remark 9.4]. For the setting of higher G-categories one needs to replace Theorem 20.45 by [172, Theorem 1.4]. □

The main result in [69, Lemma 9.3] says that mapping class groups belong to $\mathsf{AC}(\mathcal{VNIL})$ and hence satisfy the Full Farrell-Jones Conjecture 13.27, see [69, Theorem A and Remark 9.4].

20.8 Finitely Homotopy $\mathcal{F}$-Amenable Groups

Next we state the version of finitely homotopy $\mathcal{F}$-amenable groups appearing in [172, Definition 5.4] which goes back to [86, 77, 973] and is essentially the one appearing in Bartels [68, Definition 2.11 and Theorem 2.12].

An AR (= absolute retract) is a metrizable topological space such that for every embedding $i\colon X \to Y$ as a closed subspace into a metric space Y there is a retraction $r\colon Y \to X$, or, equivalently, for every metric space Z, every closed subset $Y \subseteq Z$, and every (continuous) map $f\colon Y \to X$, there exists an extension $F\colon Z \to X$ of f. An ANR is an AR if and only if it is contractible, see [488, Theorem 7.1 and Proposition 7.2 in Chapter III on page 96].

Definition 20.48 (Finitely homotopy $\mathcal{F}$-amenable group). Let G be a group and let $\mathcal{F}$ be a family of subgroups. We call G *finitely homotopy $\mathcal{F}$-amenable* if there exist

(i) A sequence $\{\Gamma_n, Z_n\}_{n\in\mathbb{N}}$ of homotopy coherent G-actions in the sense of Definition 20.32;

(ii) A sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ of abstract simplicial complexes coming with a simplicial G-action;

(iii) A sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous maps $f_n\colon Z_n \to |\Sigma_n|$,

such that the following holds:

(a) For every $n \in \mathbb{N}$ the space Z_n is a compact contractible AR;

(b) For every $n \in \mathbb{N}$ the isotropy groups of $|\Sigma_n|$ belong to $\mathcal{F}$;

(c) There exists a natural number N with $\dim(\Sigma_n) \le N$ for all $n \in \mathbb{N}$;

(d) For every $k \in \mathbb{N}$ and elements $g_0, g_1, \ldots, g_k$ in G we have

$$\lim_{n\to\infty} \sup_{\substack{(t_1,\ldots,t_k)\in[0,1]^k,\\ z\in Z_n}} d^{\Sigma_n}_{L^1}\big(f_n(\Gamma_n(g_k,t_k,\ldots,g_1,t_1,g_0,z)), g_k\ldots g_0 f_n(z)\big) = 0.$$

Note that a finitely homotopy $\mathcal{F}$-amenable group is in particular a Dress-Farrell-Hsiang-Jones group, see [172, Remark 7.2 (1)]. Hence Theorem 20.61 and Theorem 20.62 apply to homotopy $\mathcal{F}$-amenable groups, see also [172, Theorem 5.1].

Remark 20.49. The condition formulated in Definition 20.48 is slightly weaker than the assumptions in Bartels [68, Theorem 2.12] since we do not require a uniform bound on the dimension of the AR-s Z_n. In practice, however, the dimensions of the simplicial complexes Σ_n are usually bounded in terms of the dimensions of the spaces Z_n. In this case Z_n is a sequence of ER-s with uniformly bounded covering dimension.

20.9 Dress-Farrell-Hsiang Groups

Definition 20.50 (Dress group). A finite group D is called a *Dress group* if there exist (not necessarily distinct) prime numbers p and q and subgroups $P \subseteq C \subseteq D$ such that P is normal in C and C is normal in D, P is a p-group, C/P is cyclic, and D/C is a q-group.

For F a finite group, we denote the family of Dress subgroups of F by $\mathcal{D}(F)$.

Exercise 20.51. Show for a finite group F that $\mathcal{H}(F) \subseteq \mathcal{D}(F)$ holds.

The next definition is taken from [172, Definition 6.3].

Definition 20.52 (Dress-Farrell-Hsiang group). Let G be a finitely generated group and let $\mathcal{F}$ be a family of subgroups. Given a natural number N, we call G a *Dress-Farrell-Hsiang group* over $\mathcal{F}$ if there exist

(i) A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite groups;

(ii) A sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of surjective group homomorphism $\alpha_n\colon G \to F_n$;

(iii) A collection $\{(\Sigma_n, D) \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ where Σ_n is an abstract simplicial complex with a simplicial $\alpha_n^{-1}(D)$-action;

(iv) A collection $\{f_n \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ of maps of sets $f_{n,D} \colon G \to \Sigma_{n,D}$,

such that the following holds:

(a) For every $n \in \mathbb{N}$ and $D \in \mathcal{F}(F_n)$, the $\alpha_n^{-1}(D)$-isotropy groups of $|\Sigma_{n,D}|$ belong to the family $\mathcal{F}|_{\alpha_n^{-1}(D)} = \{H \cap \alpha_n^{-1}(D) \mid H \in \mathcal{F}\}$;
(b) There exists a natural number N with $\dim(\Sigma_{n,D}) \leq N$ for all $n \in \mathbb{N}$ and $D \in \mathcal{D}(F_n)$;
(c) For every $n \in \mathbb{N}$ and $D \in \mathcal{F}(F_n)$, the map $f_{n,D}$ is $\alpha_n^{-1}(D)$-equivariant where $\alpha_n^{-1}(D)$ acts on G from the left;
(d) For every $g \in G$ we have

$$\lim_{n\to\infty} \sup_{D\in\mathcal{D}(F_n),\gamma\in G} d_{L^1}^{\Sigma_{n,D}}\bigl(f_{n,D}(\gamma g), f_{n,D}(\gamma)\bigr) = 0.$$

Note that a Dress-Farrell-Hsiang group is in particular a Dress-Farrell-Hsiang-Jones group, See [172, Remark 7.2 (2)]. Hence Theorem 20.61 and Theorem 20.62 apply to Dress-Farrell-Hsiang groups, see also [172, Theorem 6.1].

The next result is due to Oliver [753, Theorem 7].

Theorem 20.53 (Fixed point free smooth actions of finite groups on disks). *Let G be a finite group. Then G is not a Dress group if and only if G acts smoothly on some disk D^n such that $(D^n)^G$ is empty.*

Exercise 20.54. Let G be a finite abelian group. Show that G admits a smooth action on some disk D^n with $(D^n)^G = \emptyset$ if and only if there are three distinct primes p_1, p_2, and p_3 such that the p_k-Sylow subgroup of G is non-cyclic for $k = 1, 2, 3$.

The following definition is equivalent to the one in [942, Definition 8.1].

Definition 20.55 (A-theoretic Swan ring $\mathrm{Sw}^A(G)$). For a group G define the *A-theoretic Swan ring* $\mathrm{Sw}^A(G)$ as follows. The underlying abelian group is defined as follows. Every compact G-CW-complexes X, or, equivalently G-CW-complexes X, whose underling CW-complex is finite, or, equivalently, G-CW-complexes X such that X has finitely many equivariant cells and the isotropy group of each $x \in X$ has finite index in G, defines an element $[X]$ in $\mathrm{Sw}^A(G)$. The relations are given as follows. If X and Y are compact G-CW-complexes and there is a G-map $f \colon X \to Y$ such that f is a homotopy equivalence (after forgetting the G-actions), then we require $[X] = [Y]$. If the compact G-CW-complex X is the union of sub G-CW-complexes X_1 and X_2 and X_0 is the intersection of X_1 and X_2, then we require $[X] = [X_1] + [X_2] - [X_0]$. The multiplication comes from the cartesian product of two compact G-CW-complexes equipped with the diagonal G-action. The zero element is represented by the empty set and the unit by G/G.

The group $\mathrm{Sw}^A(G)$ is the A-theoretic analogue of the Swan group $\mathrm{Sw}^p(G)$ introduced in Definition 12.65.

Exercise 20.56. Let G be a (not necessarily finite) group. Define for a compact G-CW-complex X the element

$$s(X) := \sum_{n \geq 0} (-1)^n \cdot [C_n^c(X)] \in \mathrm{Sw}^p(G)$$

where $C_*^c(X)$ is the cellular $\mathbb{Z}G$-chain complex of X. Show that we obtain well-defined ring homomorphism

$$s\colon \mathrm{Sw}^A(G) \to \mathrm{Sw}^p(G), \quad [X] \mapsto s(X).$$

Exercise 20.57. Let G be a (not necessarily finite) group. Let $A(G)$ be the Burnside ring defined in Example 12.63. Show that we obtain a well-defined surjective ring homomorphism $a\colon A(G) \to \mathrm{Sw}^A(G)$ by viewing a finite G-set as a compact 0-dimensional G-CW-complex.

Exercise 20.58. Let p be a prime. Then we get a well-defined sequence of isomorphisms of abelian groups

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{u} A(\mathbb{Z}/p) \xrightarrow{a} \mathrm{Sw}^A(\mathbb{Z}/p) \xrightarrow{c} \mathbb{Z} \oplus \mathbb{Z}$$

where u sends (m,n) to $m \cdot [\mathbb{Z}/p] + n \cdot [\{*\}]$, the map a has been defined in Exercise 20.57, and c sends $[M]$ to $(\mathrm{rk}_{\mathbb{Z}}(M), \mathrm{rk}_{\mathbb{Z}}(M^{\mathbb{Z}/p}))$.

The appearance of the Dress subgroups in Definition 20.60 is due to the result of Ullmann-Winges [942, Theorem 8.7] that for a finite group F and the family $\mathcal{D}(F)$ of Dress subgroup there are elements $\mu_H \in \mathrm{Sw}^A(H)$ for $H \in \mathcal{D}(F)$ satisfying

$$1_{\mathrm{Sw}^A(F)} = \sum_{H \in \mathcal{H}} \mathrm{ind}_H^F(\mu_H) \in \mathrm{Sw}^A(F) \tag{20.59}$$

where the homomorphisms $\mathrm{ind}_H^F\colon \mathrm{Sw}^A(H) \to \mathrm{Sw}^A(F)$ are induced by induction. The proof of (20.59) is based on Oliver's Theorem 20.53. Note that one needs to pass to the A-theoretic Swan ring in the context of higher categories since $\mathrm{Sw}^p(F)$ acts on $K_n(RF)$ but for instance not on $\pi_n(A(BG))$.

Note that a Dress-Farrell-Hsiang group is in particular a Dress-Farrell-Hsiang-Jones group, see [172, Remark 7.2 (2)]. Hence Theorem 20.61 and Theorem 20.62 apply to Dress-Farrell-Hsiang groups.

20.10 Dress-Farrell-Hsiang-Jones Groups

The next definition is taken from [172, Definition 7.1].

Definition 20.60 (Dress-Farrell-Hsiang-Jones group). Let G be a finitely generated group and $\mathcal{F}$ be a family of subgroups. Given a natural number N, we call G a *Dress-Farrell-Hsiang-Jones group* over $\mathcal{F}$ if there exist

(i) A sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite groups;
(ii) A sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of surjective group homomorphism $\alpha_n \colon G \to F_n$;
(iii) A collection $\{\Gamma_{n,D}, Z_{n,D} \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ of homotopy coherent G-action in the sense of Definition 20.32;
(iv) A collection $\{\Sigma_{n,D} \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ where $\Sigma_{n,D}$ is an abstract simplicial complex with a simplicial $\alpha_n^{-1}(D)$-action;
(v) A collection $\{f_{n,D} \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ of continuous maps $f_{n,D} \colon G \times Z_{n,D} \to |\Sigma_{n,D}|$,

such that the following holds:

(a) For every $n \in \mathbb{N}$ and every $D \in \mathcal{D}(F_n)$, the topological space $Z_{n,D}$ is a compact AR;
(b) For every $n \in \mathbb{N}$ and every $D \in \mathcal{D}(F_n)$, the $\alpha_n^{-1}(D)$-isotropy groups of $|\Sigma_{n,D}|$ belong to the family $\mathcal{F}|_{\alpha_n^{-1}(D)} = \{H \cap \alpha_n^{-1}(D) \mid H \in \mathcal{F}\}$;
(c) The exists a natural number N with $\dim(\Sigma_{n,D}) \leq N$ for all $n \in \mathbb{N}$ and $D \in \mathcal{D}(F_n)$;
(d) For every $n \in \mathbb{N}$ and every $D \in \mathcal{D}(F_n)$, the map $f_{n,D}$ is $\alpha_n^{-1}(D)$-equivariant where $\alpha_n^{-1}(D)$ acts on $G \times Z_{n,D}$ diagonally from the left;
(e) For every $k \in \mathbb{N}$ and elements $g_0, g_1, \ldots, g_k$ in G we have

$$\lim_{n\to\infty} \left(\sup_{\substack{D\in\mathcal{D}(F_n),\gamma\in G\\ (t_1,\ldots,t_k)\in[0,1]^k,\\ z\in Z_{n,D}}} u_n \right) = 0$$

for

$$u_n := d_{L^1}^{\Sigma_{n,D}}\big(f_{n,D}(\gamma, \Gamma_{n,D}(g_k, t_k, \ldots, g_1, t_1, g_0, z), f_{n,D}(\gamma_k g_k \ldots g_0, z)\big).$$

The next result is taken from [172, Theorem 7.4]. We will give more details of its proof in Chapter 25. **Comment 29 (by W.)**: Make the reference later more precise when Chapter 25 is written.

Theorem 20.61 (Dress-Farrell-Hsiang-Jones groups and the K-theoretic Farrell-Jones Conjecture).
Let G be a finitely generated group which is a Dress-Farrell-Hsiang-Jones group over $\mathcal{F}$ in the sense of Definition 20.60.
Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big);$$
$$H_n^G(\mathrm{pr}; \mathbf{H}_{\mathcal{C}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_{\mathcal{C}}) \to H_n^G(G/G; \mathbf{H}_{\mathcal{C}}) = \pi_n\big(\mathbf{K}_{\mathcal{C}}(I(G))\big),$$

are bijective for every additive G-category $\mathcal{A}$, every right exact G-∞-category $\mathcal{C}$, and every $n \in \mathbb{Z}$.

Theorem 20.62 (Dress-Farrell-Hsiang-Jones groups and the K-theoretic Farrell-Jones Conjecture with finite wreath products).

Let G be finitely generated group which is a Dress-Farrell-Hsiang-Jones group over $\mathcal{VCY}$ in the sense of Definition 20.60.

Then G satisfies the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products, see Conjecture 13.24 and the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products, see Conjecture 13.26.

Proof. This follows from the last paragraph starting on page 127 in [172] and Theorem 16.1 (ic). □

Remark 20.63. Note that we need for the proof of Theorem 20.61 and Theorem 20.62 the assumption that G is finitely generated. If G is finitely generated, then it suffices in Definitions 20.48, 20.52, and 20.60 to check the last requirement appearing there only for the elements $g_1, g_2, \ldots, g_k$ or g contained in one fixed finite set S of generators, since then it hold automatically for any finite subset of G or any element of G, essentially because of the tringle inquality.

20.11 Notes

It is conceivable that, if G is a Dress-Farrell-Hsiang-Jones group over $\mathcal{F}$ in the sense of Definition 20.60. the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n\bigl(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G))\bigr),$$

is bijective for every additive G-category $\mathcal{A}$ (with involution) and every $n \in \mathbb{Z}$. Details of this claim have not been checked. If this claim is true, one would get that a group G, which is a Dress-Farrell-Hsiang-Jones group over $\mathcal{VCY}$, is a Farrell-Jones group, i.e., it satisfies the Full Farrell-Jones Conjecture 13.27.

So far the L-theoretic version of the Farrell-Jones Conjecture has only been established for additive categories with involution. Christoph Winges and the author of the book expect that it can be generalized to the setting of higher categories and proved for all Dress-Farrell-Hsiang-Jones groups.

Sawicki [874] discusses the notion of *equivariant asymptotic dimension* which is closely related to the notions of a transfer $\mathcal{F}$-reducible group, see Definition 20.28, and of finitely $\mathcal{F}$-amenable group, see Definition 20.44.

There is also the notion of an *almost transfer $\mathcal{F}$-reducible group* which is introduced in [88, Definition 5.3] and is weaker than transfer $\mathcal{F}$-reducible group. Both conditions are equivalent if the family $\mathcal{F}$ is closed under the passage to overgroups of finite index, see for instance [874, Corollary 2.6].

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Chapter 21
Karoubi Filtrations

21.1 Introduction

This chapter is devoted to the notion of a Karoubi filtration, which is given by a full additive subcategory $\mathcal{U}$ of an additive category $\mathcal{A}$ satisfying certain conditions, and the existence of the associated weak homotopy fibration sequences

$$\mathbf{K}(\mathcal{U}) \to \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U});$$
$$\mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U}),$$

which induce long exact sequences of K- and L-groups. This will be a basic tool in Chapter 22, where we will define G-homology theories in terms of controlled topology and need to check the axioms of a G-homology theory such as the long exact sequence of a pair or excision. All this is presented in Section 21.2, and that is all we need for this book.

For the reader's convenience we give a mild generalization of the notion of a Karoubi fibration in Section 21.3, which may be useful in other contexts. In Section 21.2 we extend the proof of the existence of the associated weak homotopy fibration sequence $\mathbf{K}(\mathcal{U}) \to \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})$ of [197] to this setting, also taking care of a bug in [197].

For the proof presented here we use the definition of the non-connective K-theory spectrum of homotopical Waldhausen categories due to Bunke-Kasprowski-Winges [173]. Then the proof becomes conceptually very clear, it is essentially a consequence of the non-connective analogues of standard theorems such as the Fibration Theorem 21.17, Cisinski's Approximation Theorem 21.18, and the Cofinality Theorem 21.19. We also identify this notion with the one we have used so far for the non-connective K-theory of additive categories, see Theorem 21.39.

21.2 Karoubi Filtration and the Associated Weak Homotopy Fibration Sequence

If $\mathcal{U}$ is a full additive subcategory of $\mathcal{A}$, then one can define the quotient category $\mathcal{A}/\mathcal{U}$ as follows. The set of objects of $\mathcal{A}/\mathcal{U}$ agrees with the set of objects of $\mathcal{A}$. The set of morphism $\mathrm{mor}_{\mathcal{A}/\mathcal{U}}(A, A')$ for objects A and A' in

$\mathcal{A}/\mathcal{U}$ is defined to be $\mathrm{mor}_{\mathcal{A}}(A, A')/\sim$ for the equivalence relation $\sim$ where we call two morphisms $f, f' \colon A \to A'$ in $\mathcal{A}$ equivalent, if their difference $f - f' \colon A \to A'$ factorizes in $\mathcal{A}$ as a composition $A \to U \to A'$ for some object U in $\mathcal{U}$. We leave the elementary proof to the reader, that $\mathcal{A}/\mathcal{U}$ inherits from $\mathcal{A}$ the structure of an additive category such that the obvious projection $p \colon \mathcal{A} \to \mathcal{A}/\mathcal{U}$ is a functor of additive categories. For a morphism $f \colon A \to A'$ in $\mathcal{A}$, we denote by $[f] \colon A \to A'$ the morphism in $\mathcal{A}/\mathcal{U}$ represented by f.

Definition 21.1 (Quotients for additive categories). We call the additive category $\mathcal{A}/\mathcal{U}$ the *quotient category* of $\mathcal{A}$ by $\mathcal{U}$.

Definition 21.2 ($\mathcal{U}$-filtered). We say that $\mathcal{A}$ is *$\mathcal{U}$-filtered* or, equivalently, that the inclusion $\mathcal{U} \to \mathcal{A}$ is a *Karoubi filtration*, if the following holds:

The additive subcategory $\mathcal{U} \subseteq \mathcal{A}$ is full. Moreover, given an object A in $\mathcal{A}$, objects $U, V \in \mathcal{U}$, and morphisms $f \colon A \to U$ and $g \colon V \to A$ in $\mathcal{A}$, there are objects $A^{\mathcal{U}}$ in $\mathcal{U}$ and $A^{\perp}$ in $\mathcal{A}$ and morphisms $i^{\mathcal{U}} \colon A^{\mathcal{U}} \to A$ and $i^{\perp} \colon A^{\perp} \to A$ satisfying:

- $i^{\mathcal{U}} \oplus i^{\perp} \colon A^{\mathcal{U}} \oplus A^{\perp} \xrightarrow{\cong} A$ is an isomorphism in $\mathcal{A}$;
- There exists a morphism $f^{\mathcal{U}} \colon A^{\mathcal{U}} \to U$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] & U \\ A^{\mathcal{U}} \oplus A^{\perp} \arrow[u, "i^{\mathcal{U}} \oplus i^{\perp}", "\cong"'] \arrow[r, "\mathrm{pr}_{A^{\mathcal{U}}}"'] & A^{\mathcal{U}} \arrow[u, "f^{\mathcal{U}}"'] \end{tikzcd}$$

  where $\mathrm{pr}_{A^{\mathcal{U}}} \colon A^{\mathcal{U}} \oplus A^{\perp} \to A^{\mathcal{U}}$ is the canonical projection;
- There exists a morphism $g^{\mathcal{U}} \colon V \to A^{\mathcal{U}}$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "g^{\mathcal{U}}"'] & A \\ A^{\mathcal{U}} \arrow[r, "i_{A^{\mathcal{U}}}"'] & A^{\mathcal{U}} \oplus A^{\perp} \arrow[u, "\cong", "i^{\mathcal{U}} \oplus i^{\perp}"'] \end{tikzcd}$$

where $i_{A^{\mathcal{U}}} \colon A^{\mathcal{U}} \to A^{\mathcal{U}} \oplus A^{\perp}$ is the canoncial inclusion.

Exercise 21.3. Show that the morphisms $f^{\mathcal{U}}$ and $g^{\mathcal{U}}$ appearing in Definition 21.2 are uniquely determined by the desired properties.

Remark 21.4 (Relation to the classical definition of a Karoubi filtration). If one requires in Definition 21.2 additionally that $U = V$, then it reduces to [549, Definition 5.4]. One easily checks that Definition 21.2 and [549, Definition 5.4] are equivalent, the special case $U = V$ in [549, Definition 5.4] implies the general case of Definition 21.2 by considering $U \oplus V$. Note that [549, Definition 5.4] agrees with the more complicated notion of a $\mathcal{U}$-filtration due to Karoubi [530], see [549, Lemma 5.6].

The main result of this chapter is:

Theorem 21.5 (The weak homotopy fibration sequence of a Karoubi filtration). *Let $\mathcal{A}$ be an additive category and $i\colon \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory. Let $p\colon \mathcal{A} \to \mathcal{A}/\mathcal{U}$ be the canonical projection. Suppose that $\mathcal{A}$ is $\mathcal{U}$-filtered.*

(i) The sequence of spectra

$$\mathbf{K}(\mathcal{U}) \xrightarrow{\mathbf{K}(i)} \mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(p)} \mathbf{K}(\mathcal{A}/\mathcal{U})$$

is a weak homotopy fibration sequence of non-connective spectra, i.e., the composite $\mathbf{K}(p) \circ \mathbf{K}(i)$ admits a preferred nullhomotopy, since there is a preferred natural transformation from $p \circ i$ to the trivial functor, and the induced map

$$\mathbf{K}(\mathcal{U}) \to \operatorname{hofib}\bigl(\mathbf{K}(p)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\bigr)$$

is a weak homotopy equivalence.
In particular we get a long exact sequence, infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{U}) \xrightarrow{K_n(i)} K_n(\mathcal{A}) \xrightarrow{K_n(p)} K_n(\mathcal{A}/\mathcal{U}) \xrightarrow{\partial_n} K_{n-1}(\mathcal{U})$$
$$\xrightarrow{K_{n-1}(i)} K_{n-1}(\mathcal{A}) \xrightarrow{K_{n-1}(p)} K_{n-1}(\mathcal{A}/\mathcal{U}) \xrightarrow{\partial_{n-1}} \cdots;$$

(ii) Suppose additionally that $\mathcal{A}$ is an additive category with involution such that the involution induces the structure of an additive category with involution on $\mathcal{U}$.
Then $\mathcal{A}/\mathcal{U}$ inherits the structure of an additive category with involution and the sequence of spectra

$$\mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}) \xrightarrow{\mathbf{L}^{\langle -\infty \rangle}(i)} \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}) \xrightarrow{\mathbf{L}^{\langle -\infty \rangle}(p)} \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U})$$

is a weak homotopy fibration sequence of non-connective spectra.
In particular we get a long exact sequence, infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} L_n^{\langle -\infty \rangle}(\mathcal{U}) \xrightarrow{L_n^{\langle -\infty \rangle}(i)} L_n^{\langle -\infty \rangle}(\mathcal{A}) \xrightarrow{L_n^{\langle -\infty \rangle}(p)} L_n^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U})$$
$$\xrightarrow{\partial_n} L_{n-1}^{\langle -\infty \rangle}(\mathcal{U}) \xrightarrow{L_{n-1}^{\langle -\infty \rangle}(i)} L_{n-1}^{\langle -\infty \rangle}(\mathcal{A})$$
$$\xrightarrow{L_{n-1}^{\langle -\infty \rangle}(p)} L_{n-1}^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U}) \xrightarrow{\partial_{n-1}} \cdots.$$

Proof. (i) See [197, 199, 784] based on the work of Karoubi [530].
(ii) See [202, Theorem 4.2]. □

Theorem 21.5 (i) follows also from [883, Example 1.8 and Theorem 2.10].

Comment 30 (by W.): Shall we already here mention that one can prove Theorem 21.5 also in the setting of Right-Exact ∞-Categories?

Exercise 21.6. Suppose that $\mathcal{U} \to \mathcal{A}$ is a Karoubi filtration and $\mathcal{A}$ is flasque. Then there is weak homotopy equivalence $\mathbf{K}(\mathcal{U}) \xrightarrow{\simeq} \Omega\mathbf{K}(\mathcal{A}/\mathcal{U})$.

21.3 Stable Karoubi Filtration

Let $\mathcal{A}$ be an additive category and $i\colon \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory.

Definition 21.7 (Stably $\mathcal{U}$-filtered). We say that $\mathcal{A}$ is *stably $\mathcal{U}$-filtered*, or, equivalently, that the inclusion $\mathcal{U} \to \mathcal{A}$ is a *stable Karoubi filtration,*, if the following holds:

Given an object A in $\mathcal{A}$, objects $U, V \in \mathcal{U}$, and morphisms $f\colon A \to U$ and $g\colon V \to A$ in $\mathcal{A}$, there are objects $\widetilde{A}$ in $\mathcal{A}$ and $\widetilde{A}^{\mathcal{U}}$ in $\mathcal{U}$, and morphisms $\widetilde{i}\colon A \to \widetilde{A}$, $\widetilde{r}\colon \widetilde{A} \to A$, $i^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to \widetilde{A}$ and $r^{\mathcal{U}}\colon \widetilde{A} \to \widetilde{A}^{\mathcal{U}}$ satisfying:

- $\widetilde{r} \circ \widetilde{i} = \mathrm{id}_A$ and $r^{\mathcal{U}} \circ i^{\mathcal{U}} = \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}$ in $\mathcal{A}$;
- There exists a morphism $f^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to U$ such that the following diagram commutes
$$\begin{tikzcd} A \arrow[r, "f"] & U \\ \widetilde{A} \arrow[u, "\widetilde{r}"] \arrow[r, "r^{\mathcal{U}}"'] & \widetilde{A}^{\mathcal{U}}; \arrow[u, "f^{\mathcal{U}}"'] \end{tikzcd}$$
- There exists a morphism $g^{\mathcal{U}}\colon V \to \widetilde{A}^{\mathcal{U}}$ such that the following diagram commutes
$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "g^{\mathcal{U}}"] & A \arrow[d, "\widetilde{i}"] \\ \widetilde{A}^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}"'] & \widetilde{A}. \end{tikzcd}$$

Here is a stronger version where retractions are replaced by direct sum decompositions.

Definition 21.8 (Strongly stably $\mathcal{U}$-filtered). We say that $\mathcal{A}$ is *strongly stably $\mathcal{U}$-filtered*, or, equivalently, that the inclusion $\mathcal{U} \to \mathcal{A}$ is a *strongly stable Karoubi filtration*, if the following holds:

Given an object A in $\mathcal{A}$, objects $U, V \in \mathcal{U}$, and morphisms $f\colon A \to U$ and $g\colon V \to A$ in $\mathcal{A}$, there are objects $A' \in \mathcal{A}$, $(A \oplus A')^{\mathcal{U}}$ in $\mathcal{U}$ and $(A \oplus A')^{\perp}$ in $\mathcal{A}$ and morphisms $i^{\mathcal{U}}\colon (A \oplus A')^{\mathcal{U}} \to A \oplus A'$ and $i^{\perp}\colon (A \oplus A')^{\perp} \to A \oplus A'$ satisfying:

- $i^{\mathcal{U}} \oplus i^{\perp} \colon (A \oplus A')^{\mathcal{U}} \oplus (A \oplus A')^{\perp} \xrightarrow{\cong} A \oplus A'$ is an isomorphism in $\mathcal{A}$;
- There exists a morphism $f^{\mathcal{U}} \colon (A \oplus A')^{\mathcal{U}} \to \mathcal{U}$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "f"] & & U \\ A \oplus A' \arrow[u, "\mathrm{pr}_A"] & & \\ (A \oplus A')^{\mathcal{U}} \oplus (A \oplus A')^{\perp} \arrow[u, "i^{\mathcal{U}} \oplus i^{\perp}", "\cong"'] \arrow[rr, "\mathrm{pr}_{(A \oplus A')^{\mathcal{U}}}"'] & & (A \oplus A')^{\mathcal{U}} \arrow[uu, "f^{\mathcal{U}}"'] \end{tikzcd}$$

  where $\mathrm{pr}_A \colon A \oplus A' \to A$ and $\mathrm{pr}_{(A \oplus A')^{\mathcal{U}}} \colon (A \oplus A')^{\mathcal{U}} \oplus (A \oplus A')^{\perp} \to (A \oplus A')^{\mathcal{U}}$ are the canonical projections;
- There exists a morphism $g^{\mathcal{U}} \colon V \to A^{\mathcal{U}}$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[rr, "g"] \arrow[dd, "g^{\mathcal{U}}"'] & & A \arrow[d, "i_A"] \\ & & A \oplus A' \\ (A \oplus A')^{\mathcal{U}} \arrow[rr, "i_{(A \oplus A')^{\mathcal{U}}}"'] & & (A \oplus A')^{\mathcal{U}} \oplus (A \oplus A')^{\perp} \arrow[u, "\cong", "i^{\mathcal{U}} \oplus i^{\perp}"'] \end{tikzcd}$$

where $i_A \colon A \to A \oplus A'$ and $i_{(A \oplus A')^{\mathcal{U}}} \colon (A \oplus A')^{\mathcal{U}} \to (A \oplus A')^{\mathcal{U}} \oplus (A \oplus A')^{\perp}$ are the canoncial inclusions.

Lemma 21.9.

(i) We have the implications

$$\mathcal{U}\text{-filtered} \implies \text{strongly stably } \mathcal{U}\text{-filtered} \implies \text{stably } \mathcal{U}\text{-filtered};$$

(ii) If $\mathcal{A}$ is idempotent complete, then stably $\mathcal{U}$-filtered implies strongly stably $\mathcal{U}$-filtered.

Proof. (i) For $\mathcal{U}$-filtered $\implies$ strongly stably $\mathcal{U}$-filtered take $A' = \{0\}$ in the Definition 21.8 of strongly stably $\mathcal{U}$-filtered. For strongly stably $\mathcal{U}$-filtered $\implies$ stably $\mathcal{U}$-filtered take $\widehat{A}$ in Definition 21.7 to be $A \oplus A'$.

(ii) In an idempotent complete additive category every retraction comes from a direct sum decomposition. □

Let $\mathcal{A}[t, t^{-1}]$ the Laurent category associated to $\mathcal{A}$, see for instance [668, Section 1].

Lemma 21.10. *Suppose that $\mathcal{A}$ is stably $\mathcal{U}$-filtered.*

Then $\mathcal{A}[t,t^{-1}]$ is stably $\mathcal{U}[t,t^{-1}]$-filtered and there is an isomorphism of additive categories

$$\mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}] \xrightarrow{\cong} (\mathcal{A}/\mathcal{U})[t,t^{-1}].$$

Proof. Consider an object A in $\mathcal{A}[t,t^{-1}]$, objects U,V in $\mathcal{U}[t,t^{-1}]$, and morphisms $f\colon A \to U$ and $g\colon V \to A$ in $\mathcal{A}[t,t^{-1}]$. By definition A is an object in $\mathcal{A}$, U and V are objects in $\mathcal{U}$, and $f = \sum_{n\in\mathbb{Z}} f_n \cdot t^n$ and $g = \sum_{n\in\mathbb{Z}} g_n \cdot t^n$ such that $f_n\colon A \to U$ and $g_n\colon V \to A$ are morphisms in $\mathcal{A}$ and there exists a natural number N such that $f_n = 0$ and $g_n = 0$ holds for $|n| > N$. Note that in an additive category finite sums are finite direct products. We can consider the morphisms $\prod_{n=-N}^{N} f_n\colon A \to \prod_{n=-N}^{n} U$ and $\bigoplus_{n=-N}^{N} g_n\colon \bigoplus_{n=-N}^{N} V \to A$ in $\mathcal{A}$. Since $\mathcal{A}$ is stably $\mathcal{U}$-filtered, there are objects $\widetilde{A}$ in $\mathcal{A}$ and $\widetilde{A}^{\mathcal{U}}$ in $\mathcal{U}$, and morphisms $\widetilde{i}\colon A \to \widetilde{A}$, $\widetilde{r}\colon \widetilde{A} \to A$, $i^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to \widetilde{A}$ and $r^{\mathcal{U}}\colon \widetilde{A} \to \widetilde{A}^{\mathcal{U}}$ satisfying:

- $\widetilde{r} \circ \widetilde{i} = \mathrm{id}_A$ and $r^{\mathcal{U}} \circ i^{\mathcal{U}} = \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}$ in $\mathcal{A}$;
- There exists a morphism $\left(\prod_{n=-N}^{N} f_n\right)^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to \prod_{n=-N}^{n} U$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\prod_{n=-N}^{N} f_n"] & \prod_{n=-N}^{n} U \\ \widetilde{A} \arrow[u, "\widetilde{r}"] \arrow[r, "r^{\mathcal{U}}"'] & \widetilde{A}^{\mathcal{U}}; \arrow[u, "(\prod_{n=-N}^{N} f_n)^{\mathcal{U}}"'] \end{tikzcd}$$

- There exists a morphism $(\bigoplus_{n=-N}^{N} g_n)^{\mathcal{U}}\colon V \to \widetilde{A}^{\mathcal{U}}$ such that the following diagram commutes

$$\begin{tikzcd} \bigoplus_{n=-N}^{N} V \arrow[r, "\bigoplus_{n=-N}^{N} g_n"] \arrow[d, "(\bigoplus_{n=n}^{N} g_n)^{\mathcal{U}}"] & A \arrow[d, "\widetilde{i}"] \\ \widetilde{A}^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}"'] & \widetilde{A}. \end{tikzcd}$$

Define $f_n^{\mathcal{U}}\colon A \to U$ to be zero, if $|n| > N$, and to be the composite of $\left(\prod_{n=-N}^{N} f_n\right)^{\mathcal{U}}$ with the projection $\prod_{n=-N}^{n} U \to U$ onto the factor for n, if $|n| \le N$. Define $g_n^{\mathcal{U}}\colon V \to A$ to be zero, if $|n| > N$, and to be the composite of $\left(\bigoplus_{n=-N}^{N} g_n\right)^{\mathcal{U}}$ with the injection $U \to \prod_{n=-N}^{n} U$ of the factor for n, if $|n| \le N$. Now define a morphisms $f^{\mathcal{U}}\colon A \to U$ in $\mathcal{A}[t,t^{-1}]$ by $f^{\mathcal{U}} = \sum_{n\in\mathbb{Z}} f_n^{\mathcal{U}} \cdot t^n$ and a morphism $g^{\mathcal{U}}\colon V \to A$ by $\sum_{n\in\mathbb{Z}} g_n^{\mathcal{U}} \cdot t^n$. Note that the morphism $\widetilde{r}$, $\widetilde{i}$, $i^{\mathcal{U}}$ and $r^{\mathcal{U}}$ in $\mathcal{A}$ define morphisms in $\mathcal{A}[t,t^{-1}]$ by $\widetilde{r} \cdot t^0$, $\widetilde{i} \cdot t^0$, $i^{\mathcal{U}} \cdot t^0$ and $r^{\mathcal{U}} \cdot t^0$. Obviously we have $\widetilde{r} \cdot t^0 \circ \widetilde{i} \cdot t^0 = \mathrm{id}_A$ and $r^{\mathcal{U}} \cdot t^0 \circ i^{\mathcal{U}} \cdot t^0 = \mathrm{id}_{A^{\mathcal{U}}}$. The following diagrams in $\mathcal{A}[t,t^{-1}]$

$$\begin{tikzcd} A \arrow[r, "f"] & U \\ \widetilde{A} \arrow[u, "\widetilde{r}\cdot t^0"] \arrow[r, "r^{\mathcal{U}}\cdot t^0"'] & \widetilde{A}^{\mathcal{U}} \arrow[u, "f^{\mathcal{U}}"'] \end{tikzcd}$$

and

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "g^{\mathcal{U}}"] & A \arrow[d, "\widetilde{i}\cdot t^0"] \\ \widetilde{A}^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}\cdot t^0"'] & \widetilde{A} \end{tikzcd}$$

commute. Hence $\mathcal{A}[t,t^{-1}]$ is stably $\mathcal{U}[t,t^{-1}]$-filtered.

The canonical projection $\mathcal{A}[t,t^{-1}] \to \mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}]$ induces a functor of additive categories

$$F\colon \mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}] \to (\mathcal{A}/\mathcal{U})[t,t^{-1}]$$

by the following argument. Consider a morphism $f = \sum_{n\in\mathbb{Z}} f_n \cdot t^n$ whose image under the projection $\mathcal{A}[t,t^{-1}] \to \mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}]$ is zero, i.e., it can be written as a composite composite $A \xrightarrow{u} U \xrightarrow{v} B$ in $\mathcal{A}[t,t^{-1}]$ for U in $\mathcal{U}[t,t^{-1}]$. Recall that U is by definition an object in $\mathcal{U}$. If we write $u = \sum_{k\in\mathbb{Z}} u_k \cdot t^k$ and $v = \sum_{l\in\mathbb{Z}} v_l \cdot t^l$, then we get $f_n = \sum_{k\in\mathbb{Z}} v_{n-k} \circ u_k$ in $\mathcal{A}$ for each $n \in \mathbb{Z}$. This implies $[f_n] = 0$ in $\mathcal{A}/\mathcal{U}$ for every $n \in \mathbb{Z}$. Hence f is sent under the projection $\mathcal{A}[t,t^{-1}] \to (\mathcal{A}/\mathcal{U})[t,t^{-1}]$ to zero. This shows that F is well-defined.

The canonical projection $\mathcal{A}[t,t^{-1}] \to (\mathcal{A}/\mathcal{U})[t,t^{-1}]$ induces a functor of additive categories

$$G\colon (\mathcal{A}/\mathcal{U})[t,t^{-1}] \to \mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}]$$

by the following argument. Consider a morphism $f = \sum_{n\in\mathbb{Z}} f_n \cdot t^n\colon A \to B$ whose image under the projection $\mathcal{A}[t,t^{-1}] \to (\mathcal{A}/\mathcal{U})[t,t^{-1}]$ is zero, i.e., for each $n \in \mathbb{Z}$ we can write $f_n\colon A \to B$ as a composite $A \xrightarrow{u_n} U_n \xrightarrow{v_n} B$ for an object U_n in $\mathcal{U}$. We have to show that the image of f under the projection $\mathcal{A}[t,t^{-1}] \to \mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}]$ is zero. Obviously it suffices to show for each $m \in \mathbb{Z}$ that $f_m \cdot t^m$ has this property. But this follows from the equation $f_m \cdot t^m = v_m \cdot t^m \circ u_m \cdot t^0$ in $\mathcal{A}[t,t^{-1}]$ and the fact that U_m belongs to $\mathcal{U}[t,t^{-1}]$. Hence G is well-defined. One easily checks that F and G are inverse to one another. $\square$

The canonical inclusion $\mathcal{A} \to \operatorname{Idem}(\mathcal{A})$ of the additive category $\mathcal{A}$ into its idempotent completion induces a functor of unital additive categories $F\colon \mathcal{A}/\mathcal{U} \to \operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})$, which in turn induces a functor of unital additive categories

$$\text{(21.11)} \qquad \operatorname{Idem}(F)\colon \operatorname{Idem}(\mathcal{A}/\mathcal{U}) \to \operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr).$$

Lemma 21.12. *(i) The functor* $\operatorname{Idem}(F)$ *is an equivalence of unital additive categories;*
(ii) If $\mathcal{A}$ *is stably* $\mathcal{U}$*-filtered,* $\operatorname{Idem}(\mathcal{A})$ *is strongly stably* $\operatorname{Idem}(\mathcal{U})$*-filtered.*

Proof. (i) We have to show:

(a) For every object y in $\operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ there exists an object x in $\operatorname{Idem}(\mathcal{A}/\mathcal{U})$ such that $F(x)$ and y are isomorphic;
(b) For every two objects x_0 and x_1 in $\operatorname{Idem}(\mathcal{A}/\mathcal{U})$ the functor $\operatorname{Idem}(F)$ induces a bijection

$$\operatorname{mor}_{\operatorname{Idem}(\mathcal{A}/\mathcal{U})}(x_0,x_1) \xrightarrow{\cong} \operatorname{mor}_{\operatorname{Idem}(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U}))}(F(x_0),F(x_1)).$$

(a) An object (A,p) in $\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ is the same as an object in $\operatorname{Idem}(\mathcal{A})$ and hence given by an object A and a morphism $p\colon A \to A$ with $p \circ p = p$ in $\mathcal{A}$. An object $y = ((A,p),[q])$ in $\operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ is given by an object (A,p) in $\operatorname{Idem}(\mathcal{A})$ and a morphism $q\colon A \to A$ with $p \circ q \circ p = q$ in $\mathcal{A}$ such that $[q] \circ [q] = [q]$ holds in $\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})$, i.e., there exists an object $U \in \mathcal{U}$, a morphism $u\colon U \to U$ in $\mathcal{U}$ with $u \circ u = u$ in $\mathcal{U}$ and morphisms $a\colon A \to U$ and $b\colon U \to A$ in $\mathcal{A}$ with $u \circ a \circ p = a$, $p \circ b \circ u = b$, and $q \circ q - q = b \circ a$ in $\mathcal{A}$. Our candidate for the desired object $x \in \operatorname{Idem}(\mathcal{A}/\mathcal{U})$ is given by $(A,[q])$. This makes sense since $[q] \circ [q] = [q]$ in $\mathcal{A}/\mathcal{U}$ follows from $q \circ q - q = b \circ a$ in $\mathcal{A}$.

The object $F(x)$ in $\operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ is given by $((A,\operatorname{id}_A),[q])$ for the object (A,id_A) in $\operatorname{Idem}(\mathcal{A})$ and the morphism $[q]\colon (A,\operatorname{id}_A) \to (A,\operatorname{id}_A)$ in $\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})$ given by the morphism $q\colon (A,\operatorname{id}_A) \to (A,\operatorname{id}_A)$ in $\operatorname{Idem}(\mathcal{A})$. This makes sense since $[q] \circ [q] = [q]$ in $\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})$ follows from $[q] \circ [q] = [q]$ in $\mathcal{A}/\mathcal{U}$. We want to define to one another inverse isomorphisms in $\operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ between $F(x)$ and y by

$$[q]\colon ((A,\operatorname{id}_A),[q]) \to ((A,p),[q])$$

and

$$[q]\colon ((A,p),[q]) \to ((A,\operatorname{id}_A),[q]).$$

These are indeed morphisms in $\operatorname{Idem}\bigl(\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})\bigr)$ since the following diagrams commute in $\operatorname{Idem}(\mathcal{A})/\operatorname{Idem}(\mathcal{U})$

$$\begin{tikzcd}
(A,\operatorname{id}_A) \arrow[r, "{[q]}"] \arrow[d, "{[q]}"'] & (A,\operatorname{id}_A) \arrow[d, "{[q]}"] \\
(A,p) \arrow[r, "{[q]}"'] & (A,p)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A,p) \arrow[r, "{[q]}"] \arrow[d, "{[q]}"'] & (A,p) \arrow[d, "{[q]}"] \\
(A,\mathrm{id}_A) \arrow[r, "{[q]}"'] & (A,\mathrm{id}_A)
\end{tikzcd}$$

where the vertical arrows are well-defined because of the equalities $p \circ q \circ p = q$ and $p \circ p = p$ in $\mathcal{A}$. Since the identity morphisms on both $((A, \mathrm{id}_A), [q])$ and $((A,p),[q])$ in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ are given by the morphism $q\colon A \to A$ and $[q] \circ [q] = [q]$ holds in $\mathcal{A}/\mathcal{U}$, claim (a) is proved.

(b) The object $x_i = (A_i, [q_i])$ in $\mathrm{Idem}(\mathcal{A}/\mathcal{U})$ is given an object A_i in $\mathcal{A}$ and a morphism $[q_i]\colon A_i \to A_i$ in $\mathcal{A}/\mathcal{U}$ such that $[q_i] \circ [q_i] = [q_i]$ holds in $\mathcal{A}/\mathcal{U}$. A morphism $[f]\colon x_0 \to x_1$ in $\mathrm{Idem}(\mathcal{A}/\mathcal{U})$ is given by a morphism $[f]\colon A_0 \to A_1$ in $\mathcal{A}/\mathcal{U}$ such that $[q_1] \circ [f] \circ [q_0]$ holds in $\mathcal{A}/\mathcal{U}$.

As already explained above, the object $F(x_i)$ in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ is given by $((A_i, \mathrm{id}_{A_i}), [q_i])$ for the object (A_i, id_{A_i}) in $\mathrm{Idem}(\mathcal{A})$ and the morphism $[q_i]\colon (A_i, \mathrm{id}_{A_i}) \to (A_i, \mathrm{id}_{A_i})$ in $\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})$ given by the morphism $q_i\colon (A_i, \mathrm{id}_{A_i}) \to (A_i, \mathrm{id}_{A_i})$ in $\mathrm{Idem}(\mathcal{A})$. The morphism $F([f])\colon F(x_0) \to F(x_1)$ in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ is given by the morphism $[f]\colon (A_0, \mathrm{id}_{A_0}) \to (A_1, \mathrm{id}_{A_1})$ in $\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})$ represented by the morphism $f\colon (A_0, \mathrm{id}_{A_0}) \to (A_1, \mathrm{id}_{A_1})$ in $\mathrm{Idem}(\mathcal{A})$.

A morphism

$$[g]\colon F(x_0) = ((A_0, \mathrm{id}_{A_0}), [q_0]) \to F(x_1) = ((A_1, \mathrm{id}_{A_1}), [q_1])$$

in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ is given by a morphism $g\colon (A_0, \mathrm{id}_{A_0}) \to (A_1, \mathrm{id}_{A_1})$ in $\mathrm{Idem}(\mathcal{A})$ such that $[q_1] \circ [g] \circ [q_0] \ = [g]$ holds in $\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})$. Note that g is the same as a morphism $g\colon A_0 \to A_1$ since $\mathrm{id}_{A_1} \circ g \circ \mathrm{id}_{A_0} = g$ always holds. Moreover, we have $[q_1] \circ [g] \circ [q_0] \ = [g]$ in $\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})$ if and only if we have $[q_1] \circ [g] \circ [q_0] \ = [g]$ in $\mathcal{A}/\mathcal{U}$. Two morphisms $g_0, g_1\colon (A_0, \mathrm{id}_{A_0}) \to (A_1, \mathrm{id}_{A_1})$ in $\mathrm{Idem}(\mathcal{A})$ satisfying $[q_1] \circ [g_i] \circ [q_0] \ = [g_i]$ for $i = 0, 1$ in $\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})$ define the same morphism $F(x_0) \to F(x_1)$ in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ if and only if there is an object $(U,u) \in \mathrm{Idem}(\mathcal{U})$ and morphisms $a\colon (A_0, \mathrm{id}_{A_0}) \to (U,u)$ and $b\colon (U,u) \to (A_1, \mathrm{id}_{A_1})$ with $g_1 - g_0 = b \circ a$ in $\mathrm{Idem}(\mathcal{A})$. The latter condition is equivalent to the existence of an object U and morphisms $a\colon A_0 \to U$ and $b\colon U \to B$ in $\mathcal{A}$ such that $g_1 - g_0 = b \circ a$ holds in $\mathcal{A}$ since one may choose $u = \mathrm{id}_U$. Thus we have shown that a morphism $[g]\colon F(x_0) = ((A_0, \mathrm{id}_{A_0}), [q_0]) \to F(x_1) = ((A_1, \mathrm{id}_{A_1}), [q_1])$ in $\mathrm{Idem}\bigl(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})\bigr)$ is the same as a morphism $[g]\colon A_0 \to A_1$ in $\mathcal{A}/\mathcal{U}$ satisfying $[q_1] \circ [g] \circ [q_0] \ = [g]$. Now claim (b) follows. This finishes the proof of assertion (i) of Lemma 21.12.

(ii) Because of Lemma 21.9 (ii) it suffices to show that $\mathrm{Idem}(\mathcal{A})$ is stably $\mathrm{Idem}(\mathcal{U})$-filtered. We give the argument only for a morphism $f\colon (A,p) \to (U,q)$ in $\mathrm{Idem}(\mathcal{A})$ with (U,q) in $\mathrm{Idem}(\mathcal{U})$, the one for a morphism $g\colon (V,r) \to (A,p)$ in $\mathrm{Idem}(\mathcal{A})$ with with (V,r) in $\mathrm{Idem}(\mathcal{U})$ is analogous.

Since $\mathcal{A}$ is stably $\mathcal{U}$-filtered, we can find objects $\widetilde{A}$ in $\mathcal{A}$ and $\widetilde{A}^{\mathcal{U}}$ in $\mathcal{U}$, and morphisms $\widetilde{i}\colon A \to \widetilde{A}$, $\widetilde{r}\colon \widetilde{A} \to A$, $i^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to \widetilde{A}$ and $r^{\mathcal{U}}\colon \widetilde{A} \to \widetilde{A}^{\mathcal{U}}$ such that $\widetilde{r} \circ \widetilde{i} = \mathrm{id}_A$, and $r^{\mathcal{U}} \circ i^{\mathcal{U}} = \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}$ holds and a morphism $f^{\mathcal{U}}\colon \widetilde{A}^{\mathcal{U}} \to U$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & U \\ \widetilde{A} \arrow[u, "\widetilde{r}"] \arrow[r, "r^{\mathcal{U}}"'] & \widetilde{A}^{\mathcal{U}}. \arrow[u, "f^{\mathcal{U}}"'] \end{tikzcd}$$

commutes. Define objects $(\widetilde{A}, \mathrm{id}_{\widetilde{A}})$ in $\mathrm{Idem}(\mathcal{A})$ and $(\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}})$ in $\mathrm{Idem}(\mathcal{U})$. Then we get morphisms in $\mathrm{Idem}(\mathcal{A})$

$$\begin{aligned} p \circ \widetilde{r}\colon (\widetilde{A}, \mathrm{id}_{\widetilde{A}}) &\to (A, p); \\ \widetilde{i} \circ p\colon (A, p) &\to (\widetilde{A}, \mathrm{id}_{\widetilde{A}}); \\ r^{\mathcal{U}}\colon (\widetilde{A}, \mathrm{id}_{\widetilde{A}}) &\to (\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}); \\ i^{\mathcal{U}}\colon (\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}); &\to (\widetilde{A}, \mathrm{id}_{\widetilde{A}}); \\ q \circ f^{\mathcal{U}}\colon (\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}) &\to (U, q), \end{aligned}$$

such that the composite $(p \circ \widetilde{r}) \circ (\widetilde{i} \circ p)$ is the identity on (A, p), the composite $r^{\mathcal{U}} \circ i^{\mathcal{U}}$ is the identity on $(\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}})$ and the diagram in $\mathrm{Idem}(\mathcal{A})$

$$\begin{tikzcd} (A, p) \arrow[r, "f"] & (U, q) \\ (\widetilde{A}, \mathrm{id}_{\widetilde{A}}) \arrow[u, "p \circ \widetilde{r}"] \arrow[r, "r^{\mathcal{U}}"'] & (\widetilde{A}^{\mathcal{U}}, \mathrm{id}_{\widetilde{A}^{\mathcal{U}}}). \arrow[u, "q \circ f^{\mathcal{U}}"'] \end{tikzcd}$$

commutes. This finishes the proof of Lemma 21.12. □

21.4 Non-Connective *K*-Theory for Homotopical Waldhausen Categories

Recall that we have defined the negative K-theory of an additive category using the delooping construction based on the Bass-Heller-Swan decomposition of [668].

In this section we present another definition based on the non-connective K-theory spectrum associated to appropriate Waldhausen categories due to Bunke-Kasprowski-Winges [173]. These different approaches have advantages and disadvantages and we want to compare them so that finally we can use the advantages of both in the setting of additive categories. We begin with explaining the one appearing in [173].

In the sequel we use the definitions and notation of Waldhausen [960]. Given a Waldhausen category $\mathcal{W}$, Waldhausen [960] has defined its connective K-theory spectrum $\mathbf{K}^{\mathrm{W,con}}(\mathcal{W})$ and proved some basic tools such as the Approximation Theorem and the Fibration Theorem. Next we explain how one can define for a homotopical Waldhausen category $\mathcal{W}$ a non-connective K-theory spectrum $\mathbf{K}^{\mathrm{W}}(\mathcal{W})$.

The next definition is taken from [173, Definition 2.1].

Definition 21.13.

(i) The Waldhausen category $\mathcal{W}$ admits *factorizations*, if every morphism in $\mathcal{W}$ can be factorized into a cofibration followed by a weak equivalence; no functoriality of this factorization is assumed;

(ii) The Waldhausen category $\mathcal{W}$ is *homotopical*, if it admits factorizations and the weak equivalences satisfy the *two-out-of-six property*, i.e., if for composable morphisms $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3$ in $\mathcal{W}$ both $f_2 \circ f_1$ and $f_3 \circ f_2$ are weak equivalences, then also then also f_1, f_2, f_3 and $f_3 \circ f_2 \circ f_1$ are weak equivalences.

Exercise 21.14. Show that the category $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ of finitely generated projective R-modules becomes a Waldhausen category if we declare split injective R-homomorphisms to be the cofibrations and isomorphisms to be the weak equivalences. Prove that the Waldhausen category $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ is not homotopical.

Let $\mathsf{Wald}^{\mathrm{ho}}$ be the category of homotopical Waldhausen categories. In the sequel we denote by

$$\mathbf{K}^{\mathrm{W}} \colon \mathsf{Wald}^{\mathrm{ho}} \to \mathsf{SPECTRA} \tag{21.15}$$

the non-connective K-theory functor constructed in [173, Definition 2.37].

Remark 21.16. Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ becomes a Waldhausen category, if we define the weak equivalences to be the isomorphisms and the cofibration to be the morphisms $f \colon A \to B$, for which there exists an object $A^{\perp}$ and an isomorphism $u \colon A \oplus A^{\perp} \xrightarrow{\cong} B$ such that the composite of u with the canonical inclusion $A \to A \oplus A^{\perp}$ is f. Note that this Waldhausen category is *not* homotopical, as it does not satisfy factorization. So we cannot apply (21.15) to the Waldhausen category $\mathcal{A}$.

Let $\mathsf{Ch}(\mathcal{A})$ be the Waldhausen category of bounded chain complexes over $\mathcal{A}$, where a cofibration $f_* \colon C_* \to D_*$ is a chain map such that $f_n \colon C_n \to D_n$ is a cofibration in $\mathcal{A}$ and the weak equivalences are the chain homotopy equivalences. Then $\mathsf{Ch}(\mathcal{A})$ is homotopical thanks to the mapping cylinder construction. Hence we can apply (21.15) to the Waldhausen category $\mathsf{Ch}(\mathcal{A})$ and can consider its non-connective K-theory spectrum $\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A}))$.

More generally, if $\mathcal{A}$ is an exact category, then the Waldhausen category $\mathsf{Ch}(\mathcal{A})$ can be defined analogously and is homotopical.

Suppose that $\mathcal{W}$ is a category with cofibrations and that $\mathcal{W}$ is equipped with two categories of weak equivalences, one finer than the other meaning $v\mathcal{W} \subseteq w\mathcal{W}$. Thus $\mathcal{W}$ becomes a Waldhausen category in two ways. Suppose that in both cases $\mathcal{W}$ is a homotopical Walhausen category. Let $\mathcal{W}^w$ denote the full subcategory of $\mathcal{W}$ given by the objects C in $\mathcal{W}$ having the property that the map $C \to \{\bullet\}$ belongs to $w\mathcal{W}$. Then $\mathcal{W}^w$ inherits two Waldhausen structures if we put $v\mathcal{W}^w = \mathcal{W}^w \cap v\mathcal{W}$ and $w\mathcal{W}^w = \mathcal{W}^w \cap w\mathcal{W}$. Both yield homotopical Waldhausen categories.

Theorem 21.17 (Fibration Theorem). *Under the assumptions above we get a weak homotopy fibration of spectra*

$$\mathbf{K}^{\mathrm{W}}(\mathcal{W}^w, v\mathcal{W}^w) \to \mathbf{K}^{\mathrm{W}}(\mathcal{W}, v\mathcal{W}) \to \mathbf{K}^{\mathrm{W}}(\mathcal{W}, w\mathcal{W}).$$

Proof. This follows from [173, Theorem 2.35]. □

Theorem 21.18 (Cisinski's Approximation Theorem). *Let $F\colon \mathcal{W}_0 \to \mathcal{W}_1$ be an exact functor of homotopical Waldhausen categories. Assume:*

(i) An arrow in $\mathcal{W}_0$ is a weak equivalence in $\mathcal{W}_0$ if and only if its image in $\mathcal{W}_1$ is a weak equivalence in $\mathcal{W}_1$;
(ii) Given any object C_0 in $\mathcal{W}_0$ and any map $f\colon F(C_0) \to C_1$ in $\mathcal{W}_1$, there exists a commutative diagram in $\mathcal{W}_1$

$$\begin{tikzcd} F(C_0) \arrow[r, "f"] \arrow[d, "F(u)"'] & C_1 \arrow[d, "\simeq"', "v"] \\ F(D_0) \arrow[r, "\simeq", "w"'] & D_1 \end{tikzcd}$$

for a morphism $u\colon C_0 \to D_0$ in $\mathcal{W}_0$ and weak equivalences $v\colon C_1 \to D_1$ and $w\colon F(D_0) \to D_1$ in $\mathcal{W}_1$.

Then the map of spectra $\mathbf{K}^{\mathrm{W}}(F)\colon \mathbf{K}^{\mathrm{W}}(\mathcal{W}_0) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{W}}(\mathcal{W}_1)$ is a weak homotopy equivalence.

Proof. This follows from [173, Theorem 2.16]. □

Theorem 21.19 (Cofinality Theorem). *Let $I\colon \mathcal{W}_0 \to \mathcal{W}_1$ be the inclusion of a full homotopical Waldhausen subcategory $\mathcal{W}_0$ into a homotopical Waldhausen category $\mathcal{W}_1$. Assume:*

(i) The functor F admits a mapping cylinder argument, i.e., for every morphism $f\colon C_0 \to C_1$ in $\mathcal{W}_1$ such that C_0 belongs to $\mathcal{W}_0$ and C_1 is the target of a weak equivalence with some object in $\mathcal{W}_0$ as source, there is a factorization in $\mathcal{W}_1$

$$C_0 \xrightarrow{f'} C' \xrightarrow{f''} C_1$$

such that C' belongs to $\mathcal{W}_0$ and f'' is a weak equivalence;

(ii) The category $\mathcal{W}_1$ is dominated by $\mathcal{W}_0$, i.e., for any object C_1 in $\mathcal{W}_1$ there exists an object C_0 in $\mathcal{W}_0$ and an object C_1' in $\mathcal{W}_1$ and morphisms $r\colon C_0 \to C_1$ and $i\colon C_1' \to C_0$ such that $r \circ i$ is a weak equivalence.

Then $\mathbf{K}^{\mathrm{W}}(I)\colon \mathbf{K}^{\mathrm{W}}(\mathcal{W}_0) \to \mathbf{K}^{\mathrm{W}}(\mathcal{W}_1)$ is a weak homotopy equivalence.

Proof. This follows from [173, Theorem 2.30] and the fact that on the level of stable ∞-categories non-connective K-theory is inverting the passage to the idempotent completion. $\square$

There is also an Additivity Theorems in this setting, see [173, Corollary 2.36] which is the non-connective version of Theorem 7.38.

21.5 Non-Connective K-Theory and Karoubi Filtrations for Waldhausen Categories

Theorem 21.20 (The weak homotopy fibration sequence of a stable Karoubi filtration for K-theory in the setting of Waldhausen categories). *Let $\mathcal{A}$ be a additive category and $i\colon \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory. If the additive category $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered, then*

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{U})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A}/\mathcal{U}))$$

is a weak homotopy fibration of non-connective spectra, where $\mathbf{K}^{\mathrm{W}}$ has been defined in (21.15)*,*

The remainder of this section of this devoted to the proof of Theorem 21.20 which needs some preparation. We will follow the ideas of the proof of [197] and correct a bug in it. We begin with proving some lemmas, which will be needed as input.

We can consider an additive category $\mathcal{A}$ as (not necessarily homotopical) Waldhausen category as explained in Remark 21.16. Let $\mathsf{Ch}(\mathcal{A})$ be the homotopical Waldhausen category of bounded chain complexes over $\mathcal{A}$, i.e., chain complexes C_* in $\mathcal{A}$ such that there exists a natural number N (depending on C_*) satisfying $C_n = 0$ for $|n| > N$. A cofibration $i_*\colon C_* \to D_*$ is a chain map such that $i_n\colon C_n \to D_n$ is a cofibration in $\mathcal{A}$ for each $n \in \mathbb{Z}$. Weak equivalences are just the chain homotopy equivalences of bounded $\mathcal{A}$-chain complexes.

We will consider on $\mathsf{Ch}(\mathcal{A})$ also the structure of a Waldhausen category where the cofibrations are the same as before, namely chain maps which are levelwise inclusions of direct summand up to isomorphism, and weak equivalences are those chain maps $f_*\colon C_* \to D_*$ which become weak homotopy equivalences in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$. We will denote the corresponding Waldhausen category by $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))$.

Lemma 21.21. *Suppose that $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered. Then the functor*

$$F\colon \mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U})) \to \mathsf{Ch}(\mathcal{A}/\mathcal{U})$$

induced by the projection of additive categories $\mathcal{A} \to \mathcal{A}/\mathcal{U}$ is a functor of Waldhausen categories and induces a weak homotopy equivalence

$$\mathbf{K}^{\mathrm{W}}(F)\colon \mathbf{K}^{\mathrm{W}}\big(\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))\big) \to \mathbf{K}^{\mathrm{W}}\big(\mathsf{Ch}(\mathcal{A}/\mathcal{U})\big).$$

Proof. Obviously $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U})) \to \mathsf{Ch}(\mathcal{A}/\mathcal{U})$ is a functor of homotopical Waldhausen categories. We want to apply Cisinski's Approximation Theorem 21.18. A morphism in $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))$ is a weak equivalence if and only if its image under F is a weak equivalence in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$. Hence it remains to show that for any object C_* in $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))$, any object $D_* \in \mathsf{Ch}(\mathcal{A}/\mathcal{U})$ and any morphism $f_*\colon F(C_*) \to D_*$ in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$ we can construct an object C'_* in $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))$, a morphism $f'_*\colon C_* \to C'_*$ in $\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))$, and a morphism $g_*\colon F(C'_*) \to D_*$ in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$ such that g_* is a chain homotopy equivalence in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$ and the diagram in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$ commutes

(21.22)
$$\begin{tikzcd} F(C_*) \arrow[r, "f_*"] \arrow[d, "F(f'_*)"'] & D_* \\ F(C'_*) \arrow[ur, "\simeq", "g_*"'] & \end{tikzcd}$$

commutes. For this purpose we will carry out the following construction. Note that the construction here is more involved than the one in the proof of [197, Proposition 7.2] since we are only assuming strongly stably $\mathcal{U}$-filtered instead of $\mathcal{U}$-filtered. Moreover we have to fix the bug in the proof of [197, Proposition 7.2] that it is not clear that the chain map denoted there by ϕx lives already over $\mathcal{A}$, which is needed to ensure that $T(\phi x)$ lives over $\mathcal{A}$. (Note that in [197] the role of $\mathcal{A}$ and $\mathcal{U}$ are interchanged.)

Consider a sequence of morphisms in $\mathcal{A}$ of the shape

$$\cdots \xrightarrow{\begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}} C_n \oplus E_n \xrightarrow{\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}} C_{n-1} \oplus E_{n-1} \xrightarrow{\begin{pmatrix} c_{n-1} & x_{n-1} \\ 0 & e_{n-1} \end{pmatrix}} \cdots$$

such that there exists a natural number N with $C_n = E_n = 0$ for $|n| > N$, we have $c_{n+1} \circ c_n = 0$ in $\mathcal{A}$ and $\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix} \circ \begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix} = 0$ in $\mathcal{A}/\mathcal{U}$ for all $n \in \mathbb{Z}$. In other words (C_*, c_n) is a bounded chain complex over $\mathcal{A}$ and the sequence above considered in $\mathcal{A}/\mathcal{U}$ is a bounded chain complex over $\mathcal{A}/\mathcal{U}$. It is not necessarily a bounded chain complex over $\mathcal{A}$ since the composite

$\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix} \circ \begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}$ may not be zero in $\mathcal{A}$. The purpose of the following construction is to replace it by a chain complex over $\mathcal{A}$ which is $\mathcal{A}/\mathcal{U}$-chain homotopy equivalent to the given one in $\mathcal{A}/\mathcal{U}$ where C_n and c_n are not changed at all. More precisely, we want to construct a diagram in $\mathcal{A}$

$$\begin{tikzcd}[column sep=huge]
\cdots \arrow[r, "\begin{pmatrix} c_{n+1} & x'_{n+1} \\ 0 & e'_{n+1} \end{pmatrix}"] & C_n \oplus E'_n \arrow[r, "\begin{pmatrix} c_n & x'_n \\ 0 & e'_n \end{pmatrix}"] \arrow[d, "\mathrm{id}_{C_n} \oplus f_n"] & C_{n-1} \oplus E'_{n-1} \arrow[r, "\begin{pmatrix} c_{n-1} & x'_{n-1} \\ 0 & e'_{n-1} \end{pmatrix}"] \arrow[d, "\mathrm{id}_{C_{n-1}} \oplus f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}"'] & C_n \oplus E_n \arrow[r, "\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}"'] & C_{n-1} \oplus E_{n-1} \arrow[r, "\begin{pmatrix} c_{n-1} & x_{n-1} \\ 0 & e_{n-1} \end{pmatrix}"'] & \cdots
\end{tikzcd}$$

such that the upper row is a chain complex over $\mathcal{A}$, the diagram commutes over $\mathcal{A}/\mathcal{U}$ and the chain map over $\mathcal{A}/\mathcal{U}$ induced by the vertical arrows from the upper row to the lower row is a chain homotopy equivalence over $\mathcal{A}/\mathcal{U}$ and $E'_n = 0$ for $|n| > N$. We do this inductively over n, where we arrange all the desired statements hold except that we only know $\begin{pmatrix} c_{m-1} & x'_{m-1} \\ 0 & e'_{m-1} \end{pmatrix} \circ \begin{pmatrix} c_m & x'_m \\ 0 & e'_m \end{pmatrix} = 0$ in $\mathcal{A}$ for $m \le n$. The induction beginning $n = 1 - N$ is trivial, take the lower row to be the upper row. The induction step from n to $n+1$ is done as follows. By the induction hypothesis we can assume that $\begin{pmatrix} c_{m-1} & x_{m-1} \\ 0 & e_{m-1} \end{pmatrix} \circ \begin{pmatrix} c_m & x_m \\ 0 & e_m \end{pmatrix} = 0$ in $\mathcal{A}$ for $m \le n$. Recall that $c_n \circ c_{n+1} = 0$ holds in $\mathcal{A}$ by assumption. The composite

$$E_{n+1} \xrightarrow{\begin{pmatrix} x_{n+1} \\ e_{n+1} \end{pmatrix}} C_n \oplus E_n \xrightarrow{\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}} C_{n-1} \oplus E_{n-1}$$

is by assumption trivial, when considered in $\mathcal{A}/\mathcal{U}$. Hence we can find an object U in $\mathcal{U}$ and morphisms $u\colon E_{n+1} \to U$ and $v\colon U \to C_{n-1} \oplus E_{n-1}$ such that the composite above is equal to the composite $v \circ u\colon E_{n+1} \to C_{n-1} \oplus E_{n-1}$ in $\mathcal{A}$. Since by assumption $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered, we can find objects E'_{n+1} and $(E_{n+1} \oplus E'_{n+1})^{\perp}$ in $\mathcal{A}$, an object $(E_{n+1} \oplus E'_{n+1})^{\mathcal{U}}$ in $\mathcal{U}$ and morphisms $i^{\mathcal{U}}\colon (E_{n+1} \oplus E'_{n+1})^{\mathcal{U}} \to E_{n+1} \oplus E'_{n+1}$, $i^{\perp}\colon (E_{n+1} \oplus E'_{n+1})^{\perp} \to E_{n+1} \oplus E'_{n+1}$ and $u^{\mathcal{U}}\colon (E_{n+1} \oplus E'_{n+1})^{\mathcal{U}} \to U$ such that $i^{\mathcal{U}} \oplus i^{\perp}\colon (E_n \oplus E'_n)^{\mathcal{U}} \oplus (E_n \oplus E'_n)^{\perp} \to E_n \oplus E'_n$ is an isomorphism in $\mathcal{A}$ and the diagram in $\mathcal{A}$

(21.23)

$$\begin{tikzcd}
E_{n+1} \oplus E'_{n+1} \arrow[r, "\mathrm{pr}_{E_{n+1}}"] & E_{n+1} \arrow[r, "u"] & U \\
(E_{n+1} \oplus E'_{n+1})^{\mathcal{U}} \oplus (E_{n+1} \oplus E'_{n+1})^{\perp} \arrow[u, "i^{\mathcal{U}} \oplus i^{\perp}", "\cong"'] \arrow[rr, "\mathrm{pr}_{(E_{n+1} \oplus E'_{n+1})^{\mathcal{U}}}"] & & (E_{n+1} \oplus E'_{n+1})^{\mathcal{U}} \arrow[u, "u^{\mathcal{U}}"']
\end{tikzcd}$$

commutes. Now we modify the given row

$$\cdots \xrightarrow{\begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}} C_n \oplus E_n \xrightarrow{\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}} C_{n-1} \oplus E_{n-1} \xrightarrow{\begin{pmatrix} c_{n-1} & x_{n-1} \\ 0 & e_{n-1} \end{pmatrix}} \cdots$$

by first adding the elementary chain complex which is concentrated in dimension $n+1$ and n and has as $n+1$-th differential $\mathrm{id}\colon E'_{n+1} \to E'_{n+1}$ and then the $n+1$-th chain module $C_{n+1} \oplus E_{n+1} \oplus E'_{n+1}$ is cut down to the direct summand $C_{n+1} \oplus (E_{n+1} \oplus E'_{n+1})^{\perp}$. We record the result of this operation in dimensions $(n+2)$, $(n+1)$, n and $(n-1)$ only since nothing changes in the other dimensions. Namely, we get the following diagram in $\mathcal{A}$

(21.24)

$$\begin{tikzcd}
C_{n+2} \oplus E_{n+2} \arrow[r, "A_{n+2}"] \arrow[d, "\mathrm{id}"] & C_{n+1} \oplus (E_{n+1} \oplus E'_{n+1})^{\perp} \arrow[r, "A_{n+1}"] \arrow[d, "\mathrm{id}_{C_n} \oplus \mathrm{pr}_{E_{n+1}} \circ i^{\perp}"] & C_n \oplus E_n \oplus E'_{n+1} \arrow[r, "A_n"] \arrow[d, "\mathrm{id}_{C_{n-1}} \oplus \mathrm{pr}_{E_n}"] & C_{n-1} \oplus E_{n-1} \arrow[d, "\mathrm{id}"] \\
C_{n+2} \oplus E_{n+2} \arrow[r, "\begin{pmatrix} c_{n+2} & x_{n+2} \\ 0 & e_{n+2} \end{pmatrix}"'] & C_{n+1} \oplus E_{n+1} \arrow[r, "\begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}"'] & C_n \oplus E_n \arrow[r, "\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}"'] & C_{n-1} \oplus E_{n-1}
\end{tikzcd}$$

for the matrices

$$\begin{aligned}
A_{n+2} &= \begin{pmatrix} c_{n+2} & x_{n+2} \\ 0 & \mathrm{pr}_{(E_{n+1} \oplus E'_{n+1})^{\perp}} \circ (i^{\mathcal{U}} \oplus i^{\perp})^{-1} \circ i_{E_{n+1}} \circ e_{n+2} \end{pmatrix}; \\
A_{n+1} &= \begin{pmatrix} c_{n+1} & x_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \\ 0 & e_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \\ 0 & \mathrm{pr}_{E'_{n+1}} \circ i^{\perp} \end{pmatrix}; \\
A_n &= \begin{pmatrix} c_n & x_n & 0 \\ 0 & e_n & 0 \end{pmatrix}.
\end{aligned}$$

A direct computation shows that the diagram (21.24) commutes. In oder to ensure that the upper row is a $\mathcal{A}$-chain complex, we have to check that $A_n \circ A_{n+1} = 0$ holds in $\mathcal{A}$. Now $A_n \circ A_{n+1}$ is given by the matrix

$$\begin{pmatrix} c_n \circ c_{n+1} & c_n \circ x_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} + x_n \circ \circ e_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \\ 0 & e_n \circ \circ e_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \end{pmatrix}$$

We have by assumption $c_n \circ c_{n+1} = 0$. Hence it remains to show that

$$\begin{pmatrix} c_n \circ x_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} + x_n \circ \circ e_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \\ e_n \circ e_{n+1} \circ \mathrm{pr}_{E_{n+1}} \circ i^{\perp} \end{pmatrix} \colon (E_{n+1} \oplus E'_{n+1})^{\perp} \to C_{n-1} \oplus E_{n-1}$$

is the zero homomorphism in $\mathcal{A}$. This is the composite

$$(E_{n+1} \oplus E'_{n+1})^{\perp} \xrightarrow{i^{\perp}} E_{n+1} \oplus E'_{n+1} \xrightarrow{\mathrm{pr}_{E_{n+1}}} E_{n+1} \xrightarrow{\begin{pmatrix} x_{n+1} \\ e_{n+1} \end{pmatrix}} C_n \oplus E_n \xrightarrow{\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}} C_{n-1} \oplus E_{n-1}$$

and therefore agrees with the composite

$$(E_{n+1} \oplus E'_{n+1})^{\perp} \xrightarrow{i^{\perp}} E_{n+1} \oplus E'_{n+1} \xrightarrow{\mathrm{pr}_{E_{n+1}}} E_{n+1} \xrightarrow{u} U \xrightarrow{v} C_{n-1} \oplus E_{n-1}.$$

Hence it suffices to show that the composite

$$(E_{n+1} \oplus E'_{n+1})^{\perp} \xrightarrow{i^{\perp}} E_{n+1} \oplus E'_{n+1} \xrightarrow{\mathrm{pr}_{E_{n+1}}} E_{n+1} \xrightarrow{u} U$$

is trivial in $\mathcal{A}$. This follows from the diagram (21.23).

Now the diagram (21.24) yields a chain map of $\mathcal{A}$-chain complexes which is a chain homotopy equivalence when viewed in $\mathcal{A}/\mathcal{U}$ since an elementary chain complex is contractible and $i^{\perp} \colon (E_{n+1} \oplus E'_{n+1})^{\perp} \to E_{n+1} \oplus E'_{n+1}$ is an isomorphism when viewed in $\mathcal{A}/\mathcal{U}$.

This finishes the induction step in the construction above. The construction above is finished, if we have reached $n = N$.

Now we can construct the desired diagram (21.22). Consider the mapping cylinder $\mathrm{cyl}(f_*)$ of the chain map $f_* \colon F(C_*) \to D_*$ in $\mathcal{A}/\mathcal{U}$. Let $p_* \colon \mathrm{cyl}(f_*) \to D_*$ be the projection and $i_* \colon F(C_*) \to \mathrm{cyl}(f_*)$ be the inclusion. Then p_* is a chain homotopy equivalence over $\mathcal{A}/\mathcal{U}$ and $p_* \circ i_*$ agrees with f_*. If we put $E_n = C_{n-1} \oplus D_n$, then we can find a sequence in $\mathcal{A}$ of the shape

$$\cdots \xrightarrow{\begin{pmatrix} c_{n+1} & x_{n+1} \\ 0 & e_{n+1} \end{pmatrix}} C_n \oplus E_n \xrightarrow{\begin{pmatrix} c_n & x_n \\ 0 & e_n \end{pmatrix}} C_{n-1} \oplus E_{n-1} \xrightarrow{\begin{pmatrix} c_{n-1} & x_{n-1} \\ 0 & e_{n-1} \end{pmatrix}} \cdots$$

such that its image in $\mathcal{A}/\mathcal{U}$ is $\mathrm{cyl}(f_*)$. Now apply the construction above to this sequence. The result is a chain complex C'_* over $\mathcal{A}$ together with an inclusion $f'_* \colon C_* \to C'_*$ of chain complexes over $\mathcal{A}$ together with a chain homotopy equivalence $g'_* \colon F(C'_*) \to \mathrm{cyl}(f_*)$ over $\mathcal{A}/\mathcal{U}$ such that $g'_* \circ F(f'_*) = i_*$ holds over $\mathcal{A}/\mathcal{U}$. Now put $g_* = p_* \circ g'_*$. This finishes the construction of the diagram (21.22) and hence the proof of Lemma 21.21. $\square$

The following result is the extension of the corresponding result in [202, Proposition 7.4] from $\mathcal{U}$-filtered to stably $\mathcal{U}$-filtered.

Lemma 21.25. *Suppose that $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered. Then a bounded $\mathcal{A}$-chain complex C_* is finitely dominated by a bounded $\mathcal{U}$-chain complex in $\mathcal{A}$ if and only if C_* is contractible over $\mathcal{A}/\mathcal{U}$.*

Proof. If C_* is dominated in $\mathcal{A}$ by a $\mathcal{U}$-chain complex, then it is finitely dominated by 0_* and hence chain homotopy equivalent to 0_*, when considered over $\mathcal{A}/\mathcal{U}$. It remains to show for a bounded $\mathcal{A}$-chain complex C_*, which is contractible over $\mathcal{A}/\mathcal{U}$, that it is finitely dominated by a bounded $\mathcal{U}$-chain complex.

Fix a natural number N such that $C_n = 0$ for $|n| > N$. Fix morphisms $\gamma_n \colon C_n \to C_{n+1}$ in $\mathcal{A}$ for $n \in \mathbb{Z}$ such that $\mathrm{id}_{c_n} + c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n$ becomes trivial in $\mathcal{A}/\mathcal{U}$ for all $n \in \mathbb{Z}$. Suppose we have the following diagram in $\mathcal{A}$ for some $k \in \mathbb{Z}$ with $0 \leq k$

(21.26)

$$\begin{tikzcd} C^{\mathcal{U}}_N \arrow[r, "c^{\mathcal{U}}_N"] \arrow[d, "j_N"] & \cdots \arrow[r, "c^{\mathcal{U}}_{N-k+2}"] & C^{\mathcal{U}}_{n-k+1} \arrow[r, "c_{n-k+1} \circ j_{n-k+1}"] \arrow[d, "j_{n-k+1}"] & C_{n-k} \arrow[r, "c_{n-k}"] \arrow[d, "\mathrm{id}"] & \cdots \\ C_N \arrow[r, "c_N"] & \cdots \arrow[r, "c_{N-k+2}"] & C_{n-k+1} \arrow[r, "c_{n-k+1}"] & C_{n-k} \arrow[r, "c_{n-k}"] & \cdots \end{tikzcd}$$

where the lower row is the given bounded $\mathcal{A}$-chain complex C_*. Moreover, for every integer i with $0 \leq i \leq k-1$ the object $C^{\mathcal{U}}_{N-i}$ belongs to $\mathcal{U}$, $j_i \colon C^{\mathcal{U}}_{N-i} \to C_{N-i}$ is the inclusion of a direct summand up to isomorphism, and there exist morphisms $\delta_{N-i} \colon C_{N-i} \to C^{\mathcal{U}}_{N-i}$ such that the following diagram in $\mathcal{A}$ commutes

$$\begin{tikzcd} & C^{\mathcal{U}}_{N-i} \arrow[d, "j_{N-i}"] \\ C_{N-i} \arrow[ur, "\delta_{N-i}"] \arrow[r, "\mathrm{id}_{c_{N-i}} + c_{N-i+1} \circ \gamma_{N-i} + \gamma_{N-i-1} \circ c_{N-i}"'] & C_{N-i} \end{tikzcd}$$

Now we perform the following construction to improve the situation above in the sense that we can replace $k-1$ by k, where we will have to add to C_* an elementary chain complex concentrated in dimensions $(N-k)$ and $(N-k-1)$. By assumption there is an object U in $\mathcal{U}$ and morphisms $u \colon C_{N-k} \to U$ and $v \colon U \to C_{N-k}$ such that the diagram

$$\begin{tikzcd} C_{N-k} \arrow[rr, "\mathrm{id}_{c_{N-k}} + c_{n-k+1} \circ \gamma_{N-k} + \gamma_{N-k-1} \circ c_{N-k}"] \arrow[dr, "u"'] & & C_{N-k} \\ & U \arrow[ur, "v"'] & \end{tikzcd}$$

commutes. Since $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered, we can find objects A and $(C_{N-k} \oplus A)^{\perp}$ in $\mathcal{A}$, an object $(C_{N-k} \oplus A)^{\mathcal{U}}$ in $\mathcal{U}$, and morphisms

$$\begin{aligned} i^{\perp}\colon (C_{N-k} \oplus A)^{\perp} &\to C_{N-k} \oplus A; \\ i^{\mathcal{U}}\colon (C_{N-k} \oplus A)^{\mathcal{U}} &\to C_{N-k} \oplus A; \\ c^{\mathcal{U}}_{N-k}\colon C_{N-k+1} &\to (C_{N-k} \oplus A)^{\mathcal{U}}; \\ \delta'\colon U &\to (C^{\mathcal{U}}_{N-k} \oplus A)^{\mathcal{U}}, \end{aligned}$$

such that the diagram

$$\begin{tikzcd} C^{\mathcal{U}}_{N-k+1} \oplus U \arrow[r, "(c_{N-k+1} \circ j_{N-k+1}) \oplus v"] \arrow[d, "c^{\mathcal{U}}_{N-k+1} \oplus \delta'"'] & C_{N-k} \arrow[r, "i_{C_{N-k}}"] & C_{N-k} \oplus A \\ (C_{N-k} \oplus A)^{\mathcal{U}} \arrow[rr, "i_{(C_{N-k} \oplus A)^{\mathcal{U}}}"] & & (C_{N-k} \oplus A)^{\mathcal{U}} \oplus (C_{N-k} \oplus A)^{\perp} \arrow[u, "\cong"', "i^{\mathcal{U}} \oplus i^{\perp}"] \end{tikzcd}$$

commutes. Define $C^{\mathcal{U}}_{N-k}$ to be $(C_{N-k} \oplus A)^{\mathcal{U}}$. Then $c^{\mathcal{U}}_{N-k+1}$ becomes a homomorphism $C^{\mathcal{U}}_{N-k+1} \to C^{\mathcal{U}}_{N-k}$. Let $j_{N-k}\colon C^{\mathcal{U}}_{N-k} \to C_{N-k} \oplus A$ be $i^{\mathcal{U}}$. Then we obtain the following commutative diagram in $\mathcal{A}$

$$\begin{tikzcd} C^{\mathcal{U}}_{N-k+1} \arrow[r, "c^{\mathcal{U}}_{N-k+1}"] \arrow[d, "j_{N-k+1}"] & C^{\mathcal{U}}_{N-k} \arrow[r, "(c_{N-k} \oplus \mathrm{id}_A) \circ j_{N-k}"] \arrow[d, "j_{N-k}"] & C_{N-k-1} \oplus A \arrow[r, "c_{N-k-1} \circ \mathrm{pr}_{C_{N-k-1}}"] \arrow[d, "\mathrm{id}"] & C_{N-k-2} \arrow[d, "\mathrm{id}"] \\ C_{N-k+1} \arrow[r, "i_{C_{N-k}} \circ c_{N-k+1}"] & C_{N-k} \oplus A \arrow[r, "c_{N-k} \oplus \mathrm{id}_A"] & C_{N-k-1} \oplus A \arrow[r, "c_{N-k-1} \circ \mathrm{pr}_{C_{N-k-1}}"] & C_{N-k-2} \end{tikzcd}$$

which is a modification of the diagram (21.26) and agrees with it in dimensions $\geq N-k+1$ and $\leq N-k-2$.

Let C'_* be the chain complex given by the lower row. It is the direct sum of C_* and the elementary chain complex concentrated in dimensions $(N-k)$ and $(N-k-1)$ whose $(N-k)th$ differential is $\mathrm{id}\colon A \to A$. Define $\delta'_{N-i}\colon C'_{N-i} \oplus A \to C^{\mathcal{U}}_{N-i}$ to be $\delta_i\colon C_{N-i} \to C^{\mathcal{U}}_{N-i}$ for $i \leq k-1$ and to be the composite

$$\delta_{N-k}\colon C_{N-k} \oplus A \xrightarrow{\mathrm{pr}_{C_{N-k}}} C_{N-k} \xrightarrow{u} U \xrightarrow{\delta'} C^{\mathcal{U}}_{N-k} := (C_{N-k} \oplus A)^{\mathcal{U}}$$

for $i = k$.

