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# Isomorphism Conjectures in *K*- and *L*-Theory

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# Preface

The Isomorphism Conjectures due to Baum and Connes and to Farrell and Jones aim at the topological *K*-theory of reduced group  $C^*$ -algebras and the algebraic *K*-and *L*-theory of group rings. These theories are of major interest for many reasons. For instance, the algebraic *L*-groups are the recipients for various surgery obstructions and hence highly relevant for the classification of manifolds. Other important obstructions such as Wall's finiteness obstruction and Whitehead torsion take values in algebraic K-groups. The topological *K*-groups of  $C^*$ -algebras play a central role in index theory and the classification of  $C^*$ -algebras.

In general these K- and L-groups are very hard to analyze for group rings or group  $C^*$ -algebras. The Isomorphism Conjectures identify them with equivariant homology groups of classifying spaces for families of subgroups. As an illustration, let us consider the special case that G is a torsionfree group and R is a regular ring (with involution). Then the Isomorphism Conjectures predict that the so-called assembly maps

$$\begin{aligned} H_n(BG; \mathbf{K}(R)) &\xrightarrow{\cong} K_n(RG); \\ H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) &\xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG); \\ K_n(BG) &\xrightarrow{\cong} K_n(C_r^*(G)), \end{aligned}$$

are isomorphisms for all  $n \in \mathbb{Z}$ . The target is the algebraic *K*-theory of the group ring *RG*, the algebraic *L*-theory of *RG* with decoration  $\langle -\infty \rangle$ , or the topological *K*-theory of the reduced group *C*<sup>\*</sup>-algebra  $C_r^*(G)$ . The source is the evaluation of a specific homology theory on the classifying space *BG*, where  $H_n(\{\bullet\}; \mathbf{K}(R)) \cong K_n(R)$ ,  $H_n(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(R)) \cong L_n^{\langle -\infty \rangle}(R)$ , and  $K_n(\{\bullet\}) \cong K_n(\mathbb{C})$  hold for all  $n \in \mathbb{Z}$ .

Since the sources of these assembly maps are much more accessible than the targets, the Isomorphism Conjectures are key ingredients for explicit computations of the *K*-and *L*-groups of group rings and reduced group  $C^*$ -algebras. These often are motivated by and have applications to concrete problems that arise, for instance, in the classification of manifolds or  $C^*$ -algebras.

The Baum-Connes Conjecture and the Farrell-Jones Conjecture imply many other well-known conjectures. In a lot of cases these conjectures were not known to be true for certain groups until the Baum-Connes Conjecture or the Farrell-Jones Conjecture was proved for them. Examples of such prominent conjectures are the Borel Conjecture about the topological rigidity of aspherical closed manifolds, the (stable) Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature on closed Spin-manifolds, Kaplansky's Idempotent Conjecture and the Kadison Conjecture on the non-existence of non-trivial idempotents in the group ring or the reduced group  $C^*$ -algebra of torsionfree groups, the Novikov Conjecture about the homotopy invariance of higher signatures, and the conjectures about the vanishing of the reduced projective class group of  $\mathbb{Z}G$  and the Whitehead group of G for a torsionfree group G.

#### Preface

The Baum-Connes Conjecture and the Farrell-Jones Conjecture are still open in general at the time of writing. However, tremendous progress has been made on the class of groups for which they are known to be true. The techniques of the sophisticated proofs stem from algebra, dynamical systems, geometry, group theory, operator theory, and topology. The extreme broad scope of the Baum-Connes Conjecture and the Farrell-Jones Conjecture is both the main challenge and the main motivation for writing this book. We hope that, after having read parts of this monograph, the reader will share the enthusiasm of the author for the Isomorphism Conjectures.

The monograph is a guide to and gives a panorama of Isomorphism Conjectures and related topics. It presents or at least indicates the most advanced results and developments at the time of writing. It can be used by various groups of readers, such as experts on the Baum-Connes Conjecture or the Farrell-Jones Conjecture, experienced mathematicians, who may not be experts on these conjectures but want to learn or just apply them, and also, of course, advanced undergraduate and graduate students. References for further reading and information have been inserted.

We will give more information about the organization of the book and a user's guide in Section 1.11.

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# Chapter 1 Introduction

The Isomorphism Conjectures due to Paul Baum and Alain Connes and to Tom Farrell and Lowell Jones are important conjectures, which have many interesting applications and consequences. However, they are not easy to formulate and it is a priori not clear why the actual versions are the most promising ones. The current versions are the final upshot of a longer process, which has led to them step by step. They have been influenced and steered by various new results that have been proved during the last decades and given new insight into the objects, problems, and constructions at which these conjectures aim.

In this introduction we want to motivate these conjectures by explaining how one can be led to them by general considerations and certain facts. We present brief surveys about applications of these conjectures, their status, and the methods of proof. We give information about the contents of this monograph including a user's guide.

# **1.1** Why Should we Care about Isomorphism Conjectures in *K*- and *L*-Theory?

In this section we give some background and motivation for the reader who has no previous knowledge about the Baum-Connes Conjecture and the Farrell-Jones Conjecture. An expert may skip this section.

The Baum-Connes Conjecture aims at the topological K-theory of the reduced group  $C^*$ -algebra of a group, whereas the Farrell-Jones Conjecture is devoted to the algebraic K- and L-theory of the group ring of a group. K- and L-theory are rather sophisticated theories. Group rings are very difficult rings, for instance, they are in general not commutative, are not Noetherian or regular, and may have zero-divisors. So studying the algebraic K-theory and L-theory of group rings is hard and seems at first glance to be a very special problem. So why should one care?

The answer to this question is that information about the K- or L-theory of group rings or the topological K-theory of group  $C^*$ -algebras have many applications to algebra, geometry, group theory, topology, and operator algebras and that meanwhile these conjectures are known for a large class of groups.

#### 1.1.1 Projective Class Group

Let us illustrate this by considering the most prominent and easy to define *K*-group, the projective class group  $K_0(S)$  of a ring *S*. It is the abelian group which we obtain from the Grothendieck construction applied to the abelian semigroup of isomorphism classes of finitely generated projective *S*-modules under direct sum. Equivalently, it can be described as the abelian group whose generators are isomorphism classes [P] of finitely generated projective *S*-modules *P* and for every exact sequence  $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$  of finitely generated projective *S*-modules we require the relation  $[P_1] = [P_0] + [P_2]$ . The reduced projective class group  $\widetilde{K}_0(S)$  of a ring *S* is obtained from  $K_0(S)$  by dividing out the subgroup generated by all finitely generated free *S*-modules. Any finitely generated projective *S*-module *P* defines an element [P] in  $K_0(S)$  and hence also a class [P] in  $\widetilde{K}_0(S)$ . The decisive property of  $\widetilde{K}_0(S)$ is that [P] = 0 holds in  $\widetilde{K}_0(S)$  if and only if *P* is stably free, i.e., there are natural numbers *m* and *n* satisfying  $P \oplus S^m \cong_S S^n$ . So roughly speaking,  $[P] \in \widetilde{K}_0(S)$ measures the deviation of a finitely generated projective *S*-module *P* from being stably free.

Why are we especially interested in the case S = RG, where *R* is a ring, *G* is a group, and *RG* is the *group ring*? (The precise definition of *RG* can be found in Subsection 2.8.) One reason is that a representation of *G* with coefficients in *R* is the same as an *RG*-module. Another reason is that for a connected manifold or *CW*complex its universal covering comes with an action of the fundamental group  $\pi$  and the cellular  $\mathbb{Z}$ -chain complex of the universal covering is actually a free  $\mathbb{Z}\pi$ -chain complex. The latter observation opens the door to connections of algebraic *K*-theory to topological problems, as described next.

A *CW*-complex *X* is called *finitely dominated* if there is a finite *CW*-complex *Y* and maps  $i: X \to Y$  and  $r: Y \to X$  such that  $r \circ i$  is homotopic to  $id_X$ . Often one can construct a finitely dominated *CW*-complex with interesting properties but one needs to know whether it is homotopy equivalent to a finite *CW*-complex. This problem is decided by the *finiteness obstruction* of Wall. A finitely dominated connected *CW*-complex *X* with fundamental group  $\pi$  determines an element  $\tilde{o}(X) \in \tilde{K}_0(\mathbb{Z}\pi)$ , which vanishes if and only if *X* is homotopy equivalent to a finite *CW*-complex, see Theorem 2.39. So it is interesting to know whether  $\tilde{K}_0(\mathbb{Z}\pi)$  vanishes because then  $\tilde{o}(X)$  is automatically trivial. One can actually show for a finitely presented group *G* that  $\tilde{K}_0(\mathbb{Z}G)$  vanishes if and only if every finitely dominated connected *CW*-complex with fundamental group isomorphic to *G* is homotopy equivalent to a finite *CW*-complex. So we have an algebraic assertion and a topological assertion for a group *G* which turn out to be equivalent.

The question whether a finitely dominated *CW*-complex is homotopy equivalent to a finite *CW*-complex appears naturally in the construction of closed manifolds with certain properties, since a closed manifold is homotopy equivalent to a finite *CW*-complex, and one may be able to construct a finitely dominated *CW*-complex as a first approximation up to homotopy. This is explained in Section 2.5. The *Spherical Space Form Problem* 9.205 is a prominent example. It aims at the classification of 1.1 Why Should we Care about Isomorphism Conjectures in K- and L-Theory?

closed manifolds whose universal coverings are diffeomorphic or homeomorphic to the standard sphere.

If *R* is a field and the group *G* is torsionfree, then the *Idempotent Conjecture* of Kaplansky predicts that the group ring *RG* has only trivial idempotent, namely, 0 and 1. Roughly speaking, non-trivial idempotents in a ring can be used to decompose the ring into smaller pieces; think for instance of the theorems of Wedderburn and Maschke which imply that for a finite group *G* and a field *F* of characteristic zero the group ring *FG* is a product of matrix algebras over skew-fields. The Idempotent Conjecture shows that this does not apply to torsionfree groups. On the other hand, the non-existence of non-trivial idempotents gives hope that one can embed the group ring *FG* of a torsionfree group and a field *F* into a skew-field as conjectured by Malcev, which opens the door to many applications in group theory and topology, see Remark 2.85. This ring theoretic conjecture due to Kaplansky is related to the projective class group, since it is known to be true if  $\tilde{K}_0(RG)$  vanishes. There are many instances of groups where no algebraic proof is known for the Idempotent Conjecture, but one can show with geometric, homotopy theoretic, and *K*-theoretic methods that  $\tilde{K}_0(RG)$  vanishes.

A special version of the Farrell-Jones Conjecture, see Conjecture 2.60, predicts that  $\widetilde{K}_0(RG)$  vanishes if G is torsionfree and R is  $\mathbb{Z}$  or a field.

All of this is explained in detail in Chapter 2.

#### 1.1.2 The Whitehead Group

Here is another example of a nice connection between algebraic *K*-theory and topology. One can define  $K_1(S)$  of a ring *S* as the abelianization of the general linear group GL(*S*) or, equivalently, as the abelian group generated by conjugacy classes of automorphisms of finitely generated projective *S*-modules with relations concerning exact sequences and composites of such automorphisms. Given a group *G*, the *Whitehead group* Wh(*G*) is the quotient of  $K_1(\mathbb{Z}G)$  by the subgroup generated by trivial units. For more details we refer to Definition 3.1, Theorem 3.12, and Definition 3.23. This is related to topology as follows.

Given a closed manifold M, an *h*-cobordism W over M is a compact manifold W such that its boundary  $\partial W$  can be written as a disjoint union  $\partial W = \partial_0 W \amalg \partial_1 W$ , there is a preferred identification of M with  $\partial_0 W$ , and the inclusions  $\partial_k W \to W$  are homotopy equivalences for k = 0, 1. The set of isomorphism classes relative M of *h*-cobordisms over M can be identified with  $Wh(\pi)$  if M is connected, has dimension  $\geq 5$ , and  $\pi$  denotes its fundamental group, see Theorem 3.47. This is remarkable since the set of isomorphism classes of *h*-cobordism relative M over M a priori depends on M, whereas  $Wh(\pi)$  depends only on the fundamental group. In the classification of closed manifolds it is often a key step to decide whether an *h*-cobordism W over M is trivial, i.e., isomorphic relative M to  $M \times [0, 1]$ , since this has the consequence that M and  $\partial_1 W$  are isomorphic. It is not hard to show that  $Wh(\{1\})$  is trivial which, together with the results above, implies the Poincaré

Conjecture in dimensions  $\geq 5$ , see Theorem 3.51. One can show for a finitely presented group *G* and any natural number  $n \geq 5$  that Wh(*G*) is trivial if and only if for every connected *n*-dimensional closed manifold *M* with fundamental group isomorphic to *G* every *h*-cobordism over *M* is trivial. So we have again an algebraic assertion and a topological assertion for a group *G* which turn out to be equivalent.

A special version of the Farrell-Jones Conjecture, see Conjecture 3.110, predicts that Wh(G) vanishes if G is torsionfree. All of this is explained in detail in Chapter 3.

#### 1.1.3 The Borel Conjecture and the Novikov Conjecture

One of the author's favorite conjectures is the *Borel Conjecture*. It predicts that an aspherical closed manifold is topologically rigid. *Aspherical* means that the universal covering is contractible and *topologically rigid* means that every homotopy equivalence from a closed manifold to M is homotopic to a homeomorphism. In particular it implies that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic. One may view the Borel Conjecture as the topological counterpart of Mostow rigidity, see Remark 9.169.

If *G* denotes the fundamental group of an aspherical closed manifold of dimension  $\geq 5$ , then the Borel Conjecture for *M* holds if *G* satisfies both the *K*-theoretic and the *L*-theoretic Farrell-Jones Conjecture for  $\mathbb{Z}G$ , see Theorem 9.171. Moreover, all proofs of the Borel Conjecture in dimensions  $\geq 4$  are based on the Farrell-Jones Conjecture. So we see again that the Farrell-Jones Conjecture has interesting applications to topology.

*L*-theory, which one may think of as the algebraic *K*-theory of quadratic forms over finitely generated projective modules, is an important ingredient in the so-called Surgery Program 3.53, whose highlight is the Surgery Exact Sequence, see Theorem 9.127. It aims at the classification of closed manifolds, see Remark 3.53, and was initiated by the classification of exotic spheres, see Remark 3.52.

All this is explained in Chapter 9. In particular, we refer to Sections 9.12, 9.14, and 9.15.

Note that both the Baum-Connes Conjecture and the Farrell-Jones Conjecture imply the prominent *Novikov Conjecture* about the homotopy invariance of higher signatures, see Remark 9.143 and Theorem 14.29. The Novikov Conjecture and its link to both the Baum-Connes Conjecture and the Farrell-Jones Conjecture triggered a lot of interesting interactions and transfer of methods and techniques between topology and operator theory.

### 1.1.4 Further Applications

There are many more striking applications of the Farrell-Jones Conjecture and the Baum-Connes Conjecture to algebra, geometric group theory, geometry, topology, and operator algebras, which are listed in Sections 13.12 and 14.8. We hope that, by browsing through these sections, the reader will be convinced of the great interest of these conjectures.

#### 1.1.5 Status of the Full Farrell-Jones Conjecture and the Baum-Connes Conjecture with Coefficients

The Full Farrell-Jones Conjecture 13.30 implies all the variants of the Farrell-Jones Conjecture scattered in this monograph, see Theorem 13.65. A list of all the versions of the Farrell-Jones Conjecture can be found in Subsection 13.11.1.

The Baum Connes Conjecture with coefficients 14.11 is the most general variant in the Baum-Connes setting.

The class of groups for which the Full Farrell-Jones Conjecture 13.30 is known to be true is discussed in Theorem 16.1, whereas the class of groups for which the Baum Connes Conjecture with coefficients 14.11 is known to be true is discussed in Theorem 16.7. The question whether the Full Farrell-Jones Conjecture 13.30 might be true for all groups and how one might find counterexamples is treated in Section 16.10. This should convince the reader that in many interesting cases one knows that these conjectures are known to be true. Roughly speaking, in "daily life" one can expect that the Farrell-Jones Conjecture is known to be true and one can just apply it.

If one wants to figure out quickly whether a specific class of groups satisfies one of these conjectures, one should take a look at Section 16.8. Open cases are discussed in Section 16.9.

At the time of writing, no counterexamples to the Full Farrell-Jones Conjecture 13.30 are known. This is also true for the Baum-Connes Conjecture 14.11 (without coefficients). Counterexamples to the Baum Connes Conjecture with coefficients 14.11 are discussed in Remark 14.12.

### 1.1.6 Proofs

The proofs of the Farrell-Jones Conjecture or the Baum-Connes Conjecture are sophisticated and require a lot of different techniques. The proof of inheritance properties, such as the passage to subgroups, are usually based on homotopy theoretic methods. The proofs for specific classes of groups, such as hyperbolic groups, are based on transfer methods in the Farrell-Jones setting and on *KK*-theory in the Baum-Connes setting and for both conjectures require additional geometric input, which is the interesting and surprising part. For instance flow spaces play a prominent role in the proof of the Farrell-Jones Conjecture for hyperbolic groups or finite-dimensional CAT(0)-groups. It is intriguing and astonishing that the proofs of the Idempotent Conjecture of Kaplansky, which is a purely ring theoretic statement, are based for many groups on the proof of the Farrell-Jones Conjecture and thus use geometric input such as flows and compactifications of certain spaces on which the group in question acts. Often purely algebraic methods are not sufficient to prove the Idempotent Conjecture.

The reader who wants to get a first impression about the proofs should consult Chapter 19.

# **1.2** The Statement of the Baum-Connes Conjecture and of the Farrell-Jones Conjecture

Next we record the statements of the Baum-Connes Conjecture and Farrell-Jones Conjecture. Explanations and motivations will follow. The versions stated below will be generalized later.

**Conjecture 1.1 (Baum-Connes Conjecture).** Let *G* be a group. Then there is for every  $n \in \mathbb{Z}$  an isomorphism, called an *assembly map*,

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

**Conjecture 1.2.** (Farrell-Jones Conjecture for  $K_*(RG)$ ). Let *G* be a group. Let *R* be an associative ring with unit. Then there is for every  $n \in \mathbb{Z}$  an isomorphism, called an *assembly map*,

$$H_n^G(\underline{E}G;\mathbf{K}_R)\xrightarrow{\cong} K_n(RG).$$

**Conjecture 1.3. (Farrell-Jones Conjecture for**  $L_*^{\langle -\infty \rangle}(RG)$ **).** Let *G* be a group. Let *R* be an associative ring with unit and involution. Then there is for every  $n \in \mathbb{Z}$  an isomorphism, called an *assembly map*,

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

The general pattern is that the target of the assembly map is what we want to understand or to compute, namely, the K- and L-theory of group rings and group  $C^*$ -algebras, and that the source is a homological expression, which is much more accessible than the source and depends only on the values of the K- or L-groups under considerations for finite subgroups or for virtually cyclic subgroups of G. The spaces  $\underline{E}G$  and  $\underline{E}G$  are classifying spaces for the family of finite subgroups and the family of virtually cyclic subgroups, which are inserted in specific G-homology theories.

# **1.3 Motivation for and Evolution of the Baum-Connes** Conjecture

We will start with the Isomorphism Conjecture that is the easiest and most convenient to state and motivate, the Baum-Connes Conjecture for the topological K-theory of reduced group  $C^*$ -algebras. Then we will pass to the Farrell-Jones Conjecture for the algebraic K- and L-theory of group rings, which is more complicated due to the appearance of Nil-terms.

#### 1.3.1 Topological K-Theory of Reduced Group C\*-Algebras

The target of the Baum-Connes Conjecture is the topological *K*-theory of the *reduced* group  $C^*$ -algebra  $C_r^*(G)$  of a group *G*. We will consider discrete groups *G* only. One defines the topological *K*-groups  $K_n(A)$  for any Banach algebra *A* to be the abelian group  $K_n(A) = \pi_{n-1}(\operatorname{GL}(A))$  for  $n \ge 1$ . The famous Bott Periodicity Theorem gives a natural isomorphism  $K_n(A) \xrightarrow{\cong} K_{n+2}(A)$  for  $n \ge 1$ . Finally one defines  $K_n(A)$ for all  $n \in \mathbb{Z}$  so that the Bott isomorphism theorem is true for all  $n \in \mathbb{Z}$ . It turns out that  $K_0(A)$  is the same as the projective class group of the ring *A*, which is the Grothendieck group of the abelian monoid of isomorphism classes of finitely generated projective *A*-modules with the direct sum as addition. The topological *K*-theory of  $\mathbb{C} = C_r^*(\{1\})$  is trivial in odd dimensions and isomorphic to  $\mathbb{Z}$  in even dimensions. More generally, for a finite group *G* the topological *K*-theory of  $C_r^*(G)$ is the complex representation ring  $R_{\mathbb{C}}(G)$  in even dimensions and is trivial in odd dimensions.

Let P be an appropriate elliptic differential operator (or more generally an elliptic complex) on a closed *n*-dimensional Riemannian manifold M, for instance the Dirac operator or the signature operator. Then one can consider its *index* in  $K_n(\mathbb{C})$ , which is  $\dim_{\mathbb{C}}(\ker(P)) - \dim_{\mathbb{C}}(\operatorname{coker}(P)) \in \mathbb{Z}$  for even *n* and is zero for odd *n*. If *M* comes with an isometric G-action of a finite group G and P is compatible with the G-action, then ker(P) and coker(P) are complex finite-dimensional G-representations and one obtains an element in  $K_n(C_r^*(G)) = R_{\mathbb{C}}(G)$  by  $[\ker(P)] - [\operatorname{coker}(P)]$  for even *n*. Suppose that G is an arbitrary discrete group and that M is a (not necessarily compact) n-dimensional smooth manifold without boundary with a proper cocompact G-action, a G-invariant Riemannian metric, and an appropriate elliptic differential operator P compatible with the G-action. An example is the universal covering M = N of an *n*-dimensional closed Riemannian manifold N with  $G = \pi_1(N)$  and the lift  $\widetilde{P}$  to  $\widetilde{N}$  of an appropriate elliptic differential operator P on N. Then one can define an *equivariant index* of P which takes values in  $K_n(C_r^*(G))$ . Therefore the interest in  $K_*(C_r^*(G))$  comes from the fact that it is the natural recipient for indices of certain equivariant differential operators. All this will be explained in Chapter 10.

#### **1.3.2 Homological Aspects**

A first basic problem is to compute  $K_*(C_r^*(G))$  or to identify it with more familiar terms. The key idea comes from the observation that  $K_*(C_r^*(G))$  has some homological properties. More precisely, if *G* is the amalgamated free product  $G = G_1 *_{G_0} G_2$  for subgroups  $G_i \subseteq G$ , then there is a long exact sequence

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$$(1.4) \quad \cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) \xrightarrow{K_n(C_r^*(i_1)) \oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \xrightarrow{K_n(C_r^*(j_1)) - K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \xrightarrow{K_{n-1}(C_r^*(i_1)) \oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1)) \xrightarrow{K_{n-1}(C_r^*(j_1)) - K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots$$

where  $i_1, i_2, j_1$ , and  $j_2$  are the obvious inclusions, see [812, Theorem 18 on page 632]. If  $\phi: G \to G$  is a group automorphism and  $G \rtimes_{\phi} \mathbb{Z}$  is the associated semidirect product, then there is a long exact sequence

(1.5)  

$$\cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(\phi)) - \mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_{\phi} \mathbb{Z}))$$

$$\xrightarrow{\partial_n} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(\phi)) - \mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots$$

where k is the obvious inclusion, see [811, Theorem 3.1 on page 151] or more generally [812, Theorem 18 on page 632].

We compare this with group homology in order to explain the analogy with homology. Recall that the *classifying space BG* of a group *G* is an aspherical *CW*complex whose fundamental group is isomorphic to *G* and that *aspherical* means that all higher homotopy groups are trivial, or, equivalently, that the universal covering is contractible. The classifying space *BG* is unique up to homotopy. If one has an amalgamated free product  $G = G_1 *_{G_0} G_2$ , then one can find models for the classifying spaces such that  $BG_i$  is a *CW*-subcomplex of *BG* and  $BG = BG_1 \cup BG_2$ and  $BG_0 = BG_1 \cap BG_2$ . Thus we obtain a pushout of inclusions of *CW*-complexes

$$\begin{array}{c|c} BG_0 \xrightarrow{Bi_1} BG_1 \\ Bi_2 & & \downarrow Bj_1 \\ BG_2 \xrightarrow{Bj_2} BG. \end{array}$$

It yields a long Mayer-Vietoris sequence for the cellular or singular homology

$$(1.6) \quad \cdots \xrightarrow{\partial_{n+1}} H_n(BG_0) \xrightarrow{H_n(Bi_1) \oplus H_n(Bi_2)} H_n(BG_1) \oplus H_n(BG_2)$$

$$\xrightarrow{H_n(Bj_1) - H_n(Bj_2)} H_n(BG) \xrightarrow{\partial_n} H_{n-1}(BG_0)$$

$$\xrightarrow{H_{n-1}(Bi_1) \oplus H_{n-1}(Bi_2)} H_{n-1}(BG_2) \oplus H_{n-1}(BG_1)$$

$$\xrightarrow{H_{n-1}(Bj_1) - H_{n-1}(Bj_2)} H_{n-1}(BG) \xrightarrow{\partial_{n-1}} \cdots$$

If  $\phi: G \to G$  is a group automorphism, then a model for  $B(G \rtimes_{\phi} \mathbb{Z})$  is given by the mapping torus of  $B\phi: BG \to BG$ , which is obtained from the cylinder  $BG \times [0, 1]$  by identifying the bottom and the top with the map  $B\phi$ . Associated to a mapping torus, there is the long exact sequence

(1.7) 
$$\cdots \xrightarrow{\partial_{n+1}} H_n(BG) \xrightarrow{H_n(B\phi) - \mathrm{id}} H_n(BG) \xrightarrow{H_n(Bk)} H_n(B(G \rtimes_{\phi} \mathbb{Z}))$$
  
 $\xrightarrow{\partial_n} H_{n-1}(BG) \xrightarrow{H_{n-1}(B\phi) - \mathrm{id}} H_{n-1}(BG) \xrightarrow{H_n(Bk)} \cdots$ 

where k is the obvious inclusion of BG into the mapping torus.

#### 1.3.3 The Baum-Connes Conjecture for Torsionfree Groups

There is an obvious analogy between the sequences (1.4) and (1.6) and the sequences (1.5) and (1.7). On the other hand we get for the trivial group  $G = \{1\}$ that  $H_n(B\{1\}) = H_n(\{\bullet\})$  is  $\mathbb{Z}$  for n = 0 and trivial for  $n \neq 0$  so that the group homology of *BG* cannot be the same as the topological *K*-theory of  $C_r^*(\{1\})$ . But there is a better candidate, namely take the *topological K*-theory of *BG* instead of the singular homology. Topological *K*-homology is a homology theory defined for *CW*-complexes. At least we mention that for a topologist its definition is routine, namely, it is the homology theory associated to the *K*-theory spectrum which defines the topological *K*-theory of *CW*-complexes, i.e., the cohomology theory which comes from considering vector bundles over *CW*-complexes. In contrast to singular homology, the topological *K*-homology of a point  $K_n(\{\bullet\})$  is  $\mathbb{Z}$  for even *n* and is trivial for *n* odd. So we still get exact sequences (1.6) and (1.7) if we replace  $H_*$  by  $K_*$  everywhere and we have  $K_n(B\{1\}) \cong K_n(C_r^*(\{1\}))$  for all  $n \in \mathbb{Z}$ . This leads to the following conjecture.

**Conjecture 1.8 (Baum-Connes Conjecture for torsionfree groups).** Let *G* be a torsionfree group. Then there is for every  $n \in \mathbb{Z}$  an isomorphism, called an *assembly map*,

$$K_n(BG) \xrightarrow{=} K_n(C_r^*(G)).$$

This is indeed a formulation which will turn out to be equivalent to the Baum-Connes Conjecture 1.1, provided that *G* is torsionfree. Conjecture 1.8 cannot hold in general as the example of a finite group *G* already shows. Namely, if *G* is finite, then the obvious inclusion induces an isomorphism  $K_n(B\{1\}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}}$  $\mathbb{Q}$  for every  $n \in \mathbb{Z}$ , whereas  $K_0(C_r^*(\{1\}) \to K_0(C_r^*(G)))$  agrees with the map  $R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(G)$ , which is rationally bijective if and only if *G* itself is trivial. Hence Conjecture 1.8 is not true for non-trivial finite groups.

#### 1.3.4 The Baum-Connes Conjecture

What is going wrong? The sequences (1.4) and (1.5) exist regardless of whether the groups are torsionfree or not. More generally, if G acts on a tree, then they can be combined to compute the K-theory  $K_*(C_r^*(G))$  of a group G by a certain Mayer-Vietoris sequence from the stabilizers of the vertices and edges, see Pimsner [812, Theorem 18 on page 632]). In the special case where all stabilizers are finite, one sees that  $K_*(C_r^*(G))$  is built by the topological K-theory of the finite subgroups of G in a homological fashion. This leads to the idea that  $K_*(C_r^*(G))$  can be computed in a homological way, but the building blocks do not only consist of  $K_*(C_r^*(\{1\}))$  alone but of  $K_*(C_r^*(H))$  for all finite subgroups  $H \subseteq G$ . This suggests to study *equivariant* topological K-theory. It assigns to every proper G-CW-complex X a sequence of abelian groups  $K_n^G(X)$  for  $n \in \mathbb{Z}$  such that G-homotopy invariance holds and Mayer-Vietoris sequences exist. A proper G-CW-complex is a CW-complex with G-action such that for every  $g \in G$  and every open cell e with  $e \cap g \cdot e \neq \emptyset$  we have gx = x for all  $x \in e$  and all isotropy groups are finite. Two interesting features are that  $K_n^G(G/H)$ agrees with  $K_n(C_r^*(H))$  for every finite subgroup  $H \subseteq G$  and that for a free G-CWcomplex X and  $n \in \mathbb{Z}$  we have a natural isomorphism  $K_n^G(X) \xrightarrow{\cong} K_n(G \setminus X)$ . Recall that EG is a free G-CW-complex which is contractible and that  $EG \rightarrow G \setminus EG = BG$ is the universal covering of BG. We can reformulate Conjecture 1.8 by stating an isomorphism

$$K_n^G(EG) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Now suppose that G acts on a tree T with finite stabilizers. Then the computation of Pimsner [812, Theorem 18 on page 632]) mentioned above can be rephrased to the statement that there is an isomorphism

$$K_n^G(T) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular the left-hand side is independent of the tree T, on which G acts by finite stabilizers. This can be explained as follows. It is known that for every finite subgroup  $H \subseteq G$  the H-fixed point set T is again a non-empty tree and hence contractible. This implies that two trees  $T_1$  and  $T_2$ , on which G acts with finite stabilizers, are G-homotopy equivalent and hence have the same equivariant topological K-theory. The same remark applies to  $K_n(BG)$  and  $K_n^G(EG)$ , namely, two models for BG are homotopy equivalent and two models for EG are G-homotopy equivalent and two models for EG are G-homotopy equivalent and therefore  $K_n(BG)$  and  $K_n^G(EG)$  are independent of the choice of a model. This leads to the idea to look for an appropriate proper G-CW-complex  $\underline{E}G = E_{\mathcal{FIN}}(G)$ , which is characterized by a certain universal property and is unique up to G-homotopy, such that for a torsionfree group G we have  $EG = \underline{E}G$ , for a tree on which G acts with finite stabilizers, we have EG = T, and there is an isomorphism

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G)).$$

In particular for a finite group we would like to have  $\underline{E}G = G/G = \{\bullet\}$  and then the desired isomorphism above is true for trivial reasons. Recall that EG is characterized up to *G*-homotopy by the property that it is a *G*-*CW*-complex such that  $EG^H$  is empty for  $H \neq \{1\}$  and is contractible for  $H = \{1\}$ . Having the case of a tree on which *G* acts with finite stabilizers in mind, we define the *classifying space for proper G*-actions  $\underline{E}G$  to be a *G*-*CW*-complex such that  $\underline{E}G^H$  is empty for  $|H| = \infty$  and is contractible for  $|H| < \infty$ . Indeed, two models for  $\underline{E}G$  are *G*-homotopy equivalent, a tree on which *G* acts with finite stabilizers is a model for  $\underline{E}G$ , we have  $EG = \underline{E}G$  if and only if *G* is torsionfree, and  $\underline{E}G = G/G = \{\bullet\}$  if and only if *G* is finite. This leads to the Baum-Connes Conjecture, stated already as Conjecture 1.1. Classifying spaces for families will be treated in detail in Chapter 11.

The Baum-Connes Conjecture 1.1 makes sense for all groups, and no counterexamples are known at the time of writing. The Baum-Connes Conjecture 1.1 reduces in the torsionfree case to Conjecture 1.8 and is consistent with the result of Pimsner [812, Theorem 18 on page 632] for *G* acting on a tree with finite stabilizers. It is obviously true for finite groups *G*. Pimsner's result holds more generally for groups acting on trees with not necessarily finite stabilizers. So one should get the analogous result for the left-hand side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. Essentially this boils down to the question whether the analogs of the long exact sequences (1.4) and (1.5) hold for the left side of the isomorphism appearing in the Baum-Connes Conjecture 1.1. This follows for (1.4) from the fact that for  $G = G_1 *_{G_0} G_2$  one can find appropriate models for the classifying spaces for proper *G*-actions such that there is a *G*-pushout of inclusions of proper *G*-*CW*-complexes

and for a subgroup  $H \subseteq G$  and a proper *H*-*CW*-complex *X* there is a natural isomorphism

$$K_n^H(X) \xrightarrow{=} K_n^G(G \times_H X)$$

Thus the associated long exact Mayer-Vietoris sequence yields the long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n^{G_0}(\underline{E}G_0) \to K_n^{G_1}(\underline{E}G_1) \oplus K_n^{G_2}(\underline{E}G_2) \to K_n^G(\underline{E}G) \xrightarrow{\partial_n} \\ K_{n-1}^{G_0}(\underline{E}G_0) \to K_{n-1}^{G_1}(\underline{E}G_1) \oplus K_{n-1}^{G_2}(\underline{E}G_2) \to K_{n-1}^{G_0}(\underline{E}G) \to \cdots$$

which corresponds to (1.4). For (1.5) one uses the fact that for a group automorphism  $\phi: G \xrightarrow{\cong} G$  the  $G \rtimes_{\phi} \mathbb{Z}$ -*CW*-complex given by the bilaterally infinite mapping telescope of the  $\phi$ -equivariant map  $\underline{E}\phi: \underline{E}G \to \underline{E}G$  is a model for  $\underline{E}(G \rtimes_{\phi} \mathbb{Z})$ .

In general  $K_n^G(\underline{E}G)$  is much bigger than  $K_n^G(\underline{E}G) \cong K_n(BG)$  and the canonical map  $K_n^G(\underline{E}G) \to K_n^G(\underline{E}G)$  is rationally injective but not necessarily integrally injective.

#### **1.3.5 Reduced versus Maximal Group** C\*-Algebras

All the arguments above also apply to the maximal group  $C^*$ -algebra, which has even better functorial properties than the reduced group  $C^*$ -algebra. So a priori one may think that one should use the maximal group  $C^*$ -algebra instead of the reduced one. However, the version for the maximal group  $C^*$ -algebra is not true in general and the version for the reduced group  $C^*$ -algebra seems to be the right one. This will be discussed in more detail in subsection 14.5.1.

If one considers instead of the reduced group  $C^*$ -algebra the Banach group algebra  $L^1(G)$ , one obtains the *Bost Conjecture* 14.23.

#### 1.3.6 Applications of the Baum-Connes Conjecture

The assembly map appearing in the Baum-Connes Conjecture 1.1 has an indextheoretic interpretation. An element in  $K_0^G(\underline{E}G)$  can be represented by a pair  $(M, P^*)$  consisting of a cocompact proper smooth *n*-dimensional *G*-manifold *M* with a *G*-invariant Riemannian metric together with an elliptic *G*-complex  $P^*$  of differential operators of order 1 on *M* and its image under the assembly map is a certain equivariant index  $\operatorname{ind}_{C_r^*(G)}(M, P^*)$  in  $K_n(C_r^*(G))$ . There are many important consequences of the Baum-Connes Conjecture such as the *Kadison Conjecture*, see Subsection 10.4.2, the *stable Gromov-Lawson-Rosenberg Conjecture*, see Subsection 14.8.4, the *Novikov Conjecture*, see Section 9.14, and the (*Modified*) Trace *Conjecture*, see Subsections 10.4.1 and 14.8.3.

A summary of the applications of the Baum-Connes Conjecture is given in Section 14.8.

# **1.4 Motivation for and Evolution of the Farrell-Jones Conjecture** for *K*-Theory

Next we want to deal with the algebraic K-groups  $K_n(RG)$  of the group ring RG.

1.4 Motivation for and Evolution of the Farrell-Jones Conjecture for K-Theory

#### 1.4.1 Algebraic K-Theory of Group Rings

For an associative ring with unit *R* one defines  $K_0(R)$  to be the *projective class* group of *R* and  $K_1(R)$  to be the abelianization of  $GL(R) = \operatorname{colim}_{n\to\infty} GL_n(R)$ . The *higher algebraic K*-groups  $K_n(R)$  for  $n \ge 1$  are the homotopy groups of a certain *K*-theory space associated to the category of finitely generated projective *R*-modules. One can define *negative K*-groups  $K_n(R)$  for  $n \le -1$  by a certain contracting procedure applied to  $K_0(R)$ . Finally there exists a *K*-theory spectrum  $\mathbf{K}(R)$  such that  $\pi_n(\mathbf{K}(R)) = K_n(R)$  holds for every  $n \in \mathbb{Z}$ . If  $\mathbb{Z} \to R$  is the obvious ring map sending n to  $n \cdot 1_R$ , then one defines for  $n \le 1$  the *reduced K*-groups to be the cokernel of the induced map  $K_n(\mathbb{Z}) \to K_n(R)$ . The Whitehead group Wh(*G*) of a group *G* is the quotient of  $K_1(\mathbb{Z}G)$  by elements given by (1, 1)-matrices of the shape  $(\pm g)$  for  $g \in G$ .

The reduced projective class group  $\widetilde{K}_0(\mathbb{Z}G)$  is the recipient for the finiteness obstruction of a finitely dominated CW-complex X with fundamental group  $G = \pi_1(X)$ . Finitely dominated means that there is a finite CW-complex Y and maps  $i: X \to Y$  and  $r: Y \to X$  such that  $r \circ i$  is homotopic to the identity on X. The Whitehead group Wh(G) is the recipient of the Whitehead torsion of a homotopy equivalence of finite CW-complexes and of a compact h-cobordism over a closed manifold, where G is the fundamental group. An h-cobordism W over M consists of a manifold W whose boundary is the disjoint union  $\partial W = \partial_0 W [ ] \partial_1 W$ such that both inclusions  $\partial_i W \to W$  are homotopy equivalences, together with an isomorphism  $M \xrightarrow{\cong} \partial_0 W$ . The finiteness obstruction and the Whitehead torsion are very important topological obstructions whose vanishing has interesting geometric and topological consequences. The finiteness obstruction vanishes if and only if the finitely dominated CW-complex under consideration is homotopy equivalent to a finite CW-complex. The Whitehead torsion of a compact h-cobordism W over M of dimension  $\geq 6$  vanishes if and only if W is trivial, i.e., is isomorphic to a cylinder  $M \times [0, 1]$  relative  $M = M \times \{0\}$ . This explains why topologists are interested in  $K_n(\mathbb{Z}G)$  for groups G.

All these definitions and results will be explained in Chapters 2, 3, 4, 5, and 6.