Note that this elementary chain complex has a chain null homotopy which is given by $\mathrm{id}\colon A \to A$. We extend the morphisms $\gamma_i\colon C_i \to C_{i+1}$ to morphisms $\gamma'_i\colon C'_i \to C'_{i+1}$ by putting $\gamma'_{N-k} = \gamma_{N-k} \circ \mathrm{pr}_{C_{N-k}}$, $\gamma'_{N-k-1} = \gamma_{N-k} \oplus \mathrm{id}_A$, $\gamma_{N-k-2} = i_{C_{N-k-1}} \circ \gamma_{N-k-2}$ and $\gamma'_i = \gamma_i$ for $i \notin \{N-k-2, N-k-1, N-k\}$. Note that then γ_* is a chain contraction of C'_* when considered in $\mathcal{A}/\mathcal{U}$, and C_* and C'_* are chain homotopy equivalent over $\mathcal{A}$. Moreover, the diagram

$$\begin{tikzcd} & & C^{\mathcal{U}}_{N-i} \arrow[d, "j_{N-i}"] \\ C'_{N-i} \arrow[urr, "\delta'_{N-i}"] \arrow[rr, "\mathrm{id}_{c'_{N-i}} + c'_{N-i+1} \circ \gamma'_{N-i} + \gamma'_{N-i-1} \circ c'_{N-i}"'] & & C'_{N-i} \end{tikzcd}$$

commutes for $i \leq k$. Hence we have improved the situation of diagram (21.26) from $(k-1)$ to k after replacing C_* by a chain homotopy equivalent bounded chain complex C'_*.

We do this inductively starting with $k = 0$ until k reaches N. Note that $C_i^{\mathcal{U}\prime\prime} = 0$ for $i < N$, whereas $C''_i = 0$ for $i < -N-1$ and $C_{-N-1} = A$. Thus we have constructed a bounded $\mathcal{A}$ chain complex C''_* which is chain homotopy equivalent to C_* over $\mathcal{A}$, a bounded $\mathcal{U}$-chain complex $C''^{\mathcal{U}}_*$, a chain map $j''_* \colon C''^{\mathcal{U}}_* \to C''_*$ such that j_i is an inclusion of a direct summand up to isomorphism for $i \in \mathbb{Z}$, morphism $\gamma''_i \colon C''_i \to C''_{i+1}$ which fit together to a chain contraction of C''_* over $\mathcal{A}/\mathcal{U}$, and morphisms $\delta''_i \colon C''_i \to C''^{\mathcal{U}}_i$ such that for every $i \in \mathbb{Z}$ the following diagram in $\mathcal{A}$ commutes

$$\begin{tikzcd} & & C''^{\mathcal{U}}_i \arrow[d, "j_i"] \\ C''_i \arrow[urr, "\delta''_i"] \arrow[rr, "\mathrm{id}_{C''_i} + c''_{i+1} \circ \gamma''_i + \gamma''_{i-1} \circ c''_i"'] & & C''_i \end{tikzcd}$$

One easily checks that the collection of the δ''_i-s yields a chain map $\delta_* \colon C''_* \to C''^{\mathcal{U}}_*$ and the collection of the γ_i-s yields a chain homotopy $j''_* \circ \delta''_* \simeq \mathrm{id}$. Hence the bounded $\mathcal{U}$-chain complex $C''^{\mathcal{U}}_*$ dominates C''_* and hence C_*. This finishes the proof of Lemma 21.25. □

Exercise 21.27. Construct a $\mathbb{Z}[\mathbb{Z}/23]$-chain complex C_* which is finitely dominated but not $\mathbb{Z}[\mathbb{Z}/23]$-chain homotopic to a finite free $\mathbb{Z}[\mathbb{Z}/23]$-chain complex.

Proof of Theorem 21.20. Let $\mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})$ be the full homotopical Waldhausen subcategory of $\mathsf{Ch}(\mathcal{A})$ consisting of those $\mathcal{A}$-chain complexes whose the image under $\mathsf{Ch}(p) \colon \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A}/\mathcal{U})$ is contractible in $\mathsf{Ch}(\mathcal{A}/\mathcal{U})$. Let $I \colon \mathsf{Ch}(\mathcal{U}) \to \mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})$ be the inclusion of full homotopical Waldhausen categories induced by the inclusion $i \colon \mathcal{U} \to \mathcal{A}$. We conclude from the Cofinality Theorem 21.19 that we get a weak homotopy equivalence

$$\mathbf{K}(I) \colon \mathbf{K}(\mathsf{Ch}(\mathcal{U})) \to \mathbf{K}(\mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})) \tag{21.28}$$

after we have shown that the two conditions appearing in the Cofinality Theorem 21.19 are satisfied.The second condition follows from Lemma 21.25, the first one is proved as follows. Consider objects $C_*, C'_* \in \mathsf{Ch}(\mathcal{U})$ and $D \in \mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})$ together with morphisms $f \colon C \to D$, $u \colon C' \to D$, and

$v\colon D \to C'$ in $\mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})$ such that u and v are to one another chain homotopy inverse chain homotopy equivalences. Then the mapping cylinder $\operatorname{cyl}(v \circ f)$ and the canonical inclusion $i\colon C_* \to \operatorname{cyl}(v \circ f)$ live in $\mathsf{Ch}(\mathcal{U})$. Let $p\colon \operatorname{cyl}(v \circ f) \to D$ be the canonical projection, which is a chain homotopy equivalence in $\mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})$. Since $u \circ p \circ i \simeq u \circ v \circ f \simeq f$ holds, and $i\colon C_* \to \operatorname{cyl}(v \circ f)$ is a cofibration, we can change $u \circ p$ up to chain homotopy to a chain homotopy equivalence $q\colon \operatorname{cyl}(v \circ f) \to D$ satisfying $q \circ i = f$. This finishes the proof that (21.28) is a weak homotopy equivalence.

We obtain from the Fibration Theorem 21.17 a weak homotopy fibration

$$\mathbf{K}(\mathsf{Ch}(\mathcal{A}^{w(\mathcal{A}/\mathcal{U})})) \to \mathbf{K}(\mathsf{Ch}(\mathcal{A})) \to \mathbf{K}(\mathsf{Ch}(\mathcal{A}, w(\mathcal{A}/\mathcal{U}))). \tag{21.29}$$

Now Theorem 21.20 follows from Lemma 21.21, and the weak homotopy equivalences (21.28) and (21.29). □

21.6 Non-Connective K-Theory and Stable Karoubi Filtration

Theorem 21.30 (The weak homotopy fibration sequence of a stable Karoubi filtration for K-theory). *Let $\mathcal{A}$ be an additive category and $i\colon \mathcal{U} \to \mathcal{A}$ be the inclusion of a full additive subcategory. If the additive category $\mathcal{A}$ is stably $\mathcal{U}$-filtered, then*

$$\mathbf{K}(\mathcal{U}) \xrightarrow{\mathbf{K}(i)} \mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(p)} \mathbf{K}(\mathcal{A}/\mathcal{U})$$

is a weak homotopy fibration of non-connective spectra.

Proof. Firstly we reduce the claim to the special case where $\mathcal{U}$ and $\mathcal{A}$ are idempotent complete. Namely, we obtain a commutative diagram of spectra

$$\begin{tikzcd}
\mathbf{K}(\mathcal{U}) \arrow[r, "\mathbf{K}(i)"] \arrow[d, "\mathbf{K}(j)", "\simeq"'] & \mathbf{K}(\mathcal{A}) \arrow[r, "\mathbf{K}(p)"] \arrow[d, "\mathbf{K}(j)", "\simeq"'] & \mathbf{K}(\mathcal{A}/\mathcal{U}) \arrow[d, "\mathbf{K}(j)", "\simeq"'] \\
\mathbf{K}(\mathrm{Idem}(\mathcal{U})) \arrow[r, "\mathbf{K}(\mathrm{Idem}(i))"] \arrow[d, "\mathbf{K}(\mathrm{id})"] & \mathbf{K}(\mathrm{Idem}(\mathcal{A})) \arrow[r, "\mathbf{K}(\mathrm{Idem}(p))"] \arrow[d, "\mathbf{K}(\mathrm{id})"] & \mathbf{K}(\mathrm{Idem}(\mathcal{A}/\mathcal{U})) \arrow[d, "\mathbf{K}(\mathrm{Idem}(F))", "\simeq"'] \\
\mathbf{K}(\mathrm{Idem}(\mathcal{U})) \arrow[r, "\mathbf{K}(\mathrm{Idem}(i))"] & \mathbf{K}(\mathrm{Idem}(\mathcal{A})) \arrow[r, "\mathbf{K}(F \circ \mathrm{Idem}(p))"] & \mathbf{K}(\mathrm{Idem}(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U}))) \\
\mathbf{K}(\mathrm{Idem}(\mathcal{U})) \arrow[r, "\mathbf{K}(\mathrm{Idem}(i))"] \arrow[u, "\mathbf{K}(\mathrm{id})"'] & \mathbf{K}(\mathrm{Idem}(\mathcal{A})) \arrow[r, "\mathbf{K}(q)"] \arrow[u, "\mathbf{K}(\mathrm{id})"'] & \mathbf{K}(\mathrm{Idem}(\mathcal{A})/\mathrm{Idem}(\mathcal{U})) \arrow[u, "\mathbf{K}(j)"', "\simeq"]
\end{tikzcd}$$

where j denotes always the inclusion of an additive category in its idempotent completion, the functor F has been introduced in (21.11) and q is the canonial projection. By Lemma 21.12 the map $\mathbf{K}(\mathrm{Idem}(F))$ is a weak homotopy equivalence. Each map $\mathbf{K}(j)$ is a weak homotopy equivalence by see [668, Definition 4.1 and the following paragraph]. Therefore every vertical arrow in the diagram above is a weak homotopy equivalence. Hence the canonical map

$$\mathbf{K}(\mathcal{U}) \to \mathrm{hofib}\big(\mathbf{K}(p)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\big)$$

induces an isomorphism on π_n if and only if the canonical map

$$\mathbf{K}(\mathrm{Idem}(\mathcal{U})) \to \mathrm{hofib}\big(\mathbf{K}(q)\colon \mathbf{K}(\mathrm{Idem}(\mathcal{A}) \to \mathbf{K}(\mathrm{Idem}(\mathcal{A})/\operatorname{Idem}\mathcal{U})\big)$$

induces an isomorphism on π_n. Now Lemma 21.9 implies that we can assume without loss of generality that $\mathcal{U}$ and $\mathcal{A}$ are idempotent complete and that $\mathcal{A}$ is strongly stably $\mathcal{U}$-filtered.

For both $\mathbf{K}^{\mathrm{W}}$ and $\mathbf{K}$, there are natural transformations $\mathbf{K}^{\mathrm{W,con}} \to \mathbf{K}^{\mathrm{W}}$ and $\mathbf{K}^{\mathrm{con}} \to \mathbf{K}$ with their connective versions as source such that each of them induces on π_n for $n \geq 1$ an isomorphism. Recall that $\mathbf{K}^{\mathrm{W,con}}$, in contrast to $\mathbf{K}^{\mathrm{W}}$, is defined for Waldhausen categories in general, the condition homotopical is not needed, and that by definition $\mathbf{K}^{\mathrm{W,con}}(\mathcal{W}) = \mathbf{K}^{\mathrm{con}}(\mathcal{W})$ for any Waldhausen category $\mathcal{W}$.

Note that for the connective K-theory spectrum the obvious inclusion $I_0\colon \mathcal{A} \to \mathsf{Ch}(\mathcal{A})$, which assigns to an object the associated chain complex concentrated in degree 0, is an exact functor of Waldhausen categories and induces the weak homotopy equivalence on the connective K-theory by the Gillet-Waldhausen Theorem, see [931, 1.11.7] or [671, Theorem 4.1],

$$\mathbf{K}^{\mathrm{W,con}}(I_0)\colon \mathbf{K}^{\mathrm{con}}(\mathcal{A}) = \mathbf{K}^{\mathrm{W,con}}(\mathcal{A}) \to \mathbf{K}^{\mathrm{W,con}}(\mathsf{Ch}(\mathcal{A})). \tag{21.31}$$

Hence we conclude from Theorem 21.20, the long homotopy sequence associated to $\mathrm{hofib}\big(\mathbf{K}(p)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\big)$, and the Five Lemma that the canonical map

$$\mathbf{f}\colon \mathbf{K}(\mathcal{U}) \to \mathrm{hofib}\big(\mathbf{K}(p)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\big)$$

induces an isomorphism on π_n for $n \geq 1$.

Next we show by induction over $N = 1, 0, -1, -2, \dots$ that it induces an isomorphism for $n \geq N$. The induction beginning has already been explained, the induction step from N to $(N-1)$ is done as follows.

We conclude from Lemma 21.10 that $\mathcal{A}[t,t^{-1}]$ is stably $\mathcal{U}[t,t^{-1}]$-filtered and that $(\mathcal{A}/\mathcal{U})[t,t^{-1}]$ is isomorphic to $\mathcal{A}[t,t^{-1}]/\mathcal{U}[t,t^{-1}]$. Hence we obtain from the induction hypothesis that the canonical map

$$\mathbf{f}'\colon \mathbf{K}(\mathcal{U}[t,t^{-1}]) \to \mathrm{hofib}\big(\mathbf{K}(\mathcal{A}[t,t^{-1}]) \to \mathbf{K}((\mathcal{A}/\mathcal{U})[t,t^{-1}])\big)$$

is an isomorphism for $n \geq N$. We get from the Bass Heller-Swan decomposition, see for instance [668, Theorem 6.2], weak equivalences

$$\begin{aligned}
\mathbf{K}(\mathcal{U}) \wedge (S^1)_+ \vee \mathbf{N}_+\mathbf{K}(\mathcal{U}) \vee \mathbf{N}_-\mathbf{K}(\mathcal{U}) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{U}[t,t^{-1}]);\\
\mathbf{K}(\mathcal{A}) \wedge (S^1)_+ \vee \mathbf{N}_+\mathbf{K}(\mathcal{A}) \vee \mathbf{N}_-\mathbf{K}(\mathcal{A}) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{A}[t,t^{-1}]);\\
\mathbf{K}(\mathcal{A}/\mathcal{U}) \wedge (S^1)_+ \vee \mathbf{N}_+\mathbf{K}(\mathcal{A}/\mathcal{U}) \vee \mathbf{N}_-\mathbf{K}(\mathcal{A}/\mathcal{U}) &\xrightarrow{\simeq} \mathbf{K}((\mathcal{A}/\mathcal{U})[t,t^{-1}]).
\end{aligned}$$

The maps $\mathbf{f}'$ and $\mathbf{f}$ are compatible with these weak isomorphisms. This implies that the map

$$\pi_{n-1}(\mathbf{f})\colon \pi_{n-1}\big(\mathbf{K}(\mathcal{U})\big) \to \pi_{n-1}\big(\operatorname{hofib}(\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U}))\big)$$

is a direct summand of the map

$$\pi_n(\mathbf{f}')\colon \pi_n\big(\mathbf{K}(\mathcal{U}[t,t^{-1}])\big) \to \pi_n\big(\operatorname{hofib}(\mathbf{K}(\mathcal{A}[t,t^{-1}]) \to \mathbf{K}(\mathcal{A}/\mathcal{U}[t,t^{-1}]))\big)$$

Since $\pi_n(\mathbf{f}')$ is bijective for $n \geq N$, the map $\pi_{n-1}(\mathbf{f})$ is bijective for $n \geq N-1$. This finishes the proof of Theorem 21.30. $\square$

Remark 21.32. The definition of the non-connective K-theory spectrum of homotopical Waldhausen categories due to Bunke-Kasprowski-Winges [173] is based on higher categories. If one wants to avoid this, one can first prove a version of the desired weak homotopy fibration $\mathbf{K}(\mathcal{U}) \xrightarrow{\mathbf{K}(i)} \mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(p)} \mathbf{K}(\mathcal{A}/\mathcal{U})$ on the level of connective K-theory, where all the ingredients such as the Fibration Theorem of the Gillet-Waldhausen Theorem are available and then use the delooping construction of [668] to pass from the connective K-theory to the non-connective K-theory. This is more or less a variation of the proof of Theorem 21.30 described above.

21.7 Comparing the Non-Connective K-Theory Spectra

Next we want to compare the non-connective K-theory spectra $\mathbf{K}(\mathcal{A})$ and $\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A}))$ for any additive category. Note that we have compared the connective versions already in (21.31), but we also have explained in Remark 21.16, why this does not work any more for the non-connective K-theory. This will be rectified as follows.

Consider an additive category $\mathcal{A}$. Define a new additive category $\Lambda\mathcal{A}$ as follows. An object in $\Lambda\mathcal{A}$ is a sequence $\underline{A} = (A_n)_{n\in\mathbb{N}}$ of objects in $\mathcal{A}$. A morphism $\underline{f}\colon \underline{A} \to \underline{A'}$ is a collection $\{f_{n,n'}\colon A_n \to A'_{n'} \mid n,n' \in \mathbb{N}\}$ of morphisms in $\mathcal{A}$ such that there exists a natural number N (depending on $\underline{f}$) such that $f_{n,n'} \neq 0 \implies |n-n'| \leq N$ holds. Let $i\colon \mathcal{A} \to \Lambda\mathcal{A}$ be the obvious inclusion sending an object A to the object given by the sequence $\underline{A}$ with $A_0 = A$ and $A_n = 0$ for $n \geq 1$. Let $\Lambda_f\mathcal{A}$ be the full subcategory of $\Lambda\mathcal{A}$ consisting of objects $\underline{A}$ such that only finitely many of the objects A_n are different from zero. Then the inclusion $\Lambda_f\mathcal{A} \to \Lambda\mathcal{A}$ is a Karoubi filtration and we define $\Sigma\mathcal{A}$ to be the quotient $\Lambda\mathcal{A}/\Lambda_f\mathcal{A}$. The obvious inclusion $\mathcal{A} \to \Lambda_f\mathcal{A}$ is an equivalence of additive categories and hence induces a weak equivalence $\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda_f\mathcal{A}))$. There is an obvious Eilenberg swindle on $\Lambda\mathcal{A}$ coming from the shift to the right functor, which sends an object $\underline{A}$ to the object $\underline{A'}$ satisfying $A'_0 = \{0\}$ and $A'_{n+1} = A_n$ for $n \in \mathbb{N}$. The Eilenberg swindle on $\mathcal{A}$ yields an Eilenberg swindle on $\mathsf{Ch}(\mathcal{A})$. This implies that the inclusion $* \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A}))$ of the trivial trivial spectrum $*$ is a weak homotopy equivalence. Thus we get a weak homotopy equivalence

$$\Omega\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A})) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))\bigr).$$

For $k \in \mathbb{Z}$ and a spectrum $\mathbf{E}$, let $\Sigma^k\mathbf{E}$ be the spectrum obtained by shifting, i.e., $(\Sigma^k\mathbf{E})_n = \mathbf{E}_{n-k}$. Denote by $\Omega\mathbf{E}$ the spectrum with n-space ΩE_n. From the structure maps of a spectrum $\mathbf{E}$, which can be written as maps $E_n \to \Omega E_{n-1}$, we get a canonical map $\mathbf{E} \xrightarrow{\simeq} \Sigma\Omega\mathbf{E}$, which is a weak homotopy equivalence. Hence we get a weak homotopy equivalence

$$\Sigma^{-1}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A})) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))\bigr).$$

We get a weak homotopy equivalence

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda_f\mathcal{A})) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A})) \to \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))\bigr)$$

from Theorem 21.20. Composing it with the weak homotopy equivalence $\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda_f\mathcal{A}))$ yields a weak homotopy equivalence

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \to \operatorname{hofib}\bigl(\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A})) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))\bigr)$$

Thus we obtain a two stage zigzag of weak homotopy equivalence

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Lambda\mathcal{A})) \xleftarrow{\simeq} \Sigma^{-1}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))\bigr), \tag{21.33}$$

which is natural in $\mathcal{A}$. For simplicity we assume in the sequel that the two stage zigzag (21.33) is a weak homotopy equivalence

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \xrightarrow{\simeq} \Sigma^{-1}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A}))$$

and leave it to the reader to extend the argument below to the general case, which actually only requires to replace the homotopy colimits below by more sophisticated diagrams.

We can iterate this and obtain a string of weak homotopy equivalences

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \xrightarrow{\simeq} \Sigma^{-1}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma\mathcal{A})) \xrightarrow{\simeq} \Sigma^{-2}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma^2\mathcal{A})) \xrightarrow{\simeq} \cdots .$$

Define $\operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma^n\mathcal{A}))$ to be the homotopy colimit. The canonical map

$$\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A})) \xrightarrow{\simeq} \operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma^n\mathcal{A})) \tag{21.34}$$

is a weak homotopy equivalence.

There is a natural transformation from the connective K-theory to the non-connective K-theory spectrum, which induces isomorphism on homotopy groups in degree ≥ 1, see (21.31). It yields a weak homotopy equivalence of spectra, natural in $\mathcal{A}$,

$$\operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{W,con}}(\mathsf{Ch}(\Sigma^n\mathcal{A})) \xrightarrow{\simeq} \operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma^n\mathcal{A})) \tag{21.35}$$

since there is a natural isomorphism $\pi_k(\Sigma^{-n}\mathbf{E}) \xrightarrow{\cong} \pi_{k+n}(\mathbf{E})$ for any spectrum $\mathbf{E}$ and the canonical map $\operatorname{colim}_{n\to\infty} \pi_k(\mathbf{E}_n) \xrightarrow{\cong} \pi_k(\operatorname{hocolim}_{n\to\infty} \mathbf{E}_n)$ is an isomorphism for a sequence $\mathbf{E}_0 \to \mathbf{E}_1 \to \mathbf{E}_2 \to \cdots$ of spectra for every $k \in \mathbb{Z}$.

Since $\mathbf{K}^{\mathrm{W,con}}(\mathcal{W}) = \mathbf{K}^{\mathrm{con}}(\mathcal{W})$ holds for any Waldhausen category $\mathcal{W}$, we get from (21.35) the weak homotopy equivalence

$$\operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{con}}(\Sigma^n\mathcal{A}) \xrightarrow{\simeq} \operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\Sigma^n\mathcal{A})). \tag{21.36}$$

Since also the functor sending $\mathcal{A}$ to $\mathbf{K}(\mathcal{A})$ is compatible with Karoubi filtrations, see Theorem 21.30, we get analogously to (21.34) and (21.35) weak homotopy equivalences

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\simeq} \operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}(\Sigma^n\mathcal{A}) \tag{21.37}$$

and

$$\operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}^{\mathrm{con}}(\Sigma^n\mathcal{A}) \xrightarrow{\simeq} \operatorname{hocolim}_{n\to\infty} \Sigma^{-n}\mathbf{K}(\Sigma^n\mathcal{A}), \tag{21.38}$$

which are natural in $\mathcal{A}$. Putting (21.34), (21.36), (21.37), and (21.38) together shows

Theorem 21.39 (Gillet-Waldhausen zigzag for non-connective K-theory). *There is a zigzag of weak homotopy equivalences, natural in $\mathcal{A}$, from the non-connective K-theory spectrum $\mathbf{K}^{\mathrm{W}}(\mathsf{Ch}(\mathcal{A}))$ in the sense of Bunke-Kasprowski-Winges [173] to the non-connective K-theory spectrum $\mathbf{K}(\mathcal{A})$ in the sense of Lück-Steimle [668].*

Remark 21.40. Arguing as in the proof of Theorem 21.39, one can show that the definition of the non-connective K-theory spectra as they appear in [197, 668, 783] agree up to natural zigzags of weak homotopy equivalences.

21.8 Notes

We have not checked the details, but are convinced that the proof of [202, Theorem 4.2] of the existence of the long exact weak homotopy fibration $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U})$ associated to a Karoubi filtration of additive categories with involution carries over to stable Karoubi filtrations.

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Chapter 22
Controlled Topology Methods

22.1 Introduction

In this chapter we explain and prove in detail for any group G and any G-CW-complex X what we have briefly discussed in Subsection 19.4.5. We will allow more general coefficients than rings or additive G-categories, namely, categories with G-support, see Definition 22.1. This notion seems to be the most general one and illustrates nicely what is needed to successfully establish the desired constructions and theorems appearing in this chapter.

Given such a category with G-support $\mathcal{B}$, we will construct covariant functors

$$\mathcal{O}^G(-;\mathcal{B}), \mathcal{T}^G(-;\mathcal{B}), \mathcal{D}^G(-;\mathcal{B}) \colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{ADDCAT}$$

and for every G-CW-complex X in Theorem 22.19 the so-called $\mathcal{TOD}$-*sequence*,

$$K(\mathcal{T}^G(X;\mathcal{B})) \to K(\mathcal{O}^G(X;\mathcal{B})) \to K(\mathcal{D}^G(X;\mathcal{B}))$$

which is a weak homotopy fibration of spectra and natural in X.

Actually, the functor $\mathcal{D}^G(-;\mathcal{B})$ digests G-CW-pairs, and we will prove in Theorem 22.26 that we obtain a G-homology theory with values in $\mathbb{Z}$-modules in the sense of Definition 12.1 by the covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups sending (X, A) to $K_*(\mathcal{D}^G(X, A;\mathcal{B}))$. We will analyze the coefficients of this G-homology theory, namely the covariant functor

$$\mathbf{K}(\mathcal{D}^G(?;\mathcal{B})) \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{D}^G(G/H;\mathcal{B}))$$

in Section 22.8.

In Lemma 22.76 (i) we will identify the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 associated to the G-homology theory $H_*^G(-;\mathbf{K}(\mathcal{B}(?)_\oplus))$ and the family $\mathcal{F}$

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{K}(\mathcal{B}(?)_\oplus)) \to H_n^G(\{\bullet\};\mathbf{K}(\mathcal{B}(?)_\oplus)) = K_n(\mathcal{B}_\oplus)$$

with the homomorphism induced by the projection $E_{\mathcal{F}}(G) \to G/G$

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \to K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B})) = K_n(\mathcal{B}_\oplus)$$

for every $n \in \mathbb{Z}$. Moreover we show in Lemma 22.76 (ii) that the Meta-Isomorphisms Conjecture 15.2 for the G-homology theory $H_*^G(-;\mathbf{K}(\mathcal{B}(?)))$ and the family $\mathcal{F}$ is true if and only if the spectrum $\mathbf{K}(\mathcal{O}^G(E_{\mathcal{F}}(G);\mathcal{B}))$ is weakly contractible.

Note that for a G-$\mathbb{Z}$-category $\mathcal{A}$ we can define the a category with G-support $\mathcal{A}[G]$, see Example 22.2, and obtain an isomorphism

$$K_n(\mathcal{A}[H]_\oplus) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H;\mathcal{A}[G]))$$

for every $n \in \mathbb{Z}$ and every subgroup $H \subseteq G$, see Remark 22.82. This boils down for a ring R coming with a group homomorphism $\rho\colon G \to \operatorname{aut}(R)$ to an isomorphism, see Example 22.83

$$K_n(R_{\rho|H}[H]) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H;\underline{R}[G])).$$

So for an adequate choice of $\mathcal{B}$, the homomorphism $K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \to K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B}))$ can be identified with the map appearing K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories, and of course analogously for rings as coefficients. All this carries over to L-theory.

We also deal with a version $\mathcal{D}_0^G(X;\mathcal{B})$ with zero-control over $\mathbb{N}$ which also yields a G-homology, see Theorem 22.126 and is related to $\mathcal{D}^G$ by a weak homotopy pushout, see Theorem 22.109.

$$\begin{tikzcd}
\mathbf{K}(\mathcal{D}_0^G(X)) \arrow[r, "\mathbf{K}(V_{\rho_E}(X))"] \arrow[d, "\mathbf{K}(V_{\rho_O}(X))"'] & \mathbf{K}(\mathcal{D}_0^G(X)) \arrow[d] \\
\mathbf{K}(\mathcal{D}_0^G(X)) \arrow[r] & \mathbf{K}(\mathcal{D}^G(X)).
\end{tikzcd}$$

These functors $\mathcal{D}_0^G(X;\mathcal{B})$ occur in the transfer criterion for the Farrell-Jones Conjecture appearing in Theorem 24.70. The benefit of Theorem 24.70 is that it suffices to construct the transfer only on homogeneous spaces and for the functor $\mathcal{D}_0^G$ which has the pleasant feature that it is defined with zero-control in the $\mathbb{N}$-direction. This has for instance been exploited in [81, Remarks 6.14 and 7.17].

The setup with categories with G-support as coefficients is too general to expect that the Farrell-Jones Conjecture holds with them as coefficients, see Remark 22.85.

There are many different versions of the categories $\mathcal{D}^G(X)$ constructed below and also the control conditions may vary. We have decided to concentrate in this chapter on one case, namely to the setting with continuous control, as it has been established in [73], and to use the version of the setup for totally disconnected groups, see [81] reduced to discrete groups where it simplifies considerably. The hope is that the reader can easily understand the arguments in other but related situations if she or he has absorbed the cases presented in this chapter. Moreover, we give all the details, whereas in the literature the arguments are sometimes rather sketchy.

22.2 The Definition of a Category with G-Support

Let G be a discrete group. A $\mathbb{Z}$-*category* is a small category $\mathcal{A}$ enriched over the category of $\mathbb{Z}$-modules, i.e., for every two objects A and A' in $\mathcal{A}$ the set of morphisms $\operatorname{mor}_{\mathcal{A}}(A, A')$ has the structure of a $\mathbb{Z}$-module for which composition is a $\mathbb{Z}$-bilinear map.

Definition 22.1 (Category with G-support). A *category with G-support* is a pair $\mathcal{B} = (\mathcal{B}, \operatorname{supp}_G)$ consisting of:

- A $\mathbb{Z}$-category $\mathcal{B}$;
- A map called *support function*

$$\operatorname{supp}_G\colon \operatorname{mor}(\mathcal{B}) \to \{\text{finite subsets of } G\}.$$

We require that the following axioms are satisfied for all objects B in $\mathcal{B}$ and all morphisms $u, u'\colon B_1 \to B_2$, $v\colon B_2 \to B_3$ in $\mathcal{B}$:

(i) $\operatorname{supp}_G(u) = \emptyset \iff u = 0$;
(ii) $\operatorname{supp}_G(v \circ u) \subseteq \operatorname{supp}_G(v) \cdot \operatorname{supp}_G(u) =: \{gg' \mid g \in \operatorname{supp}_G(v), g' \in \operatorname{supp}_G(u)\}$;
(iii) $\operatorname{supp}_G(u + u') \subseteq \operatorname{supp}_G(u) \cup \operatorname{supp}_G(u')$;
(iv) $\operatorname{supp}_G(u) = \operatorname{supp}_G(-u)$;
(v) For every object B in $\mathcal{B}$ its support $\operatorname{supp}_G(B) := \operatorname{supp}_G(\mathrm{id}_B)$ is $\{e\}$.

Example 22.2. Let $\mathcal{A}$ be a G-$\mathbb{Z}$-category, i.e., a $\mathbb{Z}$-category with G action by isomorphisms of $\mathbb{Z}$-categories. Define the category with G-support $\mathcal{A}[G]$ as follows. The set of objects in $\mathcal{A}[G]$ is the set of objects in $\mathcal{A}$. For two objects A and A' in $\mathcal{A}$, a morphism $\phi\colon A \to A'$ in $\mathcal{A}[G]$ is a formal sum $\sum_{g \in g} \phi_g \cdot g$ where $\phi_g\colon gA \to A'$ is a morphism in $\mathcal{A}$ from gA to A' and its G-*support*

$$\operatorname{supp}_G(\phi) := \{g \in G \mid \phi_g \neq 0\}$$

is assumed to be finite. The composite of $\phi\colon A \to A'$ and $\psi\colon A' \to A''$ is given by convolution, i.e.,

$$(\psi \circ \phi)_g = \sum_{\substack{g', g'' \in G \\ g = g'g''}} \psi_{g'} \circ g'\phi_{g''}\colon gA \to A''.$$

The identity of the object $\mathcal{A}$ is given by $\sum_{g \in g} \phi_g \cdot g$ where $\phi_e = \mathrm{id}_A$ and $\phi_g = 0$ for $g \neq e$. The $\mathbb{Z}$-structure on $\operatorname{mor}_{\mathcal{A}[G]}(A, A')$ is given by

$$m \cdot \left(\sum_g \phi_g \cdot g \right) + n \cdot \left(\sum_g \psi_g \cdot g \right) = \sum_g (m \cdot \phi_g + n \cdot \psi_g) \cdot g.$$

One easily checks that $\mathcal{A}[G]$ is a $\mathbb{Z}$-category and becomes with the notion of the support above a category with G-support.

Given a $\mathbb{Z}$-category, let $\mathcal{A}_\oplus$ be the associated additive category whose objects are finite tuples of objects in $\mathcal{A}$ and whose morphisms are given by matrices of morphisms in $\mathcal{A}$ (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [671, Section 1.3].

Let R be a ring. We denote by $\underline{R}$ the $\mathbb{Z}$-category with precisely one object whose $\mathbb{Z}$-module of endomorphisms is given by R with its $\mathbb{Z}$-module structure and composition is given by the multiplication in R. Then we can consider the additive category $\underline{R}_\oplus$. It can be identified with the version of $\underline{R}_\oplus$ appearing in Section 6.6

Example 22.3. Let R be a unital ring coming with a group homomorphisms $\rho\colon G \to \operatorname{aut}(R)$ to the group of ring automorphisms of R. We can consider $\underline{R}$ as a G-$\mathbb{Z}$-category. We have defined the $\mathbb{Z}$-category $\underline{R}[G]$ in Example 22.2. It yields the additive category $\underline{R}[G]_\oplus$.

Denote by $R_\rho[G]$ the twisted group ring. We have defined the additive category $\underline{R_\rho[G]}_\oplus$ above. One easily checks that the additive categories $\underline{R}[G]_\oplus$ and $\underline{R_\rho[G]}_\oplus$ are isomorphic. Recall that $\underline{R_\rho[G]}_\oplus$ is equivalent to the category $R_\rho[G]\text{-}\mathsf{MOD}_{\mathrm{fgf}}$ of finitely generated free $R_\rho[G]$-modules, see (6.42).

22.3 The Additive Category $\mathcal{O}^G(X;\mathcal{B})$

22.3.1 The Definition of $\mathcal{O}^G(X;\mathcal{B})$

Let X be a G-CW-complex and $\mathcal{B}$ be a category with G-support in the sense of Definition 22.1. We define an additive category $\mathcal{O}^G(X;\mathcal{B})$ as follows.

Definition 22.4 ($\mathcal{O}^G(X;\mathcal{B})$). An object in $\mathcal{O}^G(X;\mathcal{B})$ is a quadruple $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ consisting of a set S and maps $\pi\colon S \to X$, $\eta\colon S \to \mathbb{N}$, and $\mathsf{B}\colon S \to \operatorname{ob}(\mathcal{B})$ satisfying:

- *Compact support over X*
  The image of $\pi\colon S \to X$ is contained in a compact subset of X;
- *Locally finiteness over $\mathbb{N}$*
  For every $t \in \mathbb{N}$ the preimage $\eta^{-1}(t)$ is a finite subset of S.

Given two objects $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ and $\mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$, a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ is given by a collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$ of morphisms in $\mathcal{B}$ satisfying the following conditions:

- *Finite G-support*
  There exists a finite subset $F \subset G$ such that
  $$\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$$
  holds for all $s \in S$ and $s' \in S'$;
- *Bounded control over* $\mathbb{N}$
  There exists a natural number n such that for $s \in S$ and $s' \in S'$ the implication
  $$\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \leq n$$
  holds;
- *Continuous control*
  For every $x \in X$ and every open G_x-invariant neighborhood $U \subseteq X$ of x, there exists an open G_x-invariant neighborhood $U' \subseteq X$ of x satisfying $U' \subseteq U$ and a natural number r' such that for $s \in S$, $s' \in S'$, and $g \in \operatorname{supp}_G(\phi_{s,s'})$ the implications
  $$g\pi(s) \in U', \eta(s) \geq r' \implies \pi'(s') \in U; \tag{22.5}$$
  $$g^{-1}\pi'(s') \in U', \eta'(s') \geq r' \implies \pi(s) \in U, \tag{22.6}$$
  hold.

Given three objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$, $\mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$, and $\mathbf{B}'' = (S'', \pi'', \eta'', \mathsf{B}'')$ and morphisms $\phi\colon \mathbf{B} \to \mathbf{B}'$ and $\phi'\colon \mathbf{B}' \to \mathbf{B}''$, define their composite $\phi' \circ \phi\colon \mathbf{B} \to \mathbf{B}''$ by
$$(\phi' \circ \phi)_{s,s''} = \sum_{s' \in S'} \phi'_{s',s''} \circ \phi_{s,s'}$$
for $s \in S$ and $s'' \in S''$.

Define the identity $\mathrm{id}_{\mathbf{B}}$ for the object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ by $(\mathrm{id}_{\mathbf{B}})_{s,s} = \mathrm{id}_{\mathsf{B}(s)}$ for $s \in S$ and by $(\mathrm{id}_{\mathbf{B}})_{s,s'} = 0$ for $s, s' \in S$ with $s \neq s'$.

Given two objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ and $\mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$ and two morphism $\phi, \phi'\colon \mathbf{B} \to \mathbf{B}'$ and $m, n \in \mathbb{Z}$, define the morphism $m \cdot \phi + n \cdot \phi'$ by
$$(m \cdot \phi + n \cdot \phi')_{s,s'} = m \cdot \phi_{s,s'} + n \cdot \phi'_{s,s'}$$
for $s \in S$ and $s' \in S'$.

We have to check that Definition 22.4 makes sense. The conditions *locally finiteness over* $\mathbb{N}$ and *bounded control over* $\mathbb{N}$ ensure that the sum occurring in the definition of the composition is indeed a finite sum, namely,
$$(\phi' \circ \phi)_{s,s''} = \sum_{\substack{s' \in S' \\ \phi'_{s',s''}, \phi_{s,s'} \neq 0}} \phi'_{s',s''} \circ \phi_{s,s'}.$$

Since ϕ and ϕ' satisfy *finite G-support*, we can choose finite subsets F and F' of G such that $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$ and $\operatorname{supp}_G(\phi'_{s',s''}) \subseteq F'$ holds for $s \in S$, $s' \in S'$, and $s'' \in S''$. We get for $s \in S$ and $s'' \in S''$

$$\begin{aligned}\operatorname{supp}_G((\phi' \circ \phi)_{s,s''}) &= \operatorname{supp}_G\left(\sum_{s' \in S'} \phi'_{s',s''} \circ \phi_{s,s'}\right)\\ &\subset \bigcup_{\substack{s' \in S'\\ \phi'_{s',s''},\phi_{s,s'} \neq 0}} \operatorname{supp}_G(\phi'_{s',s''} \circ \phi_{s,s'})\\ &\subset \bigcup_{\substack{s' \in S'\\ \phi'_{s',s''},\phi_{s,s'} \neq 0}} \operatorname{supp}_G(\phi'_{s',s''}) \cdot \operatorname{supp}_G(\phi_{s,s'})\\ &\subset \bigcup_{\substack{s' \in S'\\ \phi'_{s',s''},\phi_{s,s'} \neq 0}} F' \cdot F\\ &\subset F \cdot F'.\end{aligned}$$

Since $F' \cdot F$ is a finite subset of G, the composite $\phi' \circ \phi$ satisfies *finite G-support.*

Since both ϕ and ϕ' satisfy *bounded control over* $\mathbb{N}$, there exist natural numbers n and n' such that the implications $\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \leq n$ and $\phi'_{s',s''} \neq 0 \implies |\eta'(s') - \eta''(s'')| \leq n'$ hold for $s \in S$, $s' \in S'$ and $s'' \in S''$. Hence we have the implication $(\phi' \circ \phi)_{s,s''} \neq 0 \implies |\eta(s) - \eta''(s'')| \leq n + n'$ for $s \in S$, and $s'' \in S''$. This shows that $\phi' \circ \phi$ satisfies *bounded control over* $\mathbb{N}$

Finally we show that *continuous control* is satisfied by $\phi' \circ \phi$. Consider $x \in X$ and an open G_x-invariant neighborhood $U \subseteq X$ of x. Since ϕ' satisfies *continuous control*, we can find an open G_x-invariant neighborhood $U' \subseteq X$ of x satisfying $U' \subseteq U$ and a natural number r' such that the implication

$$g'\pi'(s') \in U', \eta'(s') \geq r' \implies \pi''(s'') \in U \tag{22.7}$$

holds for all $s' \in S'$, $s'' \in S''$ and $g' \in \operatorname{supp}_G(\phi'_{s',s''})$. Because of condition *finite G-support*, there exists a finite subset $F' \subseteq G$ with $\operatorname{supp}_G(\phi'_{s',s''}) \subseteq F'$ for $s' \in S'$, and $s'' \in S''$. Fix $g' \in F'$. Then $g'^{-1}U'$ is an open $G_{g'^{-1}x}$-invariant neighborhood of $g'^{-1}x$. Since ϕ satisfies *bounded control over* $\mathbb{N}$ and *continuous control*, we can find an open $G_{g'^{-1}x}$-invariant neighborhood $U''_{g'} \subseteq X$ of $g'^{-1}x$ satisfying $U''_{g'} \subseteq g'^{-1}U$ and a natural number $r''_{g'}$ with $r''_{g'} \geq r'$ such that the implication

$$g\pi(s) \in U''_{g'}, \eta(s) \geq r''_{g'} \implies \pi'(s') \in g'^{-1}U', \eta'(s') \geq r' \tag{22.8}$$

holds for all $s \in S'$, $s' \in S'$, and $g \in \operatorname{supp}_G(\phi_{s,s'})$. Put

$$\begin{aligned} U'' &:= \bigcap_{g' \in F'} g' U''_{g'}; \\ r'' &:= \max\{r''_{g'} \mid g' \in F'\}. \end{aligned}$$

Then $U'' \subseteq X$ is an open G_x-invariant neighborhood of x. Moreover, we get for $s \in S$, $s' \in S'$, $s'' \in S'$, and $g \in \operatorname{supp}_G(\phi_{s,s'})$, $g' \in \operatorname{supp}_G(\phi'_{s',s''})$

$$\begin{aligned} g'g\pi(s) \in U'', \eta(s) \geq r'' &\implies g\pi(s) \in g'^{-1}U'', \eta(s) \geq r'' \\ &\implies g\pi(s) \in U''_{g'}, \eta(s) \geq r''_{g'} \\ &\overset{(22.8)}{\implies} \pi'(s') \in g'^{-1}U', \eta'(s') \geq r' \\ &\implies g'\pi'(s') \in U', \eta'(s') \geq r' \\ &\overset{(22.7)}{\implies} \phi''(s'') \in U. \end{aligned}$$

Since $\operatorname{supp}_G(\phi' \circ \phi)_s^{s''} \subseteq \bigcup_{s' \in S'} \operatorname{supp}_G(\phi'_{s',s''}) \cdot \operatorname{supp}_G(\phi_{s,s'})$ holds, we have shown for $s \in S$, $s'' \in S''$ and $g'' \in \operatorname{supp}_G((\phi' \circ \phi)_{s,s'})$

$$g''\pi(s) \in U'', \eta(s) \geq r'' \implies \phi''(s'') \in U.$$

This finishes the proof of implication (22.5). We leave the analogous proof of the other implication (22.6) to the reader. This finishes the proof that $\phi' \circ \phi$ satisfies the condition *continuous control* and hence the proof that the composition is well-defined.

One easily checks that the identity morphism is well-defined.

Obviously the definition of the $\mathbb{Z}$-structure makes sense.

Given two objects $\mathbf{B} = (S, \pi, \eta, \mathrm{B})$ and $\mathbf{B}' = (S', \pi', \eta', \mathrm{B}')$, we have to define their direct sum $\mathbf{B} \oplus \mathbf{B}'$. We put

$$\mathbf{B} \oplus \mathbf{B}' = (S \amalg S', \pi \amalg \pi', \eta \amalg \eta', \mathrm{B} \amalg \mathrm{B}')$$

and define the desired morphisms $\mathbf{B} \to \mathbf{B} \oplus \mathbf{B}'$ and $\mathbf{B}' \to \mathbf{B} \oplus \mathbf{B}'$ in the obvious way. This finishes the proof that $\mathcal{O}^G(X;\mathcal{B})$ is a well-defined additive category.

Notation 22.9. When $\mathcal{B}$ is clear from the context, we will often omit it in the notation and write for instance $\mathcal{O}^G(X)$ instead of $\mathcal{O}^G(X;\mathcal{B})$.

Lemma 22.10. *(i) We can replace in Definition 22.4 the condition* (22.5) *by the condition*

$$\pi(s) \in U', \eta(s) \geq r' \implies g^{-1} \cdot \pi'(s') \in U \tag{22.11}$$

without changing $\mathcal{O}^G(X)$;

(ii) We can replace in Definition 22.4 the condition (22.6) *by the condition*

$$\pi'(s') \in U', \eta'(s') \geq r' \implies g \cdot \pi(s) \in U \tag{22.12}$$

without changing $\mathcal{O}^G(X)$*;*

(iii) We can replace in Definition 22.4 simultaneously the condition (22.5) *by the condition* (22.11) *and the condition* (22.6) *by the condition* (22.12) *without changing* $\mathcal{O}^G(X)$*.*

Proof. We give the proof only for assertion (ii), the one for the other assertions is analogous.

We first show that the condition (22.12) is automatically satisfied. Consider $x \in X$ and an open G_x-invariant neighborhood U of x. Let $\phi\colon \mathbf{B} \to \mathbf{B}'$ be a morphisms in $\mathcal{O}^G(X)$. Since it satisfies *finite G-support*, we can find a finite subset $F \subseteq G$ such that $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$ holds for all $s \in S$ and $s' \in S'$. Fix $g \in F$. We can apply condition (22.6) to the open $G_{g^{-1}x}$-invariant neighborhood $g^{-1}U$ of $g^{-1}x$, and obtain an open $G_{g^{-1}x}$-invariant neighborhood U'_g of $g^{-1}x$ with $U'_g \subseteq g^{-1}U'$ and a natural number r'_g such that for all $s \in S$, $s' \in S'$ and $g_0 \in \operatorname{supp}_G(\phi_{s,s'})$ the implication

$$g_0^{-1}\pi'(s') \in U'_g, \eta'(s') \geq r'_g \implies \pi(s) \in g^{-1}U \tag{22.13}$$

holds. Define

$$\begin{aligned} r' &= \max\{r'_g \mid g \in F\}; \\ U' &= \bigcap_{g \in G} gU'_g. \end{aligned}$$

Then U' is an open G_x-invariant neighborhood of x with $U' \subseteq U$ and condition (22.12) is satisfied since for $s \in S$, $s' \in S'$ and $g \in \operatorname{supp}_G(\phi_{s,s'}) \subseteq F$ we get

$$\begin{aligned} \pi'(s') \in U', \eta'(s') \geq r' &\implies g^{-1}\pi'(s') \in g^{-1}U', \eta'(s') \geq r' \\ &\implies g^{-1}\pi'(s') \in U'_g, \eta'(s') \geq r'_g \\ &\overset{(22.13)}{\implies} \pi(s) \in g^{-1}U \\ &\implies g\pi(s) \in U. \end{aligned}$$

The proof in the case where we replace in Definition 22.4 the condition (22.6) by the condition (22.12) and then show that condition (22.6) is satisfied, is analogous and left to the reader. □

The next result gives a criterion when we can modify the map π for an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G(X)$ without changing its isomorphism class.

Lemma 22.14. *Consider two objects in* $\mathcal{O}^G(X)$ *of the form* $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ *and* $\mathbf{B}' = (S, \pi', \eta, \mathsf{B})$*. Suppose that for every* $x \in X$ *and open* G_x*-invariant neighborhood* U *of* x *there exists an open* G_x*-invariant neighbourhood* U' *of* x *in* X *with* $U' \subseteq U$ *and a natural number* r' *such that for* $s \in S$ *the implications*

$$\pi(s) \in U', \eta(s) \geq r' \implies \pi'(s) \in U;$$
$$\pi'(s) \in U', \eta'(s) \geq r' \implies \pi(s) \in U,$$

hold.

Then $\mathbf{B}$ *and* $\mathbf{B}'$ *are isomorphic*

Proof. Define to one another inverse morphisms $\phi\colon \mathbf{B} \to \mathbf{B}'$ and $\phi'\colon \mathbf{B}' \to \mathbf{B}$ by $\phi_{s,s} = \phi'_{s,s} = \mathrm{id}_{\mathsf{B}(s)}$ for $s \in S$ and by $\phi_{s,s'} = \phi'_{s',s} = 0$ for $s, s' \in S$ with $s \neq s'$. One has to check that ϕ and ϕ' are well-defined. Note that $\mathrm{supp}_G(\phi_{s,s'})$ and $\mathrm{supp}_G(\phi_{s',s})$ are empty if $s \neq s'$ and agree with $\{e\}$ if $s = s'$. Hence ϕ and ϕ' satisfy *finite G-support* and *bounded control over* $\mathbb{N}$ for obvious reasons and the assumptions appearing in Lemma 22.14 imply *continuous control*. □

22.4 Functoriality of $\mathcal{O}^G(X;\mathcal{B})$

Consider a G-map $f\colon X \to Y$ of G-CW-complexes. Next we show that it induces a functor of additive categories

$$\mathcal{O}^G(f)\colon \mathcal{O}^G(X) \to \mathcal{O}^G(Y). \tag{22.15}$$

It sends an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G(X)$ to the object $(S, f \circ \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G(Y)$. One easily checks that the conditions *compact support over* X and *locally finiteness over* $\mathbb{N}$ are satisfied for $(S, f \circ \pi, \eta, \mathsf{B})$.

For two objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ and $\mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$ and a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ given by a collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$ in $\mathcal{O}^G(X)$, define the morphism $\mathcal{O}^G(f)(\phi)\colon \mathcal{O}^G(f)(\mathbf{B}) \to \mathcal{O}^G(f)(\mathbf{B}')$ in $\mathcal{O}^G(Y)$ by the same collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$. Obviously conditions *finite G-support* and *bounded control over* $\mathbb{N}$ are satisfied for $\mathcal{O}^G(f)(\phi)$. The hard part is the proof of *continuous control* which we will give next. We only deal with the implication (22.5), the one for the implication (22.6) is completely analogous.

Suppose that the implication (22.5) is not satisfied for $\mathcal{O}^G(f)(\phi)$. Then we can find a point $y \in Y$ and an open G_y-invariant neighborhood U of y such that for every open G_y-invariant neighborhood U' of y with $U' \subseteq U$ and natural number r' there exist elements $s \in S$ and $s' \in S$ and an element $g \in \mathrm{supp}_G(\phi_{s,s'})$ such that $g\pi(s) \in U'$, $\eta(s) \geq r'$, and $\pi'(s') \notin U$ hold. Since Y is a G-CW-complex, we can find a sequence of nested open G_y-invariant neighbourhoods $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of y such that $\bigcap_{n \geq 0} \overline{V_n} = \{y\}$. Hence we can find a sequence of nested open G_y-invariant neighbourhoods $U'_0 \supseteq U'_1 \supseteq U'_2 \supseteq \cdots$ of y satisfying $\bigcap_{n \geq 0} \overline{U'_n} = \{y\}$, a sequence of natural numbers r'_n satisfying $\lim_{n \to \infty} r'_n = \infty$, a sequence $(s_n)_{n \geq 0}$ in S, a sequence $(s'_n)_{n \geq 0}$ in S', and elements $g \in \mathrm{supp}(\phi_{s_n,s'_n})$ such that $g \cdot f \circ \pi(s_n) \in U'_n$, $\eta(s_n) \geq r'_n$, and $f \circ \pi'(s'_n) \notin U$ hold for all $n \in \mathbb{N}$.

Since ϕ satisfies *finite G-support*, we can arrange by passing to subsequences that there exists $g \in G$ such that $g = g_n$ holds for all $n \geq 0$. Since ϕ satisfies *compact support over X*, we can arrange by passing to subsequences that there exists $x \in X$ such that $\lim_{n\to\infty} \pi(s_n) = x$ holds. We get $\lim_{n\to\infty} f \circ \pi(s_n) = f(x)$. Since $g \cdot f \circ \pi(s_n) \in U'_n$ holds for all $n \geq 0$, we conclude $\lim_{n\to\infty} g \cdot f \circ \pi(s_n) = y$. This implies $f(gx) = y$. Note that $f^{-1}(U)$ is an open G_{gx}-invariant neighborhood of gx. Since ϕ satisfies *continuous control*, there exists an open G_{gx}-invariant neighborhood V'' of gx with $V'' \subseteq f^{-1}(U)$ and a natural number r'' such that for $s \in S$, $s' \in S'$, and $g'' \in \operatorname{supp}_G(\phi_{s,s'})$, the implication

$$g''\pi(s) \in V'', \eta(s) \geq r'' \implies \pi'(s') \in f^{-1}(U)$$

holds. Hence we get for all $n \in \mathbb{N}$ the implication

$$g\pi(s_n) \in V'', \eta(s_n) \geq r'' \implies \pi'(s'_n) \in f^{-1}(U).$$

Since $\lim_{n\to\infty} r'_n = \infty$, $\lim_{n\to\infty} g\pi(s_n) = gx$, and V'' is an open neighborhood of gx, we can arrange by passing to subsequences that $g\pi(s_n) \in V''$ and $\eta(s_n) \geq r''$ holds for all $n \geq 0$. Hence we get $\pi'(s'_n) \in f^{-1}(U)$ for all $n \geq 0$. This implies $f \circ \pi'(s'_n) \in U$ for all $n \geq 0$, a contradiction.

Obviously we get a covariant functor $\mathcal{O}^G(-;\mathcal{B})$ from the category of G-CW-spaces with arbitrary G-maps as morphisms to the category of additive categories.

22.5 The $\mathcal{TOD}$-Sequence

Let X be a G-CW-complex and $\mathcal{B}$ be a category with G-support in the sense of Definition 22.1.

Definition 22.16 ($\mathcal{T}^G(X)$). Let $\mathcal{T}^G(X)$ be the full additive subcategory of $\mathcal{O}^G(X)$ consisting of those objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ for which there exists a natural number n satisfying $\eta(s) \leq n$ for all $s \in S$.

Lemma 22.17. *The inclusion $\mathcal{T}^G(X) \to \mathcal{O}^G(X)$ is a Karoubi filtration in the sense of Definition 21.2.*

Proof. Consider an object $\mathbf{B} = (S_{\mathbf{B}}, \pi_{\mathbf{B}}, \eta_{\mathbf{B}}, \mathsf{B}_{\mathbf{B}})$ in $\mathcal{O}^G(X)$, two objects $\mathbf{U} = (S_{\mathbf{U}}, \pi_{\mathbf{U}}, \eta_{\mathbf{U}}, \mathsf{B}_{\mathbf{U}})$ and $\mathbf{V} = (S_{\mathbf{V}}, \pi_{\mathbf{V}}, \eta_{\mathbf{V}}, \mathsf{B}_{\mathbf{V}})$ in $\mathcal{T}^G(X)$, and morphisms $f\colon \mathbf{B} \to \mathbf{U}$ and $g\colon \mathbf{V} \to \mathbf{B}$ in $\mathcal{O}^G(X)$. By definition we can find natural numbers n_0 and n_1 such that $\eta_{\mathbf{U}}(s') \leq n_0$ for $s' \in S_{\mathbf{U}}$ and $\eta_{\mathbf{V}}(s) \leq n_0$ for $s \in S_{\mathbf{V}}$ hold and we have the implications

$$s \in S_{\mathbf{B}}, s' \in S_{\mathbf{U}}, f_{s,s'} \neq 0 \implies |\eta_{\mathbf{B}}(s) - \eta_{\mathbf{U}}(s')| \leq n_1;$$
$$s \in S_{\mathbf{V}}, s' \in S_{\mathbf{B}}, g_{s,s'} \neq 0 \implies |\eta_{\mathbf{V}}(s) - \eta_{\mathbf{B}}(s')| \leq n_1.$$

Now define objects $\mathbf{B}^{\mathcal{U}} = (S_{\mathbf{B}^{\mathcal{U}}}, \pi_{\mathbf{B}^{\mathcal{U}}}, \eta_{\mathbf{B}^{\mathcal{U}}}, \mathsf{B}_{\mathbf{B}^{\mathcal{U}}})$ in $\mathcal{T}^G(X)$, and $\mathbf{B}^{\perp} = (S_{\mathbf{B}^{\perp}}, \pi_{\mathbf{B}^{\perp}}, \eta_{\mathbf{B}^{\perp}}, \mathsf{B}_{\mathbf{B}^{\perp}})$ in $\mathcal{O}^G(X)$ by

$$S_{\mathbf{B}^{\mathcal{U}}} := \{s \in S_{\mathbf{B}} \mid \eta_{\mathbf{B}}(s) \leq n_0 + n_1\};$$
$$S_{\mathbf{B}^{\perp}} := \{s \in S_{\mathbf{B}} \mid \eta_{\mathbf{B}}(s) > n_0 + n_1\},$$

and restricting the maps $\pi_{\mathbf{B}}$, $\eta_{\mathbf{B}}$, and $\mathsf{B}_{\mathbf{B}}$. There are obvious morphisms $i^{\mathcal{U}} \colon \mathcal{B}^{\mathcal{U}} \to \mathcal{B}$ and $i^{\perp} \colon \mathcal{B}^{\perp} \to \mathcal{B}$ in $\mathcal{O}^G(X)$ such that $i^{\mathcal{U}} \oplus i^{\perp} \colon \mathcal{B}^{\mathcal{U}} \oplus \mathcal{B}^{\perp} \xrightarrow{\cong} \mathcal{B}$ is an isomorphism. We leave it to the reader to figure out the obvious definition of the maps $f^{\mathcal{U}}$ and $g^{\mathcal{U}}$ and the proof of the commutativity of the relevant diagrams. Hence inclusion $\mathcal{T}^G(X) \to \mathcal{O}^G(X)$ is a Karoubi filtration. $\square$

Definition 22.18 ($\mathcal{D}^G(X)$). Let $\mathcal{D}^G(X)$ be the additive category given by the quotient $\mathcal{O}^G(X)/\mathcal{T}^G(X)$ in the sense of Definition 21.1.

Theorem 22.19 ($\mathcal{TOD}$-sequence). *The so-called $\mathcal{TOD}$-*sequence

$$K(\mathcal{T}^G(X)) \to K(\mathcal{O}^G(X)) \to K(\mathcal{D}^G(X))$$

is a weak homotopy fibration of spectra.

Proof. This follows from Lemma 22.17 and Theorem 21.5 (i). $\square$

Given a map $f \colon X \to Y$ of G-CW-complexes, the functor of additive categories $\mathcal{O}^G(f) \colon \mathcal{O}^G(X) \to \mathcal{O}^G(Y)$ of (22.15) induces functors of additive categories

(22.20) $$\mathcal{T}^G(f) \colon \mathcal{T}^G(X) \to \mathcal{T}^G(Y);$$

(22.21) $$\mathcal{D}^G(f) \colon \mathcal{D}^G(X) \to \mathcal{D}^G(Y).$$

Lemma 22.22. *Let $f \colon X \to Y$ be a G-map between G-CW-complexes.*

Then $\tau^G(f) \colon \tau^G(X) \xrightarrow{\simeq} \tau^G(Y)$ is an equivalence of additive categories.

Proof. We can assume without loss of generality that $Y = \{\bullet\}$.

Consider an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\tau^G(\{\bullet\})$. Then S is finite. Choose any map $\pi' \colon S \to X$ and define an object $\mathbf{B}' = (S, \pi', \eta, \mathsf{B})$ in $\tau^G(X)$. Since $\mathcal{T}^G(f)(\mathbf{B}') = \mathbf{B}$, we have shown that $\tau^G(f)$ is surjective on objects. Obviously $\mathcal{T}^G(f)$ induces for two objects $\mathbf{B}_0$ and $\mathbf{B}_1$ in $\tau^G(X)$ a bijection

$$\operatorname{mor}_{\mathcal{T}^G(X)}(\mathbf{B}_0, \mathbf{B}_1) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{T}^G(\{\bullet\})}(\mathcal{T}^G(f)(\mathbf{B}_0), \mathcal{T}^G(f)(\mathbf{B}_1)), \quad \phi \mapsto \mathcal{T}^G(f)(\phi)$$

since for $\mathcal{T}^G(X)$ the conditions *finite G-support*, *bounded control over* $\mathbb{N}$, and *continuous control* are automatically satisfied. Hence $\mathcal{T}^G(f)$ is an equivalence of additive categories. $\square$

22.6 The Definition for Pairs

Let (X, A) be a G-CW-pair. Denote by $i\colon A \to X$ the inclusion.

Lemma 22.23. *(i) The functor $\mathcal{O}^G(i)\colon \mathcal{O}^G(A) \to \mathcal{O}^G(X)$ of* (22.15) *induces an isomorphism of additive categories from $\mathcal{O}^G(A)$ onto its image. The image is a full additive subcategory of $\mathcal{O}^G(X)$ which is a Karoubi filtration;*
(ii) The same statement holds for the functor $\mathcal{D}^G(i)\colon \mathcal{D}^G(A) \to \mathcal{D}^G(X)$ of (22.21).

Proof. (i) The image of $\mathcal{O}^G(i)$ can be identified with the full additive subcategory $\mathcal{O}^G(X)_A$ of $\mathcal{O}^G(X)$ whose objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ satisfy $\operatorname{im}(\pi) \subseteq A$. The functor $\mathcal{O}^G(i, \mathcal{B})$ induces an isomorphism $\mathcal{O}^G(A) \xrightarrow{\cong} \mathcal{O}^G(X)_A$ since for every $x \in A$ and open G_x-invariant neighbourhood U of x in A there exists an open G_x-invariant neighbourhood V of x in X with $U = A \cap X$. It remains to show that the inclusion $\mathcal{O}^G(X)_A \subseteq \mathcal{O}^G(X)$ is a Karoubi filtration.

Consider three objects $\mathbf{B}_0 = (S_0, \pi_0, \eta_0, \mathsf{B}_0)$, $\mathbf{B}_1 = (S_1, \pi_1, \eta_1, \mathsf{B}_1)$, and $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G(X)$ with $\operatorname{im}(\pi_0) \subseteq A$ and $\operatorname{im}(\pi_1) \subseteq A$, and two morphism $a_0\colon \mathbf{B} \to \mathbf{B}_0$ and $a_1\colon \mathbf{B}_1 \to \mathbf{B}$ in $\mathcal{O}^G(X)$. Define subsets of S which consists of those elements which are interacting with S_0 and S_1 via a_0 and a_1

$$\widehat{S}_0 := \{s \in S \mid \exists s_0 \in S_0 \text{ with } (a_0)_{s,s_0} \neq 0\};$$
$$\widehat{S}_1 := \{s' \in S \setminus \widehat{S}_0 \mid \exists s_1 \in S_1 \text{ with } (a_1)_{s_1,s} \neq 0\}.$$

Define objects $\mathbf{B}^{\mathcal{U}} = (S^{\mathcal{U}}, \pi^{\mathcal{U}}, \eta^{\mathcal{U}}, \mathsf{B}^{\mathcal{U}})$ and $\mathbf{B}^{\perp} = (S^{\perp}, \pi^{\perp}, \eta^{\perp}, \mathsf{B}^{\perp})$ by putting $S^{\mathcal{U}} := \widehat{S}_0 \amalg \widehat{S}_1$ and $S^{\perp} = S \setminus S^{\mathcal{U}}$ and defining $\pi^{\mathcal{U}}$, $\eta^{\mathcal{U}}$, $\mathsf{B}^{\mathcal{U}}$, $\pi^{\perp}$,$\eta^{\perp}$, and $\mathsf{B}^{\perp}$ by restricting π, η, and B. There are obvious morphisms $i^{\mathcal{U}}\colon \mathbf{B}^{\mathcal{U}} \to \mathbf{B}$ and $i^{\perp}\colon \mathbf{B}^{\perp} \to \mathbf{B}$ in $\mathcal{O}^G(X)_A$ such that $i^{\mathcal{U}} \oplus i^{\perp}\colon \mathbf{B}^{\mathcal{U}} \oplus \mathbf{B}^{\perp} \xrightarrow{\cong} \mathbf{B}$ is an isomorphism and morphisms $a_0^{\mathcal{U}}\colon \mathbf{B}^{\mathcal{U}} \to \mathbf{B}_0$ and $a_1^{\mathcal{U}}\colon \mathbf{B}_1^{\mathcal{U}} \to \mathbf{B}$ such that the relevant diagrams as they appear in the definition of a Karoubi filtration commute. However, we are not done since $\mathcal{B}^{\mathcal{U}}$ is not an object in $\mathcal{O}^G(X)_A$. In order to finish the proof of assertion (i) it suffices to construct an object $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathsf{B}})$ in $\mathcal{O}^G(X)_A$ together with an isomorphism $\phi\colon \widehat{\mathcal{B}} \xrightarrow{\cong} \mathcal{B}^{\mathcal{U}}$ in $\mathcal{O}^G(X)$.

Choose functions $u_0\colon \widehat{S}_0 \to S_0$, $g_0\colon \widehat{S}_0 \to G$, $u_1\colon \widehat{S}_1 \to S_1$, and $g_1\colon \widehat{S}_1 \to G$ such that $g_0(s) \in \operatorname{supp}((a_0)_{s,u_0(s)})$ holds for $s \in \widehat{S}_0$ and $g_1(s) \in \operatorname{supp}((a_1)_{u_1(s),s})$ holds for $s \in \widehat{S}_1$. Define a new object $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathsf{B}})$ in $\mathcal{O}^G(X)_A$ by

$$\begin{aligned}
\widehat{S} &:= \widehat{S}_0 \amalg \widehat{S}_1;\\
\widehat{\pi}(s) &:= \begin{cases} g_0(s)^{-1} \cdot \pi_0 \circ u_0(s) & \text{if } s \in \widehat{S}_0;\\ g_1(s) \cdot \pi_1 \circ u_1(s) & \text{if } \ s \in \widehat{S}_1; \end{cases}\\
\widehat{\eta}(s) &:= \eta(s) \quad \text{for } \ s \in \widehat{S};\\
\widehat{\mathsf{B}}(s) &:= \mathsf{B}(s) \quad \text{for } \ s \in \widehat{S}.
\end{aligned}$$

Recall $S^{\mathcal{U}} = \widehat{S}_0 \amalg \widehat{S}_1 = \widehat{S}$. In order to show that $\widehat{\mathcal{B}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic, we want to apply the criterion appearing of Lemma 22.14.

Consider an element $x \in X$ and an open G_x-invariant neighbourhood U of x in X. Since a_0 and a_1 satisfy *continuous control*, we can find an open G_x-invariant neighbourhood U' of x in X with $U' \subseteq U$ and $r' \in \mathbb{N}$ such that for $s \in S, s_0 \in S_0, g_0 \in \operatorname{supp}_G((a_0)_{s,s_0})$ the implication

$$g_0^{-1} \cdot \pi_0(s_0) \in U', \eta(s_0) \geq r' \implies \pi(s) \in U$$

and for $s_1 \in S_1$, $s \in S$, $g_1 \in \operatorname{supp}_G((a_1)_{s_1,s})$ the implication

$$g_1 \pi_1(s_1) \in U', \eta_1(s_1) \geq r' \implies \pi(s) \in U$$

hold. This implies that for $s \in \widehat{S}_0 \amalg \widehat{S}_1$ the implication

$$\widehat{\pi}(s) \in U', \widehat{\eta}(s) \geq r' \implies \pi^{\mathcal{U}}(s) \in U$$

is valid. The proof of the other implication

$$\pi^{\mathcal{U}}(s) \in U', \eta^{\mathcal{U}}(s) \geq r' \implies \widehat{\pi}(s) \in U$$

for $s \in \widehat{S}_0 \amalg \widehat{S}_1$ is analogous and left to the reader. Now Lemma 22.14 implies that $\widehat{\mathcal{B}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

(ii) The constructions appearing in the proof of assertion (i) yield the desired result for $\mathcal{D}^G(i)$ using Lemma 22.22. ◻

Definition 22.24 ($\mathcal{D}^G(X, A)$). Define the additive category $\mathcal{D}^G(X, A)$ to be the quotient of $\mathcal{D}^G(X)$ by the image of $\mathcal{D}^G(i) \colon \mathcal{D}^G(A) \to \mathcal{D}^G(X)$.

Obviously a G-map of G-CW-pairs $f \colon (X, A) \to (Y, B)$ induces a functor of additive categories

$$\mathcal{D}^G(f) \colon \mathcal{D}^G(X, A) \to \mathcal{D}^G(Y, B). \tag{22.25}$$

22.7 The Proof of the Axioms of a G-Homology Theory

The main result of this chapter is

Theorem 22.26 (The algebraic K-groups of $\mathcal{D}^G(X,A)$ yield a G-homology theory). *Let $\mathcal{B}$ be a category with G-support in the sense of Definition 22.1.*

Then we obtain a G-homology theory with values in $\mathbb{Z}$-modules in the sense of Definition 12.1 by the covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups sending (X,A) to $K_*(\mathcal{D}^G(X,A;\mathcal{B}))$.

22.7.1 The Long Exact Sequence of a Pair

Proposition 22.27. *Given a G-CW-pair (X,A), we have the inclusions $i\colon A \to X$ and $j\colon X \to (X,A)$ and obtain a long exact sequence, infinite to both sides and natural in (X,A),*

$$\begin{aligned}\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{D}^G(A)) &\xrightarrow{K_n(\mathcal{D}(i))} K_n(\mathcal{D}^G(X)) \xrightarrow{K_n(\mathcal{D}(j))} K_n(\mathcal{D}^G(X,A)) \\ &\xrightarrow{\partial_n} K_{n-1}(\mathcal{D}^G(A)) \xrightarrow{K_{n-1}(\mathcal{D}(i))} K_{n-1}(\mathcal{D}^G(X)) \\ &\qquad\xrightarrow{K_{n-1}(\mathcal{D}(j))} K_{n-1}(\mathcal{D}^G(X,A)) \xrightarrow{\partial_{n-1}} \cdots .\end{aligned}$$

Proof. This follows from Lemma 22.23 (ii) and Theorem 21.5 (i). $\square$

22.7.2 Some Eilenberg Swindles on $\mathcal{O}^G(X)$

Remark 22.28 (Eilenberg swindles on additive categories defined in terms of controlled topology). Sometimes we want to show that the algebraic K-theory of certain additive categories defined by controlled topology is weakly contractible. This is done in all cases by constructing an Eilenberg swindle. The basic strategy is illustrated for $\mathcal{O}^G(X)$ as follows.

One defines a functor $\mathrm{sh}\colon \mathcal{O}^G(X) \to \mathcal{O}^G(X)$, which shifts one position to the right over $\mathbb{N}$, as follows. It sends an object $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ to the object $\mathrm{sh}(\mathbf{B}) = (\mathrm{sh}(S),\mathrm{sh}(\pi),\mathrm{sh}(\eta),\mathrm{sh}(\mathsf{B}))$ where $\mathrm{sh}(S) = S$, $\mathrm{sh}(\mathsf{B}) = \mathsf{B}$, $\mathrm{sh}(\pi) = \pi$, $\mathrm{sh}(\eta) = \eta + 1$. Roughly speaking, nothing is changed, only the objects are moved one position to the right in the $\mathbb{N}$-direction. (Sometimes one also has to vary π.) One easily checks that $\mathrm{sh}(\mathbf{B})$ satisfies *compact support over* X and *locally finiteness over* $\mathbb{N}$. The definition of $\mathrm{sh}(\phi)$ for morphisms $\phi\colon \mathbf{B} \to \mathbf{B}'$ is the tautological one. Again it is easy to check that $\mathrm{sh}(\phi)$ will again satisfy *finite G-support*, *bounded control over* $\mathbb{N}$, and *continuous control*. Moreover there is an obvious natural equivalence $t\colon \mathrm{id} \xrightarrow{\cong} \mathrm{sh}$ of functors of additive categories $\mathcal{O}^G(X) \to \mathcal{O}^G(X)$.

The basic idea which works in some special cases, is to the define a functor $\mathrm{SH}\colon \mathcal{O}^G(X) \to \mathcal{O}^G(X)$ on objects by $\mathrm{SH}(\mathbf{B}) = \bigoplus_{n=0}^{\infty} \mathrm{sh}^n(\mathbf{B})$. This definition makes indeed sense since $\mathbf{B}$ satisfies *locally finiteness over* $\mathbb{N}$ and hence the set

$$\{(s,n) \in S \times \mathbb{N} \mid \eta(s) \le n, \mathtt{B}(s) \neq 0\} = \coprod_{k=0}^{n} \eta^{-1}(k)$$

is finite. However, the obvious definition on morphisms will not work in general. The conditions *compact support over* X and *locally finiteness over* $\mathbb{N}$ cause no difficulties, whereas conditions *continuous control* is the problem. The reason is that in $\mathrm{SH}(\mathbf{B})$ the objects are moved arbitrary far to the right concerning $\mathbb{N}$ and the continuous control condition becomes more and more restrictive the larger the position with respect to $\mathbb{N}$ is. One example where this problem does not occur is for instance the case $X = \{\bullet\}$ which we will handle in Lemma 22.29. If SH is well-defined, then one obtains the desired natural equivalence using $t\colon \mathrm{id} \xrightarrow{\cong} \mathrm{sh}$ by

$$\mathrm{id} \oplus \mathrm{SH} = \mathrm{sh}^0 \oplus \bigoplus_{n=0}^{\infty} \mathrm{sh}^n \xrightarrow{\cong} \mathrm{sh}^0 \oplus \mathrm{sh}\left(\bigoplus_{n=0}^{\infty} \mathrm{sh}^n\right) \cong \mathrm{sh}^0 \oplus \bigoplus_{n=1}^{\infty} \mathrm{sh}^n = \mathrm{SH}\,.$$

Lemma 22.29. *If $\mathcal{B}$ is a category with G-support, then $\mathcal{O}^G(\{\bullet\})$ is flasque. In particular $\mathbf{K}(\mathcal{O}^G(\{\bullet\}))$ is weakly contractible.*

Proof. The desired Eilenberg swindle described in Remark 22.28 is constructed in detail as follows. Next we define a functor of additive categories

$$\mathrm{SH}\colon \mathcal{O}^G(\{\bullet\}) \to \mathcal{O}^G(\{\bullet\}).$$

For an object $B = (S, \pi, \eta, \mathtt{B})$ in $\mathcal{O}^G(\{\bullet\})$, define $\mathrm{SH}(\mathbf{B})$ by the quadruple $(\mathrm{SH}(S), \mathrm{SH}(\pi), \mathrm{SH}(\eta), \mathrm{SH}(\mathbb{B}))$ where for $s \in S$ and $n \in \mathbb{N}$ we put

$$\begin{aligned}
\mathrm{SH}(S) &= \{(s,n) \in S \times \mathbb{N} \mid \eta(s) \le n\};\\
\mathrm{SH}(\pi)(s,n) &= \pi(s);\\
\mathrm{SH}(\eta)(s,n) &= n;\\
\mathrm{SH}(\mathtt{B})(s,n) &= \mathtt{B}(s).
\end{aligned}$$

Obviously $\mathrm{SH}(\mathbf{B})$ satisfies *compact support over* $\{\bullet\}$. Since $\mathbf{B}$ satisfies *locally finiteness* and $\mathrm{SH}(\eta')^{-1}(n) = \bigcup_{m=0}^{n} \eta^{-1}(m)$ holds for $n \in \mathbb{N}$, $\mathbf{B}'$ satisfies *locally finiteness*.

For two objects $\mathbf{B} = (S, \pi, \eta, \mathtt{B})$ and $\mathbf{B}' = (S', \pi', \eta', \mathtt{B}')$ and a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ given by a collection $\{\phi_{s,s'}\colon \mathtt{B}(s) \to \mathtt{B}'(s') \mid s \in S, s' \in S'\}$, define the morphism $\mathrm{SH}(\phi)\colon \mathrm{SH}(\mathbf{B}) \to \mathrm{SH}(\mathbf{B}')$ by the collection

$$\{\mathrm{SH}(\phi)_{(s,n),(s',n')}\colon \mathtt{B}(s) \to \mathtt{B}'(s') \mid \\ s \in S, s' \in S', n \in \mathbb{N}, n' \in \mathbb{N}, \eta(s) \le n, \eta'(s') \le n'\}$$

for $\mathrm{SH}(\phi)_{(s,n),(s',n')} = \phi_{s,s'}$ if $n - \eta(s) = n' - \eta'(s')$ and $\mathrm{SH}(\phi)_{(s,n),(s',n')} = 0$ otherwise.

Since ϕ satisfies *finite G-support*, the same is true for $\mathrm{SH}(\phi)$. Since ϕ satisfies *bounded control over* $\mathbb{N}$, we can find a natural number N such that $\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \leq N$ holds for $s \in S$ and $s' \in S'$. Now consider $(s,n) \in \mathrm{SH}(S)$ and $(s',n') \in \mathrm{SH}(S')$ with $\mathrm{SH}(\phi)_{(s,n),(s',n')} \neq 0$. Since then $n - \eta(s) = n' - \eta'(s')$ and $\phi_{s,s'} \neq 0$ hold, we get

$$|\,\mathrm{SH}(\eta)(s,n) - \mathrm{SH}(\eta')(s',n')| = |n - n'| = |\eta(s) - \eta'(s)| \leq N.$$

Hence $\mathrm{SH}(\phi)$ satisfies *bounded control over* $\mathbb{N}$. Obviously $\mathrm{SH}(\phi)$ satisfies *continuous control* since we are working over $\{\bullet\}$. One easily checks that SH is a well-defined functor of additive categories.

It remains to construct a natural equivalence $T\colon \mathrm{id} \oplus \mathrm{SH} \xrightarrow{\cong} \mathrm{SH}$ of functors of additive categories. We have to define for any object $\mathbf{B} = (S, \pi, \eta, \mathtt{B})$ an isomorphism $T(\mathbf{B})\colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \xrightarrow{\cong} \mathrm{SH}(\mathbf{B})$. We obtain a bijection of sets

$$u\colon S \coprod \mathrm{SH}(S) \xrightarrow{\cong} \mathrm{SH}(S)$$

by sending $s \in S$ to $(s, \eta(s))$ and $(s,n) \in \mathrm{SH}(S)$ to $(s, n+1)$. Note that for $s \in S$ we have $\mathtt{B}(s) = \mathrm{SH}(\mathtt{B}) \circ u(s)$ and for $(s,n) \in \mathrm{SH}(S)$ we have $\mathrm{SH}(\mathtt{B}(s,n)) = \mathtt{B}(s) = \mathrm{SH} \circ u(s,n)$. Now we can define $T(\mathbf{B})_{t,t'}$ for $t \in S \coprod \mathrm{SH}(S)$ and $t' \in \mathrm{SH}(S)$ to be $\mathrm{id}_{\mathtt{B}(t')}$ if $u(t) = t'$ and to be 0 if $u(t) \neq t'$. We leave the elementary proof to the reader to check that $T(\mathbf{B})$ is a well-defined isomorphism in $\mathcal{O}^G(\{\bullet\})$ which is natural in $\mathcal{B}$ and hence defines the desired natural equivalence $T\colon \mathrm{id} \oplus \mathrm{SH} \xrightarrow{\cong} \mathrm{SH}$.

Thus we have defined an Eilenberg swindle (SH, T) on $\mathcal{O}^G(\{\bullet\})$. The weak contractibility of $\mathbf{K}(\mathcal{O}^G(\{\bullet\}))$ follows from Theorem 6.37 (iii). $\square$

The next result generalizes Lemma 22.29. The basic idea of the proof is the same but becomes much more complicated since now we have to deal with the condition *continuous control*.

Lemma 22.30. *Let X be a G-CW-complex which is G-contractible, i.e, G-homotopy equivalent to $\{\bullet\}$.*

Then $\mathbf{K}(\mathcal{O}^G(X))$ is weakly contractible.

Proof. Denote by $\mathrm{cone}(X)$ the cone of X. As X is G-contractible, there are G-maps $i\colon X \to \mathrm{cone}(X)$ and $r\colon \mathrm{cone}(X) \to X$ with $r \circ i = \mathrm{id}_X$. Hence the composite of maps of spectra

$$\mathbf{K}(\mathcal{O}^G(X)) \xrightarrow{\mathbf{K}(\mathcal{O}^G(i))} \mathbf{K}(\mathcal{O}^G(\mathrm{cone}(X))) \xrightarrow{\mathbf{K}(\mathcal{O}^G(r))} \mathbf{K}(\mathcal{O}^G(X))$$

is the identity. Therefore it suffices to show that $\mathbf{K}(\mathrm{cone}(X))$ is weakly contractible.

We explain the basic idea of the proof before we give the details. In the construction of an Eilenberg swindle for a given object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ one assigns to $\mathbf{B}$ a new object $\mathrm{SH}(\mathbf{B})$ where one adds for $s \in S$ a copy of $\mathsf{B}(s)$ at n for each natural number $n \geq \eta(s)$. The problem is to specify where this copy over n sits in $\mathrm{cone}(X)$, i.e., to define the image of this object under π^{SH}. The idea is to move the copies of the object $\mathsf{B}(s)$ with the right speed to the cone point. This has to be done fast enough so that the obvious definition of $\mathrm{SH}(\phi)$ for a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ still defines *continuous control* but slow enough so that the desired obvious transformation $T(\mathbf{B})\colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \xrightarrow{\cong} \mathrm{SH}(\mathbf{B})$ satisfies *continuous control*. This will lead to the properties of the function ρ below.

Recall that $\mathrm{cone}(X)$ is defined as the G-pushout, where $i_0\colon X \to X \times [0,1]$ sends x to $(x, 0)$

$$\begin{array}{ccc} X & \xrightarrow{i_0} & X \times [0,1] \\ \downarrow & & \downarrow \mathrm{pr} \\ \{\bullet\} & \xrightarrow[\overline{i_0}]{} & \mathrm{cone}(X) \end{array}$$

In the sequel we write $[x,t] = \mathrm{pr}(x,t)$ for $(x,t) \in X \times [0,1]$. For $t' \in [0,1]$ we define $t' \cdot [x,t] := [x, t't]$. Denote by $*$ the cone point $[x,0]$ for any $x \in X$, or, equivalently, $* = \overline{i_0}(\{\bullet\})$. For $z = [x,t] \in \mathrm{cone}(X)$ we denote $z_I = t$. For $z = [x,t] \in \mathrm{cone}(X) \setminus \{*\}$ we denote $z_X = x$. In particular $\mathrm{pr}(x,t)_X = x$ for $x \in X$, $t \in (0,1]$, and $\mathrm{pr}(x,t)_I = t$ for $x \in X$ and $t \in [0,1]$.

Next we define a functor of additive categories

$$\mathrm{SH}\colon \mathcal{O}^G(\mathrm{cone}(X)) \to \mathcal{O}^G(\mathrm{cone}(X)).$$

For this purpose we choose a function $\rho\colon \mathbb{N} \times \mathbb{N} \to (0,1]$ with the following three properties.

- We have

  (22.31) $$\lim_{m \to \infty} \rho(m, 0) = 1;$$

- For every $m \in \mathbb{N}$, we have

  (22.32) $$\lim_{n \to \infty} \rho(m, n) = 0;$$

- For every $N \in \mathbb{N}$ and $\mu > 0$, there is $M \in \mathbb{N}$ such that for all $m, m', n \in \mathbb{N}$ the implication

  (22.33) $$m \geq M, |m - m'| \leq N \implies |\rho(m,n) - \rho(m',n)| < \mu$$

  holds;
- For every $\mu > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ the implication

$$n \geq N, m \leq n \implies 1 - \mu \leq \frac{\rho(m, n+1-m)}{\rho(m, n-m)} \leq 1 \tag{22.34}$$

holds.

If $(a_k)_{k \in \mathbb{N}}$ is any sequence of elements in $(0,1]$ such that $\lim_{k \to \infty} a_k = 0$ and $\sum_{k=0}^{\infty} a_k = \infty$, then we can take $\rho(m,n) := \exp(-\sum_{k=m}^{m+n} a_k)$. An example for $(a_k)_{k \in \mathbb{N}}$ is $a_k = 1/k$.

The functor SH sends an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ to the object $\mathrm{SH}(\mathbf{B}) = (\mathrm{SH}(S), \mathrm{SH}(\pi), \mathrm{SH}(\eta), \mathrm{SH}(\mathsf{B}))$ where for $s \in S$ we put

$$\mathrm{SH}(S) = \{(s,n) \mid s \in S, n \in \mathbb{N}, \eta(s) \leq n\}$$

and define for $(s,n) \in \mathrm{SH}(S)$

$$\begin{aligned} \mathrm{SH}(\pi)(s,n) &= \rho(\eta(s), n - \eta(s)) \cdot \pi(s); \\ \mathrm{SH}(\eta)(s,n) &= n; \\ \mathrm{SH}(\mathsf{B})(s,n) &= \mathsf{B}(s). \end{aligned}$$

Since $\mathbf{B}$ satisfies *compact support over* X, there exists a compact subset C of $\operatorname{cone}(X)$ with $\operatorname{im}(\pi) \subseteq C$. This implies

$$\operatorname{im}(\mathrm{SH}(\pi)) \subseteq [0,1] \cdot C := \{t \cdot c \mid t \in [0,1], c \in C\}.$$

Since $[0,1] \cdot C$ is compact, $\mathrm{SH}(\mathbf{B})$ satisfies *compact support over* X.

Since $\mathbf{B}$ satisfies *locally finiteness over* $\mathbb{N}$ and $\mathrm{SH}(\eta)^{-1}(m) = \coprod_{n=0}^{m} \eta^{-1}(n)$ holds, $\mathrm{SH}(\mathbf{B})$ satisfies *locally finiteness over* $\mathbb{N}$.

Consider a morphism $\phi \colon \mathbf{B} = (S, \pi, \eta, \mathsf{B}) \to \mathbf{B}' = (S', \pi', \eta, \mathsf{B}')$ given by the collection $\{\phi_{s,s'} \colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$. Define $\mathrm{SH}(\phi) \colon \mathrm{SH}(\mathbf{B}) \to \mathrm{SH}(\mathbf{B}')$ by

$$\mathrm{SH}(\phi)_{(s,n),(s',n')} = \begin{cases} \phi_{s,s'} & \text{if } n' - \eta'(s') = n - \eta(s); \\ 0 & \text{otherwise,} \end{cases}$$

for $(s,n) \in \mathrm{SH}(\mathbf{B})$ and $(s',n') \in \mathrm{SH}(\mathbf{B}')$.

Since ϕ satisfies *finite G-support*, there exists a finite subset $F \subseteq G$ such that $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$ holds for every $s \in S$, and $s' \in S'$. This implies $\operatorname{supp}_G(\mathrm{SH}(\phi)_{(s,n),(s',n')}) \subseteq F$ for every $(s,n) \in \mathrm{SH}(S)$, and $(s',n') \in \mathrm{SH}(S')$. Hence $\mathrm{SH}(\phi)$ satisfies *finite G-support*.

Since ϕ satisfies *bounded control over* $\mathbb{N}$, there exists a natural number N with $|\eta(s) - \eta'(s')| \leq N$ for all $s \in S$ and $s' \in S'$ with $\phi_{s,s'} \neq 0$. Consider $(s,n) \in \mathrm{SH}(\mathbf{B})$ and $(s',n') \in \mathrm{SH}(\mathbf{B}')$ with $\mathrm{SH}(\phi)_{(s,n),(s',n')} \neq 0$. Then $n' - \eta'(s') = n - \eta(s)$ and $|\eta(s) - \eta'(s')| \leq N$. This implies

$$|\mathrm{SH}(\eta)(s,n) - \mathrm{SH}(\eta')(s',n')| = |n - n'| = |\eta(s) - \eta'(s')| \leq N. \tag{22.35}$$

Hence $\mathrm{SH}(\phi)$ satisfies *bounded control over* $\mathbb{N}$.

The hard part is to show that $\mathrm{SH}(\phi)$ satisfies *continuous control*. We only deal with the implication (22.5). The proof for the other implication (22.6) is completely analogous.

Consider $[x,t] \in \mathrm{cone}(X)$ and an open $G_{[x,t]}$-invariant neighborhood U of $[x,t]$ in $\mathrm{cone}(X)$. We have to find an open $G_{[x,t]}$-invariant neighborhood U' of $[x,t]$ in $\mathrm{cone}(X)$ satisfying $U' \subseteq U$ and a natural number r' such that for all $(s,n) \in \mathrm{SH}(S)$, $(s',n') \in \mathrm{SH}(S')$, and $g \in \mathrm{supp}_G(\mathrm{SH}(\phi)_{(s,n),(s',n')})$ the implication

$$g \cdot \mathrm{SH}(\pi)(s,n) \in U', \mathrm{SH}(\eta)(s,n) \geq r' \implies \mathrm{SH}(\pi')(s',n') \in U \tag{22.36}$$

holds.

We begin with the case where $[x,t]$ is different from the cone point $*$, or, equivalently $0 < t \leq 1$. In the sequel we denote for $t \in (0,1]$ and $\epsilon > 0$ by $I_\epsilon(t)$ the open neighborhood of t in $[0,1]$ given by $(t-\epsilon, t+\epsilon) \cap [0,1]$.

Choose an open G_x-invariant neighbourhood V_0 of x in X and $\epsilon > 0$ satisfying

$$\mathrm{pr}(V_0 \times I_\epsilon(t)) \quad \subseteq \quad U; \tag{22.37}$$

$$\epsilon \leq t/2. \tag{22.38}$$

Since ϕ satisfies *continuous control*, we can find for $t' \in [t/2,1]$ an open G_x-invariant neighborhood $V'[t']$ of x, a real number $\delta'[t'] > 0$, and $r'[t'] \in \mathbb{N}$ such that for $s \in S$, $s' \in S'$, and $g \in \mathrm{supp}_G(\phi_{s,s'})$ the implication

$$\begin{aligned} g\pi(s)_X \in V'[t'], \pi(s)_I \in I_{\delta'[t']}(t'), \eta(s) \geq r[t'] \\ \implies \pi'(s')_X \in V_0, \pi'(s')_I \in I_{\epsilon/8}(t') \end{aligned} \tag{22.39}$$

holds. Obviously we can arrange $0 < \delta'[t'] < \epsilon/8$. Since $[t/2,1]$ is compact, we can find finitely many elements $t'_1, t'_2, \ldots, t'_l$ in $[t/2,1]$ such that for each $t' \in [t/2,0]$ there exists an element $i[t'] \in \{1,2\ldots,l\}$ satisfying $t' \in I_{\delta'[t'_i]}(t_i)$. Put

$$\begin{aligned} V' &= \bigcap_{i=1}^{l} V'[t'_i]; \\ r'_0 &= \max\{r'[t'_i] \mid i = 1,2,\ldots,l\}. \end{aligned}$$

Then V' is an open G_x-invariant neighbourhood of x in X. Moreover, for $s \in S$, $s' \in S'$, $g \in \mathrm{supp}_G(\phi_{s,s'})$, and $t' \in [t/2,1]$ the implication

$$\begin{aligned} g\pi(s)_X \in V', \pi(s)_I \geq t/2, \eta(s) \geq r'_0 \\ \implies \pi'(s')_X \in V_0, |\pi'(s')_I - \pi(s)_I| < \epsilon/4 \end{aligned} \tag{22.40}$$

holds, since $\pi(s)_I \geq t/2$ implies the existence of $i \in \{1,2\ldots,l\}$ satisfying $\pi(s)_X \in V'[t_i]$ and $\pi(s)_I \in I_{\delta'[t'_i]}(t_i)$, we conclude $\pi'(s')_X \in V_0$ and $\pi'(s')_I \in$

$I_{\epsilon/8}(t_i)$ from (22.39), and now one can apply the triangle inequality to $\pi(s)_I$, $\pi'(s')_I$, and t_i using $\delta'[t'_i] + \epsilon/8 < \epsilon/8 + \epsilon/8 = \epsilon/4$.

Let N be the number appearing in (22.35). Choose a natural number M such that (22.33) holds if we put $\mu = \epsilon/2$. Since $\lim_{n\to\infty} \rho(m,n) = 0$ holds for $m \in \{0, 1, \dots, \max\{r'_0, M\}\}$ by (22.32), we can find a natural number r' satisfying $r' \geq \max\{r'_0, M\}$ such that for every $m, n \in \mathbb{N}$ the implication

$$m \leq \max\{r'_0, M\}, n \geq r' - \max\{r'_0, M\} \implies \rho(m,n) < t/2 \tag{22.41}$$

holds. Next we show that the desired implication (22.36) holds if we put $U' := V' \times I_{\epsilon/4}(t)$ and use the number r' above.

Consider $(s,n) \in \mathrm{SH}(S)$, $(s',n') \in \mathrm{SH}(S')$, and $g \in \mathrm{supp}_G(\phi_{s,s'})$ satisfying $\mathrm{SH}(\pi)(s,n) \in U'$ and $\eta(s,n) := n \geq r'$. Since $\mathrm{SH}(\pi)(s,n) \in U'$ implies that $\mathrm{SH}(\pi)(s,n)_I = \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I$ belongs to $I_{\epsilon/4}(t)$, we get

$$\rho(\eta(s), n-\eta(s)) \geq \rho(\eta(s), n-\eta(s)) \cdot \pi(s)_I \geq t - \epsilon/4 \overset{(22.38)}{\geq} t/2. \tag{22.42}$$

We conclude from (22.41) and (22.42) that $\eta(s) > \max\{r'_0, M\}$ holds. In particular we get $\eta(s) \geq r'_0$ and $\eta(s) \geq M$.

Since $n' - \eta'(s') = n - \eta(s)$, we conclude from (22.33) and (22.35)

$$|\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \leq \epsilon/2. \tag{22.43}$$

We have $\mathrm{SH}(\pi)(s,n)_I = \rho(\eta(s), n-\eta(s)) \cdot \pi(s)_I \in I_{\epsilon/4}(t)$. This implies

$$\pi(s)_I \geq \rho(\eta(s), n-\eta(s)) \cdot \pi(s)_I \geq t - \epsilon/4 \overset{(22.38)}{\geq} t/2.$$

Since $g \cdot \mathrm{SH}(\pi)(s)_X = g\pi(s)_X \in V'$ and $\pi(s)_I \geq t/2$ hold, we get

$$\pi'(s')_X \in V_0; \tag{22.44}$$

$$|\pi'(s')_I - \pi(s)_I| < \epsilon/4, \tag{22.45}$$

from (22.40). We estimate

$$\begin{aligned}
&|\operatorname{SH}(\pi')(s')_I - t| \\
&\quad \leq \quad |\operatorname{SH}(\pi')(s')_I - \operatorname{SH}(\pi)(s)_I| + |\operatorname{SH}(\pi)(s)_I - t| \\
&\quad = \quad |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s')_I - \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I| \\
&\qquad\qquad + |\operatorname{SH}(\pi)(s)_I - t| \\
&\quad \leq \quad |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s')_I - \rho(\eta'(s'), n' - \eta'(s')) \cdot \pi(s)_I| \\
&\qquad\qquad + |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi(s)_I - \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I| \\
&\qquad\qquad + |\operatorname{SH}(\pi)(s)_I - t| \\
&\quad = \quad |\rho(\eta'(s'), n' - \eta'(s'))| \cdot |\pi'(s')_I - \pi(s)_I| \\
&\qquad\qquad + |\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \cdot |\pi(s)_I| + \epsilon/4 \\
&\quad \leq \quad |\pi'(s')_I - \pi(s)_I| + |\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| + \epsilon/4 \\
&\quad \overset{(22.43),\ (22.45)}{\leq} \quad \epsilon/4 + \epsilon/2 + \epsilon/4 \\
&\quad = \quad \epsilon.
\end{aligned}$$

This implies together with (22.37) and (22.44) that $\operatorname{SH}(\pi')(s') \in U$ holds. This finishes the proof of the implication (22.36) in the case $[x,t] \neq *$.

Next we show the implication (22.36) in the case $[x,t] = *$. Consider an open G-invariant neighborhood U of $*$. We have to find an open G-invariant neighbourhood U' of $*$ and a natural number r' such that for all $(s,n) \in \operatorname{SH}(S)$, $(s',n') \in \operatorname{SH}(S')$ and $g \in \operatorname{supp}_G(\operatorname{SH}(\phi)_{(s,n),(s',n')})$ the implication

$$g \cdot \operatorname{SH}(\pi)(s,n) \in U', \operatorname{SH}(\eta)(s,n) \geq r' \implies \operatorname{SH}(\pi')(s',n') \in U \tag{22.46}$$

holds.

For $\epsilon > 0$, we define V_ϵ to be the open G-invariant neighborhood of $*$ in $\operatorname{cone}(X)$ given by

$$V_\epsilon = \{[x,t] \mid x \in X, t < \epsilon\}.$$

Since $\mathbf{B}'$ satisfies *compact support over* $\operatorname{cone}(X)$, the subset $[0,1] \cdot \operatorname{im}(\pi')$ of $\operatorname{cone}(X)$ is compact. Hence there exists an $\epsilon > 0$ satisfying

$$V_\epsilon \cap [0,1] \cdot \operatorname{im}(\pi') \subseteq U.$$

Since $\operatorname{im}(\operatorname{SH}(\pi')) \subseteq [0,1] \cdot \operatorname{im}(\pi')$ holds, it suffices to prove (22.46) in the special case $U = V_\epsilon$.

Since ϕ satisfies *continuous control*, there exists an open G-invariant neighborhood U'_0 of $*$ in $\operatorname{cone}(X)$ and a natural number r'_2 such that for all $s \in S$, $s' \in S'$ and $g \in \operatorname{supp}_G(\phi_{s,s'})$ the implication

$$g\pi(s) \in U'_0, \eta(s) \geq r'_2 \implies \pi'(s') \in V_\epsilon \tag{22.47}$$

holds. Since $\mathbf{B}$ satisfies *compact support over* $\operatorname{cone}(X)$, there exists $\delta > 0$ satisfying

$$V_\delta \cap [0,1] \cdot \operatorname{im}(\pi) \subseteq U'_0.$$

We get from (22.47) the implication

$$\operatorname{supp}_G(\phi_{s,s'}) \neq \emptyset, \pi(s)_I < \delta, \eta(s) \geq r_2' \implies \pi'(s') \in V_\epsilon. \tag{22.48}$$

Let N be the number appearing in (22.35). Choose a natural number M such that (22.33) holds if we put $\mu = \epsilon/2$. Since $\lim_{n\to\infty} \rho(m,n) = 0$ holds for $m \in \{0, 1, \ldots, N + \max\{r', M\}\}$ by (22.32), we can find a natural number r' satisfying $r' \geq N + \max\{r_2', M\}$ such that for every $m, n \in \mathbb{N}$ the implication

$$m \leq N + \max\{r_2', M\}, n \geq r' - N - \max\{r_2', M\} \implies \rho(m,n) < \epsilon/2 \tag{22.49}$$

holds.

Next we want to prove the implication (22.46) in the special case $U = V_\epsilon$, were we take r' to be the natural number above and $U' = V_{\epsilon\delta/2}$. Consider $(s,n) \in \mathrm{SH}(s)$, $(s',n') \in \mathrm{SH}(S')$ and $g \in \operatorname{supp}_G(\mathrm{SH}(\phi)_{(s,n),(s',n')})$ satisfying $\mathrm{SH}(\pi)(s,n)_I \leq \epsilon\delta/2$ and $\mathrm{SH}(\eta)(s,n) := n \geq r'$. We have to show $\pi'(s')_I \leq \epsilon$.

If $\rho(\eta'(s'), n' - \eta'(s')) < \epsilon$ holds, then we get

$$\mathrm{SH}(\pi')(s') = \rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s') \in V_\epsilon.$$

Hence we can assume without loss of generality that

$$\rho(\eta'(s'), n' - \eta'(s')) \geq \epsilon. \tag{22.50}$$

We conclude from (22.49) and (22.50) that $\eta'(s') > N + \max\{r_2', M\}$ holds. In particular we have $\eta'(s') \geq N + r_2'$ and $\eta'(s') \geq M$. Since $n' - \eta'(s') = n - \eta(s)$ holds, we conclude from (22.33) that

$$|\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \leq \epsilon/2$$

holds. This implies together with (22.50)

$$\rho(\eta(s), n - \eta(s)) \geq \epsilon/2. \tag{22.51}$$

Hence we get

$$\pi(s)_I = \frac{\mathrm{SH}(\pi)(s)_I}{\rho(\eta(s), n - \eta(s))} \overset{(22.51)}{<} \frac{2 \cdot \mathrm{SH}(\pi)(s)_I}{\epsilon} \leq \frac{2 \cdot \delta \cdot \epsilon/2}{\epsilon} = \delta$$

Since $\eta'(s') \geq N + r_2'$, we conclude $\eta(s) \geq r_2'$ from (22.35). Finally (22.48) implies $\pi'(s') \in V_\epsilon$.

This finishes the poof that $\mathrm{SH}(\phi)$ is a well-defined morphism. Now one easily checks that SH is a well-defined functor of additive categories.

Next we define a natural equivalence of covariant functors of additive categories $\mathcal{O}^G(\operatorname{cone}(X)) \to \mathcal{O}^G(\operatorname{cone}(X))$

$$T \colon \operatorname{id} \oplus \mathrm{SH} \xrightarrow{\cong} \mathrm{SH}.$$

We have to define for an object $\mathbf{B} = (S, \pi, \eta, \mathrm{B})$ an isomorphism $T(\mathbf{B})\colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \xrightarrow{\cong} \mathrm{SH}(\mathbf{B})$ in $\mathcal{O}^G(\mathrm{cone}(X))$. Define a bijection

$$u\colon S \coprod \mathrm{SH}(S) \xrightarrow{\cong} \mathrm{SH}(S)$$

by sending $s \in S$ to $(s, \eta(s))$ and $(s, n) \in \mathrm{SH}(S)$ to $(s, n+1)$. For $z \in S \coprod \mathrm{SH}(S)$ and $(s, n) \in \mathrm{SH}(S)$ define $T(\mathbf{B})_{z,(s,n)}$ by $\mathrm{id}_{\mathrm{B}(s)}$ for $(s, n) = u(z)$ and by 0 otherwise. Note that $\mathrm{supp}_G(T(B)_{r,(s,n)})$ is empty or $\{e\}$. Obviously $T(\mathbf{B})$ satisfies *finite G-support* and *bounded control over* $\mathbb{N}$, whereas *continuous control* is proved as follows. We only deal with the implication (22.5). The proof for the other implication (22.6) is completely analogous.

Consider an element $[x, t] \in \mathrm{cone}(X)$ and a $G_{[x,t]}$-invariant neighbourhood U of $[x, t]$ in $\mathrm{cone}(X)$. It remains to construct a $G_{[x,t]}$-invariant neighbourhood U' of $[x, t]$ in $\mathrm{cone}(X)$ with $U' \subseteq U$ and a natural number r' such that for $s \in S$ the implication

$$\pi(s) \in U', \eta(s) \geq r' \implies \mathrm{SH}(\pi)(s, \eta(s)) \in U \tag{22.52}$$

and for $(s, n) \in \mathrm{SH}(S)$ the implication

$$\mathrm{SH}(\pi)(s, n) \in U', \mathrm{SH}(\eta)(s, n) \geq r' \implies \mathrm{SH}(\pi)(s, n+1) \in U \tag{22.53}$$

hold.

Next we show that we can choose $\mu \in (0, 1]$ and an open a $G_{[x,t]}$-invariant neighbourhood U' of $[x, t]$ in $\mathrm{cone}(X)$ satisfying

$$t' \cdot U' \subseteq U \text{ for all } t' \in [1 - \mu, 1]. \tag{22.54}$$

We first consider the case $[x, t] = *$. Recall that $\mathrm{pr}\colon X \times [0, 1] \to \mathrm{cone}(X)$ is the obvious projection. Let $p\colon X \to X/G$ be the canonical projection. We have $X \times \{0\} \subseteq \mathrm{pr}^{-1}(U) \subseteq X \times [0, 1]$ as $* \in U$. This implies

$$X/G \times \{0\} \subseteq p(\mathrm{pr}^{-1}(U)) \subseteq X/G \times [0, 1].$$

Since X/G is a CW-complex and hence paracompact, see [731], and $p(\mathrm{pr}^{-1}(U))$ is open, we can find a continuous map $\epsilon\colon X/G \to (0, 1)$ such that $\{(xG, t) \mid xG \in X/G, t < \epsilon(xG)\}$ is contained in $p(\mathrm{pr}^{-1}(U))$. Define

$$U' = \mathrm{pr}\big(\{(x, t) \mid x \in X, t < \epsilon \circ p(x)\}\big).$$

This is an open G-invariant neighborhood of $*$ in $\mathrm{cone}(X)$ satisfying $U' \subseteq U$ and $[0, 1] \cdot U' = U'$. Hence we choose for μ any value in $(0, 1]$.

Next we consider the case $[x, t] \neq *$, or, equivalently, $t > 0$. Let $p\colon X \to X/G_x$ be the projection. Then $\mathrm{pr}^{-1}(U)$ is an open G_x-invariant neighbourhood of $(x, t) \in X \times [0, 1]$ and $p(\mathrm{pr}^{-1}(U))$ is an open neighbourhood of $(p(x), t)$ in $X/G_x \times [0, 1]$. Choose an open neighborhood V' of $p(x)$ in X/G_x

and $\epsilon \in \mathbb{R}$ with $0 < \epsilon < t/2$ such that $V' \times (t-\epsilon, t+\epsilon)$ is contained in $p(\mathrm{pr}^{-1}(U))$. Put $V = p^{-1}(V')$. Then V is an open G_x-invariant neighborhood of x such that $V \times (t-\epsilon, t+\epsilon)$ is contained in $\mathrm{pr}^{-1}(U)$. Choose $\mu \in (0,1]$ such that $(1-\mu)\cdot(t-\epsilon/2) > t-\epsilon$ holds. Then $t't'' \in (t-\epsilon, t+\epsilon)$ holds for $t' \in [1-\mu, 1]$ and $t'' \in (t-\epsilon/2, t+\epsilon/2)$. Put

$$U' = \mathrm{pr}\big((V \times (t-\epsilon/2, t+\epsilon/2)\big).$$

This is an open G_x-invariant neighbourhood of $[x,t]$ satisfying(22.54). This finishes the proof that we can choose $\mu \in (0,1]$ and an open a $G_{[x,t]}$-invariant neighbourhood U' of $[x,t]$ in $\mathrm{cone}(X)$ satisfying (22.54).

Because of (22.31) and (22.34) we can choose a natural number r' such that for all $m \in \mathbb{N}$ with $m \geq r'$ we have

$$1 - \mu \leq \rho(m,0) \leq 1 \tag{22.55}$$

and for all $m, n \in \mathbb{N}$ with $m \leq n$ and $n \geq r'$ we have

$$1 - \mu \leq \frac{\rho(m, n+1-m)}{\rho(m, n-m)} \leq 1. \tag{22.56}$$

Now (22.52) follows from (22.54) and (22.55) since $\mathrm{SH}(\pi)(s, \eta(s)) = \rho(\eta(s), 0)\cdot \pi(s)$ holds. Moreover, (22.53) follows from (22.54) and (22.56) since we have $\mathrm{SH}(\eta)(s,n) = n$ and $\mathrm{SH}(\pi)(s, n+1) = \frac{\rho(\eta(s), n+1-\eta(s))}{\rho(\eta(s), n-\eta(s))} \cdot \mathrm{SH}(\pi)(s,n)$.

One easily checks that $T(\mathbf{B})$ is an isomorphism and the collection of the $T(\mathbf{B})$-s fit together to define the desired natural equivalence T.

Thus we have defined an Eilenberg swindle (SH, T) on $\mathcal{O}^G(\mathrm{cone}(X))$. The weak contractibility of $\mathbf{K}(\mathcal{O}^G(\mathrm{cone}(X)))$ follows from Theorem 6.37 (iii). This finishes the proof of Lemma 22.30. $\square$

22.7.3 Excision and *G*-Homotopy Invariance

Lemma 22.57. *Let (X, A) be a G-CW-pair and let $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ be an object in $\mathcal{O}^G(X)$. Choose a nested sequence of open G-invariant sets*

$$X \supseteq V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \supseteq A$$

together with a G-map $\rho\colon V_0 \to A$ such that $\bigcap_{n\geq 0} \overline{V_n} = A$ and $\rho|_A = A$ hold. Fix a non-decreasing function $\omega\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} \omega(n) = \infty$ and a natural number $w \in \mathbb{N}$. Define new objects $\mathbf{B}^{\omega,w} = (S^{\omega,w}, \pi^{\omega,w}, \eta^{\omega,w}, \mathsf{B}^{\omega,w})$ and $\mathbf{B}^{\perp} = (S^{\perp}, \pi^{\perp}, \eta^{\perp}, \mathsf{B}^{\perp})$ by

$$\begin{aligned} S^{\omega,w} &:= \{s \in S \mid \eta(s) < w \text{ or } \pi(s) \in V_{\omega\circ\eta(s)}\};\\ S^{\perp} &= S \setminus S^{\omega,w}, \end{aligned}$$

and by defining $\pi^{\omega,w}$, $\eta^{\omega,w}$, $\mathsf{B}^{\omega,w}$, $\pi^{\perp}$, $\eta^{\perp}$, *and* $\mathsf{B}^{\perp}$ *by restricting* π, η, *and* B.

(i) The desired sequences $(V_n)_{n\geq 0}$ *and the* G*-map* ρ *exist;*

(ii) There are obvious morphisms $i^{\omega,w}\colon \mathbf{B}^{\omega,w} \to \mathbf{B}$ *and* $i^{\perp}\colon \mathbf{B}^{\perp} \to \mathbf{B}$ *such that* $i^{\omega,w} \oplus i^{\perp}\colon \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \xrightarrow{\cong} \mathbf{B}$ *is an isomorphism, and* $\operatorname{im}(\pi^{\perp}) \subseteq X \setminus A$*;*

(iii) There is an object $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbb{B}})$ *in* $\mathcal{O}^G(A)$ *such that* $\mathbf{B}^{\omega,w}$ *and* $\widehat{\mathbf{B}}$ *are isomorphic in* $\mathcal{O}^G(X)$*;*

(iv) Consider an object $\mathbf{B'} = (S', \pi', \eta', \mathsf{B}')$ *in* $\mathcal{O}^G(A)$ *and a morphism* $\phi\colon \mathbf{B'} \to \mathbf{B}$ *in* $\mathcal{O}^G(X)$*. Then we can find* (ω, w) *and a morphism* $\phi'\colon \mathbf{B'} \to \mathbf{B}^{\omega,w}$ *such that* ϕ *factorizes as*

$$\phi\colon \mathbf{B'} \xrightarrow{\phi'} \mathbf{B}^{\omega,w} \xrightarrow{i^{\omega,w}} \mathbf{B}.$$

Proof. (i) The inclusion $A \to X$ is a G-cofibration, or, equivalently, a G-NDR-pair. The proofs of these facts in the non-equivariant case carry over to the equivariant case. This implies assertion (i). For some information and relevant references we refer for instance to [629, Chapter 1]. The basic ideas of the proof can also be derived from the construction appearing in the proof of [672, Theorem 7.1]. The basic idea is to use the retraction $r\colon D^n \setminus \{0\} \to S^{n-1}$ given by the radial projection and the continuous function $u\colon D^n \to [0,1]$ given by the Euclidean norm which obviously satisfies $u^{-1}(1) = S^{n-1}$ and $u^{-1}(0,1]) = D^n \setminus \{0\}$.

(ii) This is obvious.

(iii) We define $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbb{B}})$ by

$$\begin{aligned}
\widehat{S} &:= S^{\omega,w};\\
\widehat{\pi}(s) &:= \begin{cases} a_0 & \text{if } \eta(s) < w;\\ \rho \circ \pi(s) & \text{if } \eta(s) \geq w; \end{cases}\\
\widehat{\eta}(s) &:= \eta(s);\\
\widehat{\mathsf{B}}(s) &:= \mathsf{B}(s),
\end{aligned}$$

where $s \in S^{\omega,w}$ and a_0 is some point in A. In order to show that $\mathbf{B}^{\omega,w}$ and $\widehat{\mathbf{B}}$ are isomorphic in $\mathcal{O}^G(X)$, we check the criterion appearing in Lemma 22.14.

So consider $x \in X$ and an open G_x-invariant neighborhood U of x in X. Since $\mathbf{B}$ satisfies compact support over X, we can find a compact subset $C \subseteq X$ such that $\operatorname{im}(\pi) \subseteq C$ holds. Choose an open G_x-invariant neighbourhood U_0' of $x \in X$ with $\overline{U_0'} \subseteq U$. Next we show that there exists a natural number r_0' satisfying the implication

$$y \in C, \rho(y) \in U_0', y \in V_{\omega(r_0')} \implies y \in U. \tag{22.58}$$

Suppose that this is not the case. Since $V_m \subseteq V_n$ holds for $m \geq n$, and $\lim_{n\to\infty} \omega(n) = \infty$, we can find a sequence $(y_n)_{n\geq 0}$ of elements in C such that

$\rho(y_n) \in U_0'$, $y_n \in V_n$, and $y_n \notin U$ holds for $n \geq 0$. Since C is compact, there is a strictly monotone increasing function $u\colon \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} u(n) = \infty$ and an element $y \in C$ satisfying $\lim_{n\to\infty} y_{u(n)} = y$. Since for each natural number n we have $y_{u(m)} \in \overline{V_{u(n)}}$ for $m \geq n$, we get $y \in \overline{V_{u(n)}}$ for every $n \geq 0$. This implies $y \in \bigcap_{n\geq 0} \overline{V_{u(n)}} = A$ and hence $\rho(y) = y$. From $\lim_{n\to\infty} y_{u(n)} = y$ we conclude $\lim_{n\to\infty} \rho(y_{u(n)}) = \rho(y) = y$. Since $\rho(y_{u(n)}) \in \overline{U_0'}$ for $n \geq 0$, we conclude $y \in \overline{U_0'}$ and hence $y \in U$. Since $\lim_{n\to\infty} y_{u(n)} = y$ holds, there exists a natural number n_0 with $y_{u(n)} \in U$ for $n \geq n_0$, a contradiction. This finishes the proof of (22.58).

Suppose that the element $x \in X$ does not belong to A. Then we can find an open G_x-invariant neighborhood U_1' of x and a natural number r_1' satisfying the implication

$$y \in C, y \in V_{\omega(r_1')} \implies y \notin U_1'. \tag{22.59}$$

Suppose the contrary. The same ideas as they appear in the sketch of the proof of assertion (i) lead to the construction of a sequence of open G_x-invariant sets $X \supseteq W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq \{x\}$ with $\bigcap_{n\geq 0} \overline{W_n} = \{x\}$. Fix $n \in \mathbb{N}$. Since (22.59) does not hold for $U_1' = W_n$, $V_m \subseteq V_n$ holds for $m \geq n$, and $\lim_{n\to\infty} \omega(n) = \infty$, we can find an element y_n in X satisfying $y_n \in C$, $y_n \in V_n$ and $y_n \in W_n$. Since C is compact, there is a strictly monotone increasing function $u\colon \mathbb{N} \to \mathbb{N}$ and $y \in C$ with $\lim_{n\to\infty} y_{u(n)} = y$. This implies $y \in \bigcap_{n\geq 0} \overline{V_{u(n)}} = A$ and $y \in \bigcap_{n\geq 0} \overline{W_{u(n)}} = \{x\}$, a contradiction. This finishes the proof of the implication (22.59).

Now we define the desired open G_x-invariant neighborhood U' of x in X by $U_0' \cap U_1'$ and the desired natural number $r' = \max\{r_0', r_1', w\}$. We get for $s \in S^{\omega,w}$

$$\begin{aligned}
&\widehat{\pi}(s) \in U', \widehat{\eta}(s) \geq r' \\
&\implies \pi(s) \in C, \widehat{\pi}(s) \in U_0', \eta(s) \geq w, \eta(s) \geq r_0' \\
&\implies \pi(s) \in C, \rho \circ \pi(s) \in U_0', \pi(s) \in W_{\omega\circ\eta(s)}, \eta(s) \geq r_0' \\
&\implies \pi(s) \in C, \rho \circ \pi(s) \in U_0', \pi(s) \in W_{\omega(r_0')} \\
&\overset{(22.58)}{\implies} \pi^{\omega,w}(s) = \pi(s) \in U.
\end{aligned}$$

Moreover, we have for $s \in S^{\omega,w}$

$$\begin{aligned}
\eta^{\omega,w}(s) \geq r' &\implies \pi(s) \in C, \eta(s) \geq w, \eta(s) \geq r_1' \\
&\implies \pi(s) \in C, \pi(s) \in V_{\omega\circ\eta(s)}, \eta(s) \geq r_1' \\
&\implies \pi(s) \in C, \pi(s) \in V_{r_1'} \\
&\overset{(22.59)}{\implies} \pi(s) \notin U_1' \\
&\implies \pi^{\omega,w}(s) = \pi(s) \notin U'.
\end{aligned}$$

Hence there is no $s \in S^{\omega,w}$ satisfying $\pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \geq r'$ and hence the implication

$$\pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \geq r' \implies \widehat{\pi}(s) \in U$$

obviously holds. This finishes the proof of assertion (iii) in the case that $x \notin A$.

It remains to treat the case $x \in A$. Then define the desired open G_x-invariant neighborhood U' of x in X by $U'_0 \cap \rho^{-1}(U)$ and the desired natural number $r' = \max\{r'_0, w\}$. Then we get analogously to the argument above

$$\widehat{\pi}(s) \in U', \widehat{\eta}(s) \geq r' \implies \pi^{\omega,w}(s) \in U$$

and

$$\begin{aligned} \pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \geq r' &\implies \pi(s) \in \rho^{-1}(U), \eta(s) \geq w \\ &\implies \rho \circ \pi(s) \in U, \eta(s) \geq w \\ &\implies \widehat{\pi}(s) = \rho \circ \pi(s) \in U. \end{aligned}$$

This finishes the proof of assertion (iii).

(iv) Choose a compact subset $C \subseteq A$ satisfying $\operatorname{im}(\pi') \subseteq C$ and a finite subset $F \subseteq G$ such that $\operatorname{supp}_G(\phi_{s',s}) \subseteq F$ holds for all $s' \in S'$ and $s \in S$. Fix $n \in \mathbb{N}$. Consider $a \in F \cdot C$. Then V_n is an open G_a-invariant neighborhood of a in X. Since ϕ satisfies *continuous control*, we can find an open G_a-invariant neighbourhood $U_n(a)$ of a in X and a natural number $r_n(a)$ such for $s' \in S'$, $s \in S$ and $g \in \operatorname{supp}_G(\phi_{s',s})$ the implication

$$g \cdot \pi'(s') \in U_n(a), \eta'(s') \geq r_n(a) \implies \pi(s) \in V_n \tag{22.60}$$

holds. Since $F \cdot C$ is compact and contained in $\bigcup_{a \in F \cdot C} U_n(a)$, we can find a finite subset $\{a_1, a_2, \ldots, a_k\} \subseteq F \cdot C$ satisfying $F \cdot C \subseteq \bigcup_{i=1}^k U_n(a_k)$. Define a natural number

$$r_n := \max\{r_n(a_i) \mid i = 1, 2 \ldots, k\}.$$

Consider $s' \in S'$ and $s \in S$ with $\phi_{s',s} \neq 0$. Then we get the implication

$$\eta'(s') \geq r_n \implies \pi(s) \in V_n \tag{22.61}$$

by the following argument. Suppose $\eta'(s') \geq r_n$. Since $\phi_{s',s} \neq 0$, we can choose $g \in \operatorname{supp}_G(\phi_{s',s})$. Because of $g \cdot \pi'(s') \in F \cdot C$ we can find $i \in \{1, 2, \ldots, k\}$ with $g \cdot \pi'(s) \in U_n(a_i)$. Since $r_n \geq r_n(a_i)$, we conclude from the implication (22.60) that $\pi(s) \in V_n$ holds.

We can additionally arrange that $r_n < r_{n+1}$ holds for $n \in \mathbb{N}$. Since ϕ satisfies *bounded control over* $\mathbb{N}$, we can find a natural number N such that $|\eta(s') - \eta(s)| \leq N$ holds for all $s' \in S'$ and $s \in S$ with $\phi_{s',s} \neq 0$.

Now define a function

$$\omega\colon \mathbb{N} \to \mathbb{N}$$

by requiring that for $m, n \in \mathbb{N}$ with $r_n + N \leq m < r_{n+1} + N$ we have $\omega(m) = n$ and $\omega(m) = 0$ for $m < r_0 + N$. Then ω is a non-decreasing function with $\lim_{m\to\infty} = \infty$. Put $w = r_0 + N$.

Consider any $s \in S$ such that there exists $s' \in S'$ with $\phi_{s',s} \neq 0$. Next we want to show $s \in S^{\omega,w}$, or, equivalently, the implication

$$\eta(s) \geq w \implies \pi(s) \in V_{\omega\circ\eta(s)}.$$

Suppose $\eta(s) \geq w$. Then we can choose $n \in \mathbb{N}$ such that $r_n + N \leq \eta(s) < r_{n+1} + N$ holds. Then $\omega\circ\eta(s) = n$ and $\eta'(s') \geq r_n$. We conclude $\pi(s) \in V_{\omega\circ\eta(s)}$ from implication (22.61). Hence ϕ induces the desired morphism $\phi'\colon \mathbf{B}' \to \mathbf{B}^{\omega}$ by putting $\phi'_{s',s} = \phi_{s',s}$ for $s' \in S'$ and $s \in S^{\omega,w}$. This finishes the proof of Lemma 22.57. $\square$

Lemma 22.62. *Let X be G-CW-complex with sub G-CW-complexes X_0, X_1, and X_0 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$.*

(i) The inclusion $i\colon (X_2, X_0) \to (X, X_1)$ induces an equivalence of additive categories

$$\mathcal{D}^G(i)\colon \mathcal{D}^G(X_2, X_0) \xrightarrow{\simeq} \mathcal{D}^G(X, X_1);$$

(ii) The square induced by the various inclusions

$$\begin{tikzcd}
\mathbf{K}(\mathcal{D}^G(X_0)) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{D}^G(X_1)) \arrow[d] \\
\mathbf{K}(\mathcal{D}^G(X_2)) \arrow[r] & \mathbf{K}(\mathcal{D}^G(X))
\end{tikzcd}$$

is weakly homotopy cocartesian.

Proof. (i) Consider an object $\mathbf{B}$ in $\mathcal{O}^G(X)$. We get from Lemma 22.57 (ii) and (iii) applied to the pair (X, X_1) and the object $\mathbf{B}$ the decomposition $\mathbf{B} = \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp}$ such that $\mathbf{B}^{\omega,w}$ is isomorphic to an object in $\mathcal{O}^G(X_1)$ and $\operatorname{im}(\pi^{\perp}) \subseteq X \setminus X_1$ holds. Therefore the inclusion $\mathbf{B}^{\perp} \to \mathbf{B}$ yields an isomorphisms in $\mathcal{D}^G(X, X_1)$. The object $\mathbf{B}^{\perp}$ is in the image of $\mathcal{D}^G(i)$ since the inclusion $X_2 \setminus X_0 \to X \setminus X_1$ is a G-homeomorphism. We conclude that $\mathcal{D}^G(i)$ is surjective on the set of isomorphism classes of objects.

Consider a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^G(X)$. It can be written in terms of the decomposition of Lemma 22.57 (ii) applied to the pair (X, X_1) and the objects $\mathbf{B}$ and $\mathbf{B}'$ as

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \to \mathbf{B}'^{\omega,w} \oplus \mathbf{B}'^{\perp}.$$

Define a morphism in $\mathcal{O}^G(X)$ by the composite

$$\psi\colon \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \xrightarrow{\begin{pmatrix}\mathrm{id} & 0\\ 0 & b\end{pmatrix}} \mathbf{B}^{\omega,w} \oplus \mathbf{B}'^{\omega,w} \xrightarrow{\begin{pmatrix}a & \mathrm{id}\\ c & 0\end{pmatrix}} \mathbf{B}'^{\omega,w} \oplus \mathbf{B}'^{\perp}.$$

Then $\mathbf{B}^{\omega,w} \oplus \mathbf{B}'^{\omega,w}$ is isomorphic to an object in the image of $\mathcal{O}^G(X_1) \to \mathcal{O}^G(X)$ by Lemma 22.57 (iii), the morphism $\phi-\psi\colon \mathbf{B}^{\omega,w}\oplus\mathbf{B}^{\perp} \to \mathbf{B}'^{\omega,w}\oplus\mathbf{B}'^{B}$ is of the shape $\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}$, and $d\colon \mathbf{B}^{\perp} \to \mathbf{B}'^{\perp}$ is in the image of $\mathcal{O}^G(X_2) \to \mathcal{O}^G(X)$ since $\mathrm{im}(\pi^{\perp})$ and $\mathrm{im}(\pi'^{\perp})$ are contained in $X \setminus X_1 \subseteq X_2$. This implies that the morphism in $\mathcal{D}^G(X, X_1)$ represented by ϕ is in the image of $\mathcal{D}^G(i)$. Hence $\mathcal{D}^G(i)$ is full.

In order to show that $\mathcal{D}^G(i)$ is an equivalence, it remains to show that $\mathcal{D}^G(i)$ is faithful. This is done as follows.

Consider a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^G(X_2)$. Suppose that its class $[\phi]$ in $\mathcal{D}^G(X_2, X_0)$ is sent under $\mathcal{D}^G(i)$ to zero. Hence there is an object $\mathcal{B}_0$ in $\mathcal{O}^G(X_1)$, an object $\mathbf{B}_1$ in $\mathcal{T}^G(X)$, morphisms $\psi\colon \mathcal{B} \to \mathcal{B}_0$, $\psi'\colon \mathcal{B}_0 \to \mathcal{B}$, $\mu\colon \mathcal{B} \to \mathcal{B}_1$, and $\mu'\colon \mathcal{B}_1 \to \mathcal{B}'$ such that $\phi - \psi' \circ \psi$ factorizes as

$$(\phi - \psi' \circ \psi)\colon \mathcal{B} \xrightarrow{\mu} \mathcal{B}_1 \xrightarrow{\mu'} \mathcal{B}'.$$

Because of Lemma 22.22 the object $\mathbf{B}_1$ is isomorphic to an object in $\tau^G(X_0)$. Therefore we can replace ϕ by $\phi - \mu' \circ \mu$ without changing the element it represents in $\mathcal{O}^G(X_2, X_0)$. Hence we can assume without loss of generality that ϕ factorizes as

$$\phi\colon \mathcal{B} \xrightarrow{\psi} \mathcal{B}_0 \xrightarrow{\psi'} \mathcal{B}'.$$

We conclude from Lemma 22.57 (iii) and (iv) applied the pair (X, X_2) and ψ that for appropriate (ω, w) $\psi\colon \mathcal{B} \to \mathcal{B}_0$ factorizes as in $\mathcal{D}^G(X)$ as

$$\psi\colon \mathbf{B} \xrightarrow{\nu} \mathbf{B}_0^{\omega,w} \xrightarrow{i^{\omega,w}} \mathbf{B}_0$$

and there is an object $\widehat{\mathbf{B}_0}$ in $\mathcal{O}^G(X_2)$ and an isomorphism $\zeta\colon \mathbf{B}_0^{\omega,w} \xrightarrow{\cong} \widehat{\mathcal{B}_0}$ in $\mathcal{O}^G(X)$. In the construction of $\widehat{\mathbf{B}_0}$ an element $a \in X_2$ and a retraction $\rho\colon V_0 \to X_2$ occurs. One easily checks by going through the constructions appearing in Lemma 22.57 (i) and (iii) that we can pick $a \in X_0$ and can arrange that $\rho(V_0 \cap X_1) \subseteq X_0$ holds. Since $\mathbf{B}_0$ belongs to X_1 the object $\widehat{\mathcal{B}_0}$ is actually an object in $\mathcal{O}^G(X_0)$. Hence we obtain the factorization in $\mathcal{O}^G(X)$

$$\phi\colon \mathcal{B} \xrightarrow{\zeta\circ\nu} \widehat{\mathcal{B}_0} \xrightarrow{\phi'\circ i^{\omega,w}\circ\zeta^{-1}} \mathcal{B}'.$$

Since $\mathcal{O}^G(X_2) \to \mathcal{O}^G(X)$ is faithful, the factorization above can be viewed as a factorization in $\mathcal{O}^G(X_2)$. Hence the class $[\phi]$ in $\mathcal{D}^G(X_2, X_0)$ represented by ϕ is trivial.

(ii) This is a direct consequence of assertion (ii) and Proposition 22.27. This finishes the proof of Lemma 22.62 $\square$

Lemma 22.63. *The inclusion* $i\colon (X,A) \to (X,A) \times [0,1]$ *sending* x *to* $(x,0)$ *induces a weak homotopy equivalence*

$$\mathbf{K}(\mathcal{D}^G(i))\colon \mathbf{K}(\mathcal{D}^G(X,A)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G((X,A) \times [0,1])).$$

Proof. Because of the Five Lemma and Proposition 22.27 it suffices to treat the case $A = \emptyset$.

Since we can apply Lemma 22.62 (ii) to the G-CW-complex $\operatorname{cone}(X) \cup_X X \times [0,1]$ with the subcomplexes $\operatorname{cone}(X), X \times [0,1]$, and X, it suffices to show that the map induced by the obvious inclusion

$$\mathbf{K}(\mathcal{D}^G(\operatorname{cone}(X))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G(\operatorname{cone}(X) \times_X X \times [0,1]))$$

is a weak homotopy equivalence. Because of Lemma 22.17, Lemma 22.22, and Theorem 21.5 (i), it suffices to show that the map induced by the obvious inclusion

$$\mathbf{K}(\mathcal{O}^G(\operatorname{cone}(X))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{O}^G(\operatorname{cone}(X) \cup_X \times [0,1]))$$

is a weak homotopy equivalence. Since $\operatorname{cone}(X)$ and $\operatorname{cone}(X) \cup_X X \times [0,1]$ are G-homeomorphic, both its source and its target are weakly contractible by Lemma 22.30. This finishes the proof of Lemma 22.63. $\square$

Proposition 22.64. *Let* $f_0, f_1\colon (X,A) \to (Y,B)$ *be* G*-maps of* G*-*CW*-pairs which are* G*-homotopic. Then for every* $n \in \mathbb{Z}$ *the homomorphism* $K_n(\mathcal{D}^G(f_0))$ *and* $K_n(\mathcal{D}^G(f_1))$ *from* $K_n(\mathcal{D}^G(X,A))$ *to* $K_n(\mathcal{D}^G(Y,B))$ *agree.*

Proof. Let $i_k\colon (X;A) \to (X,A) \times [0,1]$ be the map sending x to (x,k) for $k = 0,1$ and let $\operatorname{pr}\colon (X,A) \times [0,1] \to (X,A)$ be the projection. Since $\operatorname{pr} \circ i_k = \operatorname{id}_{(X,A)}$ holds for $k = 0,1$, we conclude from Lemma 22.63 that the two homomorphisms $K_n(\mathcal{D}^G(i_0))$ and $K_n(\mathcal{D}^G(i_1))$ from $K_n(\mathcal{D}^G(X,A))$ to $K_n(\mathcal{D}^G((X,A) \times [0,1]))$ agree. Let $h\colon (X;A) \times [0,1] \to (Y,B)$ be a G-homotopy between f_0 and f_1. Now the claim follows from the equality $K_n(\mathcal{D}^G(f_k)) = K_n(\mathcal{D}^G(h)) \circ K_n(\mathcal{D}^G(i_k))$ for $k = 0,1$. $\square$

Proposition 22.65. *Consider a* G*-*CW*-pair* (X,A)*, a* G*-*CW*-complex* B *and a cellular* G*-map* $f\colon A \to B$*. Put* $Y = X \cup_f B$*. Then:*

(i) The pair (Y,B) *is a* G*-*CW*-pair and the canonical map* $(F,f)\colon (X,A) \to (Y,B)$ *is a cellular* G*-map;*
(ii) The functor $\mathcal{D}^G(F,f)\colon \mathcal{D}^G(X,A) \xrightarrow{\simeq} \mathcal{D}^G(Y,B)$ *is an equivalence of additive categories and induces for all* $n \in \mathbb{Z}$ *an isomorphism*

$$K_n(\mathcal{D}^G(F,f))\colon K_n(\mathcal{D}^G(X,A)) \xrightarrow{\simeq} K_n(\mathcal{D}^G(Y,B));$$

(iii) Let $i\colon A \to X$ *and* $j\colon B \to Y$ *be the inclusions. We obtain a long exact Mayer-Vietoris sequence, infinite to both sides and natural in* (X,A) *and* $f\colon A \to B$*,*

$$
\begin{aligned}
\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{D}^G(A)) & \xrightarrow{-K_n(\mathcal{D}^G(i))\times K_n(\mathcal{D}^G(f))} \\
K_n(X)\oplus K_n(B) & \xrightarrow{K_n(F)\oplus K_n(j)} K_n(\mathcal{D}^G(Y)) \xrightarrow{\partial_n} K_{n-1}(\mathcal{D}^G(A)) \\
& \xrightarrow{-K_{n-1}(\mathcal{D}^G(i))\times K_{n-1}(\mathcal{D}^G(f))} K_{n-1}(X)\oplus K_{n-1}(B) \\
& \xrightarrow{K_{n-1}(F)\oplus K_{n-1}(j)} K_{n-1}(\mathcal{D}^G(Y)) \xrightarrow{\partial_{n-1}} \cdots .
\end{aligned}
$$

Proof. (i) This is obvious.

(ii) Apply Lemma 22.62 (i) to $X \cup_A \operatorname{cyl}(f)$ and the G-subcomplexes X, $\operatorname{cyl}(f)$ and A and then Proposition 22.64 to the obvious G-homotopy equivalences $X \cup_A \operatorname{cyl}(f) \xrightarrow{\simeq} Y$ and $\operatorname{cyl}(f) \xrightarrow{\simeq} B$.

(iii) This follows from assertion (ii) and Proposition 22.27. □

22.7.4 The Disjoint Union Axiom

Proposition 22.66. *Let $\{X_i \mid i \in I\}$ be a collection of G-CW-complexes. Let $j_i\colon X_i \to \coprod_{i\in I} X_i$ be the obvious inclusion for $i \in I$.*

(i) The obvious map of additive categories

$$\bigoplus_{i\in I} \mathcal{D}^G(j_i)\colon \bigoplus_{i\in I} \mathcal{D}^G(X_i) \to \mathcal{D}^G\left(\coprod_{i\in I} X_i\right)$$

is an equivalence

(ii) The obvious map of abelian groups

$$\bigoplus_{i\in I} K_n(\mathcal{D}^G(j_i))\colon \bigoplus_{i\in I} K_n(\mathcal{D}^G(X_i)) \to K_n\left(\mathcal{D}^G\left(\coprod_{i\in I} X_i\right)\right)$$

is an isomorphism for every $n \in \mathbb{Z}$.

Proof. (i) In the sequel we put $Y = \coprod_{i\in I} X_i$. Consider an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\mathcal{D}^G(Y)$. Since it satisfies *compact support over* Y, there is a finite subset $I_0 \subseteq I$ such that $\operatorname{im}(\pi) \subseteq \coprod_{i\in I_0} Y_i$. For $i \in I_0$ define $\mathbf{B}_i = (S_i, \pi_i, \eta_i, \mathsf{B}_i)$ where $S_i = \pi^{-1}(X_i)$ and π_i, η_i, and B_i are obtained from π, η, and B by restriction. Then $\mathbf{B}$ is the finite sum $\bigoplus_{i\in I_0} \mathbf{B}_i$ and $\mathbf{B}_i$ is in the image of $\mathcal{D}^G(j_i)\colon \mathcal{D}^G(X_i) \to \mathcal{D}^G(Y)$ for $i \in I_0$. We leave it to the reader to check that this implies that the functor $\bigoplus_{i\in I} \mathcal{D}^G(j_i)$ is surjective on objects, full, and faithful. Hence $\bigoplus_{i\in I} \mathcal{D}^G(j_i)$ is an equivalence of additive categories.

(ii) This follows from assertion (i) and fact that K_n commutes with finite products, or, equivalently, with finite direct sums and is compatible with colimits over direct systems, see for instance [668, Corollary 7.2]. □

Now Theorem 22.26 follows from Propositions 22.27, 22.64, 22.65, and 22.66.

22.8 The Computation of $K_n(\mathcal{D}^G(G/H))$

In this section we analyze the coefficients $K_n(\mathcal{D}^G(G/H))$ of the G-homology theory appearing in Theorem 22.26.

22.8.1 Reduction to $K_n(\mathcal{B}(G/H))$

Consider a category with G-support $\mathcal{B}$ in the sense of Definition 22.1. Given a G-set T, define a $\mathbb{Z}$-category $\mathcal{B}(T)$ as follows. Objects are pairs (t, B) for $t \in T$ and $B \in \operatorname{ob}(\mathcal{B})$. A morphism $\phi\colon (t, B) \to (t', B')$ is a morphism $\phi\colon B \to B'$ in $\mathcal{B}$ satisfying $\operatorname{supp}_G(\phi) \subseteq G_{t,t'}$ for $G_{t,t'} := \{g \in G \mid t' = gt\}$. Composition in $\mathcal{B}(S)$ comes from the composition in $\mathcal{B}$. The identity on (t, B) is given by id_B. The structure of a $\mathbb{Z}$-category on $\mathcal{B}(S)$ comes from the one on $\mathcal{B}$. Given a map $f\colon T \to T'$, we get a functor of $\mathbb{Z}$-categories $\mathcal{B}(f)\colon \mathcal{B}(T) \to \mathcal{B}(T')$ by sending an object (t, B) to the object $(f(t), B)$ and a morphism $\phi\colon (t, B) \to (t', B')$ given by the morphism $\phi\colon B \to B'$ in $\mathcal{B}$ to the morphism $(f(s), B) \to (f(s'), B')$ in $\mathcal{B}(S')$ given by $\phi\colon B \to B'$ again. This definition makes sense as $G_{t,t'} \subseteq G_{f(t),f(t')}$ holds. Thus we obtain a covariant functor

$$\mathcal{B}(?)\colon G\text{-}\mathsf{SETS} \to \mathbb{Z}\text{-}\mathsf{CAT} \tag{22.67}$$

from the category of G-sets to the category of $\mathbb{Z}$-categories by sending T to $\mathcal{B}(T)$. It induces a covariant $\mathsf{Or}(G)$-spectrum

$$\mathbf{K}(\mathcal{B}(?)_{\oplus})\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{B}(G/H)_{\oplus}). \tag{22.68}$$

We obtain another covariant $\mathsf{Or}(G)$-spectrum

$$\mathbf{K}(\mathcal{D}^G(?;\mathcal{B}))\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{D}^G(G/H;\mathcal{B})). \tag{22.69}$$

Proposition 22.70. *There is a weak homotopy equivalence of covariant $\mathsf{Or}(G)$-spectra*

$$\mathbf{K}(\mathcal{B}(?)_{\oplus}) \xrightarrow{\simeq} \Omega\mathbf{K}(\mathcal{D}^G(?);\mathcal{B}).$$

In particular we get for $n \in \mathbb{Z}$ an isomorphism, natural in G/H,

$$K_n(\mathcal{B}(G/H)_{\oplus}) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H;\mathcal{B})).$$

Proof. Any $\mathbb{Z}$-category can be viewed as a category with G-support over the trivial group $\{1\}$. Hence we can consider for any G-set T the additive categories $\mathcal{T}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$, $\mathcal{O}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$, and $\mathcal{D}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$. Next we define a functor of additive categories

$$F(T)\colon \mathcal{O}^{\{1\}}(\{\bullet\},\mathcal{B}(T)) \to \mathcal{O}^G(T;\mathcal{B}).$$

It sends an object $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ to the object $\mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$ where $S' = S$, $\eta' = \eta$ and B' and π' are determined by the equality $\mathsf{B}(s) = (\pi'(s),\mathsf{B}'(s))$. It induces a functor of additive categories

$$\overline{F(T)}\colon \mathcal{D}^{\{1\}}(\{\bullet\},\mathcal{B}(T)) \to \mathcal{D}^{\{1\}}(T,\mathcal{B}).$$

Next we show that $\overline{F(T)}$ is full. Consider any morphism in $\mathcal{D}^{\{1\}}(T,\mathcal{B})$ from $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ to $\mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$. Choose a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^{\{1\}}(T,\mathcal{B})$ representing it. Since T is discrete and ϕ satisfies *continuous control*, we can find for every $t \in T$ a natural number $r(t)$ such that for all $s \in S$, $S' \in S'$, and $g \in \operatorname{supp}_G(\phi_{s,s'})$ the implication

$$g\cdot\pi(s) = t, \eta(s) \geq r(t) \implies \pi'(s) = t$$

holds. Since the object $\mathbf{B}$ satisfies *compact support over* T, ϕ satisfies *finite G-support*, and T is discrete, there is a finite subset $T_0 \subseteq T$ satisfying $g{\cdot}\pi(s) \in T_0$ for all $s \in S$, $s' \in S'$ and $g \in \operatorname{supp}_G(\phi_{s,s'})$. Define $r := \max\{r(t) \mid t \in T_0\}$. Then for $s \in S$, $s' \in S'$ and $g \in \operatorname{supp}_G(\phi_{s,s'})$ the implication

$$\eta(s) \geq r \implies g\pi(s) = \pi'(s')$$

is true. Since ϕ satisfies *bounded control over* $\mathbb{N}$, we can change ϕ such that $\phi_{s,s'} = 0$ holds for $s \in S$, $s' \in S'$ satisfying $\eta(s) < r$ and that the class represented by ϕ in $\mathcal{D}^{\{1\}}(T,\mathcal{B})$ is unchanged. Hence we can assume without loss of generality that $g \in G_{\pi(s),\pi'(s')}$ holds for $s \in S$, $s' \in S'$ and $g \in \operatorname{supp}_G(\phi_{s,s'})$.

Define objects $\widehat{\mathbf{B}} = (\widehat{S},\widehat{\pi},\widehat{\eta},\widehat{\mathbf{B}})$ and $\widehat{\mathbf{B}'} = (\widehat{S'},\widehat{\pi}',\widehat{\eta'},\widehat{\mathbf{B}'})$ in $\mathcal{O}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$ by requiring that $\widehat{S} = S$, $\widehat{S'} = S'$, $\widehat{\eta} = \eta$, and $\widehat{\eta'} = \eta'$ hold and we have $\widehat{\mathsf{B}}(s) = (\pi(s),\mathsf{B}(s))$ for $s \in S$ and $\widehat{\mathsf{B}'}(s) = (\pi'(s'),\mathsf{B}(s'))$ for $s' \in S$. Then $F(\widehat{\mathbf{B}}) = \mathbf{B}$ and $F(\widehat{\mathbf{B}'}) = \mathbf{B}'$. Define a morphism $\psi\colon \widehat{\mathbf{B}} \to \widehat{\mathbf{B}'}$ in $\mathcal{O}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$ by defining the morphisms $\psi_{s,s'}\colon (\pi(s),\mathsf{B}(s)) \to (\pi'(s'),\mathsf{B}(s'))$ in $\mathcal{B}(T)$ by the morphism $\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s')$ in $\mathcal{B}$. One easily checks that ψ is well-defined and sent under $F(T)$ to ϕ. Hence the class represented by ψ in $\mathcal{D}^{\{1\}}(\{\bullet\},\mathcal{B}(T))$ is sent by $\overline{F(T)}$ to the class represented by ϕ in $\mathcal{D}^{\{1\}}(T,\mathcal{B})$. This finishes the proof that $\overline{F(T)}$ is full.

Since $F(T)$ is faithful, one easily checks that $\overline{F(T)}$ is faithful. As $F(T)$ is bijective on objects, $\overline{F(T)}$ is bijective on objects. We conclude that $\overline{F(T)}\colon \mathcal{D}^{\{1\}}(\{\bullet\},\mathcal{B}(T)) \to \mathcal{D}^{\{1\}}(T,\mathcal{B})$ is an equivalence of additive categories. In particular we see that the in T natural map

$$\mathbf{K}(\overline{F(T)})\colon \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\},\mathcal{B}(T))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^{\{1\}}(T,\mathcal{B})) \tag{22.71}$$

is a weak homotopy equivalence of spectra.

The canonical map

$$\mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}(\mathcal{O}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \to \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T)))\bigr)$$

is natural in T and is a weak homotopy equivalence by Theorem 21.5 (i). The projection from $\mathbf{K}(\mathcal{O}^{\{1\}}(\{\bullet\};\mathcal{B}(T)))$ to the trivial spectrum is a weak homotopy equivalence by Lemma 22.29. It induces an in T natural weak homotopy equivalence

$$\operatorname{hofib}\bigl(\mathbf{K}(\mathcal{O}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \to \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T)))\bigr) \xrightarrow{\simeq} \Omega\mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))).$$

The composite of the two maps above gives a weak homotopy equivalence of spectra, natural in T,

$$\mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \xrightarrow{\simeq} \Omega\mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))). \tag{22.72}$$

Define the inclusion of $\mathbb{Z}$-categories $I\colon \mathcal{B}(T) \to \mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))$ by sending an object (t,B) to the object $(\{*\},\pi,\eta,\mathrm{B})$ given by $\pi(*) = \{\bullet\}$, $\eta(*) = 0$, and $\pi(*) = s$. It induces a functor of additive categories $I_\oplus\colon \mathcal{B}(T)_\oplus \to \mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))$. Obviously $I_\oplus$ is full and faithful. We leave it to the reader to show that any object in $\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))$ is isomorphic to an object in the image of $I_\oplus$. Hence $I_\oplus$ is an equivalence of additive categories and induces a weak homotopy equivalence, natural in T,

$$\mathbf{K}(I_\oplus)\colon \mathbf{K}(\mathcal{B}(T)_\oplus) \xrightarrow{\simeq} \mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \tag{22.73}$$

Now the desired weak homotopy equivalence of covariant $\mathsf{Or}(G)$-spectra from $\mathbf{K}(\mathcal{B}(?)_\oplus)$ to $\Omega\mathbf{K}(\mathcal{D}^G(?);\mathcal{B})$ comes from the maps (22.71), (22.72), and (22.73). □

We have proved in Lemma 22.29 that $\mathcal{O}^G(G/G)$ is flasque. The next exercise shows that this is not true in general for $\mathcal{O}^G(G/H)$ if $H \neq G$.

Exercise 22.74. Suppose that the category $\mathcal{O}^G(G/H)$ is flasque. Show that then the map $K_n(\mathcal{B}[G/H]_\oplus) \to K_n(\mathcal{B}_\oplus)$ induced by the projection $G/H \to G/G$ and the obvious identification $\mathcal{B}(G/G) = \mathcal{B}$ is bijective for all $n \in \mathbb{Z}$.

22.8.2 Assembly and Controlled *G*-homology

We have the G-homology theory $K_*(\mathcal{D}^G(-;\mathcal{B}))$, see Theorem 22.26. The covariant $\mathsf{Or}(G)$-spectrum $\mathbf{K}(\mathcal{B}(?))_\oplus$ of (22.68) determines a G-homology theory $H_*^G(-;\mathbf{K}(\mathcal{B}(?)))$ see Theorem 12.27.

Proposition 22.75. *There is an equivalence of G-homology theories*

$$T(-)\colon K_{*+1}(\mathcal{D}^G(-;\mathcal{B})) \xrightarrow{\cong} H_*^G(-;\mathbf{K}(\mathcal{B}(?)_\oplus)).$$

Proof. This follows from Corollary 18.16 and Proposition 22.70. □

Lemma 22.76. *Let $\mathcal{B}$ be a category with G-support and let $\mathcal{F}$ be a family of subgroups. Let $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to G/G$ be the projection.*

(i) The assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for the G-homology theory $H_*^G(-;\mathbf{K}(\mathcal{B}(?)_\oplus))$ and the family $\mathcal{F}$

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{K}(\mathcal{B}(?)_\oplus)) \to H_n^G(G/G;\mathbf{K}(\mathcal{B}(?)_\oplus)) = K_n(\mathcal{B}_\oplus)$$

can be identified for every $n \in \mathbb{Z}$ with the homomorphism induced by the projection $E_{\mathcal{F}}(G) \to G/G$

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \to K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B})) = K_n(\mathcal{B}_\oplus);$$

(ii) The Meta-Isomorphisms Conjecture 15.2 for the G-homology theory $H_*^G(-;\mathbf{K}(\mathcal{B}(?)))$ and the family $\mathcal{F}$ is true if and only if the spectrum $\mathbf{K}(\mathcal{O}^G(E_{\mathcal{F}}(G);\mathcal{B}))$ is weakly contractible.

Proof. (i) This follows from Proposition 22.75.

(ii) This follows from assertion (i), Lemma 22.22, Lemma 22.29, and the commutative diagram of spectra

$$\begin{tikzcd}
\mathcal{T}^G(E_{\mathcal{F}}(G)) \arrow[r] \arrow[d, "\mathcal{T}^G(\mathrm{pr})"] & \mathcal{O}^G(E_{\mathcal{F}}(G)) \arrow[r] \arrow[d, "\mathcal{O}^G(\mathrm{pr})"] & \mathcal{D}^G(E_{\mathcal{F}}(G)) \arrow[d, "\mathcal{D}^G(\mathrm{pr})"] \\
\mathcal{T}^G(G/G) \arrow[r] & \mathcal{O}^G(G/G) \arrow[r] & \mathcal{D}^G(G/G)
\end{tikzcd}$$

whose rows are weak homotopy fibrations by Theorem 22.19. □

Remark 22.77. The benefit of Lemma 22.76 (ii) is that the proof of the Meta-Isomorphism Conjecture is reduced to the proof of the weak contractibility of the K-theory of the specific category $\mathcal{O}^G(E_{\mathcal{F}}(G);\mathcal{B})$ defined in terms of controlled topology and not just to the weak contractibility of some abstract homotopy fiber. This will allow us to use geometric tools for a proof of the Farrell-Jones Conjecture as described in Chapter 19.

22.8.3 The Definition of a Strong Category with G-support

In this subsection we will upgrade the notion of a category with G-support of Definition 22.1 to the one of a strong category with G-support by additional implementing a G-action $\mathcal{B}$ and a homotopy trivilization for it.

Definition 22.78 (Strong category with G-support). A *strong category with G-support* over G is a triple $\mathcal{B} = (\mathcal{B}, \operatorname{supp}_G, \Omega)$ consisting of:

- A G-$\mathbb{Z}$-category $\mathcal{B}$;
- A map called *support function*

$$\operatorname{supp}_G\colon \operatorname{mor}(\mathcal{B}) \to \{\text{finite subsets of } G\};$$

- A *homotopy trivilization of the G-action on $\mathcal{B}$*, i.e., a collection $\Omega = \{\Omega_g \mid g \in G\}$ where Ω_g is a natural equivalence of functors of $\mathbb{Z}$-categories $\mathcal{B} \to \mathcal{B}$

$$\Omega_g\colon \operatorname{id}_{\mathcal{B}} \xrightarrow{\cong} \Lambda_g,$$

for $\Lambda_g\colon \mathcal{B} \to \mathcal{B}$ the functor given by multiplication with g such that condition (vii) and (viii) below are satisfied.

We require that the following axioms are satisfied for all objects B in $\mathcal{B}$, all morphisms $u, u'\colon B_1 \to B_2$, $v\colon B_2 \to B_3$ in $\mathcal{B}$, and all $g, g' \in G$:

(i) $\operatorname{supp}_G(u) = \emptyset \Longleftrightarrow u = 0$;
(ii) $\operatorname{supp}_G(v \circ u) \subseteq \operatorname{supp}_G(v) \cdot \operatorname{supp}_G(u)$;
(iii) $\operatorname{supp}_G(u + u') \subseteq \operatorname{supp}_G(u) \cup \operatorname{supp}_G(u')$;
(iv) $\operatorname{supp}_G(-u) = \operatorname{supp}_G(u)$;
(v) $\operatorname{supp}_G(B) = \{e\}$;
(vi) $\operatorname{supp}_G(gu) = g \operatorname{supp}_G(u) g^{-1}$;
(vii) $\Omega_{g'}(gB) \circ \Omega_g(B) = \Omega_{g'g}(B)$;
(viii) $\Omega_e(B) = \operatorname{id}_B$;
(ix) $\operatorname{supp}_G(\Omega_g(B)) = \{g\}$.

Remark 22.79. In Example 22.2 we actually get the structure of a strong category with G-support. Namely, for $g_0 \in G$ and object A in $\mathcal{A}[G]$ which is given by an object A in $\mathcal{A}$, we define $\Lambda_{g_0}(A)$ to be $g_0 A$ using by the given G-action on the objects of $\mathcal{A}$. For a morphism $\phi = \sum_{g \in G} \phi_g \cdot g\colon A \to A'$ in $\mathcal{A}[G]$, we define $\Lambda_{g_0}(\phi)\colon g_0 A \to g_0' A'$ by $(g_0\phi)_g = g_0 \cdot \phi_{g_0^{-1}g}$. The desired homotopy trivilization Ω is given assigning to $g_0 \in G$ the isomorphism $\Omega_{g_0}(A)\colon A \xrightarrow{\cong} \Lambda_{g_0}(A)$ in $\mathcal{A}[G]$ given by $\Omega_{g_0}(A)_{g_0} = \operatorname{id}_{g_0 A}$ and $\Omega_{g_0}(A)_{g_1} = 0$ for $g_0 \neq g_1$.

22.8.4 Reduction to $K_n(\mathcal{B}\langle H\rangle)$

Let $\mathcal{B}$ be a strong category with G-support in the sense of Definition 22.78.

Definition 22.80 ($\mathcal{B}\langle H\rangle$). For a subgroup $H \subseteq G$ define $\mathcal{B}\langle H\rangle$ to be the $\mathbb{Z}$-subcategory of $\mathcal{B}$ which has the same set of objects and for which a morphism $\phi\colon B \to B'$ of $\mathcal{B}$ belongs to $\mathcal{B}\langle H\rangle$ if $\operatorname{supp}_G(\phi) \subseteq H$ holds.

Define a functor $I\colon \mathcal{B}\langle H\rangle \to \mathcal{B}(G/H)$ of $\mathbb{Z}$-categories by sending an object B to the object (eH, B) and a morphism $\phi\colon B \to B'$ to the morphism $(eH, B) \to (eH, B')$ given by ϕ.

Proposition 22.81. *The functor $I\colon \mathcal{B}\langle H\rangle \to \mathcal{B}(G/H)$ is an equivalence of $\mathbb{Z}$-categories. In particular the homomorphism*

$$K_n(I_\oplus)\colon K_n(\mathcal{B}\langle H\rangle_\oplus) \to K_n(\mathcal{B}(G/H)_\oplus)$$

is bijective for all $n \in \mathbb{N}$.

Proof. Obviously I is full and faithful. Consider an object (gH, B) in $\mathcal{B}(G/H)$. Then $\Omega_g(g^{-1}B)\colon g^{-1}B \xrightarrow{\cong} B$ is an isomorphism in $\mathcal{B}$ with $\operatorname{supp}(\Omega_g(g^{-1}B)) = \{g\}$ and hence induces an isomorphism $(e, g^{-1}B) \xrightarrow{\cong} (g, B)$ in $\mathcal{B}(G/H)$. This shows that any object in $\mathcal{B}(G/H)$ is isomorphic to an object in the image of I. Hence I is an equivalence. □

Remark 22.82. Let $\mathcal{A}$ be a G-$\mathbb{Z}$-category. Recall from Example 22.2 and Remark 22.79 that the additive category $\mathcal{A}[G]$ is a strong category with G-support. One easily checks for any subgroup $H \subseteq G$.

$$\mathcal{A}[H] = \mathcal{A}[G]\langle H\rangle.$$

Hence we get from Proposition 22.70 and Proposition 22.81 for every $n \in \mathbb{Z}$ an isomorphism

$$K_n(\mathcal{A}[H]_\oplus) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H; \mathcal{A}[G])).$$

Example 22.83. Let R be a unital ring and let $\rho\colon G \to \operatorname{aut}(R)$ be a group homomorphism. We have defined the G-$\mathbb{Z}$-category $\underline{R}$ in Example 22.3. Denote by $R_{\rho|_H}[H]$ the twisted group ring of $H \subset G$ with respect to $\rho|_H\colon H \to \operatorname{aut}(R)$

We conclude from Example 22.3 and Remark 22.82 that there is for every $n \in \mathbb{Z}$ an isomorphisms

$$K_n(R_{\rho|_H}[H]) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H; \underline{R}[G])).$$

Exercise 22.84. Let R be a ring. Let $\mathcal{B}$ be the $\mathbb{Z}$-linear category with one object whose endomorphism ring is the group ring $R[\mathbb{Z}/2]$. Let t be the generator of $R[\mathbb{Z}/2]$. We define the support of an endomorphism $at^0 + bt^1$ to be the subset of $\mathbb{Z}$

$$\operatorname{supp}_{\mathbb{Z}}(at^0 + bt^1) = \begin{cases} \emptyset & \text{if } a = b = 0; \\ \{0\} & \text{if } a \neq 0, b = 0; \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Show that the axioms of a category with $\mathbb{Z}$-support are satisfied assembly map, we get isomorphisms

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{\mathcal{B}}) \cong K_n(R);$$
$$H_n^{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z};\mathbf{K}_{\mathcal{B}}) \cong K_n(R[\mathbb{Z}/2]),$$

and under this identification the assembly map $H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{\mathcal{B}}) \to H_n^{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z};\mathbf{K}_{\mathcal{B}})$ agrees with the map $K_n(R) \to K_n(R[\mathbb{Z}/2])$ induced by the inclusion $R \to R[\mathbb{Z}/2]$.

Remark 22.85 (Morphism Additivity). The version of the Farrell Jones Conjecture with categories with G-support is too general to expect that the Farrell-Jones Conjecture holds with them as coefficients, as Exercise 22.84 illustrates. It may hold for strong categories with G-support in the sense of Definition 22.78 if one additionally assumes

- *Morphism Additivity*
  Let $u\colon B \to B'$ be a morphism. Suppose that $\operatorname{supp}_G(u) = L_1 \sqcup L_2$ is a disjoint union. Then we require the existence of morphisms $u_i\colon B \to B'$ for $i = 1, 2$ satisfying $u = u_1 + u_2$ and $\operatorname{supp}_G(u_i) = L_i$ for $i = 1, 2$.

But then $\mathcal{B}$ is already of the shape $\mathcal{A}[G]$, see Exercise 22.88.

Exercise 22.86. Show that the two morphisms u_1 and u_2 appearing in the axiom *Morphism Additivity* stated in Remark 22.85 are unique.

Exercise 22.87. Let $\mathcal{A}$ be a G-$\mathbb{Z}$-category. Show that $\mathcal{A}[G]$ defined in Example 22.2 is a strong category with G-coefficients in the sense of Definition 22.78 satisfying *Morphism Additivity*.

Exercise 22.88. Consider a strong category $\mathcal{B}$ with G-support satisfying *Morphism Additivity*. Let $\mathcal{A}$ the G-$\mathbb{Z}$-subcategory of $\mathcal{B}$ which has the same set of objects and for which a morphism $u\colon B \to B'$ in $\mathcal{B}$ belongs to $\mathcal{A}$ if and only if $\operatorname{supp}_G(u) \subseteq \{e\}$ holds for the unit element $e \in G$. Construct an isomorphism of G-$\mathbb{Z}$-categories

$$F\colon \mathcal{A}[G] \xrightarrow{\cong} \mathcal{B}$$

which is compatible with the support functions.

In view of the last exercise it is superfluous to consider strong categories with G-support satisfying the axiom *Morphism Additivity* for discrete groups. This is different when one considers totally disconnected groups, see [81, Definition 3.2], where also a new condition *Support Cofinality* enters which is void for discrete groups.

Exercise 22.89. Show that the structure of a category with $\mathbb{Z}$-support on the category $\mathcal{B}$ of Exercise 22.84 does not extend to the structure of a strong category with $\mathbb{Z}$-support.

22.9 Induction

Let $H \subseteq G$ be a subgroup of G. Let $\mathcal{B}$ be a strong category with G-support in the sense of Definition 22.78. We have defined the $\mathbb{Z}$-category $\mathcal{B}\langle H\rangle$ in Definition 22.80. Obviously it inherits from $\mathcal{B}$ the structure of a strong category with G-support over H. Given an H-space X, we have denoted by $\operatorname{ind}_\iota X = G \times_H X$ the G-space given by induction with the inclusion $\iota\colon H \to G$, see (12.8).

Next we construct a functor of additive categories, natural in X,

$$\operatorname{ind}_\iota\colon \mathcal{O}^H(X;\mathcal{B}\langle H\rangle) \to \mathcal{O}^G(\operatorname{ind}_\iota X;\mathcal{B}). \tag{22.90}$$

Let $j\colon X \to \operatorname{ind}_\iota X$ be the ι-equivariant map sending x to (e',x). An object $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ of $\mathcal{O}^H(X;\mathcal{B}\langle H\rangle)$ is sent to the object $\operatorname{ind}_\iota \mathbf{B} = (S, j\circ\pi,\eta,\mathsf{B})$ of $\mathcal{O}^G(\operatorname{ind}_\iota X;\mathcal{B})$. Obviously $\operatorname{ind}_\iota(\mathbf{B})$ satisfies *compact support over* $\operatorname{ind}_\iota X$ and *locally finiteness over* $\mathbb{N}$ as $\mathbf{B}$ satisfies *compact support over* X and *locally finiteness over* $\mathbb{N}$. For two objects $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ and $\mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$ and a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ given by the collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$ of $\mathcal{O}^H(X;\mathcal{B}\langle H\rangle)$, define the morphism $\operatorname{ind}_\iota(\phi)\colon \operatorname{ind}_\iota(\mathbf{B}) \to \operatorname{ind}_\iota(\mathbf{B}')$ of $\mathcal{O}^G(\operatorname{ind}_\iota X;\mathcal{B})$ by the same collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S'\}$. Obviously conditions *finite G-support* and *bounded control over* $\mathbb{N}$ are satisfied for $\operatorname{ind}_\iota(\phi)$. Next we give the proof of *continuous control*. We only deal with the condition (22.5), the proof for the condition (22.6) is analogous and left to the reader.

Consider a point (g,x) in $\operatorname{ind}_\iota X$ and an open $G_{(g,x)}$-invariant neighborhood U of (g,x) in $\operatorname{ind}_\iota X$. Note for the sequel that $G_{(g,x)} = g'H_xg^{-1}$ holds and the map $j\colon X \to \operatorname{ind}_\iota X$ is an open ι-equivariant embedding. We have to find an open $G_{(g,x)}$-invariant neighborhood U' of (g,x) in $\operatorname{ind}_\iota X$ satisfying $U' \subseteq U$ and a natural number r' such that for all $s \in S$, $s' \in S'$, and $g \in \operatorname{supp}_G((\operatorname{ind}_\iota \phi)_{s,s'}) = \operatorname{supp}_H(\phi_{s,s'})$ the implication

$$g\cdot j\circ\pi(s) \in U', \eta(s) \geq r' \implies j\circ\pi'(s') \in U' \tag{22.91}$$

holds.

Suppose that $(g,x) \notin \operatorname{im}(j)$. Then $U' = g\cdot\operatorname{im}(j)$ is an open $G_{(g,x)}$-invariant neighborhood of (g,x) satisfying $U'\cap\operatorname{im}(j) = \emptyset$. Then the implication (22.91) is satisfied for trivial reasons since $\operatorname{supp}_G((\operatorname{ind}_\iota \phi)_{s,s'}) = \operatorname{supp}_H(\phi_{s,s'}) \subseteq H$ holds and $h\cdot j\circ\pi(s)$ belongs to $\operatorname{im}(j)$ and hence never belongs to U' for $h \in H$.

Next we treat the case $(g,x) \in \operatorname{im}(j)$, or, equivalently, the case $g = e$. Since ϕ satisfies *continuous control* and $j^{-1}(U)$ is an open H_x-invariant neighborhood of x, we can find an open H_x-invariant neighborhood V' of x in X with $V' \subseteq j^{-1}(U)$ such that for all $s \in S$, $s' \in S'$ and $h \in \operatorname{supp}_H(\phi_{s,s'})$ the implication

$$h\cdot\pi(s) \in U', \eta(s) \geq r' \implies \pi'(s') \in j^{-1}(U)$$

holds. Put $U' = j(V)$. Then (22.91) is satisfied for the open $G_{j(x)}$-invariant neighborhood U' of $j(x)$ in $\operatorname{ind}_\iota X$ and the number r' above.

One easily checks that the functor ind_ι of (22.90) induces for every G-CW-pair (X, A) functors of additive categories

(22.92) $\operatorname{ind}_\iota \colon \mathcal{O}^H(X, A; \mathcal{B}\langle H\rangle) \to \mathcal{O}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B})$;

(22.93) $\operatorname{ind}_\iota \colon \mathcal{T}^H(X, A; \mathcal{B}\langle H\rangle) \to \mathcal{O}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B})$;

(22.94) $\operatorname{ind}_\iota \colon \mathcal{D}^H(X, A; \mathcal{B}\langle H\rangle) \to \mathcal{D}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B})$.

Proposition 22.95. *For every G-CW-pair (X, A) and every strong category with G-support $\mathcal{B}$ over G, the functor* ind_ι *of* (22.94) *induces a weak homotopy equivalence*

$$\mathbf{K}(\operatorname{ind}_\iota) \colon \mathbf{K}(\mathcal{D}^H(X, A; \mathcal{B}\langle H\rangle)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B})).$$

Proof. We offer two proofs, a short one using basic facts about G-homology theories, and one direct proof which illustrates the role of the condition continuous control.

We can view the functors sending an H-CW-pair to the $\mathbb{Z}$-graded abelian groups $K_*(\mathcal{D}^H(X, A; \mathcal{B}\langle H\rangle))$ and $K_*(\mathcal{D}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B}))$ as H-homology theories. Then we get a natural transformation of H-homology theories by

$$\mathbf{K}_*(\operatorname{ind}_\iota) \colon K_*(\mathcal{D}^H(X, A; \mathcal{B}\langle H\rangle)) \to K_*(\mathcal{D}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B})).$$

In order to show that this is an isomorphism for every CW-pair (X, A), it suffices to do this in the special case $X = H/K$ and $A = \emptyset$ for every subgroup $K \subseteq H$, see Theorem 12.6. We have already constructed isomorphisms, see Proposition 22.70 and Proposition 22.81,

$$K_*(\mathcal{D}^H(H/K; \mathcal{B}\langle H\rangle)) \xrightarrow{\cong} K_{*-1}\big(((\mathcal{B}\langle H\rangle)\langle K\rangle)_\oplus\big) = K_{*-1}(\mathcal{B}\langle K\rangle_\oplus),$$

and

$$K_*(\mathcal{D}^G(\operatorname{ind}_\iota H/K; \mathcal{B})) = K_*(\mathcal{D}^G(G/K; \mathcal{B})) \xrightarrow{\cong} K_{*-1}(\mathcal{B}\langle K\rangle_\oplus).$$

Under these identifications

$$\mathbf{K}_*(\operatorname{ind}_\iota) \colon K_*(\mathcal{D}^H(H/K; \mathcal{B}\langle H\rangle)) \to K_*(\mathcal{D}^G(\operatorname{ind}_\iota H/K; \mathcal{B}))$$

becomes the identity on $K_{*-1}(\mathcal{B}\langle K\rangle_\oplus)$. This finishes the first proof of Proposition 22.95.

Next we present the second proof. Because of Proposition 22.27 we can assume without loss of generality $A = \emptyset$. It suffices to show that the functor of (22.94)

$$\operatorname{ind}_\iota \colon \mathcal{D}^H(X; \mathcal{B}\langle H\rangle) \to \mathcal{D}^G(\operatorname{ind}_\iota X; \mathcal{B}).$$

is an equivalence of additive categories

We first show that ind_ι is full and faithful, in other words, that for two objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ and $\mathbf{B}' = (S', \pi, \eta', \mathsf{B}')$ in $\mathcal{D}^H(X; \mathcal{B}\langle H\rangle)$ the map induced by ind_ι

$$\operatorname{mor}_{\mathcal{D}^H(X;\mathcal{B}\langle H\rangle)}(\mathbf{B}, \mathbf{B}') \to \operatorname{mor}_{\mathcal{D}^G(\operatorname{ind}_\iota X;\mathcal{B})}(\operatorname{ind}_\iota(\mathbf{B}), \operatorname{ind}_\iota(\mathbf{B}')), \tag{22.96}$$

is bijective. The elementary proof of injectivity is left to the reader. Surjectivity is proved a follows.

Recall $\operatorname{ind}_\iota(\mathbf{B}) = (S, j \circ \pi, \eta, \mathsf{B})$. Consider any element in the target of (22.96). Choose a morphism $\phi'\colon (S, j \circ \pi, \eta, \mathsf{B}) \to (S', j \circ \pi', \eta', \mathsf{B}')$ in $\mathcal{O}^G(\operatorname{ind}_\iota X; \mathcal{B})$ representing it. Next we show that we can assume without loss of generality

$$\operatorname{supp}_G(\phi'_{s,s'}) \subseteq H \text{ for } s \in S, s' \in S'. \tag{22.97}$$

Consider $x \in X$. Since ϕ' satisfies *continuous control* and $\operatorname{im}(j)$ is an open $G_{j(x)}$-invariant neighborhood of $j(x)$ in $\operatorname{ind}_\iota X$, we conclude from Lemma 22.10 (ii) that there are an open $G_{j(x)}$-invariant neighborhood U'_x of $j(x)$ in $\operatorname{ind}_\iota X$ with $U'_x \subseteq \operatorname{im}(j)$ and a natural number r'_x such that for all $s \in S$, $s' \in S'$, and $g \in \operatorname{supp}_G(\phi'_{s,s'})$ the implication

$$j \circ \pi'(s) \in U'_x, \eta'(s') \geq r'_x \implies g \cdot j \circ \pi(s) \in \operatorname{im}(j) \tag{22.98}$$

holds. Since $\mathbf{B}'$ satisfies *compact support over* X, there is a compact subset $C \subseteq X$ with $\operatorname{im}(\pi) \subseteq C$. Since $j(C) \subseteq \bigcup_{x \in C} U'_x$ and $j(C) \subseteq \operatorname{ind}_\iota X$ is compact, there is a finite subset $\{x_1, x_2, \ldots, x_m\} \subseteq C$ satisfying $j(C) \subseteq \bigcup_{i=1}^m U'_{x_i}$. Define a natural number $r' := \max\{r'_{x_i} \mid i = 1, 2 \ldots, m\}$. Then we get for all $s \in S$, $s' \in S'$, and $g \in \operatorname{supp}_G(\phi'_{s,s'})$ the implication

$$\eta'(s') \geq r' \implies g \cdot j \circ \pi(s) \in \operatorname{im}(j) \tag{22.99}$$

since for any $s' \in S'$ there exists $i \in \{1, 2, \ldots, m\}$ with $j \circ \pi(s') \in U'_{x_i}$ and $r' \geq r'_i$ and we can apply the implication (22.98). Since ϕ' satisfies *bounded control over* $\mathbb{N}$, we can modify ϕ' without changing the class which it represent in $\mathcal{O}^{G'}(\operatorname{ind}_\iota X; \mathcal{B}')$ such that for all $s \in S$, $s' \in s'$ and $g \in \operatorname{supp}_{G'}(\phi'_{s,s'})$ we have $g \cdot j \circ \pi(s) \in \operatorname{im}(j)$. Now (22.97) follows since $g \cdot j \circ \pi(s) \in \operatorname{im}(j) \implies g \in H$.