#### 1.4.2 Appearance of Nil-Terms

The situation for the algebraic *K*-theory of *RG* is more complicated than the one for the topological *K*-theory of  $C_r^*(G)$ . As a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

For algebraic K-theory the analog is the Bass-Heller-Swan decomposition

$$K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R)$$

where certain additional terms, the Nil-*terms*  $NK_n(R)$ , appear, see Subsection 6.3.4. If one replaces *R* by *RG*, one gets

$$K_n(R[G \times \mathbb{Z}]) \cong K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG).$$

Such correction terms in the form of Nil-terms also appear when one wants to get analogs of the sequences (1.4) and (1.5) for algebraic *K*-theory, see Section 6.9.

# **1.4.3** The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings and Torsionfree Groups

Let *R* be a *regular ring*, i.e., it is Noetherian and every *R*-module possesses a finite-dimensional projective resolution. For instance, any principal ideal domain is a regular ring. Then one can prove in many cases for torsionfree groups that the analogs of the sequences (1.4) and (1.5) hold for algebraic *K*-theory, see Waldhausen [974] and [977]. The same reasoning as in the Baum-Connes Conjecture for torsionfree groups leads to the following conjecture.

Conjecture 1.9. (Farrell-Jones Conjecture for  $K_*(RG)$  for torsionfree groups and regular rings). Let *G* be a torsionfree group and let *R* be a regular ring. Then there is for every  $n \in \mathbb{Z}$  an isomorphism

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{=} K_n(RG).$$

Here  $H_*(-; \mathbf{K}(R))$  is the homology theory associated to the *K*-theory spectrum of *R*. It is a homology theory with the property that  $H_n(\{\bullet\}; \mathbf{K}(R)) = \pi_n(\mathbf{K}(R)) = K_n(R)$  holds for every  $n \in \mathbb{Z}$ .

#### **1.4.4** The Farrell-Jones Conjecture for $K_*(RG)$ for Regular Rings

If one drops the condition that G is torsionfree but requires that the order of every finite subgroup of G is invertible in R, then in many cases one can still prove that the analogs of the sequences (1.4) and (1.5) hold for algebraic K-theory. The same reasoning as in the Baum-Connes Conjecture leads to the following conjecture.

**Conjecture 1.10.** (Farrell-Jones Conjecture for  $K_*(RG)$  for regular rings). Let *G* be a group. Let *R* be a regular ring such that |H| is invertible in *R* for every finite subgroup  $H \subseteq G$ . Then there is for every  $n \in \mathbb{Z}$  an isomorphism

$$H_n^G(\underline{E}G;\mathbf{K}_R) \xrightarrow{\cong} K_n(RG).$$

Here  $H_n^G(-; \mathbf{K}_R)$  is an appropriate *G*-homology theory with the property that  $H_n^G(G/H; \mathbf{K}_R) \cong H_n^H(\{\bullet\}; \mathbf{K}_R) \cong K_n(RH)$  holds for every subgroup  $H \subseteq G$  and every  $n \in \mathbb{Z}$ , and the isomorphism above is induced by the *G*-map  $\underline{E}G \to \{\bullet\}$ . Conjecture 1.10 reduces to Conjecture 1.9 if *G* is torsionfree.

#### **1.4.5** The Farrell-Jones Conjecture for $K_*(RG)$

Conjecture 1.9 can be applied in the case  $R = \mathbb{Z}$ , which is not true for Conjecture 1.10. So what is the right formulation for arbitrary rings R? The idea is that one not only needs to take all finite subgroups into account but also all virtually cyclic subgroups. A group is called *virtually cyclic* if it is finite or contains  $\mathbb{Z}$  as subgroup of finite index. Namely, let  $\underline{E}G = E_{VCY}(G)$  be the *classifying space for the family of virtually cyclic subgroups*, i.e., a *G-CW*-complex  $\underline{E}G$  such that  $\underline{E}G^H$  is contractible for every virtually cyclic subgroup  $H \subseteq G$  and is empty for every subgroup  $H \subseteq G$ which is not virtually cyclic. The *G*-space  $\underline{E}G$  is unique up to *G*-homotopy. These considerations lead to the Farrell-Jones Conjecture for  $K_*(RG)$  stated already as Conjecture 1.2.

Conjecture 1.2 makes sense for all groups and rings, and no counterexamples are known at the time of writing. We have absorbed all the Nil-phenomena into the source by replacing  $\underline{E}G$  by  $\underline{\underline{E}}G$ . There is a certain price to pay since often there are nice small geometric models for  $\underline{E}G$ , whereas the spaces  $\underline{\underline{E}}G$  are much harder to analyze and are in general huge. There are up to *G*-homotopy unique *G*-maps  $EG \rightarrow \underline{\underline{E}}G$  and  $\underline{\underline{E}}G \rightarrow \underline{\underline{E}}G$  which yield maps

$$H_n(BG; \mathbf{K}(R)) \cong H_n^G(EG; \mathbf{K}_R) \to H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(\underline{E}G; \mathbf{K}_R).$$

We will later see that there is a splitting, see Theorem 13.36,

(1.11) 
$$H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) \cong H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) \oplus H_n^G(\underline{\underline{E}}G,\underline{\underline{E}}G;\mathbf{K}_R)$$

where  $H_n^G(\underline{E}G; \mathbf{K}_R)$  is the comparatively easy homological part and all Nil-type information is contained in  $H_n^G(\underline{E}G, \underline{E}G; \mathbf{K}_R)$ . If *R* is regular and the order of any finite subgroup of *G* is invertible in *R*, then  $H_n^G(\underline{E}G, \underline{E}G; \mathbf{K}_R)$  is trivial and hence the natural map  $H_n^G(\underline{E}G; \mathbf{K}_R) \xrightarrow{\cong} H_n^G(\underline{E}G; \mathbf{K}_R)$  is bijective. Therefore Conjecture 1.2 reduces to Conjecture 1.9 and Conjecture 1.10 when they apply.

In the Baum-Connes setting the natural map  $K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(\underline{E}G)$  is always bijective.

#### **1.4.6** Applications of the Farrell-Jones Conjecture for $K_*(RG)$

We have  $K_n(\mathbb{Z}) = 0$  for  $n \leq -1$ . Both the map  $\mathbb{Z} \xrightarrow{\cong} K_0(\mathbb{Z})$  that sends n to  $n \cdot [\mathbb{Z}]$  and the map  $\{\pm 1\} \to K_1(\mathbb{Z})$  that sends  $\pm 1$  to the class of the (1, 1)-matrix  $(\pm 1)$  are bijective. Therefore an easy spectral sequence argument shows that Conjecture 1.9 implies

**Conjecture 1.12.** (Farrell-Jones Conjecture  $K_n(\mathbb{Z}G)$  in dimensions  $n \le 1$ ). Let *G* be a torsionfree group. Then  $\widetilde{K}_n(\mathbb{Z}G) = 0$  for  $n \in \mathbb{Z}, n \le 0$  and Wh(G) = 0.

In particular, the finiteness obstruction and the Whitehead torsion are always zero for torsionfree fundamental groups. This implies that every *h*-cobordism over a simply connected *d*-dimensional closed manifold for  $d \ge 5$  is trivial and thus the Poincaré Conjecture in dimensions  $\ge 6$  (and with some extra effort also in dimension d = 5). This will be explained in Section 3.5. The Farrell-Jones Conjecture for *K*-theory, see Conjecture 1.2, implies the *Bass Conjecture*, see Section 2.10. *Kaplansky's Idempotent Conjecture* follows from the Farrell-Jones Conjecture for *K*-theory for torsionfree groups and regular rings, see Conjecture 1.9, as explained in Section 2.9. Further applications of the Conjecture 1.9, e.g., to pseudoisotopy and to automorphisms of manifolds, will be discussed in Section 9.21.

A summary of the applications of the Farrell-Jones Conjecture is given in Section 13.12.

# **1.5** Motivation for and Evolution of the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

Next we want to deal with the algebraic *L*-groups  $L_n^{\epsilon}(RG)$  of the group ring *RG* of a group *G* with coefficients in an associative ring *R* with unit and involution.

#### 1.5.1 Algebraic L-Theory of Group Rings

Let *R* be an associative ring with unit. An *involution of rings*  $R \to R, r \mapsto \overline{r}$  on *R* is a map satisfying  $\overline{r+s} = \overline{r} + \overline{s}$ ,  $\overline{rs} = \overline{s}\overline{r}$ ,  $\overline{0} = 0$ ,  $\overline{1} = 1$ , and  $\overline{\overline{r}} = r$  for all  $r, s \in R$ . Given a ring with involution, the group ring *RG* inherits an involution by  $\sum_{g \in G} r_g \cdot g = \sum_{g \in G} \overline{r} \cdot g^{-1}$ . If the coefficient ring *R* is commutative, we usually use the trivial involution  $\overline{r} = r$ . Given a ring with involution, one can associate to it *quadratic L-groups*  $L_n^h(R)$  for  $n \in \mathbb{Z}$ . The abelian group  $L_0^h(R)$  can be identified with the *Witt group* of non-degenerate quadratic forms on finitely generated free *R*-modules, where every hyperbolic quadratic form represents the zero element and the addition is given by the orthogonal sum of hyperbolic quadratic forms. The abelian group  $L_2^h(R)$  is essentially given by the skew-symmetric versions. One defines
$L_1^h(R)$  and  $L_3^h(R)$  in terms of automorphism of quadratic forms. The *L*-groups are four-periodic, i.e., there is a natural isomorphism  $L_n^h(R) \xrightarrow{\cong} L_{n+4}^h(R)$  for  $n \in \mathbb{Z}$ . If one uses finitely generated projective *R*-modules instead of finitely generated free *R*-modules, one obtains the *proper quadratic L-groups*  $L_n^p(R)$  for  $n \in \mathbb{Z}$ . For every  $j \in \{-\infty\}$  II  $\{j \in \mathbb{Z} \mid j \leq 1\}$  there are versions  $L_n^{\langle j \rangle}(R)$ , where  $\langle j \rangle$  is called a *decoration*. The decorations j = 0, 1 correspond to the decorations p, h. If R is  $\mathbb{Z}G$ , one uses finitely generated based free  $\mathbb{Z}G$ -modules and takes the Whitehead torsion into account, then one obtains the *simple quadratic L-groups*  $L_n^s(\mathbb{Z}G) = L_n^{\langle 2 \rangle}(\mathbb{Z}G)$ for  $n \in \mathbb{Z}$ .

The relevance of the L-groups comes from the fact that they are the recipients for various surgery obstructions. The fundamental surgery problem is the following. Consider a map  $f: M \to X$  from a closed manifold M to a finite Poincaré complex X. We want to know whether we can change it by a process called surgery to a map  $g: N \to X$  with a closed manifold N as source and the same target such that g is a homotopy equivalence. This may answer the question whether a finite Poincaré complex X is homotopy equivalent to a closed manifold. Note that a space which is homotopy equivalent to a closed manifold must be a finite Poincaré complex, but not every finite Poincaré complex is homotopy equivalent to a closed manifold. If f comes with additional bundle data and has degree 1, we can find g if and only if the so-called surgery obstruction of f vanishes, which takes values in  $L_n^h(\mathbb{Z}G)$  for  $n = \dim(X)$  and  $G = \pi_1(X)$ . If we want g to be a simple homotopy equivalence, the obstruction lives in  $L_n^s(\mathbb{Z}G)$ . We see that, analogous to the finiteness obstruction in  $K_0(\mathbb{Z}G)$  and the Whitehead torsion in Wh(G), the algebraic L-groups are the recipients for important obstructions whose vanishing has interesting geometric and topological consequences. Also the question whether two closed manifolds are diffeomorphic or homeomorphic can be decided via surgery theory, of which the L-groups are a part.

More explanations about *L*-groups and surgery theory will be given in Chapter 9.

#### **1.5.2** The Farrell-Jones Conjecture for $L_*(RG)[1/2]$

If we invert 2, i.e., if we consider the localization  $L_n^{\langle -j \rangle}(RG)[1/2]$ , then there is no difference between the various decorations and the analogs of the sequences (1.4) and (1.5) are true for *L*-theory, see Cappell [204]. The same reasoning as for the Baum-Connes Conjecture leads to the following conjecture.

**Conjecture 1.13.** (Farrell-Jones Conjecture for  $L_*(RG)[1/2]$ ). Let *G* be a group. Let *R* be an associative ring with unit and involution. Then there is for every  $n \in \mathbb{Z}$  and every decoration *j* an isomorphism

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle j \rangle})[1/2] \xrightarrow{\cong} L_n^{\langle j \rangle}(RG)[1/2]$$

Here  $H_n^G(-; \mathbf{L}_R^{\langle j \rangle})$  is an appropriate *G*-homology theory with the property that  $H_n^G(G/H; \mathbf{L}_R^{\langle j \rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle j \rangle}) \cong L_n^{\langle j \rangle}(RH)$  holds for every subgroup  $H \subseteq G$  and every  $n \in \mathbb{Z}$  and the isomorphism above is induced by the *G*-map  $\underline{E}G \to \{\bullet\}$ .

# **1.5.3** The Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

In general the *L*-groups  $L_n^{\langle j \rangle}(RG)$  depend on the decoration and often the 2-torsion carries sophisticated information and is hard to handle. Recall that as a special case of the sequence (1.5) we obtain an isomorphism

$$K_n(C_r^*(G \times \mathbb{Z})) = K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

The *L*-theory analog is given by the Shaneson splitting [913]

$$L_n^{\langle j \rangle}(R[\mathbb{Z}]) \cong L_{n-1}^{\langle j-1 \rangle}(R) \oplus L_n^{\langle j \rangle}(R)$$

Here for the decoration  $j = -\infty$  one has to interpret j - 1 as  $-\infty$ . Since  $S^1$  is a model for  $B\mathbb{Z}$ , we get an isomorphism

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j \rangle}(R)) \cong L_{n-1}^{\langle j \rangle}(R) \oplus L_n^{\langle j \rangle}(R).$$

Therefore the decoration  $-\infty$  shows the right homological behavior and is the right candidate for the formulation of an isomorphism conjecture.

The analog of the sequence (1.4) does not hold for  $L_*^{\langle j \rangle}(RG)$ , certain correction terms, the UNil-*terms* come in, which are independent of the decoration  $\langle j \rangle$  and are always (not necessarily finitely generated) 2-primary abelian groups, see Cappell [203], [204]. As in the algebraic *K*-theory case this leads to the Farrell-Jones Conjecture for  $L_*^{\langle -\infty \rangle}(RG)$ , stated already as Conjecture 1.3. The analog of the sequence (1.5) holds for  $L_*^{\langle -\infty \rangle}(RG)$ , see Theorem 13.60.

In Conjecture 1.3 the term  $H_n^G(-; \mathbf{L}_R^{\langle -\infty \rangle})$  is an appropriate *G*-homology theory such that  $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) \cong H_n^H(\{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RH)$  holds for every subgroup  $H \subseteq G$  and every  $n \in \mathbb{Z}$ , and the assembly map is induced by the map  $\underline{EG} \to \{\bullet\}$ . Conjecture 1.3 makes sense for all groups and rings with involution, and no counterexamples are known at the time of writing.

After inverting 2 Conjecture 1.3 is equivalent to Conjecture 1.13.

There is an *L*-theory version of the splitting (1.11)

$$(1.14) \qquad H_n^G(\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^G(\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}) \oplus H_n^G(\underline{\underline{E}}G,\underline{\underline{E}}G;\mathbf{L}_R^{\langle -\infty\rangle}),$$

provided that there exists an integer  $i_0$  such that  $K_i(RV) = 0$  holds for all virtually cyclic subgroups  $V \subseteq G$  and  $i \leq i_0$ .

1.5 Motivation for and Evolution of the Farrell-Jones Conjecture for  $L_*^{\langle -\infty \rangle}(RG)$ 

# **1.5.4** Applications of the Farrell-Jones Conjecture for $L_*^{\langle -\infty \rangle}(RG)$

For applications in geometry and topology the simple *L*-groups  $L_n^s(\mathbb{Z}G)$  are the most interesting ones. The difference between the various decorations is measured by the so-called *Rothenberg sequences* and given in terms of the Tate cohomology of  $\mathbb{Z}/2$ with coefficients in  $\widetilde{K}_n(\mathbb{Z}G)$  for  $n \leq 0$  and Wh(*G*) with respect to the involution coming from the standard involution on the group ring  $\mathbb{Z}G$  sending  $\sum_{g \in G} \lambda_g \cdot g$ to  $\sum_{g \in G} \lambda_g \cdot g^{-1}$ . Hence the decorations do not matter if  $\widetilde{K}_n(\mathbb{Z}G)$  for  $n \leq 0$  and Wh(*G*) vanish. In view of Conjecture 1.12, this leads to the following version of Conjecture 1.3 for torsionfree groups

**Conjecture 1.15.** (Farrell-Jones Conjecture for  $L_*(\mathbb{Z}G)$  for torsionfree groups). Let *G* be a torsionfree group. Then there is for every  $n \in \mathbb{Z}$  and every decoration *j* an isomorphism

$$H_n(BG; \mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}G).$$

Moreover, the source, target, and the map itself are independent of the decoration *j*.

Here  $H_n(-; \mathbf{L}^{\langle j \rangle}(\mathbb{Z}))$  is the homology theory associated to the *L*-theory spectrum  $\mathbf{L}^{\langle -j \rangle}(\mathbb{Z})$  and satisfies  $H_n(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \cong \pi_n(\mathbf{L}^{\langle j \rangle}(\mathbb{Z})) \cong L_n^{\langle j \rangle}(\mathbb{Z})$ .

The *L*-theoretic assembly map appearing in Conjecture 1.15 has a topological meaning. It appears in the so-called *Surgery Exact Sequence*, which we will discuss in more detail in Section 9.12. Let  $\mathbf{L}^{s}(\mathbb{Z})\langle 1 \rangle$  be the 1-connected cover  $\mathbf{L}^{s}(\mathbb{Z})\langle 1 \rangle$  of  $\mathbf{L}^{s}(\mathbb{Z})$ . There is a canonical map  $\iota$ :  $H_{n}(BG; \mathbf{L}^{s}(\mathbb{Z})\langle 1 \rangle) \rightarrow H_{n}(BG; \mathbf{L}^{s}(\mathbb{Z}))$ . Let N be an aspherical oriented closed manifold with fundamental group G, i.e., an oriented closed manifold homotopy equivalent to BG. Then G is torsionfree, the source of the composite  $H_{n}(BG; \mathbf{L}^{s}(\mathbb{Z})\langle 1 \rangle) \rightarrow L_{n}^{s}(RG)$  of the assembly map appearing in Conjecture 1.15 with  $\iota$  consists of bordism classes of normal maps  $M \rightarrow N$  with N as target, and the composite sends such a normal map to its surgery obstruction. This is analogous to the Baum-Connes setting where the assembly map can be described by assigning to an equivariant index problem its index.

The third term in the Surgery Exact Sequence is the so-called *structure set* of N. It is the set of equivalence classes of simple homotopy equivalences  $f_0: M_0 \to N$  with a closed topological manifold as source and N as target where  $f_0: M_0 \to N$  and  $f_1: M_1 \to N$  are equivalent if there is a homeomorphism  $g: M_0 \to M_1$  such that  $f_1 \circ g$  and  $f_0$  are homotopic. Conjecture 1.15 implies that this structure set is trivial provided that the dimension of N is greater or equal to five. Hence Conjecture 1.15 implies in dimensions  $\geq 5$  the following famous conjecture if G is isomorphic to the fundamental group.

**Conjecture 1.16 (Borel Conjecture).** Let M and N be two aspherical closed topological manifolds whose fundamental groups are isomorphic. Then they are homeomorphic, and every homotopy equivalence from M to N is homotopic to a homeomorphism.

The Borel Conjecture is a topological rigidity theorem for aspherical closed manifolds and analogous to the *Mostow Rigidity Theorem*, which says that two hyperbolic closed Riemannian manifolds with isomorphic fundamental groups are isometrically diffeomorphic. The Borel Conjecture is false if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. Its connection to the Borel Conjecture is one of the main features of the Farrell-Jones Conjecture. More details will be given in Subsections 9.15.2 and 9.15.3.

The Farrell-Jones Conjecture for *L*-theory 1.3 implies the *Novikov Conjecture*, see Section 9.14. It also has applications to the problem whether Poincaré duality groups or torsionfree hyperbolic groups with spheres as boundary are fundamental groups of aspherical closed manifolds, see Sections 9.17 and 9.18. Product decompositions of aspherical closed manifolds are treated in Section 9.20.

A summary of the applications of the Farrell-Jones Conjecture is given in Section 13.12.

## **1.6 More General Versions of the Farrell-Jones Conjecture**

We will also treat versions of the Farrell-Jones Conjecture in equivariant additive categories, or more generally, in equivariant higher categories, see Sections 13.3 and 13.4. There will be versions with finite wreath products, see Section 13.5. The most general version is the Full Farrell-Jones Conjecture 13.30, see Section 13.6, which implies all other variants of the Farrell-Jones Conjecture, see Section 13.11.

## 1.7 Status of the Baum-Connes and the Farrell-Jones Conjecture

A detailed report on the groups for which these conjectures have been proved will be given in Chapter 16. For example, the Baum-Connes Conjecture 1.1 is known for a class of groups which includes amenable groups, hyperbolic groups, knot groups, fundamental groups of compact 3-manifolds (possibly with boundary), and onerelator groups, but is open for  $SL_n(\mathbb{Z})$  for  $n \ge 3$ , where for a commutative ring R we write  $SL_n(R)$  for the group of invertible (n, n)-matrices with det(A) = 1. The class of groups for which the Farrell-Jones Conjectures 1.2 and 1.3 have been proved contains hyperbolic groups, finite-dimensional CAT(0)-groups, fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary), solvable groups, lattices in almost connected Lie groups, and arithmetic groups, but they are open for amenable groups in general. If one allows coefficients, one can prove inheritance properties for the Baum-Connes Conjecture and the Farrell-Jones Conjecture, e.g., the class of groups for which they are true is closed under taking subgroups, finite direct products, free products, colimits over directed sets whose structure map are injective in the Baum-Connes case and can be arbitrary in the Farrell-Jones case. This will be explained in Sections 13.7 and 14.6.

#### 1.8 Structural Aspects

The Full Farrell-Jones Conjecture 13.30, which implies all other variants of the Farrell-Jones Conjecture, is known to be true for some groups with unusual properties, e.g., groups with expanders, Tarski monsters, lacunary hyperbolic groups, subgroups of finite products of hyperbolic groups, self-similar groups, see Theorem 16.1. At the time of writing we have no specific candidate of a group or of a general property of groups such that the Full Farrell-Jones Conjecture 13.30, or one of its consequences, e.g., the Novikov Conjecture and the Borel Conjecture, might be false. So we have no good starting point for a search for counterexamples, see Section 16.10.

At the time of writing no counterexample to the Baum-Connes Conjecture is known to the author. There exist counterexamples to the Baum-Connes Conjecture with coefficients, as explained in Section 16.10.

## **1.8 Structural Aspects**

#### 1.8.1 The Meta-Isomorphism Conjecture

The formulations of the Baum-Connes Conjecture 1.1 and of the Farrell-Jones Conjecture 1.2 and 1.3 are very similar in the homological picture. It allows a formulation of the following *Meta-Isomorphism Conjecture*, of which both conjectures are special cases and which also has other very interesting specializations, e.g., for pseudoisotopy, *A*-theory, topological Hochschild homology, and topological cyclic homology, see Section 15.2.

**Meta-Isomorphism Conjecture 1.17.** *Given a group G, a G-homology theory*  $\mathcal{H}^G_*$ *, and a family*  $\mathcal{F}$  *of subgroups of G, we say that the* Meta-Isomorphism Conjecture *is satisfied if the G-map*  $E_{\mathcal{F}}(G) \to \{\bullet\}$  *induces for every*  $n \in \mathbb{Z}$  *an isomorphism* 

$$A_{\mathcal{F}}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\{\bullet\}).$$

This general formulation is an excellent framework to construct transformations between the assembly maps appearing in different Isomorphism Conjectures. For instance, the *cyclotomic trace* relates the *K*-theoretic Farrell-Jones Conjecture with coefficients in  $\mathbb{Z}$  to the Isomorphism Conjecture for topological cyclic homology, see Subsection 15.14.3, and via *symmetric signatures* one can link the Farrell-Jones Conjecture, see Subsection 15.14.4. Moreover, basic computational tools and techniques for equivariant homology theories apply both to the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3.

#### 1.8.2 Assembly

One important idea is the assembly principle, which leads to assembly maps in a canonical and universal way by asking for the best approximation of a homotopy invariant functor from G-spaces to spectra by an equivariant homology theory. It is an important ingredient for the identification of the various descriptions of assembly maps appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture. For instance, the assembly map appearing in the Baum-Connes Conjecture 1.1 can be interpreted as assigning to an appropriate equivariant elliptic complex its equivariant index, and the assembly map appearing in the L-theoretic Farrell-Jones Conjecture 1.3 is related to the map appearing in the Surgery Exact Sequence, which assigns to a surgery problem its surgery obstruction. We have already explained above that these identifications are the basis for some of applications of the Isomorphism Conjectures, and we will see that they are also important for proofs. There is a homotopy-theoretic approach to the assembly map based on homotopy colimits over the orbit category, which motivates the name assembly. Roughly speaking, the name assembly refers to assembling the values of the K-and L-groups of the reduced group  $C^*$ -algebra or the group ring of a group G from their values on finite or virtually cyclic subgroups of G. All this will be explained in Chapter 18.

This parallel treatment of the Baum-Connes Conjecture and the Farrell-Jones Conjecture and of other variants is one of the topics of this book. However, the geometric interpretations of the assembly maps in terms of indices, surgery obstructions, or forget control are quite different. Therefore the methods of proof for the Farrell-Jones Conjecture and the Baum-Connes Conjecture use different input. Although there are some similarities in the proofs, it is not clear how to export methods of proof from one conjecture to the other.

#### **1.9 Computational Aspects**

In general the target  $K_n(C_r^*(G))$  of the assembly map appearing in the Baum-Connes Conjecture 1.1 is very hard to compute, whereas the source  $K_n^G(\underline{E}G)$  is much more accessible because one can apply standard techniques from algebraic topology such as spectral sequences and equivariant Chern characters and there are often nice small geometric models for  $\underline{E}G$ . For the Farrell-Jones Conjectures 1.2 and 1.3, this applies also to the parts  $H_n^G(\underline{E}G; \mathbf{K}_R)$  and  $H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle})$  respectively appearing in the splittings (1.11) and (1.14). The other parts  $H_n^G(\underline{E}G, \underline{E}G; \mathbf{K}_R)$  or  $H_n^G(\underline{E}G, \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle})$  are harder to handle, since they involve Nil- or UNil-terms and the  $\overline{G}$ -CW-complex  $\underline{E}G$  is not proper and in general huge. Most of the known computations of  $K_n(C_r^*(G)), K_n(RG)$ , and  $L_n^{\langle j \rangle}(RG)$  are based on the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3. Classifications of manifolds and of  $C^*$ -algebras rely on and thus motivate explicit calculations of K- and L-groups. In this context it is often important not only to determine the K- and L-groups abstractly, but to develop detection techniques so that one can identify or distinguish specific elements associated to the original classification problem or give geometric or index-theoretic interpretations to elements in the K- and L-groups.

A general guide for computations and a list of known cases including applications to classification problems will be given in Chapter 17.

# **1.10** Are the Baum-Connes Conjecture and the Farrell-Jones Conjecture True in General?

The title of this section is the central and at the time of writing unsolved question. One motivation for writing this monograph is to stimulate some very clever mathematician to work on this problem and finally find an answer. Let us speculate about the possible answer.

We are skeptical about the Baum-Connes Conjecture for two reasons: there are counterexamples for the version with coefficients, and the left side of the Baum-Connes assembly map is functorial under arbitrary group homomorphisms, whereas the right side is not. The Bost Conjecture, which predicts an isomorphism

$$K_n^G(\underline{E}G) \to K_n(L^1(G)),$$

has a much better chance to be true in general. The possible failure of the Baum-Connes Conjecture may come from the possible failure of the canonical map  $K_n(L^1(G)) \to K_n(C^r_*(G))$  to be bijective.

In spite of the Baum-Connes Conjecture, we do not see an obvious flaw with the Bost Conjecture or the Farrell-Jones Conjecture. As explained in Section 1.7 above, we have no starting point for the construction of a counterexample, and all abstract properties we know for the right side do hold for the left side of the assembly map and vice versa. In particular for the Bass Conjecture and for the Novikov Conjecture which follow from the Farrell-Jones Conjecture, the class of groups for which they are known to be true is impressive. There are some conclusions from the Farrell-Jones Conjecture which are not trivial and true for all groups. These are arguments in favor of a positive answer

The following arguments are in favor of a negative answer. The universe of groups is overwhelmingly large. We have Gromov's saying on our neck that a statement which holds for all groups is either trivial or false. We have no philosophical reason why the Bost Conjecture or the Farrell-Jones Conjecture should be true in general. Finding a counterexample will probably require some new ideas, maybe from logic or random groups. The upshot of this discussion is that the author is skeptical about the Baum-Connes Conjecture, but does not dare to make any predictions about the chances for the other conjectures, in particular for the Novikov Conjecture, to be true for all groups.

We will elaborate on this discussion in Section 16.10.

## 1.11 The Organization of the Book and a User's Guide

We have written the text in a way such that one can read small units, e.g., a single chapter, independently from the rest, concentrate on certain aspects, and extract easily and quickly specific information. Hopefully we have found the right mixture between definitions, theorems, examples, and remarks so that reading the book is entertaining and illuminating. We have successfully used parts of this book, sometimes a single chapter, for seminars, reading courses, and advanced lecture courses.

The book consists of three parts and a supplement, which we briefly review next. We will also give some further information on how to use the book.

Note that not all of the proofs are included in full. At least we convey the basic ideas and include references to sources.

#### 1.11.1 Introduction to K- and L-Theory (Part I)

In the first part "Introduction to K- and L-Theory", which encompasses Chapters 2 to 10, we introduce and motivate the relevant theories, namely, algebraic K-theory, algebraic L-theory, and topological K-theory. In these chapters we present some applications and more accessible special versions of the Baum-Connes and the Farrell-Jones Conjecture. They are rather independent of one another and one can start reading each of them without having gone though the others. If a reader just wants to get some information, for instance about Wall's finiteness obstruction, Whitehead torsion, or the projective class group, she or he can directly start reading the relevant chapter, learn the basics about these invariants, and understand the relevant special versions of the Baum-Connes Conjecture or the Farrell-Jones Conjecture without going through the other chapters. Each of these chapters is eligible for a lecture course, seminar, or reading course.

#### 1.11.2 The Isomorphism Conjectures (Part II)

In the second part "The Isomorphism Conjectures", which consists of Chapters 11 to Chapter 18, we introduce the Baum-Connes Conjecture and the Farrell-Jones Conjecture in their most general form, namely, for arbitrary groups and with coef-

ficients. We discuss further applications and in particular how they can be used for computations. We give a report about the status of these conjectures and discuss open problems.

Note that the Farrell-Jones Conjecture comes in different levels. It can be considered for rings (with involution) as coefficients and hence aims at the algebraic K-theory and L-theory of group rings. This is the most relevant version for applications, where it often suffices to treat lower and middle K-theory, torsionfree groups, and  $\mathbb{Z}$  or a field as coefficients. One may twist the group rings and allow orientation characters. The next level is to pass to equivariant additive categories (with involution) as coefficients, which has the advantage that it automatically leads to useful inheritance properties of the Farrell-Jones Conjecture and encompasses the case of rings as coefficients. For algebraic K-theory one can even allow higher categories as coefficients. This contains the version of additive categories as coefficients and also the versions of the Farrell-Jones Conjecture for Waldhausen's A-theory, for pseudoisotopy, and for Whitehead spaces as special cases. There are also versions "with finite wreath product", where the passage to overgroups of finite index is built in.

So there are many variations of the Farrell-Jones Conjecture, but the Full Farrell-Jones Conjecture 13.30 implies all of them.

We also state Meta-Conjectures, which reduce to the Baum-Connes Conjecture, the Farrell-Jones Conjecture, or other types of Isomorphism Conjectures if one feeds the right theory into them. There are versions of the Farrell-Jones Conjecture for Waldhausen's *A*-theory, pseudoisotopy, Whitehead spaces, topological Hochschild homology, topological cyclic homology, and homotopy *K*-theory.

We also briefly discuss the Farrell-Jones Conjecture for totally disconnected groups and Hecke Algebras, where for the first time a version of the Farrell-Jones Conjecture for topological groups is considered. The Baum-Connes Conjecture has already been intensively studied for topological groups. However, in this monograph we will confine ourselves to discrete groups.

#### 1.11.3 Methods of Proofs (Part III)

In the third part "Methods of Proofs", which ranges from Chapter 19 to Chapter 25, we give a survey on the background, history, philosophy, strategies, and some ingredients of the proofs. We will concentrate on the Farrell-Jones Conjecture in this part III.

The reader, who is interested in proofs, should first go through Chapter 19. There motivations for the proofs of the Farrell-Jones Conjecture and some information about their long history is given without getting lost in technical details. So it will be a soft introduction to the methods of proofs conveying ideas only. Mainly we explain why controlled topology, flows, and transfers come in, which one would not expect at first glance in view of the homotopy-theoretic nature of the Farrell-Jones Conjecture.

In Chapter 20 we isolate some conditions about a group which guarantee that it satisfies the Full Farrell-Jones Conjecture or some special version of it. Note that here K- or L-theory do not yet play any role and one can use the results of this section without any previous knowledge about them. This will be interesting for someone who is already familiar with geometric group theory but has no background in K- or L-theory.

Depending on how ambitious the reader is, she or he should go through the other chapters. We recommend to read Section 23.7, where details of the proof of the Farrell-Jones Conjecture for the surjectivity of the *K*-theoretic assembly map in dimension 1 is given, which does not use much knowledge about algebraic *K*-theory but uses all the basic ideas appearing in the proof of the Full Farrell-Conjecture.

The reader who wants to understand the proof in the most advanced setting, namely the one for higher categories as coefficients, and for the largest class of groups, namely the class of Dress-Farrell-Hsiang-Jones groups, is recommended to read through Chapter 24. For this some background in higher category theory is necessary.

We give a very brief overview of the methods of proof for the Baum-Connes Conjecture in Chapter 25.

#### 1.11.4 Supplement

The book contains a number of exercises. They are not needed for the exposition of the book, but give some illuminating insight. Moreover, the reader may test whether she or he has understood the text or improve her or his understanding by trying to solve the exercises. Hints to the solutions of the exercises are given in Chapter 26.

If one wants to find a specific topic, the extensive index of the monograph can be used to find the right spot for a specific topic. The index contains an item "Theorem", under which all theorems with their names appearing in the book are listed, and analogously there is an item "Conjecture".

#### 1.11.5 Prerequisites

We require that the reader is familiar with basic notions in topology (*CW*-complexes, chain complexes, homology, homotopy groups, manifolds, coverings, cofibrations, fibrations, ...), functional analysis (Hilbert spaces, bounded operators, differential operators, ...), algebra (groups, modules, group rings, elementary homological algebra, ...), group theory (presentations, Cayley graphs, hyperbolic groups, ...), and elementary category theory (functors, transformations, additive categories, ...).

## **1.12 Notations and Conventions**

Here is a briefing on our main conventions and notations. Details are of course discussed in the text.

- Ring will mean (not necessarily commutative) associative ring with unit unless explicitly stated otherwise;
- Module always means left module unless explicitly stated otherwise;
- Group means discrete group unless explicitly stated otherwise;
- We will always work in the category of compactly generated spaces, compare [927] and [1006, I.4]. In particular every space is automatically Hausdorff;
- For our conventions concerning spectra see Section 12.4. Spectra are denoted by boldface letters such as E;
- We use the standard symbols Z, Q, R, C, Z<sub>p</sub>, and Q<sub>p</sub> for the integers, the rational numbers, the real numbers, the complex numbers, the *p*-adic numbers, and the *p*-adic rationals;

symbol	name
$\mathbb{Z}/n$	finite cyclic group of order n
S <sub>n</sub>	symmetric group of permutations of the set $\{1, 2,, n\}$
$A_n$	alternating group of even permutations of the set $\{1, 2,, n\}$
$D_{\infty}$	infinite dihedral group
$D_{2n}$	dihedral group of order 2n

• We use the following symbols to denote various groups:

## **1.13** Acknowledgments

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## **1.14 Notes**

Further information about the Baum-Connes Conjecture and the Farrell-Jones Conjecture can be found in the survey articles [67, 88, 109, 366, 426, 484, 659, 673, 742, 846, 963].

# Chapter 2 The Projective Class Group

## 2.1 Introduction

This chapter is devoted to the *projective class group*  $K_0(R)$  of a ring R.

We give in Section 2.2 three equivalent definitions of  $K_0(R)$ , namely, by the universal additive invariant for finitely generated projective modules, by the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective modules, and by idempotent matrices, and discuss the significance of  $K_0(R)$  for the category of finitely generated projective modules. Some calculations for principal ideal domains and Dedekind rings are provided in Section 2.3.

We explain the connections to geometry. We prove *Swan's Theorem* 2.27, which identifies  $K_0(C^0(X))$  for the ring  $C^0(X)$  of continuous functions on a compact space X with the Grothendieck group of the abelian monoid of isomorphism classes of vector bundles over X, see (2.31). The relevance of  $K_0(\mathbb{Z}G)$  for topologists is illustrated by *Wall's finiteness obstruction*, which also leads to a geometric description of  $K_0(\mathbb{Z}G)$  in terms of finitely dominated spaces and is discussed in detail in Section 2.5.

We introduce variants of the *K*-theoretic Farrell-Jones Conjecture for projective class groups in Section 2.8. A prototype asserts that for a torsionfree group *G* and a regular ring *R*, e.g.,  $R = \mathbb{Z}$  or *R* a field, the change of rings map

$$K_0(R) \xrightarrow{=} K_0(RG)$$

is bijective. It implies the conjecture that for a torsionfree group *G* the reduced projective class group  $\widetilde{K}_0(\mathbb{Z}G)$  vanishes, which is for finitely presented *G* equivalent to the conjecture that every finitely dominated *CW*-complex with  $\pi_1(X) \cong G$  is homotopy equivalent to a finite *CW*-complex. We also introduce a version where the group is not necessarily torsionfree, but *R* is a regular ring with  $\mathbb{Q} \subseteq R$  or a field of prime characteristic.

In Section 2.9 we consider *Kaplansky's Idempotent Conjecture*, which asserts for a torsionfree group G and a field F that 0 and 1 are the only idempotents in FG. It is a consequence of the Farrell-Jones Conjecture. We also discuss various *Bass Conjectures*, all of which are implied by the Farrell-Jones Conjecture, in Section 2.10.

Finally, we give a survey of  $K_0(\mathbb{Z}G)$  for finite groups *G* and of  $K_0(C_r^*(G))$  in Section 2.12 and of  $K_0(\mathcal{N}(G))$  in Section 2.13, where  $C_r^*(G)$  is the reduced group  $C^*$ -algebra and  $\mathcal{N}(G)$  the group von Neumann algebra.

#### 2.2 Definition and Basic Properties of the Projective Class Group

**Definition 2.1 (Projective class group**  $K_0(R)$ ). Let *R* be an (associative) ring (with unit). Define its *projective class group*  $K_0(R)$  to be the abelian group whose generators are isomorphism classes [P] of finitely generated projective *R*-modules *P* and whose relations are  $[P_0] + [P_2] = [P_1]$  for any exact sequence  $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$  of finitely generated projective *R*-modules.

Define  $G_0(R)$  analogously but replacing finitely generated projective by finitely generated.