We conclude from (22.97) that $\phi'_{s,s'}$ belongs to $\mathcal{B}\langle H\rangle$. Define a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^H(X; \mathcal{B}\langle H\rangle)$ by $\phi_{s,s'} = \phi'_{s,s'}$ for $s \in S$ and $s' \in S'$. One easily checks that ϕ satisfies *finite support over* H, *bounded control over* $\mathbb{N}$, and *continuous control* since ϕ' satisfies *finite support over* G, *bounded control over* $\mathbb{N}$, and *continuous control*. Hence ϕ is well-defined. Its class in $\mathcal{D}^H(X; \mathcal{B}\langle H\rangle)$ is mapped by construction under the map (22.96) to the class in $\mathcal{D}^G(\operatorname{ind}_\iota X; \mathcal{B})$ represented by ϕ'. This shows that the map (22.96) is bijective.

It remains to shows show that for every object $\mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$ in $\mathcal{O}^G(\operatorname{ind}_\iota X; \mathcal{B})$ there is an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^H(X; \mathcal{B}\langle H\rangle)$ and an isomorphism $\phi\colon \operatorname{ind}_\iota(\mathbf{B}) \xrightarrow{\cong} \mathbf{B}'$ in $\mathcal{O}^G(\operatorname{ind}_\iota X;\mathcal{B})$. We put $S = S'$ and $\eta = \eta'$. Choose functions $\gamma\colon S \to G$ and $\pi\colon S \to X$ such that $\gamma(s) \cdot j \circ \pi(s) = \pi'(s)$ holds for all $s \in S$. Define $\mathsf{B}\colon S \to \operatorname{ob}(\mathcal{B})$ by sending s to $\gamma(s)^{-1} \cdot \mathsf{B}'(s)$. Then we can define the desired isomorphism ϕ by putting $\phi_{s,s'} = 0$ for $s, s' \in S$ with $s \neq s'$ and by $\phi_{s,s} = \Omega_{\gamma(s)}(\mathsf{B}(s))\colon \mathsf{B}(s) \xrightarrow{\cong} \mathsf{B}'(s)$ for $s \in S$. The proof that ϕ is well-defined is mild generalization of the proof of Lemma 22.14. This finishes the second proof of Proposition 22.95. □

22.10 The Version with Zero Control over $\mathbb{N}$

We also deal with a version $\mathcal{D}_0^G(X;\mathcal{B})$ of $\mathcal{D}_0^G(X;\mathcal{B})$ where we have zero-control over $\mathbb{N}$.

22.10.1 Control Categories with Zero Control in the $\mathbb{N}$-Direction

Definition 22.100 ($\mathcal{D}_0^G(X;\mathcal{B})$). Define $\mathcal{O}_0^G(X)$ to be the additive subcategory of $\mathcal{O}_G(X)$ which has the same set of objects and for which a morphism $\phi\colon \mathbf{B} = (S,\pi,\eta,\mathsf{B}) \to \mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$ in $\mathcal{O}^G(X)$ belongs to $\mathcal{O}_0^G(X)$ if and only if the implication

$$\phi_{s,s'} \neq 0 \implies \eta(s) = \eta(s')$$

holds for all $s \in S$ and $s' \in S'$.

Let $\mathcal{T}_0^G(X;\mathcal{B})$ be the full subcategory of $\mathcal{O}_G^0(X)$ consisting of those objects $\mathbf{B} = (\Sigma,\pi,\eta,\mathsf{B})$ for which there exists a natural number n such that $\mathsf{B}(\sigma) = 0$ holds for $\sigma \in \Sigma$ with $\eta(\sigma) \geq n$.

Define $\mathcal{D}_0^G(X)$ to be the quotient category $\mathcal{O}_0^G(X)/\mathcal{T}_0^G(X)$ in the sense of Definition 21.1.

Lemma 22.101. *The inclusion $\mathcal{T}_0^G(X) \to \mathcal{O}_0^G(X)$ is a Karoubi filtration in the sense of Definition 21.2. In particular we get a weak homotopy fibration sequence*

$$\mathcal{T}_0^G(X) \to \mathcal{O}_0^G(X) \to \mathcal{D}_0^G(X).$$

Proof. The proof of Lemma 22.17 carries directly over. Now apply Theorem 21.5 (i). □

Exercise 22.102. Show for $m \in \mathbb{Z}$

$$K_m(\mathcal{D}_0^{\{1\}}(\{\bullet\})) \cong \left(\prod_{n=0}^{\infty} K_m(\mathcal{B}_\oplus)\right) \Big/ \left(\bigoplus_{n=0}^{\infty} K_m(\mathcal{B}_\oplus)\right).$$

Let $\rho\colon \mathbb{N} \to \mathbb{N}$ be a function which is finite-to-one, i.e., the preimage of every element in $\mathbb{N}$ under ρ is finite. Next we construct a functor of additive categories

$$V'_\rho(X)\colon \mathcal{O}^G_0(X) \to \mathcal{O}^G_0(X)$$

which is essentially given by moving an object at the position n to the position $\rho(n)$ and leaving the position in X fixed. More precisely, V_ρ sends an object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ to the object $V_\rho(X)(\mathbf{B}) = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathsf{B}})$ given by

$$\begin{aligned}
\widehat{S} &= S;\\
\widehat{\pi} &= \pi;\\
\widehat{\eta} &= \rho \circ \eta;\\
\widehat{\mathsf{B}} &= \mathsf{B}.
\end{aligned}$$

Its definition on morphisms is the tautological one, i.e., a morphism $\phi\colon \mathbf{B} = (S, \pi, \eta, \mathsf{B}) \to \mathbf{B} = (S', \pi', \eta', \mathsf{B}')$ is sent to the morphism $V'_\rho(\phi)$ given by $V'_\rho(X)(\phi)_{s,s'} = \phi_{s,s'}$ for $s \in S$ and $s' \in S$.

We have to check that this is well-defined. Since ρ is finite-to-one, the new object $V'_\rho(X)(\mathbf{B})$ satisfies the conditions *compact suport over* X and *locally finiteness over* $\mathbb{N}$ as $\mathbf{B}$ does. For every natural number N, there exists a natural number N' such that the implication $\rho(n) \geq N' \implies n \geq N$ holds for every $n \in \mathbb{N}$ since ρ is finite-to-one, Hence the new morphism $V'_\rho(X)(\phi)$ satisfies *finite G-support* and and *continuous control* as ϕ does. Obviously we have for $s \in S$, $s' \in S'$

$$\begin{aligned}
V'_\rho(X)(\phi)_{s,s'} \neq 0 \implies \phi_{s,s'} \neq 0 \implies \eta(s) = \eta'(s')&\\
\implies \rho \circ \eta(s) = \rho \circ \eta'(s') \implies \widehat{\eta}(s) = \widehat{\eta'}(s')&.
\end{aligned}$$

Since $V'_\rho(X)$ maps $\mathcal{T}^G_0(X)$ to $\mathcal{T}^G_0(P; \mathcal{B})$, it induces a functor of additive categories

$$V_\rho(X)\colon \mathcal{D}^G_0(X) \to \mathcal{D}^G_0(X). \tag{22.103}$$

22.10.2 Relating the K-Theory of $\mathcal{D}^G$ and $\mathcal{D}^G_0$

We have explained in Section 22.4 that $\mathcal{D}^G(X; \mathcal{B})$ yields a covariant functor $\mathcal{D}^G\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{ADDCAT}$. One easily checks that the same construction yields a covariant functor

$$\mathcal{D}^G_0\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{ADDCAT}. \tag{22.104}$$

Composition with the functor non-connective K-theory yields the covariant functors

$$\mathbf{K} \circ \mathcal{D}^G \colon \mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}; \tag{22.105}$$

$$\mathbf{K} \circ \mathcal{D}_0^G \colon \mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}. \tag{22.106}$$

By precomposing with the inclusion $\mathsf{Or}(G) \to \mathsf{CW\text{-}COM}$, we get covariant $\mathsf{Or}(G)$-spectra

$$\mathbf{K}^{\mathcal{D}^G} \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}; \tag{22.107}$$

$$\mathbf{K}^{\mathcal{D}_0^G} \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}. \tag{22.108}$$

The main result of this section is

Theorem 22.109 (Relating the K-theory of $\mathcal{D}^G(X)$ and $\mathcal{D}_0^G$). *Define two functions $\rho_O, \rho_E \colon \mathbb{N} \to \mathbb{N}$ by*

$$\rho_O(n) = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even;} \\ \frac{n+1}{2} & \text{if } n \text{ is odd;} \end{cases}$$

$$\rho_E(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathbf{HPO}$ *be the covariant functor* G-$\mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ *given for a G-CW-complex by the homotopy pushout*

$$\begin{tikzcd} \mathbf{K}(\mathcal{D}_0^G(X)) \arrow[r, "\mathbf{K}(V_{\rho_E}(X))"] \arrow[d, "\mathbf{K}(V_{\rho_O}(X))"'] & \mathbf{K}(\mathcal{D}_0^G(X)) \arrow[d] \\ \mathbf{K}(\mathcal{D}_0^G(X)) \arrow[r] & \mathbf{HPO}(X), \end{tikzcd}$$

Then there exists a zigzag of weak homotopy equivalences of covariant functors G-$\mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ *from* $\mathbf{HPO}$ *to* $\mathbf{K} \circ \mathcal{D}^G$.

The remainder of this section is devoted to the proof of Theorem 22.109. This needs some prepration.

For a subset $J \subseteq \mathbb{N}$ define

$$\mathcal{O}_J^G(X) \subseteq \mathcal{O}_G(X); \tag{22.110}$$

$$\mathcal{D}_J^G(X) \subseteq \mathcal{D}_G(X), \tag{22.111}$$

to be the full subcategory of $\mathcal{O}^G(X)$ and $\mathcal{O}^G(X)$ respectively consisting of those objects $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ for which $\operatorname{im}(\eta) \subseteq J$ holds.

Fix a sequence of natural numbers $0 = i_0 < i_1 < i_2 < i_3 < \cdots$ such that $\lim_{\to\infty}(i_j - i_{j-1}) = \infty$ holds, for instance we can take $i_j = \frac{j(j+1)}{2}$ since then $i_0 = 0$ and $i_j - i_{j-1} = j$ holds for $j \geq 1$. Define $\mathbb{N}_j := \{i \in \mathbb{N} \mid i_j \leq i \leq i_{j+1}\}$. Put

$$
\begin{aligned}
E &:= \bigcup_{j=0}^{\infty} \mathbb{N}_{2j};\\
O &:= \bigcup_{j=0}^{\infty} \mathbb{N}_{2j+1};\\
I &:= \{i_1, i_2, \ldots\};\\
I_E &:= \{i_2, i_4, \ldots\};\\
I_O &:= \{i_1, i_3, i_5, \ldots\}.
\end{aligned}
$$

Note that $\mathcal{D}^G_{\mathbb{N}}(X) = \mathcal{D}^G(X)$, $\mathbb{N} = E \cup O$, $E \cap O = I$, $I = I_O \cup I_E$, and $I_O \cap I_E = \emptyset$ hold.

Consider the following commutative diagram of additive categories

$$
\begin{tikzcd}
\mathcal{D}^G_I(X) \arrow[r] \arrow[d] & \mathcal{D}^G_E(X) \arrow[d] \\
\mathcal{D}^G_O(X) \arrow[r] & \mathcal{D}^G(X)
\end{tikzcd}
\tag{22.112}
$$

whose arrows are all inclusions of full additive subcategories and which is natural in X.

Lemma 22.113.

(i) The following inclusions are Karoubi filtrations

$$
\begin{aligned}
\mathcal{D}^G_I(X) &\to \mathcal{D}^G_E(X);\\
\mathcal{D}^G_{I_E}(X) &\to \mathcal{D}^G_O(X);\\
\mathcal{D}^G_{I_O}(X) &\to \mathcal{D}^G_E(X);\\
\mathcal{D}^G_O(X) &\to \mathcal{D}^G(X);
\end{aligned}
$$

(ii) The functor induced on the Karoubi quotients

$$\mathcal{D}^G_E(X)/\mathcal{D}^G_I(X) \to \mathcal{D}^G(X)/\mathcal{D}^G_O(X)$$

is an equivalence of additive categories;

(iii) The diagram (22.112) *is weakly homotopy cocartesian.*

Proof. (i) We only show that the inclusion $\mathcal{D}^G_I(X) \to \mathcal{D}^G_E(X)$ is a Karoubi filtration, the proof for the other inclusions is an obvious variation. Consider an object $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G_E(X)$, objects $\mathbf{U} = (\Sigma^{\mathbf{U}}, \pi^{\mathbf{U}}, \eta^{\mathbf{U}}, \mathsf{B}^{\mathbf{U}})$ and $\mathbf{V} = (\Sigma^{\mathbf{V}}, \pi^{\mathbf{V}}, \eta^{\mathbf{V}}, \mathsf{B}^{\mathbf{V}})$ in $\mathcal{O}^G(X)_I$, and morphisms $\overline{\phi}\colon \mathbf{B} \to \mathbf{U}$ and $\overline{\psi}\colon \mathbf{V} \to \mathbf{B}$ in $\mathcal{D}^G_E(X)$. Let the morphisms $\phi\colon \mathbf{B} \to \mathbf{U}$ and $\psi\colon \mathbf{V} \to \mathbf{B}$ in $\mathcal{O}^G_E(X)$ be representatives of $\overline{\phi}$ and $\overline{\psi}$. Choose a number t such that $\phi_{\sigma,\tau} = 0$ holds for $\sigma \in \Sigma$ and $\tau \in \Sigma^{\mathbf{U}}$ with $|\eta(\sigma) - \eta^{\mathbf{U}}(\tau)| \geq t$, and $\psi_{\rho,\sigma} = 0$ holds for $\rho \in \Sigma^{\mathbf{V}}$ and $\sigma \in \Sigma$ with $|\eta(\sigma) - \eta^{\mathbf{V}}(\rho)| \geq t$. Since $\lim_{j\to\infty}(i_j - i_{j-1}) = \infty$, we can find a natural number $j_0 \geq 1$ such that $(i_j - i_{j-1}) > 2t + 1$ for $j \geq j_0$ holds. We can change the representatives ϕ and ψ such that $\phi_{\sigma,\tau} = \psi_{\rho,\sigma} = 0$ holds for $\sigma \in \Sigma$, $\tau \in \Sigma^{\mathbf{U}}$, and $\rho \in \Sigma^{\mathbf{V}}$, provided that $\pi_{\mathbb{N}}(\sigma) \leq i_{j_0}$ is true. Hence we get for every natural number j the following implications for $\sigma \in \Sigma$, $\tau \in \Sigma^{\mathbf{U}}$, and $\rho \in \Sigma^{\mathbf{V}}$

$$\begin{aligned}&\eta(\sigma) \in \mathbb{N}_{i_{2j}}, \pi_{\mathbb{N}}^{\mathbf{U}}(\tau) \in I, \phi_{\sigma,\tau} \neq 0\\ &\qquad\qquad \implies i_{2j} \leq \eta(\sigma) \leq i_{2j} + t \text{ or } i_{2j+1} - t \leq \eta(\sigma) \leq i_{2j+1};\end{aligned}$$

$$\begin{aligned}&\eta(\sigma) \in \mathbb{N}_{i_{2j}}, \pi_{\mathbb{N}}^{\mathbf{V}}(\rho) \in I, \psi_{\rho,\sigma} \neq 0\\ &\qquad\qquad \implies i_{2j} \leq \eta(\sigma) \leq i_{2j} + t \text{ or } i_{2j+1} - t \leq \eta(\sigma) \leq i_{2j+1}.\end{aligned}$$

Define new objects $\mathbf{B}^{\perp} = (\Sigma^{\perp}, \pi^{\perp}, \eta^{\perp}, \mathsf{B}^{\perp})$ and $\mathbf{B}' = (\Sigma', \pi', \eta', \mathsf{B}')$ in $\mathcal{O}_E^G(X)$ by putting

$$\begin{aligned}\Sigma^{\perp} &= \{\sigma \in \Sigma \mid \eta(\sigma) < i_{2j_0}\}\\ &\qquad \amalg \{\sigma \in \Sigma \mid i_{2j} + t < \eta(\sigma) < i_{2j+1} - t, \text{ for some } j \in \mathbb{N} \text{ with } 2j \geq j_0\};\\ \pi^{\perp} &= \pi|_{\Sigma^{\perp}};\\ \eta^{\perp} &= \eta|_{\Sigma^{\perp}};\\ \mathsf{B}^{\perp} &= \mathsf{B}|_{\Sigma^{\perp}};\\ \Sigma' &= \{\sigma \in \Sigma \mid i_{2j} \leq \eta(\sigma) \leq i_{2j} + t \text{ or } i_{2j+1} - t \leq \eta(\sigma) \leq i_{2j+1}\\ &\qquad\qquad \text{for some } j \in \mathbb{N} \text{ with } 2j \geq j_0\};\\ \pi' &= \pi|_{\Sigma'};\\ \eta' &= \eta|_{\Sigma'};\\ \mathsf{B}' &= \mathsf{B}|_{\Sigma'}.\end{aligned}$$

Since $\Sigma = \Sigma' \amalg \Sigma^{\perp}$, there are obvious morphisms $i' \colon \mathbf{B}' \to \mathbf{B}$ and $i^{\perp} \colon \mathbf{B}^{\perp} \to \mathbf{B}$ in $\mathcal{O}_E^G(X)$ given by the morphisms $\mathrm{id}_{\mathsf{B}'(\sigma')}$ and $\mathrm{id}_{\mathsf{B}^{\perp}(\sigma^{\perp})}$ for $\sigma' \in \Sigma'$ and $\sigma^{\perp} \in \Sigma^{\perp}$ such that $i \oplus i^{\perp} \colon \mathbf{B}' \oplus \mathbf{B}^{\perp} \to \mathbf{B}$ is an isomorphism. Moreover, there are morphisms $\phi' \colon \mathbf{B}' \to \mathbf{U}$ and $\psi' \colon \mathbf{U} \to \mathbf{B}'$ in $\mathcal{O}_E^G(X)$ such that $\phi \circ (i \oplus i^{\perp}) = \phi' \circ \mathrm{pr}'$ and $i' \circ \psi' = \psi$ holds where $\mathrm{pr}' \colon \mathbf{B}' \oplus \mathbf{B}^{\perp} \to \mathbf{B}'$ is the canonical projection.

Define the object $\mathbf{B}^{\mathcal{U}} = (\Sigma^{\mathcal{U}}, \pi^{\mathcal{U}}, \eta^{\mathcal{U}}, \mathsf{B}^{\mathcal{U}})$ in $\mathcal{O}^G(X)_I$ by putting for $\sigma' \in \Sigma'$

$$\begin{aligned}\Sigma^{\mathcal{U}} &= \Sigma'\\ \pi^{\mathcal{U}} &= \pi';\\ \eta^{\mathcal{U}}(\sigma') &= \begin{cases} i_{2j} & \text{if } i_{2j} \leq \eta(\sigma') \leq i_{2j} + t;\\ i_{2j+1} & \text{if } i_{2j+1} - t \leq \eta(\sigma') \leq i_{2j+1};\end{cases}\\ \mathsf{B}^{\mathcal{U}} &= \mathsf{B}'.\end{aligned}$$

We can consider $\mathbf{B}^{\mathcal{U}}$ also as an object in $\mathcal{O}^G_E(X)$. Since $\Sigma^{\mathcal{U}} = \Sigma'$ and $\mathsf{B}^{\mathcal{U}} = \mathsf{B}'$, one easily checks that taking for $\sigma \in \Sigma'$ the identity $\mathrm{id}_{\mathsf{B}'(\sigma)}$ yields well-defined to one another inverse isomorphisms $u\colon \mathbf{B}^{\mathcal{U}} \to \mathbf{B}'$ and $v\colon \mathbf{B}' \to \mathbf{B}^{\mathcal{U}}$ in $\mathcal{O}^G_E(X)$. Define morphisms in $\mathcal{O}^G_E(X)$

$$\begin{array}{lll} i^{\mathcal{U}} := i' \circ u\colon & \mathbf{B}^{\mathcal{U}} & \to \mathbf{B}; \\ \mathrm{pr}^{\mathcal{U}} := v \circ \mathrm{pr}'\colon & \mathbf{B} & \to \mathbf{B}^{\mathcal{U}}; \\ \phi^{\mathcal{U}} := \phi' \circ u\colon & \mathbf{B}^{\mathcal{U}} & \to \mathbf{U}; \\ \psi^{\mathcal{U}} := v \circ \psi'\colon & \mathbf{V} & \to \mathbf{B}^{\mathcal{U}}. \end{array}$$

One easily checks that the images of $i^{\mathcal{U}}$, $i^{\perp}$, $\mathrm{pr}^{\mathcal{U}}$, $\phi^{\mathcal{U}}$ and $\psi^{\mathcal{U}}$ under the projection $\mathcal{O}^G_E(X) \to \mathcal{D}^G_E(X)$ yield the data required for a Karoubi filtration.

(ii) Next we show that for two objects $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ and $\mathbf{B}' = (\Sigma', \pi', \eta', \mathsf{B}')$ in $\mathcal{O}^G_E(X)$ the obvious map

$$\mathrm{mor}_{\mathcal{D}^G_E(X)/\mathcal{D}^G_I(X)}(\mathbf{B}, \mathbf{B}') \to \mathrm{mor}_{\mathcal{D}^G(X)/\mathcal{D}^G_O(X)}(\mathbf{B}, \mathbf{B}') \tag{22.114}$$

is bijective.

We begin with the proof of surjectivity. It is based on the following construction. Consider a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^G(X)$. Since $\mathbb{N} = E \cup O$, one can construct objects $\mathbf{B}^E$ and $\mathbf{B}'^E$ in $\mathcal{O}^G_E(X)$ and $\mathbf{B}^O$ and $\mathbf{B}'^O$ in $\mathcal{O}^G_O(X)$ such that we get in $\mathcal{O}^G(X)$ identifications $\mathbf{B}^O \oplus \mathbf{B}^E = \mathbf{B}$ and $\mathbf{B}'^O \oplus \mathbf{B}'^E = \mathbf{B}'$. Then ϕ can be written as

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{B}^O \oplus \mathbf{B}^E \to \mathbf{B}'^O \oplus \mathbf{B}'^E.$$

Define a morphisms in $\mathcal{O}^G(X)$ by the composite

$$\psi\colon \mathbf{B}^O \oplus \mathbf{B}^E \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & b \end{pmatrix}} \mathbf{B}^O \oplus \mathbf{B}'^O \xrightarrow{\begin{pmatrix} a & \mathrm{id} \\ c & 0 \end{pmatrix}} \mathbf{B}'^O \oplus \mathbf{B}'^E.$$

Then $\mathbf{B}^O \oplus \mathbf{B}'^O$ is an object in $\mathcal{D}^G_O(X)$, the difference $\phi - \psi$ is of the shape $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$, and $d\colon \mathbf{B}^E \to \mathbf{B}'^E$ belongs to $\mathcal{O}^G_E(X)$.

It remains to prove injectivity. Consider a morphism $[\overline{\phi}]\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{D}^G_E(X)/\mathcal{D}^G_I(X)$ whose image under (22.114) is zero. We have to show that $[\overline{\phi}]$ itself is zero. Choose a representative $\overline{\phi}$ in $\mathcal{D}^G_E(X)$ of $[\overline{\phi}]$. By assumption there is an object $\mathbf{U} = (\Sigma^{\mathbf{U}}, \pi^{\mathbf{U}}\eta^{\mathbf{U}}, \mathsf{B}^{\mathbf{U}})$ in $\mathcal{O}^G_O(X)$ such that $\overline{\nu} \circ \overline{\mu} = \overline{\phi}$ holds in $\mathcal{D}^G(X)$ for appropriate morphisms $\overline{\mu}\colon \mathbf{B} \to \mathbf{U}$ and $\overline{\nu}\colon \mathbf{U} \to \mathbf{B}'$. in Choose a representative ϕ in $\mathcal{O}^G_E(X)$ of $\overline{\phi}$, and representatives μ and ν in $\mathcal{O}^G(X)$ respectively of $\overline{\mu}$ and $\overline{\nu}$ respectively. Fix a number t such that for $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ and $\tau \in \Sigma^{\mathbf{U}}$ the implications

$$\begin{aligned}
\phi_{\sigma,\sigma''} \neq 0 &\implies |\eta(\sigma) - \eta'(\sigma')| \leq t;\\
\mu_{\sigma,\tau} \neq 0 &\implies |\eta(\sigma) - \eta^{\mathbf{U}}(\tau)| \leq t;\\
\nu_{\tau,\sigma''} \neq 0 &\implies |\eta^{\mathbf{U}}(\tau) - \eta'(\sigma')| \leq t,
\end{aligned}$$

hold. Since $\lim_{j\to\infty}(i_j - i_{j-1}) = \infty$, we can find a natural number $j_0 \geq 1$ such that $(i_j - i_{j-1}) > 2t + 1$ holds for $j \geq j_0$. By possibly enlarging j_0 we can additionally arrange that $\phi_{\sigma,\sigma'} = \sum_{\tau\in\Sigma^{\mathbf{U}}} \nu_{\tau,\sigma'} \circ \mu_{\sigma,\tau}$ holds for $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ with $\eta(\sigma), \pi'_{\mathbb{N}}(\sigma') \geq i_{j_0}$. Define an object $V = (\Sigma^{\mathbf{V}}, \pi^{\mathbf{V}}, \eta^{\mathbf{V}}, \mathsf{B}^{\mathbf{V}})$ in $\mathcal{D}^G_I(X)$ by putting

$$\begin{aligned}
\Sigma^{\mathbf{V}} &= \{\tau \in \Sigma^{\mathbf{U}} \mid \eta(\tau) \geq i_{j_0} \text{ and } \exists n \in I \text{ with } |n - \eta^{\mathbf{U}}(\tau)| \leq t\};\\
\pi^{\mathbf{V}} &= \pi^{\mathbf{U}}|_{\Sigma^{\mathbf{V}}};\\
\eta^{\mathbf{V}}(\tau) &= n \quad \text{for } \tau \in \Sigma^{\mathbf{V}} \text{ and } n \in I \text{ with } |n - \eta^{\mathbf{U}}(\tau)| \leq t;\\
\mathsf{B}^{\mathbf{V}} &= \mathsf{B}^{\mathbf{U}}|_{\Sigma^{\mathbf{V}}}.
\end{aligned}$$

Define morphisms $\alpha\colon \mathbf{B} \to \mathbf{V}$ and $\beta\colon \mathbf{V} \to \mathbf{B}'$ in $\mathcal{O}^G_E(X)$ by putting for $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ and $\tau \in \Sigma^{\mathbf{V}}$

$$\begin{aligned}
\alpha_{\sigma,\tau} &= \mu_{\sigma,\tau};\\
\beta_{\tau,\sigma'} &= \nu_{\tau,\sigma'}.
\end{aligned}$$

Then $\phi_{\sigma,\sigma'} = \sum_{\tau\in\Sigma^{\mathbf{U}}} \beta_{\tau,\sigma'} \circ \alpha_{\sigma,\tau}$ holds for $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ with $\eta(\sigma), \eta(\sigma') \geq i_{j_0}$. Hence we get $\overline{\phi} = \overline{\beta} \circ \overline{\alpha}$ in $\mathcal{D}^G_E(X)$. Since $\mathbf{V}$ belongs to $\mathcal{D}^G_I(X)$, we get $[\overline{\phi}] = 0$ in $\mathcal{D}^G_E(X)/\mathcal{D}^G_I(X)$. Hence (22.114) is bijective.

It remains to construct for an object $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G(X)$ an object $\mathbf{B}'$ in $\mathcal{O}^G_E(X)$ and morphisms $i\colon \mathbf{B}' \to \mathbf{B}$ and $r\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}^G(X)$ such that $[\overline{r}] \circ [\overline{i}] = \mathrm{id}_{\mathbf{B}'}$ and $[\overline{i}] \circ [\overline{r}] = \mathrm{id}_{\mathbf{B}}$ hold in $\mathcal{D}^G(X)/\mathcal{D}^G_O(X)$. We define $\mathbf{B}' = (\Sigma', \pi', \eta', \mathsf{B}')$ by

$$\begin{aligned}
\Sigma' &= \{\sigma \in \Sigma \mid \eta(\sigma) \in E\};\\
\pi'_{\widehat{P}} &= \pi|_{\Sigma'};\\
\eta' &= \eta|_{\Sigma'};\\
\mathsf{B}' &= \mathsf{B}|_{\Sigma'},
\end{aligned}$$

and the morphisms i and r for $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ by

$$i_{\sigma',\sigma} = r_{\sigma,\sigma'} = \begin{cases} \mathrm{id}_{\mathsf{B}(\sigma')} & \text{if } \sigma = \sigma';\\ 0 & \text{otherwise.}\end{cases}$$

Obviously $r \circ i = \mathrm{id}_{\mathbf{B}'}$ holds already in $\mathcal{O}^G(X)$ which implies $[\overline{r}] \circ [\overline{i}] = \mathrm{id}_{\underline{B'}}$ in $\mathcal{D}^G(X)/\mathcal{D}^G_O(X)$. Define an object $\mathbf{U} = (\Sigma^{\mathbf{U}}, \pi^{\mathbf{U}}, \eta^{\mathbf{U}}, \mathsf{B}^{\mathbf{U}})$ in $\mathcal{O}^G_O(X)$ by

$$\begin{aligned}
\Sigma^{\mathbf{U}} &= \{\sigma \in \Sigma \mid \eta(\sigma) \notin E\};\\
\pi^{\mathbf{U}} &= \pi|_{\sigma^{\mathbf{U}}};\\
\eta^{\mathbf{U}} &= \eta|_{\sigma^{\mathbf{U}}};\\
\mathsf{B}' &= \mathsf{B}|_{\sigma^{\mathbf{U}}}.
\end{aligned}$$

Obviously $i \circ r - \mathrm{id}_{\mathbf{B}} = \mathrm{id}_{\mathbf{U}}$ holds in $\mathcal{O}^G(X)$. This implies $[\bar{i}] \circ [\bar{r}] = \mathrm{id}_{\mathbf{B}}$ in $\mathcal{D}^G(X)/\mathcal{D}^G_O(X)$.

(iii) This follows from assertions (i) and (ii) and Theorem 21.30. This finishes the proof of Lemma 22.113. □

Lemma 22.115. *The inclusions $\mathcal{D}^G_{I_O}(X) \to \mathcal{D}^G_E(X)$ and $\mathcal{D}^G_{I_E}(X) \to \mathcal{D}^G_O(X)$ induce weak equivalences*

$$\begin{aligned}
\mathbf{K}(\mathcal{D}^G_{I_O}(X)) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G_E(X));\\
\mathbf{K}(\mathcal{D}^G_{I_E}(X)) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G_O(X)).
\end{aligned}$$

Proof. We give the proof only for the first map, the one for the second is completely analogous. We have already shown in Lemma 22.113 (i) that the inclusion $\mathcal{D}^G_{I_O}(X) \to \mathcal{D}^G_E(X)$ is a Karoubi filtration. Hence it suffices to show that $\mathbf{K}\big(\mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X)\big)$ is weakly contractible. This we will do by constructing an Eilenberg swindle as follows.

Next we define a functor of additive categories

$$\mathrm{SH}\colon \mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X) \to \mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X). \tag{22.116}$$

The idea is to move the objects one position to the right in the $\mathbb{N}$-direction, to discard the objects sitting at right endpoints of the intervals $\mathbb{N}_{2j}$ since they would be moved outside the set E, and leaving the position in the X-direction fixed. AS the union of the right endpoints of the intervals $\mathbb{N}_{2j}$ for $j \geq 0$ is I_O, this gives a well-defined functor. Here are more details.

An object $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ of $\mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X)$ which is the same as an object in $\mathcal{O}^G_E(X)$, is sent to the object $\mathrm{SH}(\mathbf{B}) = (\Sigma^{\mathrm{SH}}, \pi^{\mathrm{SH}}, \eta^{\mathrm{SH}}, \mathsf{B}^{\mathrm{SH}})$ in $\mathcal{O}^G_E(X)$ given by

$$\begin{aligned}
\Sigma^{\mathrm{SH}} &= \{\sigma \in \Sigma \mid \eta(\sigma) \in E \setminus I_O\};\\
\pi^{\mathrm{SH}} &= \pi|_{\Sigma^{\mathrm{SH}}};\\
\eta^{\mathrm{SH}}(\sigma) &= \eta(\sigma) + 1 \quad \text{for } \sigma \in \Sigma^{\mathrm{SH}};\\
\mathsf{B}^{\mathrm{SH}} &= \mathsf{B}|_{\Sigma^{\mathrm{SH}}}.
\end{aligned}$$

Consider a morphism $[\phi]\colon \mathbf{B} = (\Sigma, \pi_{\mathbb{N}}, [\pi_{\widehat{P}}]_P, \mathsf{B}) \to \mathbf{B}' = (\Sigma', \pi'_{\mathbb{N}}, [\pi'_{\widehat{P}}]_P, \mathsf{B}')$ in $\mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X)$. Let $\phi\colon \mathbf{B} \to \mathbf{B}'$ be a morphism in $\mathcal{O}^G_E(X)$ representing $[\phi]$. Define a morphism $\mathrm{SH}(\phi)$ in $\mathcal{O}^G_E(X)$ by

$$\mathrm{SH}(\phi)_{\sigma,\sigma'} = \phi_{\sigma,\sigma'} \quad \text{for } \sigma \in \Sigma^{\mathrm{SH}}, \sigma' \in (\Sigma')^{\mathrm{SH}}.$$

Define $\mathrm{SH}(\phi)$ to be the class $[\mathrm{SH}(\phi)]$ of $\mathrm{SH}(\phi)$. Note that $\mathrm{SH}(\phi)$ depends on the choice of $\phi \in [\phi]$. We leave it to the reader to check that $[\mathrm{SH}(\phi)]$ depends only on $[\phi]$. Moreover, let $[\overline{\phi}]\colon \mathbf{B} \to \mathbf{B}'$ and $[\psi]\colon \mathbf{B}' \to \mathbf{B}''$ be composable morphisms in $\mathcal{D}_E^G(X)$. Choose representatives $\phi \in [\phi]$ and $\psi \in [\psi]$. Then it is not true that $\mathrm{SH}(\psi) \circ \mathrm{SH}(\phi)$ and $\mathrm{SH}(\psi \circ \phi)$ agree, but one easily checks that the classes $[\mathrm{SH}(\psi) \circ \mathrm{SH}(\phi)] = [\mathrm{SH}(\psi \circ \phi)]$ in $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ agree. Therefore the functor announced in (22.116) is well-defined.

Next we construct a natural equivalence

$$R_1\colon \mathrm{id}_{\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)} \xrightarrow{\cong} \mathrm{SH}\,. \tag{22.117}$$

of functors $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ of additive categories.

We specify for every object $\mathbf{B}$ in $\mathcal{O}_E^G(X)$ morphisms $\phi\colon \mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B}) \to \mathrm{SH}(\mathbf{B}) = (\Sigma^{\mathrm{SH}}, \pi^{\mathrm{SH}}, \eta^{\mathrm{SH}}, \mathsf{B}^{\mathrm{SH}})$ and $\psi\colon\ \mathrm{SH}(\mathbf{B}) \to \mathbf{B}$ in $\mathcal{O}_E^G(X)$ by putting for $\sigma \in \Sigma$ and $\sigma^{\mathrm{SH}} \in \Sigma^{\mathrm{SH}}$

$$\phi_{\sigma,\sigma^{\mathrm{SH}}} = \begin{cases} \mathrm{id}_{\mathsf{B}(\sigma)} & \text{if } \sigma^{\mathrm{SH}} = \sigma; \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi_{\sigma^{\mathrm{SH}},\sigma} = \begin{cases} \mathrm{id}_{\mathsf{B}(\sigma)} & \text{if } \sigma = \sigma^{\mathrm{SH}}; \\ 0 & \text{otherwise.} \end{cases}$$

We have $\phi \circ \psi = \mathrm{id}_{\mathrm{SH}(\mathbf{B})}$ in $\mathcal{O}_E^G(X)$. We do not have $\psi \circ \phi = \mathrm{id}_{\mathbf{B}}$ in $\mathcal{O}_E^G(X)$ but $[\psi \circ \phi] = [\mathrm{id}_{\mathbf{B}}]$ holds in $\mathcal{O}_E^G(X)/\mathcal{D}_{I_O}^G(X)$. Now one easily checks that the natural equivalence R_1 announced in (22.117) is well-defined.

Next we define another functor

$$S\colon \mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X). \tag{22.118}$$

The informal definition is $S(\mathbf{B}) = \bigoplus_{m=0}^{\infty} \mathrm{SH}^m(\mathbf{B})$ and analogous for morphisms where SH^m is the m-fold composite of SH. This makes sense since over a given element in $\mathbb{N}$ this direct sum is finite. Here are the more details of the definition.

An object $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ in $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ which is the same as an object in $\mathcal{O}_E^G(X)$, is sent to the object $S(\mathbf{B}) = (\Sigma^S, \pi^S, \eta^S, \mathsf{B}^S)$ in $\mathcal{D}_E^G(X)$ given by

$$\begin{aligned} \Sigma^S &= \coprod_{j \in \mathbb{N}} \coprod_{n \in \mathbb{N}_{2j}} \coprod_{k=i_{2j}}^{n} \eta^{-1}(k); \\ \pi^S &= \pi|_{\Sigma^S}; \\ \eta^S(\sigma) &= n \quad \text{for } \sigma \in \coprod_{k=i_{2j}}^{n} \eta^{-1}(k); \\ \mathsf{B}^S &= \mathsf{B}|_{\Sigma^S}. \end{aligned}$$

Consider a morphism $[\phi]\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{D}^G_E(X)/\mathcal{D}^G_{I_O}(X)$. Let the morphism $\phi\colon \mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B}) \to \mathbf{B}' = (\Sigma', \pi', \eta', \mathsf{B}')$ in $\mathcal{O}^G_E(X)$ be a representative of $[\phi]$. Define a morphism $S(\phi)$ in $\mathcal{O}^G_E(X)$ by putting for $\sigma \in \Sigma^S$ and $\sigma' \in (\Sigma')^S$

$$S(\underline{s})_{\sigma,\sigma'} = \begin{cases} \phi_{\sigma,\sigma'} & \text{if } \exists j \in \mathbb{N}, n, n' \in \mathbb{N}_{2j} \text{ with } i_{2j} \le \eta(\sigma) \le n, i_{2j} \le \pi'_{\mathbb{N}}(\sigma') \le n' \\ & \qquad\qquad \text{and } n - \eta(\sigma) = n' - \pi'_{\mathbb{N}}(\sigma'); \\ 0 & \text{otherwise.} \end{cases}$$

Now define $S([\phi])$ to be $[S(\phi)]$.

Next we construct a natural equivalence

$$R_2\colon \operatorname{id}_{\mathcal{O}^G_E(X)} \oplus (\mathrm{SH} \circ S) \xrightarrow{\cong} S \tag{22.119}$$

of functors $\mathcal{O}^G_E(X)/\mathcal{D}^G_{I_O}(X) \to \mathcal{O}^G_E(X)/\mathcal{D}^G_{I_O}(X)$ of additive categories. The idea comes from the formula

$$\begin{aligned} \mathbf{B} \oplus \mathrm{SH}(S(\mathbf{B})) &= \mathbf{B} \oplus \mathrm{SH}\left(\bigoplus_{m=0}^{\infty} \mathrm{SH}^m(S(B)) \right) \\ &= \mathbf{B} \oplus \bigoplus_{m=1}^{\infty} \mathrm{SH}^m(S(B)) = \bigoplus_{m=0}^{\infty} \mathrm{SH}^m(S(B)) = S(\mathbf{B}). \end{aligned}$$

Here are the some details of the construction. Note that for an object $\mathbf{B} = (\Sigma, \pi, \eta, \mathsf{B})$ in $\mathcal{O}^G_E(X)$ the source of $R_2(\mathbf{B})$ is given by the quadruple $(\Sigma', \pi', \eta', \mathsf{B}')$ and the target by the quadruple $(\Sigma^S, \pi^S, \eta^S, \mathsf{B}^S)$ such that

$$\begin{aligned} \Sigma' &= \Sigma \amalg (\Sigma^S)^{\mathrm{SH}} \\ &= \Sigma \amalg \{\sigma \in \Sigma^S \mid \eta^S(\sigma) \in E \setminus I_0\} \\ &= \left(\coprod_{j \in \mathbb{N}} \coprod_{n=i_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left\{ \sigma \in \coprod_{j \in \mathbb{N}} \coprod_{n \in \mathbb{N}_{2j}} \coprod_{k=i_{2j}}^{n} \eta^{-1}(k) \;\middle|\; \eta^S(\sigma) \in E \setminus I_O \right\} \\ &= \left(\coprod_{j \in \mathbb{N}} \coprod_{n=i_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left(\coprod_{j \in \mathbb{N}} \coprod_{\substack{n \in \mathbb{N}_{2j}, \\ n \ne i_{2j+1}}} \coprod_{k=i_{2j}}^{n} \pi_{\mathbb{N}}^{-1}(k) \right) \\ &= \left(\coprod_{j \in \mathbb{N}} \coprod_{n=i_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left(\coprod_{j \in \mathbb{N}} \coprod_{\substack{n \in \mathbb{N}_{2j}, \\ n \ne i_{2j}}} \coprod_{k=i_{2j}}^{n-1} \pi_{\mathbb{N}}^{-1}(k) \right) \\ &= \coprod_{j \in \mathbb{N}} \coprod_{n \in \mathbb{N}_{2j}} \coprod_{k=i_{2j}}^{n} \eta^{-1}(k) \\ &= \Sigma^S. \end{aligned}$$

Note that any element $\sigma \in \Sigma'$ belongs to Σ. Moreover, under the identification $\Sigma' = \Sigma^S$ above we have $\mathsf{B}'(\sigma) = \mathsf{B}^S(\sigma) = \mathsf{B}(\sigma)$ for $\sigma \in \Sigma'$. So we can define an isomorphism in $\mathcal{O}_E^G(X)$

$$R_2'(\mathbf{B})\colon \mathbf{B} \oplus (\mathrm{SH} \circ S)(\mathbf{B}) = (\Sigma', \pi', \eta', \mathsf{B}') \to S(\mathbf{B}) = (\Sigma^S, \pi^S, \eta^S, \mathsf{B}^S)$$

by putting $R_2'(\mathbf{B})_{\sigma_0,\sigma_1} = \mathrm{id}_{\mathsf{B}(\sigma_0)}$ if $\sigma_0 = \sigma_1$ and $R_2(\mathbf{B})_{\sigma_0,\sigma_1} = 0$ if $\sigma_0 \neq \sigma_1$ for $\sigma_0 \in \Sigma'$ and $\sigma_1 \in \Sigma^S$. Now define $R_2(\mathbf{B})$ by $[R_2'(\mathbf{B})]$. We leave it to the reader to check that the natural equivalence announced in (22.119) is well-defined.

Putting R_1 and R_2 together yields a natural equivalence of functors of additive categories $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$

$$R\colon \mathrm{id}_{\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)} \oplus S \xrightarrow{\simeq} S.$$

Theorem 6.37 (iii) implies that the spectrum $\mathbf{K}\big(\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)\big)$ is weakly contractible. This finishes the proof of Lemma 22.115. $\square$

Define injective function $\rho_I, \rho_{I_O}, \rho_{I_E}\colon \mathbb{N} \to \mathbb{N}$

$$\begin{aligned} \rho_I(j) &= i_{j+1};\\ \rho_{I_E}(j) &= i_{2j+2}.\\ \rho_{I_O}(j) &= i_{2j+1}. \end{aligned}$$

By construction they induce bijections from $\mathbb{N}$ to I, I_E, and I_O respectively.

Lemma 22.120. *Let J be I, I_O, or I_E. Then the functor $V_{\rho_J}(X)\colon \mathcal{D}_0^G(X) \to \mathcal{D}_0^G(X)$ of* (22.103) *induces an isomorphism of additive categories*

$$V_J(X)\colon \mathcal{D}_0^G(X) \xrightarrow{\cong} \mathcal{D}_J^G(X).$$

Proof. We only treat the case $J = I$, the other cases are completely analogous. The functor $V_I(X)$ is bijective on the set of objects since the function $\mathbb{N} \to I$ sending j to i_{j+1} is a bijection. Hence it remains to show for two objects $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ and $\mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$ in $\mathcal{O}_0^G(X)$ that the map induced by $V_I(X)$

$$\mathrm{mor}_{\mathcal{D}_0^G(X)}(\mathbf{B}, \mathbf{B}') \to \mathrm{mor}_{\mathcal{D}_G(X)_I}(V_I(X)(\mathbf{B}), V_I(X)(\mathbf{B}')), \quad [\phi] \mapsto V_I(X)([\phi])$$

is bijective. It is obvious that it is injective. Hence we give only more details for the proof of surjectivity. Consider a morphism $[\psi]\colon V_I(X)(\mathbf{B}) \to V_I(X)(\mathbf{B}')$ in $\mathcal{D}_I^G(X)$. Choose a representative $\psi\colon V_I(X)(\mathbf{B}) \to V_I(X)(\mathbf{B}')$ in $\mathcal{O}_I^G(X)$. There is a natural number n such that the implication $\psi_{s,s'} \neq 0 \implies |\widehat{\eta}(s) - \widehat{\eta'}(s')| \leq n$ holds for $s \in S$ and $s' \in S'$. Choose a natural number $j_0 \geq 1$ such that $i_j - i_{j-1} > n$ holds for $j \geq j_0$. Then the implication $\psi_{s,s'} \neq 0 \implies \widehat{\eta}(s) = \widehat{\eta'}(s')$ holds for $s \in S$ and $s' \in S'$ with $\widehat{\eta}(s), \widehat{\eta'}(s') \geq i_{j_0}$. We can additionally arrange without changing $[\psi]$ that that $\psi_{s,s'} = 0$ holds for $\eta(s) \le i_{j_0} + n$. Then the implication $\psi_{s,s'} \ne 0 \implies \widehat{\eta}(s) = \widehat{\eta}(s')$ holds for $s \in S$ and $s' \in S'$. Since $\widehat{\eta}(s) = \widehat{\eta}(s') \implies \eta(s) = \eta'(s')$, we can construct a morphism $\phi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{O}_0^G(X)$ satisfying $F_I'(X)(\phi) = \psi$. Note that ϕ satisfies *continuous control* as ψ satisfies *continuous control* and for every natural number N there is a natural number N' such that for all $j \in \mathbb{N}$ the implication $j \ge N' \implies i_j \ge N$ holds. This implies that $[\psi]$ is in the image of the map above. This finishes the proof of Lemma 22.120. $\square$

Next we define functors of additive categories, natural in X,

$$R_O(X)\colon \mathcal{D}_O^G(X) \to \mathcal{D}_{I_E}^G(X); \tag{22.121}$$

$$R_E(X)\colon \mathcal{D}_E^G(X) \to \mathcal{D}_{I_O}^G(X), \tag{22.122}$$

satisfying

$$R_O(X)|_{\mathcal{D}_{I_E}^G(X)} = \mathrm{id}_{\mathcal{D}_{I_E}^G(X)}; \tag{22.123}$$

$$R_E(X)|_{\mathcal{D}_{I_O}^G(X)} = \mathrm{id}_{\mathcal{D}_{I_O}^G(X)}. \tag{22.124}$$

We only explain the construction of $R_O(X)$, the one for $R_E(X)$ is completely analogous. It will be induced by the following functor of additive categories

$$R_O'(X)\colon \mathcal{O}_O^G(X) \to \mathcal{O}_{I_E}^G(X)$$

whose definition we describe next. An object $\mathbf{B} = (S, \pi, \eta, \mathsf{B})$ is sent by $R_O'(X)$ to the object $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathsf{B}})$ given by

$$\begin{aligned} \widehat{S} &= S; \\ \widehat{\pi} &= \pi; \\ \widehat{\eta}(s) &= i_{2j+2} \quad \text{if } \eta(s) \in \mathbb{N}_{2j+1}; \\ \widehat{\mathsf{B}} &= \mathsf{B}. \end{aligned}$$

The idea is to move an object with position in $\mathbb{N}_{2j+1}$ to the right endpoint of $\mathbb{N}_{2j+1}$, namely to i_{2j+2}, whereas nothing is changed concerning the X-direction. Obviously $\widehat{\mathbb{B}}$ satisfies the conditions *compact support over* X and *locally finiteness over* $\mathbb{N}$ since $\mathbf{B}$ does and $\mathbb{N}_{2j+1}$ is finite. The definition on morphisms is the tautological one. If $\phi\colon \mathbf{B} = (S, \pi, \eta, \mathsf{B}) \to \mathbf{B}' = (S', \pi', \eta', \mathsf{B}')$ is given by the collection $\{\phi_{s,s'} \mid s \in S, s' \in S'\}$, define $R_O'(\phi)$ by the same collection. Obviously $R_O(\phi)$ satisfies *finite G-support* as ϕ does. Since ϕ satisfies *bounded control over* $\mathbb{N}$, we can find a natural number n such that for $s \in S$ and $s' \in S'$ the implication $\phi_{s,s'} \ne 0 \implies |\eta(s) - \eta'(s')| \le n$ holds. Choose a natural number m such that $i_{2j+1} - i_{2j} > n$ holds for $j \ge m$. If $\eta(s) \in \mathbb{N}_{2j+1}$ for $j \ge m$, we conclude $\eta'(s') \in \mathbb{N}_{2j+1}$ and hence $\widehat{\eta}(s) = \widehat{\eta'}(s')$. Put $l = i_{2m} + n$. Then we have for $s \in S$ and $s' \in S'$ the implication

$\phi_{s,s'} \neq 0 \implies |\widehat{\eta}(s) - \widehat{\eta'}(s')| \leq l$. This shows that $R'_O(\phi)$ satisfies *bounded control over* $\mathbb{N}$. Since ϕ satisfies *continuous control* and for every natural number N there exists a natural number N' satisfying $\widehat{\eta}(s) \geq N' \implies \eta(s) \geq N$ for $s \in S$ and $\widehat{\eta'}(s') \geq N' \implies \eta'(s) \geq N$ for $s' \in S'$, *continuous control* holds also for $R_O(\phi)$.

Obviously $R'_O(X)$ induces the identity on $\mathcal{O}^G_{I_E}(X)$ and sends $\tau^G_O(X)$ to $\tau^G_{I_E}(X)$. Hence R'_ϕ induces the desired functor R_O announced in (22.121) and satisfying (22.123).

Lemma 22.125. *The functors* $R_O(X)$ *of* (22.121) *and* $R_E(X)$ *of* (22.122) *induces weak equivalences, natural in* X*,*

$$\mathbf{K}(R_O(X))\colon \mathbf{K}(\mathcal{D}^G_O(X)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G_{I_E}(X));$$
$$\mathbf{K}(R_E(X))\colon \mathbf{K}(\mathcal{D}^G_E(X)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G_{I_O}(X)).$$

Proof. Because of (22.123) and (22.124) it suffices to show that the inclusions $\mathcal{D}^G_{I_E}(X) \to \mathcal{D}^G_O(X)$ and $\mathcal{D}^G_{I_O}(X) \to \mathcal{D}^G_E(X)$ induce weak homotopy equivalences on K-theory. This has already been done, see Lemma 22.115. □

Proof of Theorem 22.109. Consider the following diagram of additive categories, natural in X,

$$\begin{tikzcd}
\mathcal{D}^G_O(X) \arrow[d, "V_{I_E}(X)^{-1}\circ R_O(X)"] & \mathcal{D}^G_I(X) \arrow[l] \arrow[r] \arrow[d, "V_I(X)^{-1}"] & \mathcal{D}^G_E(X) \arrow[d, "V_{I_O}(X)^{-1}\circ R_E(X)"] \\
\mathcal{D}^G_0(X) & \mathcal{D}^G_0(X) \arrow[l, "V_{\rho_O}(X)"'] \arrow[r, "V_{\rho_E}(X)"'] & \mathcal{D}^G_0(X)
\end{tikzcd}$$

where the upper two horizontal arrows are the inclusions, the functors $V_{\rho_O}(X)$ and $V_{\rho_E}(X)$ have been defined in (22.103), the isomorphisms of additive categories $V_{I_E}(X)$, V_I, and $V_{I_O}(X)$ come from Lemma 22.120, the functors $R_O(X)$ and $R_E(X)$ have been defined in (22.121) and (22.122). If we apply the K-theory functor, we obtain a commutative diagram of spectra, natural in X

$$\begin{tikzcd}
\mathbf{K}(\mathcal{D}^G_O(X)) \arrow[d, "\simeq"] & \mathbf{K}(\mathcal{D}^G_I(X)) \arrow[l] \arrow[r] \arrow[d, "\simeq"] & \mathbf{K}(\mathcal{D}^G_E(X)) \arrow[d, "\simeq"] \\
\mathbf{K}(\mathcal{D}^G_0(X)) & \mathbf{K}(\mathcal{D}^G_0(X)) \arrow[l, "\mathbf{K}(V_{\rho_O}(X))"'] \arrow[r, "\mathbf{K}(V_{\rho_E}(X))"'] & \mathbf{K}(\mathcal{D}^G_0(X))
\end{tikzcd}$$

whose horizontal arrows are weak homotopy equivalences by Lemma 22.125. It induces a weak homotopy equivalence from the homotopy pushout $\mathbf{HPO}(X)$ of the upper row to the lower row, natural in X. We have already constructed a weak homotopy equivalences from $\mathbf{HPO}(X)$ to $\mathbf{K}(\mathcal{D}^G(X))$, natural in X, in Lemma 22.113 (iii). This finishes the proof of Theorem 22.109. □

22.11 The Proof of the Axioms of a G-Homology Theory for $\mathcal{D}_0^G$

Next we state the main result of this section.

Theorem 22.126 (The algebraic K-groups of $\mathcal{D}_0^G(X,A)$ yield a G-homology theory). *Let $\mathcal{B}$ be a control coefficient category in the sense of Definition 22.1.*

Then we obtain a G-homology theory with values in $\mathbb{Z}$-modules in the sense of Definition 12.1 by the covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups sending (X,A) to $K_*(\mathcal{D}_0^G(X,A;\mathcal{B}))$.

First we start with G-homotopy invariance. Here the proof for $\mathcal{D}^G$ of Lemma 22.30 does not carry over since there we are shifting in the $\mathbb{N}$-direction and the construction of the natural equivalence in the relevant Eilenberg-swindle cannot be done with zero-control in the $\mathbb{N}$-direction. Therefore we have to construct a different Eilenberg-swindle where we do not move the objects in the $\mathbb{N}$-direction.

Proposition 22.127. *The inclusion $X \times \{0\} \to X \times [0,1]$ induces a weak homotopy equivalence*

$$\mathbf{K}(\mathcal{D}_0^G(X \times \{0\})) \to \mathbf{K}(\mathcal{D}_0^G(X \times [0,1])).$$

Proof. We define a functor of additive categories

$$\mathrm{SH}\colon \mathcal{O}_0^G(X \times [0,1]) \to \mathcal{O}_0^G(X \times [0,1]) \tag{22.128}$$

as follows.

Consider an object $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ in $\mathcal{O}_0^G(X \times [0,1])$. In the sequel let $\pi_X : S \to X$ and $\pi_{[0,1]} : S \to [0,1]$ be the maps for which $\pi = \pi_X \times \pi_{[0,1]}$. We define $\mathrm{SH}(\mathbf{B})$ to be the object $(\mathrm{SH}(S), \mathrm{SH}(\pi), \mathrm{SH}(\eta), \mathrm{SH}(\mathsf{B}))$ given by

$$\begin{aligned}
\mathrm{SH}(S) &= \{(s,n) \in S \times \mathbb{N} \mid n \le \eta(s) \cdot \pi_{[0,1]}(s)\};\\
\mathrm{SH}(\pi)(s,n) &= \begin{cases} \pi(s) & \text{if } \eta(s) = 0;\\ \left(\pi_X(s), \pi_{[0,1]}(s) - \frac{n}{\eta(s)}\right) & \text{if } \eta(s) \ge 1;\end{cases}\\
\mathrm{SH}(\eta)(s,n) &= \eta(s);\\
\mathrm{SH}(\mathsf{B})(s,n) &= \mathsf{B}(s).
\end{aligned}$$

The idea is to shift an object $\mathsf{B}(s)$ from position $\pi_{[0,1]}(s)$ to position $\pi_{[0,1]}(s) - \frac{1}{\eta(s)}$ if $\eta(s) \ge 1$ and $\pi_{[0,1]}(s) - \frac{1}{\eta(s)} \ge 0$ hold, to forget it if $\eta(s) \ge 1$ and $\pi_{[0,1]}(s) - \frac{1}{\eta(s)} < 0$ hold, and to leave it at $\pi_{[0,1]}(s)$ if $\eta(s) = 0$ holds, whereas $\pi_X(s)$ and $\eta(s)$ are unchanged. Then take the infinite direct sum over $k \in \mathbb{N}$

for the k-fold composition. So here we are shifting in the direction of $[0,1]$ and not in the direction of $\mathbb{N}$.

We have to check that is well-defined. Since $\operatorname{im}(\mathrm{SH}(\pi)) \subseteq \operatorname{im}(\pi_X) \times [0,1]$ and $\mathbf{B}$ satisfies *compact support over* $X \times [0,1]$, $\mathrm{SH}(\mathbf{B})$ satisfies *compact support over* $X \times [0,1]$. As $\mathbf{B}$ satisfies locally finiteness over $\mathbb{N}$, the same is true for $\mathrm{SH}(\mathbf{B})$, as we get for $m \in \mathbb{N}$

$$\begin{aligned}\mathrm{SH}(\eta)^{-1}(m) &= \{(s,n) \mid \eta(s) = m, n \leq \eta(s) \cdot \pi_{[0,1]}(s)\}\\ &\subseteq \bigcup_{s \in \eta^{-1}(m)} \{m \in \mathbb{N} \mid n \leq m\}.\end{aligned}$$

The definition on morphisms is the tautological one. If the morphism $\phi\colon \mathbf{B} = (S,\pi,\eta,\mathsf{B}) \to \mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$ is given by the collection $\{\phi_{s,s'}\colon \mathsf{B}(s) \to \mathsf{B}(s') \mid s \in S, s' \in S'\}$, then define $\mathrm{SH}(\phi)$ by the collection $\{\mathrm{SH}(\phi_{(s,n),(s',n')}) \mid (s,n) \in \mathrm{SH}(S), (s',n') \in \mathrm{SH}(S')\}$ where

$$\mathrm{SH}(\phi_{(s,n),(s',n')}) = \begin{cases} \phi_{s,s'} & \text{if } n = n';\\ 0 & \text{otherwise.}\end{cases}$$

We have to check that this is well-defined. Since ϕ satisfies *finite G-support* and $\mathrm{SH}(\phi)_{(s,n),(s',n')}$ is zero or $\phi_{s,s'}$, $\mathrm{SH}(\phi)$ satisfies *finite G-support*. If $\mathrm{SH}(\phi)_{(s,n),(s',n')} \neq 0$ holds, then $\phi_{s,s'} \neq 0$ and hence we get $\mathrm{SH}(\eta)(s,n) = \eta(s) = \eta'(s') = \mathrm{SH}(\eta')(s',n')$. The more complicated part is to show that $\mathrm{SH}(\phi)$ satisfies *continuous control* what we do next. We only deal with the implication (22.5). The proof for the other implication (22.6) is completely analogous.

Consider $(x,t) \in X \times [0,1]$ and an open $G_{(x,t)}$-invariant neighborhood U of (x,t) in $X \times [0,1]$. Choose an open G_x-invariant neighborhood V of x in X and $\epsilon > 0$ such that $V \times I_{3\epsilon}(t) \subseteq U$ holds where $I_\epsilon(t) = (t-\epsilon, t+\epsilon) \cap [0,1]$. Since ϕ satisfies *continuous control*, we can find $\delta(x,t,\epsilon) > 0$ with $\delta(x,t,\epsilon) \leq \epsilon$, $r'(x,t,\epsilon) \in \mathbb{N}$, and an open G_x-invariant neighborhood $V'(x,t,\epsilon)$ of x in X such that $V(x,t,\epsilon) \subseteq V$ and $\delta(x,t,\epsilon) \leq \epsilon$ hold and we have for every $s \in S$, $s' \in S'$ and $g \in G$ the implication

$$\begin{aligned}(22.129)\quad g\pi_X(s) \in V'(x,t,\epsilon), \pi_{[0,1]}(s) \in I_{\delta(x,t,\epsilon)}(t), \eta(s) \geq r'(x,t,\epsilon)&\\ \implies \pi'_X(s') \in V, \pi'_{[0,1]}(s') \in I_\epsilon(t).&\end{aligned}$$

Since $[0,1]$ is compact, we can find a finite subset $\{t_1, t_2, \ldots, t_l\} \subseteq [0,1]$ satisfying $\bigcup_{i=1}^{l} I_{\delta(x,t_i,\epsilon)}(t_i) = [0,1]$. Put

$$\begin{aligned} r' &= \max\{r'(x,t_i,\epsilon) \mid i = 1,2\ldots,l\};\\ V' &= \bigcap_{i=1}^{l} V'(x,t_i,\epsilon).\end{aligned}$$

Then r' is a natural number and V' is an open G_x-invariant neighborhood of x in X.

Now we are ready to prove the implication (22.5) for $\mathrm{SH}(\phi)$. Consider $(s,n) \in \mathrm{SH}(S)$, $(s',n') \in \mathrm{SH}(S')$, and $g \in \mathrm{supp}_G(\mathrm{SH}(\phi)_{(s,n),(s',n')})$. We want to show

$$g\,\mathrm{SH}(\pi)(s,n) \in V' \times I_\epsilon(t), \mathrm{SH}(\eta)(s,n) \geq r' \implies \mathrm{SH}(\pi')(s',n') \in U.$$

As $\mathrm{SH}(\phi)_{(s,n),(s',n')} = \phi_{s,s'}$, we get $g \in \mathrm{supp}_G(\phi_{s,s'})$ and $\mathrm{SH}(\phi)_{(n,n),(n',s')} \neq 0$. Choose $i \in \{1,2,\dots,l\}$ with $\pi_{[0,1]}(s) \in I_{\delta(x,t_i,\epsilon)}(t_i)$. As $V' \subseteq V'(x,t_i,\epsilon)$ and $r'(x,t_i,\epsilon) \leq r'$ hold, we get $\pi'_X(s') \in V$ and $\pi'_{[0,1]}(s') \in I_\epsilon(t_i)$ from (22.129). Since $\delta' \leq \delta(x,t_i,\epsilon) \leq \epsilon$ holds and we have $\pi_{[0,1]}(s) \in I_{\delta(x,t_i,\epsilon)}(t_i)$ and $\pi'_{[0,1]}(s') \in I_\epsilon(t_i)$, we conclude from the triangle inequality $|\pi_{[0,1]}(s) - \pi'_{[0,1]}(s')| \leq 2\epsilon$. Since $\mathrm{SH}(\phi)_{(s,n),(s',n')} = \phi_{s,s'} \neq 0$, we have $n = n'$ and $\eta(s) = \eta'(s')$. Hence we get for $\eta(n) \geq 1$

$$\begin{aligned}
&|\,\mathrm{SH}(\pi)_{[0,1]}(s,n) - \mathrm{SH}(\pi')_{[0,1]}(s',n')| \\
&= \left|\left(\pi_{[0,1]}(s) - \frac{n}{\eta(s)}\right) - \left(\pi'_{[0,1]}(s') - \frac{n'}{\eta'(s')}\right)\right| \\
&= |\pi_{[0,1]}(s) - \pi'_{[0,1]}(s')| \\
&\leq 2\epsilon.
\end{aligned}$$

If $\eta(s) = 0$, we get $|\,\mathrm{SH}(\pi)_{[0,1]}(s,n) - \mathrm{SH}(\pi')_{[0,1]}(s',n')| = |\pi_{[0,1]}(s) - \pi'_{[0,1]}(s')| \leq 2\epsilon$. Hence we get for $(s,n) \in \mathrm{SH}(S)$, $(s',n') \in \mathrm{SH}(S')$, and $g \in \mathrm{supp}_G(\mathrm{SH}(\phi)_{(s,n),(s',n')})$ satisfying $g\,\mathrm{SH}(\pi)(s,n) \in V' \times I_{\delta'}(t)$ and $\mathrm{SH}(\eta)(s,n) \geq r'$

$$\begin{aligned}
\mathrm{SH}(\pi')_X(s,n') &\in V; \\
|\,\mathrm{SH}(\pi)_{[0,1]}(s,n) - \mathrm{SH}(\pi')_{[0,1]}(s',n')| &\leq 2\epsilon.
\end{aligned}$$

The latter implies using $\mathrm{SH}(\pi)_{[0,1]}(s,n) \in I_\epsilon(t)$ and the triangle inequality $\mathrm{SH}(\pi')_{[0,1]}(s',n') \in I_{3\epsilon}(t)$. Hence we get

$$\mathrm{SH}(\pi')(s',n') \in V \times I_{3\epsilon}(t) \subseteq U.$$

This finishes the proof that $\mathrm{SH}(\phi)$ is a well-defined morphisms. One easily checks that SH is a functor of additive categories.

Consider an object $\mathbf{B} = (S,\pi,\eta,\mathsf{B})$ in $\mathcal{O}_0^G(X \times [0,1])$. Next we define two morphisms in $\mathcal{O}_0^G(X \times [0,1])$

$$\begin{aligned}
T_0(\mathbf{B})\colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) &\to \mathrm{SH}(\mathbf{B}); \\
T_1(\mathbf{B})\colon\ \mathrm{SH}(\mathbf{B}) &\to \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}).
\end{aligned}$$

Recall that $\mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) = (S \amalg \mathrm{SH}(S), \pi \amalg \mathrm{SH}(\pi), \eta \amalg \mathrm{SH}(\eta), \mathsf{B} \amalg \mathrm{SH}(\mathsf{B}))$. For $s \in S$ and $(s',n') \in \mathrm{SH}(S)$ we define

$$T_0(\mathbf{B})_{s,(s',n')} = \begin{cases} \mathrm{id}_{\mathrm{B}(s)} & \text{if } s' = s \text{ and } n' = 0; \\ 0 & \text{otherwise.} \end{cases}$$

For $(s,n) \in \mathrm{SH}(\mathbf{B})$ and $(s',n') \in \mathrm{SH}(\mathbf{B})$ define

$$T_0(\mathbf{B})_{(s,n),(s',n')} = \begin{cases} \mathrm{id}_{\mathrm{B}(s)} & \text{if } s' = s \text{ and } n' = n+1; \\ 0 & \text{otherwise.} \end{cases}$$

We have to check that this is well-defined. Note that $\mathrm{supp}_G(T_0(\mathbf{B}))$ is either empty or $\{e\}$. In particular the condition *finite G-support* is satisfied. For $s \in S$ and $(s',n') \in \mathrm{SH}(S)$ we have $T_0(\mathbf{B})_{s,(s',n')} \neq 0 \implies s = s'$ and hence $\eta(s) = \eta(s') = \mathrm{SH}(\eta')(s',n')$. For $(s,n) \in S$ and $(s',n') \in \mathrm{SH}(S)$ we have $T_0(\mathbf{B})_{(s,n),(s',n')} \neq 0 \implies s = s'$ and hence $\mathrm{SH}(\eta)(s,n) = \eta(s) = \eta(s') = \mathrm{SH}(\eta')(s',n')$. It remains to show *continuous control*. We only deal with the implication (22.5). The proof for the other implication (22.6) is completely analogous.

Consider $(x,t) \in X \times [0,1]$ and an open $G_{(x,t)}$-invariant neighborhood U of (x,t) in $X \times [0,1]$. Choose an open G_x-invariant neighborhood V of x in X and $\epsilon > 0$ such that $V \times I_{2\epsilon}(t) \subseteq U$ holds. Choose a natural number r' satisfying $r' \geq 1/\epsilon$. Then $U' := V \times I_\epsilon(t)$ is an open $G_{(x,t)}$-invariant open neighborhood of (x,t) in $X \times [0,1]$ with $U' \subseteq U$. Consider $s \in S$, $(s',n') \in \mathrm{SH}(S)$, and $g \in \mathrm{supp}_G(T_0(\mathbf{B}))_{s,(s',n')}$ such that $g\pi(s) \in U'$. Then $g = e$ and $T_0(\mathbf{B})_{s,(s',n')} \neq 0$. This implies $s' = s$ and $n' = 0$ and hence $\pi(s) = \mathrm{SH}(\pi)(s',n')$. We conclude $\mathrm{SH}(\pi)(s',n') \in U' \subseteq U$. Consider $(s,n) \in S$, $(s',n') \in \mathrm{SH}(S)$, and $g \in \mathrm{supp}_G(T_0(\mathbf{B})_{(s,n),(s',n')})$ such that $g\,\mathrm{SH}(\pi)(s) \in U'$ and $\mathrm{SH}(\eta)(S,n) \geq r'$ hold. Then $g = e$ and $T_0(\mathbf{B})_{(s,n),(s',n')} \neq 0$. This implies $s' = s$ and $n' = n+1$. We get $\mathrm{SH}(\pi)_X(s,n) = \pi_X(s) = \pi_X(s') = \mathrm{SH}(\pi)_X(s',n')$ and hence $\mathrm{SH}(\pi)_X(s',n') \in V$. Moreover

$$\begin{aligned}
&|\,\mathrm{SH}(\pi)_{[0,1]}(s,n) - \mathrm{SH}(\pi')_{[0,1]}(s',n')| \\
&= \left|\left(\pi_{[0,1]}(s) - \frac{n}{\eta(s)}\right) - \left(\pi_{[0,1]}(s') - \frac{n'}{\eta(s')}\right)\right. \\
&= \left|\left(\pi_{[0,1]}(s) - \frac{n}{\eta(s)}\right) - \left(\pi_{[0,1]}(s) - \frac{n+1}{\eta(s)}\right)\right. \\
&= \frac{1}{\eta(s)} \\
&\leq \frac{1}{r'} \\
&\leq \epsilon.
\end{aligned}$$

Since $\mathrm{SH}(\pi)_{[0,1]}(s,n) \in I_\epsilon(t)$, we get $\mathrm{SH}(\pi)_{[0,1]}(s',n') \in I_{2\epsilon}(t)$ from the triangle inequality. Hence $\mathrm{SH}(\pi)(s',n') \subseteq V \times I_{2\epsilon}(t) \subseteq U'$. This finishes the proof that $T_0(\mathbf{B})$ is well-defined.

Next we define $T_1(\mathbf{B})$. For $(s,0) \in \mathrm{SH}(S)$ and $s' \in S$ we define

$$T_1(\mathbf{B})_{(s,n),s'} = \begin{cases} \mathrm{id}_{\mathsf{B}(s)} & \text{if } s' = s \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

For $(s,n) \in \mathrm{SH}(\mathbf{B})$ and $(s',n') \in \mathrm{SH}(\mathbf{B})$ define

$$T_1(\mathbf{B})_{(s,n),(s',n')} = \begin{cases} \mathrm{id}_{\mathsf{B}(s)} & \text{if } s' = s, n \geq 1, \text{ and } n' = n-1; \\ 0 & \text{otherwise.} \end{cases}$$

We omit the proof that $T_1(\mathbf{B})$ is well-defined since it is very similar to the one for $T_0(\mathbf{B})$. Roughly speaking, $T_0(\mathbf{B})$ shifts to the right in $[0,1]$, whereas $T_1(\mathbf{B})$ shifts to the left.

Obviously $T_0(\mathbf{B}) \circ T_1(\mathbf{B}) = \mathrm{id}_{\mathrm{SH}(\mathbf{B})}$. It is not true that $T_1(\mathbf{B}) \circ T_0(\mathbf{B}) = \mathrm{id}_{\mathbf{B} \oplus \mathrm{SH}(\mathbf{B})}$. At least we can show that $\mathrm{id}_{\mathbf{B} \oplus \mathrm{SH}(\mathbf{B})} - T_1(\mathbf{B}) \circ T_0(\mathbf{B})$ factorizes as a composite

$$\mathrm{id}_{\mathbf{B} \oplus \mathrm{SH}(\mathbf{B})} - T_1(\mathbf{B}) \circ T_0(\mathbf{B}) \colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \to \mathbf{B}'_0 \to \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \tag{22.130}$$

for an object $\mathbf{B}'_0$ in $\mathcal{O}_0^G(X \times \{0\})$ as follows.

We define a kind of subobject $\mathbf{B}_0 = (S_0, \pi_0, \eta_0.\mathsf{B}_0)$ of $\mathrm{SH}(\mathbf{B})$ by

$$\begin{aligned} S_0 &= \{(s,n) \in S \times \mathbb{N} \mid \eta(s) \geq 1, n \leq \eta(s) \cdot \pi_{[0,1]}(s) < n+1; \\ \pi_0(s,n) &= \bigl(\pi_X(s), \pi_{[0,1]}(s) - \frac{n}{\eta(s)}\bigr); \\ \eta_0(s,n) &= \eta(s); \\ \mathrm{SH}(\mathsf{B})(s,n) &= \mathsf{B}(s). \end{aligned}$$

Note that $S_0 \subseteq \mathrm{SH}(\mathbf{B})$. Actually, for a given $s \in S$ with $\eta(s) \geq 1$ the element of the shape $(s,n) \in \mathrm{SH}(S)$ belongs to S_0 if and only if $(s, n+1)$ does not belong to $\mathrm{SH}(S)$ anymore. The maps η_0 and B_0 are obtained by restricting $\mathrm{SH}(\pi)$, $\mathrm{SH}(\eta)$, and $\mathrm{SH}(\mathsf{B})$ to S_0. There is an obvious subobject $\mathbf{B}^{\perp}$ of $\mathrm{SH}(\mathbf{P})$ such that $\mathbf{B}_0 \oplus \mathbf{B}^{\perp} = \mathrm{SH}(\mathbf{B})$. Moreover, there is an obvious factorization

$$\mathrm{id}_{\mathbf{B} \oplus \mathrm{SH}(\mathbf{B})} - T_1(\mathbf{B}) \circ T_0(\mathbf{B}) \colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \to \mathbf{B}_0 \to \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}).$$

Hence it suffices to show that $\mathbf{B}_0$ is isomorphic in $\mathcal{O}_0^G(X \times [0,1])$ to an object $\mathbf{B}'_0 = (S'_0, \pi'_0, \eta'_0, \mathsf{B}'_0)$ which belongs to $\mathcal{O}_0^G(X \times \{0\})$. We define $\mathbf{B}'_0$ by $S'_0 = S_0$, $\eta'_0 = \eta_0$, and $\mathsf{B}'_0 = \mathsf{B}_0$ and by putting $\pi'_0(s,n) = (\pi_X(s), 0)$. In order to show that $\mathbf{B}_0$ and $\mathbf{B}'_0$ are isomorphic in $\mathcal{O}_0^G(X \times [0,1])$ we verify the criterion occurring in Lemma 22.14.

Consider $(x,t) \in X \times [0,1]$ and an open $G_{(x,t)}$-invariant neighborhood U of (x,t) in $X \times [0,1]$. Choose an open G_x-invariant neighborhood V of x in X and $\epsilon > 0$ such that $V \times I_{2\epsilon}(t) \subseteq U$ holds. Choose a natural number r' with $r' \geq \frac{1}{\epsilon}$. Put $U' = V \times I_\epsilon(t)$. Next we prove the implication for $s \in S_0 = S'_0$

$$\pi_0(s) \in U', \eta_0(s) \geq r' \implies \pi'_0(s) \in U. \tag{22.131}$$

From $\pi_0(s) \in U'$ we get $(\pi_0)_X(s) \in V$ and $(\pi_0)_{[0,1]}(s) \in I_\epsilon(t)$. By definition we have

$$(\pi_0)_{[0,1]}(s) = \pi_{[0,1]}(s) - \frac{n}{\eta(s)} > 0 = (\pi_0')_{[0,1]}(s) > \pi_{[0,1]}(s) - \frac{n+1}{\eta(s)}.$$

This implies

$$|(\pi_0)_{[0,1]}(s) - (\pi_0')_{[0,1]}(s)| \leq \frac{1}{\eta(s)} = \frac{1}{\eta_0(s)} \leq \frac{1}{r'} \leq \epsilon.$$

Since $(\pi_0)_{[0,1]}(s) \in I_\epsilon(t)$, we conclude from the triangle inequality $(\pi_0')_{[0,1]}(s) \in I_{2\epsilon}(t)$. Since $(\pi_0)_X(s) \in V$ and $(\pi_0)_X(s) = (\pi_0')_X(s)$, we get $(\pi_0')_X(s) \in V$. This implies $(\pi_0')(s) \in V \times I_{2\epsilon}(t) \subseteq U$. This finishes the proof of (22.131). The proof of the other implication

$$\pi_0'(s) \in U', \eta_0(s) \geq r' \implies \pi_0(s) \in U.$$

is completely analogous. Thus we obtain the desired factorization (22.130).

One easily checks that SH induces a functor of additive categories

$$\overline{\mathrm{SH}} \colon \mathcal{D}_0^G(X \times [0,1], X \times \{0\}) \to \mathcal{D}_0^G(X \times [0,1], X \times \{0\})$$

and $T_0(\mathbf{B})$ and $T_1(\mathbf{B})$ induces to one another inverse isomorphisms $\overline{T_0}(\mathbf{B}) \colon \mathbf{B} \oplus \overline{\mathrm{SH}}(\mathbf{B}) \xrightarrow{\cong} \overline{\mathrm{SH}}(\mathbf{B})$ and $\overline{T_1}(\mathbf{B}) \colon \overline{\mathrm{SH}}(\mathbf{B}) \xrightarrow{\cong} \mathbf{B} \oplus \overline{\mathrm{SH}}(\mathbf{B})$. The collection of the $\overline{T_0}(\mathbf{B})$ defines a natural equivalence of functors of additive categories

$$\mathbf{T}_0 \colon \mathrm{id}_{\mathcal{D}_0^G(X \times [0,1], X \times \{0\})} \oplus \overline{\mathrm{SH}} \xrightarrow{\cong} \overline{\mathrm{SH}}.$$

We conclude from Theorem 6.37 (iii) and Proposition 22.27 that the inclusion $X \times \{0\} \to X \times [0,1]$ induces a weak homotopy equivalence

$$\mathbf{K}(\mathcal{D}_0^G(X \times \{0\})) \to \mathbf{K}(\mathcal{D}_0^G(X \times [0,1])).$$

This finishes the proof of Proposition 22.127. $\square$

Exercise 22.132. Show that the proof of the homotopy invariance for $\mathbf{K}(\mathcal{D}_0^G(-))$ of Proposition 22.127 can easily be modified to a new proof of the G-homotopy invariance for $\mathbf{K}(\mathcal{D}^G(-))$.

Proof of Theorem 22.126. The rest of the proof of Theorem 22.126 is completely analogous to the proof of Theorem 22.26, one just has to check that all constructions respect the zero-control condition appearing in the definition of $\mathcal{D}_0^G$. $\square$

22.12 Notes

We have formulated the control conditions in Definition 22.4 concretely to keep some of the arguments simple. One can also give an axiomatic approach to control conditions in terms of coarse structures as defined by Higson-Pedersen-Roe [471, Definition 2.1] by specifying subsets of X and $X \times X$ in which the supports of objects and of morphism have to take values in. This is explained for continuous control in [73, Section 2.3]. There are various modifications of this idea, see for instance [170, Definition 3.1], [81, Definition 4.8], and [172, Section 2.2].

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Chapter 23
Coverings and Flow Spaces

23.1 Introduction

In this section we want to give more details concerning the discussion in Section 19.5. Essentially we want to explain that hyperbolic and finite-dimensional CAT(0)-groups satisfy the condition strongly transfer $\mathcal{VCY}$-reducible in the sense of Definition 20.38 which implies that they satisfy the Full Farrell-Jones Conjecture 13.27. Note that this concerns only input from geometric group theory; K-theory does not play a role at this stage. K-theory will enter when we show for instance that a strongly homotopically $\mathcal{VCY}$-transfer reducible group or, more generally, a Dress-Farrell-Hsiang-Jones group satisfies the Full Farrell-Jones Conjecture 13.27, see Remark 20.3, Theorem 20.39, and Theorem 20.62. The proof of Theorem 20.62 will be discussed in Chapter 24 and Chapter 25.

The basic strategy is as follows.

- Consider an appropriate metric space X associated to a hyperbolic group or a finite-dimensional CAT(0)-group reflecting its geometry.
- Assign to X a flow space $\mathrm{FS}(X)$;
- Prove for the flow space appropriate flow estimates which reflect the negative curvature or non-positive curvature condition associated to hyperbolic or CAT(0)-groups;
- Construct specific covers of the flow space, namely long and thin $\mathcal{VCY}$-covers with finite dimension;
- Construct an appropriate map $\iota\colon G \times X \to \mathrm{FS}(X)$ and pull back the long and thin $\mathcal{VCY}$-covers of $\mathrm{FS}(X)$ to $G \times X$ using ι;
- The flow estimates will ensure that these covers on $G \times X$ are good enough to show that G is strongly homotopically $\mathcal{VCY}$-transfer reducible.

The basic ideas are carried out for closed Riemannian manifolds with negative or non-positive sectional curvature and their fundamental groups in the seminal papers of Farrell-Jones [344, 345, 352]. The papers by Bartels-Lück [77, 79] and Bartels-Lück-Reich [85, 86] transfered these ideas to more general situations such as hyperbolic or CAT(0)-spaces and hyperbolic and finite-dimensional CAT(0)-groups, going considerably beyond the world of Riemannian manifolds and diving into geometric group theory and the theory and geometry of metric spaces. Kasprowski and Rüping [554, Theorem 6.1] simplified and unified some of the arguments, see Remark 23.46.

23.2 Flow Spaces

Definition 23.1 (Flow space). A *flow space* Y is a metric space (Y, d_Y) together with a continuous $\mathbb{R}$-action $\Phi\colon Y \times \mathbb{R} \to Y$.

Notation 23.2. We will often write $\Phi_t\colon Y \to Y$ for the homeomorphism sending $y \in Y$ to $\Phi(t, y)$.

For a subset $I \subseteq \mathbb{R}$ and $y \in Y$ we put $\Phi_I(y) = \{\Phi_t(y) \mid t \in I\}$.

Note that we do not demand in Definition 23.1 that $\Phi_t\colon Y \to Y$ is isometric.

Definition 23.3 (Flow G-space for G). A *flow G-space* is a flow space (Y, d_Y, Φ) in the sense of Definition 23.1 coming with an isometric and proper G-action $\rho\colon G \times Y \to Y$ such that ρ and Φ commute, i.e., we have $\Phi_t(gy) = g\Phi_t(y)$ for all $y \in Y$, $g \in G$, and $t \in \mathbb{R}$.

Obviously a flow G-space is the same as a metric space Y with a continuous $G \times \mathbb{R}$-action such that the induced action of $G = G \times \{0\} \subseteq G \times \mathbb{R}$ on Y is isometric and proper.

23.3 The Flow Space Associated to a Metric Space

In this section we introduce the flow space $\mathrm{FS}(X)$ for arbitrary metric spaces following [79, Section 1]. This is the one used in the proof of the Farrell-Jones Conjecture for CAT(0)-groups, see [79, 77, 973].

Definition 23.4. Let X be a metric space. A continuous map $c\colon \mathbb{R} \to X$ is called a *generalized geodesic* if there are $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \coprod \{-\infty, \infty\}$ satisfying

$$c_- \le c_+, \quad c_- \ne \infty, \quad c_+ \ne -\infty,$$

such that c is locally constant on the complement of the interval $I_c := (c_-, c_+)$ and restricts to an isometry on I_c.

The numbers c_- and c_+ are uniquely determined by c, provided that c is not constant.

Definition 23.5. Let (X, d_X) be a metric space. Let $\mathrm{FS} = \mathrm{FS}(X)$ be the set of all generalized geodesics in X. We define a metric on $\mathrm{FS}(X)$ by

$$d_{\mathrm{FS}(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}}\, dt.$$

Define a flow

$$\Phi\colon \mathrm{FS}(X) \times \mathbb{R} \to \mathrm{FS}(X)$$

by $\Phi_\tau(c)(t) = c(t + \tau)$ for $\tau \in \mathbb{R}$, $c \in \mathrm{FS}(X)$ and $t \in \mathbb{R}$.

The integral $\int_{-\infty}^{+\infty} \frac{d_X(c(t),d(t))}{2e^{|t|}}\,dt$ exists as $d_X(c(t),d(t)) \leq 2|t| + d_X(c(0),d(0))$ holds by the triangle inequality. Obviously $\Phi_\tau(c)$ is a generalized geodesic with

$$\begin{aligned}\Phi_\tau(c)_- &= c_- - \tau;\\ \Phi_\tau(c)_+ &= c_+ - \tau,\end{aligned}$$

where $-\infty - \tau := -\infty$ and $\infty - \tau := \infty$.

We note that any isometry $(X,d_X) \to (Y,d_Y)$ induces an isometry $\mathrm{FS}(X) \to \mathrm{FS}(Y)$ by composition. In particular, the isometry group of (X,d_X) acts canonically on $\mathrm{FS}(X)$. Moreover, this action commutes with the flow.

For a general metric space X all generalized geodesics may be constant. In the remainder of this section we will state some properties of $\mathrm{FS}(X)$ so that the reader can get some intuition.

Lemma 23.6. *Let (X,d_X) be a metric space. The map Φ is a continuous flow and we have for $c,d \in \mathrm{FS}(X)$ and $\tau,\sigma \in \mathbb{R}$*

$$d_{\mathrm{FS}(X)}\big(\Phi_\tau(c),\Phi_\sigma(d)\big) \leq e^{|\tau|}\cdot d_{\mathrm{FS}(X)}(c,d) + |\sigma - \tau|.$$

Exercise 23.7. Give the proof of Lemma 23.6.