Given a ring homomorphism  $f: R \to S$ , we can assign to an *R*-module *M* an *S*-module  $f_*M$  by  $S \otimes_R M$  where we consider *S* as a right *R*-module using *f*. We say that  $f_*M$  is obtained by *induction with f* from *M*. If *M* is finitely generated or free or projective, the same is true for  $f_*M$ . This construction is natural, compatible with direct sums, and sends an exact sequence  $0 \to P_0 \to P_1 \to P_2 \to 0$  of finitely generated projective *R*-modules to an exact sequence  $0 \to f_*P_0 \to f_*P_1 \to f_*P_2 \to 0$  of finitely generated projective *S*-modules. Hence we get a homomorphism of abelian groups

(2.2) 
$$f_* = K_0(f) \colon K_0(R) \to K_0(S), \ [P] \mapsto [f_*P],$$

which is also called the *change of rings homomorphism*. Thus  $K_0$  becomes a covariant functor from the category of rings to the category of abelian groups.

**Remark 2.3 (The universal property of the projective class group).** One should view  $K_0(R)$  together with the assignment sending a finitely generated projective *R*-module *P* to its class [*P*] in  $K_0(R)$  as the *universal additive invariant* or the *universal dimension function* for finitely generated projective *R*-modules. Namely, suppose that we are given an abelian group and an assignment *d* that associates to a finitely generated projective *R*-module an element  $d(P) \in A$  such that  $d(P_0) + d(P_2) = d(P_1)$  holds for any exact sequence  $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective *R*-modules. Then there is precisely one homomorphism of abelian groups  $\phi: K_0(R) \rightarrow A$  such that  $\phi([P]) = d(P)$  holds for every finitely generated projective *R*-module *P*. The analogous statement holds for  $G_0(R)$ if we consider finitely generated *R*-modules instead of finitely generated projective *R*-modules.

A ring is an *integral domain* if every zero-divisor is trivial, i.e., if  $r, s \in R$  satisfy rs = 0, then r = 0 or s = 0. A *principal ideal domain* is a commutative integral domain for which every ideal is a *principal ideal*, i.e., of the form  $(r) = \{r'r \mid r' \in R\}$  for some  $r \in R$ .

**Example 2.4** ( $K_0(R)$  and  $G_0(R)$  of a principal ideal domain). Let *R* be a principal ideal domain. Then we get isomorphisms of abelian groups

$$\mathbb{Z} \xrightarrow{\cong} K_0(R), \quad n \mapsto [R^n];$$
  
$$K_0(R) \xrightarrow{\cong} G_0(R), \quad [P] \mapsto [P].$$

2.2 Definition and Basic Properties of the Projective Class Group

This follows from the structure theorem of finitely generated *R*-modules over principal ideal domains. It implies that any finitely generated *R*-module *M* can be written as a direct sum  $R^n \oplus T$  for some torsion *R*-module *T* for which there exists an exact sequence of *R*-modules of the shape  $0 \to R^s \to R^s \to T \to 0$ . Moreover, *M* is projective if and only if *T* is trivial and we have  $R^m = R^n \iff m = n$ .

**Definition 2.5 (Reduced projective class group**  $K_0(R)$ ). Define the *reduced projective class group*  $\widetilde{K}_0(R)$  to be the quotient of  $K_0(R)$  by the abelian subgroup  $\{[R^m] - [R^n] \mid n, m \in \mathbb{Z}, m, n \ge 0\}$ , which is the same as the abelian subgroup generated by the class [R].

We conclude from Example 2.4 that the reduced projective class group  $\widetilde{K}_0(R)$  is isomorphic to the cokernel of the homomorphism

$$f_*: K_0(\mathbb{Z}) \to K_0(R)$$

where f is the unique ring homomorphism  $\mathbb{Z} \to R$ ,  $n \mapsto n \cdot 1_R$ .

**Remark 2.6 (The projective class group as a Grothendieck group).** Let Proj(R) be the abelian semigroup of isomorphisms classes of finitely generated projective *R*-modules with the addition coming from the direct sum. Let  $K'_0(R)$  be the associated abelian group given by the Grothendieck construction applied to Proj(R). There is a natural homomorphism

$$\phi: K'_0(R) \xrightarrow{\cong} K_0(R)$$

sending the class of a finitely generated projective *R*-module *P* in  $K'_0(R)$  to its class in  $K_0(R)$ . This is a well-defined isomorphism of abelian groups.

The analogous definition of  $G'_0(R)$  and the construction of a homomorphism  $G'_0(R) \to G_0(R)$  makes sense, but the latter map is *not* bijective in general. It works for  $K_0(R)$  because every exact sequence of projective *R*-modules  $0 \to P_0 \to P_1 \to P_2 \to 0$  splits and thus yields an isomorphism  $P_1 \cong P_0 \oplus P_2$ . In general *K*-theory deals with exact sequences, not with direct sums. Therefore Definition 2.1 of  $K_0(R)$  reflects better the underlying idea of *K*-theory than its definition in terms of the Grothendieck construction.

**Exercise 2.7.** Prove that the homomorphism  $\phi: K'_0(R) \to K_0(R)$  appearing in Remark 2.6 is a well-defined isomorphism of abelian groups.

**Remark 2.8 (What does the reduced projective class group measure?).** Let *P* be a finitely generated projective *R*-module. Then we conclude from Remark 2.6 that its class  $[P] \in \widetilde{K}_0(R)$  is trivial if and only if *P* is *stably finitely generated free*, i.e.,  $P \oplus R^r \cong R^s$  for appropriate integers  $r, s \ge 0$ . So the reduced projective class group  $\widetilde{K}_0(R)$  measures the deviation of a finitely generated projective *R*-module from being stably finitely generated free. Note that, in general, stably finitely generated free does not imply finitely generated free, as Examples 2.9 and 2.28 will show.

**Example 2.9 (Dunwoody's example).** An interesting  $\mathbb{Z}G$ -module *P* that is stably finitely generated free but not finitely generated free is constructed by Dunwoody [317] for *G* the torsionfree one-relator group  $\langle a, b | a^2 = b^3 \rangle$ , which is the

fundamental group of the trefoil knot. Note that  $\overline{K}_0(\mathbb{Z}G)$  is known to be trivial, in other words, every finitely generated projective *RG*-module is stably finitely generated free. It is also worth mentioning that  $\mathbb{Z}G$  contains no idempotent besides 0 and 1. Hence any direct summand in  $\mathbb{Z}G$  is free.

More examples of this kind are given in Berridge-Dunwoody [134].

One basic feature of algebraic K-theory is Morita equivalence.

**Theorem 2.10 (Morita equivalence for**  $K_0(R)$ ). For every ring R and integer  $n \ge 1$ , there is a natural isomorphism

$$\mu \colon K_0(R) \xrightarrow{\equiv} K_0(\mathbf{M}_n(R)).$$

*Proof.* We can consider  $\mathbb{R}^n$  as an  $M_n(\mathbb{R})$ - $\mathbb{R}$ -bimodule, denoted by  $_{M_n(\mathbb{R})}\mathbb{R}^n_{\mathbb{R}}$ . Then  $\mu$  sends [P] to  $[_{M_n(\mathbb{R})}\mathbb{R}^n_{\mathbb{R}} \otimes_{\mathbb{R}} P]$ . We can also consider  $\mathbb{R}^n$  as an  $\mathbb{R}$ - $M_n(\mathbb{R})$ -bimodule denoted by  $_{\mathbb{R}}\mathbb{R}^n_{M_n(\mathbb{R})}$ . Define  $\nu: K_0(M_n(\mathbb{R})) \to K_0(\mathbb{R})$  by sending [Q] to  $[_{\mathbb{R}}\mathbb{R}^n_{M_n(\mathbb{R})} \otimes_{M_n(\mathbb{R})} Q]$ . Then  $\mu$  and  $\nu$  are inverse to one another.  $\Box$ 

**Exercise 2.11.** Check that  $\mu$  and  $\nu$  are inverse to one another.

We omit the easy proof of the next lemma.

**Lemma 2.12.** Let  $R_0$  and  $R_1$  be rings. Denote by  $pr_i : R_0 \times R_1 \rightarrow R_i$  for i = 0, 1 the projection. Then we obtain an isomorphism

$$(\mathrm{pr}_0)_* \times (\mathrm{pr}_1)_* \colon K_0(R_0 \times R_1) \xrightarrow{=} K_0(R_0) \times K_0(R_1).$$

**Example 2.13 (Rings with non-trivial**  $\widetilde{K}_0(R)$ ). We conclude from Example 2.4 and Lemma 2.12 that for a principal ideal domain *R* we have

$$K_0(R \times R) \cong \mathbb{Z} \oplus \mathbb{Z};$$
  
$$\widetilde{K}_0(R \times R) \cong \mathbb{Z}.$$

The  $R \times R$ -module  $R \times \{0\}$  is finitely generated projective but not stably finitely generated free. It is a generator of the infinite cyclic group  $\widetilde{K}_0(R \times R)$ .

Notation 2.14 (M(R), GL(R), and Idem(R)). Let  $M_{m,n}(R)$  be the set of (m, n)matrices over R. For  $A \in M_{m,n}(R)$ , let  $r_A \colon R^m \to R^n$ ,  $x \to xA$  be the R-homomorphism of (left) R-modules given by right multiplication by A. Let  $M_n(R)$  be the ring of (n, n)-matrices over R. Denote by  $GL_n(R)$  the group of invertible (n, n)-matrices over R. Let  $Idem_n(R)$  be the subset of  $M_n(R)$  of idempotent matrices A, i.e., (n, n)-matrices satisfying  $A^2 = A$ . There are embeddings  $i_{t,n} \colon M_n(R) \to M_{n+1}(R), A \mapsto \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix}$  for t = 0, 1 and  $n \ge 1$ . The embedding  $i_{1,n}$  induces an embedding  $GL_n(R) \to GL_{n+1}(R)$  of groups. Let GL(R) be the union of the  $GL_n(R)$ -s, which is a group again. Denote by M(R) the union of the  $M_n(R)$ -s with respect to the embeddings  $i_0$ . This is a ring without unit. Let Idem(R) be the set of idempotent elements in M(R). This is the same as the union of the

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2.2 Definition and Basic Properties of the Projective Class Group

Idem<sub>n</sub>(R)-s with respect to the embeddings Idem<sub>n</sub>(R)  $\rightarrow$  Idem<sub>n+1</sub>(R) coming from the embeddings  $i_{0,n}$ : M<sub>n</sub>(R)  $\rightarrow$  M<sub>n+1</sub>(R).

**Remark 2.15 (The projective class groups in terms of idempotent matrices).** The projective class groups  $K_0(R)$  can also be defined in terms of idempotent matrices. Namely, the conjugation action of  $GL_n(R)$  on  $M_n(R)$  induces an action of GL(R) on M(R) which leaves Idem(R) fixed. One obtains a bijection of sets

 $\phi \colon \operatorname{GL}(R) \setminus \operatorname{Idem}(R) \to \operatorname{Proj}(R), \quad [A] \mapsto \operatorname{im}(r_A \colon R^n \to R^n).$ 

This becomes a bijection of abelian semigroups if we equip the source with the addition coming from  $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  and the target with the one coming from the direct sum. So we can identify  $K_0(R)$  with the Grothendieck group associated to the abelian semigroup  $GL(R) \setminus Idem(R)$  by Remark 2.6.

**Exercise 2.16.** Show that the map  $\phi$  appearing in Remark 2.15 is a well-defined isomorphism of abelian semigroups.

**Example 2.17** (A ring *R* with trivial  $K_0(R)$ ). Let *F* be a field and let *V* be an *F*-vector space with an infinite countable basis. Consider the ring  $R = \text{end}_F(V)$ . Next we prove that  $K_0(R)$  is trivial.

By Remark 2.15 it suffices to show for every integer  $n \ge 0$  and two idempotent matrices  $A, B \in \text{Idem}_n(R)$  that the matrices  $A \oplus 0 \oplus 1$  and  $B \oplus 0 \oplus 1$  in  $M_{n+2}(R)$ are conjugate by an element in  $\text{GL}_{n+2}(R)$ . This follows from the observations that both the kernel and the image of the *F*-linear endomorphisms  $r_{A\oplus 0\oplus 1}$  and  $r_{B\oplus 0\oplus 1}$  of  $V^{n+2}$  have infinite countable dimension, two *F*-vector spaces of infinite countable dimension are isomorphic, and the inclusions induce isomorphisms  $\ker(r_{A\oplus 0\oplus 1}) \oplus \operatorname{im}(r_{A\oplus 0\oplus 1}) \xrightarrow{\cong} V^{n+2}$ .

**Lemma 2.18.** Let G be a group. Let R be a commutative integral domain with quotient field F. Then we obtain an isomorphism

$$K_0(RG) \xrightarrow{=} \widetilde{K}_0(RG) \oplus \mathbb{Z}, \quad [P] \mapsto ([P], \dim_F(F \otimes_{RG} P))$$

where *F* is considered as an *RG*-module with respect to the trivial *G*-action and the inclusion of rings  $j: R \rightarrow F$ .

*Proof.* Since  $F \otimes_{RG} P$  is a finite-dimensional *F*-vector space for finitely generated *P* and  $F \otimes_{RG} (P \oplus Q) \cong_G (F \otimes_{RG} P) \oplus (F \otimes_{RG} Q)$ , this is a well-defined homomorphism. Bijectivity follows from dim<sub>*F*</sub>( $F \otimes_{RG} RG^n$ ) = *n*.

## 2.3 The Projective Class Group of a Dedekind Domain

Let *R* be a commutative integral domain with quotient field *F*. A non-zero *R*-submodule  $I \subset F$  is called a *fractional ideal* if for some  $r \in R$  we have  $rI \subseteq R$ . A fractional ideal *I* is called *principal* if *I* is of the form  $\{\frac{ra}{b} | r \in R\}$  for some  $a, b \in R$  with  $a, b \neq 0$ .

**Definition 2.19 (Dedekind domain).** A commutative integral domain *R* is called a *Dedekind ring* if for any fractional ideal *I* there exists another fractional ideal *J* with IJ = R.

Note that in Definition 2.19 the fractional ideal J must be given by  $\{x \in F \mid x \cdot I \subseteq R\}$ .

The fractional ideals in a Dedekind ring form by definition a group under multiplication of ideals with R as unit. The principal fractional ideals form a subgroup. The *class group* C(R) is the quotient of these abelian groups.

A proof of the next theorem can be found for instance in [727, Corollary 11 on page 14] and [860, Theorem 1.4.12 on page 20].

**Theorem 2.20 (The reduced projective class group and the class group of Dedekind domains).** Let R be a Dedekind domain. Then every fractional ideal is a finitely generated projective R-module and we obtain an isomorphism of abelian groups

$$\mathbb{Z} \oplus C(R) \xrightarrow{=} K_0(R), \quad (n, [I]) \mapsto n \cdot [R] + [I] - [R].$$

In particular, we get an isomorphism

$$C(R) \xrightarrow{\equiv} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

A ring is called *hereditary* if every ideal is projective, or, equivalently, if every submodule of a projective *R*-module is projective, see [215, Theorem 5.4 in Chapter I.5 on page 14].

**Theorem 2.21 (Characterization of Dedekind domains).** *The following assertions are equivalent for a commutative integral domain with quotient field F:* 

- (i) *R* is a Dedekind domain;
- (ii) For every pair of ideals  $I \subseteq J$  of R, there exists an ideal  $K \subseteq R$  with I = JK;
- (iii) *R* is hereditary;
- (iv) Every finitely generated torsionfree R-module is projective;
- (v) *R* is Noetherian and integrally closed in its quotient field *F* and every non-zero prime ideal is maximal.

*Proof.* This follows from [271, Proposition 4.3 on page 76 and Proposition 4.6 on page 77] and the fact that a finitely generated torsionfree module over an integral domain *R* can be embedded into  $R^n$  for some integer  $n \ge 0$ . See also [57, Chapter 13].

2.4 Swan's Theorem

**Remark 2.22 (The class group in terms of ideals of** *R*). One calls two ideals *I* and *J* in *R* equivalent if there exist non-zero elements *r* and *s* in *R* with rI = sJ. Then C(R) is the same as the equivalence classes of ideals under multiplication of ideals and the class given by the principal ideals as unit. Two ideals *I* and *J* of *R* define the same element in C(R) if and only if they are isomorphic as *R*-modules, see [860, Proposition 1.4.4 on page 17].

Recall that an *algebraic number field* is a finite algebraic extension of  $\mathbb{Q}$  and the *ring of integers* in *F* is the integral closure of  $\mathbb{Z}$  in *F*.

**Theorem 2.23 (The class group of a ring of integers is finite).** Let *R* be the ring of integers in an algebraic number field. Then *R* is a Dedekind domain and its class group C(R) and hence its reduced projective class group  $\widetilde{K}_0(R)$  are finite.

*Proof.* See [860, Theorem 1.4.18 on page 22 and Theorem 1.4.19 on page 23]. □

**Remark 2.24 (Class group of**  $\mathbb{Z}[\exp(2\pi i/p)]$ ). Let *p* be a prime number. The ring of integers in the algebraic number field  $\mathbb{Q}[\exp(2\pi i/p)]$  is  $\mathbb{Z}[\exp(2\pi i/p)]$ . Its class group  $C(\mathbb{Z}[\exp(2\pi i/p)])$  is finite by Theorem 2.23. However, its structure as a finite abelian group is only known for finitely many small primes, see [727, Remark 3.4 on page 30] or [990, Tables §3 on page 352ff].

**Example 2.25** ( $\widetilde{K}_0(\mathbb{Z}[\sqrt{-5}])$ ). The reduced projective class group  $\widetilde{K}_0(\mathbb{Z}[\sqrt{-5}])$  of the Dedekind domain  $\mathbb{Z}[\sqrt{-5}]$  is cyclic of order two. A generator is given by the maximal ideal  $(3, 2 + \sqrt{-5})$  in  $\mathbb{Z}[\sqrt{-5}]$ . (For more details see [860, Exercise 1.4.20 on page 25]).

## 2.4 Swan's Theorem

Let *F* be the field  $\mathbb{R}$  or  $\mathbb{C}$ . Let *X* be a compact space. Denote by C(X, F) or briefly by C(X) the ring of continuous functions from *X* to *F*. Let  $\xi$  and  $\eta$  be (finitedimensional locally trivial) *F*-vector bundles over *X*. Denote by  $C(\xi)$  the *F*-vector space of continuous sections of  $\xi$ . This becomes a C(X)-module under pointwise multiplication. If <u>*F*</u> denotes the trivial 1-dimensional vector bundle  $X \times F \to X$ , then  $C(\underline{F})$  and C(X) are isomorphic as C(X)-modules. If  $\xi$  and  $\eta$  are isomorphic as *F*-vector bundles, then  $C(\xi)$  and  $C(\eta)$  are isomorphic as C(X)-modules. There is an obvious isomorphism of C(X)-modules

(2.26) 
$$C(\xi) \oplus C(\eta) \xrightarrow{=} C(\xi \oplus \eta)$$

Since X is compact, every *F*-vector bundle has a finite bundle atlas and admits a Riemannian metric. This implies the existence of an *F*-vector bundle  $\xi'$  such that  $\xi \oplus \xi'$  is isomorphic as an *F*-vector bundle to a trivial *F*-vector bundle  $\underline{F}^n$ . Hence  $C(\xi)$  is a finitely generated projective C(X)-module. Denote by hom $(\xi, \eta)$ the C(X)-module of morphisms of *F*-vector bundles from  $\xi$  to  $\eta$ , i.e., of continuous maps between the total spaces that commutes with the bundle projections to X and induce linear (not necessarily injective or bijective) maps between the fibers over x for all  $x \in X$ . This becomes a C(X)-module under pointwise multiplication. Such a morphism  $f: \xi \to \eta$  induces a C(X)-homomorphism  $C(f): C(\xi) \to C(\eta)$  by composition. The next result is due to Swan [939].

**Theorem 2.27 (Swan's Theorem).** *Let X be a compact space and*  $F = \mathbb{R}, \mathbb{C}$ *. Then:* 

(i) Let  $\xi$  and  $\eta$  be *F*-vector bundles. Then we obtain an isomorphism of C(X)-modules

$$\Gamma(\xi,\eta)$$
: hom $(\xi,\eta) \to \text{hom}_{C(X)}(C(\xi),C(\eta)), \quad f \mapsto C(f);$ 

- (ii) We have  $\xi \cong \eta \iff C(\xi) \cong_{C(X)} C(\eta)$ ;
- (iii) If P is a finitely generated projective C(X)-module, then there exists an F-vector bundle  $\xi$  satisfying  $C(\xi) \cong_{C(X)} P$ .

*Proof.* (i) Obviously  $\Gamma(\xi \oplus \xi', \eta)$  can be identified with  $\Gamma(\xi, \eta) \oplus \Gamma(\xi', \eta)$  and  $\Gamma(\xi, \eta \oplus \eta')$  can be identified with  $\Gamma(\xi, \eta) \oplus \Gamma(\xi, \eta'')$  under the identification (2.26). Since a direct sum of two maps is a bijection if and only if each of the maps is a bijection and for every  $\xi$  there is an  $\xi'$  such that  $\xi \oplus \xi'$  is trivial, it suffices to treat the case where  $\xi = \underline{F}^m$  and  $\eta = \underline{F}^n$  for appropriate integers  $m, n \ge 0$ . There is an obvious commutative diagram

$$\begin{array}{c} \hom(\underline{F}^{m}, \underline{F}^{n}) \xrightarrow{\Gamma(\underline{F}^{m}, \underline{F}^{n})} & \hom_{C(X)}(C(\underline{F}^{m}), C(\underline{F}^{n})) \\ \cong & \downarrow & \downarrow \\ M_{m,n}(\hom(\underline{F}, \underline{F})) \xrightarrow{M_{m,n}(\Gamma(\underline{F}, \underline{F}))} & M_{m,n}(C(\underline{F})). \end{array}$$

Hence it suffices to treat the claim for m = n = 1, which is obvious.

(ii) This follows from assertion (i).

(iii) Given a finitely generated projective C(X)-module P, choose a C(X)-map  $p: C(X)^n \to C(X)^n$  satisfying  $p^2 = p$  and  $\operatorname{im}(p) \cong_{C(X)} P$ . Because of assertion (ii) we can choose a morphism of F-vector bundles  $q: \underline{F}^n \to \underline{F}^n$  with  $\Gamma(\underline{F}^n, \underline{F}^n)(q) = p$ . We conclude  $q^2 = q$  from  $p^2 = p$  and the injectivity of  $\Gamma(\underline{F}^n, \underline{F}^n)$ . Elementary bundle theory shows that the image of q and the image of 1 - q are F-subvector bundles in  $\underline{F}^n$  satisfying  $\operatorname{im}(q) \oplus \operatorname{im}(1-q) = \underline{F}^n$ . One easily checks  $C(\operatorname{im}(q)) \cong_{C(X)} P$ .

One may summarize Theorem 2.27 by saying that we obtain an equivalence of C(X)-additive categories from the category of *F*-vector bundles over *X* to the category of finitely generated projective C(X)-modules by sending  $\xi$  to  $C(\xi)$ .

**Example 2.28** ( $C(TS^n)$ ). Consider the *n*-dimensional sphere  $S^n$ . Let  $TS^n$  be its tangent bundle. Then  $C(TS^n)$  is a finitely generated projective  $C(S^n)$ -module. It is free if and only if  $TS^n$  is trivial. This is equivalent to the condition that n = 1, 3, 7, see [155]. On the other hand  $C(TS^n)$  is always stably finitely generated free as a  $C(S^n)$ -module, since  $TS^n$  is stably finitely generated free as an *F*-vector bundle because the direct sum of  $TS^n$  and the normal bundle  $v(S^n, \mathbb{R}^{n+1})$  of the standard embedding  $S^n \subseteq \mathbb{R}^{n+1}$  is  $T\mathbb{R}^{n+1}|_{S^n}$  and both *F*-vector bundles  $v(S^n, \mathbb{R}^{n+1})$  and  $T\mathbb{R}^{n+1}|_{S^n}$  are trivial.

**Exercise 2.29.** Consider an integer  $n \ge 1$ . Show that there exists a  $C(S^n)$ -module M with  $C(TS^n) \cong_{C(S^n)} C(S^n) \oplus M$  if and only if  $S^n$  admits a nowhere vanishing vector field. (This is equivalent to requiring that  $\chi(S^n) = 0$ , or, equivalently, that n is odd.)

**Remark 2.30 (Topological** *K*-theory in dimension 0). Let *X* be a compact space. Let  $\operatorname{Vect}_F(X)$  be the abelian semigroup of isomorphism classes of *F*-vector bundles over *X* where the addition comes from the Whitney sum. Let  $K^0(X)$  be the abelian group obtained from the Grothendieck construction to it. It is called the 0-th topological *K*-group of *X*. If  $f: X \to Y$  is a map of compact spaces, the pullback construction yields a homomorphism  $K^0(f): K^0(Y) \to K^0(X)$ . Thus we obtain a contravariant functor  $K^0$  from the category of compact spaces to the category of abelian groups. Since the pullback of a vector bundle with two homotopic maps yields isomorphic vector bundles,  $K^0(f)$  depends only on the homotopy class of *f*. Actually there is a sequence of such homotopy invariant covariant functors  $K^n$  for  $n \in \mathbb{Z}$  that constitutes a generalized cohomology theory  $K^*$  called *topological K-theory*. It is 2-periodic if  $F = \mathbb{C}$ , i.e., there are natural so-called *Bott isomorphisms*  $K^n(X) \xrightarrow{\cong} K^{n+2}(X)$  for  $n \in \mathbb{Z}$ . If  $F = \mathbb{R}$ , it is 8-periodic. We will give further explanations and generalizations of topological *K*-theory later in Section 10.2

Swan's Theorem 2.27 yields an identification

(2.31) 
$$K^0(X) \cong K_0(C(X)) \quad [\xi] \mapsto [C^0(\xi)].$$

**Exercise 2.32.** Let  $f: X \to Y$  be a map of compact spaces. Composition with f yields a ring homomorphism  $C(f): C(Y) \to C(X)$ . Show that under the identification (2.31) the maps  $K^0(f): K^0(Y) \to K^0(X)$  and  $C(f)_*: K_0(C(Y)) \to K_0(C(X))$  coincide.

**Exercise 2.33.** Compute  $K_0(C(D^n))$  for the *n*-dimensional disk  $D^n$  for  $n \ge 0$ .

## 2.5 Wall's Finiteness Obstruction

We now discuss the geometric relevance of  $\widetilde{K}_0(\mathbb{Z}G)$ .

Let X be a CW-complex. It is called *finite* if it consists of finitely many cells. This is equivalent to the condition that X is compact. We call X *finitely dominated* if there exists a *finite domination* (Y, i, r), i.e., a finite CW-complex Y together with maps  $i: X \to Y$  and  $r: Y \to X$  such that  $r \circ i$  is homotopic to the identity on X. If X is finitely dominated, its set of path components  $\pi_0(X)$  is finite and the fundamental group  $\pi_1(C)$  of each component C of X is finitely presented, see Lemma 2.42.

While studying existence problems for compact manifolds with prescribed properties (like for example the existence of certain group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex with the desired property within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent finite CW-complex. If the goal is to construct a compact manifold, this is a necessary step in the construction. Wall's finiteness obstruction, which we will explain below, decides this question.

An example of such a geometric problem is the *Spherical Space Form Problem* 9.205, i.e., the classification of closed manifolds *M* whose universal coverings are diffeomorphic or homeomorphic to the standard sphere. Such examples arise as unit spheres in unitary representations of finite groups, but there are also examples that do not occur in this way. This problem initiated not only the theory of the finiteness obstruction, but also surgery theory for closed manifolds with non-trivial fundamental group. We refer to the survey articles [284] and [694] for more information about the Spherical Space Form problem. It was finally solved by Madsen-Thomas-Wall [701, 702].

The finiteness obstruction also appears in the Ph.D.-thesis [915] of Siebenmann, who dealt with the problem whether a given smooth or topological manifold can be realized as the interior of a compact manifold with boundary.

Next we explain the definition and the main properties of the finiteness obstruction, illustrating that it is a kind of Euler characteristic, but now counting elements in the projective class group instead of counting ranks of finitely generated free modules.

#### 2.5.1 Chain Complex Version of the Finiteness Obstruction

**Definition 2.34 (Types of chain complexes).** We call an *R*-chain complex *finitely* generated, free, or projective respectively if each *R*-chain module is finitely generated, free, or projective. It is called *positive* if  $C_n = 0$  for  $n \le -1$ . It is called *finite-dimensional* if there exists a natural number N such that  $C_n = 0$  for  $|n| \le N$ . It is called *finite* if it is finite-dimensional and finitely generated.

For the remainder of this section all chain complexes  $C_*$  are understood to be positive. Let *R* be a ring and  $C_*$  be an *R*-chain complex. A *finite domination*  $(F_*, i_*, p_*)$ of  $C_*$  consists of a finite free *R*-chain complex  $F_*$  and *R*-chain maps  $i_* : C_* \to F_*$  and  $r_* : F_* \to C_*$  such that  $r_* \circ i_* \simeq id_{C_*}$  holds. The existence of a finite domination is equivalent to the existence of a finite projective *R*-chain complex  $P_*$  which is *R*-chain homotopy equivalence to  $C_*$ . For a proof of this claim we refer for instance to [644, Proposition 11.11 on page 222], or to the explicit construction in Subsection 23.7.5. For any such choice of  $P_*$ , define the *finiteness obstruction*  $o(C_*) \in K_0(R)$  to be 2.5 Wall's Finiteness Obstruction

(2.35) 
$$o(C_*) := \sum_{n \ge 0} (-1)^n \cdot [P_n].$$

The *reduced finiteness obstruction*  $\tilde{o}(C_*) \in \tilde{K}_0(R)$  is the image of  $o(C_*)$  under the projection  $K_0(R) \to \tilde{K}_0(R)$ . The definition is indeed independent of the choice of  $P_*$ , since for two finite projective *R*-chain complexes  $P_*$  and  $Q_*$  coming with an *R*-chain homotopy equivalence  $f_* \colon P_* \xrightarrow{\sim} Q_*$  the mapping cone cone<sub>\*</sub>( $f_*$ ), see Definition 3.29, is contractible and hence we obtain an *R*-isomorphism

$$P_{\mathrm{odd}} \oplus Q_{\mathrm{ev}} \xrightarrow{\cong} P_{\mathrm{ev}} \oplus Q_{\mathrm{odd}}$$

from the isomorphism (3.30) and its inverse (3.31).

**Lemma 2.36.** (i) If the two *R*-chain complexes  $C_*$  and  $D_*$  are *R*-chain homotopy equivalent and one of them is finitely dominated, then both are finitely dominated and we get

$$o(C_*) = o(D_*)$$

(ii) Let  $0 \to C_* \to D_* \to E_* \to 0$  be an exact sequence of *R*-chain complexes. If two of the *R*-chain complexes  $C_*$ ,  $D_*$ , and  $E_*$  are finitely dominated, then all three are finitely dominated and we get

$$o(D_*) = o(C_*) + o(E_*);$$

(iii) Let  $C_*$  be a finitely dominated *R*-chain complex. Then it is *R*-chain homotopy equivalent to a finite free *R*-chain complex if and only if  $\tilde{o}(C_*)$  vanishes.

*Proof.* (i) This follows directly from the definitions.

(ii) One can construct a commutative diagram of *R*-chain complexes



such that the rows are exact, the upper row consists of finite projective *R*-chain complexes, and the vertical maps are *R*-chain homotopy equivalences, see for instance [644, Lemma 11.6 on page 216].

(iii) Suppose that  $\tilde{o}(C_*) = 0$ . Choose a finite projective *R*-chain complex  $P_*$  which is *R*-chain homotopy equivalent to  $C_*$ . An *elementary R*-chain complex  $E_*$  over an *R*-module *M* is an *R*-chain complex which is concentrated in two consecutive dimensions and its only non-trivial differential is given by  $id_M : M \to M$ . By adding elementary *R*-chain complexes over finitely generated free *R*-modules, one can arrange that  $P_*$  is of the shape  $\cdots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_0$  such that  $P_i$ is finitely generated free for  $i \le n-1$ . Since  $\tilde{o}(C_*) = (-1)^n \cdot [P_n] = 0$  holds in  $\tilde{K}_0(R)$ , the *R*-module  $P_n$  is stably free. Hence, by adding one further elementary chain complex over a finitely generated free *R*-module, one can arrange that  $P_*$  is finite free.

#### 2.5.2 Space Version of the Finiteness Obstruction

In the sequel we ignore base point questions. This is not a real problem since an inner automorphism of a group G induces the identity on  $K_0(RG)$ .

Given a finitely dominated connected *CW*-complex *X* with fundamental group  $\pi$ , we consider its universal covering  $\widetilde{X}$  and the associated cellular  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{X})$ . Given a finite domination (Y, i, r), we regard the  $\pi$ -covering  $\overline{Y}$  over *Y* associated to the epimorphism  $r_*: \pi_1(Y) \to \pi_1(X)$ . The pullback construction yields a  $\pi$ -covering  $i^*\overline{Y}$  over *X*. Then  $F_* = C_*(i^*\overline{Y})$  is a finite free  $\mathbb{Z}\pi$ -chain complex. The maps *i* and *r* yield  $\mathbb{Z}\pi$ -chain maps  $r_*: F_* \to C_*(\widetilde{X})$  and  $i_*: C_*(\widetilde{X}) \to F_*$  such that  $r_* \circ i_*$  is  $\mathbb{Z}\pi$ -chain homotopic to the identity on  $C_*(\widetilde{X})$ . Thus  $(F_*, i_*, r_*)$  is a finite domination of the  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{X})$ . We have defined  $o(C_*(\widetilde{X})) \in K_0(\mathbb{Z}\pi)$  in (2.35). Now define the *unreduced finiteness obstruction* 

$$(2.37) o(X) := o(C_*(\widetilde{X})) \in K_0(\mathbb{Z}\pi).$$

Define the finiteness obstruction

(2.38) 
$$\widetilde{o}(X) \in K_0(\mathbb{Z}\pi)$$

to be the image of o(X) under the canonical projection  $K_0(\mathbb{Z}\pi) \to \widetilde{K}_0(\mathbb{Z}\pi)$ . Obviously  $\widetilde{o}(X) = 0$  if X is homotopy equivalent to a finite CW-complex Z since in this case we can take  $P_* = C_*(\widetilde{Z})$  and  $C_*(\widetilde{Z})$  is a finite free  $\mathbb{Z}\pi$ -chain complex. The next result is due to Wall, see [983] and [984].

**Theorem 2.39 (Properties of the Finiteness Obstruction).** *Let X be a finitely dominated connected CW-complex.* 

- (i) The space X is homotopy equivalent to a finite CW-complex if and only if o(X) vanishes;
- (ii) Every element in  $K_0(\mathbb{Z}G)$  can be realized as the finiteness obstruction o(X) of a finitely dominated connected 3-dimensional CW-complex X with  $G = \pi_1(X)$ , provided that G is finitely presented.

Theorem 2.39 illustrates why it is important to study the algebraic object  $\widetilde{K}_0(\mathbb{Z}\pi)$  when one is dealing with geometric or topological questions. The favorite case is when  $\widetilde{K}_0(\mathbb{Z}\pi)$  vanishes because then the finiteness obstruction is obviously zero and one does not have to make a specific computation of  $\widetilde{o}(X)$  in  $\widetilde{K}_0(\mathbb{Z}\pi)$ .

**Exercise 2.40.** Let *X* be a finitely dominated connected *CW*-complex with fundamental group  $\pi$ . Define a homomorphism of abelian groups

 $\psi \colon K_0(\mathbb{Z}\pi) \to \mathbb{Z}, \quad [P] \mapsto \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}\pi} P).$ 

2.5 Wall's Finiteness Obstruction

Show that  $\psi$  sends o(X) to the Euler characteristic  $\chi(X)$ .

**Remark 2.41.** One can extend the finiteness obstruction also to not necessarily connected CW-complexes. If X is a (not necessarily connected) finitely dominated CW-complex, we define

$$\begin{split} &K_0(\mathbb{Z}[\pi_1(X)]) := \bigoplus_{C \in \pi_0(X)} K_0(\mathbb{Z}[\pi_1(C)]); \\ &\widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) := \bigoplus_{C \in \pi_0(X)} \widetilde{K}_0(\mathbb{Z}[\pi_1(C)]), \end{split}$$

and the unreduced finite obstruction and the finiteness obstruction to be

$$o(X) := \{o(C) \mid C \in \pi_0(X)\} \in K_0(\mathbb{Z}[\pi_1(X)]);$$
  
$$\widetilde{o}(X) := \{\widetilde{o}(C) \mid C \in \pi_0(X)\} \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]).$$

Note that  $K_0(\mathbb{Z}[\pi_1(X)])$  and  $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$  are covariant functors in X in the obvious way.

For more information about the finiteness obstruction we refer for instance to [380, 382, 642, 669, 740, 743, 761, 838, 965, 983, 984].

#### 2.5.3 Outline of the Proof of the Obstruction Property

In this subsection we outline the proof of Theorem 2.39.

The elementary proofs of the next two lemmas can be found in [983, Lemma 1.3] and [644, Lemma 14.8 on page 280].

**Lemma 2.42.** Let G be a finitely presented group. Let  $i: H \to G$  and  $r: G \to H$  be group homomorphisms with  $r \circ i = id_H$ . Then H is finitely presented.

**Lemma 2.43.** Let G be a finitely generated group and H be a finitely presented group. Then the kernel ker(f) of any group epimorphism  $f: G \rightarrow H$  is finitely generated as a normal subgroup, i.e., there exists a finite subset S of ker(f) such that the intersection of all normal subgroups of G containing S is ker(f).

The next Lemma 2.44 follows from Lemma 2.42 and Lemma 2.43.

**Lemma 2.44.** Let (Y, i, r) be a finite domination of the CW-complex X. Then we can arrange by attaching finitely many 2-cells to Y that the map  $\pi_1(r): \pi_1(Y) \to \pi_1(X)$  is bijective and hence r is 2-connected.

**Lemma 2.45.** *Let Y be a finitely dominated connected CW-complex whose finiteness obstruction*  $\tilde{o}(Y)$  *vanishes. Then there are:* 

(i) A finite 2-dimensional connected CW-complex Z;

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- (ii) A 2-connected map  $h: Z \to Y$ ;
- (iii) A finite free  $\mathbb{Z}\pi$ -chain complex  $C_*$  with  $C_*|_2 = C_*(\widetilde{Z})$  and a  $\mathbb{Z}\pi$ -chain homotopy equivalence  $f_* \colon C_* \to C_*(\widetilde{Y})$  with  $f_*|_2 = C_*(\widetilde{h})$ , where here and in the sequel we identify  $\pi = \pi_1(Z)$  with  $\pi_1(Y)$  using the isomorphism  $\pi_1(h) \colon \pi_1(Z) \xrightarrow{\cong} \pi_1(Y)$ .

*Proof.* By Lemma 2.44 we obtain a finite domination (Y, i, r) such that  $r: Y \to X$  is 2-connected. Take *Z* to be the 2-skeleton  $Y_2$  of *Y* and  $h: Z \to X$  to be the restriction of *r* to *Z*.

Since *h* is 2-connected, the induced  $\mathbb{Z}\pi$ -chain map  $C_*(\tilde{h}): C_*(\tilde{Z}) \to C_*(\tilde{Y})$  is 2-connected and hence  $H_n(\operatorname{cone}_*(C_*(\tilde{h}))) = 0$  for  $n \leq 2$ . Let  $P_*$  be the  $\mathbb{Z}\pi$ -subchain complex of  $\operatorname{cone}_*(C_*(\tilde{h}))$  given by

$$\dots \xrightarrow{c_5} \operatorname{cone}_4(C_*(\widetilde{h})) \xrightarrow{c_4} \operatorname{cone}_3(C_*(\widetilde{h})) \xrightarrow{c_3} \ker(c_2) \to 0 \to 0 \to 0$$

where  $c_*$  is the differential of cone $(C_*(\tilde{h}))$ . Because of the exact sequence

$$0 \to \ker(c_2) \to \operatorname{cone}_2(C_*(\widetilde{h})) \xrightarrow{c_2} \operatorname{cone}_1(C_*(\widetilde{h})) \xrightarrow{c_1} \operatorname{cone}_0(C_*(\widetilde{h})) \to 0$$

the  $\mathbb{Z}\pi$ -chain complex  $P_*$  is projective. The inclusion  $i_*: P_* \to \operatorname{cone}_*(C_*(\widetilde{h}))$  is a homology equivalence of projective  $\mathbb{Z}\pi$ -chain complexes and hence a  $\mathbb{Z}\pi$ -chain homotopy equivalence. Put  $Q_* = \Sigma^{-3}P_*$ . Then  $Q_*$  is a positive projective  $\mathbb{Z}\pi$ -chain complex such that  $\Sigma^3 Q_*$  is  $\mathbb{Z}\pi$ -chain homotopy equivalent to  $\operatorname{cone}_*(C_*(\widetilde{h}))$ .