The following lemma relates distance in X to distance in $\mathrm{FS}(X)$.

Lemma 23.8. *Let $c,d\colon \mathbb{R} \to X$ be generalized geodesics. Consider $t_0 \in \mathbb{R}$.*

(i) $d_X\big(c(t_0),d(t_0)\big) \;\leq\; e^{|t_0|}\cdot d_{\mathrm{FS}}(c,d) + 2$;
(ii) If $d_{\mathrm{FS}(X)}(c,d) \leq 2e^{-|t_0|-1}$, *then*

$$d_X\big(c(t_0),d(t_0)\big) \;\leq\; \sqrt{4e^{|t_0|+1}}\cdot\sqrt{d_{\mathrm{FS}(X)}(c,d)}.$$

In particular, $c \mapsto c(t_0)$ defines a uniform continuous map $\mathrm{FS}(X) \to X$.

Proof. See [79, Lemma 1.4]. □

Lemma 23.9. *Let (X,d_X) be a metric space. The maps*

$$\begin{aligned}FS(X) - \mathrm{FS}(X)^{\mathbb{R}} &\to \overline{\mathbb{R}}, \quad c \mapsto c_-;\\ FS(X) - \mathrm{FS}(X)^{\mathbb{R}} &\to \overline{\mathbb{R}}, \quad c \mapsto c_+,\end{aligned}$$

are continuous.

Proof. See [79, Lemma 1.6]. □

Proposition 23.10. *Let (X,d_X) be a metric space. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{FS}(X)$. Then it converges uniformly on compact subsets to $c \in \mathrm{FS}(X)$ if and only if it converges to c with respect to $d_{\mathrm{FS}(X)}$.*

Proof. See [79, Proposition 1.7]. □

Lemma 23.11. *Let (X, d_X) be a metric space. The flow space* $\mathrm{FS}(X)$ *is sequentially closed in the space of all maps $\mathbb{R} \to X$ with respect to the topology of uniform convergence on compact subsets.*

Proof. See [79, Lemma 1.8]. □

Proposition 23.12. *If (X, d_X) is a proper metric space, then* $(\mathrm{FS}(X), d_{\mathrm{FS}(X)})$ *is a proper metric space.*

Proof. See [79, Proposition 1.9]. □

Lemma 23.13. *Let (X, d_X) be a proper metric space and $t_0 \in \mathbb{R}$. Then the evaluation map* $\mathrm{FS}(X) \to X$ *defined by $c \mapsto c(t_0)$ is uniformly continuous and proper.*

Proof. See [79, Lemma 1.10]. □

Proposition 23.14. *Let G act isometrically and proper on the proper metric space (X, d_X). Then the action of G on* $(\mathrm{FS}(X), d_{\mathrm{FS}(X)})$ *is also isometric and proper. If the action of G on X is in addition cocompact, then also the G-action on* $\mathrm{FS}(X)$ *is cocompact.*

Proof. See [79, Proposition 1.11]. □

Lemma 23.15. *Let (X, d_X) be a metric space. Then* $\mathrm{FS}(X)^{\mathbb{R}}$ *is closed in* $\mathrm{FS}(X)$.

Exercise 23.16. Give the proof of Lemma 23.15.

Notation 23.17. Let X be a metric space. For $c \in \mathrm{FS}(X)$ and $T \in [0, \infty]$, define $c|_{[-T,T]} \in \mathrm{FS}(X)$ by

$$c|_{[-T,T]}(t) := \begin{cases} c(-T) & \text{if } t \leq -T; \\ c(t) & \text{if } -T \leq t \leq T; \\ c(T) & \text{if } t \geq T. \end{cases}$$

Obviously $c|_{[-\infty,\infty]} = c$ and if $c \notin \mathrm{FS}(X)^{\mathbb{R}}$ and $(-T, T) \cap (c_-, c_+) \neq \emptyset$ then $\left(c|_{[-T,T]}\right)_- = \max\{c_-, -T\}$ and $\left(c|_{[-T,T]}\right)_+ = \min\{c_+, T\}$.

We denote by

$$\mathrm{FS}(X)_f := \left\{ c \in \mathrm{FS}(X) - \mathrm{FS}(X)^{\mathbb{R}} \;\middle|\; c_- > -\infty, c_+ < \infty \right\} \cup \mathrm{FS}(X)^{\mathbb{R}}$$

the subspace of finite geodesics.

Lemma 23.18. *Let (X, d_X) be a metric space. The map $H \colon \mathrm{FS}(X) \times [0, 1] \to \mathrm{FS}(X)$ defined by $H_\tau(c) := c|_{[\ln(\tau), -\ln(\tau)]}$ is continuous and satisfies $H_0 = \mathrm{id}_{\mathrm{FS}(X)}$ and $H_\tau(c) \in \mathrm{FS}(X)_f$ for $\tau > 0$.*

Proof. See [79, Lemma 1.14]. □

23.4 The Flow Space Associated to a CAT(0)-Space

In this section we study $\mathrm{FS}(X)$ further in the case where X is a CAT(0)-space.

For the definition of a CAT(0)-space we refer to [155, Definition 1.1 in Chapter II.1 on page 158], namely to be a geodesic space all of whose geodesic triangles satisfy the CAT(0)-inequality. We will follow the notation and the description of the *bordification* $\overline{X} = X \cup \partial X$ of a CAT(0)-space X given in [155, Chapter II.8]. The definition of the topology of this bordification is briefly reviewed in Remark 23.20. In this section we will use the following convention.

- Let X be a complete CAT(0)-space;
- Let $\overline{X} := X \cup \partial X$ be the bordification of X, see [155, Chapter II.8].

23.4.1 Evaluation of Generalized Geodesics at Infinity

Definition 23.19. For $c \in \mathrm{FS}(X)$ we set $c(\pm\infty) := \lim_{t \to \pm\infty} c(t)$ where the limit is taken in $\overline{X}$.

Since X is by assumption a CAT(0)-space, we can find for $x_- \in X$ and $x_+ \in \overline{X}$ a generalized geodesic $c\colon \mathbb{R} \to X$ with $c(\pm\infty) = x_\pm$, see [155, Proposition 8.2 in Chapter II.8 on page 261]. It is not true in general that for two different points x_- and x_+ in ∂X there is a geodesic c with $c(-\infty) = x_-$ and $c(\infty) = x_+$.

Remark 23.20 (Cone topology on $\overline{X}$.). A *generalized geodesic ray* is a generalized geodesic c that is either a constant generalized geodesic or a non-constant generalized geodesic with $c_- = 0$. Fix a base point $x_0 \in X$. For every $x \in \overline{X}$, there is a unique generalized geodesic ray c_x such that $c(0) = x_0$ and $c(\infty) = x$, see [155, Proposition 8.2 in Chapter II.8 on page 261]. Define for $r > 0$

$$\rho_r = \rho_{r,x_0}\colon \overline{X} \to \overline{B}_r(x_0)$$

by $\rho_r(x) := c_x(r)$. The sets $(\rho_r)^{-1}(V)$ with $r > 0$, V an open subset of $\overline{B}_r(x_0)$ are a basis for the cone topology on $\overline{X}$, see [155, Definition 8.6 in Chapter II.8 on page 263]. A map f whose target is $\overline{X}$ is continuous if and only if $\rho_r \circ f$ is continuous for all r. The cone topology is independent of the choice of base point, see [155, Proposition 8.8 in Chapter II.8 on page 264].

Lemma 23.21. *The maps*

$$FS(X) - \mathrm{FS}(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(-\infty);$$
$$FS(X) - \mathrm{FS}(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(\infty),$$

are continuous.

Proof. See [79, Lemma 2.4]. □

Proposition 23.22. *If X is proper as a metric space, then the map*

$$E\colon \operatorname{FS}(X) - \operatorname{FS}(X)^{\mathbb{R}} \to \overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$$

defined by $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$ is injective and continuous. It is a homeomorphism onto its image.

Proof. See [79, Proposition 2.6]. □

Recall that $\operatorname{FS}(X)_f$ is the subspace of finite geodesics, see Notation 23.17.

Proposition 23.23. *Assume that X is proper as a metric space. Then the map*

$$E_f\colon \operatorname{FS}(X)_f - \operatorname{FS}(X)^{\mathbb{R}} \to \mathbb{R} \times X \times X$$

defined by $E_f(c) = \bigl(c_-, c(-\infty), c(\infty)\bigr)$ is a homeomorphism onto its image

$$\operatorname{im} E_f = \{(r, x, y) \mid x \neq y\}.$$

In particular, $\operatorname{FS}(X)_f - \operatorname{FS}(X)^{\mathbb{R}}$ is locally path connected.

Proof. See [79, Proposition 2.7]. □

23.4.2 Dimension of the Flow Space

Lemma 23.24. *If X is proper as a metric space and its dimension $\dim X$ is $\leq N$, then $\dim \overline{X} \leq N$.*

Proof. See [79, Lemma 2.8]. □

Proposition 23.25. *Assume that X is proper and that $\dim X \leq N$. Then*

$$\dim\bigl(\operatorname{FS}(X) - \operatorname{FS}(X)^{\mathbb{R}}\bigr) \leq 3N + 2.$$

Proof. See [79, Proposition 2.9]. □

23.4.3 The Example of a Complete Riemannian Manifold with Non-Positive Sectional Curvature

Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature. It is a CAT(0)-space with respect to the metric coming from the Riemannian metric, see [155, Theorem I.A.6 on page 173]. Let STM be its sphere tangent bundle. For every $x \in M$ and $v \in ST_xM$

there is precisely one geodesic $c_v\colon \mathbb{R} \to M$ for which $c_v(0) = x$ and $c_v'(0) = v$ holds. Given a geodesic $c\colon \mathbb{R} \to M$ in M and $a_-, a_+ \in \overline{\mathbb{R}}$ with $a_- \le a_+$, define the generalized geodesic $c_{[a_-,a_+]}\colon \mathbb{R} \to M$ by sending t to $c(a_-)$ if $t \le a_-$, to $c(t)$ if $a_- \le t \le a_+$, and to $c(a_+)$ if $t \ge a_+$. Obviously $c_{[-\infty,\infty]} = c$. Let $d\colon \mathbb{R} \to M$ be a generalized geodesic with $d_- < d_+$. Then there is precisely one geodesic $\widehat{d}\colon \mathbb{R} \to M$ with $\widehat{d}_{[d_-,d_+]} = d$.

Define maps

$$\alpha\colon STM \times \left\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\right\} \to \mathrm{FS}(M),\ \ (v, a_i, a_+) \mapsto c_v|_{[a_-,a_+]};$$
$$\beta\colon\ \mathrm{FS}(M) \to STM \times \left\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\right\},\ \ c \mapsto (\widehat{c}\,'(0), c_-, c_+).$$

Then α and β are to another inverse homeomorphisms. They are compatible with the flow on $\mathrm{FS}(M)$ of Definition 23.5, if one uses on $STM \times \left\{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\right\}$ the product flow given by the geodesic flow on STM and the flow on $\overline{R}$ which is at time t given by the homeomorphism $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ sending $s \in \mathbb{R}$ to $s - t$, $-\infty$ to $-\infty$, and ∞ to ∞.

23.5 The Dynamical Properties of the Flow Space Associated to a CAT(0)-Space

In Definition 23.27 we introduce the homotopy action that we will use to show that CAT(0)-groups are transfer reducible over $\mathcal{VCY}$. It will act on a large closed ball in X. (The action of G on the bordification $\overline{X}$ is not suitable, because it has to large isotropy groups.) In Theorem 23.31 which is based on Proposition 23.30, we study the dynamics of the flow with respect to the homotopy action. The analogue of Proposition 23.30 in the hyperbolic case is Theorem 23.34.

Throughout this section we fix the following convention.

- Let (X, d_X) be a CAT(0)-space which is proper as a metric space;
- Let $x_0 \in X$ be a fixed base point;
- Let G be a group with a proper isometric action on (X, d_X).

For $x, y \in X$ and $t \in [0,1]$ we will denote by $t \cdot x + (1-t) \cdot y$ the unique point z on the geodesic from x to y such that $d_X(x,z) = td_X(x,y)$ and $d_X(z,y) = (1-t)d_X(x,y)$. For $x, y \in X$ we will denote by $c_{x,y}$ the generalized geodesic determined by $(c_{x,y})_- = 0$, $c(-\infty) = x$ and $c(\infty) = y$. By [155, Proposition 1.4 (1) in Chapter II.1 on page 160] and Proposition 23.10, $(x,y) \mapsto c_{x,y}$ defines a continuous map $X \times X \to \mathrm{FS}(X)$. Note that $g \cdot c_{x,y} = c_{gx,gy}$.

23.5.1 The Homotopy Action on $\overline{B}_R(x)$

The next definition is a variation of some of the notions appearing in Section 20.5

Definition 23.26 (Homotopy S-action). Let S be a finite subset of a group G. Assume that S contains the trivial element $e \in G$. Let X be a space.

(i) A *homotopy S-action (φ, H) on X* consists of continuous maps $\varphi_g \colon X \to X$ for $g \in S$ and homotopies $H_{g,h} \colon X \times [0,1] \to X$ for $g, h \in S$ with $gh \in S$ such that $H_{g,h}(-,0) = \varphi_g \circ \varphi_h$ and $H_{g,h}(-,1) = \varphi_{gh}$ holds for $g, h \in S$ with $gh \in S$. Moreover, we require that $H_{e,e}(-,t) = \varphi_e = \mathrm{id}_X$ for all $t \in [0,1]$;

(ii) Let (φ, H) be a homotopy S-action on X. For $g \in S$ let $F_g(\varphi, H)$ be the set of all maps $X \to X$ of the form $x \mapsto H_{r,s}(x,t)$ where $t \in [0,1]$ and $r, s \in S$ with $rs = g$;

(iii) Let (φ, H) be a homotopy S-action on X. For $(g,x) \in G \times X$ and $n \in \mathbb{N}$, let $S^n_{\varphi,H}(g,x)$ be the subset of $G \times X$ consisting of all (h,y) with the following property: There are $x_0, \ldots, x_n \in X$, $a_1, b_1, \ldots, a_n, b_n \in S$, $f_1, \tilde{f}_1, \ldots, f_n, \tilde{f}_n \colon X \to X$, such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\tilde{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \tilde{f}_i(x_i)$ and $h = g a_1^{-1} b_1 \ldots a_n^{-1} b_n$;

(iv) Let (φ, H) be a homotopy S-action on X and $\mathcal{U}$ be an open cover of $G \times X$. We say that $\mathcal{U}$ is *S-long with respect to (φ, H)* if for every $(g,x) \in G \times X$ there is $U \in \mathcal{U}$ containing $S^{|S|}_{\varphi,H}(g,x)$ where $|S|$ is the cardinality of S.

Recall that for $r > 0$ and $z \in X$ we denote by $\rho_{r,z} \colon X \to \overline{B}_r(z)$ the canonical projection along geodesics, i.e., $\rho_{r,z}(x) = c_{z,x}(r)$, see also Remark 23.20. Note that $g \cdot \rho_{r,z}(x) = \rho_{r,gz}(gx)$ for $x, z \in X$ and $g \in G$.

Definition 23.27 (The homotopy S-action on $\overline{B}_R(x_0)$). Let $S \subseteq G$ be a finite subset of G with $e \in G$ and $R > 0$. Define a homotopy S-action (φ^R, H^R) on $\overline{B}_R(x)$ in the sense of Definition 23.26 (i) as follows. For $g \in S$, we define the map

$$\varphi^R_g \colon \overline{B}_R(x_0) \to \overline{B}_R(x_0)$$

by $\varphi^R_g(x) := \rho_{R,x_0}(gx)$.

For $g, h \in S$ with $gh \in S$ we define the homotopy

$$H^R_{g,h} \colon \varphi^R_g \circ \varphi^R_h \simeq \varphi^R_{gh}$$

by $H^R_{g,h}(x,t) := \rho_{R,x_0}\bigl(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx))\bigr)$.

Remark 23.28. Notice that $H^R_{g,h}$ is indeed a homotopy from $\varphi^R_g \circ \varphi^R_h$ to φ_{gh} since

$$\begin{aligned} H^R_{g,h}(x,0) &= \rho_{R,x_0}\bigl(0\cdot(ghx) + 1\cdot(g\cdot\rho_{R,x_0}(hx))\bigr) \\ &= \rho_{R,x_0}\bigl(g\cdot\rho_{R,x_0}(hx)\bigr) \\ &= \varphi^R_g \circ \varphi^R_h(x), \end{aligned}$$

and

$$\begin{aligned} H^R_{g,h}(x,1) &= \rho_{R,x_0}\bigl(1\cdot(ghx) + 0\cdot(g\cdot\rho_{R,x_0}(hx)\bigr) \\ &= \rho_{R,x_0}(ghx) \\ &= \varphi^R_{gh}(x). \end{aligned}$$

It turns out that the more obvious homotopy given by convex combination $(x,t) \mapsto t\cdot\varphi^R_{gh}(x) + (1-t)\cdot\varphi^R_g \circ \varphi^R_h(x)$ is not appropriate for our purposes.

Definition 23.29 (The map ι). Define the map

$$\iota\colon G\times X \to \mathrm{FS}(X)$$

as follows. For $(g,x)\in G\times X$ let $\iota(g,x) := c_{gx_0,gx}$.

The map ι is G-equivariant for the G-action on $G\times X$ defined by $g\cdot(h,x) = (gh,x)$.

23.5.2 The Flow Estimate

Proposition 23.30. *Let $\beta, L > 0$. For all $\delta > 0$ there are $T, r > 0$ such that for $x_1, x_2 \in X$ with $d_X(x_1,x_2) \le \beta$, $x \in \overline{B}_{r+L}(x_1)$ there is $\tau \in [-\beta,\beta]$ such that*

$$d_{\mathrm{FS}(X)}\bigl(\Phi_T(c_{x_1,\rho_{r,x_1}(x)}), \Phi_{T+\tau}(c_{x_2,\rho_{r,x_2}(x)})\bigr) \le \delta.$$

Proof. See [79, Proposition 3.5]. □

Theorem 23.31 (Flow estimates in the CAT(0)-case). *Let S be a finite subset of G (containing e). Then there is $\beta > 0$ such that the following holds:*

For all $\delta > 0$ there are $T, R > 0$ such that for every $(a,x) \in G\times\overline{B}_R(X)$, $s\in S$, and $f\in F_s(\varphi^R, H^R)$ there is $\tau\in[-\beta,\beta]$ such that

$$d_{\mathrm{FS}(X)}\left(\Phi_T(\iota(a,x)), \Phi_{T+\tau}(\iota(as^{-1}, f(x)))\right) \le \delta.$$

Proof. See [79, Proposition 3.8]. □

23.6 The Flow Space Associated to a Hyperbolic Metric Complex

In the proofs of the Farrell-Jones Conjecture for hyperbolic groups, see [85, 86] a construction of a flow space $\mathrm{FS}(X)$ based on a construction of Mineyev [716] is used. (Note that a mistake in [716] was fixed by Mole [732].) Although one does not need the construction of Mineyev anymore in the proofs and gets along with the construction presented in Section 23.3, we still briefly recall what happens in the original proofs for hyperbolic groups as an illustration for the reader and a hint how the techniques have changed over the time.

If X is hyperbolic complex with compactification $\overline{X}$ and $x_0 \in X$ a base point, there is a specific map, see [86, (8.1)],

$$\iota_{x_0} \colon X \times \overline{X} \to \mathrm{FS}(X) \tag{23.32}$$

such that the following flow estimate holds.

Lemma 23.33. *The map* $\iota_{x_0} \colon X \times \overline{X} \to \mathrm{FS}(X)$ *from* (23.32) *is continuous. It is* $\mathrm{Isom}(X)$*-equivariant with respect to the diagonal* $\mathrm{Isom}(X)$*-action on the source where* $\mathrm{Isom}(X)$ *is the group of isometric selfhomeomorphisms of* X*. For* $x \in X$ *the map* $\iota_{x_0}(x, -) \colon \overline{X} \to \mathrm{FS}(X), \quad y \mapsto \iota_{x_0}(x, y)$ *is injective.*

Proof. See [85, Lemma 8.4]. □

Theorem 23.34 (Flow estimate in the hyperbolic case). *Let* $\lambda \in (e^{-1}, 1)$ *and* $T \in [0, \infty)$ *be the constants depending only on* X *which appear in [85, Proposition 6.4]. Consider* $a, b \in X$ *and* $c \in \overline{X}$*. Put*

$$N = 2 + \frac{2}{\lambda^T \cdot (-\ln(\lambda))}.$$

Then there exists a real number τ_0 *such that*

$$|\tau_0| \leq 2 \cdot \widehat{d}(a, b) + 5$$

holds for the new metric $\widehat{d}$ *on* X *defined in [716, Lemma 2.7 on page 449 and Theorem 32 on page 446] and we get for all* $\tau \in \mathbb{R}$

$$d_{\mathrm{FS},x_0}(\phi_\tau \circ \iota_{x_0}(a, c), \phi_{\tau+\tau_0} \circ \iota_{x_0}(b, c)) \leq \frac{N}{1 - \ln(\lambda)^2} \cdot \lambda^{-\widehat{d}(a,b)} \cdot \lambda^\tau.$$

Proof. See [85, Theorem 8.6]. □

We recommend the reader to compare Theorem 23.34 with Proposition 23.30. The baby version of these two results was already discussed in Lemma 19.14.

23.7 Topological Dimension

Let X be a topological space. Let $\mathcal{U}$ be an open cover. Its *dimension* $\dim(\mathcal{U}) \in \{0,1,2,\ldots\} \amalg \{\infty\}$, sometimes also called its *order*, is the infimum over all integers $d \geq 0$ such that for any collection $U_0, U_1, \ldots, U_d$ of pairwise distinct elements in $\mathcal{U}$ the intersection $\bigcap_{i=0}^d U_i$ is empty. An open covering $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$.

Definition 23.35 ((Topological) dimension). The *dimension* (sometimes also called *topological dimension* or *covering dimension*) of a topological space X

$$\dim(X) \in \{0,1,2,\ldots\} \amalg \{\infty\}$$

is the infimum over all integers $d \geq 0$ such that any open covering $\mathcal{U}$ possesses a refinement $\mathcal{V}$ with $\dim(\mathcal{V}) \leq d$.

We state some basic properties of the dimension.

Lemma 23.36. *If A is a closed subset of X, then $\dim(A) \leq \dim(X)$.*

Exercise 23.37. Give the proof of Lemma 23.36.

Lemma 23.38. *Let X be the union $A_1 \cup A_2 \cdots A_k$ of closed subspaces $A_i \subseteq X$. Then*

$$\dim(X) = \max\{\dim(A_i) \mid i = 1, 2 \ldots, k\}.$$

Proof. See [737, Corollary 9.3 on page 304]. □

Lemma 23.39. *Let Z be a proper metric space. Suppose that G acts on Z isometrically and properly. Then we get for the topological dimensions of X and $G\backslash X$*

$$\dim(G\backslash X) \leq \dim(X).$$

Proof. See [643, Lemma 3.2]. □

Theorem 23.40. *Let X be a locally compact Hausdorff space having a countable basis for its topology. Suppose that every point of X has a neighborhood whose closure has topological dimension at most m. Then X has topological dimension at most m.*

Proof. See [737, Exercise 9 on page 315]. □

A locally compact CW-complex which is second countable, i.e., has a countable basis for its topology, has the property that its topological dimension $\dim(X)$ is the same as its dimension as CW-complex. This follows from Theorem 23.40. Note that for a connected CW-complex, locally compact, metrizable, first countable, and locally finite are equivalent conditions, see [393, Theorem B and Proposition 2.4].

Again by Theorem 23.40, a topological m-dimensional manifold M has topological dimension m.

Lemma 23.41. *Let n be an integer with $n \geq 0$. Let X be a proper metric space whose topological dimension satisfies* $\dim(X) \leq n$. *Suppose that G acts properly and isometrically on X.*

Then there exists a proper n-dimensional G-CW-complex Y together with a G-map $f\colon X \to Y$.

Proof. See [643, Lemma 3.7]. □

There is also the notion of a small inductive limit, see [328, Definition 1.1] or [555, Definition 3.1] which agrees with the notion of the topological dimension for second countable metric spaces, see [328, Theorem 1.7.7].

23.8 Long and Thin Covers

The next result is proved in [555, Theorem 1.1] based on ideas from [85, Theorem 1.4] and [79, Theorem 5.6] which in turn are motivated by [344, Proposition 7.2].

Notation 23.42. Let (X, d_X) be a metric space. For a subset $A \subseteq X$ and $\delta > 0$, we define

$$B_\delta(A) := \{y \in X \mid \exists a \in A \text{ with } d_X(y,a) < \delta\};$$
$$\overline{B}_\delta(A) := \{x \in X \mid \exists a \in ZA \text{ with } d_X(a,y) \leq \delta\}.$$

Given $x \in X$, we write

$$B_\delta(x) := B_\delta\{x\}) = \{y \in X \mid d_X(y,x) < \delta\};$$
$$\overline{B}_\delta(x) := \overline{B}_\delta(\{x\}) = \{y \in X \mid d_X(y,x) \leq \delta\}.$$

We call $B_\delta(x)$ the open and $\overline{B}_\delta(x)$ the closed ball around x of radius δ.

Note that the open ball $B_\delta(x)$ is an open subset of X, the closed ball $B_\delta(x)$ is closed subset of X, and $\overline{B}_\delta(A)$ contains the closure $\overline{B_\delta(A)}$ of $B_\delta(A)$ in X, but $\overline{B_\delta(A)}$ and $\overline{B}_\delta(A)$ are not equal in general.

Theorem 23.43 (Long and thin covers). *Let G will be a countable discrete group. Let X be a flow G-space such that the underlying topological space X is finite-dimensional, locally compact, and second countable. Let $\alpha > 0$ and $\delta > 0$ be real numbers.*

Then there exists an open $\mathcal{V}cyc$-cover $\mathcal{U}$ of X in the sense of Definition 20.18 of dimension at most $7\dim(X) + 7$ which is long and thin in the following sense:

- *(Long) For every point $x \in X$ there is $U \in \mathcal{U}$ with $\Phi_{[-\alpha,\alpha]}(x) \subseteq U$;*
- *(Thin) For every $U \in \mathcal{U}$ there is a point $x \in X$ with $U \subseteq B_\delta(\Phi_{\mathbb{R}}(x))$.*

The long and thin covers are generalizations of the long and thin cells from [344, Proposition 7.2].

A basic strategy of the proof of Theorem 23.43 presented in [555, Theorem 1.1] consists of decomposing the flow space into three parts, a part with without a short G-period, a non-periodic part with short G-period, and the periodic part with short G-period, constructing for each part an appropriate $\mathcal{V}cyc$-cover, and finally taking the union of these covers.

Definition 23.44 (Strong contracting transfers). A flow G-space Y admits *strong contracting transfers* if there is a natural number N such that for every finite subset $S \subseteq G$ and every natural number k there exists a real number $\beta > 0$ such that for every $\delta > 0$ there exists a real number $T > 0$ with the following properties:

- An N-transfer space X in the sense of Definition 20.9;
- A strong homotopy action Γ in the sense of Definition 20.32;
- A G-equivariant map $\iota\colon G \times X \to Y$, where the G-action on $G \times X$ is given by $g' \cdot (g,x) = (g'g,x)$, with the property that for every $(g,x) \in G \times X$, every $s \in S$, and every $f \in F_g(\Gamma, S, k)$ there exists $\tau \in [-\beta, \beta]$ satisfying

$$d_Y(\Phi_T \circ \iota(g,x), \Phi_{\tau+T} \circ \iota(gs^{-1}, f(x))) \leq \delta.$$

The next result follows from [555, Corollary 1.2] and Theorem 20.39.

Theorem 23.45. *Let X be a flow G-space whose underlying space X is finite-dimensional and the G-action on X is cocompact. Suppose that X admits strong contracting transfers.*

Then G is strongly transfer $\mathcal{VCY}$-reducible in the sense of Definition 20.38. In particular G satisfies the Full Farrell-Jones Conjecture 13.27.

A key ingredient in the proof that for a hyperbolic or a CAT(0)-group Theorem 23.45 applies, i.e., that the condition that the flow spaces admit strong contracting transfers, are the flow estimates as they appear for instance in Theorem 23.31 and Theorem 23.34. The basic idea of the proof of Theorem 23.45 is to pull back an appropriate $\mathcal{F}$-cover of the flow space Y coming from Theorem 23.43 back to $G \times X$ using the map ι.

Remark 23.46. Kasprowski and Rüping [554, Theorem 6.1] show using Theorem 23.45 that the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products, see Conjecture 13.24 and the L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution with finite wreath products, see Conjecture 13.25 hold for a class of group G which encompasses all hyperbolic and CAT(0)-groups. It contains for instance all groups G which acts properly and cocompactly on a finite product of hyperbolic graphs, see [554, Theorem 1.1] Their proof applies also to the Full Farrell-Conjecture 13.27 because of Remark 20.3 and Theorem 20.62. There are groups which are neither hyperbolic nor CAT(0)-groups and belong to the class of groups appearing in Kasprowski and Rüping [554, Theorem 6.1], e.g., the fundamental group $\pi_1(STF)$ of the sphere tanget bundle of a hyperbolic closed surface, see [554, Section 3]. (Note that it is well-known that $\pi_1(STF)$ satisfies the Full Farrell-Conjecture 13.27, see Theorem 16.1 (ie).)

23.9 Notes

The case of a reductive p-adic groups acting on the CAT(0)-space given by its Bruhat-Tits building is analyzed in [82]. **Comment 31 (by W.)**: Add also a reference to the preprint by Bartels-Lück-Witzel about Kac-Moody groups when it has been finished.

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Chapter 24
Transfer

24.1 Introduction

In this section we give more information about the transfer which we have already mentioned in Section 19.5 and which plays a prominent role in nearly all proofs of the Farrell-Jones Conjecture. For simplicity we refine ourselves to the Whitehead group.

24.2 The Geometric Transfer

Let $F \to E \xrightarrow{p} B$ be a fibration such that F and B have the homotopy type of a connected finite CW-complex. Then E also has the homotopy type of a connected finite CW-complex, see for instance [358, Section 3], [625], [666, Section 1].

A *simple structure* ξ on a space Y of the homotopy type of a connected finite CW-complexes is an equivalence class of homotopy equivalences $u_0\colon Z \xrightarrow{\simeq} Y$ with some connected finite CW-complex Z as source where we call two such homotopy equivalences $u_k\colon Z_k \xrightarrow{\simeq} Y$ for $k = 0, 1$ equivalent if there is a simple homotopy equivalence $v\colon Y_0 \xrightarrow{\simeq_s} Y_1$ such that $u_1 \circ v$ and u_0 are homotopic. Of course a connected finite CW-complex Y has a preferred simple structure given by id_Y.

Given a homotopy equivalence $f\colon (Y_0, \xi_0) \xrightarrow{\simeq} (Y_1, \xi_1)$ of spaces coming with simple structures, we define its Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(Y_1))$ to be the image of the Whitehead torsion $\tau(w) \in \mathrm{Wh}(\pi_1(Z_1))$ of a homotopy equivalence $w\colon Z_0 \xrightarrow{\simeq} Z_1$ under the isomorphism $(u_1)_*\colon \mathrm{Wh}(\pi_1(Z_1)) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(Y_1))$ where we have chosen representatives $u_k\colon Z_k \xrightarrow{\simeq} Y_k$ of the simple structures ξ_k for $k = 0, 1$ and require $u_1 \circ w \simeq f \circ u_0$. One easily checks that this is independent of all the choices using Theorem 3.37.

Next we define a class $\theta(p) \in H^1(B; \mathrm{Wh}(E))$. It is given after a choice of a base point $b \in B$ and a simple structure ξ_F on the fiber $F_b = p^{-1}(b)$ by the homomorphism $\pi_1(B, b) \to \mathrm{Wh}(\pi_1(E))$ which sends $w \in \pi_1(B, b)$ to the image of Whitehead torsion of the homotopy equivalence $F_b \to F_b$ given by the fiber tranport along w under the homomorphism $\mathrm{Wh}(\pi_1(F_b)) \to \mathrm{Wh}(\pi_1(E))$ coming from the inclusion $F_b \to E$. Recall that the *fiber transport* is a homomorphism of monoids $\pi_1(B, b) \to [F_b, F_b]$, see for instance [925, 15.12 on page 343], [987, page 186]. One easily checks that this well-defined, in particular that it is independent of the choice of base points and the simple structure ξ_F.

If $\theta(p) \in H^1(B; \mathrm{Wh}(E))$ vanishes and we have fixed simple structures ξ_F and ξ_B on F and B, then there is a preferred simple structure on ξ_E on E, see [358, Section 3], [625], [666, Section 1]. In the sequel we will assume that the characteristic class $\theta(p) \in H^1(B; \mathrm{Wh}(F))$ vanishes which is the case if p satisfies one of the following conditions:

- the fibration is orientable, i.e, the fiber transport $\pi_1(B) \to [F, F]$ is trivial;
- The map $\pi_1(F) \to \pi_1(E)$ is zero, or, equivalently, $\pi_1(p) \colon \pi_1(E) \to \pi_1(B)$ is bijective;
- p is a locally trivial topological bundle with a connected finite CW-complex F as fiber.

Let M be a closed topological manifold. Then, by Kirby-Siebenmann [562, Essay III, Theorem 4.1 on page 118], there is a preferred simple structure

$$\xi^{\mathrm{top}}(M) \text{ on } M, \tag{24.1}$$

which is defined by considering any triangulated closed disc bundle over M: The simple structure on the disc bundle obtained from the triangulation induces the preferred simple structure on M via the retraction onto M. This simple structure agrees with the one obtained by any triangulation or by any handlebody decomposition (more generally what they call TOP s-decomposition) of M, whenever they exist (see [562, Essay III, Theorem 5.10 on page 131 and Theorem 5.11 on page 132]). Let $F \to M \to B$ be a locally trivial bundle of connected closed topological manifolds. Then $\Theta(p)$ is trivial. If we equip B and F with the simple structures $\xi^{\mathrm{top}}(M)$ of (24.1), then the simple structure ξ_E agrees with $\xi^{\mathrm{top}}(M)$, see [358, Lemma 3.16].

Consider $\alpha \in \mathrm{Wh}(\pi_1(B))$. Let $f \colon X \to B$ be a homotopy equivalence with a connected finite CW-complex as source satisfying $\tau(f) = \alpha$. Consider the following pullback

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{f}"] \arrow[d, "\overline{p}"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

We conclude $\Theta(\overline{p}) = 0$ from our assumption $\Theta(p) = 0$. Hence there is a preferred simple structure on both $\overline{X}$ and E and the Whitehead torsion $\tau(\overline{f}) \in \mathrm{Wh}(\pi_1(E))$ is defined. The *geometric transfer*

$$p^* \colon \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E)) \tag{24.2}$$

is defined by the equality $p^*(\alpha) = \tau(\overline{f})$. The proof that this construction is well-defined can be found in Anderson [28, 29] for locally trivial PL-bundles and in the more general setting above in [625], [358, Section 3]. The equivariant version of this construction is presented in detail in [629, Section 15].

Example 24.3. Let $F \to M \to B$ be a locally trivial bundle of connected closed topological manifolds. Let $W = (W, B, B', f, f')$ be a topological h-cobordism over B_0. Choose a retraction r of the homotopy equivalence $B \xrightarrow{f} \partial_0 W \to W$. Consider the pullback

$$\begin{tikzcd} \overline{W} \arrow[r, "\overline{r}"] \arrow[d, "\overline{p}"'] & E \arrow[d, "p"] \\ W \arrow[r, "r"'] & B. \end{tikzcd}$$

Then $\overline{W}$ is a topological h-cobordism over E and the transfer homomorphism $p^*\colon \operatorname{Wh}(\pi_1(B)) \to \operatorname{Wh}(\pi_1(E))$ of (24.2) sends the Whitehead torsion of W, see (3.48), to the Whitehead torsion of $\overline{W}$.

24.3 The Algebraic Transfer

Next we describe the algebraic version of the geometric transfer. Let R and S be unital rings R.

Definition 24.4 (Chain homotopy representation). A *chain homotopy representation* (C_*, U) consists of an S-chain complex C_* and a ring homomorphism $U_*\colon R \to [C_*, C_*]_S$ to the ring of S-chain homotopy classes of S-selfchain homotopy equivalences $C_* \to C_*$ where the multiplicative structure comes from composition.

Given a matrix A in $GL_n(R)$, we obtain a well defined S-chain homotopy class of S-chain homotopy equivalences $U_*(A)\colon \bigoplus_{i=1}^n C_* \xrightarrow{\simeq} \bigoplus_{i=1}^n C_*$ by the (n,n)-matrix $(U(a_{i,j}))_{i,j}$ of S-chain homotopy classes S-chain maps $U(a_{i,j})\colon C_* \to C_*$. Suppose that C_* is a finite free S-chain complex. Choose a basis for C_*. Then $\bigoplus_{i=1}^n C_*$ is a finite free S-chain complex which comes with a basis, and hence the Whitehead torsion $\tau(U_*(A)) \in \widetilde{K}_1(S)$ of $U_*(A)\colon \bigoplus_{i=1}^n C_* \xrightarrow{\simeq} \bigoplus_{i=1}^n C_*$ is defined, see (3.33). One easily checks that $\tau(U_*(A))$ is independent of the choice of the basis on C_*. We obtain a well-defined homomorphism of abelian groups

$$p_U^*\colon \widetilde{K}_1(R) \to \widetilde{K}_1(S) \tag{24.5}$$

by sending the class of $[A]$ of A to $\tau(U_*(A))$. Although it is not relevant for us here, we mention that using the selftorsion of Subsection 24.7.3 one can define a map $p_U^*\colon K_1(R) \to K_1(S)$ which induces the map (24.5) and is defined in the more general case where C_* is a finitely dominated projective S-chain complex. All of this is explained in [626, Section 4].

Given a fibration $F \to E \xrightarrow{p} B$ such that F is a connected finite CW-complex and B is path connected, one can assign to it using the fiber transport a chain homotopy representation $U(p)$ for $R = \mathbb{Z}[\pi_1(B)]$ and $S = \mathbb{Z}[\pi_1(E)]$ whose underlying $\mathbb{Z}[\pi_1(E)]$-chain complex is finite free. In the special case that $\pi_1(p)\colon \pi_1(E) \xrightarrow{\cong} \pi_1(B)$ is bijective, it is defines as follows. Take $C_*(F)$ to be the cellular $\mathbb{Z}$-chain complex of F. Put $C_* = \mathbb{Z}[\pi_1(E)] \otimes_{\mathbb{Z}} C_*(F)$. This is obviously a finite free $\mathbb{Z}[\pi_1(E)]$-chain complex. For $w \in \pi_1(B)$ the fiber transport defines a $\mathbb{Z}$-chain map $t(w)_*\colon C_* \to C_*$ which is well-defined up to $\mathbb{Z}$-chain homotopy. Choose $\widetilde{w} \in \pi_1(E)$ whose image under $\pi_1(p)$ is w. Define $U(w)_*\colon C_* \to C_*$ by

$$U(w)_*(v \otimes x) = v\widetilde{w}^{-1} \otimes t(w)_*(x). \tag{24.6}$$

This extends to a ring homomorphism $U_*\colon \mathbb{Z}[\pi_1(B)] \to [C_*, C_*]_{\mathbb{Z}[\pi_1(E)]}$ by $\mathbb{Z}$-linearity. So the transfer of (24.5) is defined. One easily checks that it induces a homomorphism of abelian groups

$$p^*_{U(p)}\colon \operatorname{Wh}(\mathbb{Z}[\pi_1(B)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(E)]), \tag{24.7}$$

provided that $\Theta(p) = 0$ holds. The next theorem is taken from [626, Theorem 5.4].

Theorem 24.8 (The geometric and algebraic transfer agree). *In the situation where the geometric transfer p^* of* (24.2) *is defined, the algebraic transfer $p^*_{U(p)}$ of* (24.7) *is defined and p^* and $p^*_{U(p)}$ agree.*

In view of Theorem 24.8 we abbreviate $p^*_{U(p)}$ by p^* in the sequel.

24.4 The Down-Up Formula

Consider a fibration $F \to E \xrightarrow{p} B$ such that F is a connected finite CW-complex and B is path connected. The group homomorphism $\pi_1(p)\colon \pi_1(E) \to \pi_1(B)$ induces a map $p_*\colon \operatorname{Wh}(\mathbb{Z}[\pi_1(E)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(B)])$. Next we investigate the composite $p_* \circ p^*\colon \operatorname{Wh}(\mathbb{Z}[\pi_1(B)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(B)])$.

For a group G let $\operatorname{Sw}^p(G)$ be the Grothendieck groups of $\mathbb{Z}G$-modules M that are finitely generated free as abelian groups, see Definition 12.65. There is a pairing, see (12.69)

$$s\colon \operatorname{Sw}^p(G) \otimes K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}G) \tag{24.9}$$

It induces a pairing

$$s\colon \operatorname{Sw}^p(G) \otimes \operatorname{Wh}(G) \to \operatorname{Wh}(G). \tag{24.10}$$

Exercise 24.11. Show that the pairing (24.9) induces a well-defined pairing (24.10).

The fiber transport induces a homotopy $\pi_1(B)$-action on F. So $H_n(F;\mathbb{Z})$ becomes a $\mathbb{Z}[\pi_1(B)]$-module. Thus we obtain a $\mathbb{Z}[\pi_1(B)]$-module $H_n(F;\mathbb{Z})$ that is finitely generated as abelian group. Define the element

$$h(p) = \sum_{n \geq 0} (-1)^n \cdot [H_1(F;\mathbb{Z})] \in \mathrm{Sw}(\pi_1(B)). \tag{24.12}$$

for the Swan ring $\mathrm{Sw}(G)$ given by $\mathbb{Z}G$-module that are finitely generated as abelian groups, see Definition 12.65.

Theorem 24.13 (Down-up formula).

(i) The composite

$$p_* \circ p^* \colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(B)])$$

agrees with $s(e^{-1}(h(p)), -)$ *for the pairing* s *defined in* (24.10)*, the element* $h(p)$ *defined in* (24.12)*, and the isomorphism* $e\colon \mathrm{Sw}^p(\pi_1(B)) \xrightarrow{\cong} \mathrm{Sw}(\pi_1(B))$ *from Lemma 12.66;*

(ii) If p *is orientable, i.e., its fiber transport is trivial, then the composite* $p_* \circ p^* \colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(B)])$ *is multiplication with the Euler characteristic* $\chi(F)$*;*

(iii) If the fiber F *is contractible, then* $p_* \colon \mathrm{Wh}(\mathbb{Z}[\pi_1(E)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(B)])$ *is an isomorphism whose inverse is* $p^* \colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(E)])$*.*

Proof. (i) See [628, Corollary 6.4].

(ii) This follows from assertion (i).

(iii) This follows from assertion (ii). □

Example 24.14. Let M be a connected closed smooth manifold of dimension $d \geq 5$. Then we have the locally trivial bundle $p\colon STM \to M$ given by the sphere bundle associated to the tangent bundle. For it the transfer $p^*\colon \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$ above is defined. Let W be an h-cobordism over B. Choose a retraction $r\colon W \to B$ of the composite $B \xrightarrow{f} \partial_0 W \to W$. As explained in Example 24.3, the pullback construction associated to r yields an h-cobordism $\overline{W}$ over E and we have the equality

$$p^*(\tau(W)) = \tau(\overline{W}).$$

The map $p_*\colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(E)])$ is bijective. The composite $p_* \circ p^*\colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(B)])$ is multiplication by $(1 + (-1)^{d-1})$ provided that M is orientable. If $p_+\colon S_+TM \to M$ is the hemisphere bundle appearing in Section 19.5, then the transfer map

$$(p_+)^*\colon \mathrm{Wh}(\mathbb{Z}[\pi_1(B)]) \to \mathrm{Wh}(\mathbb{Z}[\pi_1(S_+TM)])$$

is an isomorphism with inverse $(p_+)_*\colon \operatorname{Wh}(\mathbb{Z}[\pi_1(S_+TM)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(B)])$. All these claims follow from Theorem 24.13

There is also a more complicated up-down-formulas in favourite cases, see [628, Theorem 8.2]. It leads to interesting computations of the transfer map p^*, see [628, Sections 8 and 9], but these are not needed for the purposes of this book.

24.5 The Transfer for Finitely Dominated $\mathbb{Z}$-Chain Complexes with Homotopy G-Action

Let G be a group and C_* be a $\mathbb{Z}$-chain complex C_* together with a homotopy G-action, i.e., a group homomorphism $\rho\colon G \to [C_*, C_*]_{\mathbb{Z}}$ to the group of $\mathbb{Z}$-chain homotopy classes of self $\mathbb{Z}$-chain maps $C_* \to C_*$. It induces a $\mathbb{Z}G$-$\mathbb{Z}G$ chain homotopy representation $U_*\colon \mathbb{Z}G \to [\mathbb{Z}G \otimes_{\mathbb{Z}} C_*, \mathbb{Z}G \otimes_{\mathbb{Z}} C_*]_{\mathbb{Z}G}$ in the sense of Definition 24.4 by

$$U(g_0)_*(g \otimes x) = gg_0^{-1} \otimes \rho(g_0)_*(x). \tag{24.15}$$

Suppose additionally that C_* is a finite free $\mathbb{Z}$-chain complex. So the transfer of (24.5) is defined

$$p^*_{C_*,\rho}\colon \operatorname{Wh}(G) \to \operatorname{Wh}(G) \tag{24.16}$$

An R-chain complex C_* is called *finitely dominated*, if there exists a finite free R chain complex D_* together with R-chain maps $i_*\colon C_* \to D_*$ and $r_*\colon D_* \to C_*$ satisfying $r_* \circ i_* \simeq_R \operatorname{id}_{C_*}$.

Lemma 24.17. *The definition of the transfer* (24.16) *extends to the case where we weaken the condition that C_* is a finite free $\mathbb{Z}$-chain complex to the condition that C_* is finitely dominated.*

Proof. First we consider two finite free $\mathbb{Z}$-chain complexes C_* and C'_*, two homotopy G-actions $\rho\colon G \to [C_*, C_*]_{\mathbb{Z}}$ and $\rho'\colon G \to [C'_*, C'_*]_{\mathbb{Z}}$, and a $\mathbb{Z}$-chain homotopy equivalence $f_*\colon C_* \to C'_*$ such that $\rho'(g) \circ f_* \simeq_{\mathbb{Z}} f_* \circ \rho(g)$ holds for all $g \in G$ and then prove

$$p^*_{C_*,\rho} = p^*_{C'_*,\rho'} \tag{24.18}$$

Let U_* and U'_* be the $\mathbb{Z}G$-$\mathbb{Z}G$-chain representations associated to (C_*, ρ) and (C'_*, ρ'), see (24.15). From f_* we obtain a $\mathbb{Z}G$-chain homotopy equivalence

$$u := \bigoplus_{i=1}^n \operatorname{id}_{\mathbb{Z}G} \otimes_{\mathbb{Z}} f\colon \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C'_* \xrightarrow{\simeq} \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C_*$$

Given any A in $GL_n(R)$, we get a diagram of finite free $\mathbb{Z}G$-chain complexes

$$\begin{tikzcd}
\bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C_* \arrow[r, "U_*(A)"] \arrow[d, "u"'] & \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C_* \arrow[d, "u"] \\
\bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C'_* \arrow[r, "U'_*(A)"'] & \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C_*
\end{tikzcd}$$

which commutes up to $\mathbb{Z}G$-chain homotopy and where all arrows are $\mathbb{Z}G$-chain homotopy equivalences. Equip C_* and C'_* with some $\mathbb{Z}$-basis and use in the sequel the induced $\mathbb{Z}G$-basis on the $\mathbb{Z}G$-chain complexes appearing in the diagram above. We get for the Whitehead torsion in $\mathrm{Wh}(G)$

$$\begin{aligned}
\tau(U_*(A)) &= \tau(u) + \tau(U_*(A)) - \tau(u) = \tau(u \circ U_*(A)) - \tau(u) \\
&= \tau(U'_*(A) \circ u) - \tau(u) = \tau(U'_*(A)) + \tau(u) - \tau(u) = \tau(U'_*(A)).
\end{aligned}$$

Now (24.18) follows from the definitions.

Next we define for a $\mathbb{Z}$-chain complex C_* which is $\mathbb{Z}$-chain homotopy equivalent to some finite free $\mathbb{Z}$-chain complex and comes with a homotopy G-action its transfer map (24.7).Choose a finite free $\mathbb{Z}$-chain complex C'_* together with a $\mathbb{Z}$-chain homotopy equivalence $f_*\colon C'_* \xrightarrow{\simeq} C_*$. Then there is precisely one homotopy G-action $\rho'\colon G \to [C'_*, C'_*]_{\mathbb{Z}}$ such that $f_* \circ \rho(g) \simeq \rho'(g) \circ f_*$ holds for every $g \in G$. Now define

$$p^*_{C_*,\rho} := p^*_{C'_*,\rho'}$$

This is independent of the choice of C'_* and f_* by (24.18).

Finally we mention that any finitely dominated $\mathbb{Z}$-complex C_* is $\mathbb{Z}$-chain homotopy equivalent to a finite projective $\mathbb{Z}$-chain complex, see for instance [629, Proposition 11.11 on page 222], and hence to a finite free $\mathbb{Z}$-chain complex since $\mathbb{Z}$ is a principal ideal domain. □

24.6 The Transfer for Finitely Dominated Spaces with Homotopy G-Action

Let X be a finitely dominated space, i.e., that there is a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. Suppose that X comes with a homotopy G-action $\rho\colon G \to [X, X]$ in the sense of Definition 20.27. Then we obtain by passing to the singular $\mathbb{Z}$-chain complex $C^{\mathrm{sing}}_*(X)$ a homotopy G-action $\rho^{\mathrm{sing}}\colon G \to [C^{\mathrm{sing}}_*(X), C^{\mathrm{sing}}_*(X)]_{\mathbb{Z}}$. Since X is finitely dominated and the singular chain complex of a finite CW-complex Y is $\mathbb{Z}$-chain homotopy equivalent to the finite free cellular $\mathbb{Z}$-chain complex $C^c_*(Y)$, see for instance [629, Proposition 13.10 on page 264] the $\mathbb{Z}$-chain complex $C_*^{\mathrm{sing}}(X)$ is finitely dominated. Hence we get from Lemma 24.17 a transfer map

$$p_{X,\rho}^*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G) \tag{24.19}$$

Remark 24.20. One easily checks that it still satisfies the Down-Up Formula 24.13 taking into account that $H_n(C_*^{\mathrm{sing}}(X))$ is finitely generated as abelian group since X is finitely dominated. More precisely, we get an element

$$h(X;\rho) = \sum_{n\geq 0} (-1)^n \cdot [H_1(X;\mathbb{Z})] \in \mathrm{Sw}(G) \tag{24.21}$$

and the equality in $\mathrm{Wh}(G)$

$$p_{X,\rho}^*(u) = s(e^{-1}(h(X;\rho)), u) \tag{24.22}$$

for the pairing s defined in (24.10), the element $h(X;\rho)$ defined in (24.21), and the isomorphism $e\colon \mathrm{Sw}^p(\pi_1(B)) \xrightarrow{\cong} \mathrm{Sw}(\pi_1(B))$ from Lemma 12.66;

Suppose additionally that X is contractible, Then $H_n(X;\mathbb{Z}) = 0$ for $n \geq 1$ and $H_0(X;\mathbb{Z})$ is the $\mathbb{Z}G$-module given by $\mathbb{Z}$ with the trivial G-action. Since $[\mathbb{Z}] = e^{-1}(h(X;\rho))$ is the unit in $\mathrm{Sw}^p(G)$, the down up formula implies that $p_{X,\rho}^*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ is the identity.

Example 24.23. Let G be a hyperbolic group group in the sense of Gromov, see for instance [149, 155, 408, 423]. Let $X = P_d(G)$ be the associated Rips complex for some number $d > 16\delta + 8$ if G is δ-hyperbolic space with respect to some choice S of a finite set of generators, see Subsection 11.6.7. Such $\delta > 0$ exists by the definition of hyperbolic. The obvious simplicial G-action on X is cocompact and proper. The barycentric subdivision of X is a cocompact model for the classifying space $E_{\mathcal{FIN}}(G)$, see Theorem 11.29. Now take $\overline{X} = X \cup \partial X$ to be the compactification of X in the sense of Gromov, see [423], [155, Section 3 in Chapter III.H]. Then $\overline{X}$ is a contractible compact metrizable G-space.

Then the transfer map (24.19) is defined and yields an isomorphism

$$p^*\colon \mathrm{Wh}(G) \xrightarrow{\cong} \mathrm{Wh}(G).$$

A controlled version of this transfer, which works for the K-groups in all dimensions and is described in [86, Section 6], is a key ingredient in the proof of the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories for a hyperbolic group in [86, Main Theorem].

Analogously a controlled version of the transfer map above described in [77, Section 7] is a key ingredient in the proof of the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories for a finite-dimensional CAT(0)-group G in [77, Theorem B]. Here $\overline{X}$ is a bordification defined in Bridson-Haefliger [155, Chapter II.8] of a finite-dimensional

CAT(0)-space X on which the CAT(0)-group G acts properly, cocompactly, and isometrically.

24.7 Proof of Surjectivity of the Assembly map in Dimension 1

In this section we give the proof of a special case of Theorem 20.12 as an illustration of the methods and results described so far. Reducing to this special case avoids some formidable purely technical input which will make the exposition much harder but will be discussed later in Chapter 25.

Proposition 24.24. *Let G be a finitely generated group. Let $\mathcal{F}$ be a family of subgroups such that G is strictly $\mathcal{F}$-transfer reducible in the sense of Definition 20.11.*

Then the assembly map

$$H_1^G(\mathrm{pr}; \mathbf{K}_{\mathbb{Z}}) \colon H_1^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_1^G(G/G; \mathbf{K}_{\mathbb{Z}}) = K_1(\mathbb{Z}G)$$

is surjective.

The rest of this section is devoted to the proof of Proposition 24.24. In this section R will always be $\mathbb{Z}$ for simplicity.

24.7.1 Basic Strategy of the Proof of Proposition 24.24

For the remainder of this section fix an element

$$u \in K_1(\mathbb{Z}G).$$

We want to show that u is in the image of $H_1^G(\mathrm{pr}; \mathbf{K}_{\mathbb{Z}})$.

For an element $a = \sum_{g \in g} \lambda_g \cdot g \in \mathbb{Z}G$, define the $\mathbb{Z}G$-homomorphism $V(a) \colon \mathbb{Z}G \to \mathbb{Z}G$ by sending x to $\sum_{g \in G} \lambda \cdot xg^{-1}$. Given a matrix $A = (a_{i,j})_{i,j} \in M_{m,n}(\mathbb{Z}G)$, define a $\mathbb{Z}G$-homomorphism
(24.25)

$$V(A) \colon \mathbb{Z}G^m \to \mathbb{Z}G^n, \quad (x_1, x_2, \dots, x_m) \mapsto \bigl(\sum_{i=1}^{m} V(a_{i,j})(x_i)\bigr)_{j=1,\dots,n}.$$

One easily checks that $V(AB) = V(A) \circ V(B)$ holds for $A \in M_{m,n}(\mathbb{Z}G)$ and $B \in M_{n,o}(\mathbb{Z}G)$ and $V(I_n) = \mathrm{id}_{\mathbb{Z}G^n}$ holds for the identity matrix $I_n \in GL_n(\mathbb{Z}G)$.

Choose a natural number n and an element $A = (a_{i,j}) \in GL_n(\mathbb{Z}G)$ such that u is represented by the $\mathbb{Z}G$-automorphism $V(A) \colon \mathbb{Z}G^n \xrightarrow{\cong} \mathbb{Z}G^n$ given by right multiplication with A. Choose a finite subset $T \subseteq G$ such that for any $i,j \in \{1,2,\ldots,n\}$ the elements $a_{i,j}, b_{i,j} \in \mathbb{Z}G$ are of the form $\sum_{g \in T} \lambda_g \cdot g$ and $e \in T$ holds. By possibly enlarging T we can additionally arrange that T is a finite set of generators of G.

Next let us recall what we get from the assumption that G is strictly $\mathcal{F}$-transfer reducible. Let N be the number appearing Definition 20.11. Then the following holds by assumption:

- We have an N-transfer space X in the sense of Definition 20.9, that is a compact contractible metric space (X, d_X) with the property that for any $\delta > 0$ there exists an N-dimensional simplicial complex K, maps $i\colon X \to |K|$ and $r\colon |K| \to X$, and a homotopy $h\colon X \times [0,1] \to X$ from $p \circ i$ to id_X which is δ-controlled, i.e., for every $x \in X$ the diameter of the subset $h(\{x\} \times [0,1])$ of X is smaller than δ;
- The N-transfer space X comes with a G-action;
- For every $\epsilon > 0$ there exists
  - an abstract simplicial $(G, \mathcal{F})$-complex Σ of dimension $\le N$;
  - a map $v\colon X \to |\Sigma|$ that is (μ, T)-*almost* G-*equivariant*, i.e., we have $d^{L^1}(v(gx), gv(x)) \le \epsilon$ for every $g \in T$ and every $x \in X$.

Next we formulate what we need to prove Proposition 24.24. In view of the Algebraic Thin h-Cobordism Theorem 19.8 we have to construct for the number ϵ_N appearing in Theorem 19.8 and the element $u \in K_1(\mathbb{Z}G)$

- An abstract simplicial $(G, \mathcal{F})$-complex Z of dimension $\le N$;
- A free G-space Y together with a map $w\colon Y \to |Z|$;
- An ϵ_N automorphism $a\colon M \to M$ in $\mathsf{GM}^G(Y)$, i.e., an automorphism $a\colon M \to M$ in $\mathsf{GM}^G(Y)$ such that both a and a^{-1} are ϵ_N-controlled with respect to w and the L^1-metric d^{L^1} on $|Z|$. Recall that a morphism $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in X\}\colon M \to N$ in $\mathsf{GM}^G(Y)$ is ϵ_N-controlled with respect to w and the L^1-metric d^{L^1} on $|Z|$ if the implication $x, y \in X, f_{x,y} \neq 0 \implies d^{L^1}(p(x), p(y)) \le \epsilon_N$ holds;
- The class $[F^f(a)] \in K_1(\mathbb{Z}G)$ of the RG-automorphism $F^f(a)\colon F^f(M) \xrightarrow{\cong} F^f(M)$ of the finitely generated free RG-module $F^f(M)$ for the functor F^f defined in (19.6) is u.

Put

$$\epsilon = \frac{\epsilon_N}{5(72N + 181)}. \tag{24.26}$$

Now make the choice of the data (X, d), Σ, and v described above for this choice of ϵ.

Next we construct the desired data mentioned above. We will take for Y the G-space $G \times X$ where the G-action is given by $g' \cdot (g, x) = (g'g, x)$ for

$g' \in G$ and $(g,x) \in Y$. We take $Z = \Sigma$. Define the G-map $w\colon Y \to |\Sigma|$ by sending (g,x) to $gv(x)$.

So for the rest of this section we have fixed $u \in K_1(\mathbb{Z}G)$, $A \in GL_n(\mathbb{Z}G)$, the finite subset $T \subset G$, numbers N and ϵ_N, the abstract simplicial G-complex Z, the G-spaces X and Y, metrics d on X and d^{L^1} on $|Z|$, the map v, and the G-map $w\colon Y \to |Z|$ and will only consider $R = \mathbb{Z}$. Recall that the G-action on X is not necessarily isometric, whereas the G-action on $|Z|$ is isometric, and that v is (ϵ, T)-almost G-equivariant.

Note that so far we have not used the G-action on X which will enter in the construction of the desired ϵ_N-controlled automorphism $a\colon M \to M$ in $\mathsf{GM}^G(Y)$ satisfying $u = [F^f(a)]$. The only thing that remains to be done is the construction of a which will occupy the rest of this section.

24.7.2 The Width

Definition 24.27 (Width function). Let $\mathcal{A}$ be an additive category. A *width function* $\mathrm{wd} = \mathrm{wd}_{\mathcal{A}}$ on $\mathcal{A}$ is a function

$$\mathrm{wd}\colon \mathrm{mor}(\mathcal{A}) \to \mathbb{R}^{\geq 0} \amalg \{-\infty, \infty\}$$

satisfying the following axioms.

(i) Consider finitely many objects $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ and morphisms $f_{i,j}\colon A_i \to B_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ in $\mathcal{A}$. Let $f\colon \bigoplus_{i=1}^m A_i \to \bigoplus_{j=1}^n B_j$ be the morphism given by the collection of the $f_{i,j}$-s. Then

$$\mathrm{wd}(f) \leq \max\{\mathrm{wd}(f_{i,j}) \mid i = 1, \ldots, m, j = 1, \ldots, n\};$$

(ii) Consider morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{A}$. Then we get

$$\mathrm{wd}(g \circ f) \leq \mathrm{wd}(f) + \mathrm{wd}(g);$$

(iii) Consider morphisms $f, g\colon A \to B$ and and $\lambda, \mu \in \mathbb{Z}$. Then

$$\mathrm{wd}(\lambda \cdot f + \mu \cdot g) \leq \max\{\mathrm{wd}(f), \mathrm{wd}(g)\};$$

(iv) We have $f = 0 \Longleftrightarrow \mathrm{wd}(f) = -\infty$ for every morphism $f\colon M \to N$ in $\mathcal{A}$.

We define the width $\mathrm{wd}(A)$ of an object to be the width $\mathrm{wd}(\mathrm{id}_A)$ if the identity on A.

We call wd *trivial on objects* if for every object A we have $\mathrm{wd}(A) = 0$.

Remark 24.28 (Passage to idempotent completion). Let $\mathcal{A}$ be an additive category with width function wd. Then its idempotent completion inherits a width function $\widehat{\mathrm{wd}}$ which assigns to a morphism $f\colon (A,p) \to (B,q)$ in $\mathrm{Idem}(\mathcal{A})$ the width $\mathrm{wd}_{\mathcal{A}}(f)$ of the underlying morphism $f\colon A \to B$ in $\mathcal{A}$.

Note that the identity of an object (A, P) in $\mathrm{Idem}(\mathcal{A})$ is given by $p\colon A \to A$ and hence $\widehat{\mathrm{wd}}(A, p) = \mathrm{wd}(p)$. So, even if wd is trivial on objects, $\widehat{\mathrm{wd}}$ is not necessarily trivial on objects.

Next we present our main example of a width function.

Example 24.29 (Width function on $\mathsf{GM}^G(Y)$). We define a width function on the additive category $\mathsf{GM}^G(Y)$ from Definition 19.4 as follows, where we use the data fixed in Subsection 24.7.1.

Given two objects $M = \{M_x \mid x \in Y\}$ and $N = \{N_y \mid y \in Y\}$ and a morphism $f\colon M \to N$ in $\mathsf{GM}^G(Y)$ which consists of a collection of morphisms $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in Y\}$ in $\mathcal{F}^\kappa(\mathbb{Z})$, we define the *width*

$$\mathrm{wd}_Z(f) \in \mathbb{R}^{\geq 0} \amalg \{-\infty\}$$

to be the supremum of the set $\{d^{L^1}(w(x), w(y)) \mid x, y \in X, f_{x,y} \neq 0\}$ if f is not the zero homomorphism and to be $-\infty$ otherwise. Note that this width function is trivial on objects.

Note that for $\epsilon \geq 0$ a morphism $f\colon M \to N$ in $\mathsf{GM}^G(Y)$ is ϵ-controlled in the sense of Subsection 19.4.3 if and only if $\mathrm{wd}_Z(f) \leq \epsilon$ holds.

Exercise 24.30. Show that the axioms of a width function which is trivial on objects are satisfied in Example 24.29.

Example 24.31 (Width function on $\mathsf{GM}(X)$). Let (X, d) be any metric space. Let $\mathsf{GM}(X)$ be $\mathsf{GM}^G(X)$ for $G = \{1\}$. We will equip it the following width function wd_X.

Given two objects $M = \{M_x \mid x \in X\}$ and $N = \{N_y \mid y \in X\}$ and a morphism $f\colon M \to N$ $\mathsf{GM}^G(X)$, consisting of a collection of morphisms $f = \{f_{x,y}\colon M_x \to N_y \mid x, y \in X\}$ in $\mathcal{F}^\kappa(\mathbb{Z})$, we define the *width*

$$\mathrm{wd}(f) \in \mathbb{R}^{\geq 0} \amalg \{-\infty\}$$

to be the supremum of the set $\{d(x, y) \mid x, y \in X, f_{x,y} \neq 0\}$ if f is not the zero homomorphism and to be $-\infty$ otherwise. Note that this width function is trivial on objects.

24.7.3 Selftorsion

Let $\mathcal{A}$ be an additive category. Let $C_* = (C_*, c_*)$ be a *bounded* $\mathcal{A}$-chain complex i.e., a sequence of morphisms in $\mathcal{A}$

$$\cdots \xrightarrow{c_{n+2}} C_{n+2} \xrightarrow{c_{n+1}} C_{n+1} \xrightarrow{c_n} C_n \xrightarrow{c_{n-1}} C_{n-1} \xrightarrow{c_{n-2}}$$

such that $c_{n+1} \circ c_n = 0$ holds for $n \in \mathbb{Z}$ and there exists a natural number N with $C_n = 0$ for $n \in \mathbb{Z}$ with $|n| \geq N$. There are obvious notions of a chain map, a chain homotopy, and a chain contraction. Let $f_* \colon C_* \xrightarrow{\simeq} D_*$ be a chain homotopy equivalence of bounded $\mathcal{A}$-chain complexes. Denote by $\operatorname{cone}(f_*)$ its mapping cone whose n-th differential e_* is given by

$$e_n \colon C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}. \tag{24.32}$$

Given an $\mathcal{A}$-chain map $g_* \colon D_* \to C_*$ and $\mathcal{A}$-chain homotopies $h_* \colon g_* \circ f_* \simeq \mathrm{id}_{C_*}$ and $k_* \colon f_* \circ g_* \simeq \mathrm{id}_{D_*}$, define an $\mathcal{A}$-chain isomorphism $u_* \colon \operatorname{cone}(f_*) \xrightarrow{\cong} \operatorname{cone}(f_*)$ by $u_n = \begin{pmatrix} \mathrm{id}_{C_{n-1}} & 0 \\ f_n \circ h_{n-1} - k_{n-1} \circ f_{n-1} & \mathrm{id}_{D_n} \end{pmatrix}$ and an $\mathcal{A}$-chain homotopy $\delta_* \colon u_* \simeq 0_*$ by $\delta_n = \begin{pmatrix} h_{n-1} & g_n \\ 0 & -k_n \end{pmatrix}$. Then we obtain a chain contraction γ_* of $\operatorname{cone}(f_*)$ by

$$\gamma_n = u_{n+1}^{-1} \circ \delta_n \tag{24.33}$$

Now consider a selfchain homotopy equivalence $f_* \colon C_* \xrightarrow{\simeq} C_*$ of the bounded $\mathcal{A}$-chain complex C_*. Define objects in $\mathcal{A}$ by

$$\begin{aligned} C_{\mathrm{all}} &= \bigoplus_{n \in \mathbb{Z}} C_n; \\ \operatorname{cone}(f)_{\mathrm{odd}} &= \bigoplus_{n \in \mathbb{Z}} \operatorname{cone}(f_*)_{2n+1}; \\ \operatorname{cone}(f)_{\mathrm{ev}} &= \bigoplus_{n \in \mathbb{Z}} \operatorname{cone}(f_*)_{2n}. \end{aligned}$$

We obtain isomorphisms

$$(e+\gamma)_{\mathrm{odd}} \colon \operatorname{cone}(f_*)_{\mathrm{odd}} \xrightarrow{\cong} \operatorname{cone}(f_*)_{\mathrm{ev}}; \tag{24.34}$$

$$(e+\gamma)_{\mathrm{ev}} \colon \operatorname{cone}(f_*)_{\mathrm{ev}} \xrightarrow{\cong} \operatorname{cone}(f_*)_{\mathrm{odd}}, \tag{24.35}$$

satisfying

$$(e+\gamma)_{\mathrm{ev}} \circ (e+\gamma)_{\mathrm{odd}} = (\mathrm{id} + \gamma \circ \gamma); \tag{24.36}$$

$$(e+\gamma)_{\mathrm{odd}} \circ (e+\gamma)_{\mathrm{ev}} = (\mathrm{id} + \gamma \circ \gamma). \tag{24.37}$$

Let

$$I_{\mathrm{odd}}\colon \operatorname{cone}(f_*)_{\mathrm{odd}} \xrightarrow{\cong} C_{\mathrm{all}};$$
$$I_{\mathrm{ev}}\colon \operatorname{cone}(f_*)_{\mathrm{ev}} \xrightarrow{\cong} C_{\mathrm{all}},$$

be the obvious isomorphisms coming from $\operatorname{cone}(f_*)_n = C_{n-1} \oplus C_n$.

Thus we obtain an automorphism

$$I_{\mathrm{ev}} \circ (e+\gamma) \circ I_{\mathrm{odd}}^{-1}\colon C_{\mathrm{all}} \xrightarrow{\cong} C_{\mathrm{all}}.$$

Its class

$$t(f_*) := [I_{\mathrm{ev}} \circ (e+\gamma) \circ I_{\mathrm{odd}}^{-1}] \in K_1(\mathcal{A}) \tag{24.38}$$

is called the *selftorsion* of f_*. The proof that this element is well-defined and has the following properties in [629, Section 12] for $R\Gamma$-modules carries directly over to additive categories.

Lemma 24.39.

(i) Let $f_*, g_*\colon C_* \xrightarrow{\simeq} C_*$ be selfchain homotopy equivalences of the bounded $\mathcal{A}$-chain complex C_*. If they are chain homotopic, then

$$t(g_*) = t(f_*);$$

(ii) Consider a commutative diagram of bounded $\mathcal{A}$-chain complexes with selfchain homotopy equivalences as vertical arrows

$$\begin{tikzcd}
0 \arrow[r] & C_* \arrow[r, "i_*", "\simeq"'] \arrow[d, "f_*", "\simeq"'] & D_* \arrow[r, "p_*"] \arrow[d, "g_*", "\simeq"'] & E_* \arrow[r] \arrow[d, "h_*"] & 0 \\
0 \arrow[r] & C_* \arrow[r, "i_*"] & D_* \arrow[r, "p_*"] & E_* \arrow[r] & 0
\end{tikzcd}$$

where for each $n \in \mathbb{Z}$ the sequence $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{p_n} E_n \to 0$ is split exact, i.e., there exists a morphism $s_n\colon E_n \to D_n$ such that $p_n \circ s_n = \mathrm{id}_{E_n}$ holds and $i_n \oplus s_n\colon C_n \oplus E_n \xrightarrow{\cong} D_n$ is an isomorphism. Then we get

$$t(g_*) = t(f_*) + t(h_*);$$

(iii) Let $f_*, g_*\colon C_* \xrightarrow{\simeq} C_*$ be selfchain homotopy equivalences of the bounded $\mathcal{A}$-chain complex C_*. Then we get

$$t(g_* \circ f_*) = t(f_*) + t(g_*).$$

Exercise 24.40. Let $f_*\colon C_* \xrightarrow{\cong} C_*$ be a chain automorphism of a bounded $\mathcal{A}$-chain complex. Show

$$t(f_*) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot [f_n] \in K_1(\mathcal{A}).$$

24.7.4 Selftorsion and Width functions

Let $\mathcal{A}$ be an additive category coming with a width function wd in the sense of Definition 24.27. We define the *width* $\operatorname{wd}(C_*)$ of a bounded $\mathcal{A}$-chain complex $C_* = (C_*, c_*)$ to be

$$\operatorname{wd}(C_*) = \max\{\operatorname{wd}(C_n), \operatorname{wd}(c_n) \mid n \in \mathbb{Z}\}. \tag{24.41}$$

We define the *width* $\operatorname{wd}(f_*)$ of a $\mathcal{A}$-chain map $f_* \colon C_* \to D_*$ of a bounded $\mathcal{A}$-chain complex $C_* = (C_*, c_*)$ to be

$$\operatorname{wd}(f_*) = \max\{\operatorname{wd}(f_n) \mid n \in \mathbb{Z}\}, \tag{24.42}$$

and the *width* $\operatorname{wd}(h_*)$ of a $\mathcal{A}$-chain homotopy $h_* \colon C_* \to D_{*+1}$ of bounded $\mathcal{A}$-chain complexes to be

$$\operatorname{wd}(H_*) = \max\{\operatorname{wd}(H_n) \mid n \in \mathbb{Z}\}. \tag{24.43}$$

Notation 24.44. For $\epsilon > 0$ and two $\mathcal{A}$-chain maps $f_*, g_* \colon C_* \to D_*$ we write

$$f_* \simeq_\epsilon g_*$$

if there exists a $\mathcal{A}$-chain homotopy $h_* \colon f_* \simeq g_*$ with $\operatorname{wd}(h_*) \le \epsilon$.

Definition 24.45 (ϵ-controlled isomorphism). An *ϵ-controlled isomorphism* $f \colon A \xrightarrow{\cong} B$ in $\mathcal{A}$ is an isomorphism $f \colon A \xrightarrow{\cong} B$ such that

$$\operatorname{wd}(A), \operatorname{wd}(B), \operatorname{wd}(f), \operatorname{wd}(f^{-1}) \le \epsilon$$

hold.

If $A = B$, we talk of an *ϵ-controlled automorphism*

Exercise 24.46. Let $f \colon A \to B$ be an ϵ-controlled $\mathcal{A}$-isomorphism and $g \colon B \to C$ be a δ-controlled $\mathcal{A}$-isomorphism.

Show that $g \circ f \colon A \to C$ is an $(\epsilon + \delta)$-controlled $\mathcal{A}$-isomorphism.

Definition 24.47 (ϵ-controlled chain homotopy equivalence). Consider $\epsilon > 0$ and a $\mathcal{A}$-chain map $f_* \colon C_* \to D_*$. We call f_* an *ϵ-controlled $\mathcal{A}$-chain homotopy equivalence* if there is a $\mathcal{A}$-chain map $g_* \colon D_* \to C_*$ satisfying

$$\operatorname{wd}(C_*), \operatorname{wd}(D_*), \operatorname{wd}(f_*), \operatorname{wd}(g_*) \le \epsilon$$

and

$$g_* \circ f_* \simeq_\epsilon \mathrm{id}_{C_*};$$
$$f_* \circ g_* \simeq_\epsilon \mathrm{id}_{D_*}.$$

If $C_* = D_*$, we talk of an *ϵ-controlled $\mathcal{A}$-selfchain homotopy equivalence.*

The next lemma is a direct consequence of the axioms appearing in Definition 24.27.

Lemma 24.48. *Consider $\delta, \epsilon > 0$.*

(i) Let $f_*, g_*, h_* \colon C_* \to D_*$ be $\mathcal{A}$-chain maps of bounded $\mathcal{A}$-chain complexes and $\lambda, \mu \in \mathbb{Z}$. Then

$$\mathrm{wd}(\lambda \cdot f_* + \mu \cdot g_*) \le \max\{\mathrm{wd}(f_*), \mathrm{wd}(g_*)\}$$

and

$$f_* \simeq_\epsilon g_*, g_* \simeq_\epsilon h_* \implies f_* \simeq_\epsilon h_*;$$

(ii) Let $f_*, f'_* \colon D_* \to E_*$, $u_* \colon C_* \to D_*$, and $v_* \colon E_* \to F_*$ be $\mathcal{A}$-chain maps of bounded $\mathcal{A}$-chain complexes satisfying $f_* \simeq_\epsilon f'_*$, $\mathrm{wd}(u_*) \le \delta$, and $\mathrm{wd}(v_*) \le \delta$. Then

$$v_* \circ f_* \simeq_{\delta+\epsilon} v_* \circ f'_*;$$
$$f_* \circ u_* \simeq_{\delta+\epsilon} f'_* \circ u_*;$$

(iii) Let $f_* \colon C_* \to D_*$ and $g_* \colon D_* \to E_*$ be ϵ-controlled $\mathcal{A}$-chain homotopy equivalence of bounded $\mathcal{A}$-chain complexes.
Then $g_* \circ f_* \colon C_* \to E_*$ is a 3ϵ-controlled $\mathcal{A}$-chain homotopy equivalences of bounded $\mathcal{A}$-chain complexes.

Exercise 24.49. Give the proof of Lemma 24.48.

Proposition 24.50. *Let $\mathcal{A}$ be an additive category coming with a width function* wd*. Consider $\epsilon > 0$. Let $f_* \colon C_* \xrightarrow{\simeq} C_*$ be an* $\mathrm{Idem}(\mathcal{A})$*-selfchain homotopy equivalences which is ϵ-controlled.*

Then there is an 5ϵ-controlled $\mathcal{A}$-automorphism $a \colon A \xrightarrow{\cong} A$ such that the selftorsion $t(F_*) \in K_1(\mathcal{A}) = K_1(\mathrm{Idem}(\mathcal{A}))$ satisfies

$$t(u_*) = [a].$$

Proof. Recall that we have defined a width function $\widehat{\mathrm{wd}}$ on $\mathrm{Idem}(\mathcal{A})$ in Remark 24.28. By assumption we have $\mathrm{Idem}(\mathcal{A})$-chain homotopy equivalences $f_* \colon C_* \to C_*$ and $g_* \colon C_* \to C_*$ and $\mathrm{Idem}(\mathcal{A})$-chain homotopies $h_* \colon f_* \circ g_* \simeq \mathrm{id}_{C_*}$ and $k_* \colon g_* \circ f_* \simeq \mathrm{id}_{C_*}$ such that $\widehat{\mathrm{wd}}(C_*)$, $\widehat{\mathrm{wd}}(f_*)$, $\widehat{\mathrm{wd}}(g_*)$, $\widehat{\mathrm{wd}}(h_*)$, and $\widehat{\mathrm{wd}}(k_*)$ are less or equal to ϵ. Let $\mathrm{cone}(f_*)$ be the mapping cone of f_*, see (24.32). In the sequel we will apply over and over again the axioms appearing Definition 24.27 and Lemma 24.48. One easily checks

$\widehat{\mathrm{wd}}(\mathrm{cone}(f_*)) \leq \epsilon$. We have constructed a chain contraction γ for $\mathrm{cone}(f_*)$ in (24.33). We get $\widehat{\mathrm{wd}}(\gamma_*) \leq 3\epsilon$. We conclude from (24.36) and (24.37) that the $\mathrm{Idem}(\mathcal{A})$-automorphisms $I_{\mathrm{ev}} \circ (e+\gamma) \circ I_{\mathrm{odd}}^{-1} \colon C_{\mathrm{all}} \xrightarrow{\cong} C_{\mathrm{all}}$ is 3ϵ-controlled. Its class in $K_1(\mathrm{Idem}(\mathcal{A}))$ is by definition $t(f_*)$.

For each object $C_n = (A_n, p_n)$ in $\mathrm{Idem}(\mathcal{A})$ we can consider the object $C_n^\perp = (A_n, \mathrm{id} - p_n)$ in $\mathrm{Idem}(\mathcal{A})$. Obviously we have $\widehat{\mathrm{wd}}(C_n) = \widehat{\mathrm{wd}}(C_n^\perp) = \widehat{\mathrm{wd}}(p_n) = \widehat{\mathrm{wd}}(\mathrm{id}_{A_n} - p_n)$. The $\mathrm{Idem}(\mathcal{A})$-isomorphisms

$$a_n = p_n \oplus (\mathrm{id}_{A_n} - p_n) \colon C_n \oplus C_n^\perp \xrightarrow{\cong} A_n = (A_n, \mathrm{id}_{A_n})$$

and

$$b_n = (p_n \oplus (\mathrm{id}_{A_n} - p_n)) \colon A_n = (A_n, \mathrm{id}_{A_n}) \xrightarrow{\cong} C_n \oplus C_n^\perp$$

are inverse to one another and satisfy $\widehat{\mathrm{wd}}(a_n), \widehat{\mathrm{wd}}(b_n) \leq \epsilon$. Put $A_{\mathrm{all}} = \bigoplus_{n\in\mathbb{Z}} A_n$ and $C_{\mathrm{all}}^\perp = \bigoplus_{n\in\mathbb{Z}} C_n^\perp$. The collection of the isomorphisms a_n-s and b_n-s yields to one another inverse $\mathrm{Idem}(\mathcal{A})$-isomorphisms $a_{\mathrm{all}} \colon C_{\mathrm{all}} \oplus C_{\mathrm{all}}^\perp \xrightarrow{\cong} A_{\mathrm{all}}$ and $b_{\mathrm{all}} \colon A_{\mathrm{all}} \xrightarrow{\cong} C_{\mathrm{all}} \oplus C_{\mathrm{all}}^\perp$ with $\widehat{\mathrm{wd}}(a_{\mathrm{all}}), \widehat{\mathrm{wd}}(b_{\mathrm{all}}) \leq \epsilon$. Define the $\mathcal{A}$-automorphism

$$a \colon A_{\mathrm{all}} \xrightarrow{b_{\mathrm{all}}} C_{\mathrm{all}} \oplus C_{\mathrm{all}}^\perp \xrightarrow{(I_{\mathrm{ev}}\circ(e+\gamma)\circ I_{\mathrm{odd}}^{-1})\oplus \mathrm{id}_{C_{\mathrm{all}}^\perp}} C_{\mathrm{all}} \oplus C_{\mathrm{all}}^\perp \xrightarrow{a_{\mathrm{all}}} A_{\mathrm{all}}.$$

One easily checks that $t(f_*) = [a]$ in $K_1(\mathcal{A})$ and a is a 5ϵ-controlled $\mathcal{A}$-automorphism. $\square$

24.7.5 Finite Domination

Consider a full and faithful inclusion $\mathcal{A} \to \mathcal{B}$ of additive categories, e.g., the inclusion of the category of finitely generated free R-modules into the category of free R-modules for a ring R. Let C_* be a (not necessarily finite dimensional) positive $\mathcal{B}$-chain complex. Consider a *finite domination* $(D_*, i_*, r_*.h_*)$ of C_* over $\mathcal{A}$, i.e, a finite dimensional positive $\mathcal{A}$-chain complex D_*, $\mathcal{B}$-chain maps $i_* \colon C_* \to D_*$ and $r_* \colon D_* \to C_*$, and a $\mathcal{B}$-chain homotopy $h_* \colon r_* \circ i_* \simeq D_*$. From these data we construct an explicite finite dimensional positive chain complex P_* over $\mathrm{Idem}(\mathcal{A})$ with $\dim(P_*) = \dim(D_*)$ together with $\mathrm{Idem}(\mathcal{B})$-chain homotopy equivalences $f_* \colon C_* \to P_*$ and $g_* \colon P_* \to C_*$ and $\mathrm{Idem}(\mathcal{B})$-chain homotopies $k_* \colon f_* \circ g_* \simeq \mathrm{id}_{P_*}$ and $l_* \colon g_* \circ f_* \simeq \mathrm{id}_{C_*}$ following [822] and [77, Remark 8.3].

Define the chain complex C' over $\mathcal{A}$ by defining its m-th chain object to be

$$C'_m = \bigoplus_{j=0}^{m} D_j$$

and its m-th differential to be

$$c'_m \colon C'_m = \bigoplus_{j=0}^{m} D_j \to C'_{m-1} = \bigoplus_{k=0}^{m-1} D_k$$

where the (j,k)-entry $d_{j,k} \colon D_j \to D_k$ for $j \in \{0,1,2\ldots,m\}$ and $k \in \{0,1,2\ldots,m-1\}$ is given by

$$d_{j,k} := \begin{cases} 0 & \text{if } j \geq k+2; \\ (-1)^{m+k} \cdot d_j & \text{if } j = k+1; \\ \mathrm{id} - r_j \circ i_j & \text{if } j = k, j \equiv m \mod 2; \\ r_j \circ i_j & \text{if } j = k, j \equiv m+1 \mod 2; \\ (-1)^{m+k+1} \cdot i_k \circ h_{k-1} \circ \ldots \circ h_j \circ r_j & \text{if } j \leq k-1. \end{cases}$$

Define chain maps $f'_* \colon C_* \to C'_*$ and $g'_* \colon C'_* \to C_*$ by

$$f'_m \colon C_m \to C'_m = D_0 \oplus D_1 \oplus \cdots \oplus D_m, \quad x \mapsto (0, 0, \ldots, i_m(x))$$

and

$$g'_m \colon C'_m = D_0 \oplus \cdots \oplus D_m \to C_m, \quad (x_0, x_1, \cdots x_m) \mapsto \sum_{j=0}^{m} h_{m-1} \circ \cdots \circ h_j \circ r_j(x_j).$$

We have $g'_* \circ f'_* = r_* \circ i_*$ and hence h_* is a chain homotopy $g'_* \circ f'_* \simeq \mathrm{id}_{C_*}$. We obtain a chain homotopy $k'_* \colon f'_* \circ g'_* \simeq \mathrm{id}_{C'_*}$ if we define

$$k'_m \colon C'_m = D_0 \oplus D_1 \oplus \cdots \oplus D_m \to C'_{m+1} = D_0 \oplus D_1 \oplus \cdots \oplus D_m \oplus D_{m+1}$$

to be the obvious inclusion.

Let N be the dimension of D_*. Thus we get $C'_m = C'_N$ for $m \geq N$ and $c'_{m+1} = \mathrm{id} - c'_m$ for $m \geq N+1$. Since $c'_{m+1} \circ c'_m = 0$ for all m, we conclude $c'_m \circ c'_m = c'_m$ for $m \geq N+1$. Hence C' has the form

$$\cdots \to C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{\mathrm{id} - c'_{N+1}} C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{c'_N} C'_{N-1} \xrightarrow{c'_{N-1}} \ldots \xrightarrow{c'_1} C'_0 \to 0 \to \ldots.$$

Define the desired N-dimensional chain complex P_* over $\mathrm{Idem}(\mathcal{A})$ by

$$0 \to 0 \to (C'_N, \mathrm{id} - c'_{N+1}) \xrightarrow{c'_N \circ i} C'_{N-1} \xrightarrow{c'_{N-1}} \ldots \xrightarrow{c'_1} C'_0 \to 0 \to \cdots$$

where $i \colon (C'_N, \mathrm{id} - c'_{N+1}) \to C'_N$ is the obvious morphism in $\mathrm{Idem}(\mathcal{A})$ that is given by $\mathrm{id} - c'_{N+1} \colon C'_N \to C'_N$. Let

$$u_* \colon P_* \to C'$$

be the $\mathrm{Idem}(\mathcal{A})$-chain map for which u_m is the identity for $m \leq N-1$, u_N is $i\colon (C'_N, \mathrm{id} - c'_{N+1}) \to C'_N$, and $u_m\colon 0 \to C_m$ is the canonical map for $m \geq N+1$. Let

$$v_*\colon C'_* \to P_*$$

be the $\mathrm{Idem}(\mathcal{A})$-chain map which is given by the identity for $m \leq N-1$, by the canonical projection $C'_m \to 0$ for $m \geq N+1$ and for $m = N$ by the morphism $C_N \to (C'_N, \mathrm{id} - c'_{N+1})$ defined by $\mathrm{id} - c'_{N+1}\colon C'_N \to C'_N$. Obviously $v_* \circ u_* = \mathrm{id}_{P_*}$. We obtain a chain homotopy $l'_*\colon \mathrm{id}_{C'_*} \sim u_* \circ v_*$ if we take $l'_m = 0$ for $m \leq N$, $l'_m = c'_{N+1}$ for $m \geq N, m - N \equiv 0 \mod 2$, and $l'_m = 1 - c'_{N+1}$ for $m \geq N, m - N \equiv 1 \mod 2$.

Define the desired $\mathrm{Idem}(\mathcal{B})$-chain map

$$f_*\colon C_* \to P_*$$

to be the composite $v_* \circ f'_*$ and the desired $\mathrm{Idem}(\mathcal{B})$-chain map

$$g_*\colon P_* \to C_*$$

to be the composite $g'_* \circ u_*$. We obtain desired $\mathrm{Idem}(\mathcal{B})$-chain homotopies

$$k_* = v_* \circ h_* \circ u_*\colon f_* \circ g_* \simeq \mathrm{id}_{P_*}$$

and

$$l_* = -g'_* \circ l'_* \circ f'_* + h_*\colon g_* \circ f_* \simeq \mathrm{id}_{C_*}.$$

24.7.6 Finite Domination and Width functions

Consider a full and faithful inclusion $\mathcal{A} \to \mathcal{B}$ of additive categories. Suppose that $\mathcal{B}$ comes with a width function wd. Consider a $\mathcal{B}$-chain complex C_* together with a *finite domination* $(D_*, i_*, r_*.h_*)$ of C_* over $\mathcal{A}$. For $\epsilon > 0$ we call it *ϵ-controlled* if $\mathrm{wd}(r_* \circ i_*), \mathrm{wd}(h_*) \leq \epsilon$ hold.

Proposition 24.51. *Fix a natural number N.*

Then, for every $\epsilon > 0$ and every ϵ-controlled domination of $(D_*, i_*, r_*.h_*)$ of a $\mathcal{B}$-chain complex C_* over $\mathcal{A}$ with $\dim(D_*) \leq N$ and $\mathrm{wd}(C_*), \mathrm{wd}(D_*) \leq \epsilon$, there is an N-dimensional $\mathrm{Idem}(\mathcal{A})$-chain complex P_* with $\mathrm{wd}(P_*) \leq (N+2)\epsilon$ together with an $(2N+5)\epsilon$-controlled $\mathrm{Idem}(\mathcal{B})$-chain homotopy equivalence $f_*\colon P_* \to C_*$.

Proof. This follows from the explicite constructions of the N-dimensional $\mathcal{A}$-chain complex P_*, the $\mathrm{Idem}(\mathcal{B})$-chain maps $f_*\colon C_* \to P_*$ and $g_*\colon P_* \to C_*$ and the $\mathrm{Idem}(\mathcal{B})$-chain homotopies $k_*\colon f_* \circ g_* \simeq \mathrm{id}_{P_*}$ and $l_*\colon g_* \circ f_* \simeq \mathrm{id}_{C_*}$ of Subsection 24.7.5, the axioms appearing Definition 24.27, and Lemma 24.48. □

24.7.7 Comparing Singular and Simplicial Chain Complexes

Let $X = (X, d)$ be a metric space. As before we denote the singular chain complex of X by $C_*^{\text{sing}}(X)$. For $\delta > 0$ we define

$$C_*^{\text{sing},\delta}(X) \subset C_*^{\text{sing}}(X)$$

as the chain subcomplex generated by all singular n-simplices $\sigma\colon \Delta_n \to X$ for $n \geq 0$ for which the diameter of $\sigma(\Delta_n)$ is less or equal to δ, i.e., for all $y, z \in \Delta_n$ we have $d(\sigma(y), \sigma(z)) \leq \delta$.

We have defined the additive category $\mathsf{GM}(X)$ and its width function wd_X in Example 24.31. The $\mathbb{Z}$-chain complex $C_*^{\text{sing},\delta}(X)$ can be considered as a $\mathsf{GM}(X)$-chain complex, denoted again by $C_*^{\text{sing},\delta}(X)$, via the barycenter map, i.e., for $x \in X$ the module $C_n^{\text{sing},\delta}(X)_x$ is generated by all singular n-simplices which satisfy the condition above and map the barycenter to x. Obviously $\text{wd}_X(C_*^{\text{sing},\delta}(X)) \leq \delta$ holds. Note that the image of the $\mathsf{GM}(X)$-chain complex $C_*^{\text{sing},\delta}(X)$ under the functor F of (19.5) can be identified with the $\mathbb{Z}$-chain complex $C_*^{\text{sing},\delta}(X)$.

The proof of the next result can be found in [86, Lemma 6.7].

Lemma 24.52. *Let $X = (X, d)$ be a metric space.*

(i) For $\delta' > \delta > 0$ the inclusion

$$\text{inc}_*^{\delta,\delta'}\colon C_*^{\text{sing},\delta}(X, d) \to C_*^{\text{sing},\delta'}(X, d)$$

is a δ'-controlled $\mathsf{GM}(X)$-chain homotopy equivalence;

(ii) For every $\delta > 0$ the inclusion

$$i\colon C_*^{\text{sing},\delta}(X, d) \to C_*^{\text{sing}}(X)$$

is a $\mathsf{GM}(X)$-chain homotopy equivalence;

(iii) Suppose $X = |L|$ for the geometric realization $|L|$ of an abstract simplicial complex L. Let $C_*(T)$ denote the simplicial chain complex considered as a $\mathsf{GM}(X)$-chain complex via the barycenters. Suppose all simplices of T have diameter smaller than δ. Then $\text{wd}_X(C_*(L)) \leq \delta$ and realization defines a $\mathsf{GM}(X)$-chain map

$$C_*(L) \to C_*^{\text{sing},\delta}(X)$$

which is a δ-controlled $\mathsf{GM}(X)$-chain homotopy equivalence.

The next result is proved in [77, Lemma 8.5].

Lemma 24.53. *Let $X = (X, d)$ be a metric space. Consider $\mu, \nu > 0$. Let $\varphi, \varphi'\colon X \to X$ be maps satisfying*

$$d(x, y) \leq \mu \implies d(\varphi(x), \varphi(y)), d(\varphi'(x), \varphi'(y)) \leq \nu$$

for all $x, y \in X$. *Let* $h\colon \varphi \simeq \varphi'$ *be a homotopy.*

Then there is a $\mathsf{GM}(X)$*-chain homotopy* $H_*\colon C_*^{\mathrm{sing},\mu,\nu}(\varphi)_* \simeq C_*^{\mathrm{sing},\mu,\nu}(\varphi')_*$ *of* $\mathsf{GM}(X)$*-chain maps* $C_*^{\mathrm{sing},\mu}(X) \to C_*^{\mathrm{sing},\nu}(X)$ *satisfying*

$$\operatorname{supp} H_* \subseteq \{(h_t(x), y) \mid t \in [0,1], d(x,y) \le \mu\}.$$

Proposition 24.54. *Consider a natural number* N *and an* N*-transfer space* $X = (X,d)$ *in the sense of Definition 20.9.*

Then for every $\epsilon > 0$ *there is an* N*-dimensional* $\mathrm{Idem}(\mathsf{GM}(X)^f)$*-chain complex* P_* *with* $\mathrm{wd}_X(P_*) \le (12N+24)\epsilon$ *together with a* $(24N+60)\epsilon$*-controlled* $\mathrm{Idem}(\mathsf{GM}(X))$*-chain homotopy equivalence* $f_*\colon P_* \to C_*^{\mathrm{sing},\epsilon}(X)$.

Proof. Fix $\epsilon > 0$. We can choose an N-dimensional abstract simplicial complex K, maps $i\colon X \to |K|$ and $r\colon |K| \to X$, and a homotopy $h\colon X \times [0,1] \to X$ from $r \circ i$ to id_X which is ϵ-controlled, i.e., for every $x \in X$ the diameter of the subset $h(\{x\} \times [0,1])$ of X is smaller than ϵ. By subdividing K we can arrange that for any simplex $\sigma \in K$ the diameter of the subset $r(|\sigma|)$ of X is less or equal to ϵ. This implies $\mathrm{wd}_X(C_*(K)) \le \epsilon$ where we consider $C_*(K)$ as a $GM(X)$-chain complex using the image of the barycenters of simplices under r. Analogously we can consider $C_*^{\mathrm{sing},3\epsilon}(|K|)$ as a $\mathsf{GM}(X)$-chain complex with $\mathrm{wd}_X(C_*^{\mathrm{sing},3\epsilon}(|K|)) \le 3\epsilon$. We get for any $x, y \in X$

$$\begin{aligned} d(r \circ i(x), r \circ i(y)) &= d(h_0(x), h_0(y)) \\ &\le d(h_0(x), h_1(x)) + d(h_1(x), h_1(y)) + d(h_1(y), h_0(y)) \\ &\le \epsilon + d(x,y) + \epsilon = 2\epsilon + d(x,y). \end{aligned}$$

Hence $i\colon X \to K$ induces a $\mathsf{GM}(X)$-chain map

$$C_*^{\mathrm{sing},\epsilon,3\epsilon}(i)\colon C_*^{\mathrm{sing},\epsilon}(X) \to C_*^{\mathrm{sing},3\epsilon}(|K|).$$

Obviously r induces a $\mathsf{GM}(X)$-chain map

$$C_*^{\mathrm{sing},3\epsilon}(r)\colon C_*^{\mathrm{sing},3\epsilon}(|K|) \to C_*^{\mathrm{sing},3\epsilon}(X).$$

Let

$$\mathrm{inc}^{\epsilon,3\epsilon}\colon C_*^{\mathrm{sing},\epsilon}(X) \to C_*^{\mathrm{sing},3\epsilon}(X)$$

be the inclusion. We conclude from Lemma 24.53 applied in the case $\mu = \epsilon$ and $\nu = 3\epsilon$ to $h\colon r \circ i \simeq \mathrm{id}_X$ that there is a $\mathsf{GM}(X)$-chain homotopy of $\mathsf{GM}(X)$-chain maps from $C_*^{\mathrm{sing},\epsilon}(X)$ to $C_*^{\mathrm{sing},3\epsilon}(X)$

$$H_*\colon C_*^{\mathrm{sing},\epsilon}(r) \circ C_*^{\mathrm{sing},\epsilon,3\epsilon}(i) \simeq \mathrm{inc}^{\epsilon,3\epsilon}$$

with $\mathrm{wd}_X(H_*) \le 2\epsilon$, since for $t \in [0,1]$ and $x, y \in X$ with $d(x,y) \le \epsilon$ we get

$$\begin{aligned} d(h_t(x),y) &\le d(h_t(x),h_1(x)) + d(h_1(x),y) \\ &= d(h_t(x),h_1(x)) + d(x,y) \le \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence we get

$$C_*^{\mathrm{sing},\epsilon}(r) \circ C_*^{\mathrm{sing},\epsilon,3\epsilon}(i) \simeq_{3\epsilon} \mathrm{inc}^{\epsilon,3\epsilon}.$$

From Lemma 24.52 (iii) we get 3ϵ-controlled $\mathsf{GM}(X)$-chain homotopy equivalences

$$\begin{aligned} a_* \colon C_*(K) &\to C_*^{\mathrm{sing},3\epsilon}(|K|); \\ b_* \colon C_*^{\mathrm{sing},3\epsilon}(|K|) &\to C_*(K), \end{aligned}$$

satisfying

$$\begin{aligned} b_* \circ a_* &\simeq_{3\epsilon} \mathrm{id}_{C_*(K)}; \\ a_* \circ b_* &\simeq_{3\epsilon} \mathrm{id}_{C_*^{\mathrm{sing},3\epsilon}(|K|)}. \end{aligned}$$

We conclude from Lemma 24.52 (i) that $\mathrm{inc}^{\epsilon,3\epsilon} \colon C_*^{\mathrm{sing},\epsilon}(X) \to C_*^{\mathrm{sing},3\epsilon}(X)$ is a 3ϵ-controlled $\mathsf{GM}(X)$-chain homotopy equivalence and we can choose a 3ϵ-controlled $\mathsf{GM}(X)$-chain homotopy equivalence $\mathrm{inc}_{\epsilon,3\epsilon}^{-1} \colon C_*^{\mathrm{sing},3\epsilon}(X) \to C_*^{\mathrm{sing},\epsilon}(X)$ satisfying

$$\begin{aligned} \mathrm{inc}_{\epsilon,3\epsilon}^{-1} \circ \mathrm{inc}_{\epsilon,3\epsilon} &\simeq_{3\epsilon} \mathrm{id}_{C_*^{\mathrm{sing},\epsilon}(X)}; \\ \mathrm{inc}_{\epsilon,3\epsilon}^{-1} \circ \mathrm{inc}_{\epsilon,3\epsilon}^{-1} &\simeq_{3\epsilon} \mathrm{id}_{C_*^{\mathrm{sing},3\epsilon}(X)}. \end{aligned}$$

Now define $\mathsf{GM}(X)$-chain maps

$$j_* = b_* \circ C_*^{\mathrm{sing},\epsilon,3\epsilon}(i) \colon C_*^{\mathrm{sing},\epsilon}(X) \to C_*(K)$$

and

$$p_* = \mathrm{inc}_{\epsilon,3\epsilon}^{-1} \circ C_*^{\mathrm{sing},3\epsilon}(r) \circ a_* \colon C_*(K) \to C_*^{\mathrm{sing},\epsilon}(X).$$

We conclude using Lemma 24.48

$$p_* \circ j_* \simeq_{12\epsilon} \mathrm{id}_{C_*^{\mathrm{sing},\epsilon}(X)}.$$

Now we get from Proposition 24.51 applied to the inclusion $\mathrm{Idem}(\mathsf{GM}(X)^f) \to \mathrm{Idem}(\mathsf{GM}(X))$ and the domination of the $\mathrm{Idem}(\mathsf{GM}(X))$-chain complex $C_*^{\mathrm{sing},\epsilon}(X)$ by the $\mathsf{GM}(X)^f$-chain complex $C_*(K)$ above an N-dimensional $\mathrm{Idem}(\mathsf{GM}(X))^f$-chain complex P_* with $\mathrm{wd}_X(P_*) \le (12N+16)\epsilon$ together a $(16N+40)\epsilon$-controlled $\mathrm{Idem}(\mathsf{GM}(X))$-chain homotopy equivalence $f_* \colon P_* \to C_*^{\mathrm{sing},\epsilon}(X)$. □

24.7.8 Taking the Group Action on X into Account

Consider a natural number N and an N-transfer space $X = (X, d)$ in the sense of Definition 20.9. Suppose that X comes with a (not necessarily isometric) G-action. Let $T \subseteq G$ be a finite subset. Fix $\epsilon > 0$.

Since T is finite and X is compact, there exists a real number δ with $0 < \delta < \epsilon$ such that the implication $d(x,y) \le \delta \implies d(gx, gy) \le \epsilon$ holds for all $g \in T$, $x \in X$ and $y \in X$. By the same argument applied to δ instead of ϵ there exists a real number γ with $0 < \gamma < \delta$ such that the implication $d(x,y) \le \gamma \implies d(gx, gy) \le \delta$ holds for all $g \in T$, $x \in X$ and $y \in X$. Now we get from Proposition 24.54 an N-dimensional $\operatorname{Idem}(\mathsf{GM}(X)^f)$-chain complex P_* together with $\operatorname{Idem}(\mathsf{GM}(X))$-chain homotopy equivalences $f_* \colon P_* \to C_*^{\mathrm{sing},\gamma}(X)$ and $g_* \colon C_*^{\mathrm{sing},\gamma}(X) \to P_*$ together with $\operatorname{Idem}(\mathsf{GM}(X))$-chain homotopies $h_* \colon g_* \circ f_* \simeq \mathrm{id}_{P_*}$ and $k_* \colon f_* \circ g_* \simeq \mathrm{id}_{C_*^{\mathrm{sing},\gamma}(X)}$ such that $\mathrm{wd}_X(P_*)$, $\mathrm{wd}_X(f_*)$, $\mathrm{wd}_X(g_*)$, $\mathrm{wd}_X(h_*)$, and $\mathrm{wd}_X(k_*)$ are less or equal to $24(N+60)\gamma$.

Define the finite subset T^2 of G by

$$T^2 = \{g \in G \mid \exists g_1, g_2 \in T, g = g_1, g_2\}.$$

Since every $g \in T^2$ satisfies $d(x,y) \le \gamma \implies d(gx, gy) \le \epsilon$, the map $l_g \colon X \to X$ sending x to gx induces a $\mathsf{GM}(X)$-chain map

$$C_*^{\mathrm{sing},\gamma,\epsilon}(l_g) \colon C_*^{\mathrm{sing},\gamma}(X) \to C_*^{\mathrm{sing},\epsilon}(X).$$

Now define for $g \in T^2$ a $\operatorname{Idem}(\mathsf{GM}(X)^f)$-chain homotopy equivalence

$$\varphi[g]_* \colon P_* \to P_* \tag{24.55}$$

by the composite

$$P_* \xrightarrow{f_*} C_*^{\mathrm{sing},\gamma}(X) \xrightarrow{C_*^{\mathrm{sing},\gamma,\epsilon}(l_g)} C_*^{\mathrm{sing},\epsilon}(X) \xrightarrow{\mathrm{inc}_{\gamma,\epsilon}^{-1}} C_*^{\mathrm{sing},\gamma}(X) \xrightarrow{g_*} P_*$$

where $\mathrm{inc}_{\gamma,\epsilon}^{-1}$ is an ϵ-controlled $\mathsf{GM}(X)$-chain homotopy equivalence coming from Lemma 24.52 (i) which is up to ϵ-controlled homotopy an chain homotopy inverse of the ϵ-controlled $\mathsf{GM}(X)$-chain homotopy equivalence $\mathrm{inc}_{\gamma,\epsilon}$.

Recall that $\mathrm{inc}_*^{\gamma} \colon C_*^{\mathrm{sing},\gamma} \to C_*^{\mathrm{sing}}(X)$ is the inclusion of $\operatorname{Idem}(\mathsf{GM}(X))$-chain complexes.

Proposition 24.56. *We get with the choices above:*

(i) We obtain for every $g \in T^2$

$$(x,y) \in \operatorname{supp}(\varphi_g) \implies d(gx, y) \le (48N + 121)\epsilon;$$

(ii) For $g, h \in T$ *there exists an* $\operatorname{Idem}(\mathsf{GM}(X))$*-chain homotopy*

$$\Phi[g,h]_* \colon \varphi[gh]_* \simeq \varphi[g]_* \circ \varphi[h]_*$$

satisfying

$$(x,y) \in \operatorname{supp}(\varphi_g) \implies d(ghx,y) \leq (72N+181)\epsilon;$$

(iii) We obtain for every $g \in T^2$ an up to $\operatorname{Idem}(\mathsf{GM}(X))$*-chain homotopy commutative dagram whose vertical arrows are* $\operatorname{Idem}(\mathsf{GM}(X))$*-chain homotopy equivalences*

$$\begin{tikzcd} P_* \arrow[r, "\operatorname{inc}_*^{\gamma} \circ f_*", "\simeq"'] \arrow[d, "\varphi[g]_*"'] & C_*^{\text{sing}}(X) \arrow[d, "C_*^{\text{sing}}(l_g)"] \\ P_* \arrow[r, "\operatorname{inc}_*^{\gamma} \circ f_*", "\simeq"'] & C_*^{\text{sing}}(X). \end{tikzcd}$$

Proof. (i) We get $\operatorname{wd}_X(f_*) \leq (24N+60)\gamma$ and $\operatorname{wd}_X(g_*) \leq (24N+60)\gamma$ from Proposition 24.54 and $\operatorname{wd}_X(\operatorname{inc}_{\gamma,\epsilon}^{-1}) \leq \epsilon$ from Lemma 24.52 (i). Obviously we have

$$(x,y) \in \operatorname{supp}(C_*^{\text{sing},\gamma,\epsilon}(l_g)) \implies y = gx. \tag{24.57}$$

One easily checks for $(x,y) \in \operatorname{supp}(\varphi_g)$

$$\begin{aligned} d(gx,y) &\leq \operatorname{wd}_X(f_*)+\operatorname{wd}_X(\operatorname{inc}_{\gamma,\epsilon}^{-1})+\operatorname{wd}_X(g_*) \leq (24N+60)\gamma+\epsilon+(24N+60)\gamma \\ &\leq (24N+60)\epsilon + \epsilon + (24N+60)\gamma\epsilon = (48N+121)\epsilon. \end{aligned}$$

(ii) The desired homotopy $\Phi[gh]_*$ is given by the composite of the following homotopies and identities

$$\begin{aligned} &\varphi[gh] \\ &= g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_{gh}) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\delta,\epsilon}(l_g) \circ C_*^{\text{sing},\gamma,\delta}(l_h) \circ f_* \\ &\overset{(1)}{\simeq} g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\delta,\epsilon}(l_g) \circ \operatorname{inc}^{\gamma,\delta} \circ (\operatorname{inc}^{\gamma,\delta})^{-1} \circ C_*^{\text{sing},\gamma,\delta}(l_h) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_g) \circ (\operatorname{inc}^{\gamma,\delta})^{-1} \circ C_*^{\text{sing},\gamma,\delta}(l_h) \circ f_* \\ &\overset{(2)}{\simeq} g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_g) \circ (\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1} \circ \operatorname{inc}^{\delta,\epsilon} \circ C_*^{\text{sing},\gamma,\delta}(l_h) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_g) \circ (\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_h) \circ f_* \\ &\overset{(3)}{\simeq} g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_g) \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_h) \circ f_* \\ &\overset{(4)}{\simeq} g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_g) \circ f_* \circ g_* \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ C_*^{\text{sing},\gamma,\epsilon}(l_h) \circ f_* \\ &= \varphi[g] \circ \varphi[h]. \end{aligned}$$

In the sequel we will apply (an obvious variation of) Lemma 24.48 (ii) over and over again. Here the homotopy (1) comes from the δ-controlled homotopy id $\simeq \mathrm{inc}^{\gamma,\delta} \circ (\mathrm{inc}^{\gamma,\delta})^{-1}$ of Lemma 24.52 (i). The homotopy (2) comes from the ϵ-controlled homotopy id $\simeq \mathrm{inc}^{\delta,\epsilon} \circ (\mathrm{inc}^{\delta,\epsilon})^{-1}$ of Lemma 24.52 (i). The homotopy (3) comes from the sequence homotopies each of which comes from of Lemma 24.52 (i)

$$\begin{aligned}
&(\mathrm{inc}^{\gamma,\delta})^{-1} \circ (\mathrm{inc}^{\delta,\epsilon})^{-1}\\
&\simeq (\mathrm{inc}^{\gamma,\delta})^{-1} \circ (\mathrm{inc}^{\delta,\epsilon})^{-1} \circ \mathrm{inc}^{\gamma,\epsilon} \circ (\mathrm{inc}^{\gamma,\epsilon})^{-1}\\
&= (\mathrm{inc}^{\gamma,\delta})^{-1} \circ (\mathrm{inc}^{\delta,\epsilon})^{-1} \circ \mathrm{inc}^{\delta,\epsilon} \circ \mathrm{inc}^{\gamma,\delta} \circ (\mathrm{inc}^{\gamma,\epsilon})^{-1}\\
&\simeq (\mathrm{inc}^{\gamma,\delta})^{-1} \circ \mathrm{inc}^{\gamma,\delta} \circ (\mathrm{inc}^{\gamma,\epsilon})^{-1}\\
&\simeq (\mathrm{inc}^{\gamma,\epsilon})^{-1}.
\end{aligned}$$

One easily checks that latter chain homotopy from $(\mathrm{inc}^{\gamma,\delta})^{-1} \circ (\mathrm{inc}^{\delta,\epsilon})^{-1}$ to $(\mathrm{inc}^{\gamma,\epsilon})^{-1}$ is 9ϵ-controlled. The homotopy (4) comes from Proposition 24.54. Note that $\mathrm{wd}_X(f_*) \le (24N+60)\gamma$, $\mathrm{wd}_X(g_*) \le (24N+60)\gamma$ holds by Proposition 24.54 and $(\mathrm{inc}^{\gamma,\epsilon})^{-1} \le \epsilon$, $\mathrm{wd}_X((\mathrm{inc}^{\gamma,\delta})^{-1}) \le \delta$, and $\mathrm{wd}_X((\mathrm{inc}^{\gamma,\delta})^{-1} \le \delta$ holds by Lemma 24.52 (i). We have $\gamma \le \delta \le \epsilon$. Recall the implication (24.57) which we can apply to l_g and l_h and l_{gh}. One easily checks that for all $(x,y) \in \mathrm{supp}(\Phi[g,h])$ we have

$$d(ghx,y) \le (72N+181)\epsilon.$$

(iii). The chain map $\mathrm{inc}_*^{\gamma} \colon C_*^{\mathrm{sing},\gamma} \to C_*^{\mathrm{sing}}(X)$ is a $\mathsf{GM}(X)$-chain homotopy equivalence by Lemma 24.52 (ii). The $\mathsf{GM}(X)$-chain maps $f_* \colon P_* \to C_*^{\mathrm{sing},\gamma}(X)$ and $g_* \colon P_* \to C_*^{\mathrm{sing},\gamma}(X)$ are chain homotopy inverses of one another. We have $\mathrm{inc}_*^{\gamma,\epsilon} \circ \mathrm{inc}_*^{\epsilon} = \mathrm{inc}_*^{\gamma}$ and $(\mathrm{inc}^{\gamma,\epsilon})^{-1}$ is a $\mathsf{GM}(X)$ chain homotopy inverse of $\mathrm{inc}^{\gamma,\epsilon})^{-1}$. This finishes the proof of Proposition 24.56. □

24.7.9 Passing to $Y = G \times X$

Now we consider the data we have fixed in Subsection 24.7.1. Recall that $Y = G \times X$ with the G-action given by $g'(g,x) = (g'g,x)$. We define a functor of additive categories

$$\mathrm{ind} \colon \mathsf{GM}(X) \to \mathsf{GM}^G(Y) \tag{24.58}$$

as follows. An object $M = \{M_x \mid x \in X\}$ is sent to the object

$$\mathrm{ind}(M) = \{\mathrm{ind}(M)_{(g,x)} \mid (g,x) \in Y\}$$

given by $\mathrm{ind}(M)_{(g,x)} = M_x$. A morphism $f = \{f_{x,y} \colon M_x \to N_y \mid x,y \in X\}$ from $M = \{M_x \mid x \in X\}$ to $N = \{N_y \mid y \in X\}$ is sent to the morphism

$$\mathrm{ind}(f) = \{\mathrm{ind}(f)_{(g,x),(h,y)} \colon M_x \to N_y \mid (g,x),(h,y) \in Y\}$$

given by $\operatorname{ind}(f)_{(g,x),(h,y)} = f_{x,y}$ if $g = h$ and by $\operatorname{ind}(f)_{(g,x),(h,y)} = 0$ if $g \neq h$. In the sequel we consider on $\mathsf{GM}^G(Y)$ the width function wd_Z of Example 24.29. respect to the map $w\colon G \times X \to Z$ that we have defined by $w(g,x) = gv(x)$ for the given (ϵ, T)-almost G-equivariant map $v\colon X \to Z = |\Sigma|$ in Subsection 24.7.1. Obviously ind induces a functor of additive categories

$$\operatorname{Idem}(\operatorname{ind})\colon \operatorname{Idem}(\mathsf{GM}(X)) \to \operatorname{Idem}(\mathsf{GM}^G(Y)). \tag{24.59}$$

Fix $\epsilon > 0$. Since $v\colon X \to |Z|$ has a compact metric space as source, we can find a real number $\xi > 0$ such that the implication

$$d(x,y) \leq (12N+24)\xi \implies d^{L^1}(v(x), v(y)) \leq \epsilon$$

holds for $x, y \in X$. From Proposition 24.54 we get $\operatorname{Idem}(\mathsf{GM}(X))$-chain maps $f_*\colon P_* \to C_*^{\mathrm{sing},\xi}(X)$ and $g_*\colon C_*^{\mathrm{sing},\xi}(X) \to P_*$, and $\operatorname{Idem}(\mathsf{GM}(X))$-chain homotopies $h_*\colon g_* \circ f_* \simeq \mathrm{id}_{P_*}$ and $k_*\colon f_* \circ g_* \simeq \mathrm{id}_{C_*^{\mathrm{sing},\xi}(X)}$ such that

$$\mathrm{wd}_X(P_*), \mathrm{wd}_X(f_*), \mathrm{wd}_X(g_*), \mathrm{wd}_X(h_*), \mathrm{wd}_X(k_*) \leq (12N+24)\xi$$

holds where wd_X is understood to be over the metric space X. They induce $\operatorname{Idem}(\mathsf{GM}(X))$-chain maps

$$\begin{aligned}\operatorname{Idem}(\operatorname{ind})(f_*)\colon \operatorname{Idem}(\operatorname{ind})(P_*) &\to \operatorname{Idem}(\operatorname{ind})(C_*^{\epsilon}(X));\\ \operatorname{Idem}(\operatorname{ind})(g_*)\colon \operatorname{Idem}(\operatorname{ind})(C_*^{\epsilon}(X)) &\to \operatorname{Idem}(\operatorname{ind})(P_*),\end{aligned}$$

and $\operatorname{Idem}(\mathsf{GM}(X))$-chain homotopies

$$\begin{aligned}\operatorname{Idem}(\operatorname{ind})(h_*)\colon \operatorname{Idem}(\operatorname{ind})(g_*) \circ \operatorname{Idem}(\operatorname{ind})(f_*) &\simeq \mathrm{id}_{\operatorname{Idem}(\operatorname{ind})(P_*)};\\ \operatorname{Idem}(\operatorname{ind})(k_*)\colon \operatorname{Idem}(\operatorname{ind})(f_*) \circ \operatorname{Idem}(\operatorname{ind})(g_*) &\simeq \mathrm{id}_{\operatorname{Idem}(\operatorname{ind})(C_*^{\mathrm{sing},\epsilon}(X))},\end{aligned}$$

such that

$$\begin{aligned}\mathrm{wd}(\operatorname{Idem}(\operatorname{ind})(P_*)), \mathrm{wd}(\operatorname{Idem}(\operatorname{ind})(f_*)), \mathrm{wd}(\operatorname{Idem}(\operatorname{ind})(g_*)),&\\ \mathrm{wd}(\operatorname{Idem}(\operatorname{ind})(h_*)), \mathrm{wd}(\operatorname{Idem}(\operatorname{ind})(k_*)) &\leq \epsilon\end{aligned}$$

holds. We give the proof for the width of $\operatorname{Idem}(\operatorname{ind})(f_*)$, the one for other terms is analogous. Consider $(g,x),(h,y) \in \operatorname{supp}(\operatorname{Idem}(\operatorname{ind})(f_*))$. Then we have $g = h$ and $x, y \in \operatorname{supp}(f_*)$. The latter implies $d(x,y) \leq (12N+24)\xi$ and hence $d(v(x), v(y)) \leq \epsilon$. We compute

$$\begin{aligned}d^{L^1}(w(g,x), w(h,y)) = d^{L^1}(w(g,x), w(g,y)) = d^{L^1}&(gv(x), gv(y))\\ &= d^{L^1}(v(x), v(y)) \leq \epsilon.\end{aligned}$$

Given $g \in T$, we define a $\operatorname{Idem}(\mathsf{GM}^G(Y))$-chain map

$$U[g]_*\colon \operatorname{Idem}(\operatorname{ind})(P_*) \to \operatorname{Idem}(\operatorname{ind})(P_*)$$

by putting $(U(g)_n)_{(g_1,x_1),(g_1,x_2)}\colon (P_n)_{x_1} \to (P_n)_{x_2}$ to be $(\varphi[g]_n)_{x_1,x_2}$ if $g_2 = g_1 g^{-1}$ and to be zero otherwise, where $\varphi[g]_*$ has been introduced in (24.55). For $g,h \in T$ we define $\operatorname{Idem}(\mathsf{GM}^G(G \times X))$-chain homotopies

$$H[g,h]_*\colon U(gh) \simeq U(g) \circ U(h)$$

by putting $(H[g,h]_n)_{(g_1,x_1,g_2,x_2)}\colon \operatorname{Idem}(\operatorname{ind})(P_n)_{x_1} \to \operatorname{Idem}(\operatorname{ind})(P_{n+1})_{x_2}$ to be $(\Phi[g,h]_n)_{x_1,x_2}$ if $g_2 = g_1(gh)^{-1}$ and to be zero otherwise, where $\Phi[g,h]$ has been defined in the proof Proposition 24.56 (ii). Proposition 24.56 implies for $g,h \in T$

$$\operatorname{wd}(U[g]_*) \leq (48N + 121)\epsilon; \tag{24.60}$$

$$\operatorname{wd}(H[g,h]) \leq (72N + 181)\epsilon. \tag{24.61}$$

For $a = \sum_{g \in T} \lambda_g \cdot g \in \mathbb{Z}G$, we define an $\operatorname{Idem}(\mathsf{GM}^G(G \times X))$-chain map

$$U[a]_* = \sum_{g \in T} \lambda_g \cdot U[g]_*\colon \operatorname{Idem}(\operatorname{ind})(P_*) \to \operatorname{Idem}(\operatorname{ind})(P_*).$$

For elements $a = \sum_{g \in T} \lambda_g \cdot g$ and $b = \sum_{g \in T} \mu_h \cdot h$ in RG, we define a $\operatorname{Idem}(\mathsf{GM}^G(Y))$-chain homotopy

$$H[a.b]_*\colon U[ab]_* \simeq U[a]_* \circ U[b]_*$$

by $H[a.b]_* = \sum_{g,h \in T} \lambda_g \mu_h \cdot \Phi[g,h]_*\colon \operatorname{Idem}(\operatorname{ind})(P_*) \to \operatorname{Idem}(\operatorname{ind})(P_{*+1})$. For the matrix $A = (a_{i,j}) \in GL_n(\mathbb{Z}G)$ and its inverse $B = (b_{i,j})$, we define $\operatorname{Idem}(\mathsf{GM}^G(Y))$-chain maps for $\operatorname{Idem}(\operatorname{ind})(P_*)^n = \bigoplus_{i=1}^n \operatorname{Idem}(\operatorname{ind})(P_*)$

$$U[A]_* = (U(a_{i,j}))_{i,j}\colon \operatorname{Idem}(\operatorname{ind})(P_*)^n \to \operatorname{Idem}(\operatorname{ind})(P_*)^n;$$
$$U[B]_* = (U(a_{i,j}))_{i,j}\colon \operatorname{Idem}(\operatorname{ind})(P_*)^n \to \operatorname{Idem}(\operatorname{ind})(P_*)^n.$$

Define $\operatorname{Idem}(\mathsf{GM}^G(Y))$-chain homotopies

$$K_*\colon U(A)_* \circ U(B)_* \simeq \operatorname{id}_{\operatorname{Idem}(\operatorname{ind})(P_*)^n};$$
$$L_*\colon U(B)_* \circ U(A)_* \simeq \operatorname{id}_{\operatorname{Idem}(\operatorname{ind})(P_*)^n},$$

by $(K_n)_{i,k} = \sum_{j=1}^n H(a_{i,j} b_{j,k})$ and $(L_n)_{i,k} = \sum_{j=1}^n H(b_{i,j} a_{j,k})$ for $i,k \in \{1,2\ldots,n\}$. We conclude from the axioms appearing Definition 24.27 and the inequalities (24.60) and (24.61)

$$\text{wd}(U[A]_*) \le (48N+121)\epsilon; \tag{24.62}$$
$$\text{wd}(U[B]_*) \le (48N+121)\epsilon; \tag{24.63}$$
$$\text{wd}(K) \le (72N+181)\epsilon; \tag{24.64}$$
$$\text{wd}(L) \le (72N+181)\epsilon. \tag{24.65}$$

There is an obvious identification of $\mathsf{GM}^G(Y)$-chain complex

$$\text{ind}(C_*^{\text{sing}}(X)) = C_*^{\text{sing}}(Y)$$

as G is discrete. Under this identification the $\mathsf{GM}^G(Y)$-chain maps $\text{ind}(C_*^{\text{sing}}(l_g))$ and $C_*^{\text{sing}}(L_g)$ agree where $L_g\colon Y \to Y$ sends (h,x) to (hg^{-1}, gx). Under this identification we conclude from Proposition 24.56 (iii) that we obtain an up to chain homotopy commutative diagram of chain homotopy equivalences of $\text{Idem}(\mathsf{GM}^G(X))$-chain complexes

$$\begin{tikzcd}
\text{ind}(P_*) \arrow[r, "\text{ind}(\text{inc}_*^\epsilon \circ f_*)", "\simeq"'] \arrow[d, "U[g]_*"] & C_*^{\text{sing}}(Y) \arrow[d, "C_*^{\text{sing}}(L_g)"] \\
\text{ind}(P_*) \arrow[r, "\text{ind}(\text{inc}_*^\epsilon \circ f_*)", "\simeq"'] & C_*^{\text{sing}}(Y).
\end{tikzcd}$$

If we apply the functor F of (19.5) to the diagram above and use the identification of $F(C_*^{\text{sing}}(G\times X))$ with the singular $\mathbb{Z}$-chain complex $C_*^{\text{sing}}(G\times X)$, we obtain an up to $\mathbb{Z}G$-chain homotopy commutative diagram of $\mathbb{Z}G$-chain homotopy equivalences

$$\begin{tikzcd}
F(\text{ind}(P_*)) \arrow[r, "F(\text{ind}(\text{inc}_*^\epsilon \circ f_*))", "\simeq"'] \arrow[d, "F(U[g]_*)"] & C_*^{\text{sing}}(Y) \arrow[d, "C_*^{\text{sing}}(L_g)"] \\
F(\text{ind}(P_*)) \arrow[r, "F(\text{ind}(\text{inc}_*^\epsilon \circ f_*))", "\simeq"'] & C_*^{\text{sing}}(Y).
\end{tikzcd}$$

Since X is contractible and $Y = G\times X$ is equipped with the G-action $g'\cdot(g,x) = (g'g,x)$, the projection $\text{pr}\colon Y = G\times X \to G$ is a G-homotopy equivalences and induces a $\mathbb{Z}G$-chain homotopy equivalence $C_*^{\text{sing}}(Y) \to C_*^{\text{sing}}(G)$. There is an obvious $\mathbb{Z}G$-chain homotopy equivalence $a_*\colon C_*^{\text{sing}}(G) \to 0[\mathbb{Z}G]_*$ onto the $\mathbb{Z}$-chain complex concentrated in dimension 0 whose 0-th chain module is $\mathbb{Z}G$. We obtain a $\mathbb{Z}G$-chain homotopy equivalence

$$b_*\colon \text{ind}(P_*) \xrightarrow{\text{ind}(\text{inc}_*^\epsilon \circ f_*)} C_*^{\text{sing}}(Y) \xrightarrow{C_*^{\text{sing}}(\text{pr})} C_*^{\text{sing}}(G) \xrightarrow{a_*} 0[\mathbb{Z}G]_*$$

such that the following diagram of finite free $\mathbb{Z}G$-chain complexes commutes up to $\mathbb{Z}G$-chain homotopy for every $g\in T$ where $r_{g^{-1}}\colon G\to G$ sends g' to $g'g^{-1}$.

$$\begin{tikzcd}
F(\mathrm{ind}(P_*)) \arrow[r, "b_*", "\simeq"'] \arrow[d, "F(U[g]_*)"] & 0[\mathbb{Z}G]_* \arrow[d, "0[r_{g^{-1}}]_*"] \\
F(\mathrm{ind}(P_*)) \arrow[r, "b_*", "\simeq"'] & 0[\mathbb{Z}G]_*.
\end{tikzcd}$$

One easily checks that following diagram of finite free $\mathbb{Z}G$-chain complexes commutes up to $\mathbb{Z}G$-chain homotopy where $V(A)$ has been defined in (24.25).

$$\begin{tikzcd}
F(\mathrm{ind}(P_*))^n \arrow[r, "b_*", "\simeq"'] \arrow[d, "F(U[A]_*)"] & 0[\mathbb{Z}G^n]_* \arrow[d, "0[V(A)]_*"] \\
F(\mathrm{ind}(P_*))^n \arrow[r, "b_*", "\simeq"'] & 0[\mathbb{Z}G^n]_*.
\end{tikzcd}$$

We conclude from Lemma 24.48

(24.66) $$u = [V(A)] = t(0[V(A)]_*) = t(F(U(A)_*)) \in K_1(\mathbb{Z}G).$$

(Note that (24.66) is closely related to up-down-formula, see Remark 24.20.) Recall that $U(A)_*$ is a $(72N+181)\epsilon$-controlled $\mathrm{Idem}(\mathsf{GM}^G(Y))$-chain homotopy equivalence, see (24.62), (24.63), (24.64), and (24.65). Hence Proposition 24.50 together with (24.66) implies

$$u = [F^f(a)] \in K_1(\mathbb{Z}G)$$

for some $5(72N+181)\epsilon$-controlled automorphism a in $\mathsf{GM}^G(Y)$. By our choice $\epsilon = \frac{\epsilon_N}{5(48N+181)}$, see (24.26) we have $5(72N+181)\epsilon = \epsilon_N$. This finishes the proof of Proposition 24.24.

24.8 The Strategy Theorem

Consider a covariant functor

$$\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}.$$

Given a G-CW-complex space Z, we obtain from $\mathbf{E}$ a new covariant functor

(24.67) $$\mathbf{E}_Z\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}, \quad X \mapsto \mathbf{E}(X \times Z).$$

The canonical projection $q\colon X \times Z \to X$ yields a transformation of covariant functors G-CW-$\mathsf{COM} \to \mathsf{SPECTRA}$.

(24.68) $$\mathbf{pr}\colon \mathbf{E}_Z \to \mathbf{E}.$$

Let $L\colon \mathsf{Or}(G) \to G$-$\mathsf{CW}$-$\mathsf{COM}$ be the obvious inclusion.

Theorem 24.69 (Strategy Theorem). *Suppose that the following conditions hold:*

(i) The covariant functor

$$\mathbf{E}\colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{SPECTRA}$$

is excisive;

(ii) There exists a map of covariant $\mathsf{Or}(G)$*-spectra*

$$\mathbf{trf}\colon L^*\mathbf{E} \to L^*\mathbf{E}_Z$$

such that the composite $L^*\mathbf{pr} \circ \mathbf{trf}\colon L^*\mathbf{E} \to L^*\mathbf{E}$ *is a weak homotopy equivalence of covariant* $\mathsf{Or}(G)$*-spectra;*

(iii) The projection onto the second factor $\mathrm{pr}_2\colon Z \times Z \to Z$ *is a homotopy equivalence of* G*-CW-complexes.*

Then

$$H_n^G(\mathrm{pr}; L^*\mathbf{E})\colon H_n^G(Z; L^*\mathbf{E}) \to H_n^G(\{\bullet\}, L^*\mathbf{E})$$

is bijective for all $n \in \mathbb{Z}$ *where* $\mathrm{pr}\colon Z \to \{\bullet\}$ *is the projection. Moreover, we obtain for all* $n \in \mathbb{Z}$ *a commutative diagram of isomorphisms*

$$\begin{tikzcd} H_n^G(Z; L^*\mathbf{E}) \arrow[r, "H_n^G(\mathrm{pr}; L^*\mathbf{E})", "\cong"'] \arrow[d, "\cong"'] & H_n^G(\{\bullet\}; L^*\mathbf{E}) \arrow[d, "\cong"'] \\ \pi_n(\mathbf{E}(Z)) \arrow[r, "\cong", "\mathbf{E}(\mathrm{pr})"'] & \pi_n(\mathbf{E}(\{\bullet\})). \end{tikzcd}$$

Proof. The desired commutative diagram comes from Theorem 18.11 applied to $\mathbf{E}$. Moreover, by Theorem 18.11 the vertical arrows are bijective for all $n \in \mathbb{Z}$. It remains to prove the bijectivity of $H_n^G(\mathrm{pr}; L^*\mathbf{E})\colon H_n^G(Z; L^*\mathbf{E}) \to H_n^G(\{\bullet\}, L^*\mathbf{E})$ for all $n \in \mathbb{Z}$.

We have the following commutative diagram

$$\begin{tikzcd} H_n^G(Z; L^*\mathbf{E}) \arrow[r, "H_n^G(\mathrm{pr}; L^*\mathbf{E})"] \arrow[d, "H_n(Z; \mathbf{trf})"'] & H_n^G(\{\bullet\}; L^*\mathbf{E}) \arrow[d, "H_n(\{\bullet\}; \mathbf{trf})"] \\ H_n^G(Z; L^*\mathbf{E}_Z) \arrow[r, "H_n^G(\mathrm{pr}; L^*\mathbf{E}_Z)"] \arrow[d, "H_n(Z; L^*\mathbf{pr})"'] & H_n^G(\{\bullet\}; L^*\mathbf{E}_Z) \arrow[d, "H_n(\{\bullet\}; L^*\mathbf{pr})"] \\ H_n^G(Z; L^*\mathbf{E}) \arrow[r, "H_n^G(\mathrm{pr}; L^*\mathbf{E})"] & H_n^G(\{\bullet\}; L^*\mathbf{E}) \end{tikzcd}$$

for which the composites of the vertical arrows are in both cases isomorphisms by Lemma 12.6. Hence it suffices to show that $H_n^G(\mathrm{pr}; L^*\mathbf{E}_Z)$ is bijective for all $n \in \mathbb{Z}$. From Theorem 18.11 applied to $\mathbf{E}_Z$, we obtain a commutative diagram

$$\begin{tikzcd} H_n^G(Z; L^*\mathbf{E}_Z) \arrow[r, "H_n^G(\mathrm{pr}; L^*\mathbf{E}_Z)"] \arrow[d, "\cong"'] & H_n^G(\{\bullet\}, L^*\mathbf{E}_Z) \arrow[d, "\cong"] \\ \pi_n(\mathbf{E}(Z \times Z)) \arrow[r, "\pi_n(\mathbf{E}(\mathrm{pr}_2))"] & \pi_n(\mathbf{E}(Z)) \end{tikzcd}$$

whose vertical arrows are bijective. Since pr_2 is by assumption a G-homotopy equivalence, $H_n^G(\mathrm{pr}; L^*\mathbf{E}_Z)$ is bijective for all $n \in \mathbb{Z}$. $\square$

Let $\mathcal{A}$ be any additive G-category. We have defined the additive category $\mathcal{A}[G]$ in Example 22.2 and explained in Remark 22.79 that it comes with the structure of a strong category with G-support in the sense of Definition 22.78. So we can consider the covariant $\mathsf{Or}(G)$-spectra $\mathbf{K}^{\mathcal{D}^G}$ of (22.107) and $\mathbf{K}^{\mathcal{D}_0^G}$ of (22.108). We get another covariant $\mathsf{Or}(G)$-spectrum $\mathbf{K}^{\mathcal{D}_0^G}_{E_{\mathcal{VCY}}(G)}$ by sending an object G/H to $\mathbf{K}(\mathcal{D}_0^G(G/H \times E_{\mathcal{VCY}}(G)))$, see (24.67).

Theorem 24.70 (Transfer criterion for the Farrell-Jones Conjecture). *Suppose that there is a map of covariant $\mathsf{Or}(G)$-spectra*

$$\mathbf{trf} \colon \mathbf{K}^{\mathcal{D}_0^G} \to \mathbf{K}^{\mathcal{D}_0^G}_{E_{\mathcal{VCY}}(G)}$$

such that $\mathbf{pr} \circ \mathbf{trf}$ is a weak homotopy equivalence of covariant $\mathsf{Or}(G)$-spectra, where $\mathbf{pr}$ has been defined in (24.68).

Then the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories 13.11 holds for G.

The analogous statement holds for the L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution 13.16.

Proof. We give the proof for K-theory only, the one for L-theory is completely analogous.

The projection onto the second factor $\mathrm{pr}_2 \colon E_{\mathcal{VCY}}(G) \times E_{\mathcal{VCY}}(G) \to E_{\mathcal{VCY}}(G)$ is a G-homotopy equivalence by Theorem 11.19. The functor $\mathbf{K} \circ \mathcal{D}_0^G \colon \mathsf{CW\text{-}COM} \to \mathsf{SPECTRA}$ of (22.106) is excisive by Theorem 22.126. We conclude from Theorem 24.69 applied to it that

$$H_n^G(\mathrm{pr}; \mathbf{K}^{\mathcal{D}_0^G}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}^{\mathcal{D}_0^G}) \to H_n^G(\{\bullet\}, \mathbf{K}^{\mathcal{D}_0^G})$$

is bijective for all $n \in \mathbb{Z}$. Now Theorem 22.109 using Mayer-Vietoris sequences and the Five Lemma implies that

$$H_n^G(\mathrm{pr}; \mathbf{K}^{\mathcal{D}^G}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}^{\mathcal{D}^G}) \to H_n^G(\{\bullet\}, \mathbf{K}^{\mathcal{D}^G})$$

is bijective for all $n \in \mathbb{Z}$. We conclude from Lemma 22.76 (i), Remark 22.82, and Lemma 12.6 that the assembly map appearing in K-theoretic Farrell-

Jones Conjecture with coefficients in additive G-categories 13.11 is bijective for all $n \in \mathbb{Z}$. $\square$

The benefit of Theorem 24.70 is that it suffices to construct the transfer only on homogeneous spaces and for the functor $\mathcal{D}_0^G$ which has the pleasant feature that it is defined with zero-control in the $\mathbb{N}$-direction. This has for instance been exploited in [81, Remarks 6.14 and 7.17]

24.9 Notes

There seems to be no construction of a transfer in the Baum-Connes setting. That is the reason why some of the spectacular results about the validity of the Farrell-Jones Conjecture for certain groups, for instance all lattices in second countable locally compact Hausdorff groups, do not carry over to the Baum-Connes Conjecture. This might be different if one replaces the group C^*-algebra by the group Frechet algebra.

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Chapter 25
Higher Categories as Coefficients

25.1 Introduction

Comment 32 (by W.): This chapter should be filled in by Christoph and should be an extract of the proofs appearing in [172].

In this chapter we give more information and details about the proofs of Theorem 20.61 and Theorem 20.62 following [172]. We have discussed already in Chapter 20 that this are the most general results about the (K-theoretic) Farrell Jones Conjecture, which imply and uniform all the ones proved so far. Some basic strategies for and some of the history of their proofs have already been discussed in special cases in Chapters 19 and Chapters 24. Roughly speaking, the main achievement in [172] is to generalize the formulations and poofs of the K-theoretic Farrell-Jones Conjecture as they appear for additive categories for instance in [71, 78, 77, 85, 526, 973, 974] to higher categories, where a first step in this direction was already taken for A-theory in [330].

Comment 33 (by W.): This is only a first suggestion for a beginning of the introduction.

25.2 Section 1

25.2.1 Subsection 1.1

25.2.2 Subsection 1.2

25.2.3 Subsection 1.3

25.3 Section 2

25.4 Section 3

Exercise 25.1. Test. (There is a section with solutions to the exercises.)

25.5 Notes

Comment 34 (by W.): There is a preprint by Reis [834]. Shall we mention it in this chapter? Its abstract says:

The Farrell-Jones conjecture for lax monoidal finitary localising invariants was recently proved by Bunke-Kasprowski- Winges. In this short note, making use of the theory of noncommutative motives, we prove that the lax monoidal assumption is not necessary.

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Chapter 26
Analytic Methods

26.1 Introduction

The methods of proofs for the Farrell-Jones Conjecture and the Baum-Conjecture are rather different. But both use controlled methods, see Section 19.4, Chapter 22, and [471]. In the Farrell-Jones setting transfers were a key ingredient, see Section 19.5 and Chapter 24, which do not seem to exist in the Baum-Connes setting. In the Baum-Connes setting KK-theory, see Section 10.5, is the main tool which does not work out in the Farrell-Jones setting. This has for instance the consequence that the Full Farrell-Jones Conjecture is known for every (not necessarily cocompact) lattices in path connected second countable locally compact Hausdorff groups, whereas as the Baum-Connes Conjecture is not known for $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. On the other hand the Baum-Connes Conjecture with coefficients is known for a-T-menable groups, whereas the Farrell-Jones Conjecture has not been proved for elementary amenable groups. We have given status reports for the Farrell-Jones Conjecture and the Baum-Conjecture in Sections 16.2 and 16.4 and discussed open case in Section 16.9. We have linked these two conjectures in Subsection 15.14.4.

We give only a very brief survey over the methods used in the Baum-Connes Conjecture. More information can be found for instance in the survey articles [409, 727, 945].

26.2 The Dirac-Dual Dirac Method

Next we briefly discuss the *Dirac-dual Dirac method* which is the key strategy in many of the proofs of the Baum-Connes Conjecture 14.9 or the Baum-Connes Conjecture 14.11 with coefficients, see for instance [469, Theorem 7.1].

A G-C^*-algebra A is called *proper*, if there exists a locally compact proper G-space X and a G-homomorphism $\sigma\colon C_0(X) \to \mathcal{B}(A)$, $f \mapsto \sigma_f$ satisfying $\sigma_f(ab) = a\sigma_f(b) = \sigma_f(a)b$ for $f \in C_0(X)$, $a, b \in A$ and for every net $\{f_i \mid i \in I\}$, which converges to 1 uniformly on compact subsets of X, we have $\lim_{i\in I} \parallel \sigma_{f_i}(a) - a \parallel = 0$ for all $a \in A$. A locally compact G-space X is proper if and only if $C_0(X)$ is proper as a G-C^*-algebra.

The following result is proved in Tu [938] extending results of Kasparov-Skandalis [548, 540].

Theorem 26.1 (The Baum-Connes Conjecture with coefficients for proper G-C^*-algebras). *The Baum-Connes Conjecture 14.11 with coefficients holds for a proper G-C^*-algebras B.*

Theorem 26.2 (Dirac-dual Dirac method). *Let G be a countable (discrete) group. Let F be $\mathbb{R}$ or $\mathbb{C}$. Suppose that there exist a proper G-C^*-algebra A, elements $\alpha \in KK_i^G(A, F)$, called the* Dirac element*, and $\beta \in KK_i^G(F, A)$, called the* dual Dirac element*, satisfying*

$$\beta \otimes_A \alpha = 1 \quad \in KK_0^G(F, F).$$

Then the Baum-Connes Conjecture 14.9 and the Baum-Connes Conjecture 26.1 with coefficients are true over F.

Proof. We only treat the case $F = \mathbb{C}$ and the case of trivial coefficients. The assembly map appearing in Theorem 14.9 is a retract of the bijective assembly map from Theorem 26.1. This follows from the following commutative diagram for any cocompact G-CW-subcomplex $C \subseteq \underline{E}G$

$$\begin{tikzcd}
K_*^G(C) \arrow[r, "-\widehat{\otimes}_{\mathbb{C}}\beta"] \arrow[d, "\mathrm{asmb}^{G,\mathbb{C}}(C)_*"] & K_*^G(C;A) \arrow[r, "-\widehat{\otimes}_A\alpha"] \arrow[d, "\mathrm{asmb}_A^{G,\mathbb{C}}(C)_*"] & K_*^G(C) \arrow[d, "\mathrm{asmb}^{G,\mathbb{C}}(C)_*"] \\
K_*(C_r^*(G)) \arrow[r, "-\widehat{\otimes}_{C_r^*(G)} j_G(\beta)"'] & K_*(A \rtimes_r G) \arrow[r, "-\widehat{\otimes}_{A \rtimes_r} j_G(\alpha)"'] & K_*(C_r^*(G))
\end{tikzcd}$$

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective. $\square$

The reader should note the formal similarity between the proof of Theorem 26.2 and the proof of the Strategy Theorem 24.69.

In order to give a glimpse of the basic ideas from operator theory, we briefly describe how to define the Dirac element α in the case where G acts on a complete Riemannian manifold M. Let $T_{\mathbb{C}}M$ be the complexified tangent bundle and let $\mathrm{Cliff}(T_{\mathbb{C}}M)$ be the associated Clifford bundle. Let A be the proper G-C^*-algebra given by the sections of $\mathrm{Cliff}(T_{\mathbb{C}}M)$ which vanish at infinity. Let H be the Hilbert space $L^2(\wedge^* T_{\mathbb{C}}^*M)$ of L^2-integrable differential forms on $T_{\mathbb{C}}M$ with the obvious $\mathbb{Z}/2$-grading coming from even and odd forms. Let U be the obvious G-representation on H coming from the G-action on M. For a 1-form ω on M and $u \in H$ define a homomorphism of C^*-algebras $\rho\colon A \to \mathcal{B}(H)$ by

$$\rho_\omega(u) := \omega \wedge u + i_\omega(u).$$

Now $D = (d + d^*)$ is a symmetric densely defined operator $H \to H$ and defines a bounded selfadjoint operator $F\colon H \to H$ by putting $F = \frac{D}{\sqrt{1+D^2}}$. Then (U, ρ, F) is an even cocycle and defines an element $\alpha \in K_0^G(M) =$

$KK_0^G(C_0(M), \mathbb{C})$. More details of this construction and the construction of the dual Dirac element β under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [945, Chapter 9].

26.3 Banach KK-Theory

Skandalis [900] showed that the Dirac-dual Dirac method cannot work for all groups as long as one works with KK-theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [519]. This problem can be circumvented if one drops the condition unitary and works with a variant of KK-theory for Banach algebras as worked out by Lafforgue [582, 584, 586].

26.4 Notes

Nishikawa [743] describes a variation of the Dirac-dual-Dirac method which was used by Brodzki-Guentner-Higson-Nishikawa [158] to give a new proof the Baum-Connes Conjecture for groups which act properly and cocompactly on a finite-dimensional CAT(0)-cubical complex with bounded geometry.

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Chapter 27
Solutions of the Exercises

Chapter 2

2.7. Check that the homomorphism $\psi\colon K_0(R) \to K_0'(R)$, $[P] \to [P]$ is well-defined using the fact that every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules splits. Obviously ψ is the inverse of ϕ.

2.11. Show that the R-R-bimodule $\left({}_R R^n{}_{M_n(R)}\right) \otimes_{M_n(R)} \left({}_{M_n(R)} R^n{}_R\right)$ is isomorphic as R-R-bimodule to R and that the $M_n(R)$-$M_n(R)$-bimodule $\left({}_{M_n(R)} R^n{}_R\right) \otimes_R \left({}_R R^n{}_{M_n(R)}\right)$ is isomorphic as $M_n(R)$-$M_n(R)$-bimodule to $M_n(R)$.

2.16. See [844, Theorem 1.2.3 on page 8].

2.29. There exists a nowhere vanishing vector field on S^n if and only if there exists F-subbundles ξ and η in TS^n such that $TS^n = \xi \oplus \eta$ and ξ is a 1-dimensional trivial F-vector bundle. Now apply Theorem 2.27.

2.32. Let ξ be a vector bundle over Y. It suffices to construct a $C^0(X)$-isomorphism

$$\alpha(\xi)\colon C^0(X) \otimes_{C^0(Y)} C^0(\xi) \xrightarrow{\cong} C^0(f^*\xi).$$

Given $\phi \in C^0(X)$ and $s \in C^0(\xi)$, define $\alpha(\xi)(\phi \otimes s)$ to be the section of $f^*\xi$ which sends $x \in X$ to $\phi(x) \cdot s \circ f(x) \in \xi_{f(x)} = (f^*\xi)_x$. Since $\alpha(\xi \oplus \eta)$ can be identified with $\alpha(\xi) \oplus \alpha(\eta)$ and $\alpha(\underline{F})$ is obviously bijective, $\alpha(\xi)$ is bijective for all F-vector bundles ξ over Y.

2.33. Because of the identification (2.31) and the homotopy invariance of the functor $K^0(X)$ we get

$$K_0(C(D^n)) \cong K^0(D^n) \cong K^0(\{\bullet\}) \cong \mathbb{Z}.$$

2.40. This follows from the fact that $\mathbb{Z} \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ is isomorphic as $\mathbb{Z}$-chain complex to $C_*(X)$.

2.49. We can assume without loss of generality that X is connected, otherwise treat any component of X separately. Put $\pi = \pi_1(X)$. For $i = 0, 1, 2$

let $\overline{X_i} \to X_i$ be the π-covering obtained from the the universal covering $\widetilde{X} \to X$ by the pull back construction associated to $j_i \colon X_i \to X$. Since X_i is finite dominated, we conclude from Lemma 2.48 that $C_*(\overline{X_i})$ is finitely dominated as a $\mathbb{Z}\pi$-chain complex and directly from the definitions that $(j_i)_*(o(X_i)) = o(C_*(\widetilde{X_i}))$ holds in $K_0(\mathbb{Z}[\pi])$ for $i = 0, 1, 2$. There is an exact sequence of $\mathbb{Z}\pi$-chain complexes

$$0 \to C_*(\overline{X_0}) \to C_*(\overline{X_1}) \oplus C_*(\overline{X_2}) \to C_*(\widetilde{X}) \to 0.$$

Since $\pi_1(C)$ is finitely presented for each $C \in \pi_0(X_i)$ and $i \in \{0, 1, 2\}$, $\pi_1(X)$ is finitely presented. This follows essentially from the Seifert-van Kampen Theorem. We conclude from Lemma 2.36 (ii) and Lemma 2.48 that $C_*(\widetilde{X})$ is finitely dominated and hence X is finitely dominated and we get in $K_0(\mathbb{Z}[\pi])$

$$\begin{aligned} o(X) = o(C_*(\widetilde{X})) &= o(C_*(\overline{X_1})) + o(C_*(\overline{X_2})) - o(C_*(\overline{X_0})) \\ &= (j_1)_*(o(X_1)) + (j_2)_*(o(X_2)) - (j_0)_*(o(X_1)). \end{aligned}$$

2.50. Recall that we have chosen a finite domination (Z, i, r) of X. Construct an extension $g \colon \operatorname{cyl}(r) \cup_Z \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \to X$ of $\mathrm{id}_X \coprod F \cup_X F \colon X \coprod \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \to X$ and a homotopy equivalence $h \colon Z \to \operatorname{cyl}(r) \cup_Z \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i)$. Now the claim follows from the commutative diagram

$$\begin{tikzcd} X \coprod (\operatorname{cyl}(i) \cup_X \operatorname{cyl}(i)) \arrow[r, "j"] \arrow[dr, "f \coprod f'"'] & \operatorname{cyl}(r) \cup_Z \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \arrow[d, "f \circ g"] & Z \arrow[l, "h"'] \arrow[dl, "f \circ g \circ h"'] & \emptyset \arrow[l] \arrow[dll] \\ & Y & & \end{tikzcd}$$

where j is the inclusion.

2.56. Let (B, b) be a functorial additive invariant for finite CW-complexes. Define a natural transformation $T(X) \colon \bigoplus_{C \in \pi_0(C)} \mathbb{Z} \to B(X)$ by sending $\{n_C \mid C \in \pi_0(X)\}$ to $\sum_{C \in \pi_0(X)} n_c \cdot A(i_C)(a(\{\bullet\}))$ where $i_C \colon \{\bullet\} \to X$ is any map whose image is contained in C. Obviously it is the only possible natural transformation satisfying $T(\{\bullet\})(\chi(\{\bullet\})) = b(\{\bullet\})$. Using the additivity and homotopy invariance one proves by induction over the number of cells for a finite CW-complex X that $T(X)(\{\chi(C) \mid C \in \pi_0(X)\}) = b(X)$ holds. More details can be found in [627, Theorem 4.1].

2.58. (i) Fix a finitely dominated CW-complex Y. Define a functor A from finitely dominated CW-complexes to abelian groups by $A(X) := U(X \times Y)$.

Define $a(X) \in A(X)$ to be $u(X \times Y)$. Check that (A, a) is a functorial additive invariant for finitely dominated CW-complexes. Hence there exists a unique transformation $T_Y : U(?) \to U(? \times Y)$ sending $u(X)$ to $u(X \times Y)$. Define $B(Y)$ as the abelian group of transformations $U(?) \to U(? \times Y)$ and $b(Y) := T_Y$. Show that (B, b) is a functorial additive invariant for finitely dominated CW-complexes. Hence there is a natural transformation $S \colon U \to B$ satisfying $S(Y)(u(Y)) = b(Y)$ for all finitely dominated CW-complexes Y. This S gives the desired natural pairing $P(X, Y)$.

(ii) If Y is a finite CW-complex with $\chi(C) = 0$ for all $C \in \pi_0(Y)$, then $o(C) = 0$ for every $C \in \pi_0(C)$ by Lemma 2.18 and Theorem 2.39. Theorem 2.57 implies $u(Y) = 0$. We conclude from (i) that $u(X \times Y) = P(X, Y)(u(X) \otimes u(Y)) = 0$. Hence $X \times Y$ is homotopy equivalent to a finite CW-complex by Theorem 2.39 and Theorem 2.57.

2.66. We define a functor $F \colon \mathtt{Or}_{\mathcal{F}}(G) \to \mathtt{Sub}_{\mathcal{F}}(G)$ as follows. It sends an object G/H to the subgroup H. Consider a G-map $f \colon G/H \to G/K$. Choose $g \in G$ with $f(1H) = gK$. Since f is a G-map, we get $hgK = hf(1H) = f(hH) = f(1H) = gK$ and hence $g^{-1}hg \in K$ for all $h \in H$. Hence we can define $F(f)$ to be the class of the homomorphism $c(g^{-1}) \colon H \to K, h \mapsto g^{-1}hg$. The morphism $F(f)$ does not depend on the choice of g since any other choice of g is of the form gk for some $k \in K$ and we have $c((gk)^{-1}) = c(k^{-1}) \circ c(g^{-1})$ and $c(k^{-1}) \in \operatorname{inn}(K)$.

Obviously F is bijective on objects and surjective on morphisms.

2.76. Let $t \in \mathbb{Z}/2$ be the generator. Let $a + bt \in R[\mathbb{Z}/2]$ be an idempotent. Since $(a + bt)^2 = (a^2 + b^2) + (ab + ba)t$ holds, we conclude $a^2 + b^2 = a$ and $ab + ba = b$. This implies

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + b^2 + ab + ba = a + b.$$

Since by assumption 0 and 1 are the only idempotents in R, we get $a + b = 0$ or $a + b = 1$.

Suppose that $b = -a$. Then we get

$$(2a)^2 = 4a^2 = 2(a^2 + (-a)^2) = 2(a^2 + b^2) = 2a.$$

Hence $2a = 0$ or $2a = 1$. Since 2 is not a unit in R we conclude $2a = 0$. Hence we get

$$a = a^2 + b^2 = a^2 + a^2 = 2a^2 = (2a)a = 0a = 0.$$

This implies $a + bt = 0$.

It remains to treat the case $b = 1 - a$. Then we get

$$(a - 1) \cdot (2a - 1) = 2a^2 - 3a + 1 = a^2 + (1 - a)^2 - a = a^2 + b^2 - a = a - a = 0.$$

Because R is an integral domain, we get either $(a-1)=0$ or $(2a-1)=0$. Since 2 is not invertible in R, the case $(2a-1)=0$ cannot occur and hence $a+bt=1$.

2.77. (i) The ring homomorphism $\epsilon\colon \mathbb{Z}[x] \to \mathbb{Z}$ sending x to 1 induces a ring homomorphism $R \to \mathbb{Z}$. Since 2 is not a unit in $\mathbb{Z}$, the element $\overline{2} = 2 \cdot \overline{1}$ is not invertible in R.

(ii) Let $u \in R$ be an idempotent. We can choose an element of the form $a+bx+cx^2 \in \mathbb{Z}[x]$ with $c \in \{0,1\}$ such that $u = \overline{a+bx+cx^2}$. Since $\epsilon(u) = a+b+c$ and all idempotents in $\mathbb{Z}$ are trivial, we have either $a+b+c=0$ or $a+b+c=1$. The ring homomorphism $\delta\colon \mathbb{Z}[x] \to \mathbb{Z}[1/2]$ sending x to $1/2$ sends u to $a+b/2+c/4$. Since $\mathbb{Z}[1/2]$ has only trivial idempotents, either $4a+2b+c=0$ or $4a+2b+c=4$ holds. AS this implies $c = 0 \mod 2$ and we have $c \in \{0,1\}$, we must have $c=0$. This implies $a+b \in \{0,1\}$ and $2a+b \in \{0,2\}$. Since $a = (a+2b)-(a+b)$ holds, we conclude $a \in \{-1,0,1,2\}$. Hence only the following four cases can occur for (a,b), namely $(0,0)$, $(1,0)$, $(2,-2)$, and $(-1,2)$. Obviously the first two cases correspond to the trivial idempotent. Obviously elements $u = \overline{2-2x}$ and $u = \overline{-1+2x}$ are different in R. They are idempotents since we get in both cases $u^2 - u = \overline{2-6x+4x^2} = \overline{2 \cdot (x^2-3x+1)} = 0$.

(iii) The element $\overline{x} + (1-\overline{x}) \cdot t$ in $R[t]$ is an idempotent by the following computation

$$\begin{aligned}
(\overline{x} + (1-\overline{x}) \cdot t)^2 &= \overline{x}^2 + (1-\overline{x})^2 + 2 \cdot (\overline{x} \cdot (1-\overline{x})) \cdot t \\
&= \overline{x^2 + (1-x)^2} + \overline{2 \cdot (x \cdot (1-x))} \cdot t \\
&= \overline{2x^2 - 2x + 1} + \overline{-2x^2 + 2x} \cdot t \\
&= \overline{x} + (1-\overline{x}) \cdot t.
\end{aligned}$$

2.90. Choose an integer $n \geq 0$ and a matrix $A \in M_n(FH)$ such that $A^2 = A$ and $\operatorname{im}(r_A\colon FH^n \to FH^n) \cong_{FH} V$. We compute for $h \in G$, if $l_h\colon V \to V$ is given by left multiplication with h

$$\begin{aligned}
\chi_F(V)(h^{-1}) &= \operatorname{tr}_F(l_{h^{-1}}\colon V \to V) \\
&= \operatorname{tr}_F\left(l_{h^{-1}} \circ r_A\colon FH^n \to FH^n\right) \\
&= \sum_{i=1}^n \operatorname{tr}_F\left(FH \to FH,\ u \mapsto h^{-1}ua_{i,i}\right) \\
&= \operatorname{tr}_F\left(FH \to FH,\ u \mapsto h^{-1}u\left(\sum_{i=1}^n a_{i,i}\right)\right)
\end{aligned}$$

Write $\sum_{i=1}^n a_{i,i} = \sum_{k \in H} \lambda_k \cdot k$. Then we get

$$\begin{aligned}
\chi_F(V)(h^{-1}) &:= \operatorname{tr}_F\left(FH \to FH,\ u \mapsto h^{-1}u\left(\sum_{k\in H}\lambda_k\cdot k\right)\right)\\
&= \sum_{k\in H}\lambda_k\cdot\operatorname{tr}_F\left(FH\to FH,\ u\mapsto h^{-1}uk\right)\\
&= \sum_{k\in H}\lambda_k\cdot\left|\{u\in H \mid u=h^{-1}uk\}\right|\\
&= \sum_{k\in(h)}\lambda_k\cdot\left|\{u\in H\mid h=uku^{-1}\}\right|\\
&= \sum_{k\in(h)}\lambda_k\cdot|C_H\langle h\rangle|\\
&= |C_H\langle h\rangle|\cdot\sum_{k\in(h)}\lambda_k\\
&= |C_H\langle h\rangle|\cdot\operatorname{HS}_{FH}(V)(h).
\end{aligned}$$

2.94. Suppose that $\widetilde{K}_0(FG)$ is a torsion group. This is equivalent to the statement that $\widetilde{K}_0(FG)\otimes_{\mathbb{Z}} F$ is trivial. Lemma 2.18 and Lemma 2.92 imply that $\operatorname{class}_F(G)_f \cong_F F$ and hence $\operatorname{con}_F(G)_f$ consists only of one element. Hence every element in G of finite order is trivial.

2.96. Because of the commutative diagram appearing in the proof of Lemma 2.92, it suffices to prove the claim in the case that G is finite. In this case one computes that $\operatorname{HS}(P)$ evaluated at the unit $e\in G$ is $\frac{\dim_F(P)}{|H|}$.

2.99. Show $\sum_{g\in G}\operatorname{HS}_{\mathbb{Z}G}(P)(g) = \operatorname{HS}_{\mathbb{Z}}(\mathbb{Z}\otimes_{\mathbb{Z}G}P) = \dim_{\mathbb{Z}}(\mathbb{Z}\otimes_{\mathbb{Z}G}P)$.

2.113. The list of finite groups of order ≤ 9 consists of the cyclic groups $\mathbb{Z}/n$ for $n=1,2,3,\dots,9$, the abelian non-cyclic groups $\mathbb{Z}/2\times\mathbb{Z}/2$, $\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$, $\mathbb{Z}/2\times\mathbb{Z}/4$, and $\mathbb{Z}/3\times\mathbb{Z}/3$, and the following non-abelian groups $S_3=D_6$, D_8 and Q_8. Now inspecting Theorem 2.112 gives the answer:

$$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/3\times\mathbb{Z}/3,\ Q_8.$$

2.115. Theorem 2.114 implies that $K_0(FD_8)$ is $\mathbb{Z}^n$ for some n. We conclude from Theorem 2.88 that $n=|\operatorname{con}_F(D_8)_f|$. A presentation for D_8 is $\langle x,y \mid x^4=1, y^2=1, yxy^{-1}=x^{-1}\rangle$. In particular D_8 is a semidirect product $\mathbb{Z}/4\rtimes\mathbb{Z}/2$ if $\mathbb{Z}/4$ is the group generated by x and $\mathbb{Z}/2$ the subgroup generated by y. The elements x^2, y, xy, x^2y and x^3y have order 2, the elements x and x^{-1} have order four. We have one conjugacy class of elements of order 4, namely (x) and three conjugacy classes of elements of order two, namely (x^2), (y) and

(yx). As we also have the conjugacy class of the unit, we see $|\operatorname{con}_{\mathbb{C}}(D_8)_f| = 5$. Since x is conjugated to x^{-1}, we conclude $|\operatorname{con}_{\mathbb{R}}(D_8)_f| = 5$. Since every cyclic subgroup of order 4 is conjugated to $\langle x \rangle$, we get $|\operatorname{con}_{\mathbb{Q}}(D_8)_f| = 5$. This shows

$$K_0(FD_8) \cong \mathbb{Z}^5 \quad \text{for } F = \mathbb{Q}, \mathbb{R}, \mathbb{C}.$$

2.118. Recall that a hyperbolic group does not contain $\mathbb{Z}^2$ as subgroup. Because of Remark 2.117 it suffices to show for a torsionfree hyperbolic group G that it is cyclic if there exists an element g different from the unit element with finite (g). The finiteness of (g) is equivalent to the condition that the centralizer $C_g\langle g \rangle = \{h \in G \mid hg = gh\}$ has finite index in G. Since $\langle g \rangle$ is infinite, hyperbolic implies that $C_G\langle g \rangle$ is virtually cyclic. Hence G is a torsionfree virtually cyclic group and therefore cyclic.

2.123. Write $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$ for distinct primes $p_1, p_2, \dots, p_r$ and integers $n_i \geq 1$. Then Lemma 2.12 implies

$$K_0(\mathbb{Z}/n) = \prod_{i=1}^{r} K_0(\mathbb{Z}/p_i^{n_i}).$$

Since $\mathbb{Z}/p_i^{n_i}$ is local, the claim follows from Theorem 2.122.

2.126. A counterexample is given by $G_1 = G_2 = \mathbb{Z}/3$ since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3]) = \{0\}$ and $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}/3]) \neq \{0\}$ by Theorem 2.112.

Chapter 3

3.3. Let $f\colon R^n \to R^n$ be an R-automorphism. This is the same as a K-linear isomorphism $V^n \to V^n$. Since V is a K-vector space with infinite countable basis, we can choose a K-isomorphism $\alpha\colon \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} V$. Let $a\colon \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} V^n$ be the R-isomorphism given by $\oplus_{k=0}^{\infty} f$. Let $\gamma\colon V^n \oplus \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} V^n$ be the R-automorphism which sends $v \oplus (v_0, v_1, v_2, \dots)$ to $(v, v_0, v_1, v_2 \dots)$. One easily checks $\gamma^{-1} \circ a \circ \gamma = f \oplus a$ Define an R-automorphism $b\colon V \to V$ by $\alpha \circ a \circ \alpha^{-1}$. This is the same as an R-automorphism $b\colon R \to R$. Now one computes

$$[f] + [b] = [f \oplus b] = [f \oplus a] = [a] = [b]$$

in $K_1(R)$ using the fact that conjugated automorphisms define the same element in $K_1(R)$. This implies $[f] = 0$.

3.7. We get from Theorem 3.6 an isomorphism $i\colon \mathbb{H}^\times/[\mathbb{H}^\times,\mathbb{H}^\times] \xrightarrow{\cong} K_1(\mathbb{H})$. Obviously the collection of maps μ_n defines a homomorphism $\mu\colon K_1(\mathbb{H}) \to \mathbb{R}$. The norm of a quaternion $z = a + bi + cj + dk$ is defined by $N(z) := \sqrt{a^2+b^2+c^2+d^2}$. Let $N\colon \mathbb{H}^\times/[\mathbb{H}^\times,\mathbb{H}^\times] \to \mathbb{R}^{>0}$ be the induced homomorphism of abelian groups. Its restriction to $\mathbb{R}^{>0} \subseteq \mathbb{H}$ is the identity. Since $\mu_1(z) = |z|^4$ for $z \in \mathbb{H}$, it remains to prove that $N^{-1}(1) \subseteq [\mathbb{H}^\times,\mathbb{H}^\times]$.

Since $ie^{j\theta}i^{-1} = e^{-j\theta}$ holds for $\theta \in \mathbb{R}$ and similarly with i, j and k cyclically permuted, $e^{2i\theta_1}$, $e^{2j\theta_2}$, and $e^{2k\theta_3}$ are all commutators. These generate an open neighborhood of 1 in $S^3 = N^{-1}(1)$. Since S^3 is connected, the claim follows.

3.18. Take the norm on $\mathbb{Z}[i]$ sending $a+bi$ to $\sqrt{a^2+b^2}$. It yields an Euclidean algorithm. A direct calculation shows $\mathbb{Z}[i]^\times = \{1,-1,i,-i\}$. Now apply Theorem 3.17.

3.22. This follows from Theorem 3.20 and Theorem 3.21.

3.25. The map ϕ is induced by the composite

$$K_1(\mathbb{Z}[\mathbb{Z}/5]) \xrightarrow{f_*} K_1(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \xrightarrow{|\;|} \mathbb{R}^{>0}.$$

Since $(1-t-t^{-1})\cdot(1-t^2-t^3) = 1$, the element $1-t-t^{-1}$ is a unit in $\mathbb{Z}[\mathbb{Z}/5]$ and defines an element in $\mathrm{Wh}(\mathbb{Z}/5)$. Its image under ϕ is $(1-2\cdot\cos(2\pi/5))$ and hence different from 1.

3.36. Let ϵ_* be a chain contraction for E_*. Choose for any $n \in \mathbb{Z}$ an R-homomorphism $\sigma_n\colon E_n \to D_n$ satisfying $p_n \circ \sigma_n = \mathrm{id}_{E_n}$. Define $s_n\colon E_n \to D_n$ by $d_{n+1} \circ \sigma_{n+1} \circ \epsilon_n + \sigma_n \circ \epsilon_{n-1} \circ e_n$.

There are examples of short exact sequences of R-chain complexes whose boundary operator in the associated long homology sequence is not trivial and hence for which $H_n(p_*)$ is not surjective for all $n \in \mathbb{Z}$.

3.41. This is done by the following sequence of expansions. We describe the simplicial complexes obtained after each step:

(i) The standard 2-simplex spanned by v_0, v_1, v_2;
(ii) Three vertices v_0, v_1, v_2 and two edges $\{v_0,v_1\}$ and $\{v_0,v_2\}$;
(iii) The standard 1-simplex spanned by v_0, v_1;
(iv) The standard 0-simplex given by v_0.

3.46. This follows from $\widetilde{K}_0(\mathbb{Z}) = 0$, see Example 2.4, and $\mathrm{Wh}(\{1\}) = 0$, see Theorem 3.17, together with Theorem 2.39 and Theorem 3.45.

3.49. Choose a non-trivial element in $\mathrm{Wh}(\mathbb{Z}/5)$, see Exercise 3.25. By Theorem 3.47 we can find an h-cobordism (W, M_0, M_1) whose Whitehead torsion is x. Hence it is non-trivial. In order to show that $(W \times S^3; M_0 \times S^3, M_1 \times S^3)$

is trivial, we have to show $\tau(i_0 \times \mathrm{id}_{S^3}) = 0$ for $i_0 \colon M_0 \to W$ the inclusion. This follows from Theorem 3.37 (iv) since both $\tau(\mathrm{id}_{S^3})$ and $\chi(S^3)$ vanish.

3.57. By definition $\mathbb{RP}^3$ is the lens space $L(V)$ for the cyclic group $\mathbb{Z}/2$ where V has as underlying unitary vector space $\mathbb{C}^2$ and the generator s of $\mathbb{Z}/2$ acts on V by $-\mathrm{id}$. The cellular $\mathbb{Z}[\mathbb{Z}/2]$-chain complex $C_*(SV)$ is concentrated in dimensions $0, 1, 2, 3$ and is given by

$$\ldots \to 0 \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s-1} \mathbb{Z}[\mathbb{Z}/2] \to 0 \to \ldots .$$

Hence $\mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_*(SV)$ is the $\mathbb{R}$-chain complex

$$\ldots \to 0 \to \mathbb{R} \xrightarrow{2\cdot\mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{2\cdot\mathrm{id}} \mathbb{R} \to 0 \to \ldots$$

It is contractible, a chain contraction γ_* is given by $\gamma_0 = \gamma_2 = 1/2{\cdot}\mathrm{id}$ and $\gamma_n = 0$ for $n \neq 0, 2$. Hence $(c+\gamma)_{\mathrm{odd}} \colon \mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_{\mathrm{odd}}(SV) \to \mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_{\mathrm{odd}}(SV)$ is given by

$$\begin{pmatrix} 2 & 1/2 \\ 0 & 2 \end{pmatrix} \colon \mathbb{R}^2 \to \mathbb{R}^2$$

This implies $\rho(\mathbb{RP}^3; V) = 4$.

3.77. We use induction over $n \geq 0$. The case $n = 0$, i.e., the trivial group, follows from Example 2.4. The induction step from n to $n+1$ is a direct consequence of Theorem 3.76 (i) since $R[\mathbb{Z}^n][\mathbb{Z}]$ is isomorphic to $R[\mathbb{Z}^{n+1}]$.

3.82. Because of Theorem 3.80 (ii) the ring $\mathbb{Z}[\mathbb{Z}^n]$ is regular. Hence we get from Exercise 3.77 and Lemma 3.85 that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^n]) = 0$.

To show $\mathrm{Wh}(\mathbb{Z}^n) = 0$, we use induction over $n \geq 0$. The case $n = 0$, i.e., the trivial group, follows from Example 2.4 and Theorem 3.17. The induction step from n to $n+1$ follows from Theorem 3.81 since $\mathbb{Z}[\mathbb{Z}^n][\mathbb{Z}]$ is isomorphic to $\mathbb{Z}[\mathbb{Z}^{n+1}]$.

3.92. Obviously $(2) \xrightarrow{\cong} (N_{\mathbb{Z}/2})$ sending 2 to $N_{\mathbb{Z}/2}$ is an isomorphism of rings without unit.

Theorem 3.89 together with Lemma 3.91 yields exact sequences

$$K_1(\mathbb{Z}) \to K_1(\mathbb{Z}/n) \to K_0((n)) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/n);$$
$$K_1(\mathbb{Z}[\mathbb{Z}/2]) \to K_1(\mathbb{Z}) \to K_0((N_{\mathbb{Z}/2})) \to K_0(\mathbb{Z}[\mathbb{Z}/2]) \to K_0(\mathbb{Z}),$$

since the ring homomorphism $\mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}$ sending $a + bt$ to $a - b$ induces an isomorphism of rings $\mathbb{Z}[\mathbb{Z}/2]/(N_{\mathbb{Z}/2}) \xrightarrow{\cong} \mathbb{Z}$. Because of Theorem 3.6 and Theorem 3.17 the determinant induces isomorphisms

$$\det\colon K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\};$$
$$\det\colon K_1(\mathbb{Z}/n) \xrightarrow{\cong} \mathbb{Z}/n^{\times}.$$

The map $K_k(\mathbb{Z}[\mathbb{Z}/2]) \to K_k(\mathbb{Z})$ is surjective for $k = 0, 1$ because its composition with $K_k(\mathbb{Z}) \to K_k(\mathbb{Z}[\mathbb{Z}/2])$ is the identity. The map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/n)$ is injective since its composition with the map $K_0(\mathbb{Z}/n) \to \mathbb{Z},\ [P] \mapsto |P|$ is injective by Theorem 2.4. This implies

$$\begin{aligned} K_0((n)) &\cong \begin{cases} 0 & \text{if } n = 2; \\ (\mathbb{Z}/n)^{\times}/\{\pm 1\} & \text{if } n \geq 3; \end{cases} \\ K_0((N_{\mathbb{Z}/2})) &= \{0\}; \\ \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) &= \{0\}. \end{aligned}$$

3.96. Because of Remark 3.95 it suffices to show for each two-sided ideal $I \subseteq F$ that $\mathrm{E}(F, I) = \mathrm{SL}(F, I)$. This is trivial if $I = 0$. If $I = F$, this follows from Theorem 3.17.