The mapping cylinder cyl( $C_*(\tilde{h})$ ), see Definition 3.29, is  $\mathbb{Z}\pi$ -chain homotopy equivalent to  $C_*(\tilde{Y})$  and there is an obvious short exact sequence of  $\mathbb{Z}\pi$ -chain complexes

$$0 \to C_*(\widetilde{Z}) \to \operatorname{cyl}_*(C_*(\widetilde{h})) \to \operatorname{cone}(C_*(\widetilde{h})) \to 0.$$

Since  $C_*(\widetilde{Z})$  is finite free and  $C_*(\widetilde{Y})$  is finitely dominated, we conclude from Lemma 2.36 (i) and (ii) that  $Q_*$  is finitely dominated and that we get in  $\widetilde{K}_0(\mathbb{Z}\pi)$ 

$$\begin{split} \widetilde{o}(Q_*) &= -\widetilde{o}(P_*) = -\widetilde{o}(\operatorname{cone}_*(C_*(\widetilde{h}))) = \widetilde{o}(\operatorname{cyl}_*(C_*(\widetilde{h}))) - \widetilde{o}(C_*(\widetilde{Z})) \\ &= \widetilde{o}(C_*(\widetilde{Y})) - \widetilde{o}(C_*(\widetilde{Z})) = 0 - 0 = 0. \end{split}$$

Lemma 2.36 (iii) implies that  $Q_*$  is  $\mathbb{Z}\pi$ -chain homotopy equivalent to a finite free positive  $\mathbb{Z}\pi$ -chain complex  $F_*$ . Choose a  $\mathbb{Z}\pi$ -chain homotopy equivalence  $g_*: \Sigma^3 F_* \to \operatorname{cone}_*(C_*(\widetilde{h}))$ . We get a commutative diagram of  $\mathbb{Z}\pi$ -chain complexes with exact rows and  $\mathbb{Z}\pi$ -chain homotopy equivalences as vertical arrows

$$0 \longrightarrow C_{*}(\widetilde{Z}) \longrightarrow C_{*} \longrightarrow \Sigma^{3}F_{*} \longrightarrow 0$$

$$\downarrow^{\text{id}} \qquad \qquad \downarrow^{g'_{*}} \qquad \qquad \downarrow^{g_{*}} \qquad \qquad \downarrow^{g_{*}}$$

$$0 \longrightarrow C_{*}(\widetilde{Z}) \longrightarrow \text{cyl}_{*}(C_{*}(\widetilde{h})) \longrightarrow \text{cone}_{*}(C_{*}(\widetilde{h})) \longrightarrow 0$$

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by requiring that the right square is a pull back. Now define the desired  $\mathbb{Z}\pi$ -chain map  $f_*: C_* \to C_*(\widetilde{Y})$  to be the composite of  $g'_*$  with the canonical  $\mathbb{Z}\pi$ -chain homotopy equivalence  $\operatorname{cyl}_*(C_*(\widetilde{h})) \to C_*(\widetilde{Y})$ .

Next we present the main tool to pass from chain complexes to *CW*-complexes. Its proof can be found in [984, Theorem 2] or in the more general equivariant setting in [644, Theorem 13.19 on page 268].

**Theorem 2.46 (Realization Theorem).** Let  $h: Z \to Y$  be a map between connected *CW*-complexes such that  $\pi_1(h): \pi_1(Z) \to \pi_1(Y)$  is an isomorphism. In the sequel we identify  $\pi = \pi_1(Y)$  with  $\pi_1(Z)$  using  $\pi_1(h)$ . Put  $d = \dim(Z)$  and suppose  $2 \le d < \infty$ . Assume the existence of a free  $\mathbb{Z}\pi$ -chain complex  $C_*$  with a preferred  $\mathbb{Z}\pi$ -basis and a  $\mathbb{Z}\pi$ -chain homotopy equivalence  $f_*: C_* \to C_*(\widetilde{Y})$  such that the restriction  $C_*|_d$  to dimensions 0, 1, ..., d agrees with  $C_*(\widetilde{Z})$  and  $f_*|_d = C_*(\widetilde{h})$ .

Then we can construct a CW-complex X such that its d-skeleton  $X_d$  agrees with Z and a cellular homotopy equivalence  $g: X \to Z$  satisfying under the obvious identification  $\pi = \pi_1(X) = \pi_1(Y) = \pi_1(Z)$ :

- (i) We have  $g|_Z = h$ ;
- (ii) There is a  $\mathbb{Z}\pi$ -chain isomorphism  $u_* \colon C \xrightarrow{\cong} C_*(\widetilde{X})$  such that the given  $\mathbb{Z}\pi$ -basis on  $C_*$  is mapped bijectively to the cellular  $\mathbb{Z}\pi$ -basis of  $\widetilde{X}$ ;
- (iii) We have  $C_*(g) \circ u_* = f_*$ .

**Remark 2.47.** Note that there is no absolute version of the Realization Theorem 2.46 in the sense that, for a *d*-dimensional *CW*-complex *Z* with fundamental group  $\pi$  and dimension  $d \ge 2$  and a based free  $\mathbb{Z}\pi$ -chain complex  $C_*$  with  $C_*|_d = C_*(\widetilde{Z})$ , we can find a *CW*-complex *X* with  $X_d = Z$  and  $C_*(\widetilde{X}) = C_*$ . Moreover, the assumption dim $(Z) \ge 2$  cannot be dropped in the Realization Theorem 2.46.

**Lemma 2.48.** Let X be a connected CW-complex. Then it is finitely dominated if and only if  $\pi_1(X)$  is finitely presented and the  $\mathbb{Z}[\pi_1(X)]$ -chain complex  $C_*(\widetilde{X})$  is finitely dominated.

*Proof.* This follows essentially from Theorem 2.46; details of the proof can be found in [984, Corollary 5.1] or in the more general equivariant setting in [644, Proposition 14.6 (a) on page 282].  $\Box$ 

Next we can give the proof of Theorem 2.39.

*Proof of Theorem* 2.39. (i) If the finitely dominated connected *CW*-complex *Y* is homotopy equivalent to a finite *CW*-complex, we get  $\tilde{o}(Y) = 0$  directly from the definitions. Now suppose that *Y* is a finitely dominated connected *CW*-complex with  $\tilde{o}(Y) = 0$ . We conclude from Lemma 2.45 and Theorem 2.46 that *Y* is homotopy equivalent to a *CW*-complex *X* for which its cellular  $\mathbb{Z}\pi$ -chain complex  $C_*(\tilde{X})$  is finite free. The latter implies that *X* is finite.

(ii). Since *G* is finitely presented, we can choose a connected finite 2-dimensional *CW*-complex *Z* with  $\pi_1(Z) = G$ . Consider any element  $\xi \in \widetilde{K}_0(\mathbb{Z}\pi)$ . Choose a finitely generated projective *R*-module *P* and a natural number *n* such that  $\xi = [P] - [\mathbb{Z}\pi^n]$  holds. Choose an exact sequence  $0 \to \bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \to P \to 0$ . Now consider  $X' = X \lor \bigvee_{i_2 \in I} S^2$ . For each  $i_3 \in I_3$  we attach a 3-cell to X' with an attaching map  $q_{i_3} \colon S^2 \to X'$  such that  $[q_{i_3}] \in \pi_2(X')$  corresponds to the image of the basis element in  $\bigoplus_{I_3} \mathbb{Z}\pi$  associated to  $i_3$  under the composite

$$\bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \xrightarrow{j} \pi_2(X')$$

where *j* sends the basis element associated to  $i_2 \in I_2$  to the element in  $\pi_2(X')$  given by the obvious inclusion of  $S^2 \to X'$  associated to  $i_2$ . Call the resulting 3-dimensional *CW*-complex *Y*. Note that we can identify  $\pi$  with  $\pi_1(Y)$ . We obtain an exact sequence of free  $\mathbb{Z}\pi$ -chain complexes

$$0 \to C_*(\widetilde{X}) \to C_*(\widetilde{Y}) \to C_*(\widetilde{Y}, \widetilde{X}) \to 0.$$

The  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{Y}, \widetilde{X})$  is concentrated in dimensions 2 and 3 and its third differential is *u*. This implies that  $C_*(\widetilde{Y}, \widetilde{X})$  is  $\mathbb{Z}\pi$ -chain homotopy equivalent to the  $\mathbb{Z}\pi$ -chain complex concentrated in dimension 2 with *P* as second  $\mathbb{Z}\pi$ -chain module. Hence  $C_*(\widetilde{Y}, \widetilde{X})$  is finitely dominated and  $o(C_*(\widetilde{Y}, \widetilde{X})) = [P]$  by Lemma 2.36 (i). Lemma 2.36 (ii) implies that  $C_*(\widetilde{Y})$  is finitely dominated. Then *Y* is finitely dominated as a *CW*-complex by Lemma 2.48. Lemma 2.36 (ii) implies that we get for some integer *m* 

$$o(C_*(\widetilde{Y})) = o(C_*(\widetilde{Z})) + o(C_*(\widetilde{Y},\widetilde{X})) = m \cdot [\mathbb{Z}\pi] + [P].$$

By attaching to *Y* finitely many trivial 2 and 3-cells, we can arrange that *Y* is a finitely dominated connected *CW*-complex with  $\pi_1(Y) = G$  and  $o(Y) = [P] - [\mathbb{Z}\pi^n] = \xi$ .

Exercise 2.49. Let

$$\begin{array}{c|c} X_0 & \xrightarrow{\iota_1} & X_1 \\ \vdots_2 & & \downarrow_{j_0} & \downarrow_{j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a cellular pushout, i.e., the diagram is a pushout, the map  $i_1$  is an inclusion of *CW*-complexes, the map  $i_2$  is cellular and *X* carries the induced *CW*-structure. Suppose that  $X_0$ ,  $X_1$ ,  $X_2$  are finitely dominated.

Then X is finitely dominated and we get in  $K_0(\mathbb{Z}[\pi_1(X)])$ 

$$o(X) = (j_1)_*(o(X_1)) + (j_2)_*(o(X_2)) - (j_0)_*(o(X_1)).$$

# 2.6 Geometric Interpretation of Projective Class Group and Finiteness Obstruction

Next we give a geometric construction of  $\widetilde{K}_0(\mathbb{Z}\pi)$  that is in the spirit of the wellknown interpretation of the Whitehead group in terms of deformation retractions, which we will present later in Section 3.4. The material of this section is taken from [642], where more information and details of the proofs can be found.

Given a space *Y*, we want to define an abelian group Wa(*Y*). The underlying set is the set of equivalence classes of an equivalence relation ~ defined on the set of maps  $f: X \to Y$  with finitely dominated *CW*-complexes as source and the given space *Y* as target. We call  $f_0: X_0 \to Y$  and  $f_4: X_4 \to Y$  equivalent if there exists a commutative diagram



such that  $j_1$  and  $j_3$  are homotopy equivalences and  $i_0$  and  $i_4$  are inclusions of *CW*complexes with the property that the larger one is obtained from the smaller one by attaching finitely many cells. Obviously this relation is symmetric and reflexive. It needs some work to show transitivity and hence that it is an equivalence relation. The addition in Wa(*Y*) is given by the disjoint sum, i.e., define the sum of the class of  $f_0: X_0 \to Y$  and  $f_1: X_1 \to Y$  to be the class of  $f_0 \coprod f_1: X_0 \coprod X_1 \to Y$ . It is easy to check that this is compatible with the equivalence relation. The neutral element is represented by  $\emptyset \to Y$ . The inverse of the class [f] of  $f: X \to Y$  is constructed as follows. Choose a finite domination (Z, i, r) of X. Construct a map  $F: cyl(i) \to X$ from the mapping cylinder of *i* to Y such that  $F|_X = id_X$  and  $F|_Z = r$ . Then an inverse of [f] is given by the class [f'] of the composite

$$f': \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \xrightarrow{F \cup_{\operatorname{id}_X} F} X \xrightarrow{f} Y.$$

This finishes the definition of the abelian group Wa(Y). A map  $f: Y_0 \to Y_1$  induces a homomorphism of abelian groups  $Wa(f): Wa(Y_0) \to Wa(Y_1)$  by composition. Thus Wa defines a functor from the category of spaces to the category of abelian groups.

**Exercise 2.50.** Show that [f] + [f'] = 0 holds for the composite f' above.

Given a finitely dominated *CW*-complex *X*, define its *geometric finiteness obstruction*  $o_{geo}(X) \in Wa(X)$  by the class of  $id_X$ .

**Theorem 2.51 (The geometric finiteness obstruction).** *Let* X *be a finitely dominated CW-complex. Then* X *is homotopy equivalent to a finite CW-complex if and only if*  $o_{geo}(X) = 0$  *in* Wa(X).

*Proof.* Obviously  $o_{geo}(X) = 0$  if X is homotopy equivalent to a finite CW-complex. Suppose  $o_{geo}(X) = 0$ . Hence there are a CW-complex Y, a map  $r: Y \to X$  and a homotopy equivalence  $h: Y \to Z$  to a finite CW-complex Z such that Y is obtained from X by attaching finitely many cells and  $r \circ i = id_X$  holds for the inclusion  $i: X \to Y$ . The mapping cylinder cyl(r) is built from the mapping cylinder cyl(i) by attaching a finite number of cells and is homotopy equivalent to X. Choose a homotopy equivalence  $g: cyl(i) \to Z$ . Consider the push-out



where *i* is the inclusion. Since *g* is a homotopy equivalence, the same is true for g'. Hence *X* is homotopy equivalent to the finite *CW*-complex *Z'*.

**Theorem 2.52 (Identifying the finiteness obstruction with its geometric counterpart).** Let Y be a space. Then there is a natural isomorphism of abelian groups

$$\Phi\colon \operatorname{Wa}(Y) \xrightarrow{\cong} \bigoplus_{C \in \pi_0(Y)} \widetilde{K_0}(\mathbb{Z}\pi_1(C)).$$

*Proof.* We only explain the definition of  $\Phi$ . Consider an element  $[f] \in Wa(Y)$  represented by a map  $f: X \to Y$  from a finitely dominated *CW*-complex *X* to *Y*. Given a path component *C* of *X*, let  $C_f$  be the path component of *Y* containing f(C). The map *f* induces a map  $f|_C: C \to C_f$  and hence a map  $(f|_C)_*: \widetilde{K}_0(\mathbb{Z}\pi_1(C)) \to \widetilde{K}_0(\mathbb{Z}\pi_1(C_f))$ . Since *X* is finitely dominated, every path component *C* of *X* is finitely dominated, and we can consider its finiteness obstruction  $\widetilde{o}(C) \in \widetilde{K}_0(\mathbb{Z}\pi_1(C))$ . Let  $\phi([f])_C$  be the image of  $\widetilde{o}(C)$  under the composite

$$\widetilde{K}_0(\mathbb{Z}\pi_1(C)) \xrightarrow{(f|_C)_*} \widetilde{K}_0(\mathbb{Z}\pi_1(C_f)) \to \bigoplus_{C \in \pi_0(Y)} \widetilde{K}_0(\mathbb{Z}\pi_1(C)).$$

Since  $\pi_0(X)$  is finite, we can define

$$\phi([f]) := \sum_{C \in \pi_0(X)} \phi([f])_C.$$

We omit the easy proof that this is compatible with the equivalence relation appearing in the definition of Wa(Y), that  $\phi$  is a homomorphism of abelian groups and that Theorem 2.39 implies that  $\Phi$  is bijective.

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## **2.7 Universal Functorial Additive Invariants**

In this section we describe the pair  $(K_0(\mathbb{Z}\pi_1(X)), o(X))$  by an abstract property.

**Definition 2.53 (Functorial additive invariant for finitely dominated** *CW*-complexes). A *functorial additive invariant for finitely dominated CW*-complexes consists of a covariant functor *A* from the category of finitely dominated *CW*-complexes to the category of abelian groups together with an assignment *a* that associates to every finitely dominated *CW*-complex *X* an element  $a(X) \in A(X)$  such that the following axioms are satisfied:

• Homotopy invariance of A

If  $f, g: X \to Y$  are homotopic maps between finitely dominated *CW*-complexes, then A(f) = A(g);

- Homotopy invariance of a(X)
   If f: X → Y is a homotopy equivalence of finitely dominated CW-complexes, then A(f)(a(X)) = a(Y);
- Additivity Let

 $\begin{array}{c|c} X_0 & \xrightarrow{i_1} & X_1 \\ \vdots & & & \downarrow \\ i_2 & & & \downarrow \\ i_2 & & & \downarrow \\ X_2 & \xrightarrow{j_2} & X \end{array}$ 

be a *cellular pushout*, i.e., the diagram is a pushout, the map  $i_1$  is an inclusion of *CW*-complexes, the map  $i_2$  is cellular and *X* carries the induced *CW*-structure. Suppose that  $X_0$ ,  $X_1$ ,  $X_2$  are finitely dominated. Then *X* is finitely dominated and

ten X is initely dominated and

$$a(X) = A(j_1)(a(X_1)) + A(j_2)(a(X_2)) - A(j_0)(a(X_0));$$

• Normalization  $a(\emptyset) = 0.$ 

**Example 2.54 (Componentwise Euler characteristic).** Let *A* be the covariant functor sending a finitely dominated *CW*-complex *X* to  $H_0(X; \mathbb{Z}) = \bigoplus_{C \in \pi_0(X)} \mathbb{Z}$ . Let  $a(X) \in A(X)$  be the componentwise Euler characteristic, i.e., the collection of integers  $\{\chi(C) \mid C \in \pi_0(X)\}$ . Then (A, a) is a functorial additive invariant for finitely dominated *CW*-complexes.

**Definition 2.55 (Universal functorial additive invariant for finitely dominated** *CW*-complexes). A universal functorial additive invariant for finitely dominated *CW*-complexes (U, u) is a functorial additive invariant with the property that for any functorial additive invariant (A, a) there is precisely one natural transformation  $T: U \rightarrow A$  with the property that T(X)(u(X)) = a(X) holds for every finitely dominated *CW*-complex *X*.

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**Exercise 2.56.** Show that the functorial additive invariant defined in Example 2.54 is the universal one if we restrict to finite *CW*-complexes.

Obviously the universal additive functorial invariant is unique (up to unique natural equivalence) if it exists. It is also easy to construct it. However, it turns out that there exists a concrete model, namely, the following theorem is proved in [642, Theorem 4.1].

**Theorem 2.57 (The finiteness obstruction is the universal functorial additive invariant).** The covariant functor  $X \mapsto \bigoplus_{C \in \pi_0(X)} K_0(\mathbb{Z}\pi_1(C))$  together with the componentwise finiteness obstruction  $\{o(C) \mid C \in \pi_0(X)\}$  is the universal functorial additive invariant for finitely dominated CW-complexes.

**Exercise 2.58.** (i) Construct for finitely dominated *CW*-complexes *X* and *Y* a natural bilinear pairing

$$P(X,Y): U(X) \times U(Y) \to U(X \times Y)$$

sending (u(X), u(Y)) to  $u(X \times Y)$  where (U, u) is the universal functorial additive invariant for finitely dominated *CW*-complexes;

(ii) Let X be a finitely dominated CW-complex. Let Y be a finite CW-complex such that  $\chi(C) = 0$  for every component C of Y. Show that  $X \times Y$  is homotopy equivalent to a finite CW-complex.

## **2.8** Variants of the Farrell-Jones Conjecture for $K_0(RG)$

In this section we state variants of the Farrell-Jones Conjecture for  $K_0(RG)$ , where RG, sometimes also written as R[G], is the *group ring* of a group G with coefficients in an associative ring R with unit. Elements in RG are given by formal finite sums  $\sum_{g \in G} r_g \cdot g$ , and addition and multiplication is given by

$$\begin{pmatrix} \sum_{g \in G} r_g \cdot g \end{pmatrix} + \begin{pmatrix} \sum_{g \in G} s_g \cdot g \end{pmatrix} := \sum_{g \in G} (r_g + s_g) \cdot g; \\ \begin{pmatrix} \sum_{g \in G} r_g \cdot g \end{pmatrix} \cdot \begin{pmatrix} \sum_{g \in G} s_g \cdot g \end{pmatrix} := \sum_{\substack{g \in G}} \begin{pmatrix} \sum_{\substack{h, k \in G, \\ g = hk}} r_h \cdot s_k \end{pmatrix} \cdot g$$

The Farrell-Jones Conjecture itself will give a complete answer for arbitrary groups and rings, but to formulate the full version some additional effort will be needed. If one assumes that *R* is regular and *G* is torsionfree or that *R* is regular and  $\mathbb{Q} \subseteq R$ , then the conjecture reduces to easy to formulate statements, which we will present next. Moreover, these special cases are already very interesting.

**Definition 2.59 (Projective resolution).** Let *M* be an *R*-module. A *projective resolution*  $(P_*, \phi)$  of *M* is a positive projective *R*-chain complex  $P_*$  with  $H_n(P_*) = 0$  for  $n \ge 1$  together with an *R*-isomorphism  $\phi: H_0(P_*) \xrightarrow{\cong} M$ . It is called *finite, finitely* 

2.8 Variants of the Farrell-Jones Conjecture for  $K_0(RG)$ 

generated, free, finite-dimensional, or d-dimensional if the R-chain complex  $P_*$  has this property.

A ring *R* is *Noetherian* if any submodule of a finitely generated *R*-module is again finitely generated. A ring *R* is called *regular* if it is Noetherian and any finitely generated *R*-module has a finite-dimensional projective resolution. Any principal ideal domain such as  $\mathbb{Z}$ , any field, and, more generally, any Dedekind domain is regular, see Theorem 2.21.

**Conjecture 2.60 (Farrell-Jones Conjecture for**  $K_0(R)$  **for torsionfree** *G* **and regular** *R***).** Let *G* be a torsionfree group and let *R* be a regular ring. Then the map induced by the inclusion of the trivial group into *G* 

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective.

In particular we get for any principal ideal domain R and torsionfree G

$$\widetilde{K}_0(RG)=0.$$

**Remark 2.61 (Relevance of Conjecture 2.60).** In view of Remark 2.8 Conjecture 2.60 is equivalent to the statement that for a torsionfree group *G* and a regular ring *R* every finitely generated projective *RG*-module is stably finitely generated free. This is the algebraic relevance of this conjecture. Its geometric meaning comes from the following conclusion of Theorem 2.39. Namely, if  $R = \mathbb{Z}$  and *G* is a finitely presented torsionfree group, it is equivalent to the statement that every finitely dominated *CW*-complex with  $\pi_1(X) \cong G$  is homotopy equivalent to a finite *CW*-complex.

**Definition 2.62 (Family of subgroups).** A *family*  $\mathcal{F}$  *of subgroups* of a group *G* is a set of subgroups that is closed under conjugation with elements of *G* and under passing to subgroups.

Our main examples of families are listed below

notation	subgroups
$\mathcal{TR}$	trivial group
<i>FСУ</i>	finite cyclic subgroups
$\mathcal{F}IN$	finite subgroups
СУС	cyclic subgroups
VCУ	virtually cyclic subgroups
ALL	all subgroups

**Definition 2.64 (Orbit category).** The *orbit category* Or(G) has as objects homogeneous spaces G/H and as morphisms G-maps. Given a family  $\mathcal{F}$  of subgroups of G, let the  $\mathcal{F}$ -restricted orbit category  $Or_{\mathcal{F}}(G)$  be the full subcategory of Or(G) whose objects are homogeneous spaces G/H with  $H \in \mathcal{F}$ .

**Definition 2.65 (Subgroup category).** The *subgroup category* Sub(G) has as objects subgroups H of G. For  $H, K \subseteq G$ , let  $conhom_G(H, K)$  be the set of all group homomorphisms  $f: H \to K$  for which there exists a group element  $g \in G$  such that f is given by conjugation with g. The group of inner automorphisms inn(K) consists of those automorphisms  $K \to K$  that are given by conjugation with an element  $k \in K$ . It acts on conhom(H, K) from the left by composition. Define the set of morphisms in Sub(G) from H to K to be  $inn(K) \setminus conhom(H, K)$ . Composition of group homomorphisms defines the composition of morphisms in Sub(G).

Given a family  $\mathcal{F}$ , define the  $\mathcal{F}$ -restricted category of subgroups  $\operatorname{Sub}_{\mathcal{F}}(G)$  to be the full subcategory of  $\operatorname{Sub}(G)$  that is given by objects H belonging to  $\mathcal{F}$ .

#### **Exercise 2.66.** Show that $Sub_{\mathcal{F}}(G)$ is a quotient category of $Or_{\mathcal{F}}(G)$ .

Note that there is a morphism from *H* to *K* only if *H* is conjugate to a subgroup of *K*. Clearly  $K_0(R(-))$  yields a functor from  $\text{Sub}_{\mathcal{F}}(G)$  to abelian groups since inner automorphisms on a group *K* induce the identity on  $K_0(RK)$ . Using the inclusions into *G*, one obtains a map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{F}}(G)} K_0(RH) \to K_0(RG).$$

We briefly recall the notion of a colimit of a covariant functor  $F: C \to \mathbb{Z}$ -MOD from a small category *C* into the category of abelian groups, where *small* means that the objects of *C* form a set. Given an abelian group *A*, let  $C_A$  be the constant functor  $C \to \mathbb{Z}$ -MOD that sends every object in *C* to *A* and every morphism in *C* to id<sub>A</sub>. Given a homomorphism  $f: A \to B$  of abelian groups, let  $C_f: C_A \to C_B$  be the obvious transformation. The *colimit*, sometimes also called the *direct limit*, of *F* consists of an abelian group colim<sub>C</sub> *F* together with a transformation  $T_F: F \to C_{colim_C} F$  such that for any abelian group *B* and transformation  $T: F \to C_B$  there exists precisely one homomorphism of abelian groups  $\phi: \operatorname{colim}_C F \to B$  satisfying  $C_{\phi} \circ T_F = T$ . The colimit is unique (up to unique isomorphism) and always exists. If we replace abelian group by ring or by *R*-module respectively, we get the notion of a *colimit*, sometimes also called a *direct limit*, of functors from a small category to rings or *R*-modules respectively.

Conjecture 2.67 (Farrell-Jones Conjecture for  $K_0(RG)$  for regular R with  $\mathbb{Q} \subseteq R$ ). Let R be a regular ring with  $\mathbb{Q} \subseteq R$  and let G be a group.

Then the homomorphism

(2.68)  $I_{\mathcal{FIN}}(G,F)$ :  $\operatorname{colim}_{H\in\operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$ 

coming from the various inclusions of finite subgroups of G into G is a bijection.

One can also ask for the following stronger version of Conjecture 2.67, which also encompasses Conjecture 2.60.

**Conjecture 2.69 (Farrell-Jones Conjecture for**  $K_0(RG)$  **for regular** R**).** Let R be a regular ring and let G be a group. Let  $\mathcal{P}(G, R)$  be the set of primes which are not invertible in R and for which G contains an element of order p.

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Then the homomorphism

$$I_{\mathcal{FIN}}(G,F)$$
: colim <sub>$H \in Sub_{\mathcal{FIN}}(G)$</sub>   $K_0(RH) \to K_0(RG)$ 

coming from the various inclusions of finite subgroups of G into G is a  $\mathcal{P}(G, R)$ -isomorphism, i.e., an isomorphism after inverting all primes in  $\mathcal{P}(G, R)$ .

We mention that the surjectivity of the map  $I_{\mathcal{FIN}}(G, F)$  is equivalent to the surjectivity of the map induced by the various inclusions of subgroups  $H \in \mathcal{FIN}$  into G

$$\bigoplus_{H \in \mathcal{FIN}} K_0(RH) \to K_0(RG),$$

because this map factorizes as

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$$\bigoplus_{I \in \mathcal{FIN}} K_0(RH) \xrightarrow{\psi} \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \xrightarrow{I_{\mathcal{FIN}}(G,F)} K_0(RG),$$

where the first map  $\psi$  is surjective.

**Remark 2.70 (Module-theoretic relevance of Conjecture 2.67).** Conjecture 2.67 implies that for a regular ring R with  $\mathbb{Q} \subseteq R$  every finitely generated projective R-module is, up to adding finitely generated free RG-modules, a direct sum of finitely many RG-modules of the shape  $RG \otimes_{RH} P$  for a finite subgroup  $H \subseteq G$  and a finitely generated projective RH-module P. So it predicts the (stable) structure of finitely generated projective RG-modules in the most elementary way. We mention, however, that the situation is much more complicated in the case where we drop the assumption that R is regular and  $\mathbb{Q} \subseteq R$ . In particular, for  $R = \mathbb{Z}$  new phenomena will occur, as explained later, which are related to so-called negative K-groups and Nil-groups. For instance, the obvious inclusion  $\mathbb{Z}/6 \to \mathbb{Z} \times \mathbb{Z}/6$  does not induce a surjection  $K_0(\mathbb{Z}[\mathbb{Z}/6]) \to K_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/6])$ , since  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0$  and  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/6]) \cong \mathbb{Z}$ , whereas by  $K_0(\mathbb{Q}[\mathbb{Z}/6]) \to K_0(\mathbb{Q}[\mathbb{Z} \times \mathbb{Z}/6])$  is known to be bijective as predicted by Conjecture 2.67.

**Remark 2.71 (Conjecture 2.67 and the Atiyah Conjecture).** Conjecture 2.67 plays a role in a program aiming at a proof of the Atiyah Conjecture about  $L^2$ -Betti numbers, as explained in [650, Section 10.2]. Atiyah defined the *n*-th  $L^2$ -Betti number of the universal covering  $\widetilde{M}$  of a closed Riemannian manifold M to be the non-negative real number

$$b_n^{(2)}(\widetilde{M}) := \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}\left(\mathrm{e}^{-t\Delta_n(\widetilde{x},\widetilde{x})}\right) \mathrm{d}\widetilde{x}$$

where  $\mathcal{F}$  is a fundamental domain for the  $\pi_1(M)$ -action and  $e^{-t\Delta_n(\tilde{x},\tilde{x})}$  denotes the heat kernel on  $\tilde{M}$ . The version of the *Atiyah Conjecture* which we are interested in and which is at the time of writing open says that  $d \cdot b_n^{(2)}(\tilde{M})$  is an integer if d is an integer such that the order of any finite subgroup of  $\pi_1(M)$  divides d. In particular  $b_n^{(2)}(\tilde{M})$  is expected to be an integer if  $\pi_1(M)$  is torsionfree. This gives an interesting

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connection between the analysis of heat kernels and the projective class group of complex group rings  $\mathbb{C}G$ .

If one drops the condition that there exists a bound on the order of finite subgroups of  $\pi_1(M)$ , then also transcendental real numbers can occur as the  $L^2$ -Betti number of the universal covering  $\tilde{M}$  of a closed Riemannian manifold M, see [58, 433, 809].

An *R*-module *M* is called *Artinian* if for any descending series of submodules  $M_1 \supseteq M_2 \supseteq \cdots$  there exists an integer *k* such that  $M_k = M_{k+1} = M_{k+2} = \cdots$  holds. An *R*-module *M* is called *simple* or *irreducible* if  $M \neq \{0\}$  and *M* contains only  $\{0\}$  and *M* as submodules. A ring *R* is called *Artinian* if both *R* considered as a left *R*-module is Artinian and *R* considered as a right *R*-module is Artinian, or, equivalently, every finitely generated left *R*-module and every finitely generated right *R*-module is Artinian. Skew-fields and finite rings are Artinian, whereas  $\mathbb{Z}$  is not Artinian.

**Conjecture 2.72 (Farrell-Jones Conjecture for**  $K_0(RG)$  **for an Artinian ring** *R***).** Let *G* be a group and *R* be an Artinian ring.

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Then the canonical map

 $I_{\mathcal{FIN}}(G, R)$ : colim<sub> $H \in Sub_{\mathcal{FIN}}(G)$ </sub>  $K_0(RH) \to K_0(RG)$ 

is an isomorphism

# 2.9 Kaplansky's Idempotent Conjecture

In this section we discuss the following conjecture.

**Conjecture 2.73 (Kaplansky's Idempotent Conjecture).** Let R be an integral domain and let G be a torsionfree group. Then all idempotents of RG are trivial, i.e., equal to 0 or 1.

**Remark 2.74 (Kaplansky's Idempotent Conjecture for prime characteristic).** There is a reasonable more general version of Conjecture 2.73 where one replaces the condition that *G* is torsionfree by the weaker condition that any prime *p* which divides the order of some finite subgroup  $H \subseteq G$  is not invertible in the integral domain *R*. If *R* is a skew-field of prime characteristic *p*, then this condition reduces to the condition that any finite subgroup  $H \circ G$  is a *p*-group.

The version of Kaplansky's Idempotent Conjecture 2.73 described in Remark 2.74 is consistent with the observation that the only known idempotents in a group ring RG come from idempotents in R or by the following construction.

**Example 2.75 (Construction of idempotents).** Let *G* be a group and  $g \in G$  be an element of finite order. Suppose that the order |g| is invertible in *R*. Define an element  $x := |g|^{-1} \cdot \sum_{i=1}^{|g|} g^i$ . Then  $x^2 = x$ , i.e., *x* is an idempotent in *RG*.

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2.9 Kaplansky's Idempotent Conjecture

**Exercise 2.76.** Show that the version of Kaplansky's Idempotent Conjecture of Remark 2.74 holds for  $G = \mathbb{Z}/2$ .

**Exercise 2.77.** Consider the ring  $R = \mathbb{Z}[x]/(2x^2 - 3x + 1)$ . In the sequel denote by  $\overline{u}$  the class of  $u \in \mathbb{Z}[x]$  in R. Show:

- (i) 2 is not invertible in *R*;
- (ii) There are precisely two non-trivial idempotents in *R*, namely  $\overline{2-2x}$  and  $\overline{-1+2x}$ ;
- (iii) The element  $\overline{x} + (1 \overline{x}) \cdot t$  is a non-trivial idempotent in  $R[\mathbb{Z}/2]$ .

**Remark 2.78 (Sofic groups).** In the next theorem we will use the notion of a *sofic group* that was introduced by Gromov and originally called *subamenable group*. Every residually amenable group is sofic but the converse is not true. The class of sofic groups is closed under taking subgroups, direct products, amalgamated free products, colimits and inverse limits, and, if *H* is a sofic normal subgroup of *G* with amenable quotient G/H, then *G* is sofic. To the authors' knowledge there is no example of a group that is not sofic. There is a note by Dave Witte Morris [752] following Deligne [300] where a central extension  $1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow SP(2n, \mathbb{R}) \rightarrow 1$  is constructed such that *G* is not residually finite. The group *G* is viewed as a candidate for a group which is not sofic. It is unknown but likely to be true that all hyperbolic groups are sofic. For more information about the notion of a sofic group we refer to [332].

**Definition 2.79 (Directly finite).** An *R*-module *M* is called *directly finite* if every *R*-module *N* satisfying  $M \cong_R M \oplus N$  is trivial. A ring *R* is called *directly finite* (or *von Neumann finite*) if it is directly finite as a module over itself, or, equivalently, if  $r, s \in R$  satisfy rs = 1, then sr = 1. A ring is called *stably finite* if the matrix algebra  $M_n(R)$  is directly finite for all  $n \ge 1$ .

**Remark 2.80 (Stable finiteness).** Stable finiteness for a ring *R* is equivalent to the following statement. Every finitely generated projective *R*-module *P* whose class in  $K_0(R)$  is zero is already the trivial module, i.e.,  $0 = [P] \in K_0(R)$  implies  $P \cong 0$ .

If *F* is a field of characteristic zero, then *FG* is stably finite for every group *G*. This is proved by Kaplansky [544], see also Passman [791, Corollary 1.9 on page 38]. If *R* is a skew-field and *G* is a sofic group, then *RG* is stably finite. This is proved for free-by-amenable groups by Ara-Meara-Perera [35] and extended to sofic groups by Elek-Szabo [331, Corollary 4.7]. These results have been extended to extensions with a finitely generated residually finite groups as kernel and a sofic finitely generated group as quotient by Berlai [128].

The next theorem is taken from [88, Theorem 1.12].

**Theorem 2.81 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture).** Let G be a group. Let R be a ring whose idempotents are all trivial. Suppose that

$$K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Then 0 and 1 are the only idempotents in RG if one of the following conditions is satisfied:

- (i) *RG* is stably finite;
- (ii) *R* is a field of characteristic zero;
- (iii) *R* is a skew-field and *G* is sofic.

**Remark 2.82 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture).** Theorem 2.81 implies that for a skew-field D of characteristic zero and a torsionfree group G Kaplansky's Idempotent Conjecture 2.73 is true for DG, provided that Conjecture 2.60 holds and that D is commutative or G is sofic.

Remark 2.83 (The Farrell-Jones Conjecture and the Kaplansky's Idempotent Conjecture for prime characteristic). Suppose that D is a skew-field of prime characteristic p, that Conjecture 2.72 holds for G and D, and that all finite subgroups of G are p-groups. Then  $K_0(D) \xrightarrow{\cong} K_0(DG)$  is an isomorphism since for a finite p-group H the group ring DH is a local ring, see [271, Theorem 5.24 on page 114], and hence  $\widetilde{K}_0(DH) = 0$  by Lemma 2.123. If we furthermore assume that G is sofic, then Theorem 2.81 implies that all idempotents in DG are trivial.

**Remark 2.84 (Reducing the case of a field of characteristic zero to**  $\mathbb{C}$ ). Let *F* be a field of characteristic zero and let  $u = \sum_{g \in G} x_g \cdot g \in FG$  be an element. Let *K* be the finitely generated field extension of  $\mathbb{Q}$  given by  $K = \mathbb{Q}(x_g \mid g \in G) \subset F$ . Then *u* is already an element in *KG*. The field *K* embeds into  $\mathbb{C}$  since *K* is finitely generated, it is a finite algebraic extension of a transcendental extension *K'* of  $\mathbb{Q}$ , see [617, Theorem 1.1 on p. 356], and *K'* has finite transcendence degree over  $\mathbb{Q}$ . Since the transcendence degree of  $\mathbb{C}$  over  $\mathbb{Q}$  is infinite, there exists an embedding  $K' \hookrightarrow \mathbb{C}$ induced by an injection of a transcendence basis of *K* over  $\mathbb{Q}$  into a transcendence basis of  $\mathbb{C}$  over  $\mathbb{Q}$ . It extends to an embedding  $K \hookrightarrow \mathbb{C}$  because  $\mathbb{C}$  is algebraically closed. Hence *u* can be viewed as an element in  $\mathbb{C}G$ . This reduces the case of fields *F* of characteristic zero to the case  $F = \mathbb{C}$ .

Next we mention some further results.

Formanek [398, Theorem 9], see also [189, Proposition 4.2], has shown that all idempotents of *FG* are trivial, provided that *F* is a field of characteristic zero and there are infinitely many primes *p* for which there do not exist an element  $g \in G, g \neq 1$  and an integer  $k \ge 1$  such that *g* and  $g^{p^k}$  are conjugate. Torsionfree hyperbolic groups satisfy these conditions. Hence Formanek's results imply that all idempotents in *FG* are trivial if *G* is torsionfree hyperbolic and *F* is a field of characteristic zero.

Delzant [301] has proved the Kaplansky's Idempotent Conjecture 2.73 for all integral domains R for a torsionfree hyperbolic group G, provided that G admits an appropriate action with large enough injectivity radius. Delzant actually deals with zero-divisors and units as well.

2.10 The Bass Conjectures

**Remark 2.85 (Conjectures related to the Idempotent conjecture).** There are also the Zero-Divisor Conjecture due to Kaplansky, which predicts for an integral domain R and a torsionfree group G that RG has no non-trivial zero-divisors, and the Embedding Conjecture due to Malcev, which predicts for an integral domain Rand a torsionfree group G that RG can be emdedded into a skew-field. Obviously the Embedding Conjecture implies the Zero-Divisor Conjecture, which in turn implies the Idempotent Conjecture 2.73. The Zero-Divisor Conjecture does *not* follow from Conjecture 2.60. For a ring R with  $\mathbb{Q} \subseteq R = \mathbb{C}$  the Zero-Divisor Conjecture follows from the Atiyah Conjecture about the integrality of  $L^2$ -Betti numbers for torsionfree groups, see [650, Lemma 10.15 on page 376]. There is also the Unit-Conjecture 3.125, which implies the Zero-Divisor Conjecture, see [610, (6.20) on page 95], and is discussed in Section 3.14.

# 2.10 The Bass Conjectures

### 2.10.1 The Bass Conjecture for Fields of Characteristic Zero as Coefficients

Let *G* be a group. Let con(G) be the set of conjugacy classes (g) of elements  $g \in G$ . Denote by  $con(G)_f$  the subset of con(G) consisting of those conjugacy classes (g) for which each representative *g* has finite order. Let *R* be a commutative ring. Let class(G, R) and  $class(G, R)_f$  be the free *R*-module with the set con(G) and  $con(G)_f$  as basis. This is the same as the *R*-module of *R*-valued functions on con(G) and  $con(G)_f$  with finite support. Define the *universal R-trace* 

(2.86) 
$$\operatorname{tr}_{RG}^{u} \colon RG \to \operatorname{class}(G, R), \quad \sum_{g \in G} r_{g} \cdot g \mapsto \sum_{g \in G} r_{g} \cdot (g).$$

It extends to a function  $\operatorname{tr}_{RG}^{u}$ :  $\operatorname{M}_{n}(RG) \to \operatorname{class}(G, R)$  on (n, n)-matrices over RG by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective RG-module. Choose a matrix  $A \in \operatorname{M}_{n}(RG)$  such that  $A^{2} = A$  and the image of the RG-map  $r_{A}: RG^{n} \to RG^{n}$  given by right multiplication with A is RG-isomorphic to P. Define the *Hattori-Stallings rank* of P to be

(2.87) 
$$\operatorname{HS}_{RG}(P) = \operatorname{tr}_{RG}^{u}(A) \in \operatorname{class}(G, R).$$

The Hattori-Stallings rank depends only on the isomorphism class of the *RG*-module *P*. It induces an *R*-homomorphism, the *Hattori-Stallings homomorphism*,

(2.88) 
$$\operatorname{HS}_{RG}: K_0(RG) \otimes_{\mathbb{Z}} R \to \operatorname{class}(G, R), \quad [P] \otimes r \mapsto r \cdot \operatorname{HS}_{RG}(P).$$

Let *F* be a field of characteristic zero. Fix an integer  $m \ge 1$ . Let  $F(\zeta_m) \supset F$  be the Galois extension given by adjoining the primitive *m*-th root of unity  $\zeta_m$  to *F*. Denote by  $\Gamma(m, F)$  the Galois group of this extension of fields, i.e., the group of automorphisms  $\sigma: F(\zeta_m) \to F(\zeta_m)$  that induce the identity on *F*. It can be identified

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with a subgroup of  $\mathbb{Z}/m^*$  by sending  $\sigma$  to the unique element  $u(\sigma) \in \mathbb{Z}/m^*$  for which  $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$  holds. Let  $g_1$  and  $g_2$  be two elements of G of finite order. We call them *F*-conjugate if for some (and hence all) positive integers m with  $g_1^m = g_2^m = 1$  there exists an element  $\sigma$  in the Galois group  $\Gamma(m, F)$  with the property that  $g_1^{u(\sigma)}$  and  $g_2$  are conjugate. Two elements  $g_1$  and  $g_2$  are *F*-conjugate for  $F = \mathbb{Q}$ ,  $\mathbb{R}$ , or  $\mathbb{C}$ , if the cyclic subgroups  $\langle g_1 \rangle$  and  $\langle g_2 \rangle$  are conjugate if  $g_1$  and  $g_2$ , or  $g_1$  and  $g_2^{-1}$ , or  $g_1$  and  $g_2$  are conjugate, respectively.

Denote by  $\operatorname{con}_F(G)_f$  the set of *F*-conjugacy classes  $(g)_F$  of elements  $g \in G$  of finite order. Let  $\operatorname{class}_F(G)_f$  be the *F*-vector space with the set  $\operatorname{con}_F(G)_f$  as basis, or, equivalently, the *F*-vector space of functions  $\operatorname{con}_F(G)_f \to F$  with finite support. There are obvious inclusions of *F*-modules

$$class_F(G)_f \subseteq class(G, F)_f \subseteq class(G, F).$$

**Lemma 2.89.** Suppose that *F* is a field of characteristic zero and *H* is a finite group. Then the Hattori-Stallings homomorphism, see (2.88), induces an isomorphism

$$\mathrm{HS}_{FH} \colon K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{\cong} \mathrm{class}_F(H)_f.$$

*Proof.* Since *H* is finite, an *FH*-module is a finitely generated projective *FH*-module if and only if it is a (finite-dimensional) *H*-representation with coefficients in *F* and  $K_0(FH)$  is the same as the representation ring  $\text{Rep}_F(H)$ . The Hattori-Stallings rank  $\text{HS}_{FH}(V)$  and the character  $\chi_V$  of a *G*-representation *V* with coefficients in *F* are related by the formula

(2.90) 
$$\chi_V(h^{-1}) = |C_G\langle h\rangle| \cdot \mathrm{HS}_{FH}(V)(h)$$

for  $h \in H$  where  $C_G \langle h \rangle$  is the centralizer of h in G. Hence Lemma 2.89 follows from representation theory, see for instance [908, Corollary 1 in Chapter 12 on page 96].

### Exercise 2.91. Prove formula (2.90).

The following conjecture is the obvious generalization of Lemma 2.89 to infinite groups.

Conjecture 2.92 (Bass Conjecture for fields of characteristic zero as coefficients). Let F be a field of characteristic zero and let G be a group. The Hattori-Stallings homomorphism of (2.88) induces an isomorphism

$$\operatorname{HS}_{FG}$$
:  $K_0(FG) \otimes_{\mathbb{Z}} F \to \operatorname{class}_F(G)_f$ .

2.10 The Bass Conjectures

**Lemma 2.93.** Suppose that *F* is a field of characteristic zero and *G* is a group. Then the composite

(2.94) 
$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}(G)}} K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{I_{\mathcal{FIN}(G,F) \otimes_{\mathbb{Z}} \operatorname{id}_F}} K_0(FG) \otimes_{\mathbb{Z}} F \xrightarrow{\operatorname{HS}_{FG}} \operatorname{class}(G,F)$$

is injective and has as image  $class_F(G)_f$  where  $I_{\mathcal{FIN}}(G,F)$  is the map defined in (2.68).

*Proof.* This follows from the commutative diagram below, compare [646, Lemma 2.15 on page 220].

Here the isomorphism *j* is the direct limit over the obvious maps  $class_F(H)_f \rightarrow class_F(G)_f$  given by extending a class function in the trivial way and the map *i* is the natural inclusion and in particular injective.

**Exercise 2.95.** Let *F* be a field of characteristic zero. Show that the group *G* must be torsionfree if  $\widetilde{K}_0(FG)$  is a torsion group.

**Theorem 2.96 (The Farrell-Jones Conjecture and the Bass Conjecture for fields of characteristic zero).** The Farrell-Jones Conjecture 2.67 for  $K_0(RG)$  for regular R and  $\mathbb{Q} \subseteq R$  implies the Bass Conjecture 2.92 for fields of characteristic zero as coefficients.

*Proof.* This follows from Lemma 2.93.

The Bost Conjecture 14.23 implies the Bass Conjecture 2.92 for fields of characteristic zero as coefficients, provided that  $F = \mathbb{C}$ , see [131, Theorem 1.4 and Lemma 1.5].

**Exercise 2.97.** Let *F* be field of characteristic zero and let *G* be a group. Suppose that the Farrell-Jones Conjecture 2.67 for  $K_0(RG)$  for regular *R* and  $\mathbb{Q} \subseteq R$  holds for R = F. Consider any finitely generated projective *FG*-module *P*. Then the Hattori-Stallings rank  $HS_{FG}(P)$  evaluated at the unit  $e \in G$  belongs to  $\mathbb{Q} \subseteq F$ .

**Remark 2.98 (Zalesskii's Theorem).** Zalesskii [1031], see also [189, Theorem 3.1], has shown for every field *F*, every group *G*, and every idempotent  $x \in FG$  that  $HS_{FG}((x))$  evaluated at the unit  $e \in G$  belongs to the prime field of *F*, where (*x*) is the finitely generated projective *FG*-module given by the two-sided ideal (x)  $\subseteq$  *FG* spanned by *x*.

#### 2.10.2 The Bass Conjecture for Integral Domains as Coefficients

**Conjecture 2.99 (Bass Conjecture for integral domains as coefficients).** Let *R* be a commutative integral domain and let *G* be a group. Let  $g \in G$  be an element in *G*. Suppose that either the order |g| is infinite or that the order |g| is finite and not invertible in *R*.

Then for every finitely generated projective RG-module P the value of its Hattori-Stallings rank  $HS_{RG}(P)$  at (g) is trivial.

Sometimes the Bass Conjecture 2.99 for integral domains as coefficients is called the *Strong Bass Conjecture*, see [104, 4.5]. The *Weak Bass Conjecture*, see [104, 4.4], states for a finitely generated projective  $\mathbb{Z}G$ -module *P* that the evaluation of its Hattori-Stallings rank at the unit  $HS_{\mathbb{Z}G}(P)(1)$  agrees with  $\dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P)$ . Note that  $HS_{\mathbb{Z}G}(P)(1)$  is the same as the von Neumann dimension  $\dim_{N(G)}(N(G) \otimes_{\mathbb{Z}G} P)$  for a finitely generated projective  $\mathbb{Z}G$ -module *P*, see [650, Corollary 9.61 on page 362].

**Exercise 2.100.** Show that the Weak Bass Conjecture follows from the Bass Conjecture 2.99 for integral domains as coefficients.

The Bass Conjecture 2.99 can be interpreted topologically. Namely, the Bass Conjecture 2.99 is true for a finitely presented group *G* in the case  $R = \mathbb{Z}$  if and only if every homotopy idempotent self-map of an oriented smooth closed manifold whose dimension is greater than 2 and whose fundamental group is isomorphic to *G*, is homotopic to one that has precisely one fixed point, see [132]. The Bass Conjecture 2.99 for *G* in the case  $R = \mathbb{Z}$  (or  $R = \mathbb{C}$ ) also implies for a finitely dominated *CW*-complex with fundamental group *G* that its Euler characteristic agrees with the  $L^2$ -Euler characteristic of its universal covering, see [327, 0.3].

The next results follows from the argument in [372, Section 5].

**Theorem 2.101 (The Farrell-Jones Conjecture and the Bass Conjecture for integral domains).** Let G be a group. Suppose that

 $I(G, F) \otimes_{\mathbb{Z}} \mathbb{Q}$ : colim<sub>Or<sub>*FTN</sub>(G)</sub> K\_0(FH) \otimes\_{\mathbb{Z}} \mathbb{Q} \to K\_0(FG) \otimes\_{\mathbb{Z}} \mathbb{Q}</sub></sub>* 

is surjective for all fields F of prime characteristic.

*Then the Bass Conjecture* 2.99 *is satisfied for G and every commutative integral domain R.* 

In particular, the Bass Conjecture 2.99 follows from the Farrell-Jones Conjecture 2.72.

For finite *G* and *R* an integral domain such that no prime dividing the order of |G| is a unit in *R*, Conjecture 2.99 was proved by Swan [937, Theorem 8.1], see also [104, Corollary 4.2]. The Bass Conjecture 2.99 has been proved by Bass [104, Proposition 6.2 and Theorem 6.3] for  $R = \mathbb{C}$  and *G* a torsionfree linear group and by Eckmann [325, Theorem 3.3] for  $R = \mathbb{Q}$ , provided that *G* has at most cohomological dimension 2 over  $\mathbb{Q}$ .

The following result is due to Linnell [632, Lemma 4.1].

2.11 The Passage from the Integral to the Rational Group Ring

**Theorem 2.102 (The Bass Conjecture for integral domains and elements of finite order).** *Let G be a group.* 

- (i) Let p be a prime, and let P be a finitely generated projective  $\mathbb{Z}_{(p)}G$ -module. Suppose for  $g \in G$  that  $HS(P)(g) \neq 0$ . Then there exists an integer  $n \geq 1$  such that g and  $g^{p^n}$  are conjugate in G and we get for the Hattori-Stallings rank  $HS(P)(g) = HS(P)(g^{p^n})$ ;
- (ii) Let P be a finitely generated projective  $\mathbb{Z}G$ -module. Suppose for  $g \in G$  that  $g \neq 1$  and  $HS(P)(g) \neq 0$ . Then there exist subgroups C, H of G such that  $g \in C$ ,  $C \subseteq H$ , C is isomorphic to the additive group  $\mathbb{Q}$ , H is finitely generated, and the elements of C lie in finitely many H-conjugacy classes. In particular the order of g is infinite.

More information about the Bass Conjectures can be found in [103, 131, 133, 189, 234, 336, 337, 338, 546, 650, 788, 893, 894].

# 2.11 The Passage from the Integral to the Rational Group Ring

The following conjecture is taken from [673, Conjecture 85 on page 754].

**Conjecture 2.103 (The rational**  $\widetilde{K}_0(\mathbb{Z}G)$ **-to-** $\widetilde{K}_0(\mathbb{Q}G)$ **-Conjecture).** The change of ring maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Z}G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Q}G)$$

is trivial.

If *G* satisfies the Farrell-Jones Conjecture 2.67 for  $K_0(RG)$  for regular *R* with  $\mathbb{Q} \subseteq R$ , then it satisfies the rational  $\widetilde{K}_0(\mathbb{Z}G)$ -to- $\widetilde{K}_0(\mathbb{Q}G)$ -Conjecture 2.103, see [673, Proposition 87 on page 754].

**Remark 2.104.** The question whether an integral version of Conjecture 2.103 holds, i.e., whether the change of ring maps

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$$

is trivial, is discussed in [673, Remark 89 on page 756].

The answer is no in general. Counterexamples have been constructed by Lehner [625], who actually carefully analyzes the image of the map  $\widetilde{K}_0(\mathbb{Z}G) \rightarrow \widetilde{K}_0(\mathbb{Q}G)$ . The group  $G = QD_{32} *_{Q_{16}} QD_{32}$  is a counterexample, where  $QD_{32}$  is the quasi-dihedral group of order 32, and  $Q_{16}$  is the generalized quaternion group of order 16, see [625, Theorem 1.5].

# 2.12 Survey on Computations of $K_0(RG)$ for Finite Groups

In this section we give a brief survey about computations of  $K_0(RG)$  for finite groups G and certain rings R. The upshot will be that the reduced projective class group  $\widetilde{K}_0(\mathbb{Z}G)$  is a finite abelian group, but in most cases it is non-trivial and unknown, and that for F a field of characteristic zero  $K_0(FG)$  is a well-known finitely generated free abelian group.

The following result is due to Swan [937, Theorem 8.1 and Proposition 9.1].

**Theorem 2.105** ( $\tilde{K}_0(RG)$ ) is finite for finite *G* and *R* the ring of integers in an algebraic number field). Let *G* be a finite group. Let *R* be the ring of algebraic integers in an algebraic number field, e.g.,  $R = \mathbb{Z}$ . Then  $\tilde{K}_0(RG)$  is finite.

A proof of the next theorem will be given in Section 3.8. It was originally proved by Rim [852].

**Theorem 2.106 (Rim's Theorem).** Let p be a prime number. The homomorphism induced by the ring homomorphism  $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\exp(2\pi i/p)]$  sending the generator of  $\mathbb{Z}/p$  to the primitive p-th root of unity  $\exp(2\pi i/p)$ 

$$K_0(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{=} K_0(\mathbb{Z}[\exp(2\pi i/p)])$$

is a bijection.

**Example 2.107**  $(\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p]))$ . Let *p* be a prime. We have already mentioned in Remark 2.23 that  $\mathbb{Z}[\exp(2\pi i/p)]$  is the ring of integers in the algebraic number field  $\mathbb{Q}[\exp(2\pi i/p)]$  and hence a Dedekind domain and that the structure of its ideal class group  $C(\mathbb{Z}[\exp(2\pi i/p)])$  is only known for a few primes. Thus the message of Rim's Theorem 2.106 is that we know the structure of the finite abelian group  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$  only for a few primes. Here is a table taken from [727, page 30] or [990, Tables §3 on page 352ff].

р	$\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$
≤ 19	{0}
23	$\mathbb{Z}/3$
29	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
31	$\mathbb{Z}/9$
37	$\mathbb{Z}/37$
41	$\mathbb{Z}/11 \oplus \mathbb{Z}/11$
43	Z/211
47	$\mathbb{Z}/5 \oplus \mathbb{Z}/139$

**Remark 2.108** (Strategy to study  $\widetilde{K}_0(\mathbb{Z}G)$  for finite *G*). A  $\mathbb{Z}$ -order  $\Lambda$  is a  $\mathbb{Z}$ -algebra that is finitely generated projective over  $\mathbb{Z}$ . Its *locally free class group* is defined as the subgroup of  $K_0(\Lambda)$ 

$$(2.109) \qquad Cl(\Lambda) := \left\{ [P] - [Q] \mid P_{(p)} \cong_{\Lambda_{(p)}} Q_{(p)} \text{ for all primes } p \right\}$$

where (p) denotes localization at the prime p. This is the part of  $K_0(\Lambda)$  that can be described by localization sequences. Its significance for  $\Lambda = \mathbb{Z}G$  lies in the result of Swan [937], see also Curtis-Reiner [271, Theorem 32.11 on page 676] and [272, (49.12 on page 221], that  $\widetilde{K}_0(\mathbb{Z}G) \cong Cl(\mathbb{Z}G)$  for every finite group G. Now fix a maximal  $\mathbb{Z}$ -order  $\mathbb{Z}G \subseteq \mathcal{M} \subseteq \mathbb{Q}G$ . Such a maximal order has better ring properties than  $\mathbb{Z}G$ , namely, it is a hereditary ring. The map  $i_*: Cl(\mathbb{Z}G) \to Cl(\mathcal{M})$  induced by the inclusion  $i: \mathbb{Z}G \to \mathcal{M}$  is surjective. Define

$$(2.110) D(\mathbb{Z}G) = \ker (i_* : Cl(\mathbb{Z}G) \to Cl(\mathcal{M}))$$

The definition of  $D(\mathbb{Z}G)$  is known to be independent of the choice of the maximal order  $\mathcal{M}$ . Thus the study of  $\widetilde{K}_0(\mathbb{Z}G)$  splits into the study of  $D(\mathbb{Z}G)$  and  $Cl(\mathcal{M})$ . The analysis of  $Cl(\mathcal{M})$  can be intractable and involves studying cyclotomic fields, whereas the analysis of  $D(\mathbb{Z}G)$  essentially uses *p*-adic logarithms.

**Remark 2.111 (Finiteness obstructions and**  $D(\mathbb{Z}G)$ ). Often calculations concerning finiteness obstructions are done by first showing that its image in  $Cl(\mathcal{M}) = \widetilde{K}_0(\mathbb{Z}G)/D(\mathbb{Z}G)$  is trivial, and then determining it in  $D(\mathbb{Z}G)$ . For instance, Mislin [739] proved that the finiteness obstruction for every finitely dominated homologically nilpotent space with the finite group *G* as fundamental group lies in  $D(\mathbb{Z}G)$ , but that not every element in  $D(\mathbb{Z}G)$  occurs this way. Questions concerning the Spherical Space Form Problem involve direct computations in  $D(\mathbb{Z}G)$ , see for instance Bentzen [122], Bentzen-Madsen [123], and Milgram [719]. The group  $D(\mathbb{Z}G)$  enters also in the work of Oliver on actions of finite groups on disks, see [771, 772].

For computations of  $D(\mathbb{Z}G)$  for finite *p*-groups we refer to Oliver [773, 774] and Oliver-Taylor [777].

A survey on  $D(\mathbb{Z}G)$  and the methods of its computations can be found in Oliver [775].

### Theorem 2.112 (Vanishing results for $D(\mathbb{Z}G)$ ).

- (i) Let G be a finite abelian group G. Then  $D(\mathbb{Z}G) = 0$  holds if and only if G satisfies one of the conditions:
  - (a) *G* has prime order;
  - (b) *G* is cyclic of order 4, 6, 8, 9, 10, 14;
  - (c) *G* is  $\mathbb{Z}/2 \times \mathbb{Z}/2$ ;
- (ii) If G is a finite group that is not abelian and satisfies  $D(\mathbb{Z}G) = 0$ , then it is  $D_{2n}$  for  $n \ge 3$ ,  $A_4$ ,  $A_5$ , or  $S_4$ ;
- (iii) One has  $D(\mathbb{Z}G) = 0$  if G is  $A_4$ ,  $A_5$  or  $S_4$ ;
- (iv)  $D(\mathbb{Z}D_{2n}) = 0$  for n < 60 and  $D(\mathbb{Z}D_{120}) = \mathbb{Z}/2$ ;
- (v)  $D(\mathbb{Z}D_{2n}) = 0$  if n satisfies one of the following conditions:
  - (a) *n* is an odd prime;
  - (b) *n* is a power of a regular odd prime;
  - (c) n is a power of 2.

*Proof.* (i) This is proved by Cassou-Nogués [218], see also [272, Theorem 50.16 on page 253].

(ii) This is proved in Endo-Hironaka [339], see also [272, Theorem 50.29 on page 266].

(iii) This follows from Reiner-Ulom [849], see also [272, Theorem 50.29 on page 266].

(iv) This is proved in Endo-Miyata [340], see [272, Theorem 50.30 on page 266].

(v) This is proved in Endo-Hironaka [339], see also [272, Theorem 50.29 on page 266].  $\Box$ 

# Theorem 2.113 (Finite groups with vanishing $\widetilde{K}_0(\mathbb{Z}G)$ ).

(i) Let G be a finite abelian group G. Then  $\widetilde{K}_0(\mathbb{Z}G) = 0$  holds if and only if G satisfies one of the conditions:

(a) *G* is cyclic of order *n* for 1 ≤ *n* ≤ 11;
(b) *G* is cyclic of order 13, 14, 17, 19;
(c) *G* is Z/2 × Z/2;

- (ii) If G is a non-abelian finite group with  $\widetilde{K}_0(\mathbb{Z}G) = 0$ , then G is  $D_{2n}$  for  $n \ge 3$ ,  $A_4, A_5$ , or  $S_4$ ;
- (iii) We have  $\widetilde{K_0}(\mathbb{Z}G) = 0$  for  $G = A_4, S_4, D_6, D_8, D_{12}$ .

*Proof.* (i) This is proved by Cassou-Nogués [218], see also [272, Corollary 50.17 on page 253].

(ii) This follows from Theorem 2.112 (ii).

(iii) The cases  $G = A_4, S_4, D_6, D_8$  are already treated in [848, Theorem 6.4 and Theorem 8.2]. Because of Theorem 2.112 (iii) it suffices to show for the maximal order  $\mathcal{M}$  for the groups  $G = A_4, S_4, D_6, D_8, D_{12}$  that  $Cl(\mathcal{M}) = 0$ . This follows from the fact that  $\mathbb{Q}G$  is a products of matrix algebras over  $\mathbb{Q}$  and hence the maximal  $\mathbb{Z}$ -order  $\mathcal{M}$  is a products of matrix rings over  $\mathbb{Z}$ .

**Exercise 2.114.** Determine all finite groups G of order  $\leq 9$  for which  $\widetilde{K}_0(\mathbb{Z}G)$  is non-trivial.

**Theorem 2.115** ( $K_0(RG)$  for finite G and an Artinian ring R). Let R be an Artinian ring. Let G be a finite group. Then RG is also an Artinian ring. There are only finitely many isomorphism classes  $[P_1], [P_2], \ldots, [P_n]$  of irreducible finitely generated projective RG-modules, and we obtain an isomorphism of abelian groups

$$\mathbb{Z}^n \xrightarrow{\cong} K_0(RG), \quad (k_1, k_2, \dots k_n) \mapsto \sum_{i=1}^n k_i \cdot [P_i].$$

*Proof.* This follows from [271, Proposition 16.7 on page 406 and the paragraph after Corollary 6.22 on page 132].

Let *F* be a field of characteristic zero or of characteristic *p* for a prime number *p* not dividing |G|. Then  $K_0(FG)$  is the same as the representation ring  $\operatorname{Rep}_F(G)$  of *G* with coefficients in the field *F* since the ring *FG* is *semisimple* i.e., every submodule of a module is a direct summand. If *F* is a field of characteristic zero, then representations are detected by their characters, see Lemma 2.89. For more information about modules over *FG* for a finite group *G* and a field *F* we refer for instance to Curtis-Reiner [271, Chapter 1 and Chapter 2] and Serre [908].

**Exercise 2.116.** Compute  $K_0(FD_8)$  for  $F = \mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$ .

# **2.13** Survey on Computations of $K_0(C_r^*(G))$ and $K_0(\mathcal{N}(G))$

Let *G* be a group. Let  $\mathcal{B}(L^2(G))$  denote the algebra of bounded linear operators on the Hilbert space  $L^2(G)$  whose orthonormal basis is *G*. The *reduced group*  $C^*$ -algebra  $C_r^*(G)$  is the closure in the norm topology of the image of the regular representation  $\mathbb{C}G \to \mathcal{B}(L^2(G))$  that sends an element  $u \in \mathbb{C}G$  to the (left) *G*-equivariant bounded operator  $L^2(G) \to L^2(G)$  given by right multiplication with  $u^{-1}$ . The group von Neumann algebra  $\mathcal{N}(G)$  is the closure in the weak topology. There is an identification  $\mathcal{N}(G) = \mathcal{B}(L^2(G))^G$ . One has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{N}(G) \subseteq \mathcal{B}(L^2(G)).$$

We have  $\mathbb{C}G = C_r^*(G) = \mathcal{N}(G)$  if and only if G is finite. If  $G = \mathbb{Z}$ , then the Fourier transform gives identifications  $C_r^*(\mathbb{Z}) = C(S^1)$  and  $\mathcal{N}(\mathbb{Z}) = L^{\infty}(S^1)$ .

**Remark 2.117** ( $K_0(C_r^*(G))$ ) versus  $K_0(\mathbb{C}G)$ ). We will later see that the study of  $K_0(C_r^*(G))$  is not done according to its algebraic nature. Instead we will introduce and analyze the topological *K*-theory of  $C_r^*(G)$  and explain that in dimension 0 the algebraic and the topological *K*-theory of  $C_r^*(G)$  agree. In order to explain the different flavor of  $K_0(C_r^*(G))$  in comparison with  $K_0(\mathbb{C}G)$ , we mention the conclusion of the Baum-Connes Conjecture for torsionfree groups 10.44 that for torsionfree *G* there exists an isomorphism

$$\bigoplus_{n\geq 0} H_{2n}(BG;\mathbb{Q}) \xrightarrow{\cong} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The space *BG* is the *classifying space of the group G*, which is up to homotopy characterized by the property that it is a *CW*-complex with  $\pi_1(BG) \cong G$  whose universal covering is contractible. We denote by  $H_*(X, R)$  the singular or cellular homology of a space or *CW*-complex *X* with coefficient in a commutative ring *R*. We can identify  $H_*(BG; R)$  with the group homology of *G* with coefficients in *R*.

We see that  $K_0(C_r^*(G))$  can be huge also for torsionfree groups, whereas  $K_0(\mathbb{C}G) \cong \mathbb{Z}$  for torsionfree *G* is a conclusion of the Farrell-Jones Conjecture 2.60 for  $K_0(R)$  for torsionfree *G* and regular *R*. We see already here a homological be-

havior of  $K_0(C_r^*(G))$ , which is not yet evident in the case of group rings so far and will become clear later.

**Remark 2.118** ( $K_0(\mathcal{N}(G))$ ). The projective class group  $K_0(\mathcal{A})$  can be computed for any von Neumann algebra  $\mathcal{A}$  using the center-valued universal trace, see for instance [650, Section 9.2]. In particular one gets for a finitely generated group Gthat does not contain  $\mathbb{Z}^n$  as subgroup of finite index an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}.$$

Here  $\mathcal{Z}(\mathcal{N}(G))$  is the center of the group von Neumann algebra and the  $\mathbb{Z}/2$ -action comes from taking the adjoint of an operator in  $\mathcal{B}(L^2(G))$ , see [650, Example 9.34 on page 353]. If *G* is a finitely generated group that does not contain  $\mathbb{Z}^n$  as subgroup of finite index and for which the conjugacy class (*g*) of an element *g* different from the unit is always infinite, then  $\mathcal{Z}(\mathcal{N}(G)) = \mathbb{C}$  and one obtains an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathbb{R}.$$

A pleasant feature of  $\mathcal{N}(G)$  is that there is no difference between stably isomorphic and isomorphic in the sense that for three finitely generated projective  $\mathcal{N}(G)$ -modules  $P_0, P_1$ , and Q we have  $P_0 \oplus Q \cong_{\mathcal{N}(G)} P_1 \oplus Q$  if and only if  $P_0 \cong_{\mathcal{N}(G)} P_1$ .

We see that in the case of the group von Neumann algebra we can compute  $K_0(\mathcal{N}(G))$  completely, but the answer does not show any homological behavior in *G*. In fact, the Farrell-Jones Conjecture and the Baum-Connes Conjecture have no analog for group von Neumann algebras.

**Exercise 2.119.** Let *G* be a torsionfree hyperbolic group that is not cyclic. Prove  $K_0(\mathcal{N}(G)) \cong \mathbb{R}$ .

**Remark 2.120 (Change of rings homomorphisms for**  $\widetilde{K}_0$  **for**  $\mathbb{Z}G \to \mathbb{C}G \to C^*_r(G) \to \mathcal{N}(G)$ ). We summarize what is conjectured or known about the string of change of rings homomorphism

$$\widetilde{K}_0(\mathbb{Z}G) \xrightarrow{i_1} \widetilde{K}_0(\mathbb{C}G) \xrightarrow{i_2} \widetilde{K}_0(C_r^*(G)) \xrightarrow{i_3} \widetilde{K}_0(\mathcal{N}(G))$$

coming from the various inclusion of rings. The first map  $i_1$  is conjectured to be rationally trivial, see [673, Conjecture 85 on page 754], but is not integrally trivial, see [625, Theorem 5.1]. The second map  $i_2$  is conjectured to be rationally injective, compare [649, Theorem 0.5], but is not surjective in general. The map  $i_3$  is in general not injective, not surjective, and not trivial. It is known that the composite  $i_3 \circ i_2 \circ i_1$  is trivial, see for instance [650, Theorem 9.62 on page 362].

2.14 Notes

# **2.14 Notes**

Algebraic K-theory is compatible with direct limits, as explained for the projective class group next. A *directed set I* is a non-empty set with a partial ordering  $\leq$  such that for two elements  $i_0$  and  $i_1$  there exists an element i with  $i_0 \le i$  and  $i_1 \le i$ . A *directed system of rings* is a set of rings  $\{R_i \mid i \in I\}$  indexed by a directed set I together with a choice of a ring homomorphism  $\phi_{i,j} \colon R_i \to R_j$  for  $i, j \in I$  with  $i \leq j$ such that  $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$  holds for  $i, j, k \in I$  with  $i \leq j \leq k$  and  $\phi_{i,i} = id$  holds for  $i \in I$ . The *colimit*, sometimes also called the *direct limit*, of  $\{R_i \mid i \in I\}$  is a ring denoted by  $\operatorname{colim}_{i \in I} R_i$  together with ring homomorphisms  $\psi_i \colon R_i \to \operatorname{colim}_{i \in I} R_i$ for every  $j \in I$  such that  $\psi_i \circ \phi_{i,j} = \psi_i$  holds for  $i, j \in I$  with  $i \leq j$  and the following universal property is satisfied: For every ring S and every system of ring homomorphisms  $\{\mu_i : R_i \to S \mid i \in I\}$  such that  $\mu_i \circ \phi_{i,j} = \mu_i$  holds for  $i, j \in I$  with  $i \leq j$ , there is precisely one ring homomorphism  $\mu$ :  $\operatorname{colim}_{i \in I} R_i \to S$  satisfying  $\mu \circ \psi_i = \mu_i$  for every  $i \in I$ . If we replace ring by group or module everywhere, we get the notion of directed system and direct limit of groups or modules respectively. This is a special case of the direct limit of a functor, namely, consider I as category with the set I as objects and precisely one morphism from i to j if  $i \leq j$ , and no other morphisms.

**Remark 2.121 (Filtered categories).** One may consider instead of a directed set a *filtered category*, i.e, a nonempty category I such that for every two objects i and j there is an object k together with two morphisms  $i \rightarrow k$  and  $j \rightarrow k$  and for two morphism  $f, g: i \rightarrow j$  with the same source and target there is a morphism  $h: j \rightarrow k$  with  $hj \circ f = h \circ k$ , and all the results about colimits over directed sets stay true if one considers colimits over filtered categories. Then one talks about filtered systems instead of filtered sets.

Let  $\{R_i \mid i \in I\}$  be a direct system of rings. For every  $i \in I$ , we obtain a change of rings homomorphism  $(\psi_i)_* \colon K_0(R_i) \to K_0(R)$ . The universal property of the direct limit yields a homomorphism

(2.122)  $\operatorname{colim}_{i \in I}(\psi_i)_* \colon \operatorname{colim}_{i \in I} K_0(R_i) \xrightarrow{\cong} K_0(R),$ 

which turns out to be an isomorphism, see [860, Theorem 1.2.5].

We denote by  $R^{\times}$  the group of units in *R*. A ring *R* is called *local* if the set  $I := R - R^{\times}$  forms a (left) ideal. If *I* is a left ideal, it is automatically a two-sided ideal and it is maximal both as a left ideal and as a right ideal. A ring *R* is local if and only if it has a unique maximal left ideal and a unique maximal right ideal and these two coincide. An example of a local ring is the ring of formal power series F[[t]] with coefficients in a field *F*. If *R* is a commutative ring and *I* is a prime ideal, then the localization  $R_I$  of *R* at *I* is a local ring.

**Theorem 2.123** ( $K_0(R)$  of local rings). Let R be a local ring. Then every finitely generated projective R-module is free and  $K_0(R)$  is infinite cyclic with [R] as generator.

*Proof.* See for instance [727, Lemma 1.2 on page 5] or [860, Theorem 1.3.11 on page 14].  $\Box$ 

The proof is based on *Nakayama's Lemma*, which says for a ring *R* and a finitely generated *R*-module *M* that  $rad(R)M = M \iff M = 0$  holds. Here rad(R) is the *radical*, or *Jacobson radical*, i.e., the two-sided ideal that is given by the intersection of all maximal left ideals, or, equivalently, of all maximal right ideals of *R*. The radical is the same as the set of elements  $r \in R$  for which there exists an  $s \in S$  such that 1 - rs has a left inverse in *R*.

If *R* is a commutative ring and *spec*(*R*) is its *spectrum* consisting of its prime ideals and equipped with the Zariski topology, then we obtain for every finitely generated projective *R*-module *P* a continuous rank function  $\text{Spec}(R) \to \mathbb{Z}$  by sending a prime ideal *I* to the rank of the finitely generated free  $R_I$ -module  $P_I = P \otimes_R R_I$ . This makes sense because of Theorem 2.123 since  $R_I$  is local. If *R* is a commutative integral domain, this rank function is constant. For more details we refer for instance to [860, Proposition 1.3.12 on page 15].

**Exercise 2.124.** Prove for an integer  $n \ge 1$  that  $K_0(\mathbb{Z}/n)$  is the free abelian group whose rank is the number of prime numbers dividing *n*.

A ring is called *semilocal* if R/rad(R) is Artinian, or, equivalently, R/rad(R) is semisimple. If R is commutative, then R is semilocal if and only if it has only finitely many maximal ideas, see [916, page 69]. For a semilocal ring R, the projective class group  $K_0(R)$  is a finitely generated free abelian group, see [916, Proposition 14 on page 28]. More information about semilocal rings can be found for instance in [610, § 20].

**Lemma 2.125.** For any ring R and nilpotent two-sided ideal  $I \subseteq R$ , the map  $K_0(R) \rightarrow K_0(R/I)$  induced by the projection  $R \rightarrow R/I$  is bijective.

*Proof.* See [998, Lemma 2.2 in Section II.2 on page 70].

Given two groups  $G_1$  and  $G_2$ , let  $G_1 * G_2$  by the amalgamated free product. Then the natural maps  $G_k \to G_0 * G_1$  for k = 1, 2 induce an isomorphism, see [421, Theorem 1.1],

(2.126) 
$$\widetilde{K}_0(\mathbb{Z}[G_1]) \oplus \widetilde{K}_0(\mathbb{Z}[G_1]) \cong \widetilde{K}_0(\mathbb{Z}[G_1 * G_2]).$$

This is a first glimpse of a homological behavior of  $K_0$  if one compares this with the corresponding isomorphism of group homology

$$\widetilde{H}_n(G_1) \oplus \widetilde{H}_n(G_1) \cong \widetilde{H}_n(G_1 * G_2).$$

**Exercise 2.127.** Show that the projections  $pr_k: G_1 \times G_2 \rightarrow G_k$  for k = 1, 2 do *not* in general induce isomorphisms

$$\widetilde{K}_0(\mathbb{Z}[G_1 \times G_2]) \to \widetilde{K}_0(\mathbb{Z}[G_1]) \times \widetilde{K}_0(\mathbb{Z}[G_2]).$$

### 2.14 Notes

There are also equivariant versions of the finiteness obstructions, see for instance [32], [642], and [644, Chapter 3 and 11]. Finiteness obstructions for categories are investigated in [391, 390].

Andrej Jaikin-Zapirain pointed out that he and Pablo Sánchez-Peralta have proved the following result confirming Conjecture 2.60 in a special case.

A presentation  $G = \langle X | R \rangle$  is called a Cohen–Lyndon presentation if for each  $r \in R$ , there exists a transversal  $T_r$  of the normal subgroup  $N = \langle \langle R \rangle \rangle$ , such that N is freely generated by the set  $\{r^g | r \in R, g \in T_r\}$ .

They prove that if G has a Cohen-Lyndon presentation and S is a regular ring, then the natural map

$$K_0(S) \rightarrow K_0(S[G])$$

is an isomorphism.

# Chapter 3 The Whitehead Group

# **3.1 Introduction**

This chapter is devoted to the *first K-group*  $K_1(R)$  of a ring *R* and the *Whitehead group* Wh(*G*) of a group *G*.

We give two equivalent definitions of  $K_1(R)$ , namely, as the universal determinant and in terms of invertible matrices. We explain some basic calculations of  $K_1(R)$  for rings with Euclidian algorithm, local rings, and rings of integers in algebraic number fields.

We introduce the *Whitehead group* of a group and the *Whitehead torsion* of a homotopy equivalence of finite *CW*-complexes algebraically and geometrically. The relevance of these notions are illustrated by the *s*-*Cobordism Theorem* and its applications to the classification of manifolds and by the classification of *lens spaces* by *Reidemeister torsion*.

The next topic is the *Bass-Heller-Swan decomposition* and the *long exact sequence* associated to a pullback of rings and to a two-sided ideal. These are important tools for computations and relate  $K_0(R)$  and  $K_1(R)$ .

We discuss *Swan homomorphisms* and *free homotopy representations*. Thus we provide a link between torsion invariants and finiteness obstructions.