3.101. Consider $k \in \mathbb{Z}$ with $(k, |G|) = 1$. Choose $l \in \mathbb{Z}$ with $kl = 1 \mod |G|$. Choose a generator $t \in G$. Define elements $u, v \in \mathbb{Z}G$.

$$\begin{aligned} u &= 1 + t + t^2 + \cdots + t^{k-1}; \\ v &= 1 + t^k + t^{2k} + \cdots + t^{(l-1)k}. \end{aligned}$$

Then $(t-1) \cdot (t^k - 1) \cdot uv = uv$ holds in $\mathbb{Z}G$. One easily checks that $(t-1) \cdot (t^k - 1) \cdot w = 0 \iff w \in (N_G)$ for $w \in \mathbb{Z}G$. Hence $\overline{u} \in \mathbb{Z}G/(N_G)$ is a unit and maps to $\overline{k}$ under the map $j_1\colon \mathbb{Z}G/(N_G) \to \mathbb{Z}/|G|$. Now the claim follows from the Mayer-Vietoris sequence associated to the diagram (3.97).

3.103. Since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) = 0$, see Theorem 2.112 (i), we can assume without loss of generality $|G| \geq 3$.

Suppose $d = 1$. Then $G\backslash X$ is a connected finitely dominated 1-dimensional CW-complex. Since its homology is finitely generated and it is homotopy equivalent to a 1-dimensional CW-complex Y with precisely one 0-cell, the CW-complex Y is finite.

Suppose that $d \geq 2$. Then d is odd by Theorem 3.102 (i). The unit sphere S in $\mathbb{C}^{(d+1)/2}$ with the G-action for which the generator acts by multiplication with $\exp(2\pi i/|G|)$ is a free d-dimensional G-homotopy representation such that $G\backslash S$ is compact and hence finite. By elementary obstruction theory there exists a G-map $X \to S$. Now apply Theorem 2.39 (i), Lemma 3.102 (ii) and Exercise 3.101.

3.111. Obviously the image of the map $\operatorname{dirlim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G\times\mathbb{Z})} K_1(RH) \to K_1(R[G\times\mathbb{Z}])$ is contained in the image of the map $K_1(RG) \to K_1(R[G\times\mathbb{Z}])$. Theorem 3.72 implies $K_0(\mathbb{Z}G) = \{0\}$ if $K_1(RG) \to K_1(R[G\times\mathbb{Z}])$ is surjective.

If R is a commutative integral domain, $K_0(R)$ and hence $K_0(RG)$ cannot be zero. Namely, if F is its quotient field, the homomorphism $K_0(R) \to \mathbb{Z}$, $[P] \mapsto \dim_F(F \otimes_R P)$ is a well-defined surjective map.

3.118. This follows from Theorem 3.115 and Theorem 3.116 (iv).

3.123. A counterexample is given by $G_1 = G_2 = \mathbb{Z}/3$ since $\mathrm{Wh}(\mathbb{Z}/3) = \{0\}$ and $\mathrm{Wh}(\mathbb{Z}/3 \times \mathbb{Z}/3) \neq \{0\}$ by Theorem 3.116.

Chapter 4

4.5. Apply Remark 4.4 to the obvious pullback of rings

$$\begin{tikzcd} R \times S \arrow[r, "\mathrm{pr}_R"] \arrow[d, "\mathrm{pr}_S"'] & R \arrow[d] \\ S \arrow[r] & \{0\} \end{tikzcd}$$

Or, if one does not like the ring $\{0\}$ consisting of one element, use Lemma 3.9 and the Bass-Heller-Swan decomposition 4.3.

4.9. This follows by induction over k using Theorem 4.7.

4.11. The exact Mayer-Vietoris sequence appearing in Example 4.10 yields for a prime p the exact sequence

$$\begin{aligned} &K_1(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]) \to K_1(\mathbb{F}_p[\mathbb{Z}^k]) \\ &\qquad \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \to 0 \end{aligned}$$

We have $\widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/3)]) = \{0\}$ by Theorem 2.105 and Example 2.106. Hence it suffices to show that the map $K_1(\mathbb{Z}[\exp(2\pi i/3)][\mathbb{Z}^k]) \to K_1(\mathbb{F}_3[\mathbb{Z}^k])$ is surjective. Because of the Bass-Heller-Swan decomposition 4.3 it suffices to prove the surjectivity of $K_i(\mathbb{Z}[\exp(2\pi i/3)]) \to K_i(\mathbb{F}_3)$ for $i = 0, 1$. The case $i = 0$ follows from the fact that $K_0(\mathbb{F}_3)$ is generated by $[\mathbb{F}_3]$. It remains to treat $i = 1$. Let $f\colon \mathbb{Z}[\exp(2\pi i/3)] \to \mathbb{F}_3$ be the ring homomorphism which is uniquely determined by the property that it sends $\exp(2\pi i/3)$ to 1. Because of Theorem 3.17 it suffices to show that for every unit u in $\mathbb{F}_3$ we can find a unit u' in $\mathbb{Z}[\exp(2\pi i/3)]$ which is mapped to u under f. Since $\pm\exp(2\pi i/3)$ is a unit in $\mathbb{Z}[\exp(2\pi i/3)]$ and $f(\pm\exp(2\pi i/3)) = \pm 1$, we conclude $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$ for $k \geq 0$.

Now all other claims follow from Theorem 4.3.

4.13. The pullback of rings appearing in Example 4.12 yields a pullback of rings

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k] \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k] \arrow[d, "j_1"] \\ \mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k] \arrow[r, "j_2"'] & (\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k] \end{tikzcd}$$

where $j_2 = j_1$. Put $j := j_1 = j_2$. We obtain from Remark 4.4 the exact sequence

$$\begin{aligned} K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) &\xrightarrow{j_* \oplus j_*} K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) \\ \to K_0(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) &\to K_0(\mathbb{Z}[\mathbb{Z}/3] \times \mathbb{Z}^k) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \\ &\xrightarrow{j_* \oplus j_*} K_0((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) \to \cdots \end{aligned}$$

The following facts are consequences of Theorem 3.80 (i), Exercise 4.5, Exercise 4.11, and Theorem 4.7. We have $K_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = K_n((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) = \{0\}$ for $n \le -1$ and $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = \{0\}$. We can identify the map

$$j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_0((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$$

with the map $j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4)$ which in turn can be identified the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ sending (a, b) to $(a + b, a + b)$. The map

$$j_* \oplus j_* \colon K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$$

can be identified with the direct sum of the map $j_* \oplus j_* \colon K_1(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3]) \to K_1(\mathbb{F}_2 \times \mathbb{F}_4)$ with k fold direct sum of copies of the map $j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4)$. In order to prove

$$K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z}^{k+1} & \text{for } n = 0; \\ \mathbb{Z} & \text{for } n = -1; \\ 0 & \text{for } n \le -2, \end{cases}$$

it remains to show that the map $j_* \colon K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$ is surjective.

Recall that we established an identification of rings $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times \mathbb{F}_4$ in Example 4.12. Because of Lemma 3.9 and Theorem 3.17 the determinant induces an isomorphism $K_1(\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]) \xrightarrow{\cong} (\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3])^{\times}$. Hence it suffices to show that for every unit u in $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$ we can find a unit u' in $\mathbb{Z}[\mathbb{Z}/3]$ which is mapped under the obvious projection $\mathrm{pr} \colon \mathbb{Z}[\mathbb{Z}/3] \to$

$\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$ to u. There are three units in $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times \mathbb{F}_4$, namely, $1 \otimes 1$, $1 \otimes t$, and $1 \otimes t^2$. Obviously they are images of units under pr.

We conclude $N^p K_n(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k])$ for $n \leq 0$, $p \geq 1$ and $k \geq 0$ from Theorem 4.3 since $K_0(\mathbb{Z}[\mathbb{Z}/6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6])^{k-1} \cong \mathbb{Z}^k \cong K_0(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6])$, $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong K_{-1}(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6])$ and $K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) = \{0\}$ for $n \leq -1$ holds.

4.15. This follows from Lemma 4.14 since the assumptions imply that $K_m(R) \to K_m(R[\mathbb{Z}^n])$ induced by the inclusion $R \to R[\mathbb{Z}^n]$ is bijective.

4.19. Because of Theorem 4.7. it suffices to prove that RG is regular, provided that R is regular, G is a finite group, and the order $|G|$ of G is invertible in R. Since R is Noetherian and G is finite, RG is Noetherian. Let M be any finitely generated RG-module. Then the RG-module M is a direct summand in the RG-module $M' := RG \otimes_R M$ where g acts on $x \otimes m$ by $gx \otimes m$. So M' does not see the G-action on M. The injection $M \to M'$ is given by $m \mapsto \frac{1}{|G|} \cdot \sum_{g \in G} g \otimes g^{-1}m$ and the retraction $M' \to M$ by $g \otimes m \mapsto gm$. Let P_* be a finite projective R-resolution of the finitely generated R-module M. Then P'_* is a finite RG-resolution of M'. Since M is a direct RG-summand in M', it possesses a finite projective RG-resolution as well.

Chapter 5

5.6. This follows from Lemma 3.11, Theorem 3.12 and Definition 5.1.

5.8. This follows from Theorem 5.7 and the fact that $H_2(\mathrm{E}(R))$ is the kernel of the universal central extensions $\phi^R\colon \mathrm{St}(R) \to \mathrm{E}(R)$ of the perfect group $\mathrm{E}(R)$.

5.17. Obviously the matrices $d_{1,2}(u)$ and $d_{1,3}(v)$ represent the trivial element in $K_1(R)$. Hence they belong to $\mathrm{E}(R)$ by Lemma 3.11 and Theorem 3.12. Let $\widetilde{d}_{1,2}(u)$ and $\widetilde{d}_{1,3}(v)$ be fixed preimages of $d_{1,2}(u)$ and $d_{1,3}(v)$ under the canonical map $\phi^R\colon \mathrm{St}(R) \to \mathrm{E}(R)$. Then any other lifts are of the form $\widetilde{d}_{1,2}(u) \cdot x$ and $\widetilde{d}_{1,3}(v) \cdot y$ for elements in the center of $\mathrm{St}(R)$. One easily checks $[\widetilde{d}_{1,2}(u), \widetilde{d}_{1,3}(v)] = [\widetilde{d}_{1,2}(u) \cdot x, \widetilde{d}_{1,3}(v) \cdot y]$.

5.20. We get $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ with generator $\{-1, -1\}$ from Theorem 5.18 (vi).

5.23. We obtain $\mathrm{Wh}_2(\mathbb{Z}/n) = 0$ for $n = 1, 2, 3, 4$ from Section 5.8. By Theorem 3.115 the Whitehead group $\mathrm{Wh}(\mathbb{Z}/n)$ vanishes if and only if $n = 1, 2, 3, 4, 6$. Theorem 2.112 (i) implies $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/n]) = 0$ for $n = 1, 2, 3, 4$. We conclude $K_i(\mathbb{Z}[\mathbb{Z}/n]) = 0$ for $n = 1, 2, 3, 4$ and all $i \leq -1$ from Theorem 4.22 (i) and (v) We conclude $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \neq 0$ from Example 4.12. Hence the answer is $n = 1, 2, 3, 4$.

Chapter 6

6.1. Let Z be acyclic. Since $H_0(Z)$ is the free abelian group with $\pi_0(Z)$ as $\mathbb{Z}$-basis, Z is path connected. Since the classifying map $f\colon Z \to B\pi$ for $\pi = \pi_1(Z)$ is 2-connected, it induces by the Hurewicz Theorem an isomorphism $H_1(Z) \to H_1(\pi)$ and an epimorphism $H_2(Z) \to H_2(\pi)$.

6.7. If P_1 and P_2 are two perfect subgroups of G, then the subgroup $\langle P_1, P_2\rangle$ generated by $P_1 \cup P_2$ is again a perfect subgroup of G.

6.8. Recall that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ by Lemma 3.11. We know already because of Theorem 5.7 that $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ is perfect since only a perfect group possesses a universal central extension. Since the image of a perfect subgroup under an epimorphism of groups is perfect and the only perfect subgroup of the abelian group $\mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$ is the trivial group, every perfect subgroup of $\mathrm{GL}(R)$ is contained in $\mathrm{E}(R)$.

6.10. Since $B\mathrm{GL}(R)$ and hence $B\mathrm{GL}(R)^+$ is path connected, this follows directly from the definitions in the case $n = 0$. If $n = 1$, this follows from Theorem 3.12, Theorem 6.5 (iv), and Exercise 6.8.

6.22. This follows by induction over k from Theorem 4.3, Theorem 4.22 (i) and Theorem 6.21.

6.26. We conclude from Example 2.4 and Theorem 3.17 that the sequence looks like

$$\{\pm 1\} \xrightarrow{j_1} \mathbb{Q}^\times \xrightarrow{\partial_1} \bigoplus_p \mathbb{Z} \xrightarrow{i_0} \mathbb{Z} \xrightarrow{j_0} \mathbb{Z} \to 0$$

where p runs through all prime numbers, that j_1 is the inclusion and that j_0 the identity. Hence the map i_0 is the zero map. The map ∂_1 sends a rational number of the shape $\pm p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k}$ for pairwise distinct primes p_1, p_2, $\dots$, p_k and integers n_1, n_2, $\dots$, n_k to the element $(n_p)_p$ whose entry for $p = p_i$ is n_i for $i = 1, 2, \dots, k$ and is 0 for any prime p which is not contained in $\{p_1, p_2, \dots, p_k\}$.

6.27. We get from Corollary 6.25 the exact sequence for $n \geq 1$.

$$\bigoplus_p K_n(\mathbb{F}_p) \to K_n(\mathbb{Z}) \to K_n(\mathbb{Q}) \to \bigoplus_p K_{n-1}(\mathbb{F}_p).$$

By Theorem 6.23 $K_n(\mathbb{F}_p) = 0$ for $n = 2k$ for $k \geq 1$, and $K_n(\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ holds for all $n \geq 1$.

6.31. From the analogue of the sequence (6.28) for K^{TOP} and the assumption that k is odd, we conclude

$$K_n^{\mathsf{TOP}}(\mathbb{R};\mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \equiv 0 \mod 4; \\ \{0\} & n = 1,2,3 \mod 4. \end{cases}$$

We know $K_n(\mathbb{R}) = \{0\}$ for $n \leq -1$ from Theorem 4.7. Now the sequence (6.28) and Theorem 6.30 imply

$$K_n(\mathbb{R};\mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \geq 0 \quad \text{and} \quad n \equiv 0 \mod 4; \\ \{0\} & n \geq 0 \quad \text{and} \quad n = 1,2,3 \mod 4; \\ \{0\} & n \leq -1. \end{cases}$$

6.38. Generators for $K_0(\mathcal{A})$ are isomorphism classes of objects. Relations are $[P_1] + [P_2] = [P_1 \oplus P_2]$ for any objects P_1, P_2.

The generators of $K_1(\mathcal{A})$ are conjugacy classes of objects of $\mathcal{A}$. Relations are $[g \circ f] = [g] + [f]$ for any automorphisms f, g of the same object and $\left[\begin{pmatrix} f_1 & f_0 \\ 0 & f_2 \end{pmatrix}\right] = [f_1] + [f_2]$ for any automorphisms $f_i \colon P_i \to P_i$ for $i = 1,2$ and a any morphism $f_0 \colon P_2 \to P_1$.

The functor S induces homomorphism $S_i \colon K_i(\mathcal{A}) \to K_i(\mathcal{A})$ for $i = 1,2$. The existence of the natural transformation T implies that the two homomorphism $S_i + \mathrm{id}_{K_i(\mathcal{A})}$ and S_i coincide. Hence $\mathrm{id}_{K_i(\mathcal{A})}$ is the zero-homomorphism which means $K_i(\mathcal{A}) = 0$.

6.39. Let $\mathcal{A}$ be the additive category of countably generated projective R-modules. Let S be the functor sending an object P to $(P \oplus P \oplus \cdots)$. Then we obtain a natural transformation $T \colon \mathrm{id} \oplus S \to S$ by rebracketing, i.e. $(P \oplus P \oplus \cdots) = P \oplus (P \oplus P \oplus \cdots)$. Hence $\mathcal{A}$ is flasque and we can apply Theorem 6.37 (iii).

6.46. This follows directly from the resolution Theorem 6.45.

6.51. Since the rings $\mathbb{Z}$, $\mathbb{Z}[1/p]$, and $\mathbb{F}_p$ are regular, this follows from Example 6.50 using Exercise 6.46.

6.55. Because of Conjecture 6.53 it suffices to construct the corresponding sequence for $H_*(-;\mathbf{K})$

$$\begin{aligned} \cdots \to H_n(BG_0;\mathbf{K}(R)) &\to H_n(BG_1;\mathbf{K}(R)) \oplus H_n(BG_2;\mathbf{K}(R)) \\ &\to H_n(BG;\mathbf{K}(R)) \to H_{n-1}(BG_0;\mathbf{K}(R)) \\ &\to H_{n-1}(BG_1;\mathbf{K}(R)) \oplus H_{n-1}(BG_2;\mathbf{K}(R)) \to \cdots . \end{aligned}$$

One can arrange that BG_i is a sub CW-complex of BG and $BG = BG_1 \cup BG_2$ and $BG_0 = BG_1 \cap BG_2$. Now the desired sequence above is the associated Mayer-Vietoris sequence.

6.56. Because of Conjecture 6.53 it suffices to construct the corresponding sequence for $H_*(-;\mathbf{K})$

$$\cdots \to H_n(BG;\mathbf{K}(R)) \xrightarrow{\mathrm{id}-\phi_*} H_n(BG;\mathbf{K}(R)) \to H_n(B(G \rtimes_\phi \mathbb{Z});\mathbf{K}(R)) \\ \to H_{n-1}(BG;\mathbf{K}(R)) \xrightarrow{\mathrm{id}-\phi_*} H_{n-1}(BG;\mathbf{K}(R)) \to \cdots .$$

The automorphism ϕ induces a homotopy equivalence $B\phi\colon BG \to BG$. The mapping torus of $B\phi$ is a model for $B(G \rtimes_\phi \mathbb{Z})$. Now the desired long exact sequence comes from the Wang sequence associated to the fibration $BG \to B(G \rtimes_\phi \mathbb{Z}) \to S^1$.

6.60. If R is regular, then $R[t]$ is regular. There is an obvious identification $(R[t])G = (RG)[t]$. Hence we obtain a commutative diagram

$$\begin{tikzcd} H_n(BG;\mathbf{K}(R)) \arrow[r, "\cong"] \arrow[d] & K_n(RG) \arrow[d] \\ H_n(BG;\mathbf{K}(R[t])) \arrow[r, "\cong"'] & K_n((RG)[t]) \end{tikzcd}$$

where the vertical arrows are induced by the canonical inclusions $R \to R[t]$ and $RG \to (RG)[t]$. The horizontal arrows are bijective by assumption. Since R is regular, the left vertical arrow is bijective because of Theorem 6.16 (ii) and the Atiyah-Hirzebruch spectral sequence. Hence also the right vertical arrow is bijective. This implies $NK_n(RG) = 0$ for all $n \in \mathbb{Z}$.

6.63. Consider the following commutative diagram

$$
\begin{tikzcd}
 & & \mathrm{Nil}_n \arrow[d, "i_n \oplus l_n"] & & \\
 & & K_n(RG_0) \oplus K_n(RG_1) \oplus K_n(RG_2) \arrow[dl, dashed, "\alpha"] \arrow[d, "K_n(i_0) \oplus K_n(i_1) \oplus K_n(i_2)"] & & \\
0 \arrow[r] & \ker(p_n) \arrow[r, "\iota"] \arrow[d, dashed, "\beta"] & K_n(RG) \arrow[r, "p_n"] \arrow[d, "\partial_n"] & NK_n \arrow[r] \arrow[d, "0 \oplus \mathrm{id}"] & 0 \\
0 \arrow[r] & K_{n-1}(RG_0) \arrow[r, "f_{n-1} \circ (1,-1)"] \arrow[dr, "0 \oplus K_{n-1}(j_1) \oplus -K_{n-1}(j_2)"'] & \mathrm{Nil}_{n-1} \arrow[r, "i_{n-1} \oplus \pi"] \arrow[d, "i_{n-1} \oplus l_{n-1}"] & K_{n-1}(RG_0) \oplus NK_n \arrow[r] & 0 \\
 & & K_{n-1}(RG_0) \oplus K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \arrow[d, "K_{n-1}(i_0) \oplus K_{n-1}(i_1) \oplus K_{n-1}(i_2)"] & & \\
 & & K_{n-1}(RG). & &
\end{tikzcd}
$$

Here Nil_n stands for the n-homotopy group of $\mathbf{Nil}(RG_0; RG_1, RG_2)$, NK_n stands for $NK_n(RG_0; RG_1, RG_2)$, the letters ι and π denote obvious inclusions or projections. The maps i_*, l_*, and f_* are induced by the map of spectra $\mathbf{i}$, $\mathbf{l}$, and $\mathbf{f}$. The middle column is the long exact sequence associated to the homotopy cartesian square appearing in Theorem 6.61 (i) with boundary operator ∂_n. Theorem 6.61 implies that the two horizontal short sequences are (split) exact and the diagram (without the dashed arrows) commutes.

Now an easy diagram chase shows that exists dotted arrows uniquely determined by the property that the diagram remains commutative.

Define the desired long exact Mayer-Vietoris sequence by the homomorphism $\alpha' \colon K_0(RG_1) \oplus K_n(RG_2) \to \ker(p_n)$ which is the restriction of α, the homomorphism β, and the homomorphism $(j_1)_* \oplus (j_2)_* \colon K_n(RG) \to K_n(RG_1) \oplus K_n(RG_2)$. We leave it to the reader to check using the diagram above that this sequence is indeed exact.

6.73. There is an obvious projection $\mathrm{pr} \colon R \to R_0$. Since $\mathrm{pr} \circ i = \mathrm{id}_{R_0}$ for the inclusion $i \colon R_0 \to R$, it suffices to prove that $KH_n(i) \circ KH_n(\mathrm{pr}) \colon KH_n(R) \to KH_n(R)$ is surjective. Define a map $\varphi \colon R \to R[t]$ by sending $r_n \in R_n$ to $r_n \cdot t^n$. For $k = 0, 1$ let $\mathrm{ev}_k \colon R[t] \to R$ be the ring homomorphism given by putting $t = 0$ for $k = 0$ and $t = 1$ for $k = 1$. Then $\mathrm{ev}_1 \circ \varphi = \mathrm{id}_R$ and $KH_n(\mathrm{ev}_k)$ is bijective for $k = 0, 1$ and $n \in \mathbb{Z}$ by homotopy invariance. Hence $KH_n(\mathrm{ev}_0)$ and $KH_n(\varphi)$ are isomorphisms. Since $\mathrm{ev}_0 \circ \varphi$ agrees with $i \circ \mathrm{pr}$, the claim follows.

Chapter 7

7.7. The composite of two cofibrations is again a cofibration. The same is true for weak equivalences. Hence $co\mathcal{C}$ and $w\mathcal{C}$ are indeed subcategories of $\mathcal{C}$.

Axioms (i), (ii) and (iv) appearing in Definition 7.5 are obviously satisfied.

Consider chain maps $i_*\colon A_* \to B_*$ and $f_*\colon A_* \to C_*$ of finite projective R-chain complexes such that $i_n\colon A_n \to B_n$ is split injective for all $n \in \mathbb{Z}$. Define D_* to be the cokernel of the chain map $i_* \oplus f_*\colon A_* \to B_* \oplus C_*$. Then we obtain a short exact sequence of finite projective R-chain complexes $0 \to A_* \xrightarrow{i_* \oplus f_*} B_* \oplus C_* \xrightarrow{\mathrm{pr}_*} D_* \to 0$ since for every $n \geq 0$ the sequence of R-modules $0 \to A_n \xrightarrow{i_n \oplus f_n} B_n \oplus C_n \xrightarrow{\mathrm{pr}_n} D_n \to 0$ is split exact because i_n is split injective. One easily checks that we obtain a pushout of finite projective R-chain complexes

$$\begin{tikzcd} A_* \arrow[r, "i_*"] \arrow[d, "f_*"] & B_* \arrow[d, "\mathrm{pr}_* |_{B_*}"] \\ C_* \arrow[r, "\mathrm{pr}_* |_{C_*}"] & D \end{tikzcd}$$

such that the lower horizontal arrow is a cofibration. Hence axiom (iii) is true.

Axiom (v) follows from the long exact homology sequences associated to a short exact sequence of R-chain complexes and the Five Lemma.

7.19. This follows the property of the map $\mathbf{i}$ of (7.14) that $\pi_n(\mathbf{i})$ is bijective for $n \geq 1$, from Remark 7.15, and from Theorem 7.18, since $K_n(\mathbb{Z})$ vanishes for $n \leq -1$ and is $\mathbb{Z}$ for $n = 0$.

7.23. Since $\mathrm{Wh}_2(\mathbb{Z})$ is trivial, one easily checks that under the isomorphism (7.22) the kernel of $L_2(S^1)$ is isomorphic to $NA_2(\{\bullet\}) \oplus NA_2(\{\bullet\})$ and hence non-trivial.

7.28. We obtain from the fibration (7.24) the exact sequence

$$\begin{aligned} \pi_1(BG_+ \wedge A(\{\bullet\})) \to \pi_1(A(BG)) \to \pi_1(\mathrm{Wh}(BG)) \\ \to \pi_0(BG_+ \wedge A(\{\bullet\})) \to \pi_0(A(BG)). \end{aligned}$$

Since $A(\{\bullet\})$ is connected, the Atiyah-Hirzebruch spectral sequence shows that $\pi_0(\{\bullet\}_+ \wedge A(\{\bullet\})) \xrightarrow{\cong} \pi_0(BG_+ \wedge A(\{\bullet\}))$ is bijective. Since the homomorphism $\pi_0(\{\bullet\}_+ \wedge A(\{\bullet\})) \to \pi_0(BG_+ \wedge A(\{\bullet\}))$ is split injective, the map $\pi_0(BG_+ \wedge A(\{\bullet\})) \to \pi_0(A(BG))$ is injective. Using diagram (7.25), we obtain a short exact sequence

$$H_1(B\pi_1(BG); \mathbf{K}(\mathbb{Z})) \to K_1(\mathbb{Z}\pi_1(BG)) \to \pi_1(\mathrm{Wh}(BG)) \to 0.$$

Again by the Atiyah-Hirzebruch spectral sequence we obtain an isomorphism $H_1(B\pi_1(BG); \mathbf{K}(\mathbb{Z})) \cong G/[G,G] \times \{\pm 1\}$. Hence the image of the map $H_1(B\pi_1(BG);\mathbf{K}(\mathbb{Z})) \to K_1(\mathbb{Z}\pi_1(BG))$ is the subgroup of $K_1(\mathbb{Z}G) = K_1(\mathbb{Z}\pi_1(BG))$ given by the trivial units $\{\pm g \mid g \in G\}$. This implies $\mathrm{Wh}(G) \cong \pi_1(\mathrm{Wh}(BG))$.

7.33. Suppose such M exists. The long exact homotopy sequence of the fibration (7.24) looks like

$$\cdot \to \pi_n(M_+ \wedge A(\{\bullet\})) \to A_n(M) \to \mathrm{Wh}_n(M) \to \cdots .$$

The splitting (7.31) yields isomorphisms

$$A_n(M) \cong \mathrm{Wh}_n^{\mathsf{DIFF}}(M) \oplus \pi_n(\Sigma^\infty M).$$

Rationally the Atiyah-Hirzebruch sequence always collapses. Hence we obtain from Theorem 6.24 and Theorem 7.18 isomorphisms

$$\pi_n(M_+ \wedge A(\{\bullet\})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(M;\mathbb{Q}) \oplus \bigoplus_{k\geq 1} H_{n-4k-1}(M;\mathbb{Q}).$$

Hence we obtain the long exact sequence of $\mathbb{Q}$-modules

$$\begin{aligned}\cdot \to \mathrm{Wh}_{n+1}^{\mathsf{PL}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n(M;\mathbb{Q}) \oplus \bigoplus_{k\geq 1} H_{n-4k-1}(M;\mathbb{Q})&\\ \to H_n(M;\mathbb{Q}) \oplus \mathrm{Wh}_n^{\mathsf{DIFF}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Wh}_n^{\mathsf{PL}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \cdots&\end{aligned}$$

Since by assumption the map $\mathrm{Wh}_n^{\mathsf{DIFF}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Wh}_n(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ is bijective for $n \geq 0$, we obtain for every $n \geq 0$ isomorphisms

$$H_n(M;\mathbb{Q}) \oplus \bigoplus_{k\geq 1} H_{n-4k-1}(M;\mathbb{Q}) \cong H_n(M;\mathbb{Q}).$$

This implies for every $n \geq 0$ and $k \geq 1$ that $H_{n-4k-1}(M;\mathbb{Q}) = 0$, a contradiction to $H_0(M;\mathbb{Q}) = \mathbb{Q}$.

Chapter 8

8.3. Test

Chapter 9

9.6. It is straightforward to check that $e(P)$ is a well-defined R-homomorphism, compatible with direct sums and natural. It remains to show that it is bijective for a finitely generated projective R-module P. Let Q be another finitely generated projective R-module. Since $e(P \oplus Q)$ is up to isomorphism $e(P) \oplus e(Q)$, the map $e(P \oplus Q)$ is bijective if and only if both $e(P)$ and $e(Q)$ are bijective. Since we can find Q such that $P \oplus Q \cong R^n$, it suffices to consider the case $P = R$ which follows from a direct computation.

9.14. Let $b_i(M) := \dim_{\mathbb{R}}(H_i(M;\mathbb{R}))$ be the i-th-Betti number. Poincaré duality implies $b_i(M) = b_{4k-i}(M)$ for all $i \geq 0$. We conclude directly from the definition of the signature that $\operatorname{sign}(M) \equiv b_{2k}(M) \mod 2$. We get modulo 2

$$\begin{aligned}
\chi(M) &\equiv \sum_{i=0}^{4k} (-1)^i \cdot b_i(M) \\
&\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=2k+1}^{4k} (-1)^i \cdot b_i(M) \\
&\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=2k+1}^{4k} (-1)^i \cdot b_{4k-i}(M) \\
&\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) \\
&\equiv b_{2k}(M) \\
&\equiv \operatorname{sign}(M).
\end{aligned}$$

9.16.

(i) If n is odd, then $\dim(\mathbb{CP}^n)$ is not divisible by four and hence $\operatorname{sign}(\mathbb{CP}^n) = 0$. If n is even, then the intersection pairing of $\mathbb{CP}^n$ looks like $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $(a,b) \mapsto ab$ and hence $\operatorname{sign}(\mathbb{CP}^n) = 1$.
(ii) Since STM is the boundary of the total space DTM of the disk tangent bundle, Theorem 9.15 (i) implies $\operatorname{sign}(STM) = 0$.
(iii) We get $\operatorname{sign}(M) = 0$ from assertions (v) and (vi) of Theorem 9.15.

9.23. Note in the situation under consideration that $\epsilon = 1$ and the involution on $\mathbb{Z}$ is the trivial involution. Hence the projection $\mathrm{pr}\colon R \to Q_\epsilon(R)$ is the identity. We conclude from Remark 9.21 that (P,λ) admits a quadratic refinement if and only if there exists a map $\mu\colon P \to \mathbb{Z}$ such that $\mu(nx) = n^2\mu(x)$ holds for all $n \in \mathbb{Z}$ and $x \in P$, $\mu(x+y) - \mu(x) - \mu(y) = \lambda(x,y)$ is true for all $x, y \in P$ and $\lambda(x,x) = 2 \cdot \mu(x)$ is valid for all $x \in P$. Obviously the existence of μ implies $\lambda(x,x)$ to be even for all $x \in P$. Suppose that $\lambda(x,x)$ to is even for all $x \in P$. Then we can define $\mu(x) := \lambda(x,x)/2$ and μ has all desired properties.

9.27. Show that the diagonal in $P \oplus P$ is a Lagrangian for the non-degenerate ϵ-quadratic form $(P \oplus P, \psi \oplus -\psi)$ and then apply Lemma 9.26.

9.28. This follows from Lemma 9.11, Remark 9.24, and Lemma 9.26.

9.31. A non-degenerate quadratic form on V is a map $\mu\colon V \to \mathbb{F}_2$ such that $\mu(0) = 0$, we obtain a non-degenerate symmetric pairing $\lambda\colon V \times V \to \mathbb{F}_2$ by $\lambda(p,q) = \mu(p+q) + \mu(p) + \mu(q)$ and that $\lambda(p,p) = 0$ for all $p \in V$, see Remark 9.21. Fix a basis $\{e_1, e_2\}$ for V. Then $\lambda(e_i, e_j) = 1$ for $i \neq j$, since λ is non-degenerate and we already know $\lambda(e_1,e_1) = \lambda(e_2,e_2) = \lambda(e_1+e_2, e_1+e_2) = 0$ and $\lambda(e_1,e_2) = \lambda(e_2,e_1)$. This implies that either $\mu(e_1) = \mu(e_2) = \mu(e_1+e_2) = 1$ or that precisely one of the elements $\mu(e_1), \mu(e_2), \mu(e_1+e_2)$ is 1. By possibly replacing the basis $\{e_1,e_2\}$ by the basis $\{e_1, e_1+e_2\}$ or $\{e_2, e_1+e_2\}$, we can arrange that either $\mu(e_1) = \mu(e_2) = \mu(e_1+e_2) = 1$ or that $\mu(e_1) = \mu(e_2) = 0$ and $\mu(e_1+e_2) = 1$. The first one has Arf invariant 1, the second 0. Hence there are up two isomorphism precisely two non-degenerate quadratic forms on V.

9.44. By the definition of the selfintersection number it suffices to show $\mu(f) \neq 0$ in $Q_\epsilon(\mathbb{Z}\pi)$. The map $\mathbb{Z}\pi \to \mathbb{Z}/2$ sending $\sum_{g\in\pi} n_g \cdot g$ to $\sum_{g\in G} \overline{n_g}$ induces a map of abelian groups $Q_\epsilon(\mathbb{Z}\pi) \to \mathbb{Z}/2$. Since the set of double points consists of precisely one element, it sends $\mu(f)$ to $\overline{1}$ and hence $\mu(f) \neq 0$.

9.45. Consider the inclusion $i\colon S^1 \to S^1 \times S^1$ onto the first factor. One easily changes it locally by to an immersion $j\colon S^1 \to S^1 \times S^1$ in general position with exactly one double point such that i and j are homotopic. We conclude from Exercise 9.44 that i and j are not regularly homotopic.

9.52. Denote by $C^{n-*}(\widetilde{X})_{\mathrm{untw}}$ the $\mathbb{Z}\pi$-chain complex which is analogously defined as $C^{n-*}(\widetilde{X})$, but now with respect to the untwisted involution. Its n-th homology $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$ depends only on the homotopy type of X. If X carries the structure of a Poincaré complex with respect to $w\colon \pi_1(X) \to \{\pm 1\}$, then the Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence induces a $\mathbb{Z}\pi$-isomorphism $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}}) \cong \mathbb{Z}^w$. Thus we rediscover w from $H_n(C^{n-*}(\widetilde{X})_{\mathrm{untw}})$.

9.60. This follows from fact that two embeddings $M \to \mathbb{R}^{n+m}$ for large enough m are diffeotopic.

9.69. It suffices to show that f is l-connected for $l = k+1, k+2, \ldots$. By assumption this holds for $l = k+1$. In the induction step f is l-connected for some $l \geq k+1$ and we have to show that f is $(l+1)$-connected, i.e., $\pi_{l+1}(f) = 0$. By Lemma 9.64 (ii) which applies also to the case where M is only a finite Poincaré complex, it suffices to show that $K_l(\widetilde{M}) = 0$. By Lemma 9.64 (i), which applies also to the case where M is only a finite

Poincaré complex, it suffices to show $K_{n-l}(\widetilde{M}) = 0$. Since f is $(k+1)$-connected and $n - l \leq k$, $K_{n-l}(\widetilde{M}) = 0$ vanishes by Lemma 9.64 (ii).

9.75. Let $f\colon M \to S^{4k+2}$ be any map of degree one. Choose an embedding $i\colon M \to \mathbb{R}^{4k+2+m}$ for large enough m. Then the given stable trivialization of the tangent bundle defines a trivialization of the normal bundle. It can be viewed as bundle map $\overline{f}\colon \mu(i) \to \underline{\mathbb{R}^m}$ covering f. Thus we obtain a normal map of degree one $(\overline{f}, f)$. It defines a surgery obstruction $\sigma(\overline{f}, f) \in L_{4k+2}(\mathbb{Z})$. Since $L_{4k+2}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$, this is the same as an element $\alpha(M) \in \mathbb{Z}/2$. It is independent by the choice of f and $\overline{f}$ and depends only on the stably framed bordism class of M since by a theorem due to Hopf the homotopy class of f is uniquely determined by its degree and the surgery obstruction is a invariant under normal bordism.

9.100. Because of Theorem 9.99 (ii) we can assume that $F\colon W \to X \times [0,1]$ is a simple homotopy equivalence. We conclude from Theorem 3.37 (iii) that both inclusions $M \to W$ and $N \to W$ are simple homotopy equivalence. By Theorem 3.47 there exists a diffeomorphism $W \to M \times [0,1]$. Hence the restriction of this diffeomorphism to N is a diffeomorphism $N \to M \times \{1\} = M$.

9.107. This follows from the various Rothenberg sequence since the $\mathbb{Z}/2$-Tate cohomology of any $\mathbb{Z}[\mathbb{Z}/2]$-module is annihilated by multiplication with 2.

9.112. Since $\operatorname{Wh}(\mathbb{Z})$, $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}])$ and $K_n(\mathbb{Z}[\mathbb{Z}])$ for $n \leq -1$ vanish, see Example 2.4, Theorem 3.17 and Theorem 4.7, the decoration does not matter by Theorem 9.106. We conclude from (9.109) and the computations of $L_n(\mathbb{Z})$ in Theorem 9.29, Theorem 9.32 and Theorem 9.82:

$$L_n(\mathbb{Z}[\mathbb{Z}]) \cong L_{n-1}(\mathbb{Z}) \oplus L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \equiv 0,1 \mod 4; \\ \mathbb{Z}/2 & n \equiv 2,3 \mod 4. \end{cases}$$

9.115. We conclude from Conjectures 3.110, 4.18, and 9.114 and Theorem 9.106 that the decoration does nor matter. If $g = 0$, $\pi_1(F_g)$ is trivial and hence $L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]) = L_n^{\langle -\infty \rangle}(\mathbb{Z})$. Suppose $g \geq 1$. Then F_g itself is a model for $B\pi_1(F_g)$. Because of Conjecture 9.114 we get

$$H_n(F_g; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]).$$

Now we use the Atiyah-Hirzebruch spectral sequence to compute $H_n(F_g; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))$. This is rather easy since F_g is 2-dimensional, the edge homomorphism which describes $H_n(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}) \to H_n(F_g; \mathbf{L}^{\langle -\infty \rangle})$ is split injective and $L_n^{\langle -\infty \rangle}(\mathbb{Z})$

is $\mathbb{Z}$ if $n \equiv 0 \mod 4$, $\mathbb{Z}/2$ if $n \equiv 0 \mod 4$, and $\{0\}$ otherwise. The result is

$$L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0 \mod 4; \\ \mathbb{Z}^{2g} & n \equiv 1 \mod 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 2 \mod 4; \\ (\mathbb{Z}/2)^{2g} & n \equiv 3 \mod 4. \end{cases}$$

9.142. Because of Poincare duality it suffices to show $f_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) = \mathcal{L}(N) \cap [N]_{\mathbb{Q}}$. But this follows from the Novikov Conjecture 9.137 because of Remark 9.141 since we can put $N = BG$.

9.148. This follows from the long exact homotopy sequence associated to a fibration.

9.149. Let $C \subseteq \pi_1(X)$ be any finite cyclic subgroup. Since the universal covering $\widetilde{X}$ is a model for $E\pi_1(X)$, it is also a model for EC after restricting the group action. Hence $C\backslash\widetilde{X}$ is a finite dimensional CW-model for BC. This implies that the group homology $H_n(C)$ of C is trivial in dimensions $n > \dim(X)$. It is known that the homology of C is C in all odd dimensions. Hence C must be trivial. This shows that $\pi_1(X)$ is torsionfree.

9.151. The top homology group $H_n(M;\mathbb{F}_2)$ with $\mathbb{F}_2$-coefficients of any closed n-dimensional manifold M is known to be isomorphic to $\mathbb{F}_2$. If M is simply connected and aspherical it is homotopy equivalent to the one-point-space $\{\bullet\}$. This implies $n = 0$ and hence $M = \{\bullet\}$ for a simply connected aspherical manifold.

9.152. See [645, Lemma 3.2].

9.163. See Example 3.62.

9.164. Let k and n be natural numbers such at least one of them is even. Then S^k and S^n are topologically rigid but $S^k \times S^n$ is not. See Remark 9.162.

9.175. Let G_k be a n_k-dimensional Poincaré duality group for $k = 0, 1$. Let P_*^k be a n_k-dimensional finite projective $\mathbb{Z}[G_k]$-resolution of the trivial $\mathbb{Z}[G_k]$-module $\mathbb{Z}$. Then $P_*^0 \otimes_{\mathbb{Z}} P_*^1$ is a $(n_0 + n_1)$-dimensional finite projective $\mathbb{Z}[G_0 \times G_1]$-resolution of the trivial $\mathbb{Z}[G_0 \times G_1]$-module $\mathbb{Z}$. The obvious chain map given by the tensor product over $\mathbb{Z}$ and the obvious identification $\mathbb{Z}[G_0] \otimes_{\mathbb{Z}} \mathbb{Z}[G_1] = \mathbb{Z}[G_0 \times G_1]$

$$\hom_{\mathbb{Z}[G_0]}(P_*^0, \mathbb{Z}[G_0]) \otimes_{\mathbb{Z}} \hom_{\mathbb{Z}[G_1]}(P_*^1, \mathbb{Z}[G_1]) \xrightarrow{\cong} \hom_{\mathbb{Z}[G_0 \times G_1]}\bigl(P_*^0 \otimes_{\mathbb{Z}} P_*^1, \mathbb{Z}[G_0 \times G_1]\bigr)$$

is an isomorphism of $\mathbb{Z}$-cochain complexes. Since $\hom_{\mathbb{Z}[G_0]}(P_*^0, \mathbb{Z}[G_0])$ is a free $\mathbb{Z}$-cochain complex whose cohomology is concentrated in dimension n_k and given there by $\mathbb{Z}$, there exists a $\mathbb{Z}$-chain homotopy equivalence from $[n_k](\mathbb{Z})$ which is the $\mathbb{Z}$-chain complex concentrated in dimension n_k and having $\mathbb{Z}$ as n_k-th chain module, to $\hom_{\mathbb{Z}[G_0]}(P_*^0, \mathbb{Z}[G_0])$. Hence $\hom_{\mathbb{Z}[G_0]}(P_*^0, \mathbb{Z}[G_0]) \otimes_{\mathbb{Z}} \hom_{\mathbb{Z}[G_1]}(P_*^1, \mathbb{Z}[G_1])$ is $\mathbb{Z}$-chain homotopy equivalent to $[n_0](\mathbb{Z}) \otimes [n_1](\mathbb{Z}) \cong [n_0 + n_1](\mathbb{Z})$. This implies that $H^n\bigl(\hom_{\mathbb{Z}[G_0 \times G_1]}(P_*^0 \otimes_{\mathbb{Z}} P_*^1, \mathbb{Z}[G_0 \times G_1])\bigr)$ is $\mathbb{Z}$ in dimension $(n_0 + n_1)$ and trivial otherwise. Hence $G_0 \times G_1$ is a $(n_0 + n_1)$-dimensional Poincaré duality group.

9.190. This follows from Theorem 9.168, Theorem 13.29 (iv), and Theorem 16.1 (ia).

Chapter 10

10.3. Since

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot \dim_{\mathbb{Q}}(H_n(X;\mathbb{Q})) = \sum_{n \geq 0} (-1)^n \cdot \dim_{\mathbb{Q}}(H^n(X;\mathbb{Q}))$$

holds, this follows directly from (10.1) and (10.2).

10.17. We obtain from (10.16) an isomorphism

$$K_G^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{C \in \mathcal{C}(G)} H^*(X^C/G; \mathbb{Q}(\zeta_C)).$$

Now use the fact $\dim_{\mathbb{Q}}(\mathbb{Q}(\zeta_C)) = \varphi(|C|)$.

10.21. Without loss of generality we can assume that $K_j^H(Y,B)$ is torsionfree for all $j \in \mathbb{Z}$. Now check that we obtain two G-homology theories on pairs of finite proper G-CW-complexes (X,A) by putting

$$\mathcal{H}_G^*(X,A) := \bigoplus_{i+j=n} K_i^G(X,A) \otimes_{\mathbb{Z}} K_j^H(Y,B);$$
$$\mathcal{K}_H^*(X,A) := K_{G\times H}^*((X,A) \times (Y,B)).$$

(When one wants to check the exactness of the long exact sequence of a pair for $\mathcal{H}_G^*$, we need that assumption that $K_i^H(Y,B)$ is torsionfree and hence the functor $- \otimes_{\mathbb{Z}} K_j^H(Y,B)$ is exact for all $j \in \mathbb{Z}$). The external multiplication defines a natural transformation $T_G^*\colon \mathcal{H}_G^* \to \mathcal{K}_G^*$ of G-cohomology theories for pairs of finite proper G-CW-complexes. One checks that $T_G^*(G/H)\colon \mathcal{H}_G^*(G/H) \to \mathcal{K}_G^*(G/H)$ is bijective for all finite subgroups $H \subseteq G$. Now prove by induction over the number of equivariant cells using the Five Lemma, the long exact sequence of a pair, excision and G-homotopy invariance that $T_G^n(X,A)$ is bijective for all pairs of finite proper G-CW-complexes (X,A) and all $n \in \mathbb{Z}$.

10.23. This follows from the long exact sequence of the pair (DE, SE), the Thom isomorphisms (10.22) and the commutativity of the following diagram which is a consequence of the naturality of the product

$$\begin{tikzcd}
K_G^*(X) \arrow[r, "-\cup e(p)"] \arrow[d, "K_G^*(p_{DE})"', "\cong"] & K_G^*(X) \arrow[d, "\cong"', "K_G^*(p_{DE})"] \\
K_G^*(DE) \arrow[r, "-\cup K_G^0(j)(\lambda_E)"] & K_G^*(DE) \\
K_G^*(DE) \arrow[u, "\mathrm{id}", "\cong"'] \arrow[r, "-\cup\lambda_E", "\cong"'] & K_G^*(DE,SE) \arrow[u, "K_G^*(j)"]
\end{tikzcd}$$

10.29. If G contains an element g of order ≥ 3, then show $||xx^*|| \neq ||x||^2$ for $x = g + 1 - g^{-1}$. If G contains an element g of order 2, then show $||xx^*|| \neq ||x||^2$ for $x = g + i \in L^1(G, \mathbb{C})$. Finally one checks directly that $L^1(G, F)$ is a C^*-algebra if G is trivial or if G has order 2 and $F = \mathbb{R}$.

10.35. Since $\mathcal{K}$ is the colimit $\operatorname{colim}_{n\to\infty} M_n(\mathbb{C})$, we conclude from Morita equivalence and the compatibility with colimits over directed systems that the obvious inclusion of C^*-algebras $\mathbb{C} \to \mathcal{K}$, induces an isomorphisms $K_0(\mathbb{C}) \xrightarrow{\cong} K_*(\mathcal{K})$. The C^*-algebra $\mathcal{B}$ is contractible, i.e., the zero homomorphism is homotopic to the identity $\mathcal{B} \to \mathcal{B}$, a homotopy is given by $F_t(x) = F(tx)$. Homotopy invariance implies the vanishing of $K_*(\mathcal{B})$. Now the long exact sequence of the ideal $\mathcal{K} \subseteq \mathcal{B}$ yields an isomorphism $K_n(\mathcal{B}/\mathcal{K}) \cong K_{n-1}(\mathcal{K})$ for all $n \in \mathbb{Z}$. To finish the calculation, one directly proves that $K_n(\mathbb{C})$ is $\mathbb{Z}$ for $n = 0$ and trivial for $n = 1$ and applies Bott periodicity.

10.45. Since G is by assumption is finite, $H_n(BG;\mathbb{Q})$ is $\mathbb{Q}$ if $n = 0$ and is trivial for $n \neq 0$. We conclude from the Chern characters (10.1) and (10.7) that $\dim_{\mathbb{Q}}(K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(KO_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$. We have $K_0(C_r^*(G)) \cong \operatorname{Rep}_{\mathbb{C}}(G)$ and $KO_0(C_r^*(G)) \cong \operatorname{Rep}_{\mathbb{R}}(G)$. Now use the obvious fact that $\dim_{\mathbb{Q}}(\operatorname{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1 \iff \dim_{\mathbb{Q}}(\operatorname{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1 \iff G = \{1\}$ holds.

10.47. Let $c\colon S^1 \to S^1$ be the automorphism of S^1 sending $z \in S^1$ to z^{-1}. Let T_c be the mapping torus. One easily checks that T_c is a model for

BG. Elementary considerations about homology theories lead to the so-called Wang sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(S^1) \xrightarrow{\mathrm{id} - K_n(c)} K_n(S^1) \xrightarrow{K_n(i)} K_n(T_c)$$
$$\xrightarrow{\partial_n} K_{n-1}(S^1) \xrightarrow{\mathrm{id} - K_{n-1}(c)} K_{n-1}(S^1) \xrightarrow{K_{n-1}(i)} \cdots .$$

We know that $K_n(S^1) \cong \mathbb{Z}$ for all $n \in \mathbb{Z}$. Elementary considerations about homology theories imply that $K_n(c) = -\mathrm{id}_{K_n(S^1)}$ for odd n and $K_n(c) = \mathrm{id}_{K_0(S^1)}$ for even n. Hence the Wang sequence reduces to

$$\cdots \to \mathbb{Z} \xrightarrow{2\cdot\mathrm{id}} \mathbb{Z} \xrightarrow{K_1(i)} K_1(T_c) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{K_0(i)} K_0(T_c) \to \mathbb{Z} \xrightarrow{2\cdot\mathrm{id}} \mathbb{Z} \to \cdots .$$

This implies

$$K_n(C_r^*(G)) \cong K_n(T_c) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd.} \end{cases}$$

10.62. Obviously $\hom_{\{1\}}(F, i^*F) \cong F$. Since $i_*F = C_0(G,F)$, all homomorphisms of G-C^*-algebras from i_*F to F are zero and hence $\hom_G(i_*F, F)$ vanishes.

10.71. Put $G = \mathbb{Z}/p$. Since p is an odd prime, we have $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V^G) \equiv 0 \mod 2$ and hence $\dim(SV) = \dim(SV^G) \equiv 0 \mod 2$. Since $\dim(SV) = d-1$, we get $K_n(SV) = K_n(SV^G) = K_n(S^{d-1})$ for all $n \in \mathbb{Z}$. Since $\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathbb{Z}^p$, we get $\mathrm{im}(\theta_G) = \mathbb{Z}[1/p]$ and $\mathrm{im}(\theta_{\{1\}}) \cong \mathbb{Z}[1/p]^{p-1}$. We conclude from Theorem 10.69

$$\begin{aligned} &\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV) \\ &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n\big(SV^G/C_G G\big) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/C_G\{1\}) \\ &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/G). \end{aligned}$$

The Atiyah-Hirzebruch spectral sequence converges to $K_n(SV/G)$ and has as E^2-term $E^2_{r,s} = H_r(SV/G; K_s(\{\bullet\}))$. Since $|G|$ is a p-power, we get a $\mathbb{Z}[1/p]$-isomorphism $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} H_r(SV/G) \cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}[1/p]G} H_r(SV)$. Since p is odd, the G-operation on $H_i(SV)$ is trivial. Hence we get a $\mathbb{Z}[1/p]$-isomorphism

$$\begin{aligned} \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E^2_{r,s} &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} H_r(SV; K_s(\{\bullet\})) \\ &\cong \begin{cases} \mathbb{Z}[1/p] & \text{if } r = 0, d-1 \text{ and } s \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E^2_{r,s}$ is a finitely generated free $\mathbb{Z}[1/p]$-module for each (r,s) and we conclude from the isomorphism (10.1) for each $n \in \mathbb{Z}$

$$\sum_{r+s=n} \mathrm{rk}_{\mathbb{Z}[1/p]}\big(\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E^2_{r,s}\big) = \mathrm{rk}_{\mathbb{Z}[1/p]}\big(\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(SV/G)\big).$$

This implies that all differentials in the Atiyah-Hirzebruch spectral sequence are trivial after inverting p and we get

$$\begin{aligned}\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(SV/G) &\cong \begin{cases} \mathbb{Z}[1/p] & \text{if } d \text{ is even;} \\ \mathbb{Z}[1/p]^2 & \text{if } d \text{ is odd and } n \text{ is even;} \\ 0 & \text{if } d \text{ is odd and } n \text{ is odd.} \end{cases} \\ &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}).\end{aligned}$$

Now the claim follows from

$$\begin{aligned}\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV) &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/G) \\ &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(S^{d-1}) \\ &\cong \mathbb{Z}[1/p]^p \otimes_{\mathbb{Z}} K_n(S^{d-1}).\end{aligned}$$

10.74. The abelian group $K_1(C(X))$ is not finitely generated because of Theorem 3.120, whereas as $K_1(C(X)) \cong K^1(X)$ is finitely generated.

Chapter 11

11.6. Define the n-skeleton of $\widetilde{X}$ to be $p^{-1}(X_n)$. Use the facts that a covering over a contractible space such as D^n is trivial and a covering is a local homeomorphism.

11.7. The Euler characteristic of a compact CW-complex can be computed by counting cells. Each equivariant cell in $X - X^{\mathbb{Z}/p}$ contributes p (non-equivariant) cells.

11.12. Choose an irrational number θ. Let $\phi\colon S^1 \to S^1$ be the homeomorphism given by multiplication with the complex number $\exp(2\pi i\theta)$. The space S^1 with the associated $\mathbb{Z}$-action is free but not proper.

11.14. Suppose that there is a free smooth $\mathbb{Z}/p$-action on S^{2n}. By Remark 11.13 we obtain a free $\mathbb{Z}/p$-CW-structure on S^{2n}. By a previous exercise we get the contradiction

$$0 \equiv \chi(\emptyset) \equiv \chi\big((S^{2n})^{\mathbb{Z}/p}\big) \equiv \chi(S^{2n}) \equiv 2 \mod p.$$

11.17. This follows from Theorem 11.16 (i).

11.20. Suppose that $E_{\mathcal{F}}(G)$ has a zero-dimensional model. Hence it is a disjoint union of spaces of the shape G/H. Since $E_{\mathcal{F}}(G)$ is path connected, it must be G/G. This implies $G \in \mathcal{F}$.

If $G \in \mathcal{F}$ holds, G/G is a 0-dimensional G-CW-model for $E_{\mathcal{F}}(G)$.

An example for L is $\mathbb{R}$.

11.34. We obtain from Subsection 11.6.13 that there is a G-CW-model for $\underline{E}G$ which is obtained from G/M by attaching free cells of dimensions ≤ 2. Let $i \colon G/M \to \underline{E}G$ be the inclusion. Consider the map

$$j = \mathrm{id}_{EG} \times_G i \colon EG \times_G G/M \to EG \times_G \underline{E}G$$

Since for a space Y the canonical projection $EG \times_G (G \times Y) \to Y$ is a homotopy equivalence, we conclude by a Mayer-Vietoris argument that $H_n(j)$ is bijective for $n \geq 3$. The canonical projections $EG \times_G G/M \to EG/M = BM$ and $EG \times_G \underline{E}G \to EG \times_G \{\bullet\} = BG$ are homotopy equivalences since $\underline{E}G$ is (after forgetting the group action) contractible.

11.35. Since hyperbolic groups, arithmetic groups, mapping class groups, $\mathrm{Out}(F_n)$, and one-relator groups have a finite dimensional model for $\underline{E}G$ by Subsections 11.6.7, 11.6.8, 11.6.9, 11.6.10, and 11.6.13, it suffices to show for group G with a d-dimensional model for $\underline{E}G$ that $H_k(BG;\mathbb{Q}) = 0$ holds for $k > d$.

The cellular $\mathbb{Q}$-chain complex $C_*(X)$ of a proper G-CW-complex X consists of projective $\mathbb{Q}G$-modules since for any finite subgroup $H \subseteq G$ the $\mathbb{Q}G$-module $\mathbb{Q}[G/H]$ is projective. Since $\underline{E}G$ is contractible (after forgetting the group action), its cellular $\mathbb{Q}$-chain complex yields a $\dim(\underline{E}G)$-dimensional projective $\mathbb{Q}G$-resolution of the trivial $\mathbb{Q}G$-module $\mathbb{Q}$.

11.47. Since H is infinite and countable, its cardinality is $\aleph_0$. We conclude $\underline{\mathrm{gd}}(H) = 1$ from Remark 11.46. Theorem 11.45 implies that $\underline{\mathrm{gd}}(H \rtimes \mathbb{Z}) \leq 2$. Since $H \rtimes \mathbb{Z}$ is finitely generated and does not contain a finitely generated free group of finite index, we cannot have $\underline{\mathrm{gd}}(H \rtimes \mathbb{Z}) \leq 1$.

11.54. The universal covering on M is the hyperbolic space and hence contractible. Therefore N is a model for BG. Since $H_n(BG;\mathbb{Z}^w) \cong \mathbb{Z}$ for $w \colon G = \pi_1(N) \to \{\pm\}$ given by the first Stiefel-Whitney class of N, we conclude $\mathrm{cd}(G) = \dim(N) = \underline{\mathrm{gd}}(G)$. Since G is hyperbolic and hence satisfies conditions ($\underline{\underline{\mathrm{M}}}$) and ($\underline{\underline{\mathrm{NM}}}$), we conclude $\underline{\mathrm{gd}}(G) = \underline{\underline{\mathrm{gd}}}(G)$ from Theorem 11.53 (iii).

11.60. This follows from Theorem 11.59.

11.63. Apply Theorem 11.62 to X and take $Y = H\backslash \underline{E}G$ for a torsionfree subgroup H of G with $[G:H] = 2$.

Chapter 12

12.4. The desired $\mathbb{Z}/2$-pushout for the 1-skeleton is obvious and for the 2-skeleton given by

$$\begin{tikzcd} \mathbb{Z}/2 \times S^1 \arrow[r, "\mathrm{pr}"] \arrow[d] & S^1 \arrow[d] \\ \mathbb{Z}/2 \times D^2 \arrow[r] & S^2 \end{tikzcd}$$

where pr is the projection. Now one easily checks that $C_*(S^2) \otimes_{\mathsf{Or}(\mathbb{Z}/2)} R_{\mathbb{C}}$ is given by the $\mathbb{Z}$-chain complex concentrated in dimensions 0,1 and 2

$$\cdots \to \{0\} \to \{0\} \to R_{\mathbb{C}}(\{1\}) \xrightarrow{c_2} R_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{0} R_{\mathbb{C}}(\mathbb{Z}/2) \to \{0\} \to \cdots$$

where c_2 is induction with the inclusion $\{1\} \to \mathbb{Z}/2$. This implies

$$H_n^{\mathbb{Z}/2}(S^2; R_{\mathbb{C}}) \cong \begin{cases} \mathbb{Z}^2 & n = 0; \\ \mathbb{Z} & n = 1; \\ \{0\} & \text{otherwise.} \end{cases}$$

12.7. By applying Lemma 12.5 to the skeletal filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq \cup_{n\ge 0} X_n = X$, the claim can be reduced to finite dimensional pairs. Using the axioms of a G-homology theory, the Five Lemma, and induction over the dimension one reduces the proof to the special case $(X,A) = (G/H, \emptyset)$.

12.11. This follows directly from the axiom about the compatibility with conjugation applied in the case $X = \{\bullet\}$.

12.16. The real line $\mathbb{R}$ with the obvious action of $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$ is a D_∞-CW-model for $\underline{E}D_\infty$, see Theorem 11.25. Up to conjugacy there are two subgroups H_0 and H_1 of order two in D_∞. One obtains a D_∞-pushout

$$\begin{tikzcd} D_\infty \times S^0 = D_\infty \amalg D_\infty \arrow[r, "\mathrm{pr}_0 \amalg \mathrm{pr}_1"] \arrow[d, "i"'] & D_\infty/H_0 \amalg D_\infty/H_1 \arrow[d] \\ D_\infty \times D^1 \arrow[r] & \underline{E}D_\infty \end{tikzcd}$$

where pr_0 and pr_1 are the obvious projections and i is the obvious inclusion. Hence the associated long exact Mayer-Vietoris sequence reduces to

$$0 \to K_1^{D_\infty}(\underline{E}D_\infty) \to R_{\mathbb{C}}(\{1\}) \oplus R_{\mathbb{C}}(\{1\}) \\ \xrightarrow{f} R_{\mathbb{C}}(\{1\}) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \to K_0^{D_\infty}(\underline{E}D_\infty) \to 0$$

where f sends (v, w) to $(v + w, i_*(v), -i_*(w))$ for i_* the map induced by the inclusion $i\colon \{1\} \to \mathbb{Z}/2$. This implies

$$K_n^{D_\infty}(\underline{E}D_\infty) \cong \begin{cases} \mathbb{Z}^3 & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

12.36. One easily checks that for a given group G and every subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$ the map $H_n^G(G/H; \mathbf{t})\colon H_n^G(G/H; \mathbf{E}) \to H_n^G(G/H; \mathbf{F})$ can be identified with the map $\pi_n(\mathbf{t}(t^G(G/H)))\colon \pi_n(\mathbf{E}(t^G(G/H))) \to \pi_n(\mathbf{F}(t^G(G/H)))$ and hence is bijective by assumption. Now apply Lemma 12.6.

12.47. The argument appearing in the solution of Exercise 12.16 yields a long exact Mayer- Vietoris sequence

$$\cdots \to K_0(R) \oplus K_0(R) \to K_0(R) \oplus K_0(R[\mathbb{Z}/2]) \oplus K_0(R[\mathbb{Z}/2]) \\ \to H_0^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_R) \to K_{-1}(R) \oplus K_{-1}(R) \\ \to K_{-1}(R) \oplus K_{-1}(R[\mathbb{Z}/2]) \oplus K_{-1}(R[\mathbb{Z}/2]) \to \cdots .$$

Since the obvious map $K_n(R) \to K_n(R[\mathbb{Z}/2])$ is split injective, we obtain for $n \in \mathbb{Z}$ isomorphisms

$$K_n(R[\mathbb{Z}/2]) \oplus \operatorname{coker}(K_n(R) \to K_n(R[\mathbb{Z}/2])) \cong H_n^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_R).$$

If $n \leq -1$, then $K_n(R[\mathbb{Z}/2]) = 0$ for $R = \mathbb{Z}, \mathbb{C}$ by Theorem 4.16 and Theorem 4.22. Hence

$$H_n^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_R) \cong \{0\} \quad \text{for } n \leq -1$$

The map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\mathbb{Z}/2])$ is bijective by Example 2.106 and $K_0(\mathbb{Z}) = \mathbb{Z}$ by Example 2.4. Hence

$$H_0^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_{\mathbb{Z}}) \cong \mathbb{Z}.$$

Since $K_0(\mathbb{C}H) \cong R_{\mathbb{C}}(H)$ for a finite group H, one easily checks

$$H_0^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_{\mathbb{C}}) \cong \mathbb{Z}^3.$$

12.49. Since X/G has no odd dimensional cells, X has no odd dimensional equivariant cells. Moreover, if X/G is finite, then X has only finitely many equivariant cells. We conclude for any coefficients system M that the Bredon homology $H_p(X;M)$ vanishes, if p is odd, or if p is larger then the dimension of X. If X has only finitely many equivariant cells and $M(G/H)$ is a finitely generated free abelian groups for any finite subgroup $H \subseteq G$, then $H_p(X;M)$is finitely generated free abelian for all $p \in \mathbb{Z}$. Since $K_q^G(G/H) = 0$ for odd q and is a finitely generated free abelian group for even q for every finite subgroup $H \subseteq G$, and all isotropy groups of X are by assumption finite, we conclude for the E^2-terms of the equivariant Atiyah-Hirzebruch spectral sequence of Theorem 12.48 that $E^2_{p,q} = 0$ if $p+q$ is odd. If X has only finitely many equivariant cells, then $E^2_{p,q}$ is finitely generated free if $p+q$ is even and vanishes for large enough q. Now the claim follows from this spectral sequence.

12.54. Consider the long exact sequence of the pair $(EG \times_G X, EG \times_G X^G)$ and of the pair $(X/G, X^G/G) = (X/G, X^G)$ and the map between them induced by the projection $(EG \times_G X, EG \times_G X^G) \to (X/G, X^G/G)$, and use the fact that (X, X^G) is relatively free and hence $\mathcal{H}_n(EG \times_G X, EG \times_G X^G) \to \mathcal{H}_n(X/G, X^G/G)$ is bijective.

12.59. From Theorem 12.58 we get a natural isomorphism of spectral sequences from the equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathcal{BH}^G(X)$ to the equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathcal{H}^G_*(X)$. One easily checks that all the differentials in the equivariant Atiyah-Hirzebruch spectral sequence converging to $\mathcal{BH}^G(X)$ vanish.

12.60. For every finite group $H \subseteq G$ the group $W_G H$ is finite and hence $\mathbb{Q}[W_G H]$ is semisimple. Therefore every $\mathbb{Q}[W_G H]$-module is flat. Because of Theorem 12.58 it suffices to show that for every finite subgroup $H \subseteq G$ and every $n \in \mathbb{Z}$ the map

$$H_p(C_G H \backslash \iota^H_{\mathcal{F} \subseteq \mathcal{G}}; \mathbb{Q}) \colon H_p(C_G H \backslash E_{\mathcal{F}}(G)^H; \mathbb{Q}) \to H_p(C_G H \backslash E_{\mathcal{G}}(G)^H; \mathbb{Q})$$

is bijective. This is obviously true if $H \notin \mathcal{F}$. Suppose $H \in \mathcal{F}$. Then the claim follows from fact that both $C_*(E_{\mathcal{F}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $C_*(E_{\mathcal{G}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q}$ are projective $\mathbb{Q}[C_G H]$-resolutions of the trivial $\mathbb{Q}[C_G H]$-module $\mathbb{Q}$ which implies that

$$C_*(\iota^H_{\mathcal{F} \subseteq \mathcal{G}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to C_*(E_{\mathcal{F}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q} \to C_*(E_{\mathcal{G}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a $\mathbb{Q}[C_G H]$-chain homotopy equivalence and hence induces after applying $\mathbb{Q} \otimes_{\mathbb{Q}[C_G H]} -$ a $\mathbb{Q}$-chain homotopy equivalence.

12.64. The desired pairing is given by

$$A(G) \times M(G) \to M(G), \quad ([G/H], x) \mapsto \operatorname{ind}_H^G \circ \operatorname{res}_G^H(x).$$

12.86. Every subgroup $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ is conjugated to one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators given by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

So we shall restrict from now on to the study of the actions of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ given by the actions of the above described generators. The case $\mathbb{Z}/4$ has been carried out in Example 12.85. The computations for the other cases is analogous. We get in all cases that $G \backslash \underline{E}G$ is homeomorphic to S^2. There are up to conjugacy four non-trivial finite subgroups, which are all isomorphic to $\mathbb{Z}/2$, in G in the case $F = \mathbb{Z}/2$. There up to conjugacy three non-trivial finite subgroups, which are all isomorphic to $\mathbb{Z}/3$ in G in the case $F = \mathbb{Z}/3$. In the case $F = \mathbb{Z}/6$ there are up to conjugacy three non-trivial finite subgroups, the first is isomorphic to $\mathbb{Z}/2$, the second to $\mathbb{Z}/3$ and the third to $\mathbb{Z}/6$. Hence we get in all cases $K_1^G(\underline{E}G) = 0$ and

$$\begin{aligned}
K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/2}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_2) &\cong \mathbb{Z}^6; \\
K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/3}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_3) &\cong \mathbb{Z}^8; \\
K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/4}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_4) &\cong \mathbb{Z}^9; \\
K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/6}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_6) &\cong \mathbb{Z}^{10}.
\end{aligned}$$

12.94. Since G is finite, an easy spectral sequence argument shows that there is an isomorphism $u\colon \mathbb{Q} \otimes_{\mathbb{Z}G} H_n^{G,\xi}(EG; \mathbf{K}_R) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_n^{G,\xi}(EG; \mathbf{K}_R)$, where G acts on $K_n(R)$ via $\alpha \circ \xi$ and trivially on $\mathbb{Q}$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_n^{G,\xi}(EG; \mathbf{K}_R) \arrow[r] \arrow[d, "\cong"', "u^{-1}"] & \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_n^{G,\xi}(\underline{E}G; \mathbf{K}_R) = \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R_{\alpha\circ\xi}G) \arrow[d, "\mathrm{res}"] \\
\mathbb{Q} \otimes_{\mathbb{Z}G} K_n(R) \arrow[r, "N"'] & \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R)^G
\end{tikzcd}$$

where the lower horizontal arrow N is the norm map and the right vertical arrow res is restriction with the inclusion $R \to R_{\alpha\circ\xi}G$. Since the lower vertical arrow N is an isomorphism, the upper horizontal arrow is injective.

Chapter 13

13.3. If we replace in Conjecture 13.1 the family $\mathcal{VCY}$ by $\mathcal{FIN}$, then the Conjecture 13.2 for $\mathbb{Z}$ reduces to the statement that for any ring R the map induced by the projection $E\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) \to H_n^G(\mathbb{Z}/\mathbb{Z};\mathbf{K}_R) = K_n(R\mathbb{Z})$$

is an isomorphism. Since $\mathbb{Z}$ acts freely on $\mathbb{Z}$ and $(E\mathbb{Z})/\mathbb{Z} = S^1$, we get an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) = H^{\{1\}}(S^1;\mathbf{K}_R) = K_n(R) \oplus K_{n-1}(R).$$

Under this identification the assembly map above becomes the restriction of the Bass-Heller-Swan isomorphism of Theorem 6.16

$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R\mathbb{Z}).$$

to $K_n(RG) \oplus K_{n-1}(RG)$. This implies that $NK_n(R)$ vanishes for all $n \in \mathbb{Z}$ and all rings R, a contradiction by Example 3.69.

13.5. If we replace in Conjecture 13.4 the family $\mathcal{VCY}$ by $\mathcal{FIN}$, then Conjecture 13.7 for $\mathbb{Z}$ reduces to the statement that for any ring R with involution the map induced by the projection $E\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(\mathbb{Z}/\mathbb{Z};\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R\mathbb{Z})$$

is an isomorphism. Since $\mathbb{Z}$ acts freely on $\mathbb{Z}$ and $(E\mathbb{Z})/\mathbb{Z} = S^1$, we get an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R\mathbb{Z}) \oplus L_{n-1}^{\langle -\infty \rangle}(R\mathbb{Z})$$

Under this identification the assembly map above can be identified with the isomorphism appearing in the Shaneson splitting (9.109).

13.13. The structure of an abelian group on each set of morphisms comes from the obvious structure of an abelian group on $M_{m,n}(R)$. The direct sum of $[m]$ and $[n]$ is $[m+n]$. The direct sum on morphisms is given by taking block matrices. The zero object is $[0]$. We obtain a natural equivalence from $\underline{R}_{\oplus}$ to the additive category of finitely generated free R-modules by sending $[m]$ to R^m and a morphism $[m] \to [n]$ given by a (m,n)-matrix A to the R-linear map $R^m \to R^n$ given by right multiplication with A.

13.18. We only present the proof of the harder implication. Suppose that for every two objects A and B in $\mathcal{A}$ the induced map $\operatorname{mor}_{\mathcal{A}}(A_0,A_1) \to \operatorname{mor}_{\mathcal{B}}(F(A_0),F(A_1))$ sending f to $F(f)$ is bijective and for each object B in $\mathcal{B}$ there exists an object A in $\mathcal{A}$ such that $F(A)$ and B are isomorphic in $\mathcal{B}$. Choose for any object $B \in \mathcal{B}$ an object $A(B) \in \mathcal{A}$ and an isomorphism

$u(B)\colon B \xrightarrow{\cong} F(A(B))$ in $\mathcal{B}$. Next we define a functor $F'\colon \mathcal{B} \to \mathcal{A}$ of additive categories. It sends an object B to $A(B)$. A morphism $f\colon B_0 \to B_1$ is send to the morphism $F'(f)\colon A(B_0) \to A(B_1)$ which is uniquely determined by the property that $F(F'(f)) = u(B_1) \circ f \circ u(B_0)^{-1}$. One easily checks that $F'(g \circ f) = F'(g) \circ F'(f)$ and $F'(f_0 + f_1) = F'(f_0) + F'(f_1)$ holds. Consider two objects B_0 and B_1. We have to show that for the natural inclusions $j_i\colon B_i \to B_0 \oplus B_1$ for $i = 0, 1$ the morphism $F'(j_0) \oplus F'(j_1)\colon F'(B_0) \oplus F'(B_1) \to F(B_0 \oplus B_1)$ is an isomorphism. This follows from the following diagram that commutes by definition of F' and whose lower left vertical arrow is an isomorphism since F is compatible with direct sums

$$\begin{tikzcd}
B_0 \oplus B_1 \arrow[r, "j_0 \oplus j_1 = \mathrm{id}", "\cong"'] \arrow[d, "u(B_0)\oplus u(B_1)", "\cong"'] & B_0 \oplus B_1 \arrow[d, "\cong", "u(B_0 \oplus B_1)"'] \\
F(A(B_0)) \oplus F(A(B_1)) \arrow[r, "F(F'(j_0))\oplus F(F'(j_1))"] \arrow[d, "\cong"] & F(A(B_0 \oplus B_1)) \arrow[d, "\cong", "\mathrm{id}"'] \\
F(A(B_0) \oplus A(B_1)) \arrow[r, "F(F'(j_0)\oplus F'(j_1))"] & F(A(B_0 \oplus B_1))
\end{tikzcd}$$

Hence F' is a functor of additive categories. Natural transformations of functors of additive categories $S\colon F \circ F' \to \mathrm{id}_{\mathcal{B}}$ and $T\colon F' \circ F \to \mathrm{id}_{\mathcal{A}}$ are determined by $S(B) = u(B)$ and $F(T(A)) = u(F(A))$.

13.30. This follows from Theorem 13.29 (v) since G is virtually cyclic if Q is virtually cyclic.

13.31. Let G be a group. It is the disjoint union of its finitely generated subgroups. Hence by Theorem 13.29 (vi) the Full Farrell-Jones Conjecture 13.27 holds for all groups if and only if it holds for all finitely generated groups. Any finitely generated group can be written as a colimit over a directed set of finitely presented groups. Hence by Theorem 13.29 (vi) the Full Farrell-Jones Conjecture 13.27 holds for all finitely generated groups if and only if holds for all finitely presented groups. Finally note that group G is finitely presented if and only if it occurs as the fundamental group of a connected orientable closed 4-manifold.

13.40. This follows from Lemma 13.39 by the following argument. Since K_W is finite and the image of ϕ is by assumption infinite, the composite $p_W \circ \phi\colon V \to Q_W$ has infinite image. Since Q_W is isomorphic to $\mathbb{Z}$ or D_∞, the same is true for the image of $p_W \circ \phi\colon V \to Q_W$. By assertion (v) of Lemma 13.39 the kernel of $p_W \circ \phi\colon V \to Q_W$ is K_V. Hence $\phi(K_V) \subseteq K_W$ and ϕ induces maps ϕ_K and ϕ_Q making the diagram of interest commutative. Since the image of $p_W \circ \phi\colon V \to Q_W$ is infinite, $\phi_Q(Q_V)$ is infinite. This implies that ϕ_Q is injective since both Q_V and Q_W are isomorphic to D_∞ or $\mathbb{Z}$.

13.41. Suppose that G admits a proper cocompact isometric action on $\mathbb{R}$. Since the action is cocompact and $\mathbb{R}$ is not compact, the group G must be infinite. Let K be the kernel of the homomorphism $\rho\colon G \to \operatorname{aut}(\mathbb{R})$ coming from the G-action. Since the action is proper, K must be finite. Let $Q \subseteq \operatorname{aut}(\mathbb{R})$ be the image of ρ. The group of isometries of $\mathbb{R}$ is $\mathbb{R} \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ corresponds to $\{\pm \mathrm{id}\}$ and $\mathbb{R}$ to translations $l_r\colon \mathbb{R} \to \mathbb{R}$ with elements $r \in \mathbb{R}$. Let $r_0 := \inf\{r \in \mathbb{R} \mid r > 0, l_r \in Q\}$. Since Q acts properly, we have $r_0 > 0$ and $Q \cap \mathbb{R} \subseteq \mathbb{R}$ is the infinite cyclic group generated by r_0. Now one easily checks that Q is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}/2$. Hence G is virtually cyclic.

If G is virtually cyclic, then it admits an epimorphism with finite kernel onto $\mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}/2$ by Lemma 13.39 (i). These two groups and hence G admit proper cocompact isometric actions on $\mathbb{R}$.

13.46. Suppose that H is infinite and belongs to $\mathcal{HE}_p \cap \mathcal{VCY}_I$. Then there are exact sequences $1 \to \mathbb{Z} \to H \xrightarrow{q} Q \to 1$ and $1 \to P \xrightarrow{i} H \to \mathbb{Z} \to 1$ where $i\colon P \to H$ is the inclusion of a finite normal subgroup P, and Q is a finite p-group. The restriction $q|_P\colon P \to Q$ is injective since P is a finite subgroup of H and the kernel of q is infinite cyclic. Hence P is a finite p-group. Fix an element $t \in H$ whose image under the epimorphism $H \to \mathbb{Z}$ is a generator. Then $t \in N_G P$. Let p^m be the order of Q. Consider any $x \in P$. We have $q(t^{p^m} x t^{-p^m}) = q(t)^{p^m} q(x) q(t)^{-p^m} = q(x)$. Since $q|_P\colon P \to Q$ is injective, we get $t^{p^m} x t^{-p^m} = x$. In particular $H \cong P \rtimes_\phi \mathbb{Z}$ for the automorphism $\phi\colon P \xrightarrow{\cong} P$ of p-power order given by conjugation with t.

Suppose H is isomorphic to $P' \rtimes_\phi \mathbb{Z}$ for some finite p-group P and automorphism $\phi\colon P \to P$ whose order is p^m for some natural number m. Then obviously H belongs to $\mathcal{VCY}_I$. The exact sequence $1 \to \mathbb{Z} \xrightarrow{p^m \cdot \mathrm{id}} \mathbb{Z} \to \mathbb{Z}/p^m \to 1$ induces an exact sequence $1 \to \mathbb{Z} \to P \rtimes_\phi \mathbb{Z} \to P \rtimes_\phi \mathbb{Z}/p^m \to 1$. Since $P \rtimes_\phi \mathbb{Z}/p^m$ is a finite p-group, H belongs to $\mathcal{HE}_p$.

13.47. Because of Exercise 13.46 there exists a finite p-group P and an automorphism $\phi\colon P \to P$ whose order is a p-power such that G is isomorphic to $P \rtimes_\phi \mathbb{Z}$. Note that a model for $E_{\mathcal{FIN}}(G)$ is $E\mathbb{Z}$ considered as G-CW-complex by restriction with the canonical epimorphism $G \to \mathbb{Z}$. We conclude from Theorem 6.64 and Remark 6.65 that

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ for all $n \in \mathbb{Z}$ if and only if we have $N_\pm K_n(RP; \phi)[1/p] = 0$ for all $n \in \mathbb{Z}$. This follows from Theorem 6.66.

13.49. This follows directly from Theorem 13.48 since $\mathcal{P}(G, R)$ is empty if $\mathbb{Q} \subseteq R$ holds or if G is torsionfree.

13.58. The group G satisfies the Full Farrell Jones Conjecture 13.27 by Theorem 13.29 (iv) and (v). Since every virtually cyclic subgroup of G is of type I, Theorem 13.57 implies that the projection pr induces for every additive G-category with involution $\mathcal{A}$ and all $n \in \mathbb{Z}$ an isomorphism

$$H_n^G\bigl(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}\bigr) \colon H_n^G\bigl(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}\bigr) \\ \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n\bigl(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G))\bigr).$$

Hence we get from Remark 13.17 that the projection pr induces for all $n \in \mathbb{Z}$ an isomorphism

$$H_n^G\bigl(\mathrm{pr}; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr) \colon H_n^G\bigl(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr) \to H_n^G(G/G; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

Recall that we have an extension $1 \to F \to G \xrightarrow{f} \mathbb{Z}^d \to 1$ for a finite group F. Hence the restriction $f^* E\mathbb{Z}^d$ with f of $E\mathbb{Z}^d$ is a model for $E_{\mathcal{FIN}}(G)$. Hence it suffices to construct for any free $\mathbb{Z}^n$-CW-complex X an appropriate spectral sequence converging to $H_n^G\bigl(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr)$. Since the assignment sending X to $H_n^G\bigl(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr)$ is a $\mathbb{Z}^d$-homology theory in the sense of Definition 12.1 and X is assumed to be a free $\mathbb{Z}^d$-CW-complex, the equivariant Atiyah-Hirzebruch spectral sequence of Theorem 12.48 converges to $H_n^G\bigl(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr)$ and has as E^2-term $H_p\bigl(C_*(X) \otimes_{\mathbb{Z}^d} H_q^G(G/F; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})\bigr)$. Using the induction structure on $H_*^?(-; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ and Lemma 12.12, one can identify the $\mathbb{Z}^d$-modules $H_q^G\bigl(G/F; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\bigr)$ and $L_q^{\langle -\infty \rangle}(\mathbb{Z}F)$.

13.66. Induction with $i \colon H \to G$ and restriction with $f \colon H \to \mathbb{Z}$ induces homomorphisms $i_* \colon G_0(\mathbb{C}H) \to G_0(\mathbb{C}G)$ and $f^* \colon G_0(\mathbb{C}\mathbb{Z}) \to G_0(\mathbb{C}H)$. The class $[\mathbb{C}]$ of the trivial $\mathbb{C}\mathbb{Z}$-module $\mathbb{C}$ is sent under $i_* \circ f^*$ to the class of $\mathbb{C}[G/H]$. Since there exists a short exact sequence $0 \to \mathbb{C}\mathbb{Z} \to \mathbb{C}\mathbb{Z} \to \mathbb{C} \to 0$, we have $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}\mathbb{Z})$.

Chapter 14

14.10. Equip $\mathbb{R}$ with the $G = \mathbb{Z} \times \mathbb{Z}/k$-action where $\mathbb{Z}$ acts by translation and $\mathbb{Z}/k$ acts trivial. There is a G-pushout

$$\begin{tikzcd} \mathbb{Z} \times \{0,1\} \arrow[r, "j"] \arrow[d, "i"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \times [0,1] \arrow[r] & \mathbb{R} \end{tikzcd}$$

where we think of $\mathbb{Z}$ as the G-space $G/(\mathbb{Z}/k)$, the map i is the inclusion and j sends $(n,0)$ to n and $(n,1)$ to $n+1$. Hence $\mathbb{R}$ is a G-CW-complex whose isotropy groups are all finite and whose H-fixed point set is contractible for every finite subgroup $H \subseteq G$. We conclude that $\mathbb{R}$ is a model for $\underline{E}G$. The Mayer-Vietoris sequence associated to the G-pushout looks like

$$\cdots \to K_n^G(\mathbb{Z}) \oplus K_n^G(\mathbb{Z}) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} K_n^G(\mathbb{Z}) \oplus K_n^G(\mathbb{Z}) \to K_n^G(\underline{E}G) = K_n^G(\mathbb{R})$$
$$\to K_{n-1}^G(\mathbb{Z}) \oplus K_{n-1}^G(\mathbb{Z}) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} K_{n-1}^G(\mathbb{Z}) \oplus K_{n-1}^G(\mathbb{Z}) \to \cdots$$

where we identify $K_n^G(\mathbb{Z} \times [0,1]) \cong K_n^G(\mathbb{Z})$ via the isomorphism induced by the projection $\mathbb{Z} \times [0,1] \to \mathbb{Z}$. Since $K_n^G(\mathbb{Z}) \cong K_n^G(G/(\mathbb{Z}/k))$ is $\operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/k)$ for n even and zero for n odd, we conclude for all $n \in \mathbb{Z}$

$$K_n^G(\underline{E}G) \cong \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/k) \cong \mathbb{Z}^k.$$

14.15. We only treat the case $F = \mathbb{C}$, the case $F = \mathbb{R}$ is analogous. Since H and G are torsionfree and satisfy the Baum-Connes Conjecture 14.9, they also satisfy the Baum-Connes Conjecture for torsionfree groups 10.44 by Remark 14.14. Hence it suffices to show that the homomorphism $K_n(Bf)\colon K_n(BH) \to K_n(BG)$ is bijective for all $n \in \mathbb{Z}$. This follows from the Atiyah-Hirzebruch spectral sequence converging to $K_n(BH)$ and $K_n(BG)$, since $H_n(Bf;\mathbb{Z})\colon H_n(BH;\mathbb{Z}) \to H_n(BG;\mathbb{Z})$ is bijective for all $n \in \mathbb{Z}$ by assumption and hence f induces isomorphisms between the E^2-pages.