We explain the variant of the *Farrell-Jones Conjecture* that for a torsionfree group G the reduced projective class group  $\widetilde{K}_0(\mathbb{Z}G)$  and the Whitehead group Wh(G) vanish. It implies that any *h*-cobordism with torsionfree fundamental group and dimension  $\geq 6$  is trivial.

Finally, we give a survey of computations of  $K_1(\mathbb{Z}G)$  for finite groups G and of the algebraic  $K_1$ -group of commutative Banach algebras, commutative  $C^*$ -algebras, and of some group von Neumann algebras.

# **3.2** Definition and Basic Properties of $K_1(R)$

**Definition 3.1** ( $K_1$ -group  $K_1(R)$ ). Let R be a ring. Define the  $K_1$ -group of a ring  $K_1(R)$  to be the abelian group whose generators are conjugacy classes [f] of automorphisms  $f: P \to P$  of finitely generated projective R-modules with the following relations:

### • Additivity

Given a commutative diagram of finitely generated projective R-modules

3 The Whitehead Group

$$0 \longrightarrow P_1 \xrightarrow{i} P_2 \xrightarrow{p} P_3 \longrightarrow 0$$
$$\downarrow f_1 \qquad \downarrow f_2 \qquad \downarrow f_3$$
$$0 \longrightarrow P_1 \xrightarrow{i} P_2 \xrightarrow{p} P_3 \longrightarrow 0$$

with exact rows and automorphisms as vertical arrows, we get

$$[f_1] + [f_3] = [f_2];$$

#### • Composition formula

Given automorphisms  $f, g: P \rightarrow P$  of a finitely generated projective *R*-module *P*, we get

$$[g \circ f] = [f] + [g].$$

Define  $G_1(R)$  analogously but replacing finitely generated projective by finitely generated everywhere.

Given a ring homomorphism  $f: R \to S$ , we obtain a change of rings homomorphism

$$(3.2) \quad f_* = K_1(f) \colon K_1(R) \to K_1(S), \ [g \colon P \to P] \mapsto [f_*g \colon f_*P \to f_*P]$$

analogously as we have defined it for the projective class group in (2.2). Thus  $K_1$  becomes a covariant functor from the category of rings to the category of abelian groups.

**Exercise 3.3.** Show that  $K_1(R) = 0$  holds for the ring *R* appearing in Example 2.17.

**Remark 3.4 (The universal property of**  $K_1(R)$ ). One should view  $K_1(R)$  together with the assignment sending an automorphism  $f: P \to P$  of a finitely generated projective *R*-module *P* to its class  $[f] \in K_1(R)$  as the *universal determinant*. Namely, for any abelian group *A* and assignment *a* which sends the automorphism *f* of a finitely generated projective *R*-module to  $a(f) \in A$  such that (A, a) satisfies additivity and the composition formula appearing in Definition 3.1, there exists precisely one homomorphism of abelian groups  $\phi: K_1(R) \to A$  such that  $\phi([f]) =$ a(f) holds for every automorphism *f* of a finitely generated projective *R*-module.

We always have the following map of abelian groups

$$(3.5) i: \mathbb{R}^{\times}/[\mathbb{R}^{\times}, \mathbb{R}^{\times}] \to K_1(\mathbb{R}), \quad [x] \mapsto [r_x: \mathbb{R} \to \mathbb{R}]$$

where  $r_x$  denotes right multiplication by x. It is neither injective nor surjective in general. However, we have

**Theorem 3.6**  $(K_1(F)$  of skew-fields). The map *i* defined in (3.5) is an isomorphism if *R* is a skew-field or, more generally, a local ring. It is surjective (with an explicitly described kernel) if *R* is a semilocal ring.

3.2 Definition and Basic Properties of  $K_1(R)$ 

*Proof.* See for instance [916, Corollary 43 on page 133], [860, Corollary 2.2.6 on page 69], and [916, Proposition 53 on page 140].  $\Box$ 

**Exercise 3.7.** Let  $\mathbb{H}$  be the skew-field of quaternions  $\{a+bi+cj+dk \mid a, b, c, d \in \mathbb{R}\}$ . Since  $\mathbb{H}$  is a 4-dimensional vector space, there is an embedding  $GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$ . Its composite with the determinant over  $\mathbb{R}$  yields a homomorphism  $\mu_n \colon GL_n(\mathbb{H}) \to \mathbb{R}^{>0}$  to the multiplicative group of positive real numbers. Show that the system of homomorphisms  $\mu_n$  induces an isomorphism

$$\mu \colon K_1(\mathbb{H}) \xrightarrow{\cong} \mathbb{R}^{>0}.$$

The proofs of the next two results are analogous to those of Theorem 2.10 and Lemma 2.12.

**Theorem 3.8 (Morita equivalence for**  $K_1(R)$ ). For every ring R and integer  $n \ge 1$ , there is natural isomorphism

$$\mu \colon K_1(R) \xrightarrow{\equiv} K_1(\mathcal{M}_n(R)).$$

**Lemma 3.9.** Let  $R_0$  and  $R_1$  be rings. Denote by  $pr_i: R_0 \times R_1 \rightarrow R_i$  for i = 0, 1 the projection. Then we obtain an isomorphism

$$(\mathrm{pr}_0)_* \times (\mathrm{pr}_1)_* \colon K_1(R_0 \times R_1) \xrightarrow{=} K_1(R_0) \times K_1(R_1).$$

**Lemma 3.10.** Define the abelian group  $K_1^f(R)$  analogous to  $K_1(R)$  but with finitely generated projective replaced by finitely generated free everywhere. Then the canonical homomorphism

$$\alpha \colon K_1^f(R) \xrightarrow{=} K_1(R), \quad [f] \mapsto [f]$$

is an isomorphism.

*Proof.* Given an automorphism  $f: P \to P$  of a finitely generated projective *R*-module *P*, we can choose a finitely generated projective *R*-module *Q*, a finitely generated free *R*-module *F* and an *R*-isomorphism  $\phi: P \oplus Q \xrightarrow{\cong} F$ . We obtain an automorphism  $\phi \circ (f \oplus id_Q) \circ \phi^{-1}: F \to F$  and thus an element  $[\phi \circ (f \oplus id_Q) \circ \phi^{-1}] \in K_1^f(R)$ . One easily checks that it is independent of the choice of *Q* and  $\phi$  and then that it depends only on  $[f] \in K_1(R)$ . Thus we obtain a homomorphism of abelian groups  $\beta: K_1(R) \to K_1^f(R)$ . One easily checks that  $\alpha$  and  $\beta$  are inverse to one another.

Next we give a matrix description of  $K_1(R)$ . Denote by  $E_n(i, j)$  for  $n \ge 1$  and  $1 \le i, j \le n$  the (n, n)-matrix whose entry at (i, j) is one and is zero elsewhere. Denote by  $I_n$  the identity matrix of size n. An *elementary* (n, n)-matrix is a matrix of the form  $I_n + r \cdot E_n(i, j)$  for  $n \ge 1, 1 \le i, j \le n, i \ne j$  and  $r \in R$ . Let A be an (n, n)-matrix. The matrix  $B = A \cdot (I_n + r \cdot E_n(i, j))$  is obtained from A by adding the *i*-th column multiplied by *r* from the right to the *j*-th column. The matrix  $C = (I_n + r \cdot E_n(i, j)) \cdot A$  is obtained from *A* by adding the *j*-th row multiplied by *r* from the left to the *i*-th row. Let  $E(R) \subset GL(R)$  be the subgroup generated by all elements in GL(R) that are represented by elementary matrices.

**Lemma 3.11.** The subgroup E(R) of GL(R) coincides with the commutator subgroup [GL(R), GL(R)].

*Proof.* For  $n \ge 3$ , pairwise distinct numbers  $1 \le i, j, k \le n$ , and  $r \in R$ , we can write  $I_n + r \cdot E_n(i, k)$  as a commutator in  $GL_n(R)$ , namely,

$$I_n + r \cdot E_n(i,k) = (I_n + r \cdot E_n(i,j)) \cdot (I_n + E_n(j,k)) \cdot (I_n + r \cdot E_n(i,j))^{-1} \cdot (I_n + E_n(j,k))^{-1}.$$

This implies  $E(R) \subset [GL(R), GL(R)]$ .

Let *A* and *B* be two elements in  $GL_n(R)$ . Let [*A*] and [*B*] be the elements in GL(R) represented by *A* and *B*. Given two elements *x* and *y* in GL(R), we write  $x \sim y$  if there are elements  $e_1$  and  $e_2$  in E(R) with  $x = e_1ye_2$ , in other words, if the classes of *x* and *y* in  $E(R) \setminus GL(R) / E(R)$  agree. One easily checks

$$[AB] \sim \left[ \begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix} \right] \sim \left[ \begin{pmatrix} AB & A \\ 0 & I_n \end{pmatrix} \right] \sim \left[ \begin{pmatrix} 0 & A \\ -B & I_n \end{pmatrix} \right] \sim \left[ \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \right]$$

since each step is given by multiplication from the right or left by a block matrix of the form  $\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}$  or  $\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix}$  and such a block matrix is obviously obtained from  $I_{2n}$  by a sequence of column and row operations and hence its class in GL(*R*) belongs to E(*R*). Analogously we get

$$[BA] \sim \left[ \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix} \right].$$

Since the element in GL(*R*) represented by  $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$  belongs to E(*R*), we conclude

$$\left[ \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \right] \sim \left[ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right] \sim \left[ \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix} \right]$$

and hence

$$[AB] \sim [BA].$$

This implies for any element  $x \in GL(R)$  and  $e \in E(R)$  that  $xex^{-1} \sim ex^{-1}x = e$  and hence  $xex^{-1} \in E(R)$ . Therefore E(R) is normal. Given a commutator  $xyx^{-1}y^{-1}$  for  $x, y \in GL(R)$ , we conclude for appropriate elements  $e_1, e_2, e_3$  in E(R)

$$xyx^{-1}y^{-1} = e_1yxe_2x^{-1}y^{-1} = e_1(yx)e_2(yx)^{-1} = e_1e_3 \in E(R).$$

3.2 Definition and Basic Properties of  $K_1(R)$ 

**Theorem 3.12**  $(K_1(R)$  equals GL(R)/[GL(R), GL(R)]). There is a natural isomorphism

$$\operatorname{GL}(R)/[\operatorname{GL}(R),\operatorname{GL}(R)] \xrightarrow{\cong} K_1(R).$$

Proof. Because of Lemma 3.10 it suffices to construct mutually inverse homomorphisms of abelian groups  $\alpha$ :  $\operatorname{GL}(R)/[\operatorname{GL}(R), \operatorname{GL}(R)] \to K_1^f(R)$  and  $\beta$ :  $K_1^f(R) \to \operatorname{GL}(R)/[\operatorname{GL}(R), \operatorname{GL}(R)]$ . The map  $\alpha$  sends the class [A] of  $A \in \operatorname{GL}_n(R)$  to the class  $[r_A]$  of  $r_A \colon \mathbb{R}^n \to \mathbb{R}^n$ ,  $x \mapsto xA$ . This is a well-defined homomorphism of abelian groups since  $[r_{AB}] = [r_A] + [r_B], [r_{A \oplus I_1}] = [r_A]$  holds for all  $n \in \mathbb{Z}, n \ge 1$ and  $A, B \in GL_n(R)$ , and  $K_1(R)$  is abelian. The map  $\beta$  sends the class [f] of an automorphism f of a finitely generated free R-module F to the class [A(f, B)] of the invertible (n, n)-matrix A(f, B) associated to f after a choice of some ordered R-basis B for F. This class is independent of the choice of B, since for another choice of an ordered bases B' there exists a  $U \in GL_n(R)$  with  $UA(f, B)U^{-1} = A(f, B')$ , which implies

$$[A(f, B')] = [UA(f, B)U^{-1}] = [U][A(f, B)][U]^{-1}$$
$$= [U][U]^{-1}[A(f, B)] = [A(f, B)].$$

Thus we have defined  $\beta$  on generators. It remains to check the relations. Obviously the composition formula is satisfied. Additivity is satisfied because of the following calculation in GL(R)/[GL(R), GL(R)] for  $A \in GL_m(R)$ ,  $B \in GL_n(R)$  and  $C \in$  $M_{m,n}(R)$  based on Lemma 3.11

$$\begin{bmatrix} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} I_m & 0 \\ C^{-1}B & I_n \end{pmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \end{bmatrix} \cdot \begin{bmatrix} \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix} \end{bmatrix} \cdot \begin{bmatrix} \begin{pmatrix} I_m & 0 \\ C^{-1}B & I_n \end{pmatrix} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} C \end{bmatrix} \cdot \begin{bmatrix} I_{m+n} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} C \end{bmatrix}.$$

One easily checks that  $\alpha$  and  $\beta$  are inverse to one another.

**Remark 3.13 (What**  $K_1(R)$  **measures).** We conclude from Lemma 3.11 and Theorem 3.12 that two matrices  $A \in GL_m(R)$  and  $B \in GL_n(R)$  represent the same class in  $K_1(R)$  if and only if B can be obtained from A by a sequence of the following operations:

(i) Elementary row operation

B is obtained from A by adding the k-th row multiplied by r from the left to the *l*-th row for  $r \in R$  and  $k \neq l$ ;

(ii) Elementary column operation

B is obtained from A by adding the k-th column multiplied by r from the right to the *l*-th row for  $r \in R$  and  $k \neq l$ ;

(iii) Stabilization

B is obtained by taking the direct sum of A and  $I_1$ , i.e., B looks like the block matrix  $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ ;

(iv) *Destabilization* 

A is the direct sum of B and  $I_1$ . (This is the inverse operation to (iii).)

Since multiplication from the left or right by an elementary matrix corresponds to the operation (i) or the operation (ii), the abelian group  $K_1(R)$  is trivial if and only if any invertible matrix  $A \in GL_n(R)$  can be reduced by a sequence of the operations above to the empty matrix.

One could delete the operation (i) or the operation (ii) from the list above without changing the conclusion. This follows from the fact that E(R) is a normal subgroup of GL(R).

The elementary proof of the next lemma is left to the reader.

**Lemma 3.14.** *Let R be a commutative ring. Then the determinant defines a homomorphism of abelian groups* 

det: 
$$K_1(R) \to R^{\times}$$
,  $[f] \mapsto \det(f)$ .

It satisfies det  $\circ i = id_{R^{\times}}$  for the map *i* defined in (3.5). In particular the map det is surjective.

**Definition 3.15** ( $SK_1(R)$  of a commutative ring *R*). Let *R* be a commutative ring. Define

$$SK_1(R) := \ker (\det : K_1(R) \to R^{\times})$$

We will see in Section 3.12 that there are commutative group rings  $\mathbb{Z}G$  for which the surjective map det:  $K_1(\mathbb{Z}G) \to \mathbb{Z}G^{\times}$  is not injective, or, equivalently, with non-trivial  $SK_1(\mathbb{Z}G)$ . Here is another example.

**Example 3.16.** The following example is taken from [106, Example 4.4], see also [860, Exercise 2.3.11 on page 82]. Let  $\Lambda$  be obtained from the polynomial ring  $\mathbb{R}[x, y]$  by dividing out the ideal generated by  $x^2 + y^2 - 1$ . This is a Dedekind domain. The matrix

$$M := \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{SL}_2(\Lambda)$$

represents a non-trivial element in  $SK_1(\Lambda)$ . The proof uses Mennicke symbols and is based on the observation that the function

$$S^1 \to \operatorname{SL}_n(\mathbb{R}), \quad (x, y) \mapsto \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

represents a non-trivial element in  $\pi_1(SL_n(\mathbb{R})) \cong \pi_1(SO(n)) \cong \mathbb{Z}/2$  for  $n \ge 3$ .

**Theorem 3.17** ( $K_1(R) = R^{\times}$  for commutative rings with Euclidean algorithm). Let *R* be a commutative ring with Euclidean algorithm in the sense of [860, 2.3.1 on page 74], for instance a field or  $\mathbb{Z}$ .

Then the determinant induces an isomorphism

det: 
$$K_1(R) \xrightarrow{\cong} R^{\times}$$
.

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*Proof.* Because of Lemma 3.14 it suffices to show for  $A \in GL_n(R)$  with det(A) = 1 that it can be reduced to the empty matrix by a sequence of operations appearing in Remark 3.13. But this is a well-known result of elementary algebra, see for instance [860, Theorem 2.3.2 on page 74].

**Exercise 3.18.** Prove  $K_1(\mathbb{Z}[i]) \cong \{1, -1, i, -i\} \cong \mathbb{Z}/4$ .

**Remark 3.19** ( $K_1(R)$  of principal ideal domains). There exist principal ideal domains R such that det:  $K_1(R) \rightarrow R^{\times}$  is not bijective. For instance Grayson [435] gives such an example, namely, take  $\mathbb{Z}[x]$  and invert x and all polynomials of the shape  $x^m - 1$  for  $m \ge 1$ . Other examples can be found in Ischebeck [517].

**Theorem 3.20 (Vanishing of**  $SK_1$  **of ring of integers in an algebraic number field).** Let *R* be the ring of integers in an algebraic number field. Then the determinant induces an isomorphism

det: 
$$K_1(R) \xrightarrow{\cong} R^{\times}$$
.

*Proof.* See [106, page 77] or [727, Corollary 16.3 on page 159].

The proof of the next classical result can be found for instance in [859, Theorem 2.3.8 on page 79].

**Theorem 3.21 (Dirichlet Unit Theorem).** Let R be the ring of integers in an algebraic number field F. Let  $r_1$  be the number of distinct embeddings of F into  $\mathbb{R}$  and let  $r_2$  be the number of distinct conjugate pairs of embeddings of F into  $\mathbb{C}$  with image not contained in  $\mathbb{R}$ . Then:

- (i)  $r_1 + 2r_2$  is the degree  $[F : \mathbb{Q}]$  of the extension  $\mathbb{Q} \subseteq F$ ;
- (ii) The abelian group  $R^{\times}$  is finitely generated:

(iii) The torsion subgroup of  $R^{\times}$  is the finite cyclic group of roots of unity in F; (iv) The rank of  $R^{\times}$  is  $r_1 + r_2 - 1$ .

**Exercise 3.22.** Let *R* be the ring of integers in an algebraic number field *F*. Then  $K_1(R)$  is finite if and only if *F* is  $\mathbb{Q}$  or an imaginary quadratic field.

### **3.3 Whitehead Group and Whitehead Torsion**

In this section we will assign to a homotopy equivalence  $f: X \to Y$  of finite *CW*-complexes its Whitehead torsion  $\tau(f)$  in the Whitehead group Wh $(\pi(Y))$  associated to *Y*. A basic feature is that the Whitehead torsion can distinguish manifolds or spaces that are homotopy equivalent. The notion of Whitehead torsion goes back to the papers by J.H.C. Whitehead [1007, 1008, 1009].

The *reduced*  $K_1$ -group  $\widetilde{K}_1(R)$  is defined to be the cokernel of the map  $K_1(\mathbb{Z}) \to K_1(R)$  induced by the unique ring homomorphism  $\mathbb{Z} \to R$ . The homomorphism det:  $K_1(\mathbb{Z}) \to \{\pm 1\}$  is a bijection, because  $\mathbb{Z}$  is a ring with Euclidean algorithm, see Theorem 3.17. Hence  $\widetilde{K}_1(R)$  is the same as the quotient of  $K_1(R)$  by the cyclic subgroup of at most order two generated by the class of the (1, 1)-matrix (-1).

**Definition 3.23 (Whitehead group).** Define the *Whitehead group* Wh(*G*) of a group *G* to be the cokernel of the map  $G \times \{\pm 1\} \rightarrow K_1(\mathbb{Z}G)$  that sends  $(g, \pm 1)$  to the class of the invertible (1, 1)-matrix  $(\pm g)$ .

Obviously a group homomorphism  $u: G \to H$  induces a homomorphism of abelian groups

(3.24) 
$$u_* = \operatorname{Wh}(u) \colon \operatorname{Wh}(G) \to \operatorname{Wh}(H).$$

**Exercise 3.25.** Using the ring homomorphism  $f : \mathbb{Z}[\mathbb{Z}/5] \to \mathbb{C}$  that sends the generator of  $\mathbb{Z}/5$  to  $\exp(2\pi i/5)$  and the norm of a complex number, define a homomorphism of abelian groups

$$\phi: \operatorname{Wh}(\mathbb{Z}/5) \to \mathbb{R}^{>0}.$$

Show that  $1 - t - t^{-1}$  is a unit in  $\mathbb{Z}[\mathbb{Z}/5]$  whose class in Wh( $\mathbb{Z}/5$ ) is an element of infinite order. (Actually Wh( $\mathbb{Z}/5$ ) is an infinite cyclic group with this class as generator.)

For a ring R and a group G we denote by

$$(3.26) A_0 = K_0(i) \colon K_0(R) \to K_0(RG)$$

the map induced by the inclusion  $i: R \to RG$ . Sending  $(g, [P]) \in G \times K_0(R)$  to the class of the *RG*-automorphism  $R[G] \otimes_R P \to R[G] \otimes_R P$ ,  $u \otimes x \mapsto ug^{-1} \otimes x$ defines a map  $\Phi: G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \to K_1(RG)$ . Define a homomorphism

 $(3.27) \quad A_1 := \Phi \oplus K_1(i) \colon (G/[G,G] \otimes_{\mathbb{Z}} K_0(R)) \oplus K_1(R) \to K_1(RG).$ 

**Definition 3.28 (Generalized Whitehead group).** For a regular ring *R* and a group *G* we define the *generalized Whitehead group*  $Wh_1^R(G)$  as the cokernel of the map  $A_1$  introduced in (3.27). Denote by  $Wh_0^R(G)$  the cokernel of the map  $A_0$  defined in (3.26).

Note that the abelian group  $Wh_1^{\mathbb{Z}}(G)$  of Definition 3.28 agrees with the abelian group Wh(G) of Definition 3.23.

Next we will define torsion invariants on the level of chain complexes.

We begin with some input about chain complexes. Let  $f_*: C_* \to D_*$  be a chain map of *R*-chain complexes for some ring *R*. Define  $cyl_*(f_*)$  to be the chain complex with *n*-th differential

$$C_{n-1} \oplus C_n \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0 & 0\\ -id & c_n & 0\\ f_{n-1} & 0 & d_n \end{pmatrix}} C_{n-2} \oplus C_{n-1} \oplus D_{n-1}.$$

Define  $cone_*(f_*)$  to be the quotient of  $cyl_*(f_*)$  by the obvious copy of  $C_*$ . Hence the *n*-th differential of  $cone_*(f_*)$  is

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$$C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0\\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}.$$

Given a chain complex  $C_*$ , define  $\Sigma C_*$  to be the quotient of cone<sub>\*</sub>(id<sub>C\*</sub>) by the obvious copy of  $C_*$ , i.e., the chain complex with *n*-th differential

$$C_{n-1} \xrightarrow{-c_{n-1}} C_{n-2}.$$

**Definition 3.29 (Mapping cylinder and mapping cone).** Given a chain map  $f_*: C_* \to D_*$ , we call  $cyl_*(f_*)$  the *mapping cylinder* and  $cone_*(f_*)$  the *mapping cone*. For a chain complex  $C_*$ , we call  $\Sigma C_*$  the *suspension*.

These algebraic notions of mapping cylinder, mapping cone, and suspension are modelled on their geometric counterparts. Namely, the cellular chain complex of a mapping cylinder of a cellular map f of CW-complexes is the mapping cylinder of the chain map induced by f. As suggested already from the geometric picture, there exists obvious exact sequences such as  $0 \rightarrow C_* \rightarrow \text{cyl}_*(f_*) \rightarrow \text{cone}_*(f_*) \rightarrow 0$  and  $0 \rightarrow D_* \rightarrow \text{cone}_*(f_*) \rightarrow \Sigma C_* \rightarrow 0$ .

A *chain contraction*  $\gamma_*$  for an *R*-chain complex  $C_*$  is a collection of *R*-homomorphisms  $\gamma_n : C_n \to C_{n+1}$  for  $n \in \mathbb{Z}$  satisfying  $c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n = \operatorname{id}_{C_n}$  for all  $n \in \mathbb{Z}$ . We call a finite free *R*-chain complex *based free* if each *R*-chain module  $C_n$  comes with a preferred basis. Suppose that  $C_*$  is a finite based free *R*-chain complex which is *contractible*, i.e., which possesses a chain contraction. Put  $C_{\text{odd}} = \bigoplus_{n \in \mathbb{Z}} C_{2n+1}$  and  $C_{\text{ev}} = \bigoplus_{n \in \mathbb{Z}} C_{2n}$ . Let  $\gamma_*$  and  $\delta_*$  be two chain contractions. Define *R*-homomorphisms

$$(3.30) (c_* + \gamma_*)_{\text{odd}} : C_{\text{odd}} \to C_{\text{ev}};$$

$$(3.31) (c_* + \delta_*)_{\rm ev} : C_{\rm ev} \to C_{\rm odd}.$$

Choose on each of the bases an ordering. Let *A* be the matrix of  $(c_* + \gamma_*)_{odd}$ with respect to the given ordered bases. Let *B* be the matrix of  $(c_* + \delta_*)_{ev}$  with respect to the given ordered bases. We define  $\mu_n := (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n$  and  $\nu_n := (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n$ . One easily checks that the endomorphisms  $(id + \mu_*)_{odd}$ ,  $(id + \nu_*)_{ev}$ ,  $(c_* + \gamma_*)_{odd} \circ (id + \mu_*)_{odd} \circ (c_* + \delta_*)_{ev}$ , and  $(c_* + \delta_*)_{ev} \circ (id + \nu_*)_{ev} \circ (c_* + \gamma_*)_{odd}$  are given by upper triangular matrices whose diagonal entries are identity maps. Hence *A* and *B* are invertible and their classes  $[A], [B] \in \widetilde{K}_1(R)$  satisfy [A] = -[B]. Since [B] is independent of the choice of  $\gamma_*$ , the same is true for [A]. Moreover [A] is independent of the choice of orderings on the bases, since the class of any permutation automorphism of a finitely generated free *R*-module in  $K_1(R)$  is in the image of the homomorphism  $K_1(\mathbb{Z}) \to K_1(R)$ . Thus we can associate to a finite based free contractible *R*-chain complex  $C_*$  an element

(3.32) 
$$\tau(C_*) := [A] \in \widetilde{K}_1(R).$$

Let  $f_*: C_* \to D_*$  be a homotopy equivalence of finite based free *R*-chain complexes. Its mapping cone cone $(f_*)$  is a contractible finite based free *R*-chain complex. Define the *Whitehead torsion* of  $f_*$  by

(3.33) 
$$\tau(f_*) := \tau(\operatorname{cone}_*(f_*)) \in K_1(R).$$

Now we can pass to *CW*-complexes. Let  $f: X \to Y$  be a cellular homotopy equivalence of connected finite *CW*-complexes. Let  $p_X: \widetilde{X} \to X$  and  $p_Y: \widetilde{Y} \to Y$ be the universal coverings. Identify  $\pi_1(Y)$  with  $\pi_1(X)$  using the isomorphism induced by f. (We ignore base point questions here and in the sequel. This can be done since an inner automorphism of a group G induces the identity on  $K_1(\mathbb{Z}G)$  and hence also on Wh(G).) There is a lift  $\widetilde{f}: \widetilde{X} \to \widetilde{Y}$  which is  $\pi_1(Y)$ -equivariant. It induces a  $\mathbb{Z}\pi_1(Y)$ -chain homotopy equivalence  $C_*(\widetilde{f}): C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ . The *CW*-structure defines a basis for each  $\mathbb{Z}\pi_1(Y)$ -chain module  $C_n(\widetilde{X})$  and  $C_n(\widetilde{Y})$  which is unique up to multiplying each basis element by a unit of the form  $\pm g \in \mathbb{Z}\pi_1(Y)$  and permuting the elements of the basis. Pick such a *cellular basis* for each chain module. We can apply (3.33) to it and thus obtain an element in  $\widetilde{K}_1(\mathbb{Z}\pi_1(Y))$ . Its image under the projection  $\widetilde{K}_1(\mathbb{Z}\pi_1(Y)) \to Wh(\pi_1(Y))$  is denoted by

(3.34) 
$$\tau(f) \in \mathrm{Wh}(\pi_1(Y)).$$

Since we consider  $\tau(f)$  in Wh( $\pi_1(Y)$ ), the choice of the cellular basis does not matter anymore.

Given a (not necessarily cellular) homotopy equivalence of connected finite *CW*-complexes  $f: X \to Y$ , we can define its Whitehead torsion  $\tau(f)$  as follows. We can choose by the Cellular Approximation Theorem a cellular map  $f': X \to Y$  that is homotopic to f, and define the Whitehead torsion  $\tau(f)$  by  $\tau(f')$ . Since the Whitehead torsion of two cellular maps which are homotopic, and hence even cellularly homotopic by the Cellular Approximation Theorem, agrees, it is independent of the choice of f'.

If  $f: X \to Y$  is a homotopy equivalence of finite *CW*-complexes, then define Wh $(\pi_1(Y)) := \bigoplus_{C \in \pi_0(Y)} Wh(\pi_1(C))$  and  $\tau(f) \in Wh(\pi_1(Y))$  by the collection of the Whitehead torsions of the homotopy equivalences induced between path components. Obviously a map  $g: Y_1 \to Y_2$  induces a homomorphism of abelian groups  $g_*: Wh(\pi_1(Y_1)) \to Wh(\pi_1(Y_2))$  by the homomorphisms between the various fundamental groups of the path components induced by g.

**Definition 3.35 (Whitehead torsion).** We call  $\tau(f)$  the (algebraic) *Whitehead torsion* of the homotopy equivalence  $f: X \to Y$  of finite *CW*-complexes.

**Exercise 3.36.** Let  $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$  be an exact sequence of projective *R*-chain complexes. Suppose that  $E_*$  is contractible. Construct an *R*-chain map  $s_* \colon E_* \to D_*$  such that  $p_* \circ s_* = \operatorname{id}_{E_*}$ . Show that  $i_* \oplus s_* \colon C_* \oplus E_* \to D_*$  is an isomorphism of *R*-chain complexes. Give a counterexample to the conclusion if one drops the condition that  $E_*$  is contractible.

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The basic properties of this invariant are summarized in the following theorem, whose proof can be found for instance in [247, (22.1), (22.4), (23.1), and (23.2)], [667, Chapter 3], or [648, Chapter 2].

### Theorem 3.37 (Basic properties of Whitehead torsion).

(i) Sum formula

Let the following two diagrams be pushouts of finite CW-complexes



where the left vertical arrows are inclusions of CW-complexes, the upper horizontal maps are cellular, and X and Y are equipped with the induced CWstructure. Let  $f_i: X_i \to Y_i$  be homotopy equivalences for i = 0, 1, 2 satisfying  $f_1 \circ i_1 = k_1 \circ f_0$  and  $f_2 \circ i_2 = k_2 \circ f_0$ . Put  $l_0 = l_1 \circ k_1 = l_2 \circ k_2$ . Denote by  $f: X \to Y$  the map induced by  $f_0, f_1$ , and  $f_2$  and the pushout property. Then f is a homotopy equivalence and

$$\tau(f) = (l_1)_* \tau(f_1) + (l_2)_* \tau(f_2) - (l_0)_* \tau(f_0);$$

(ii) Homotopy invariance

Let  $f \simeq g: X \to Y$  be homotopic maps of finite CW-complexes. Then the homomorphisms  $f_*, g_*: Wh(\pi_1(X)) \to Wh(\pi_1(Y))$  agree. If additionally fand g are homotopy equivalences, then we obtain

$$\tau(g) = \tau(f);$$

(iii) Composition formula

Let  $f: X \to Y$  and  $g: Y \to Z$  be homotopy equivalences of finite CWcomplexes. Then we get

$$\tau(g \circ f) = g_* \tau(f) + \tau(g);$$

(iv) Product formula

Let  $f: X' \to X$  and  $g: Y' \to Y$  be homotopy equivalences of connected finite *CW*-complexes. Then

$$\tau(f \times g) = \chi(X) \cdot j_* \tau(g) + \chi(Y) \cdot i_* \tau(f)$$

where  $\chi(X), \chi(Y) \in \mathbb{Z}$  denote the Euler characteristics,  $j_*$ : Wh $(\pi_1(Y)) \rightarrow$ Wh $(\pi_1(X \times Y))$  is the homomorphism induced by  $j: Y \rightarrow X \times Y, y \mapsto (y, x_0)$ for some base point  $x_0 \in X$  and  $i_*$  is defined analogously. Let X be a finite simplicial complex. Let X' be its barycentric subdivision. Then one can show  $\tau(f) = 0$  for the map  $f: X \to X'$  whose underlying map of spaces is the identity. However, if  $X_1$  and  $X_2$  are two finite CW-complexes with the same underlying space, it is not at all clear that  $\tau(f) = 0$  holds for the map  $f: X_1 \to X_2$ whose underlying map of spaces is the identity. This problem is solved by the following (in comparison with the other statements above much deeper) result due to Chapman [227], [228], see also [247, Appendix] and [798, Section 5].

**Theorem 3.38 (Topological invariance of Whitehead torsion).** *The Whitehead torsion of a homeomorphism*  $f: X \rightarrow Y$  *of finite CW-complexes vanishes.* 

# 3.4 Geometric Interpretation of Whitehead Group and Whitehead Torsion

In this section we introduce the concept of a simple homotopy equivalence  $f: X \to Y$  of finite *CW*-complexes geometrically. We will show that the obstruction for a homotopy equivalence  $f: X \to Y$  of finite *CW*-complexes to be simple is the Whitehead torsion.

We have the inclusion of spaces  $S^{n-2} 
ightharpoonrightarrow S^{n-1} 
ightharpoonrightarrow D^n$  where  $S^{n-1}_+ 
ightharpoonrightarrow D^n$  where  $S^{n-1}_+ 
ightharpoonrightarrow S^{n-1}_+$  is the upper hemisphere. The pair  $(D^n, S^{n-1}_+)$  carries an obvious relative CW-structure. Namely, attach an (n-1)-cell to  $S^{n-1}_+$  by the attaching map id:  $S^{n-2} 
ightarrow S^{n-2}$  to obtain  $S^{n-1}$ . Then we attach to  $S^{n-1}$  an *n*-cell by the attaching map id:  $S^{n-1} 
ightarrow S^{n-1}$  to obtain  $D^n$ . Let X be a CW-complex. Let  $q: S^{n-1}_+ 
ightarrow X$  be a map satisfying  $q(S^{n-2}) 
ightarrow X_{n-2}$  and  $q(S^{n-1}_+) 
ightarrow X_{n-1}$ . Let Y be the space  $D^n \cup_q X$ , i.e., the pushout



where *i* is the inclusion. Then *Y* inherits a *CW*-structure by putting  $Y_k = j(X_k)$  for  $k \le n-2$ ,  $Y_{n-1} = j(X_{n-1}) \cup g(S^{n-1})$  and  $Y_k = j(X_k) \cup g(D^n)$  for  $k \ge n$ . Note that *Y* is obtained from *X* by attaching one (n-1)-cell and one *n*-cell. Since the map  $i: S_+^{n-1} \to D^n$  is a homotopy equivalence and cofibration, the map  $j: X \to Y$  is a homotopy equivalence and a cofibration. We call *j* an *elementary expansion* and say that *Y* is obtained from *X* by an elementary expansion. There is a map  $r: Y \to X$  with  $r \circ j = id_X$ . This map is unique up to homotopy relative j(X). We call any such map an *elementary collapse* and say that *X* is obtained from *Y* by an elementary collapse.

**Definition 3.39 (Simple homotopy equivalence).** Let  $f: X \to Y$  be a map of finite *CW*-complexes. We call it a simple *homotopy equivalence* if there is a sequence of maps

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$$X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X[n] = Y$$

such that each  $f_i$  is an elementary expansion or elementary collapse and f is homotopic to the composite of the maps  $f_i$ .

**Remark 3.40 (Combinatorial meaning of simple homotopy equivalence).** The idea of the definition of a simple homotopy equivalence is that such a map can be written as a composite of elementary maps, namely, elementary expansions and collapses, which are obviously homotopy equivalences and in some sense the smallest and most elementary steps to pass from one finite *CW*-complex to another without changing the homotopy type. If one works with simplicial complexes, an elementary map has a purely combinatorial description. An elementary collapse means to delete a simplex and one of its faces that is not shared by another simplex. So one can describe the passage from one finite simplicial complex to another coming from a simple homotopy equivalence by finitely many combinatorial data. This does not work for two finite simplicial complexes that are homotopy equivalent but not simple homotopy equivalent.

This approach is similar to the idea in knot theory that two knots are equivalent if one can pass from one knot to the other by a sequence of elementary moves, the so-called Reidemeister moves. A Reidemeister move obviously does not change the equivalence class of a knot and, indeed, it turns out that one can pass from one knot to a second knot by a sequence of Reidemeister moves if and only if the two knots are equivalent, see for instance [187, Chapter 1] or [989]. The analogous statement is not true for homotopy equivalences  $f: X \to Y$  of finite *CW*-complexes because there is an obstruction for f to be simple, namely, its Whitehead torsion.

**Exercise 3.41.** Consider the simplicial complex *X* with four vertices  $v_0$ ,  $v_1$ ,  $v_2$ , and  $v_3$ , the edges  $\{v_0, v_1\}$ ,  $\{v_1, v_2\}$ ,  $\{v_0, v_2\}$ , and  $\{v_2, v_3\}$  and one 2-simplex  $\{v_0, v_1, v_2\}$ . Describe a sequence of elementary collapses and expansions transforming it to the one-point-space  $\{\bullet\}$ .

Recall that the *mapping cylinder* cyl(f) of a map  $f: X \to Y$  is defined by the pushout

$$\begin{array}{c} X \times \{0\} \xrightarrow{f} Y \\ \downarrow \\ \downarrow \\ X \times [0,1] \longrightarrow \operatorname{cyl}(f). \end{array}$$

There are natural inclusions  $i_X: X = X \times \{1\} \rightarrow \text{cyl}(f)$  and  $i_Y: Y \rightarrow \text{cyl}(f)$  and a natural projection  $p: \text{cyl}(f) \rightarrow Y$ . Note that  $i_X$  is a cofibration and  $p \circ i_X = f$  and  $p_Y \circ i_Y = \text{id}_Y$ . Define *the mapping cone* cone(f) by the quotient  $\text{cyl}(f)/i_X(X)$ .

**Lemma 3.42.** Let  $f: X \to Y$  be a cellular map of finite CW-complexes and  $A \subset X$  be a CW-subcomplex. Then the inclusion  $cyl(f|_A) \to cyl(f)$  is a composite of elementary expansions and hence a simple homotopy equivalence. In particular the inclusion  $i_Y: Y \to cyl(f)$  is a simple homotopy equivalence.

*Proof.* It suffices to treat the case where X is obtained from A by attaching an *n*-cell by an attaching map  $q: S^{n-1} \to X$ . Then there is an obvious pushout

and an obvious homeomorphism

$$(D^n \times [0,1], S^{n-1} \times [0,1] \cup_{S^{n-1} \times \{0\}} D^n \times \{0\}) \to (D^{n+1}, S^n_+).$$

**Lemma 3.43.** A map  $f: X \to Y$  of finite CW-complexes is a simple homotopy equivalence if and only if  $i_X: X \to cyl(f)$  is a simple homotopy equivalence.

*Proof.* This follows from Lemma 3.42 since the composite of a simple homotopy equivalence and a homotopy inverse of a simple homotopy equivalence is again a simple homotopy equivalence.

We only sketch the proof of the next result. More details can be found for instance in [247, (22.2)] or [648, Chapter 2]. However, we try to give enough information about its proof to illustrate why the geometric problem to decide whether a homotopy equivalence is simple is equivalent to a question about an invertible matrix A, which has a positive answer precisely if the class of A vanishes in the Whitehead group. Then the key will be Remark 3.13.

### Theorem 3.44 (Whitehead torsion and simple homotopy equivalences).

- (i) Let X be a finite CW-complex. Then for any element  $x \in Wh(\pi_1(X))$  there is an inclusion  $i: X \to Y$  of finite CW-complexes such that i is a homotopy equivalence and  $i_*^{-1}(\tau(i)) = x$ ;
- (ii) Let  $f: X \to Y$  be a homotopy equivalence of finite CW-complexes. Then its Whitehead torsion  $\tau(f) \in Wh(\pi_1(Y))$  vanishes if and only if f is a simple homotopy equivalence.

*Proof.* (i) We can assume without loss of generality that X is connected. Put  $\pi = \pi_1(X)$ . Choose an element  $A \in \operatorname{GL}_n(\mathbb{Z}\pi)$  representing  $x \in \operatorname{Wh}(\pi)$ . Choose  $n \ge 2$ . In the sequel we fix a zero-cell in X as base point. Put  $X' = X \vee \bigvee_{j=1}^{n} S^n$ . Let  $b_j \in \pi_n(X')$  be the element represented by the inclusion of the *j*-th copy of  $S^n$  into X' for j = 1, 2, ..., n. Recall that  $\pi_n(X')$  is a  $\mathbb{Z}\pi$ -module. Choose for i = 1, 2, ..., na map  $f_i: S^n \to X'_n$  such that  $[f_i] = \sum_{j=1}^n a_{i,j} \cdot b_j$  holds in  $\pi_n(X')$ . Attach to X' for each  $i \in \{1, 2, ..., n\}$  an (n + 1)-cell by  $f_i: S^n \to X'_n$ . Let Y be the resulting CW-complex and  $i: X \to Y$  be the inclusion. Then i is an inclusion of finite CW-complexes and induces an isomorphism on the fundamental groups. In the sequel we identify  $\pi$  and  $\pi_1(Y)$  by  $\pi_1(i)$ . The cellular  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{Y}, \widetilde{X})$  is

concentrated in dimensions *n* and (n + 1) and its (n + 1)-differential is given by the matrix *A* with respect to the cellular basis. Hence  $C_*(\widetilde{Y}, \widetilde{X})$  is a contractible finite based free  $\mathbb{Z}\pi$ -chain complex with  $\tau(C_*(\widetilde{Y}, \widetilde{X})) = [A]$  in Wh $(\pi)$ . This implies that  $i: X \to Y$  is a homotopy equivalence with  $i_*^{-1}(\tau(i)) = x$ .

(ii) Suppose that *f* is a simple homotopy equivalence. We want to show  $\tau(f) = 0$ . Because of Theorem 3.37 (iii) it suffices to prove for an elementary expansion  $j: X \to Y$  that its Whitehead torsion  $\tau(j) \in Wh(Y)$  vanishes. We can assume without loss of generality that *Y* is connected. In the sequel we write  $\pi = \pi_1(Y)$  and identify  $\pi$  with  $\pi_1(X)$  by  $\pi_1(f)$ . The following diagram of based free finite  $\mathbb{Z}\pi$ -chain complexes

$$0 \longrightarrow C_{*}(\widetilde{X}) \xrightarrow{C_{*}(\widetilde{J})} C_{*}(\widetilde{Y}) \xrightarrow{\operatorname{pr}_{*}} C_{*}(\widetilde{Y}, \widetilde{X}) \longrightarrow 0$$
  
$$\downarrow d_{*} \uparrow \qquad C_{*}(\widetilde{J}) \uparrow \qquad 0_{*} \uparrow$$
  
$$0 \longrightarrow C_{*}(\widetilde{X}) \xrightarrow{\operatorname{id}_{*}} C_{*}(\widetilde{X}) \xrightarrow{\operatorname{pr}_{*}} 0 \longrightarrow 0$$

has based exact rows and  $\mathbb{Z}\pi$ -chain homotopy equivalences as vertical arrows. Elementary facts about chain complexes, in particular the conclusion from Exercise 3.36, imply

$$\tau(C_*(\widetilde{j})) = \tau(\mathrm{id}_* \colon C_*(\widetilde{X}) \to C_*(\widetilde{X})) + \tau(0_* \colon 0 \to C_*(\widetilde{Y}, \widetilde{X}))$$
$$= 0 + \tau(C_*(\widetilde{Y}, \widetilde{X})) = \tau(C_*(\widetilde{Y}, \widetilde{X})).$$

The  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{Y}, \widetilde{X})$  is concentrated in two consecutive dimensions and its only non-trivial differential is id:  $\mathbb{Z}\pi \to \mathbb{Z}\pi$  if we identify the two non-trivial  $\mathbb{Z}\pi$ -chain modules with  $\mathbb{Z}\pi$  using the cellular basis. This implies  $\tau(C_*(\widetilde{Y}, \widetilde{X})) = 0$ and hence  $\tau(j) := \tau(C_*(\widetilde{j})) = 0$ .

Now suppose that  $\tau(f) = 0$ . We want to show that f is simple. We can assume without loss of generality that X is connected, otherwise treat each path component separately. Because of Lemma 3.43 we can assume that f is an inclusion  $i: X \to Y$  of connected finite *CW*-complexes which is a homotopy equivalence. We have to show that we can achieve by a sequence of elementary collapses and expansions that Y = X, i.e., we must get rid of all the cells in Y - X.

Since  $\chi(X) = \chi(Y)$ , it is clear that one cannot remove a single cell, this always has to be done in pairs. In the first step one shows for an *n*-dimensional cell  $e_n$  that one can attach one new (n+1)-cell  $e_{n+1}$  and a new (n+2)-cell  $e_{n+2}$  by an elementary expansion and then get rid of  $e_n$  and  $e_{n+1}$  by an elementary collapse. The outcome is that one can replace an *n*-cell by an (n + 2)-cell. Analogously one can show that one can replace an (n + 2)-cell by an *n*-cell. Thus one can arrange for some integer  $n \ge 2$  that Y is obtained from X by attaching k cells of dimension n trivially and then attaching k cells of dimension (n + 1). Hence the cellular  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{Y}, \widetilde{X})$  is concentrated in dimension n and (n + 1). After we have picked a cellular basis, its (n + 1)-differential is given by an invertible (k, k)-matrix A. By definition  $\tau(f)$  is the class of this matrix in Wh( $\pi$ ). In Remark 3.13 we have described what  $\tau(f) = [A] = 0$  means, namely, there is a sequence of operations that transform A to the empty matrix. Note that X = Y holds if and only if A is the empty matrix. Now the main idea is to show that each of these operations can be realized by elementary expansions and collapses.

Next we describe the Whitehead group geometrically. Fix a finite *CW*-complex *X*. Consider two pairs of finite *CW*-complexes (Y, X) and (Z, X) such that the inclusions of *X* into *Y* and *Z* are homotopy equivalences. We call them equivalent if there is a chain of pairs of finite *CW*-complexes

$$(Y, X) = (Y[0], X), (Y[1], X), (Y[2], X), \dots, (Y[n], X) = (Z, X)$$

such that for each  $k \in \{1, 2, ..., n\}$  either Y[k] is obtained from Y[k-1] by an elementary expansion or Y[k-1] is obtained from Y[k] by an elementary expansion. Denote by Wh<sup>geo</sup>(X) the equivalence classes [Y, X] of such pairs (Y, X). This becomes an abelian group under the addition  $[Y, X] + [Z, X] := [Y \cup_X Z, X]$ . The zero element is given by [X, X]. The inverse of [Y, X] is constructed as follows. Choose a map  $r: Y \to X$  with  $r_X = id$ . Let  $p: X \times [0, 1] \to X$  be the projection. Then  $[(cyl(r) \cup_p X) \cup_r X, X] + [Y, X] = 0$ .

A map  $g: X \to X'$  induces a homomorphism  $g_*: Wh^{geo}(X) \to Wh^{geo}(X')$  by sending [Y, X] to  $[Y \cup_g X', X']$ . We obviously have  $id_* = id$  and  $(g \circ h)_* = g_* \circ h_*$ . In other words, we obtain a covariant functor on the category of finite *CW*-complexes with values in abelian groups. More information about this construction can be found for instance in [247, § 6 in Chapter II].

Given a homotopy equivalence of finite *CW*-complexes  $f: X \to Y$ , define its *geometric Whitehead torsion*  $\tau^{\text{geo}}(f) \in \text{Wh}^{\text{geo}}(X)$  to be the class of (cyl(f), X). Because of Lemma 3.43 we have  $\tau^{\text{geo}}(f) = 0$  if and only f is a simple homotopy equivalence

The next result is essentially a consequence of Theorem 3.44. Details of its proof can be found in [247, §21].

### Theorem 3.45 (Geometric and algebraic Whitehead groups).

(i) Let X be a finite CW-complex. The map

$$\tau: \operatorname{Wh}^{\operatorname{geo}}(X) \to \operatorname{Wh}(\pi_1(X))$$

sending [Y, X] to  $i_*^{-1}\tau(i)$  for the inclusion  $i: X \to Y$  is a natural isomorphism of abelian groups.

It sends  $\tau^{\text{geo}}(f)$  to  $f_*^{-1}\tau(f)$  for a homotopy equivalence  $f: X \to Y$  of finite *CW*-complexes.

(ii) A homotopy equivalence  $f: X \to Y$  is a simple homotopy equivalence if and only if  $\tau(f) \in Wh(Y)$  vanishes.

**Exercise 3.46.** Let *Y* be a simply connected finitely dominated *CW*-complex. Show that there exists a finite *CW*-complex *X* and a homotopy equivalence  $f: X \to Y$ . Prove that for any two finite *CW*-complexes  $X_0$  and  $X_1$  and homotopy equivalences

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 $f_i: X_i \to Y$  for i = 0, 1 there exists a simple homotopy equivalence  $g: X_0 \to X_1$  with  $f_1 \circ g \simeq f_0$ .

### 3.5 The *s*-Cobordism Theorem

One of the main applications of Whitehead torsion is the theorem below.

**Theorem 3.47** (*s*-Cobordism Theorem). Let  $M_0$  be a connected closed manifold of dimension  $n \ge 5$  with fundamental group  $\pi = \pi_1(M_0)$ . Then:

- (i) Let  $(W; M_0, f_0, M_1, f_1)$  be an h-cobordism over  $M_0$ . Then W is trivial over  $M_0$  if and only if its Whitehead torsion  $\tau(W, M_0) \in Wh(\pi)$  vanishes;
- (ii) For any  $x \in Wh(\pi)$  there is an h-cobordism  $(W; M_0, f_0, M_1, f_1)$  over  $M_0$  with  $\tau(W, M_0) = x \in Wh(\pi)$ ;
- (iii) The function assigning to an h-cobordism  $(W; M_0, f_0, M_1, f_1)$  over  $M_0$  its Whitehead torsion yields a bijection from the diffeomorphism classes relative  $M_0$  of h-cobordisms over  $M_0$  to the Whitehead group  $Wh(\pi)$ .

Here are some definitions. An *n*-dimensional cobordism (sometimes also called just a bordism)  $(W; M_0, f_0, M_1, f_1)$  consists of a compact *n*-dimensional manifold W, closed (n - 1)-dimensional manifolds  $M_0$  and  $M_1$ , a disjoint decomposition  $\partial W = \partial_0 W \coprod \partial_1 W$  of the boundary  $\partial W$  of W, and diffeomorphisms  $f_0: M_0 \to \partial W_0$ and  $f_1: M_1 \to \partial W_1$ . If we want to specify  $M_0$ , we say that W is a *cobordism over*  $M_0$ . If  $f_0$  and  $f_1$  are obvious from the context, we briefly write  $(W; \partial_0 W, \partial_1 W)$ . We call a cobordism  $(W; M_0, f_0, M_1, f_1)$  an *h*-cobordism if the inclusions  $\partial_i W \to W$  for i = 0, 1 are homotopy equivalences and an *s*-cobordism if the inclusions  $\partial_i W \to W$ for i = 0, 1 are simple homotopy equivalences. Two cobordisms  $(W; M_0, f_0, M_1, f_1)$ and  $(W'; M_0, f'_0, M'_1, f'_1)$  over  $M_0$  are diffeomorphic relative  $M_0$  if there is a diffeomorphism  $F: W \to W'$  with  $F \circ f_0 = f'_0$ . We call an *h*-cobordism over  $M_0$ trivial if it is diffeomorphic relative  $M_0 = M_0 \times \{0\}$  to the trivial *h*-cobordism  $(M_0 \times [0, 1]; M_0 \times \{0\}, (M_0 \times \{1\}))$ . Note that the choice of the diffeomorphisms  $f_i$ do play a role although they are often suppressed in the notation.

The Whitehead torsion of an h-cobordism  $(W; M_0, f_0, M_1, f_1)$  over  $M_0$ 

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

is defined to be the preimage of the Whitehead torsion, see Definition 3.35,

$$\tau\left(M_0 \xrightarrow{f_0} \partial_0 W \xrightarrow{i_0} W\right) \in \mathrm{Wh}(\pi_1(W))$$

under the isomorphism  $(i_0 \circ f_0)_*$ : Wh $(\pi_1(M_0)) \xrightarrow{\cong}$  Wh $(\pi_1(W))$  where the map  $i_0: \partial_0 W \to W$  is the inclusion. Here we use the fact that each smooth closed manifold has a *CW*-structure, which comes for instance from a smooth triangulation,

or that each closed topological manifold of dimension different from 4 has a CW-structure, which comes from a handlebody decomposition, and that the choice of CW-structure does not matter by the topological invariance of the Whitehead torsion, see Theorem 3.38.

The idea of the proof of Theorem 3.47 is analogous to that of Theorem 3.44, but now one uses a handlebody decomposition instead of a *CW*-structure and carries out the manipulation for handlebodies instead of cells. Here a handlebody of index kcorresponds to a k-dimensional cell. More details can be found for instance in [667, Chapter 2].

The *h*-Cobordism Theorem 3.50 is due to Smale [920]. The *s*-Cobordism Theorem 3.47 is due to Barden, Mazur, and Stallings, see [66, 711]. In the PL category proofs can be found in [877, 6.19 on page 88]. Its topological version follows from Kirby and Siebenmann [579, Conclusion 7.4 on page 320]. More information about the *s*-Cobordism Theorem can be found for instance in [575], [725], [877, page 87-90]. The *s*-Cobordism Theorem is known to be false for dim( $M_0$ ) = 4 in general, by the work of Donaldson [312], but it is true for  $n = \dim(M_0) = 4$  for good fundamental groups in the topological category by results of Quinn and Freedman [118, 401, 402, 403]. Counterexamples in the case dim( $M_0$ ) = 3 are constructed by Matsumoto and Siebenmann [710] and Cappell and Shaneson [206] where the relevant 4-dimensional *s*-cobordism to be smooth in these counterexamples. It follows from Kwasik and Schultz [598] and Perelman's proof of the Thurston Geometrization Conjecture, see [580, 751], that every *h*-cobordism between two orientable closed 3-manifolds is an *s*-cobordism.

**Exercise 3.49.** Show for  $n \ge 6$  that there exists an *n*-dimensional *h*-cobordism  $(W; M_0, M_1)$  which is not trivial such that the *h*-cobordism  $(W \times S^3; M_0 \times S^3, M_1 \times S^3)$  is trivial.

Since the Whitehead group of the trivial group vanishes, see Theorem 3.17, the *s*-Cobordism Theorem 3.47 implies, see also [725],

**Theorem 3.50** (*h*-Cobordism Theorem). Let  $M_0$  be a simply connected closed *n*-dimensional manifold with dim $(M_0) \ge 5$ . Then every *h*-cobordism  $(W; M_0, f_0, M_1, f_1)$  over  $M_0$  is trivial.

**Theorem 3.51 (Poincaré Conjecture).** The Poincaré Conjecture is true for a closed *n*-dimensional manifold M with dim $(M) \ge 5$ , namely, if M is simply connected and its homology  $H_p(M)$  is isomorphic to  $H_p(S^n)$  for all  $p \in \mathbb{Z}$ , then M is homeomorphic to  $S^n$ .

*Proof.* We only give the proof for dim $(M) \ge 6$ . Since M is simply connected and  $H_*(M) \cong H_*(S^n)$ , one can conclude from the Hurewicz Theorem and Whitehead Theorem [1006, Theorem IV.7.13 on page 181 and Theorem IV.7.17 on page 182] that there is a homotopy equivalence  $f: M \to S^n$ . Let  $D_i^n \subset M$  for i = 0, 1 be two embedded disjoint disks. Let W be obtained from M by removing the interior of the two disks  $D_0^n$  and  $D_1^n$ . Then W turns out to be a

#### 3.5 The s-Cobordism Theorem

simply connected *h*-cobordism. Hence we can find because of Theorem 3.50 a homeomorphism  $F: (\partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$ that is the identity on  $\partial D_0^n = \partial D_0^n \times \{0\}$  and induces some (unknown) homeomorphism  $f_1: \partial D_0^n \times \{1\} \to \partial D_1^n$ . By the *Alexander trick* one can extend  $f_1: \partial D_0^n = \partial D_0^n \times \{1\} \to \partial D_1^n$  to a homeomorphism  $\overline{f_1}: D_0^n \to D_1^n$ . Namely, any homeomorphism  $f: S^{n-1} \to S^{n-1}$  extends to a homeomorphism  $\overline{f}: D^n \to D^n$ by sending  $t \cdot x$  for  $t \in [0, 1]$  and  $x \in S^{n-1}$  to  $t \cdot f(x)$ . Now define a homeomorphism  $h: D_0^n \times \{0\} \cup_{i_0} \partial D_0^n \times [0, 1] \cup_{i_1} D_0^n \times \{1\} \to M$  for the canonical inclusions  $i_k: \partial D_0^n \times \{k\} \to \partial D_0^n \times [0, 1]$  for k = 0, 1 by  $h|_{D_0^n \times \{0\}} = \text{id}, h|_{\partial D_0^n \times [0, 1]} = F$ and  $h|_{D_0^n \times \{1\}} = \overline{f_1}$ . Since the source of *h* is obviously homeomorphic to  $S^n$ , Theorem 3.51 follows.

In the case dim(M) = 5 one uses the fact that M is the boundary of a contractible 6-dimensional manifold W and applies Theorem 3.50 to W with an embedded disk removed.

The Poincaré Conjecture, see Theorem 3.51, is known in all dimensions, where dimension 3 is due to the work of Perelman, see [580, 750, 751, 803, 804, 805], and dimension 4 is due to Freedman, see [118, 401, 402, 403]. The first proof of the Poincaré Conjecture in the topological category in dimension  $\geq 5$  was given by Newman [758] using engulfing theory. The smooth version of the Poincaré Conjecture holds in dimensions  $\leq 3$ , is open in dimension 4, and holds in dimensions 5, 6, 12, 56, and 61. It is conjectured that it holds only in finitely many dimensions and that it is actually false in all dimensions except 1, 2, 3, 4, 5, 6, 12, 56, and 61. This is discussed for instance in [667, Remark 12.36 on page 445].

**Remark 3.52 (Exotic Spheres).** Note that the proof of the Poincaré Conjecture in Theorem 3.51 works only in the topological category but not in the smooth category. In other words, we cannot conclude the existence of a diffeomorphism  $h: S^n \to M$ . The proof in the smooth case breaks down when we apply the Alexander trick. The construction of  $\overline{f}$  given by coning f yields only a homeomorphism  $\overline{f}$  and not a diffeomorphism, even if we start with a diffeomorphism f. The map  $\overline{f}$  is smooth outside the origin of  $D^n$  but not necessarily at the origin. Indeed, not every diffeomorphism  $f: S^{n-1} \to S^{n-1}$  can be extended to a diffeomorphism  $D^n \to D^n$  and there exist so-called *exotic spheres*, i.e., closed manifolds that are homeomorphic to  $S^n$  but not diffeomorphic to  $S^n$ . The classification of these exotic spheres is one of the early very important achievements of surgery theory and one motivation for its further development. For more information about exotic spheres we refer for instance to [576], [611], [628], [648, Chapter 6] and [667, Chapter 12].

**Remark 3.53 (The Surgery Program).** In some sense the *s*-Cobordism Theorem 3.47 is one of the first theorems where diffeomorphism classes of certain manifolds are determined by an algebraic invariant, namely, the Whitehead torsion. Moreover, the Whitehead group  $Wh(\pi)$  depends only on the fundamental group  $\pi = \pi_1(M_0)$ , whereas the diffeomorphism classes of *h*-cobordisms over  $M_0$  a priori depend on  $M_0$  itself. The *s*-Cobordism Theorem 3.47 is one step in a program to decide whether two closed manifolds *M* and *N* are diffeomorphic, which is in general

a very hard question. The idea is to construct an *h*-cobordism (W; M, f, N, g) with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial *h*-cobordism over M, which implies that M and N are diffeomorphic. So the Surgery Program is:

- (i) Construct a simple homotopy equivalence  $f: M \to N$ ;
- (ii) Construct a cobordism (W; M, N) and a map (F, f, id):  $(W; M, N) \rightarrow (N \times [0, 1], N \times \{0\}, N \times \{1\});$
- (iii) Modify W and F relative boundary by so-called surgery such that F becomes a simple homotopy equivalence and thus W becomes an h-cobordism whose Whitehead torsion is trivial.

The advantage of this approach will be that it can be reduced to problems in homotopy theory and algebra, which can sometimes be handled by well-known techniques. In particular one will sometimes get computable obstructions for two homotopy equivalent manifolds to be diffeomorphic. Often surgery theory has proved to be very useful when one wants to distinguish two closed manifolds which have very similar properties. The classification of homotopy spheres is one example. Moreover, surgery techniques can be applied to problems that are of a different nature than of diffeomorphism or homeomorphism classifications, for instance for the construction of group actions.

More information about surgery theory will be given in Chapter 9.

### 3.6 Reidemeister Torsion and Lens Spaces

In this section we briefly deal with Reidemeister torsion, which was defined earlier than (and motivated the definition of) Whitehead torsion. Reidemeister torsion was the first invariant in algebraic topology that could distinguish between spaces which are homotopy equivalent but not homeomorphic. Namely, it can be used to classify lens spaces up to homeomorphism, see Reidemeister [847]. We will give no proofs. More information and complete proofs can be found in [247, Chapter V], [648, Section 2.4], and [667, Section 3.5].

Let *X* be a finite *CW*-complex with fundamental group  $\pi$ . Let *U* be an orthogonal finite-dimensional  $\pi$ -representation. Denote by  $H_*(X; U)$  the homology of *X* with coefficients in *U*, i.e., the homology of the  $\mathbb{R}$ -chain complex  $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$ . Suppose that *X* is *U*-acyclic, i.e.,  $H_n(X; U) = 0$  for all  $n \ge 0$ . If we fix a cellular basis for  $C_*(\widetilde{X})$  and some orthogonal  $\mathbb{R}$ -basis for *U*, then  $U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$  is a contractible based free finite  $\mathbb{R}$ -chain complex and yields an element  $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$ , see (3.32). Define the *Reidemeister torsion* 

$$(3.54) \qquad \qquad \rho(X;U) \in \mathbb{R}^{>0}$$

to be the image of  $\tau(U \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})) \in \widetilde{K}_1(\mathbb{R})$  under the homomorphism  $\widetilde{K}_1(\mathbb{R}) \to \mathbb{R}^{>0}$ sending the class [A] of  $A \in GL_n(\mathbb{R})$  to  $|\det(A)|$ . Note that for any trivial unit  $\pm \gamma$  the automorphism of U given by multiplication by  $\pm \gamma$  is orthogonal and that the absolute value of the determinant of any orthogonal automorphism of U is 1.
Therefore  $\rho(X; U) \in \mathbb{R}^{>0}$  is independent of the choice of cellular basis for  $C_*(\widetilde{X})$  and the orthogonal basis for U, and hence is an invariant of the *CW*-complex X and the orthogonal representation U.

We state without proof the next result, which essentially says that the Whitehead torsion of a homotopy equivalence is related to the difference of the Reidemeister torsion of the target and the source when defined.

**Lemma 3.55.** Let  $f: X \to Y$  be a homotopy equivalence of connected finite CWcomplexes and let U be an orthogonal finite-dimensional  $\pi = \pi_1(Y)$ -representation. Suppose that Y is U-acyclic. Let  $f^*U$  be the orthogonal  $\pi_1(X)$ -representation obtained from U by restriction with the isomorphism  $\pi_1(f)$ .

Let  $d_U$ : Wh $(\pi(Y)) \to \mathbb{R}^{>0}$  be the map sending the class [A] of  $A \in$ GL<sub>n</sub>( $\mathbb{Z}\pi_1(Y)$ ) to  $|\det(\mathrm{id}_U \otimes_{\mathbb{Z}\pi} r_A : U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n \to U \otimes_{\mathbb{Z}\pi} \mathbb{Z}\pi^n)|.$ 

*Then* X *is*  $f^*U$ *-acyclic and we get* 

$$\frac{\rho(Y;U)}{\rho(X;f^*U)} = d_U(\tau(f)).$$

Next we introduce lens spaces. Let *G* be a cyclic group of finite order |G|. Let *V* be a unitary finite-dimensional *G*-representation. Define its *unit sphere SV* and its *unit disk DV* to be the *G*-subspaces  $SV = \{v \in V \mid ||v|| = 1\}$  and  $DV = \{v \in V \mid ||u|| \le 1\}$  of *V*. Note that a complex finite-dimensional vector space has a preferred orientation as real vector space, namely, the one given by the  $\mathbb{R}$ -basis  $\{b_1, ib_1, b_2, ib_2, \ldots, b_n, ib_n\}$  for any  $\mathbb{C}$ -basis  $\{b_1, b_2, \ldots, b_n\}$ . Any  $\mathbb{C}$ -linear automorphism of a complex finite-dimensional vector space preserves this orientation. Thus *SV* and *DV* are oriented compact Riemannian manifolds with isometric orientation preserving *G*-action. We call a unitary *G*-representation *V free* if the induced *G*-action on its unit sphere *SV* is free. Then  $SV \to G \setminus SV$  is a covering and the quotient space  $L(V) := G \setminus SV$  inherits from *SV* the structure of an oriented closed Riemannian manifold.

**Definition 3.56 (Lens space).** We call the closed oriented Riemannian manifold L(V) the *lens space* associated to the free finite-dimensional unitary representation V of the finite cyclic group G.

**Exercise 3.57.** Show that the 3-dimensional real projective space  $\mathbb{RP}^3$  is a lens space. Let  $\mathbb{R}^-$  be the non-trivial orthogonal  $\mathbb{Z}/2$ -representation. Show that  $\mathbb{RP}^3$  is  $\mathbb{R}^-$ -acyclic and compute the Reidemeister torsion  $\rho(\mathbb{RP}^3; \mathbb{R}^-)$ .

One can also specify these lens spaces by numbers as follows.

**Notation 3.58.** Let  $\mathbb{Z}/t$  be the cyclic group of order  $t \ge 2$ . The 1-dimensional unitary representation  $V_k$  for  $k \in \mathbb{Z}/t$  has as underlying vector space  $\mathbb{C}$  and  $l \in \mathbb{Z}/t$  acts on it by multiplication with  $\exp(2\pi i k l/t)$ . Note that  $V_k$  is free if and only if  $k \in (\mathbb{Z}/t)^{\times}$ , and is trivial if and only if k = 0 in  $\mathbb{Z}/t$ . Define the lens space  $L(t; k_1, \ldots, k_c)$  for an integer  $c \ge 1$  and elements  $k_1, \ldots, k_c$  in  $(\mathbb{Z}/t)^{\times}$  by  $L(\bigoplus_{i=1}^c V_{k_i})$ .

Lens spaces form a very interesting family of manifolds, which can be completely classified as we will see. Two lens spaces L(V) and L(W) of the same dimension  $n \ge 3$  have the same homotopy groups, namely, their fundamental group is Gand their p-th homotopy group is isomorphic to  $\pi_p(S^n)$  for  $p \ge 2$ . They also have the same homology with integral coefficients, namely  $H_p(L(V)) \cong \mathbb{Z}$  for  $p \in \{0, n\}, H_p(L(V)) \cong G$  for p odd and  $1 \le p < n$ , and  $H_p(L(V)) = 0$  for all other values of p. Also their cohomology groups agree. Nevertheless not all of them are homotopy equivalent. Moreover, there are homotopy equivalent lens spaces that are not diffeomorphic, see Example 3.62.

We state without proof the following result.

**Theorem 3.59 (Homotopy Classification of lens spaces).** The lens spaces  $L(t; k_1, ..., k_c)$  and  $L(t; l_1, ..., l_c)$  are homotopy equivalent if and only if there are  $e \in (\mathbb{Z}/t)^{\times}$  and  $\epsilon \in \{\pm 1\}$  satisfying  $\prod_{i=1}^{c} k_i = \epsilon \cdot e^c \cdot \prod_{i=1}^{c} l_i$  in  $(\mathbb{Z}/t)^{\times}$ .

The lens spaces  $L(t; k_1, ..., k_c)$  and  $L(t; l_1, ..., l_c)$  are oriented homotopy equivalent if and only if there is an  $e \in (\mathbb{Z}/t)^{\times}$  satisfying  $\prod_{i=1}^{c} k_i = e^c \cdot \prod_{i=1}^{c} l_i$  in  $(\mathbb{Z}/t)^{\times}$ .

#### Theorem 3.60 (Diffeomorphism Classification of Lens Spaces).

- (i) Let G be a finite cyclic group. Let L(V) and L(W) be two lens spaces of the same dimension n ≥ 3. Then the following statements are equivalent:
  - (a) There is an automorphism  $\alpha : G \to G$  such that V and  $\alpha^*W$  are isomorphic as orthogonal G-representations;
  - (b) There is an isometric diffeomorphism  $L(V) \rightarrow L(W)$ ;
  - (c) There is a diffeomorphism  $L(V) \rightarrow L(W)$ ;
  - (d) There is a homeomorphism  $L(V) \rightarrow L(W)$ ;
  - (e) There is a simple homotopy equivalence  $L(V) \rightarrow L(W)$ ;
  - (f) There is an automorphism  $\alpha : G \to G$  such that for any orthogonal finitedimensional representation U with  $U^G = 0$

$$\rho(L(W); U) = \rho(L(V); \alpha^* U)$$

holds;

(g) There is an automorphism  $\alpha: G \to G$  such that for any non-trivial 1-dimensional unitary G-representation U

$$\rho(L(W); \operatorname{res} U) = \rho(L(V); \alpha^* \operatorname{res} U)$$

holds where the orthogonal representation res U is obtained from U by restricting the scalar multiplication from  $\mathbb{C}$  to  $\mathbb{R}$ ;

(ii) Two lens spaces L(t; k<sub>1</sub>,..., k<sub>c</sub>) and L(t; l<sub>1</sub>,..., l<sub>c</sub>) are homeomorphic if and only if there are e ∈ (ℤ/t)<sup>×</sup>, signs ε<sub>i</sub> ∈ {±1} and a permutation σ ∈ Σ<sub>c</sub> such that k<sub>i</sub> = ε<sub>i</sub> · e · l<sub>σ(i)</sub> holds in (ℤ/t)<sup>×</sup> for i = 1, 2, ..., c.

*Proof.* We give only a sketch of the proof of assertion (i). Assertion (ii) is a direct consequence of assertion (i).

The implications (ia)  $\Rightarrow$  (ib)  $\Rightarrow$  (ic)  $\Rightarrow$  (id) and (if)  $\Rightarrow$  (ig) are obvious. The implication (id)  $\Rightarrow$  (ie) follows from Theorem 3.38. The implication (ie)  $\Rightarrow$  (if) follows from Lemma 3.55. The hard part of the proof is the implication (ig)  $\Rightarrow$  (ia). It involves proving the formula

$$\rho(L(V \oplus W); \operatorname{res} U) = \rho(L(V); \operatorname{res} U) \cdot \rho(L(W); \operatorname{res} U)$$

for two free unitary *G*-representations *V* and *W* and then directly computing  $\rho(L(V); \operatorname{res} U)$  for every free 1-dimensional unitary representation *V*. Finally one has to show that the values of the Reidemeister torsion distinguish *V* and *W* viewed as orthogonal representations up to automorphisms of *G*. This proof is based on the number-theoretic result mentioned below, whose proof can be found for instance in [294] or [399].

**Lemma 3.61 (Franz' Independence Lemma).** Let  $t \ge 2$  be an integer and  $S = \{j \in \mathbb{Z} \mid 0 < j < t, (j, t) = 1\}$ . Let  $(a_j)_{j \in S}$  be a sequence of integers indexed by S such that  $\sum_{j \in S} a_j = 0$ ,  $a_j = a_{t-j}$  for  $j \in S$  and  $\prod_{j \in S} (\zeta^j - 1)^{a_j} = 1$  holds for every *t*-th root of unity  $\zeta \neq 1$ . Then  $a_j = 0$  for  $j \in S$ .

**Example 3.62.** We conclude from Theorem 3.59 and Theorem 3.60 (ii) the following facts:

- (i) Any homotopy equivalence  $L(7; k_1, k_2) \rightarrow L(7; k_1, k_2)$  has degree 1. Thus  $L(7; k_1, k_2)$  possesses no orientation reversing self-diffeomorphism;
- (ii) L(5; 1, 1) and L(5; 2, 1) have the same homotopy groups, homology groups and cohomology groups, but they are not homotopy equivalent;

(iii) L(7; 1, 1) and L(7; 2, 1) are homotopy equivalent, but not homeomorphic.

**Example 3.63** (*h*-cobordisms between lens spaces). The rigidity of lens spaces is illustrated by the following fact. Let (W, L, L') be an *h*-cobordism of lens spaces that is compatible with the orientations and the identifications of  $\pi_1(L)$  and  $\pi_1(L')$  with *G*. Then *W* is diffeomorphic relative *L* to  $L \times [0, 1]$  and *L* and *L'* are diffeomorphic, see [726, Corollary 12.13 on page 410].

**Remark 3.64 (Differential geometric characterization of lens spaces).** Lens spaces with their preferred Riemannian metric have constant positive sectional curvature. A closed Riemannian manifold with constant positive sectional curvature and cyclic fundamental group is isometrically diffeomorphic to a lens space after possibly rescaling the Riemannian metric by a constant [1018].

**Remark 3.65 (de Rham's Theorem).** The results above when interpreted as statements about unit spheres in free representations are generalized by De Rham's Theorem [293], see also [639, Proposition 3.2 on page 478], [645, page 317], and [876, section 4], as follows. It says for a finite group *G* and two orthogonal *G*-representations *V* and *W* whose unit spheres *SV* and *SW* are *G*-diffeomorphic that *V* and *W* are isomorphic as orthogonal *G*-representations. This remains true if one replaces *G*-diffeomorphic by *G*-homeomorphic provided that *G* has odd order, see [504], [699], but not for any finite group *G*, see [205, 207, 460, 462, 463].

We refer to [247], [648, Chapter 2], and [726] for more information about Reidemeister torsion and lens spaces.

**Remark 3.66 (Further appearance of Reidemeister torsion).** The Alexander polynomial of a knot can be interpreted as a kind of Reidemeister torsion of the canonical infinite cyclic covering of the knot complement, see [724], [959]. Reidemeister torsion appears naturally in surgery theory [695]. Counterexamples to the (polyhedral) Hauptvermutung that two homeomorphic simplicial complexes are already PL-homeomorphic are given by Milnor [723], see also [842], and detected by Reidemeister torsion. Seiberg-Witten invariants for 3-manifolds are closely related to torsion invariants, see Turaev [958].

**Remark 3.67 (Analytic Reidemeister torsion).** Ray-Singer [845] defined the analytic counterpart of topological Reidemeister torsion using a regularization of the zeta-function. Ray and Singer conjectured that the analytic and topological Reidemeister torsion agree. This conjecture was proved independently by Cheeger [236] and Müller [753]. Manifolds with boundary and manifolds with symmetries, sum (= glueing) formulas and fibration formulas are treated in [179, 273, 639, 645, 682, 966]. For a survey on analytic and topological torsion we refer for instance to [661]. There are also  $L^2$ -versions of these notions, see for instance [190, 210, 637], [650, Chapter 3], [680, 707].

# 3.7 The Bass-Heller-Swan Theorem for K<sub>1</sub>

In the section we want to compute  $K_1(R[\mathbb{Z}])$  for a ring *R*. This computation, the so-called Bass-Heller-Swan decomposition, marks the beginning of the (long) way towards the final formulation of the Farrell-Jones Conjecture for algebraic *K*-theory.

# 3.7.1 The Bass-Heller-Swan Decomposition for K<sub>1</sub>

We need some preparation to formulate it. In the sequel we write  $R[\mathbb{Z}]$  as the ring  $R[t, t^{-1}]$  of finite Laurent polynomials in *t* with coefficients in *R*. Obviously the ring R[t] of polynomials in *t* with coefficients in *R* is a subring of  $R[t, t^{-1}]$ . Define the ring homomorphisms

**Definition 3.68** ( $NK_n(R)$ ). Define for n = 0, 1

$$NK_n(R) := \ker((\operatorname{ev}_0)_* \colon K_n(R[t]) \to K_n(R)).$$

#### 3.7 The Bass-Heller-Swan Theorem for $K_1$

**Example 3.69.** Let *F* be a field. Put  $R = F[t]/(t^2)$ . Every element in *R* can be uniquely written as a + bt for  $a, b \in F$ . We have  $(1 + bt) \cdot (1 - bt) = 1 - b^2t^2 = 1$  in *R*. Hence the element  $a + bt \in R$  is a unit if and only if  $a \neq 0$ . We conclude that *R* is a local ring with  $(t) = \{bt \mid b \in F\}$  as the unique maximal ideal. Since *R* is commutative, the homomorphism

$$i_R \colon R^{\times} \xrightarrow{=} K_1(R), \quad [x] \mapsto [r_x \colon R \to R]$$

is bijective by Theorem 3.6. Let  $ev_0: R \to F$  be the ring homomorphism sending a + bt to a. Its kernel is (t). It induces a group homomorphism  $R[x]^{\times} \to F[x]^{\times}$ . Since  $F[x]^{\times}$  is the multiplicative group of non-trivial polynomials over F of degree 0 and  $(1 + tvx) \cdot (1 - tvx) = 1 - v^2 t^2 x^2 = 1$  holds in R[x] for all  $v \in F[x]$ , we obtain an isomorphism of abelian groups

$$\phi \colon R^{\times} \oplus F[x] \xrightarrow{=} R[x]^{\times}, \quad (u, v) \mapsto u \cdot (1 + tvx).$$

Since R[x] is commutative, the map  $i_{R[x]} \colon R[x]^{\times} \xrightarrow{\cong} K_1(R[x])$  is injective, a retraction is given by the determinant. We conclude that the following composite is an injection of abelian groups

$$F[x] \xrightarrow{\phi|_{F[x]}} \operatorname{coker} \left( R^{\times} \to R[x]^{\times} \right) \xrightarrow{i} \operatorname{coker} \left( K_1(R) \to K_1(R[x]) \right) \cong NK_1(R)$$

where *i* is the map induced by  $i_R$  and  $i_{R[x]}$ . This implies that  $NK_1(R)$  is an abelian group which is not finitely generated.

Example 3.69 illustrates the following fact. If *R* is any ring, then  $NK_1(R)$  is either trivial or infinitely generated as an abelian group, see Theorem 6.20. So in general  $NK_1(R)$  is hard to compute. At least we have the following useful results. If *R* is a ring of finite characteristic *N*, then we obtain  $NK_n(R)[1/N] = 0$  for n = 0, 1, see Theorem 6.17. If  $NK_n(R) = 0$  and *G* is finite, then  $NK_n(RG)[1/|G|] = 0$  for n = 0, 1, see Theorem 6.18.

Recall that an endomorphism  $f: P \rightarrow P$  of an *R*-module *P* is called *nilpotent* if there exists a positive integer *n* with  $f^n = 0$ .