14.28. Take $G = \mathbb{Z}/2$. Consider the Atiyah-Hirzebruch spectral sequence converging to $K_{p+q}(BG)$ with E^2-term $E^2_{p,q} = H_p(BG; K_q(\{\bullet\}))$. Its E^2-term looks like

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Because of the checkerboard pattern and by a standard edge argument applied to the split injection $K_*(\{\bullet\}) \to K_n(BG)$ coming from the inclusion $\{\bullet\} \to BG$, all differentials are trivial and the E^2-term is the E^∞-term. Hence $K_1(BG)$ is non-trivial. (Actually it is $\mathbb{Z}/2^\infty \cong \mathbb{Z}[1/2]/\mathbb{Z}$). On the other hand $K_1(C_r^*(\mathbb{Z}/2))$ is trivial.

14.32. Since any group is the directed union of its finitely generated subgroups, it suffices to consider finitely generated free abelian groups and finitely generated free groups by Theorem 14.31 (iv). Since a finitely generated abelian group is the direct product of finitely many copies of $\mathbb{Z}$ and of a finite group, Theorem 14.31 (iii) and (vi) imply that it suffices to prove the claim for $\mathbb{Z}$ and any finite group. The Baum-Connes Conjecture 14.11 with coefficients holds obviously for finite groups. It holds for $\mathbb{Z}$ by Theorem 14.31 (vii).

14.33. Let F_g be the surface of genus $g \geq 1$. Let $\overline{F} \to F$ be the covering associated to the epimorphism $\pi_1(F_g) \to H_1(F_g)$. Then $\overline{F_g}$ is a non-compact 2-manifold and hence homotopy equivalent to a 1-dimensional CW-complex. Hence $\pi_1(\overline{F_g})$ is free, $H_1(F_g)$ is a finitely generated free abelian group and we have the exact sequence $1 \to \pi_1(\overline{F_g}) \to G \to H_1(F_g) \to 1$. We conclude from Theorem 14.31 that G satisfies the Baum-Connes Conjecture 14.9. Since F_g itself is a model for BG, we get $K_n(F_g) \cong K_n(C_r^*(G))$. Now an easy application of the Atiyah-Hirzebruch spectral sequence yields the claim.

14.43. Note that $K_0(C_r^*(G)) = R_{\mathbb{C}}(G)$. One easily checks by inspecting the definition of (10.48) of the trace for a finite dimensional complex representation V that $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$ sends the class of $[V]$ to $|G|^{-1} \cdot \dim_{\mathbb{C}}(V)$.

14.55. Suppose that F is S^2. Since M is spin and hence in particular orientable and any orientation preserving selfdiffeomorphism of S^2 is isotopic to the identity, M must be $S^1 \times S^2$ and hence carries a Riemannian metric of positive scalar curvature.

Suppose that F is not S^2. Then F and hence M are aspherical. Hence it suffices to show by Lemma 14.54 that the Baum-Connes Conjecture 14.11 with coefficients holds for $\pi_1(M)$. The Baum-Connes Conjecture 14.11 with coefficients holds for all finitely generated free groups and for $\mathbb{Z}$ by Theorem 14.31 (v). Hence it holds for every free group and every finitely generated abelian group by Theorem 14.31 (iii) and (iv). Let $\overline{F} \to F$ be the covering associated to the epimorphism $\pi_1(F) \to H_1(F)$. Then $\overline{F}$ is a non-compact 2-manifold and hence homotopy equivalent to a 1-dimensional CW-complex. Hence $\pi_1(\overline{F})$ is free. Now apply Theorem 14.31 (ii) to the short exact sequences $1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(S^1) \to 1$ and $1 \to \pi_1(\overline{F}) \to \pi_1(F) \to H_1(F) \to 1$.

Chapter 15

15.3. Let $\alpha \colon H \to G$ be a group homomorphism. Then $\alpha_* E_{\mathcal{C}(H)}(H)$ is a G-CW-complex whose G-isotropy groups are of the shape $\alpha(L)$ for $L \in \mathcal{C}(H)$ and hence all belong to $\mathcal{C}(G)$. This implies that there is up to G-homotopy precisely one G-map $f \colon \alpha_* E_{\mathcal{C}(H)}(H) \to E_{\mathcal{C}(G)}(G)$. The following diagram commutes

$$\begin{tikzcd} \alpha_* E_{\mathcal{C}(H)}(H) \arrow[r, "\alpha_* \mathrm{pr}"] \arrow[d, "f"'] & \alpha_* H/H = G/\alpha(H) \arrow[d, "\mathrm{pr}"] \\ E_{\mathcal{C}(G)}(G) \arrow[r, "\mathrm{pr}"'] & G/G \end{tikzcd}$$

Now apply $\mathcal{H}_*^G$ to this diagram and combine it with the following commutative diagram coming from the induction structure applied to α

$$\begin{tikzcd}
\mathcal{H}^H_n(E_{\mathcal{C}(H)}(H)) \arrow[r, "\mathcal{H}^H_n(\mathrm{pr})"] \arrow[d] & \mathcal{H}^H_n(H/H) \arrow[d] \\
\mathcal{H}^G_n(\alpha_* E_{\mathcal{C}(H)}(H)) \arrow[r, "\mathcal{H}^G_n(\alpha_* \mathrm{pr})"'] & \mathcal{H}^G(\alpha_* H/H)
\end{tikzcd}$$

15.7. The restriction of a G-CW-complex X to K with ϕ is a K-CW-complex $\phi^* X$ by Remark 11.3. For a point $x \in X$ the K-isotropy group K_x of $\phi^* X$ is $\phi^{-1}(G_x)$ where G_x is the G-isotropy group of X. In particular we get $\phi(K_x) = G_x$ and hence every K-isotropy group of $\phi^* X$ belongs to $\phi^* \mathcal{F}$. Consider a subgroup $H \subset K$. Then $(\phi^* X)^H = X^{\phi(H)}$. Now apply these assertions to $X = E_{\mathcal{F}}(G)$.

15.15. Let G be any group. Denote by $\mathrm{pr}\colon G \times \mathbb{Z} \to \mathbb{Z}$ the projection. The Fibered Meta-Isomorphism Conjecture 15.8 predicts that the assembly map

$$\begin{aligned} H_n(\mathrm{pr}; \mathbf{K}_R)\colon H^{G\times\mathbb{Z}}_n(E_{\mathrm{pr}^* \mathcal{FIN}}(G \times \mathbb{Z}); \mathbf{K}_R) & \\ \to H^{G\times\mathbb{Z}}_n(G \times \mathbb{Z}/G \times \mathbb{Z}; \mathbf{K}_R) &= K_n(R[G \times \mathbb{Z}]) \end{aligned}$$

is bijective for all $n \in \mathbb{Z}$. A model for $E_{\mathrm{pr}^* \mathcal{FIN}}(G \times \mathbb{Z})$ is $\mathrm{pr}^* E\mathbb{Z}$. Since $\mathbb{Z}$ acts freely on $E\mathbb{Z}$ and $(E\mathbb{Z})/\mathbb{Z} = S^1$, the left side of the map above can be identified with

$$\begin{aligned} H^{G\times\mathbb{Z}}_n(E_{\mathrm{pr}^* \mathcal{FIN}}(G \times \mathbb{Z}); \mathbf{K}_R) &= H^{G\times\mathbb{Z}}_n(\mathrm{pr}^* E\mathbb{Z}; \mathbf{K}_R) \\ &= H^G_n(G/G \times S^1) \\ &= H_n(G/G; \mathbf{K}_R) \oplus H_{n-1}(G/G; \mathbf{K}_R) \\ &= K_n(RG) \oplus K_{n-1}(RG). \end{aligned}$$

Under this identification the assembly map above becomes the restriction of the Bass-Heller-Swan isomorphism of Theorem 6.16

$$K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG) \xrightarrow{\cong} K_n(RG[t, t^{-1}]).$$

to $K_n(RG) \oplus K_{n-1}(RG)$. Hence the Fibered Meta-Isomorphism Conjecture 15.8 implies that for every group G and $n \in \mathbb{Z}$ we have $NK_n(RG) = 0$.

15.17. This follows from Lemma 15.16 applied to the inclusion $i\colon H \to G$ since $\mathcal{C}(H) = i^* \mathcal{C}(G)$.

15.34. Put $\Gamma = G \times_\phi \mathbb{Z}$. The proof is completely analogous to the one in Example 15.30 but now applied to a 1-dimensional Γ-CW-complex T which is a tree and whose 1-skeleton is obtained from the 0-skeleton by the Γ-pushout

$$\begin{tikzcd}
\Gamma/G \times S^0 \arrow[r, "q"] \arrow[d] & \Gamma/G \arrow[d] \\
\Gamma/G \times D^1 \arrow[r] & T
\end{tikzcd}$$

Here q is the disjoint union of identity $\mathrm{id}\colon \Gamma/G \to \Gamma/G$ and the Γ-map $\Gamma/G \to \Gamma/G$ sending γG to $\gamma t G$ for $t \in \Gamma$ a lift of the generator in $\mathbb{Z}$.

15.40. Put $\pi = \pi_1(X)$. Conjecture 15.39 yields a weak homotopy equivalence $E\pi_+ \wedge_\pi \mathbf{S}(\widetilde{X}) \to \mathbf{S}(X)$ because of the identifications $X = \pi\backslash\widetilde{X}$ and $E_{\mathcal{TR}}(\pi)_+ \wedge_{\mathsf{Or}(\pi)} \mathbf{S}^\pi_{\widetilde{X}} = E\pi_+ \wedge_\pi \mathbf{S}(\widetilde{X})$.

Suppose that $\mathbf{S}$ is of the shape $X \mapsto X_+ \wedge \mathbf{H}_\mathbb{Z}$ for $\mathbf{H}_\mathbb{Z}$ the Eilenberg-spectrum of $\mathbb{Z}$. Recall that the homology theory associated to $\mathbf{H}_\mathbb{Z}$ is singular homology H_n. Then

$$\pi_n\big((E\pi)_+ \wedge_\pi \mathbf{S}(\widetilde{X})\big) \cong H_n(E\pi \times_\pi \widetilde{X});$$
$$\pi_n\big((B\pi)_+ \wedge_\pi \mathbf{S}(\{\bullet\})\big) \cong H_n(B\pi),$$

and $H_n(E\pi \times_\pi \widetilde{X})$ and $H_n(B\pi)$ are not isomorphic in general.

In the sequel we equip $\mathbf{S}(\{\bullet\})$ with the trivial π-action. Suppose that $\widetilde{X}$ is contractible or $\mathbf{S}$ is of the shape $Y \mapsto \mathbf{T}(\Pi(Y))$ for some covariant functor $\mathbf{T}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$. Then the projection $\widetilde{X} \to \{\bullet\}$ induces a π-map $\mathbf{f}\colon \mathbf{S}(\widetilde{X}) \to \mathbf{S}(\{\bullet\})$ such that after forgetting the group action $\mathbf{f}$ is a weak homotopy equivalence. Hence we obtain a weak homotopy equivalence

$$E\pi_+ \wedge_\pi \mathbf{S}(\widetilde{X}) \xrightarrow{(\mathrm{id}_{E\pi})_+ \wedge_\pi \mathbf{f}} E\pi_+ \wedge_\pi \mathbf{S}(\{\bullet\}) = B\pi_+ \wedge \mathbf{S}(\{\bullet\}).$$

If X is simply connected, then $E\pi_+ \wedge_\pi \mathbf{S}(\widetilde{X})$ is $\mathbf{S}(X)$, and $\pi_n(\mathbf{S}(X))$ is in general not isomorphic to $\pi_n(\mathbf{S}(\{\bullet\}))$.

15.46. We conclude from the assumptions that for two groups $H_1, H_2 \in \mathcal{C}$ Conjecture 15.39 holds for $(H_1 \times H_2, \mathcal{C}(H_1 \times H_2))$. Hence Theorem 15.45 (iii) applies. By assumption also Theorem 15.45 (iv) applies. Hence Conjecture 15.39 holds for $(F \wr H, \mathcal{C}(F \wr H))$ if F is a free group and H is a finite group since this is true by assumption for every finitely generated free group F.

If Conjecture 15.39 is true for $(*_{i \in I} G_i, \mathcal{C}(*_{i\in I} G_i))$, it is also true for $(G_i, \mathcal{C}(G_i))$ for every $i \in I$ by Theorem 15.45 (i).

Suppose that Conjecture 15.39 holds for $(G_i, \mathcal{C}(G_i))$ for every $i \in I$. Now we can proceed as in the proof of assertion (vii) of Theorem 13.29 using Theorem 15.45 (ii) to show that Conjecture 15.39 holds for $(*_{i \in I} G_i, \mathcal{C}(*_{i\in I} G_i))$.

15.51. The key ingredient is to construct for a group homomorphism $\phi\colon K \to G$ and a subgroup $H \subseteq K$ a natural weak homotopy equivalence of spaces

$$E\mathcal{G}^K(K/H) \times_{\mathcal{G}^K(K/H)} p^*\phi^*Z \xrightarrow{\simeq} K/H \times_K (EK \times \phi^*Z)$$

where $p\colon \mathcal{G}^K(K/H) \to \mathcal{G}^K(K/K) = I(K)$ is induced by the projection $K/H \to K/K$. Because of the third isomorphism appearing in [265, Lemma 1.9], it suffices to construct a map

$$u\colon p_*E\mathcal{G}^K(K/H) \times_K \phi^*Z \xrightarrow{\simeq} K/H \times_K (EK \times \phi^*Z)$$

where here and in the sequel we consider a K-space as a $\mathcal{G}^K(K/K) = I(K)$-space and vice versa in the obvious way. Since $(K/H \times EK) \times_K \phi^*Z = K/H \times_K (EK \times \phi^*Z)$, it suffices to construct for every K-set S a natural K-homotopy equivalence

$$v\colon p_*E\mathcal{G}^K(S) \xrightarrow{\simeq} S \times EK,$$

since we then can define $u = v \times_K \mathrm{id}_{\phi^*Z}$ for $S = K/H$. Unravelling the definition we see that the source of v is given by

$$p_*E\mathcal{G}^K(S) = \coprod_{s\in S} K \times E\mathcal{G}^K(S)(s)/\sim$$

for the equivalence relation $\sim$ given by

$$(k,x) \sim (k(k')^{-1}, E\mathcal{G}^K(S)(k'\colon s \to k's)(u)).$$

Define a K-map $\coprod_{s\in S} K \times E\mathcal{G}^K(S)(s) \to S \times E\mathcal{G}^K(K/K)$ by sending the element (k,x) in the summand $K \times E\mathcal{G}^K(S)(s)$ belonging to $s \in S$ to the element $\bigl(ks, E\mathcal{K}^K(k \cdot K/K \to K/K)(u)\bigr)$. On easily checks that it is compatible with $\sim$ and induces the desired K-map

$$v\colon p_*E\mathcal{G}^K(S) = \coprod_{s\in S} K \times E\mathcal{G}^K(S)(s)/\sim \;\to\; S \times EK.$$

It remains to show that v is a K-homotopy equivalence. Since the source and target of v are free K-CW-complexes, it suffices to show that v is a homotopy equivalence (after forgetting the K-action). We obtain a (non-equivariant) homeomorphism

$$\coprod_{s\in S} E\mathcal{G}^K(S)(s) \xrightarrow{\cong} \coprod_{s\in S} K \times E\mathcal{G}^K(S)(s)/\sim$$

by sending the element $x \in E\mathcal{G}^K(S)(s)$ belong to the summand of $s \in S$ to the element represented by $(1,s)$. Hence the both the source and the target of v have the property that each path component is contractible. Since v is a bijection on the path component, it is a homotopy equivalence.

15.60. We get $\pi_n(\mathbf{A}(\{\bullet\})) \cong K_n(\mathbb{Z}) \cong 0$ for $n \leq -1$ and $\pi_0(\mathbf{A}(\{\bullet\})) \cong K_0(\mathbb{Z}) \cong \mathbb{Z}$ from Example 2.4, Theorem 3.17, and Theorem 7.18 (i). Now apply the Atiyah-Hirzebruch spectral sequence to $H_n(BG;\mathbf{A}(\{\bullet\}))$ for $n \leq 0$.

15.68. This follows from the p-chain spectral sequence, see Subsection 12.6.2, and Theorem 15.67 by an inspection of the resulting long exact sequence. See also [660, Proposition 1.2].

15.98. Consider the commutative diagram appearing in Remark 15.96. The two vertical arrows are bijective as explained in Remark 15.96. The upper horizontal arrow is bijective by assumption. The lower horizontal arrow is bijective by Theorem 15.95. Hence the right vertical arrow is bijective.

15.99. Since $R[G \times \mathbb{Z}] = R[\mathbb{Z}][G]$ and $R[\mathbb{Z}]$ is regular, Conjecture 15.97 is true for G and $G \times \mathbb{Z}$. We obtain from the Bass-Heller Swan decompositions for K-theory, see Theorem 6.16, and homotopy K-theory, see Theorem 15.74, the commutative diagram with isomorphisms as horizontal arrows

$$\begin{tikzcd}
K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG) \arrow[r, "\cong"] \arrow[d, "\begin{pmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \end{pmatrix}"'] & K_n(R[G \times \mathbb{Z}]) \arrow[d, "h"] \\
KH_n(RG) \oplus KH_{n-1}(RG) \arrow[r, "\cong"] & KH_n(R[G \times \mathbb{Z}])
\end{tikzcd}$$

where the maps denoted by h are induced by the canoncial map $\mathbf{K} \to \mathbf{KH}$ and bijective.

15.102. (P) $\implies$ (I): Let $d\colon G \to \prod_{i \in I} G$ be the diagonal embedding. Then $(\prod_{i \in I} G, \prod_{i \in I} \mathcal{F}_i)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 because of (P). Hence $(G, d^* \prod_{i \in I} \mathcal{F}_i)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 by Lemma 15.16. One easily checks $d^* \prod_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i$.

(I) $\implies$ (P): Consider the projection $\mathrm{pr}_j\colon \prod_{i \in I} G_i \to G_j$ for $j \in I$. We conclude from Lemma 15.16 that $(\prod_{i \in I} G_i, \mathrm{pr}_j^* \mathcal{F}_j)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every $j \in I$. Hence $(\prod_{i \in I} G_i, \bigcap_{j \in I} \mathrm{pr}_j^* \mathcal{F}_j)$ satisfies the Fibered Meta Isomorphism Conjecture 15.8 because of (I). One easily checks $\prod_{i \in I} \mathcal{F}_i = \bigcap_{j \in I} \mathrm{pr}_j^* \mathcal{F}_j$.

Chapter 16

16.2. We know that L/K is a smooth manifold, which is diffeomorphic to $\mathbb{R}^{\dim(L/K)}$, and, equipped with the obvious left G action, it is a model for the classifying space for proper G-actions, see Theorem 11.24. Hence $G\backslash L/K$ is an aspherical closed smooth manifold of dimension ≥ 5. Since G satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (id), the claim follows from Theorem 9.168.

16.3. Let G be a group. It is the disjoint union of its finitely generated subgroups. Hence by Theorem 16.1 (iie) the Full Farrell-Jones Conjecture 13.27 holds for all groups if and only if it holds for all finitely generated groups. Any finitely generated group can be written as a directed colimit of finitely presented groups. Hence by Theorem 16.1 (iie) the Full Farrell-Jones Conjecture 13.27 holds for all finitely generated groups if and only if holds for all finitely presented groups. Now the claim follow from Theorem 16.1 (iia) since every finitely presented group is a subgroup of U.

16.4. We obtain an embedding of rings $R \to \mathrm{end}_S(R)$ by sending $r \in R$ to the S-homomorphism of right S-modules $l_r \colon R \to R,\ r' \mapsto rr'$. Since R is finitely generated free as right S-module, we obtain for some natural number k an identification of rings $\mathrm{end}_S(R) = \mathrm{M}_k(S)$. The inclusion of rings $R \to \mathrm{M}_k(S)$ yields an inclusion of rings $\mathrm{M}_n(R) \to \mathrm{M}_n(\mathrm{M}_k(S)) = \mathrm{M}_{kn}(S)$. By passing to units we obtain an inclusion of groups $\mathrm{GL}_n(R) \to \mathrm{GL}_{kn}(S)$. Now the claim follows from Theorem 16.1 (iia).

16.6. This follows from the commutative diagram

$$\begin{tikzcd} H_n(BG;\mathbf{K}(R)) \arrow[r] \arrow[d, "\cong"] & K_n(RG) \arrow[d] \\ H_n(BG;\mathbf{KH}(R)) \arrow[r, "\cong"] & KH_n(RG) \end{tikzcd}$$

whose horizontal arrows are assembly maps and whose vertical arrows are change of theory maps. Moreover, the left vertical arrow is bijective since $K_n(R) \to KH_n(R)$ is bijective for all $n \in \mathbb{Z}$ and all regular rings R, and the lower horizontal arrow is bijective because of Theorem 16.5 (i).

16.8. This follows directly from Theorem 16.7 (iic).

16.9. This follows from Theorem 2.81, Lemma 10.51, Lemma 10.53, Theorem 13.62, Theorem 16.1, and Theorem 16.7.

16.16. By Lemma 15.23 (ii) it suffices to prove the injectivity for any finitely generated subgroup of G since G is the directed union of its finitely generated subgroups. The relevant equivariant homology theories are (strongly) continuous by [70, Lemma 6.2].

16.22. We have $G/[G,G] = H_1(G) \cong \mathbb{Z}$, and the projection $\mathrm{pr}\colon G \to H_1(G)$ induces an isomorphisms on the group homology $H_n(G) \to H_n(G/[G,G])$ for all $n \in \mathbb{Z}$. This follows from Alexander-Lefschetz duality. The Atiyah-Hirzebruch spectral sequence implies that $H_n(\mathrm{pr};\mathbf{K}(R))\colon H_n(G;\mathbf{K}(R)) \to H_n(G/[G,G];\mathbf{K}(R))$ is bijective for all $n \in \mathbb{Z}$. Since G satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1 (ie) and hence the K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings 6.53 by Theorem 13.62 ii, the map $K_n(RG) \to K_n(R[G/[G,G]])$ induced by pr is a bijection. Since $G/[G,G] \cong \mathbb{Z}$, we get $K_n(R[G/[G,G]]) \cong K_n(R) \oplus K_{n-1}(R)$ from the Bass-Heller-Swan decomposition for algebraic K-theory, see Theorem 6.16.

The L-theory case is treated analogously, but not replacing Theorem 6.16 by (9.109).

16.25. Show by induction over $i = 1, 2, \ldots, d$ that G_i is torsionfree and satisfies the Baum-Connes Conjecture 14.11 with coefficients using Theorem 16.7 (if) and (iic).

16.27. We want to apply Theorem 16.1 (iic). So we need to show that K satisfies the Full Farrell-Jones Conjecture 13.27 and that for any any extension $1 \to K \to G \to \mathbb{Z} \to 1$ the group G satisfies the Full Farrell-Jones Conjecture 13.27. Since K is either the fundamental group of a closed surface or a free group, see [991, Lemma 2.1], both K and G are strongly poly-surface groups or normally poly-free groups and hence satisfy the Full Farrell-Jones Conjecture 13.27 by Theorem 16.24 or Theorem 16.26.

16.35. We only treat the K-theory case, the argument for L-theory is completely analogous. Let G be a group with a finite model for BG and let R be a regular ring. Choose M, i and r as they appear in Theorem 16.34. We obtain a commutative diagram

$$\begin{tikzcd}
H_n(BG;\mathbf{K}_R) \arrow[r] \arrow[d, "H_n(i;\mathbf{K}(R))"'] & K_n(RG) \arrow[d, "i_*"] \\
H_n(M;\mathbf{K}_R) \arrow[r] \arrow[d, "H_n(p;\mathbf{K}(R))"'] & K_n(R[\pi_1(M)]) \arrow[d, "p_*"] \\
H_n(BG;\mathbf{K}_R) \arrow[r] & K_n(RG)
\end{tikzcd}$$

The right vertical arrows are assembly maps. The composite of the two vertical arrows of the left column and the right column are the identity. Since the middle horizontal arrow is bijective, the same is true for the upper horizontal arrow.

16.39. We conclude from Theorem 3.115 that for a natural number n the vanishing of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Wh}(\mathbb{Z}/n) = 0$ implies $n = 1, 2, 3, 4, 6$. Now apply Theorem 16.38.

Chapter 17

17.2. We get from Theorem 10.79 (i)

$$K_n(C_r^*(\mathbb{Z}/m, \mathbb{C})) \cong \begin{cases} \mathbb{Z}^m & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Since $K_n(C_r^*(\mathbb{Z}/2, \mathbb{C})) \to K_n(C_r^*(\mathbb{Z}/6, \mathbb{C}))$ is split injective, the computation for $K_n(C_r^*(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}))$ follows.

We have $C_r^*(\mathbb{Z}/2, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$, $C_r^*(\mathbb{Z}/3, \mathbb{R}) \cong \mathbb{R} \times \mathbb{C}$, and $C_r^*(\mathbb{Z}/6, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}$. We get from Theorem 10.79 (ii)

$$\begin{aligned} KO_n(C_r^*(\mathbb{Z}/2, \mathbb{R})) &\cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R}); \\ KO_n(C_r^*(\mathbb{Z}/4, \mathbb{R})) &\cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R}) \oplus K_n(\mathbb{C}); \\ KO_n(C_r^*(\mathbb{Z}/6, \mathbb{R})) &\cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R}) \oplus K_n(\mathbb{C}) \oplus K_n(\mathbb{C}). \end{aligned}$$

Since $KO_n(C_r^*(\mathbb{Z}/2, \mathbb{R})) \to K_n(C_r^*(\mathbb{Z}/6, \mathbb{R}))$ is split injective, the computation for $KO_n(C_r^*(\mathrm{SL}_2(\mathbb{Z}), \mathbb{R}))$ follows by inspecting the values of $KO_n(\mathbb{R})$ and $K_n(\mathbb{C})$.

17.3. The group G contains a finitely generated free group. Hence it satisfies the Full Farrell-Jones Conjecture 13.27 by Theorem 16.1. It satisfies the Baum-Connes 14.11 with coefficients by Theorem 16.7. We conclude from Theorem 13.48 that the assembly maps

$$\begin{aligned} H_0^G(\underline{E}G; \mathbf{K}_{\mathbb{C}}) &\xrightarrow{\cong} K_0(\mathbb{C}G); \\ K_n^G(\underline{E}G) &\xrightarrow{\cong} K_n(C_*^r(G)), \end{aligned}$$

are isomorphisms. Since for a finite group H we have $K_0(\mathbb{C}H) = K_0(C_r^*(H)) = R_{\mathbb{C}}(H)$ and $K_{-1}(\mathbb{C}H) = K_1(C_r^*(H)) = \{0\}$, we get from Example 15.30 exact sequences

$$R_{\mathbb{C}}(C) \xrightarrow{i_* \oplus i_*} R_{\mathbb{C}}(D_8) \oplus R_{\mathbb{C}}(D_8) \to K_0(\mathbb{C}G) \to 0,$$

and

$$0 \to K_1(C_r^*(G)) \to R_{\mathbb{C}}(C) \xrightarrow{i_* \oplus i_*} R_{\mathbb{C}}(D_8) \oplus R_{\mathbb{C}}(D_8) \to K_0(\mathbb{C}G) \to 0$$

where $i\colon C \to D_8$ is the inclusion. The group C has two irreducible complex representations, the trivial 1-dimensional representation $\mathbb{C}$ and the non-trivial

1-dimensional representation $\mathbb{C}^-$. The group D_8 has four 1-dimensional irreducible representations and one 2-dimensional irreducible representation. The homomorphism $i_*\colon R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(D_8)$ sends the class of $\mathbb{C}$ to the class of the sum of the four 1-dimensional irreducible representations and $\mathbb{C}^-$ to the sum of two copies of the 2-dimensional irreducible representation, see [892, Subsections 3.3, 5.1 and 5.3]. Hence $i_*\colon R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(D_8)$ looks like

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^{10}$$

We conclude that i_* is injective and its cokernel is isomorphic to $\mathbb{Z}^8 \oplus \mathbb{Z}/2$.

17.5. This follows from Theorem 12.79 and Theorem 13.33.

17.7. This follows from Theorem 4.22 (v) and Example 17.6.

17.9. Since G is elementary amenable, it satisfies the L-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2 see [458, Theorem 5.2.1]. So we can apply Theorem 17.8.

For every non-trivial finite cyclic subgroup $C \subseteq G$ we have $C \subseteq C_G C \subseteq \bigoplus_{\mathbb{Z}} F$ and hence $H_p(C_G C;\mathbb{Q}) = 0$ for $p \neq 0$ and $H_0(C_G C;\mathbb{Q}) \cong \mathbb{Q}$. Hence we get from Theorem 17.8 for all $n \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{p+q=n} H_p(G;\mathbb{Q}) \otimes_{\mathbb{Z}} L_q(\mathbb{Z}) \oplus \bigoplus_{(C)\in J, C\neq\{1\}} \mathbb{Q} \otimes_{\mathbb{Q}[W_G C]} \Theta_C \cdot \big(\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}C)\big) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G).$$

We get $L_n^{\langle j \rangle}(\mathbb{Z}C) = 0$ for odd n from Theorem 9.196 (iv) since F and hence C has odd order. From the Lyndon-Serre spectral sequence applied to $\bigoplus_{\mathbb{Z}} F \to G \to \mathbb{Z}$ we conclude $H_*(G;\mathbb{Q}) \cong H_*(\mathbb{Z};\mathbb{Q})$. Hence we obtain for odd n an isomorphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}) \oplus \mathbb{Q} \otimes_{\mathbb{Z}} L_{n-1}^{\langle j \rangle}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G).$$

This implies

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Q} & n \equiv 1 \mod 4; \\ \{0\} & n \equiv 3 \mod (4). \end{cases}$$

17.15. This follows from Theorem 11.36.

17.17. This follows from Theorem 17.16 by the following facts. The group G satisfies conditions (M) and (NM) and has up to conjugacy precisely one maximal finite subgroup, see Subsection 11.6.12. It is a hyperbolic group and satisfies the Full Farrell-Jones Conjecture, see Subsection 16.8.15. Since m is odd, all virtually cyclic subgroups of G are of type I. We conclude from Section 17.3 that the infinite virtually cyclic subgroups of type I are orientable.

17.22. Since $H_1(G)$ is the abelianization of G, we obtain a short exact sequence $\mathbb{Z} \xrightarrow{D} \bigoplus_{i=1}^n \mathbb{Z} \to H_1(G) \to 0$ where D sends $x \in \mathbb{Z}$ to $(d_1x, d_2x, \ldots, d_nx)$.

17.23. One easily checks that G is torsionfree and the word $s_1s_2s_1s_2^{-1}s_1^{-2} \in F$ is a commutator. Put $R = \mathbb{C}[\mathbb{Z}/m]$. Then $R \cong \prod_{i=1}^m \mathbb{C}$ is semisimple and in particular regular and we obtain from Lemma 17.21 (i) an isomorphism for $n \in \mathbb{Z}$

$$\begin{aligned} &K_n(\mathbb{C}[\mathbb{Z}/m \times G]) \cong K_n(\mathbb{C}[\mathbb{Z}/m][G]) \\ &\quad \cong K_n(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-1}(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-1}(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-2}(\mathbb{C}[\mathbb{Z}/m]). \end{aligned}$$

We get from Example 2.4, Lemma 2.12, Theorem 3.6, Lemma 3.9, and Theorem 4.7

$$K_n(\mathbb{C}[\mathbb{Z}/m]) = \begin{cases} \mathbb{C}[\mathbb{Z}/m]^\times & n = 1; \\ \mathbb{Z}^m & n = 0; \\ 0 & n \leq -1. \end{cases}$$

17.24. The group G is solvable and torsionfree and hence satisfies Conjecture 3.110, Conjecture 4.18, and the Farrell-Jones Conjecture 9.114 for torsionfree groups for L-theory. We conclude from Theorem 9.106 that $L_n^s(\mathbb{Z}[G]) = L_n^{\langle -\infty \rangle}(\mathbb{Z}[G])$. The group G is a one-relator groups with presentation $\langle s_1, s_2 \mid s_1s_2s_1^{-1}s_2 \rangle$. The word $s_1s_2s_1^{-1}s_2 \in F$ is not a commutator. Hence we get from Lemma 17.21 (ii) a short exact sequence

$$\begin{aligned} 0 \to H_1(BG) \otimes_{\mathbb{Z}} L_{n-1}^{\langle -\infty \rangle}(\mathbb{Z}) \to H_n(BG, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \\ \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(BG); L_{n-2}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0, \end{aligned}$$

and an isomorphism

$$L_n^s(\mathbb{Z}G) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}) \oplus H_n(BG, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})).$$

We have $H_1(BG) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$ and $L_n^{\langle -\infty \rangle}(\mathbb{Z}) \cong \mathbb{Z}, 0, \mathbb{Z}/2, 0$ for $n \equiv 0, 1, 2, 3 \mod 4$. Hence we get

$$L_n^s(\mathbb{Z}G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0 \mod 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 1 \mod 4; \\ \mathbb{Z}/2 & n \equiv 2 \mod 4; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod 4. \end{cases}$$

17.31. This follows from Theorem 3.115, Theorem 3.116 (iv) and Theorem 17.30 (ii).

17.36. This follows from Theorem 9.106 and the Shaneson splitting, see Theorem 9.108, if we can construct an orientable closed aspherical smooth 3-manifold N such that $L_i^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(M)])$ contains p-torsion for at least one $i \in \mathbb{Z}$. Namely, then we can $M = N \times T^{n-3}$.

If $p = 2$, take $N = T^3$. If p is odd, this follows from Example 17.35.

17.40. The $\mathbb{Z}/3$-action given by ϕ on $\mathbb{Z}^2$ is free outside the origin. Now apply Theorem 17.38 (iii) together with (17.39).

17.46. Note that G is the right angled Artin group associated to the simplicial graph X consisting of 3 vertices e_0,e_1, and e_2 and to edges $[e_0, e_1]$ and $[e_1, e_2]$. Note that $\sigma = X$ in this case. Hence we get $r_{-1} = 1$, $r_0 = 3$ and $r_1 = 2$. We get from (17.43) and Theorem 17.45

$$H_n(G) \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}^3 & n = 1; \\ \mathbb{Z}^2 & n = 2; \\ \{0\} & n \geq 3, \end{cases}$$

$$K_n(C_r^*(G)) \cong \mathbb{Z}^3 \text{ for } n \in \mathbb{Z},$$

and

$$KO_n(C_r^*(G,\mathbb{R})) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (8); \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2 & n \equiv 1 \mod (8); \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^4 & n \equiv 2 \mod (8); \\ (\mathbb{Z}/2)^5 & n \equiv 3 \mod (8); \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & n \equiv 4 \mod (8); \\ \mathbb{Z}^3 & n \equiv 5 \mod (8); \\ \mathbb{Z}^2 & n \equiv 6 \mod (8); \\ \{0\} & n \equiv 7 \mod (8). \end{cases}$$

17.48. We can arrange without changing the isomorphism type of $(\mathbb{Z}/2)^3 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$ that the inclusions of $\mathbb{Z}/2$ into $(\mathbb{Z}/2)^2$ and $(\mathbb{Z}/2)^3$ are given by sending x to $(x,0)$ and $(x,0,0)$. Hence G is isomorphic to the right-angled Coxeter group associated to the simplicial graph with vertices e_0, e_1, e_2, e_3 and edges $[e_0,e_1]$, $[e_0,e_2]$, $[e_1,e_2]$, and $[e_2,e_3]$. Then the associated flag complex Σ is obtained from X by adding the 2-simplex $[e_0,e_1,e_2]$. Hence the number of the simplices of X is $r = 10$. Now apply Theorem 17.47.

17.50. Recall from the proof of Theorem 17.49 that M is aspherical. In particular π is torsionfree and we get for any abelian group A using Poincaré duality and the Universal Coefficient Theorem

$$\begin{aligned} H_n(B\pi;A) &\cong H_n(M;A); \quad \text{for } n \geq 0; \\ H_1(B\pi;A) &\cong \pi/[\pi,\pi] \otimes_{\mathbb{Z}} A; \\ H_2(B\pi;A) &\cong \hom_{\mathbb{Z}}(\pi,A); \\ H_3(B\pi;A) &\cong A; \\ H_n(B\pi;A) &\cong \{0\} \quad \text{for } n \notin \{0,1,2,3\}. \end{aligned}$$

The independence of $L_n^{\langle i \rangle}(\mathbb{Z}\pi)$ from the decoration follows from Theorem 9.106 and from Conjectures 3.110 and 4.18, which hold for π by Theorem 13.62 (xii). We obtain from Theorem 17.49 (ii) an isomorphism

$$H_n(B\pi;\mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}\pi).$$

Next we apply the Atyiah-Hirzebruch spectral sequence to $H_n(B\pi;\mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ Recall that $L_n(\mathbb{Z})$ is $\mathbb{Z}, \{0\}, \mathbb{Z}/2, \{0\}$ for $n \equiv 0,1,2,3 \mod 4$, see Theorem 9.196 (i). Since the composite $L_n(\mathbb{Z}) \to L_n(\mathbb{Z}\pi) \to L_n(\mathbb{Z})$ is the identity, all differentials in the Atiyah-Hirzebruch spectral sequence are trivial. Hence we obtain isomorphisms

$$\begin{aligned} L_0(\mathbb{Z}\pi) &\cong H_0(B\pi;L_0(\mathbb{Z})) \oplus H_2(B\pi;L_2(\mathbb{Z}/2)); \\ L_2(\mathbb{Z}\pi) &\cong H_0(B\pi;L_2(\mathbb{Z})) \oplus H_2(B\pi;L_0(\mathbb{Z}/2)), \end{aligned}$$

and two short exact sequences

$$0 \to H_1(B\pi; L_2(\mathbb{Z})) \to L_3(\mathbb{Z}\pi) \to H_3(B\pi; L_0(\mathbb{Z})) \to 0;$$

$$0 \to H_1(B\pi; L_0(\mathbb{Z})) \to L_1(\mathbb{Z}\pi) \to H_3(B\pi; L_2(\mathbb{Z})) \to 0.$$

The first one splits because of $H_3(B\pi; L_0(\mathbb{Z})) \cong \mathbb{Z}$. In order to show that the second one splits it suffices to show that it splits after localization at 2 since $H_3(B\pi; L_2(\mathbb{Z})) \cong \mathbb{Z}/2$. This follows from Lemma 9.116 (i).

Chapter 18

18.9. This follows from the following facts. We have $\emptyset/G = \emptyset$. If $f\colon X \to Y$ is a G-homotopy equivalence, $f/G\colon X/G \to Y/G$ is a homotopy equivalence. If the G-CW-complex X is the union of G-CW-subcomplexes X_1 and X_2 with intersection X_0, then CW-complex X/G is the union of CW-subcomplexes X_1/G and X_2/G with intersection X_0/G. If $\{X_i \mid i \in I\}$ is a collection of G-CW-complexes, then the canonical map $(\coprod_{i\in I} X_i)/G \to \coprod_{i \in I} X_i/G$ is a homeomorphism.

18.12. Suppose that $\mathbf{E}$ is weakly $\mathcal{F}$-excisive. Theorem 18.11 (ii) and (iv) imply that the assignment sending (X, A) to $\operatorname{coker}\bigl(\pi_n(\emptyset_+) \to \pi_n(\mathbf{E}(X/A))\bigr)$ is a G-homology theory.

Now suppose that the assignment sending (X, A) to $\operatorname{coker}\bigl(\pi_n(\mathbf{E}(\emptyset_+)) \to \pi_n(\mathbf{E}(X/A))\bigr)$ is a G-homology theory. Then we get from Theorem 18.11 (ii) and (iv) and from Lemma 12.6 applied to $\mathbf{E}^{\%} \to \mathbf{E}$ that $\mathbf{E}$ is weakly $\mathcal{F}$-excisive.

18.13. We use induction over the dimension $d = \dim(X)$ of X. The induction beginning $d = 0$ follows from the fact that X is a finite union of homogenous spaces $\coprod_{i=1}^k G/H_i$ and hence we get an isomorphisms $\bigoplus_{i=1}^k \pi_n(\mathbf{E}(G/H_i)) \xrightarrow{\cong} \pi_n(\mathbf{E}(X))$. The induction step from $(d-1)$ to $d \geq 1$ is done as follows. Choose a G-pushout

$$\begin{tikzcd} \coprod_{i=1}^l G/H_i \times S^{d-1} \arrow[r] \arrow[d] & X_{d-1} \arrow[d] \\ \coprod_{i=1}^l G/H_i \times D^d \arrow[r] & X. \end{tikzcd}$$

Because of the associated Mayer-Vietoris sequence, it suffices to show for all $n \in \mathbb{Z}$ and $i \in \{1, 2, \ldots, l\}$ that $\pi_n(G/H_i \times S^{d-1})$, $\pi_n(G/H_i \times D^d)$, and $\pi_n(X_{d-1})$ are finitely generated. This follows from the induction hypothesis and the fact that $\pi_n(G/H_i \times D^d) \cong \pi_n(G/H_i)$ holds by weak G-homotopy invariance.

18.17. Since the projection $E_{\mathcal{F}}(G) \to G/G$ induces a homotopy equivalence $EG \times_G E_{\mathcal{F}}(G) \to EG \times_G G/G$, the map induced by the projection $E_{\mathcal{F}}(G) \to G/G$ induces a weak homotopy equivalence $\mathbf{E}(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathbf{E}(G/G)$. Obviously we get a weak equivalence of $\mathsf{Or}(G)$ spectra from $\mathbf{K}_R \circ \mathcal{G}^G$ for $\mathcal{G}^G \colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}$ defined in (12.29) to $\mathbf{E}|_{\mathsf{Or}(G)}$ since there is an equivalence of groupoids $\mathcal{G}^G(G/H) \xrightarrow{\simeq} \Pi(EG \times_G G/H)$ which is natural in G/H. Now apply Lemma 12.6 and Corollary 18.16.

Chapter 19

19.7. One easily checks that F and F^f are compatible with the structures of an additive category, is fully faithful, and every object of $RG\text{-}\mathsf{MOD}_{\mathrm{fgf}}$ is isomorphic to some object in the image of F.

19.12. Define a functor of additive categories $F \colon \mathcal{T}(X) \to \mathsf{GM}^{\{1\}}(X)$ by sending an object $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$ to the object $F(M) = \{F(M)_x \mid x \in X\}$ given by $F(M)_x = \bigoplus_{s \in \mathbb{N}} M_{(x,s)}$ and a morphism $f = \{f_{(x,s),(y,t)} \mid (x,s),(y,t) \in X \times \mathbb{N}\}$ from $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$ to $N = \{N_{(y,t)} \mid (y,t) \in X \times \mathbb{N}\}$ to the morphism $F(f) \colon F(M) \to F(N)$ which is defined for $x, y \in X$ by the morphism $\bigoplus_{s \in \mathbb{N}} M_{(x,s)} \to \bigoplus_{t \in \mathbb{N}} N_{(y,t)}$ given by the collection of R-homomorphisms $\{f_{(x,s),(y,s)} \colon M_{(x,s)} \to N_{(y,t)} \mid s,t \in \mathbb{N}\}$. Obviously $F \circ I$ is the identity on $\mathsf{GM}^{\{1\}}(X)$. It remains to show that $I \circ F$ is naturally equivalent to the identity on $\mathcal{T}(X)$. For this purpose we have to construct for every object M in $\mathcal{T}(X)$ a natural isomorphism $u \colon I \circ F(M) \xrightarrow{\cong} M$ in $\mathcal{T}(X)$. For (x,s) and (y,t) in $X \times \mathbb{N}$ we define $u_{(x,s),(y,t)} \colon I \circ F(M)_{(x,s)} \to M_{(y,t)}$ be the projection $\bigoplus_{s \in \mathbb{N}} M_{(x,s)} \to M_{(y,t)}$ to the summand belonging to (y,t) if $s = 0$ and $x = y$ and to be zero otherwise. For (x,s) and (y,t) in $X \times \mathbb{N}$ we define $(u^{-1})_{(x,s),(y,t)} \colon M_{(x,s)} \to (I \circ F(M))_{(y,t)}$ to be the inclusion $M_{(x,s)} \to \bigoplus_{t \in \mathbb{N}} M_{(y,t)}$ of the summand belonging to s if $x = y$ and to be zero otherwise.

19.13. The hyperbolic metric is given by

$$\begin{aligned} &d_{\mathrm{hyp}}((x_1,y_1),(x_2,y_2)) \\ &\qquad = 2 \cdot \ln\left(\frac{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}+\sqrt{(x_2-x_1)^2+(y_2+y_1)^2}}{2\sqrt{y_1y_2}}\right). \end{aligned}$$

Hence $\gamma_{(x_k,y_k)} \colon [0,\infty) \to \mathbb{H}^2$ sends t to $(x_k, \exp(t) \cdot y_k)$ since for $t, s \in \mathbb{R}$ we get $d_{\mathrm{hyp}}(\gamma_{(x_k,y_k)}(t), \gamma_{(x_k,y_k)}(s)) = |t-s|$. We compute for $t \geq 0$

$$\begin{aligned}
&\lim_{t\to\infty} d_{\mathrm{hyp}}(\gamma_{(x_1,y_1)}(t),\gamma_{x_2,y_2}(t))\\
&= \lim_{t\to\infty} d_{\mathrm{hyp}}(\gamma_{(x_1,\exp(t)\cdot y_1)}(t),\gamma_{x_2,\exp(t)\cdot y_2}(t))\\
&= \lim_{t\to\infty} 2\cdot\ln\Bigg(\frac{\sqrt{(x_2-x_1)^2+(\exp(t)\cdot y_2-\exp(t)\cdot y_1)^2}}{2\sqrt{\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}}\\
&\qquad\qquad + \frac{\sqrt{(x_2-x_1)^2+(\exp(t)\cdot y_2+\exp(t)\cdot y_1)^2}}{2\sqrt{\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}}\Bigg)\\
&= \lim_{t\to\infty} 2\cdot\ln\Bigg(\sqrt{\frac{(x_2-x_1)^2}{4\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}+\frac{(\exp(t)\cdot y_2-\exp(t)\cdot y_1)^2}{4\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}}\\
&\qquad\qquad +\sqrt{\frac{(x_2-x_1)^2}{4\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}+\frac{(\exp(t)\cdot y_2+\exp(t)\cdot y_1)^2}{4\exp(t)\cdot y_1\cdot\exp(t)\cdot y_2}}\Bigg)\\
&= \lim_{t\to\infty} 2\cdot\ln\Bigg(\sqrt{\frac{(x_2-x_1)^2}{4y_1y_2\cdot\exp(2t)}+\frac{(y_2-y_1)^2}{4y_1y_2}}\\
&\qquad\qquad +\sqrt{\frac{(x_2-x_1)^2}{4y_1y_2\cdot\exp(2t)}+\frac{(y_2+y_1)^2}{4y_1y_2}}\Bigg)\\
&= 2\cdot\ln\left(\sqrt{\frac{(y_2-y_1)^2}{4y_1y_2}}+\sqrt{\frac{(y_2+y_1)^2}{4y_1y_2}}\right)\\
&= 2\cdot\ln\left(\frac{|y_2-y_1|+(y_2+y_1)}{\sqrt{4y_1y_2}}\right)\\
&= 2\cdot\ln\left(\frac{2\max\{y_1,y_2\}}{\sqrt{4y_1y_2}}\right)\\
&= 2\cdot\left(\ln(\max\{y_1,y_2\})-\frac{\ln(y_1)}{2}-\frac{\ln(y_2)}{2}\right)\\
&= |\ln(y_1)-\ln(y_2)|.
\end{aligned}$$

19.15. Let $t\in[T-r',T+r']$. We have $|\tau|\le d(x_1,x_2)\le 2\beta$. From $T-r'=r''>2\beta$ we conclude $t,t+\tau>0$. If $t\ge d(x,x_1)$, then $t+\tau\ge d(x,x_2)$ holds and hence we get $c_{x_1,x}(t)=x=c_{x_2,x}(t+\tau)$ so that the assertion follows in this case. Hence we can assume without loss of generality $0<t<d(x,x_1)$. One easily checks that $0<t+\tau<d(x,x_2)$ and $d(c_{x_1,x}(t),x)=d(x,x_1)-t=d(x,x_2)-(t+\tau)=d(c_{x_2,x}(t+\tau),x)$ hold. We can suppose without loss of generality $d(x,x_1)\le d(x,x_2)$, the proof in the other case is analogous interchanging the role of x_1 and x_2. We have $d(x,x_1)=d(x,c_{x_2,x}(d(x,x_2)-d(x,x_1)))$. The Intercept Theorem implies

$$d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) = \frac{d(x_1, c_{x_2,x}(d(x,x_2) - d(x,x_1))) \cdot (d(x,x_1) - t)}{d(x,x_1)}.$$

We have $d(x_2, c_{x_2,x}(d(x,x_2) - d(x,x_1))) = d(x,x_1)$. Hence the triangle inequality implies

$$\begin{aligned} d(x_1, c_{x_2,x}(d(x,x_2) - d(x,x_1))) &\leq d(x_1,x_2) - d(x_2, c_{x_2,x}(d(x,x_2) - d(x,x_1))) \\ &= d(x_1,x_2) - d(x,x_1) \\ &\leq d(x_1,x_2) - (d(x,x_2) - d(x_1,x_2)) \\ &= 2d(x_1,x_2) - d(x,x_2) \\ &\leq 2d(x_1,x_2) \\ &\leq 4\beta. \end{aligned}$$

Hence we get

$$d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \leq \frac{4 \cdot \beta \cdot (d(x,x_1) - t)}{d(x,x_1)}.$$

Since we have $r'' = T - r' \leq t < d(x,x_1)$ and

$$\begin{aligned} d(x,x_1) - t \leq d(x,x_0) + d(x_0,x_1) - t &\leq r' + r'' + L + \beta - (T - r') \\ &= r' + r'' + \beta + L - r'' = r' + \beta + L, \end{aligned}$$

the asserted inequality $d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \leq \frac{4 \cdot \beta \cdot (r' + \beta + L)}{r''}$ follows.

We have $t \leq T + r' = 2r' + r''$ We have already shown $|\tau| \leq 2\beta$ and hence $t + \tau \leq 2r' + r'' + 2\beta$. Since this implies $t, t+\tau \in [0, 2r' + r'' + 2\beta]$, we get $c_{x_1,x}(t) \in B_{2r'+r''+2\beta}(x_1)$ and $c_{x_1,x}(t+\tau) \in B_{2r'+r''+2\beta}(x_2)$.

Chapter 20

20.14. The assertion follows from Theorem 20.12 applied to $N = \dim(\Sigma)$, $X = |\Sigma|$, $f = \mathrm{id}_{|\Sigma|}\, E$, and any $\epsilon > 0$. Since Σ is finite, the group of simplicial automorphisms of Σ is also finite. Therefore G contains a normal subgroup of finite index which acts trivially on Σ and hence on $|\Sigma|$.

20.20. Define a G-homeomorphism $f\colon G \times_1 X \xrightarrow{\cong} G \times_d X$ by sending (g,x) to (g, gx).

20.37. The assertion for $F_g(\Gamma, S, k)$ is a consequence of the equality $\Gamma(g_k, t_k, \ldots, g_1, t_1, g_0, z) = g_k \cdots g_0 z$.

The assertion for $S^1_{\Gamma,S,k}(g,x)$ is proved as follows. Consider $(h,y) \in G \times X$ with the property that there are $a, b \in S$, $f \in F_a(\Gamma, S, k)$, and $f' \in F_b(\Gamma, S, k)$ satisfying both $f(x) = f'(y)$ and $h = ga^{-1}b$. We conclude from the assertion for $F_g(\Gamma, S, k)$ that this is equivalent to the condition that for some a, b in S we have $ax = by$ and $h = ga^{-1}b$.

The claim for $S^n_{\Gamma,S,k}(g,x)$ follows by induction of n since $S[k,n] = \{u_1 \cdots u_n \mid u_i \in S[k,1]\}$ holds.

20.43. Suppose that the condition is satisfied for S_1 and $\epsilon > 0$. Choose a natural number k such that each element in S_2 can be written as a word in the generators of S_2 consisting of at most k elements. Then the conditions is satisfied for S_2 and $k \cdot \epsilon > 0$, since for an element g which can be written as a word in l-elements of S_1 we conclude from the triangle inequality and the G-invariance of the L^1-metric that $d_{L^1}(f(gx), gf(x)) \leq l \cdot \epsilon$ holds.

20.54. Because of Theorem 20.53 we have to show for any finite abelian group G that it is a Dress group if and only if the set of primes for which the p-Sylow group is non-cyclic consists of at most two elements. This follows from the fact that G is the direct product of its p-Sylow subgroups and G is cyclic if and only if all its p-Sylow subgroups are cyclic.

20.51. Consider $H \in \mathcal{D}(H)$. Choose a prime q and a normal subgroup $H \subseteq F$ such that H is cyclic and F/H is a q-group. Now take p to be any prime, $P = \{1\}$, $C = H$ and $D = F$ in the Definition 20.50.

20.56. Obviously $C_n(X)$ is $\mathbb{Z}G$-module whose underlying abelian groups is finitely generated free. Hence $s(X)$ is well-defined.

Suppose that $f\colon X \to Y$ is a G-map which is (after forgetting the G-action) a homotopy equivalence. Then we obtain an exact sequence of finite $\mathbb{Z}G$-chain complexes $0 \to C_*(Y) \to \mathrm{cone}(C_*(f)) \to \Sigma C_*(X) \to 0$ such that each chain module is finitely generated free as abelian group and $\mathrm{cone}(C_*(f))$ has trivial homology. Define for a finite $\mathbb{Z}G$-chain complex E_* whose chain modules are finitely generated free as abelian groups the element $s(E_*) = \sum_{n \geq 0} (-1)^n \cdot [E_n]$ in $\mathrm{Sw}^p(G)$. Now one easily checks

$$\begin{aligned} s(X) &= s(C^c_*(X)); \\ s(Y) &= s(C^c_*(Y)); \\ s(\mathrm{cone}(C_*(f))) &= 0; \\ s(\Sigma C_*(X)) &= -s(C_*(X)); \\ s(\mathrm{cone}(C_*(f))) &= s(C_*(Y)) + s(\Sigma C_*(X)). \end{aligned}$$

This implies $s(X) = s(Y)$.

Suppose that the compact G-CW-complex X is the union of sub G-CW-complexes X_1 and X_2 and X_0 is the intersection of X_1 and X_1. Then we conclude from the short exact sequence of $\mathbb{Z}G$ chain -complexes

$$0 \to C_*(X_0) \to C_*(X_1) \oplus C_*(X_2) \to C_*(X) \to 0$$

that $s(X) = s(X_1) + s(X_2) - s(X_0)$ holds. Hence the map $s\colon \mathrm{Sw}^A(G) \to \mathrm{Sw}^p(G)$ sending $[X]$ to $s(X)$ is a well-defined map of abelian groups. It is compatible with the multiplication since there is a $\mathbb{Z}G$-chain isomorphism $C_*(X) \otimes C_*(Y) \xrightarrow{\cong} C_*(X \times Y)$ for any two compact G-CW-complexes X and Y.

20.57. Since for two finite G-sets S and S' we can view $S \amalg S'$ as the G-pushout of $S \leftarrow \emptyset \to S'$, we get a well-defined homomorphisms of abelian groups $a\colon A(G) \to \mathrm{Sw}^A(G)$ by sending $[S]$ to $[S]$. It is compatible with the multiplication since it is defined on $A(G)$ and $\mathrm{Sw}^A(G)$ by the cartesian product equipped with the diagonal G-action.

In order to show that the homomorphism a is surjective, we show by induction over $d = 0, 1, 2, \ldots$ that for any cocompact finite G-CW-complex X of dimension $\leq d$ the class $[X]$ is in the image of a. The induction beginning $d = 0$ is obvious since a cocompact 0-dimension G-CW-complex is the same as a finite G-set. The induction step from $(d-1)$ to $d \geq 1$ is done as follows. We can write X as a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_d} G/H_i \times S^{d-1} & \xrightarrow{q} & X_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d & \longrightarrow & X \end{array}$$

for a finite set I and subgroups $H_i \subseteq G$ of finite index. Since we can replace X_d by the mapping cylinder $\mathrm{cyl}(q)$ and the projections $\mathrm{cyl}(q) \to X_d$ and $G/H_i \times D^d \to G/H_i$ are G-homotopy equivalence, we obtain in $\mathrm{Sw}^A(G)$

$$[X] = \sum_{i \in I_d} [G/H_i] + [X_{d-1}] - \sum_{i \in I_d} [G/H_i \times S^{d-1}].$$

Since by induction hypothesis $[X_{d-1}]$, $[G/H_i]$, and $[G/H_i \times S^{d-1}]$ lie in the image of a, the same is true for $[X]$.

20.58. The map u is obviously an isomorphisms of abelian group. The homomorphism a is surjective by Exercise 20.57. The map c is well-defined since for an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules $0 \to M_0 \to M_1 \to M_2 \to 0$ the sequence of $\mathbb{Q}$-modules $0 \to \mathbb{Q} \otimes_{\mathbb{Z}} M_0^{\mathbb{Z}/p} \to \mathbb{Q} \otimes_{\mathbb{Z}} M_1^{\mathbb{Z}/p} \to \mathbb{Q} \otimes_{\mathbb{Z}} M_2^{\mathbb{Z}/p} \to 0$ is exact. The composite $c \circ a \circ u$ sends (m, n) to $(m, m + n)$ and hence is bijective. Hence all three maps u, a, and b are bijective.

Chapter 21

21.3. If $f^{\mathcal{U}}$ and $g^{\mathcal{U}}$ exists, then $f^{\mathcal{U}} = f \circ i^{\mathcal{U}}$ and $g^{\mathcal{U}} = \mathrm{pr}^{\mathcal{U}} \circ (i^{\mathcal{U}} \oplus i^{\perp})^{-1} \circ g$.

21.6. We get a weak homotopy equivalence

$$\mathbf{K}(\mathcal{U}) \xrightarrow{\simeq} \operatorname{hofib}\bigl(\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\bigr)$$

from Theorem 21.5 (i). The projection $\mathbf{K}(\mathcal{A}) \to *$ to the trivial spectrum $*$ is a weak homotopy equivalence by Theorem 6.37 (iii) and hence induces a weak homotopy equivalence

$$\operatorname{hofib}\bigl(\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})\bigr) \xrightarrow{\simeq} \operatorname{hofib}\bigl(* \to \mathbf{K}(\mathcal{A}/\mathcal{U})\bigr) = \Omega\mathbf{K}(\mathcal{A}/\mathcal{U}).$$

21.14. We leave the elementary proof that $R\text{-}\mathsf{MOD}_{\mathrm{fgp}}$ is a Waldhausen category to the reader. Obviously it cannot be homotopical since the zero homomorphism $R \to R$ is not injective and hence can not factorize into a split injective map followed by an isomorphism.

21.27. Since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/23])$ is non-trivial, see Example 2.106, there exists a finitely generated projective $\mathbb{Z}[\mathbb{Z}/23]$-module P which is not stably free. Let $0[P]$ be the $\mathbb{Z}[\mathbb{Z}/23]$-chain complex concentrated in dimension zero whose 0-th chain module is P. Then $0[P]$ is finitely dominated but cannot be $\mathbb{Z}[\mathbb{Z}/23]$-chain homotopy equivalent to a finite free $\mathbb{Z}[\mathbb{Z}/23]$-chain complex.

Chapter 22

22.74. Since $\mathcal{O}^G(G/H)$ is flasque, we get $K_n(\mathcal{O}^G(G/H)) = 0$ for all $n \in \mathbb{Z}$ from Lemma 6.37 (iii). We conclude from the $\mathcal{TOD}$-sequence of Theorem 22.19 that the canonical map $K_{n+1}(\mathcal{D}^G(G/H)) \xrightarrow{\cong} K_n(\mathcal{T}^G(G/H))$ is an isomorphism for all $n \in \mathbb{Z}$. We have already constructed a natural isomorphism $K_n(\mathcal{B}[G/H]_\oplus) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H))$ for every $n \in \mathbb{Z}$ in in Proposition 22.70. We get a canonical isomorphism $K_n(\mathcal{T}^G(G/H)) \xrightarrow{\cong} K_n(\mathcal{T}^G(G/G))$ from Lemma 22.22. These three isomorphism can be combined to an isomorphism $K_n(\mathcal{T}^G(G/G)) \xrightarrow{\cong} K_n(\mathcal{B}[G/H]_\oplus)$. There is an obvious identification $\mathcal{T}^G(G/G) = \mathcal{B}_\oplus$. Under it we get an isomorphism $K_n(\mathcal{B}[G/H]_\oplus) \xrightarrow{\cong} K_n(\mathcal{B}_\oplus)$ which comes from the obvious projection $G/H \to G/G$ and the obvious identification $\mathcal{B}(G/G) = \mathcal{B}$.

22.84. We leave the elementary proof that $(\mathcal{B}, \operatorname{supp}_{\mathbb{Z}})$ satisfies the axioms appearing in Definition 22.1 to the reader.

The category $\mathcal{B}(\mathbb{Z}/\mathbb{Z}) \cong \mathcal{B}$ is $\underline{R[\mathbb{Z}/2]}$ and hence $H_n^G(\mathbb{Z}/\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \cong K_n(R[\mathbb{Z}/2])$.

The category $\mathcal{B}(\mathbb{Z})$ can be identified with the category $\coprod_{n\in\mathbb{Z}} \underline{R}$. Hence the obvious functor of additive categories $\bigoplus_{n\in\mathbb{Z}} \underline{R}_\oplus \to \mathcal{B}(\mathbb{Z})_\oplus$ is an equivalence. Thus we get an isomorphism

$$\alpha\colon \bigoplus_{n\in\mathbb{Z}} K_n(R) \xrightarrow{\cong} K_n(\mathcal{B}(\mathbb{Z}))$$

since algebraic K-theory of additive categories is compatible with direct sums over arbitrary index sets. Let $s\colon \mathbb{Z}\to\mathbb{Z}$ be the automorphism sending n to $n+1$ and $\mathrm{sh}\colon \bigoplus_{n\in\mathbb{Z}} K_n(R) \to \bigoplus_{n\in\mathbb{Z}} K_n(R)$ the shift automorphism sending $(x_n)_{n\in\mathbb{N}}$ to $(x_{n-1})_{n\in\mathbb{N}}$. Then the following diagram commutes

$$\begin{tikzcd}
\bigoplus_{n\in\mathbb{Z}} K_n(R) \arrow[r, "\alpha", "\cong"'] \arrow[d, "\mathrm{sh}"'] & K_n(\mathcal{B}[\mathbb{Z}]_{\oplus}) \arrow[d, "K_n(\mathcal{B}(s)_{\oplus})", "\cong"'] \\
\bigoplus_{n\in\mathbb{Z}} K_n(R) \arrow[r, "\cong", "\alpha"'] & K_n(\mathcal{B}(\mathbb{Z})_{\oplus}).
\end{tikzcd}$$

The following sequence of abelian group is exact

$$0 \to \bigoplus_{n\in\mathbb{Z}} K_n(R) \xrightarrow{\mathrm{id}-\mathrm{sh}} \bigoplus_{n\in\mathbb{Z}} K_n(R) \xrightarrow{\epsilon} K_n(R) \to 0$$

where ϵ sends $(x_n)_{n\geq 0}$ to $\sum_{n\in\mathbb{Z}} x_n$. Since a model for $E\mathbb{Z}$ is $\mathbb{R}$ with the standard $\mathbb{Z}$-action, we obtain a long exact sequence

$$\begin{aligned}
\cdots \to K_n(\mathcal{B}(\mathbb{Z})_{\oplus}) &\xrightarrow{\mathrm{id}-K_n(\mathcal{B}(s)_{\oplus})} K_n(\mathcal{B}(\mathbb{Z})_{\oplus}) \to H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{\mathcal{B}}) \\
&\to K_{n-1}(\mathcal{B}(\mathbb{Z})_{\oplus}) \xrightarrow{\mathrm{id}-K_{n-1}(\mathcal{B}(s)_{\oplus})} K_{n-1}(\mathcal{B}(\mathbb{Z})_{\oplus}) \to \cdots .
\end{aligned}$$

Hence we obtain an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{\mathcal{B}}) \cong K_n(R).$$

We leave the elementary proof of the claim about the identification of the assembly map to the reader.

22.86. Suppose that we have two other morphisms $u_i'\colon B\to B'$ for $i=1,2$ satisfying $u=u_1'+u_2'$ and $\mathrm{supp}_G(u_i')=L_i$ for $i=1,2$. Put $v_i := u_i - u_i'$ for $i=1,2$. Then we have $0=v_1+v_2$ and hence $\mathrm{supp}_G(v_1+v_2)=\emptyset$. We conclude

$$\begin{aligned}
\mathrm{supp}_G(v_1) &= \mathrm{supp}_G((v_1+v_2)+(-v_2)) \subseteq \mathrm{supp}_G(v_1+v_2)\cup\mathrm{supp}_G(-v_2) \\
&= \emptyset + \mathrm{supp}_G(v_2) = \mathrm{supp}_G(u_2+(-u_2')) \subseteq \mathrm{supp}_G(u_2)\cup\mathrm{supp}_G(-u_2') \\
&\subseteq \mathrm{supp}_G(u_2)\cup\mathrm{supp}_G(u_2') \subseteq L_2\cup L_2 = L_2
\end{aligned}$$

and

$$\begin{aligned}\operatorname{supp}_G(v_1) = \operatorname{supp}_G(u_1 + (-u_1')) &\subseteq \operatorname{supp}_G(u_1) \cup \operatorname{supp}_G(-u_1')\\ &\subseteq \operatorname{supp}_G(u_1) \cup \operatorname{supp}_G(u_1') \subseteq L_1 \cup L_1 = L_1.\end{aligned}$$

Since $L_1 \cap L_2 = \emptyset$, we conclude $\operatorname{supp}_G(v_1) = \emptyset$ and hence $v_1 = 0$. This implies $u_1 = u_1'$. Analogously one shows $u_2 = u_2'$.

22.87. Obviously $\mathcal{A}[G]$ with the support defined in Example 22.2 is a category with G-support. It is a strong category with G-support since $\mathcal{A}$ comes with a G-action and we can define the desired homotopy trivilization $\Omega_g \colon \operatorname{id}_{\mathcal{B}} \xrightarrow{\cong} \Lambda_g$ by the isomorphisms $\operatorname{id}_{gA} \cdot g \colon A \to gA$ in $\mathcal{A}[G]$ for any object $A \in \mathcal{A}$. *Morphism Additivity* obviously holds.

22.88. Define a functor of G-$\mathbb{Z}$-categories

$$F \colon \mathcal{A}[G] \xrightarrow{\cong} \mathcal{B}$$

by requiring that F is the identity on the set of objects and sends a morphism $\sum_{g \in G}(f_g \colon gA \to A) \cdot g$ from A to A' in $\mathcal{A}[G]$ to the morphism $\sum_{g \in G}(f_g \circ \Omega_g(A))$ from A to A' in $\mathcal{A}[G]$. Using Exercise 22.86 one easily checks that F is full and faithful and hence an isomorphism of G-$\mathbb{Z}$-categories. One easily checks that F is compatibile with the support functions.

22.89. Suppose such an extension to the structure of a strong category with $\mathbb{Z}$-support exists. The natural transformation Ω_1 for the generator $1 \in \mathbb{Z}$ is an isomorphism in $\mathcal{B}$ with support $\{1\}$. This is a contradiction since no morphism in $\mathcal{B}$ has support $\{1\}$.

22.102. Because of Theorem 21.5 (i) and Lemma 22.101 its suffices to show

$$\begin{aligned}K_m(\mathcal{T}_0^{\{1\}}(\{\bullet\})) &\cong \bigoplus_{n=0}^{\infty} K_m(\mathcal{B}_\oplus);\\ K_m(\mathcal{O}_0^{\{1\}}(\{\bullet\})) &\cong \prod_{n=0}^{\infty} K_m(\mathcal{B}_\oplus).\end{aligned}$$

Non-connective K-theory is compatible with infinite direct products of additive categories, by [200], see also [556, Theorem 1.2]. It is also compatible with directed unions, see for instance [668, Corollary 7.2], and hence with infinite direct sums. Since the obvious functors

$$\begin{aligned}\bigoplus_{n=0}^{\infty} \mathcal{B}_\oplus &\xrightarrow{\simeq} \mathcal{T}_0^{\{1\}}(\{\bullet\});\\ \mathcal{O}_0^{\{1\}}(\{\bullet\}) &\xrightarrow{\simeq} \prod_{n=0}^{\infty} \mathcal{B}_\oplus,\end{aligned}$$

are equivalences of additive categories, the claim follows.

22.132. The key observation is the following. Given a morphism $\phi\colon \mathbf{B} = (S,\pi,\eta,\mathsf{B}) \to \mathbf{B}' = (S',\pi',\eta',\mathsf{B}')$, there exists because of *bounded control over* $\mathbb{N}$ a natural number n such that for $s \in S$ and $s' \in S'$ the implication $\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \leq n$ holds. Hence for any natural number r with $r > n$ we conclude that

$$\left|\frac{1}{\eta(s)} - \frac{1}{\eta'(s')}\right| = \left|\frac{\eta(s) - \eta'(s')}{\eta(s) \cdot \eta(s')}\right| \leq \frac{n}{r \cdot (r-n)}$$

holds for $s \in S$ and $s' \in S'$ with $\phi_{s,s'} \neq 0$, provided that $\eta(s) \geq r$ or $\eta(s') \geq r$. Obviously we have $\lim_{r\to\infty} \frac{n}{r\cdot(r-n)} = 0$.

Chapter 23

23.7. Obviously $\Phi_\tau \circ \Phi_\sigma = \Phi_{\tau+\sigma}$ for $\tau,\sigma \in \mathbb{R}$ and $\Phi_0 = \mathrm{id}_{\mathrm{FS}(X)}$. The main task is to show that $\Phi\colon \mathrm{FS}(X) \times \mathbb{R} \to \mathrm{FS}(X)$ is continuous.

We estimate for $c \in \mathrm{FS}(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{\mathrm{FS}(X)}\big(c, \Phi_\tau(c)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t), c(t+\tau)\big)}{2e^{|t|}}\, dt \\ &\leq \int_{\mathbb{R}} \frac{|\tau|}{2e^{|t|}}\, dt \\ &= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|t|}}\, dt \\ &= |\tau|. \end{aligned}$$

We estimate for $c,d \in \mathrm{FS}(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_\tau(d)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t+\tau), d(t+\tau)\big)}{2e^{|t|}}\, dt \\ &= \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t-\tau|}}\, dt \\ &\leq \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|-|\tau|}}\, dt \\ &= e^{|\tau|} \cdot \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|}}\, dt \\ &= e^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c,d). \end{aligned}$$

The two inequalities above together with the triangle inequality imply for $c,d \in \mathrm{FS}(X)$ and $\tau,\sigma \in \mathbb{R}$

$$\begin{aligned}
&d_{\mathrm{FS}(X)}\big(\Phi_\tau(c),\Phi_\sigma(d)\big)\\
&= d_{\mathrm{FS}(X)}\big(\Phi_\tau(c),\Phi_{\sigma-\tau}\circ\Phi_\tau(d)\big)\\
&\le d_{\mathrm{FS}(X)}\big(\Phi_\tau(c),\Phi_\tau(d)\big)+d_{\mathrm{FS}(X)}\big(\Phi_\tau(d),\Phi_{\sigma-\tau}\circ\Phi_\tau(d)\big)\\
&\le e^{|\tau|}\cdot d_{\mathrm{FS}(X)}(c,d)+|\sigma-\tau|.
\end{aligned}$$

This implies that Φ is continuous at (c,τ).

23.16. Note that $\mathrm{FS}(X)^{\mathbb{R}}$ is the space of constant generalized geodesics. Let $c\in\mathrm{FS}(X)-\mathrm{FS}(X)^{\mathbb{R}}$. Pick $t_0,\ t_1\in\mathbb{R}$ such that $c(t_0)\neq c(t_1)$. Set $\delta:=d_X(c(t_0),c(t_1))/2$. For $x\in X$ then $d_X(x,c(t_0))\ge\delta$ or $d_X(x,c(t_1))\ge\delta$. Denote by c_x the constant generalized geodesic at x. If $d_X(x,c(t_0))\ge\delta$, then $d_X(x,c(t))\ge\delta/2$ if $t\in[t_0-\delta/2,t_0+\delta/2]$. Thus in this case

$$d_{\mathrm{FS}(X)}(c_x,c)\ge\int_{t_0-\delta/2}^{t_0+\delta/2}\frac{\delta/2}{2e^{|t|}}\,dt=:\epsilon_0.$$

Similarly,

$$d_{\mathrm{FS}(X)}(c_x,c)\ge\int_{t_1-\delta/2}^{t_1+\delta/2}\frac{\delta/2}{2e^{|t|}}\,dt=:\epsilon_1,$$

if $d_X(x,c(t_1))\ge\delta/2$. Hence $B_\epsilon(c)\cap\mathrm{FS}(X)^{\mathbb{R}}=\emptyset$ if we set $\epsilon:=\min\{\epsilon_0/2,\epsilon_1/2\}$.

23.37. If $\dim(X)=\infty$, the claim is obviously true. So we can assume in the sequel that $\dim(X)$ is a natural number.

Let $\mathcal{U}$ be an open covering of A. For $U\in\mathcal{U}$ choose an open subset $U'\subseteq X$ satisfying $U=A\cap U'$. Then $\mathcal{U}'=\{U'\mid u\in\mathcal{U}\}\amalg\{X\setminus A\}$ is an open covering of X. Let $\mathcal{V}'$ be a refinement of $\mathcal{U}'$ with $\dim(\mathcal{V})\le\dim(X)$. Then $\mathcal{V}=\{V'\cap A\mid V'\in\mathcal{V}'\}$ is an open covering of A with $\dim(\mathcal{V})\le\dim(\mathcal{V}')\le\dim(X)$. This implies $\dim(A)\le\dim(X)$.

Chapter 24

24.11. We have to show for $\epsilon\in\{\pm1\}$ and $g\in G$ that, for any $\mathbb{Z}G$-module M which is finitely generated free as abelian groups, the element $s([M],(\epsilon\cdot g))$ lies in the kernel of the projection $K_1(\mathbb{Z}G)\to\mathrm{Wh}(G)$ for the element $(\epsilon\cdot g)\in K_1(\mathbb{Z}G)$ represented by the trivial unit $\epsilon\cdot g\in\mathbb{Z}g^\times$ and the element $[M]\in\mathrm{Sw}(G)$ represented by M. It is not hard to check that $s([M],(\epsilon g))$ is represented by the composite of the automorphisms $(\epsilon\cdot l_g)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Z}G}$ and $\mathrm{id}_M\otimes r_g$ of $M\otimes_2\mathbb{Z}G$ where $l_g\colon M\to M$ is left multiplication and $r_g\colon\mathbb{Z}G\to\mathbb{Z}G$ is right multiplication. One easily checks that the class of $(\epsilon\cdot l_g)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{Z}G}$ in $K_1(\mathbb{Z}G)$ lies in the image of $K_1(\mathbb{Z})\to K_1(\mathbb{Z}G)$ and the class of $\mathrm{id}_M\otimes r_g$ in $K_1(\mathbb{Z}G)$ is $\mathrm{rk}_{\mathbb{Z}}(M)\cdot(g)$.

24.30. (i) For $x, y \in Z$ with $f_{x,y} \neq 0$ there exist $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ with $(f_{i,j})_{x,y} \neq 0$ which implies $d^{L^1}(w(x), w(y)) \leq \mathrm{wd}(f_{i,j})$.

(ii) For $x, y \in Z$ with $(g \circ f)_{x,y} \neq 0$ there exists $z \in Z$ with $f_{x,z} \neq 0$ and $g_{z,y} \neq 0$ and hence we get

$$d^{L^1}(w(x), w(y)) \leq d^{L^1}(w(x), w(,z)) + d^{L^1}(w(z), w(y)) \leq \mathrm{wd}(f) + \mathrm{wd}(g).$$

(iii) Suppose that $(\lambda \cdot f + \mu \cdot g)_{x,y} \neq 0$ holds. Then $f_{x,y} \neq 0$ or $g_{x,y} \neq 0$ holds. This implies $d^{L^1}(w(x), w(y)) \leq \mathrm{wd}(f)$ or $d^{L^1}(w(x), w(y)) \leq \mathrm{wd}(g)$.

(iv) This follows from the definition of the width.

It is trivial on objects since we have $(\mathrm{id}_M)_{x,y} \neq 0 \implies x = y \implies d^{L^1}(w(x), w(y)) = 0$.

24.40. Define for a bounded $\mathcal{A}$-chain complex C_* the number $d(C_*)$ to be the minimum over those numbers d for which there exists integers a and b such that $a \leq b$ holds, we have $C_n = 0$ for $n < a$ and $n > b$, and $d = b - a$ holds. Then we use induction over $d(C_*)$. In the induction beginning $d(C_*) = 0$ the $\mathcal{A}$-chain complex C_* is concentrated in one dimension and the claim follows directly from the definition. The induction step follows from Additivity.

24.46. The inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$. One easily checks using the axioms appearing in Definition 24.27 that $\mathrm{wd}(g \circ f), \mathrm{wd}((g \circ f)^{-1}) \leq \epsilon + \delta$ holds.

24.49. In the sequel we will apply the axioms appearing in Definition 24.27 over and over again.

(i) The equality $\mathrm{wd}(\lambda \cdot f_* + \mu \cdot g_*) \leq \max\{\mathrm{wd}(f_*), \mathrm{wd}(g_*)\}$ follows directly from these axioms. If $h_*\colon f_* \simeq g_*$ and $k_*\colon g_* \simeq h_*$ are $\mathcal{A}$-chain homotopies, then $h_* + k_*$ is a $\mathcal{A}$-chain homotopy $f_* \simeq h_*$.

(ii) If $h_*\colon f_* \simeq f'_*$ is a $\mathcal{A}$-chain homotopy, then we obtain $\mathcal{A}$-chain homotopies $v_{*+1} \circ h_*\colon v_* \circ f_* \simeq v_* \circ f'_*$ and $h_* \circ u_*\colon f_* \circ u_* \simeq f'_* \circ u_*$

(iii) Choose $\mathcal{A}$-chain maps $u_*\colon D_* \to C_*$ and $\mathcal{A}$-chain map $v_*\colon E_* \to D_*$ satisfying

$$\begin{aligned}
\mathrm{wd}(u_*) &\leq \epsilon; \\
\mathrm{wd}(v_*) &\leq \epsilon; \\
u_* \circ f_* &\simeq_\epsilon \mathrm{id}_{C_*}; \\
f_* \circ u_* &\simeq_\epsilon \mathrm{id}_{D_*}; \\
v_* \circ g_* &\simeq_\epsilon \mathrm{id}_{D_*}; \\
g_* \circ v_* &\simeq_\epsilon \mathrm{id}_{E_*}.
\end{aligned}$$

Now assertions (i) and (ii) imply

$$\begin{aligned}
\mathrm{wd}(g_* \circ f_*) &\leq 2\epsilon; \\
\mathrm{wd}(u_* \circ v_*) &\leq 2\epsilon; \\
(u_* \circ v_*) \circ (g_* \circ f_*) &\simeq_{3\epsilon} \mathrm{id}_{C_*}; \\
(g_* \circ f_*) \circ (u_* \circ v_*) &\simeq_{3\epsilon} \mathrm{id}_{E_*}.
\end{aligned}$$

Chapter 25

25.1. Test.

Chapter 26

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Notation

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Chapter 28
Comments (temporary chapter)

Comment 35 (by W.): This chapter has to be taken out in the final version.

28.1 Comments about Notation and Terminology

19.10.23 Here are some Makros used in the surgery book.

(a) G, O, PL, SG, SO, SPL, STOP, TOP, G/O, G/PL, PL/O, G/TOP.
(b) BO, BSO, BPL, BSPL, BTOP, BSTOP, BGL, BSL.
(c) BG, EG, DIFF, BDIFF(M), BSDIFF(M), DIFF(M), SDIFF, SDIFF(M), BO(k), BSO(k), BG(k), BSG(k), BGL(k), BGL(n, R), BSL(k), BPL(k), BSPL(k), BTOP(k), BSTOP(k), EG, E(R), G(k), O(k), SO(k), G(k), SG(k), GL(k), GL(n, R), SL(n, R), SU(n, R), PL(k), RU(n, R), SPL(k), TOP(k), STOP(k), GL(n, R), SL(n, R), SU(n, R), TU(n, R), UU(n, R), M($m, n; R$).
(d) TOP, PL, DIFF.
(e) B(G/O).

Some we are using here as well but not all of them.
Here are some conventions we will use to indicate the topological category or the smooth category or the PL-category $\mathbf{K}^{\mathsf{TOP}}$, K^{TOP}, K^{ALG}, $\mathcal{P}^{\mathsf{TOP}}$, $\mathcal{P}^{\mathsf{PL}}$, $\mathcal{P}^{\mathsf{DIFF}}$.
Automorphisms groups are denoted by $\mathrm{Top}(M)$ and $\mathrm{Diff}(M)$.
For matrices we use $\mathrm{M}_{m,n}(R)$, $\mathrm{M}_n(R)$, $\mathrm{GL}_n(R)$, $M(R)$, and $\mathrm{GL}(R)$.
What shall be the standard for the classifying space of a group. Shall we just write \$BG\$ which gives BG. Sometimes with this convention there is a rather large space between B and G. We could use \Bpar{G}, which gives $\mathrm{B}G$. The same discussion applies to EG.

28.2 Comments about Grammar, English and Spelling

27.11.11 torsionfree (One word).
27.11.11 semisimple (One word).
27.11.11 pseudoisotopy (One word).
27.11.11 prove, proved proven.
27.11.11 choose, chose, chosen.

27.11.11 We should agree on a unified use of the words any, each and every. I am not certain about the rules.
27.11.11 Write semigroup, semisimple, semidirect and so on.
27.11.11 Before R and S it must be “an”, before h it depends on the pronouncation e.g., “an h-cobordism”, but “a homomorphism”.
27.11.11 Write “generalisation” and “generalise”.
27.11.11 Write “well-known” and “well-defined”.
27.11.11 Write “simply connected” and “path connected”.
27.11.11 Insert always a comma after “namely”.
27.11.11 Write selfmap, selfdiffeomorphism, and so on.
27.11.11 Write finite dimensional and not finite-dimensional.
27.11.11 Write “aspherical closed manifolds”, i.e., “aspherical” before “closed”
27.11.11 Write “one-relator group”.
27.11.11 Write “hyperbolic group” and not “word-hyperbolic group”.
27.11.11 Write “handlebody” and not “handle body”.
30.03.17 Write “neighbourhood” and not “neighborhood”.
27.11.11 Always a comma after i.e. and e.g., so in the text it should be e.g., and i.e.,.
05.03.14 Section, Theorems etc are proper names and so have capitals.
06.07.14 We have to unify the spelling concerning the choice of s or z, e.g., realize versus realise, parallizable versus parallisable, generalize versus generalise, and so on.
23.03.15 Write “non-singular” not “nonsingular” and “non-degenerate” not “non-degenerated”.
23.03.15 Write “an n-manifold” and “an m-manifold”, not “a n-manifold” or “a m-manifold”.
23.03.15 Use “transverse” as an adjective and “transversal” as a noun.
23.03.15 Use “isotope” for the present tense of the verb “to isotope”, i.e. perform an isotopy.
02.03.17 Sometimes we write lift or lifting. Hatcher is indeed using both words.
18.07.17 Write “regularly homotopic” not “regular homotopic”.
18.07.17 Sentences ending or pausing in diagrams show have fullstops or commas on the bottom right hand symbol of the diagram. When this is not practical, another option is to end the sentence with a colon before the diagram and leave the diagram without punctuation; e.g. “We consider the following commutative diagram:”.
19.08.17 It is “local coefficient system” or “infinite cyclic local coefficient system”, but “homology with coefficients in R”. “
27.09.21 Do not start a sentence with a mathematical symbol. So instead of starting with “$H_n(C)$” start with “The homology group $H_n(C)$”.
09.11.21 Surgery Program in capital letters.
29.11.21 Replace “Notice” by “Note”
16.01.2023 Use “highly connected” and not “highly-connected”.
09.10.2023 Use “a priori” and not “a-priori”.
01.09.2023 No comma before “if and only if”.

15.09.2023 Use the Oxford comma.

15.09.2023 Relative sentence that are important for the context and cannot be left out shall start with “that” and not with “which” and there is no comma. Relative sentences that do contain only some additonal information and could be left out, start with “which” and not with “that” and are usually separated by commas.

17.01.2023 I found the following rues in the Internet concerning commas. We should follow them (to some extend).

(a) Use commas in a series of three or more items;
(b) Use a comma to separate independent clauses introduced by and, or, nor, but, yet, for, so (in the sense of “as a result”), as (in the sense of “because”), and while (in the sense of “at the same time”).
(c) Use a comma after an introductory phrase/word.
(d) Use a comma after a dependent clause preceding an independent clause.
(e) Use commas before and after parts of the sentence that are not essential to its meaning.
(f) Note that a dependent clause should be separated from an independent clause by a comma only when the dependent clause precedes the independent one. If the dependent clause follows the independent one, no comma is places before if, where, whether, because, although, since, when, while, unless, etc.

23.09.2023 Insert a page break if a Lemma, Definition, or so appears only with one or two lines at the bottom of a page.

28.3 Mathematical Comments and Problems

15.05.2023 There is a new preprint *An improvement of the Farrell-Jones conjecture for localising invariants* by Reis [834] on the arXive under arXiv:2211.15523 [math.KT]. It improves the paper Bunke-Kasprowski-Winges [172].

23.03.2024 It is unlikely that for the following preprints there will ever be a published version: [39, 121, 137, 309, 321, 578, 732, 787, 788, 789, 914, 998].

28.4 Possible Additional References

[834],

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