**Definition 3.70 (Nil-group** Nil<sub>0</sub>(*R*)). Define the 0-th Nil-group Nil<sub>0</sub>(*R*) to be the abelian group whose generators are conjugacy classes [f] of nilpotent endomorphisms  $f: P \rightarrow P$  of finitely generated projective *R*-modules with the following relation. Given a commutative diagram of finitely generated projective *R*-modules



with exact rows and nilpotent endomorphisms as vertical arrows, we get

$$[f_1] + [f_3] = [f_2].$$

Let  $\iota: K_0(R) \to \operatorname{Nil}_0(R)$  be the homomorphism sending the class [P] of a finitely generated projective *R*-module *P* to the class  $[0: P \to P]$  of the trivial endomorphism of *P*.

**Definition 3.71 (Reduced Nil-group**  $\widetilde{Nil}_0(R)$ ). Define the reduced 0-th Nil-groups  $\widetilde{Nil}_0(R)$  to be the cokernel of the map  $\iota$ .

The homomorphism  $\operatorname{Nil}_0(R) \to K_0(R)$ ,  $[f: P \to P] \mapsto [P]$  is a retraction of the map  $\iota$ . So we get a natural splitting

$$\operatorname{Nil}_0(R) \xrightarrow{\cong} \widetilde{\operatorname{Nil}}_0(R) \oplus K_0(R).$$

Denote by

$$i: NK_1(R) \to K_1(R[t])$$

the inclusion. Let

$$l_{\pm} \colon R[t] \to R[t, t^{-1}]$$

be the inclusion of rings sending *t* to  $t^{\pm 1}$ . Define

$$j_{\pm} := K_1(l_{\pm}) \circ j : NK_1(R) \to K_1(R[t, t^{-1}]).$$

The homomorphism

$$B: K_0(R) \to K_1(R[t, t^{-1}])$$

sends the class [P] of a finitely generated projective *R*-module *P* to the class  $[r_t \otimes_R \operatorname{id}_P]$  of the  $R[t, t^{-1}]$ -automorphism  $r_t \otimes_R \operatorname{id}_P : R[t, t^{-1}] \otimes_R P \to R[t, t^{-1}] \otimes_R P$  that maps  $u \otimes p$  to  $ut \otimes p$ . The homomorphism

$$N': \operatorname{Nil}_0(R) \to K_1(R[t])$$

sends the class [f] of the nilpotent endomorphism  $f: P \to P$  of the finitely generated projective *R*-module *P* to the class  $[id - r_t \otimes_R f]$  of the R[t]-automorphism

$$\operatorname{id} -r_t \otimes_R f \colon R[t] \otimes_R P \to R[t] \otimes_R P, \quad u \otimes p \mapsto u \otimes p - ut \otimes f(p).$$

This is indeed an automorphism. Namely, if  $f^{n+1} = 0$ , then an inverse is given by  $\sum_{k=0}^{n} (r_t \otimes_R f)^k$ . The composite of N' with both  $(ev_0)_* \colon K_1(R[t]) \to K_1(R)$  and  $\iota \colon K_0(R) \to Nil_0(R)$  is trivial. Hence N' induces a homomorphism

$$N: \operatorname{Nil}_0(R) \to NK_1(R).$$

The proof of the following theorem can be found for instance in [105] (for regular rings), [102, Chapter XII], [860, Theorem 3.2.22 on page 149], and [998, 3.6 in Section III.3 on page 205].

3.7 The Bass-Heller-Swan Theorem for  $K_1$ 

**Theorem 3.72 (Bass-Heller-Swan decomposition for**  $K_1$ ). *The following maps are isomorphisms of abelian groups, natural in* R,

$$N: \widetilde{\operatorname{Nil}}_0(R) \xrightarrow{=} NK_1(R);$$

$$j \oplus K_1(i'): NK_1(R) \oplus K_1(R) \xrightarrow{\cong} K_1(R[t]);$$

$$B \oplus K_1(i) \oplus j_+ \oplus j_-: K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]).$$

One easily checks that Theorem 3.72 applied to  $R = \mathbb{Z}G$  implies the following reduced version

**Theorem 3.73 (Bass-Heller-Swan decomposition for**  $Wh(G \times \mathbb{Z})$ ). *Let G be a group. Then there is an isomorphism of abelian groups, natural in G* 

 $\overline{B} \oplus \operatorname{Wh}(i) \oplus \overline{j_+} \oplus \overline{j_-} \colon \widetilde{K}_0(\mathbb{Z}G) \oplus \operatorname{Wh}(G) \oplus NK_1(\mathbb{Z}G) \oplus NK_1(\mathbb{Z}G) \xrightarrow{\cong} \operatorname{Wh}(G \times \mathbb{Z}).$ 

**Example 3.74** ( $\widetilde{K}_0(\mathbb{Z}G)$  affects Wh(*G*)). The Whitehead group Wh( $S_n$ ) of the symmetric group  $S_n$  is trivial, see Theorem 3.116 (iii), whereas  $\widetilde{K}_0(\mathbb{Z}[S_n])$  is a finite non-trivial group for  $n \ge 5$ , see Theorem 2.113 (ii). In the sequel we let  $n \ge 5$ . We conclude from Theorem 3.73 that Wh( $S_n \times \mathbb{Z}$ ) is non-trivial for  $n \ge 5$ , whereas the obvious map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{T}\mathcal{N}}(S_n \times \mathbb{Z})} \operatorname{Wh}(H) \to \operatorname{Wh}(S_n \times \mathbb{Z})$$

is the zero map and hence not surjective. Also the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{T}N}(G)} K_1(\mathbb{Z}H) \to K_1(\mathbb{Z}[S_n \times \mathbb{Z}])$$

cannot be surjective. Hence there is no hope that a formula which computes  $K_n(RG)$  in terms of the values  $K_n(RH)$  for all finite or all virtually cyclic subgroups H of G (such as appearing in Conjecture 2.67) is true in general. The general picture will be that a computation of a K or L-group of RG in dimension n involves K- and L-groups of RH in all dimensions  $\leq n$  where H runs through all virtually cyclic subgroups of G.

Denote by

$$k_{\pm} \colon R \to R[t^{\pm 1}]$$

the ring homomorphism sending r to  $r \cdot t^0$ . Obviously  $l_{\pm} \circ k_{\pm} = i$ . Define a map

$$C: K_1(R[t, t^{-1}]) \to K_0(R)$$

by sending the class [f] of an  $R[t, t^{-1}]$ -automorphism  $f : R[t, t^{-1}]^n \to R[t, t^{-1}]^n$ to the element  $[P(f, l)] - l \cdot [R]$  where l is a large enough positive integer and P(f, l)is the finitely generated projective R-module  $f(t^{l-1} \cdot R[t^{-1}]) \cap R[t]$ . We omit the proof that P(f, l) is a finitely generated projective R-module for large enough l, that the class  $[P(f, l)] - l \cdot [R]$  is independent of l and depends only on [f], and that the map C is a well-defined homomorphism of abelian groups. **Theorem 3.75 (Fundamental Theorem of** *K***-theory in dimension** 1). *There is a sequence which is natural in R and exact* 

$$0 \to K_1(R) \xrightarrow{K_1(k_+) \oplus -K_1(k_-)} K_1(R[t]) \oplus K_1(R[t^{-1}])$$
$$\xrightarrow{K_1(l_+)_* \oplus K_1(l_-)} K_1(R[t,t^{-1}]) \xrightarrow{C} K_0(R) \to 0$$

where  $k_+$ ,  $k_-$ ,  $l_+$ , and  $l_-$  are the obvious inclusions.

If we regard it as an acyclic  $\mathbb{Z}$ -chain complex, there exists a chain contraction, natural in R.

*Proof.* One checks  $C \circ B = id_{K_0(R)}$  and  $C \circ i_* = C \circ j_- = C \circ j_+ = 0$ . Now apply Theorem 3.72.

# 3.7.2 The Grothendieck Decomposition for G<sub>0</sub> and G<sub>1</sub>

There is also a G-theory version of the Bass-Heller-Swan decomposition, which is due to Grothendieck. Its proof can be found in [105] or [860, Theorem 3.2.12 on page 141, Theorem 3.2.16 on page 143 and Theorem 3.2.19 on page 147].

**Theorem 3.76 (Grothendieck decomposition for**  $G_0$  **and**  $G_1$ ). Let *R* be a Noetherian ring.

(i) The inclusions  $R \to R[t]$  and  $R \to R[t, t^{-1}]$  induce isomorphisms of abelian groups

$$G_0(R) \xrightarrow{\cong} G_0(R[t]);$$
  

$$G_0(R) \xrightarrow{\cong} G_0(R[t, t^{-1}]);$$

(ii) There are natural isomorphisms

$$i'_* \colon G_1(R) \xrightarrow{\cong} G_1(R[t]);$$
  
$$B \oplus i_* \colon G_0(R) \oplus G_1(R) \xrightarrow{\cong} G_1(R[t, t^{-1}]),$$

where  $i'_*$ , *B*, and  $i_*$  are defined analogously to the maps appearing in Theorem 3.72.

**Exercise 3.77.** Show that the map  $\mathbb{Z} \xrightarrow{\cong} G_0(R[\mathbb{Z}^n])$  sending *n* to  $n \cdot [R[\mathbb{Z}^n]]$  is an isomorphism for a principal ideal domain *R* and  $n \ge 0$ .

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#### 3.7.3 Regular Rings

**Theorem 3.78 (Hilbert Basis Theorem).** *If* R *is Noetherian, then* R[t] *and*  $R[t, t^{-1}]$  *are Noetherian.* 

*Proof.* See for instance [860, Theorem 3.2.1 on page 133 and Corollary 3.2.2 on page 134].

Let (*P*) be a property of groups, e.g., being finite or being cyclic. A group *G* is called *virtually* (*P*) if *G* contains a subgroup  $H \subset G$  of finite index such that *H* has property (P). A group *G* is *poly-(P)* if there is a finite sequence of subgroups  $\{1\} = G_0 \subset G_1 \subset G_2 \subset \ldots G_r = G$  such that  $G_i$  is normal in  $G_{i+1}$  and the quotient  $G_{i+1}/G_i$  has property (P) for  $i = 0, 1, 2, \ldots, r - 1$ . Thus the notions of *virtually finitely generated abelian, virtually free, virtually nilpotent, poly-cyclic, poly-Z, and virtually poly-cyclic* are defined, where poly-Z stands for poly-(infinite cyclic).

**Theorem 3.79 (Noetherian group rings).** *If R is a Noetherian ring and G is a virtually poly-cyclic group, then RG is Noetherian.* 

*Proof.* See for instance [650, Lemma 10.55 on page 397].  $\Box$ 

No counterexample is known to the conjecture that  $\mathbb{C}G$  is Noetherian if and only if *G* is virtually poly-cyclic.

# Theorem 3.80 (Regular group rings).

- (i) The rings R[t] and  $R[t, t^{-1}]$  are regular if R is regular;
- (ii) The ring RG is regular if R is regular and G is poly- $\mathbb{Z}$ ;
- (iii) The ring RG is regular if R is regular,  $\mathbb{Q} \subseteq R$  and G is virtually poly-cyclic;

*Proof.* (i) This is proved for instance in [860, Theorem 3.2.3 on page 134 and Corollary 3.2.4 on page 136].

(ii) This follows from [880, Theorem 8.2.2 on page 533 and Theorem 8.2.18 on page 537] in the case where R is a field.

(iii) This follows from [880, Theorem 8.2.2 on page 533 and Theorem 8.2.20 on page 538] in the case where R is a field.

A ring is called *semihereditary* if every finitely generated ideal is projective, or, equivalently, if every finitely generated submodule of a projective *R*-module is projective, see [215, Proposition 6.2 in Chapter I.6 on page 15].

**Theorem 3.81 (Bass-Heller-Swan decomposition for**  $K_1$  **for regular rings).** *Suppose that* R *is semihereditary or regular. Then we get* 

$$\widetilde{\mathrm{Nil}}_0(R) = NK_1(R) = 0,$$

and the Bass-Heller-Swan decomposition of Theorem 3.72 reduces to the isomorphism

 $B \oplus i_* \colon K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]).$ 

3 The Whitehead Group

*Proof.* The proof for regular *R* can be found for instance in [860, Exercise 3.2.25 on page 152] or [940, Corollary 16.5 on page 226].

Suppose that *R* is semihereditary. Consider a nilpotent endomorphism  $f: P \to P$  of the finitely generated projective *R*-module *P*. Define  $I_1(f) = \text{im}(f)$  and  $K_1(f) = \text{ker}(f)$ . Let  $f|_{I_1(f)}: I_1(f) \to I_1(f)$  be the endomorphism induced by *f*. Since *R* is semihereditary,  $I_1(f)$  is a finitely generated projective *R*-module. We obtain a commutative diagram

$$0 \longrightarrow K_{1}(f) \xrightarrow{i} P \xrightarrow{f} I_{1}(f) \longrightarrow 0$$
$$\downarrow 0 \qquad \qquad \downarrow f \qquad \qquad \downarrow f I_{I_{1}(f)}$$
$$0 \longrightarrow K_{1}(f) \xrightarrow{i} P \xrightarrow{f} I_{1}(f) \longrightarrow 0$$

with exact rows and nilpotent endomorphisms of finitely generated projective *R*-modules as vertical arrows. Hence we get  $[f: P \rightarrow P] = [I_1(f): I_1(f) \rightarrow I_1(f)]$  in  $\widetilde{\text{Nil}}_0(R)$ . Define inductively  $I_{n+1}(f) = I_1(f|_{I_n(f)})$ . Hence we get for all  $n \ge 1$ 

$$[f: P \to P] = [f|_{I_n(f)}: I_n(f) \to I_n(f)].$$

Since *f* is nilpotent, there exists some *n* with  $I_n(f) = 0$ . This implies [f] = 0 in  $\widetilde{\text{Nil}}_0(R)$ . Now apply Theorem 3.72.

**Exercise 3.82.** Prove that  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^n]) = Wh(\mathbb{Z}^n) = 0$  for all  $n \ge 0$ .

**Remark 3.83 (Glimpse of a homological behavior of** *K***-theory).** In the case when R is regular, Theorem 3.81 imbues a homological flavor into *K*-theory. Just observe the analogy between the two formulas

$$K_1(R[\mathbb{Z}]) \cong K_0(R[\{1\}]) \oplus K_1(R[\{1\}]);$$
  
$$H_1(\mathbb{Z}; A) \cong H_0(\{1\}; A) \oplus H_1(\{1\}; A),$$

where in the second line we consider group homology with coefficients in some abelian group A, which corresponds to the role of R in the first line.

**Remark 3.84 (Von Neumann algebras are semihereditary but not Noetherian).** Note that any von Neumann algebra is semihereditary. This follows from the facts that any von Neumann algebra is a Baer \*-ring and hence in particular a Rickart  $C^*$ -algebra [124, Definition 1, Definition 2 and Proposition 9 in Chapter 1.4] and that a  $C^*$ -algebra is semihereditary if and only if it is Rickart [34, Corollary 3.7 on page 270]. The group von Neumann algebra  $\mathcal{N}(G)$  is Noetherian if and only if *G* is finite, see [650, Exercise 9.11 on page 367].

Lemma 3.85. If R is regular, then the canonical homomorphism

$$f: K_0(R) \xrightarrow{=} G_0(R), [P] \mapsto [P]$$

is a bijection.

*Proof.* We have to define an inverse homomorphism

$$r: G_0(R) \to K_0(R).$$

Given a finitely generated *R*-module *M*, we can choose a finite projective resolution  $P_* = (P_*, \phi)$  since *R* is by assumption regular. We want to define

$$r([M]) := \sum_{n \ge 0} (-1)^n \cdot [P_n].$$

The Fundamental Lemma of Homological Algebra implies for two projective resolutions  $P_*$  and  $Q_*$  of M the existence of an R-chain homotopy equivalence  $f_*: P_* \to Q_*$ , see for instance [997, Comparison Theorem 2.2.6 on page 35]. We conclude from Lemma 2.36 (i)

$$\sum_{n\geq 0} (-1)^n \cdot [P_n] = o(P_*) = o(Q_*) = \sum_{n\geq 0} (-1)^n \cdot [Q_n].$$

Hence the choice of projective resolution does not matter in the definition of r([M]). It remains to show for an exact sequence of finitely generated *R*-modules  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  that r(M) - r(M') + r(M'') = 0 holds. This follows from Lemma 2.36 (ii) since we can construct from finite projective *R*-resolutions  $P_*$  of *M* and  $P''_*$  of *M''* a finite projective *R*-resolution  $P'_*$  of *M* such that there exists a short exact sequence of *R*-chain complexes  $0 \rightarrow P_* \rightarrow P'_* \rightarrow P''_* \rightarrow 0$ , see [644, Lemma 11.6 on page 216]. Hence *r* is well-defined. One easily checks that *r* and *f* are inverse to one another.

# 3.8 The Mayer-Vietoris K-Theory Sequence of a Pullback of Rings

**Theorem 3.86 (Mayer-Vietoris sequence for middle** *K***-theory of a pullback of rings).** *Consider a pullback of rings* 

$$\begin{array}{c|c} R & \xrightarrow{i_1} & R_1 \\ \vdots & & & & \downarrow \\ i_2 & & & & \downarrow \\ R_2 & \xrightarrow{j_2} & R_0 \end{array}$$

such that  $j_1$  or  $j_2$  is surjective. Then there exists a natural exact sequence of six terms

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$$K_1(R) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_1(R_1) \oplus K_1(R_2) \xrightarrow{(j_1)_* - (j_2)_*} K_1(R_0)$$
$$\xrightarrow{\partial_1} K_0(R) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_0(R_1) \oplus K_0(R_2) \xrightarrow{(j_1)_* - (j_2)_*} K_0(R_0)$$

Its construction and its proof requires some preparation. In particular we need the following basic construction due to Milnor [727, page 20]. Let  $\overline{j_k}: P_k \to (j_k)_* P_k$  be the map sending  $x \in P_k$  to  $1 \otimes x \in R_0 \otimes_{j_k} P_k$  for k = 1, 2. Define a ring homomorphism  $i_0 = j_1 \circ i_1 = j_2 \circ i_2: R \to R_0$ . Given  $R_k$ -modules  $P_k$  for k = 0, 1, 2 and isomorphisms of  $R_0$ -modules  $f_k: (j_k)_* P_k \xrightarrow{\cong} P_0$  for k = 1, 2, define an R-module  $M = M(P_1, P_2, f_1, f_2)$  by the pullback of abelian groups



together with the *R*-multiplication on *M* induced by the *R*-actions on  $P_k$  that comes from the ring homomorphisms  $i_k : R \to R_k$  for k = 0, 1, 2.

- **Lemma 3.87.** (i) *The R-module M is projective if*  $P_0$  *and*  $P_1$  *are projective. The R-module M is finitely generated projective if*  $P_0$  *and*  $P_1$  *are finitely generated projective;*
- (ii) Every projective R-module P can be realized up to isomorphism as M for appropriate projective R<sub>k</sub>-modules P<sub>k</sub> for k = 0, 1, 2 and isomorphisms of R<sub>0</sub>-modules f<sub>k</sub>: (j<sub>k</sub>)<sub>\*</sub>P<sub>k</sub> <sup>≅</sup> → P<sub>0</sub> for k = 1, 2;
- (iii) The  $R_k$ -modules  $(i_k)_*M$  and  $P_k$  are isomorphic for k = 1, 2.

*Proof.* This is proved in Milnor [727, Theorems 2.1, 2.2 and 2.3 on page 20] or in [916, Proposition 59 on page 155, Proposition 60 on page 157, Proposition 61 on page 158].

Now we can give the proof of Theorem 3.86

*Proof.* The main step is to construct the boundary homomorphism  $\partial_1$ . Given an element  $x \in K_1(R_0)$ , we can find an automorphism  $f: R_0^n \xrightarrow{\cong} R_0^n$  of a finitely generated free *R*-module with x = [f], see Lemma 3.10. The  $R_0$ -module  $M(R_1^n, R_2^n, \mathrm{id}_{R_0^n}, f)$  is a finitely generated projective  $R_0$ -module by Lemma 3.87 (i). Define

$$\partial_1(x) := [M(R_1^n, R_2^n, \mathrm{id}_{R_0^n}, f)] - [R_0^n].$$

This is a well-defined homomorphism of abelian groups, see [916, page 164]. The elementary proof of the exactness of the sequence of six terms can be found in [916, Proposition 63 on page 164].

Now we are ready to give the promised proof of Rim's Theorem 2.106.

*Proof.* Consider the pullback of rings



where here and in the sequel  $\mathbb{F}_q$  denotes the field with q elements,  $i_1$  sends the generator of  $\mathbb{Z}/p$  to  $\exp(2\pi i/p)$ , the map  $i_2$  sends the generator of  $\mathbb{Z}/p$  to  $1 \in \mathbb{Z}$ , the map  $j_2$  is the projection and the homomorphism  $j_1$  sends  $\exp(2\pi i/p)$  to 1. Obviously  $j_1$  and  $j_2$  are surjective. Hence we get from Theorem 3.86 an exact sequence

$$K_1(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_1(\mathbb{Z}[\exp(2\pi i/p)]) \oplus K_1(\mathbb{Z}) \xrightarrow{(j_1)_* - (j_2)_*} K_1(\mathbb{F}_p) \xrightarrow{\partial_1} K_0(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{(i_1)_* \oplus (i_2)_*} K_0(\mathbb{Z}[\exp(2\pi i/p)]) \oplus K_0(\mathbb{Z}) \xrightarrow{(j_1)_* - (j_2)_*} K_0(\mathbb{F}_p).$$

The map  $(j_2)_*: K_0(\mathbb{Z}) \to K_0(\mathbb{F}_p)$  is bijective by Example 2.4. Hence it remains to prove that  $(j_1)_*: K_1(\mathbb{Z}[\exp(2\pi i/p)]) \to K_1(\mathbb{F}_p)$  is surjective. Because of Theorem 3.17 we have to find for each integer k with  $1 \le k \le p - 1$  a unit  $u \in \mathbb{Z}[\exp(2\pi i/p)]^{\times}$  satisfying  $j_1(u) = \overline{k}$ . Put  $\xi = \exp(2\pi i/p)$ . Choose an integer l such that  $kl = 1 \mod p$ . Define

$$u := 1 + \xi + \xi^{2} + \dots + \xi^{k-1};$$
  
$$v := 1 + \xi^{k} + \xi^{2k} + \dots + \xi^{(l-1)k}.$$

Since  $(\xi - 1)u = \xi^k - 1$  and  $(\xi^k - 1) \cdot v = \xi - 1$  and  $\mathbb{Z}[\exp(2\pi i/p)]$  is an integral domain, we get uv = 1 and hence  $u \in \mathbb{Z}[\exp(2\pi i/p)]^{\times}$ . Obviously  $j_1(u) = \overline{k}$ .  $\Box$ 

# 3.9 The K-Theory Sequence of a Two-Sided Ideal

Let  $I \subseteq R$  be a two-sided ideal in the ring R. The double of the ring R along the ideal I is the subring D(R, I) of  $R \times R$  consisting of pairs  $(r_1, r_2)$  satisfying  $r_1 - r_2 \in I$ . Let  $p_k : D(R, I) \to R$  send  $(r_1, r_2)$  to  $r_k$  for k = 1, 2.

**Definition 3.88** ( $K_n(R, I)$ ). Define for n = 0, 1 the abelian group  $K_n(R, I)$  to be the kernel of the homomorphism

$$(p_1)_*: K_n(D(R, I)) \to K_n(R).$$

**Theorem 3.89** (Exact sequence of a two-sided ideal for middle *K*-theory). We obtain an exact sequence, natural in  $I \subseteq R$ ,

$$K_1(R,I) \xrightarrow{j_1} K_1(R) \xrightarrow{pr_1} K_1(R/I) \xrightarrow{\partial_1} K_0(R,I) \xrightarrow{j_1} K_0(R) \xrightarrow{pr_0} K_0(R/I).$$

*Proof.* We obtain a pullback of rings

$$\begin{array}{c|c} D(R,I) & \xrightarrow{p_1} & R \\ p_2 & & \downarrow \\ R & \xrightarrow{p_r} & R/I \end{array}$$

such that pr is surjective. We get from Theorem 3.86 the exact sequence

$$\begin{split} K_1(D(R,I)) \xrightarrow{(p_1)_* \oplus (p_2)_*} K_1(R) \oplus K_1(R) \xrightarrow{-\operatorname{pr}_* + \operatorname{pr}_*} K_1(R/I) \xrightarrow{\partial} \\ K_0(D(R,I)) \xrightarrow{(p_1)_* \oplus (p_2)_*} K_0(R) \oplus K_0(R) \xrightarrow{-\operatorname{pr}_* + \operatorname{pr}_*} K_0(R/I). \end{split}$$

This yields the desired exact sequence if we define  $j_n: K_n(R, I) \to K_n(R)$  to be the restriction of  $(p_2)_*: K_n(D(R, I)) \to K_n(R)$  to  $K_n(R, I)$  for n = 0, 1 and let  $\partial_1$  be the map induced by  $\partial$ .

Next we give alternative descriptions of  $K_0(R, I)$ .

Let *S* be a ring, but now for some time we do not require that it has a unit. If we want to emphasize that we do not require this, we say that *S* is a ring without unit, although it may have one. The point is that a homomorphism of rings without units  $f: S \rightarrow S'$  is a map compatible with the abelian group structure and the multiplication but no requirement about the unit is made. The *ring obtained from S* by adjoining a unit  $S_+$  has as underlying group  $S \oplus \mathbb{Z}$ . The multiplication is given by

$$(s_1, n_1) \cdot (s_2, n_2) := (s_1 s_2 + n_1 s_2 + n_2 s_1, n_1 n_2).$$

The unit in  $S_+$  is given by (0, 1). We obtain a natural embedding  $i_S : S \to S_+$  by sending *s* to (s, 0). Let  $p_S : S_+ \to \mathbb{Z}$  be the homomorphism of rings with unit sending (s, n) to *n*. We obtain an exact sequence of rings without unit  $0 \to S \xrightarrow{i_S} S_+ \xrightarrow{p_S} \mathbb{Z} \to 0$ . If  $f : S \to S'$  is a homomorphism of rings without unit, we obtain a homomorphism  $f_+ : S_+ \to S'_+$  of rings with unit by sending (s, n) to (f(s), n). If *S* has a unit  $1_S$ , then we obtain an isomorphism of rings with unit  $u_S : S_+ \xrightarrow{\cong} S \times \mathbb{Z}$  by sending (s, n) to  $(s + n \cdot 1_S, n)$ .

**Definition 3.90** ( $K_n(S)$  for rings without unit). Let *S* be a ring without unit. Define for n = 0, 1

$$K_n(S) := \ker ((p_S)_* : K_n(S_+) \to K_n(\mathbb{Z})).$$

Given a homomorphism  $f: S \to S'$  of rings without unit, the homomorphism  $(f_+)_*: K_n(S_+) \to K_n(S'_+)$  induces a homomorphism of abelian groups  $f_*: K_n(S) \to K_n(S')$ . Thus we obtain a covariant functor from the category of rings without unit to the category of abelian groups by sending S to  $K_n(S)$ .

If *S* happens to already have a unit, we get back the old definition (up to natural isomorphism). Namely, the isomorphism  $K_0(u_S): K_n(S_+) \xrightarrow{\cong} K_n(S \times \mathbb{Z})$  sends  $\ker((p_S)_*)$  to the kernel of the map  $(\operatorname{pr}_{\mathbb{Z}})_*: K_n(S \times \mathbb{Z}) \to K_n(\mathbb{Z})$  given by the projection  $\operatorname{pr}_{\mathbb{Z}}: S \times \mathbb{Z} \to \mathbb{Z}$  and the inclusion  $j: S \to S \times \mathbb{Z}$ ,  $s \mapsto (s, 0)$  induces an isomorphism of  $K_n(S)$  to the kernel of the map  $\operatorname{pr}_{\mathbb{Z}}$  by Theorem 2.12 and Theorem 3.9.

**Lemma 3.91.** Let I be a two-sided ideal in the ring R. Let  $K_0(I)$  be the projective class group of the ring I without unit, see Definition 3.90. Then there is a natural isomorphism

$$K_0(I) \xrightarrow{=} K_0(R, I).$$

In particular,  $K_0(R, I)$  depends only on the ring without unit I but not on R.

*Proof.* The isomorphism is induced by the homomorphism of rings with unit  $I_+ \rightarrow D(R, I)$  sending (s, n) to  $(n \cdot 1_R, n \cdot 1_R + s)$ . The proof that it is bijective can be found for instance in [860, Theorem 1.5.9 on page 30].

Exercise 3.92. Let *n* be a positive integer. Compute

$$K_0((n)) \cong \begin{cases} 0 & \text{if } n = 2; \\ (\mathbb{Z}/n)^{\times}/\{\pm 1\} & \text{if } n \ge 3, \end{cases}$$

for the ideal  $(n) = \{mn \mid m \in \mathbb{Z}\} \subseteq \mathbb{Z}$ . Prove for the ideal  $(N_{\mathbb{Z}/2}) \subseteq \mathbb{Z}[\mathbb{Z}/2]$  generated by the norm element that  $(N_{\mathbb{Z}/2})$  and  $(2\mathbb{Z})$  are isomorphic as rings without unit. Conclude

$$K_0(\mathbb{Z}[\mathbb{Z}/2]) = 0$$

Next we give an alternative description of  $K_1(R, I)$ . Define GL(R, I) to be the kernel of the map  $GL(R) \rightarrow GL(R/I)$  induced by the projection  $R \rightarrow R/I$ . Let E(R, I) be the smallest normal subgroup of E(R) that contains all matrices of the shape  $I_n + r \cdot E_{i,j}^n$  for  $n \in \mathbb{Z}, n \ge 1, i, j \in \{1, 2, ..., n\}, i \ne j, r \in I$ . Note that  $E(R, I) \subseteq GL(R, I)$ . The proof of the next result can be found for instance in [860, Theorem 2.5.3 on page 93].

**Theorem 3.93 (Relative Whitehead Lemma).** Let  $I \subseteq R$  be a two-sided ideal. *Then:* 

(i) The subgroup E(R, I) of GL(R) is normal;

(ii) There is an isomorphism, natural in (R, I)

$$\operatorname{GL}(R, I)/\operatorname{E}(R, I) \xrightarrow{=} K_1(R, I);$$

(iii) The center of GL(R)/E(R, I) is GL(R, I)/E(R, I);

(iv) We have E(R, I) = [E(R), E(R, I)] = [GL(R), E(R, I)].

**Example 3.94** ( $K_1(R, I)$  depends on R). In contrast to  $K_0(R, I)$  it is not true that  $K_1(R, I)$  is independent of R, as shown by Swan [942, Section 1]. Let S be a ring and put

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in S \right\};$$
$$R' = \left\{ \begin{pmatrix} n & b \\ 0 & n \end{pmatrix} \mid n \in \mathbb{Z}, b \in S \right\};$$
$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}.$$

Then  $K_1(R, I) = \{0\}$  and  $K_1(R', I) \cong S$ .

**Remark 3.95 (Congruence Subgroup Problem).** Given a commutative ring R, the *Congruence Subgroup Problem* asks if every normal subgroup of GL(R) is of the form  $SL(R, I) := \{A \in GL(R, I) \mid det(A) = 1\}$  for some two-sided ideal  $I \subseteq R$ . Bass has shown that for any normal subgroup  $H \subseteq GL(R)$  there exists an ideal  $I \subseteq R$  satisfying  $E(R, I) \subseteq H \subseteq GL(R, I)$ , see [102, Theorem 2.1 (a) on page 229] or [859, Exercise 2.5.21 on page 106]. Hence the Congruence Subgroup Problem has a positive answer if and only for every two-sided ideal  $I \subseteq R$  we have E(R, I) = SL(R, I), see [859, Exercise 2.5.21 on page 106]. More information about this problem can be found for instance in [106].

**Exercise 3.96.** Show that the Congruence Subgroup Problem has a positive answer for every field *F*.

# 3.10 Swan Homomorphisms

# 3.10.1 The Classical Swan Homomorphism

The definitions and results of this subsection are taken from Swan [938]. This paper marked the beginning of a development that finally leads to a solution of the Spherical Space Form Problem 9.205, which we have also discussed in Section 2.5. It presents a nice and illuminating interaction between geometry, group theory, and algebraic K-theory.

Let *G* be a finite group. Let  $N_G \in \mathbb{Z}G$  be the *norm element*, i.e.,  $N_G := \sum_{g \in G} g$ . Consider the following pullback of rings

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(3.97)

$$\mathbb{Z}G \xrightarrow{i_1} \mathbb{Z}G/(N_G)$$

$$\downarrow j_1$$

$$\mathbb{Z} \xrightarrow{j_2} \mathbb{Z}/|G|$$

where  $(N_G) \subseteq \mathbb{Z}G$  is the ideal generated by  $N_G$ ,  $i_1$  and  $j_2$  are the obvious projections,  $i_2$  is induced by the group homomorphism  $G \to \{1\}$ , and  $j_1$  is the unique ring homomorphism for which the diagram above commutes. One easily checks that it is a pullback and that the maps  $i_1$  and  $j_1$  are surjective. Hence we can apply Theorem 3.86 and obtain a boundary homomorphism  $\partial \colon K_1(\mathbb{Z}/|G|) \to K_0(\mathbb{Z}G)$ . The obvious homomorphism  $i \colon \mathbb{Z}/|G|^{\times} \to K_1(\mathbb{Z}/|G|)$  is an isomorphism by Theorem 3.6, since the commutative finite ring  $\mathbb{Z}/|G|$  is a commutative semilocal ring and hence the determinant det:  $K_1(\mathbb{Z}/|G|) \to \mathbb{Z}/|G|^{\times}$  is an inverse of i.

**Definition 3.98 (Swan homomorphism).** The (*classical*) *Swan homomorphism* is the composite

$$\operatorname{sw}^G \colon \mathbb{Z}/|G|^{\times} \xrightarrow{i} K_1(\mathbb{Z}/|G|) \xrightarrow{\partial} K_0(\mathbb{Z}G).$$

**Lemma 3.99.** Let  $\overline{n} \in \mathbb{Z}/|G|^{\times}$  be an element represented by  $n \in \mathbb{Z}$ . Then the ideal  $(n, N_G) \subseteq \mathbb{Z}G$  generated by n and  $N_G$  is a finitely generated projective  $\mathbb{Z}G$ -module and

$$\operatorname{sw}(\overline{n}) = [(n, N_G)] - [\mathbb{Z}G].$$

*Proof.* Let  $P_1$  be the  $\mathbb{Z}$ -module  $\mathbb{Z}$ ,  $P_2$  be the  $\mathbb{Z}G/(N_G)$ -module  $\mathbb{Z}G/(N_G)$ , and  $P_0$  be the  $\mathbb{Z}/|G|$ -module  $\mathbb{Z}/|G|$ . Consider the automorphism  $r_{\overline{n}} \colon \mathbb{Z}/|G| \to \mathbb{Z}/|G|$  given by multiplication by *n*. Define a  $\mathbb{Z}G$ -module *P* by the pullback

$$P \longrightarrow \mathbb{Z}G/(N_G)$$

$$i_2 \downarrow \qquad \qquad \downarrow r_{\overline{n}} \circ j_1$$

$$\mathbb{Z} \longrightarrow \mathbb{Z}/|G|.$$

One easily checks that the  $\mathbb{Z}G$  map  $(n, N_G) \to \mathbb{Z}$  which sends n to n and  $N_G$  to |G| and the  $\mathbb{Z}G$  map  $(n, N_G) \to \mathbb{Z}G/(N_G)$  which sends n to the class of 1 and  $N_G$  to 0 induce an isomorphism of  $\mathbb{Z}G$ -modules  $(n, N_G) \xrightarrow{\cong} P$ . We conclude from Lemma 3.87 (i) that  $(r, N_G)$  is a finitely generated projective  $\mathbb{Z}G$ -module and that  $\operatorname{sw}(\overline{n}) = [(n, N_G)] - [\mathbb{Z}G].$ 

**Remark 3.100 (Another description of the Swan homomorphism).** For every  $n \in \mathbb{Z}$  with (n, |G|) = 1, the abelian group  $\mathbb{Z}/n$  with the trivial *G*-action is a  $\mathbb{Z}G$ -module that possesses a finite projective resolution  $P_*$ , see [171, Theorem VI.8.12 on page 152]. Since two finite projective resolutions of  $\mathbb{Z}/n$  are  $\mathbb{Z}G$ -chain homotopic, their finiteness obstructions agree, Lemma 2.36 (i). Thus we can define  $[\mathbb{Z}/n] \in K_0(\mathbb{Z}G)$  by  $o(P_*) = \sum_{n \ge 0} (-1)^n \cdot [P_n]$  for any finite projective resolution  $P_*$ . We get

$$\operatorname{sw}(\overline{n}) = -[\mathbb{Z}/n]$$

for any integer  $n \in \mathbb{Z}$  with (n, |G|) = 1. This follows essentially from [938, Lemma 6.2] and Lemma 3.99.

**Exercise 3.101.** Show that  $sw^G$  is trivial for a finite cyclic group G.

# 3.10.2 The Classical Swan Homomorphism and Free Homotopy Representations

Let *G* be a finite group. A *free d-dimensional G-homotopy representation X* is a *d*-dimensional *CW*-complex *X* together with a *G*-action such that for any open cell *e* we have  $ge \cap e \neq \emptyset \Rightarrow g = 1$  and the space *X* is homotopy equivalent to  $S^d$ . Then  $G \setminus X$  is a finitely dominated *CW*-complex, see [644, Lemma 20.2 on page 392]. Let  $f: X \to Y$  be a *G*-map of free *d*-dimensional *G*-homotopy representations for  $d \ge 2$ . Let  $n \ge 0$  be the integer such that the homomorphism of infinite cyclic groups  $H_d(f): H_d(X) \to H_d(Y)$  sends a generator of  $H_d(X)$  to  $\pm n$ -times the generator of  $H_d(Y)$ . Let  $o(G \setminus X), o(G \setminus Y) \in K_0(\mathbb{Z}G)$  be the finiteness obstructions of *X* and *Y* with respect to the obvious identification  $G = \pi_1(X) = \pi_1(Y)$ .

**Lemma 3.102.** Let G be a finite group of order  $\geq 3$ .

(i) The *G*-action on  $H_m(X)$  is trivial for  $m \ge 0$  and *d* is odd; (ii) We have  $n \ge 1$ , (n, |G|) = 1, and

$$sw^G(\overline{n}) = o(G \setminus Y) - o(G \setminus X).$$

*Proof.* (i) Let  $C_*(X)$  be the cellular  $\mathbb{Z}G$ -chain complex. The conditions about the *G*-actions imply that  $C_*(X)$  is a free  $\mathbb{Z}G$ -chain complex and is the same as  $C_*(\overline{G\setminus X})$ . Since  $G\setminus X$  is finitely dominated, we can find a finite projective  $\mathbb{Z}G$ -chain complex  $P_*$  that is  $\mathbb{Z}G$ -chain homotopy equivalent to  $C_*(X)$ , see [644, Proposition 11.11 on page 222] or Subsection 23.7.5. Since  $\mathbb{C}G$  is semisimple, every submodule of a finitely generated  $\mathbb{C}G$ -module is finitely generated projective again. This implies the following equality in  $K_0(\mathbb{C}G) = R_{\mathbb{C}}(G)$ :

$$\sum_{m\geq 0} (-1)^m \cdot [P_m \otimes_{\mathbb{Z}G} \mathbb{C}G] = [H_0(X;\mathbb{C})] + (-1)^d \cdot [H_d(X;\mathbb{C})].$$

The Bass Conjecture for integral domains 2.99 has been proved for finite groups and  $R = \mathbb{Z}$  by Swan [937, Theorem 8.1]. This implies that  $P_n \otimes_{\mathbb{Z}G} \mathbb{C}G$  is a finitely generated free  $\mathbb{C}G$ -module for every *n*. Since  $P_* \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq C_*(G \setminus X)$ , we conclude  $\sum_{m \ge 0} (-1)^m \cdot [P_m \otimes_{\mathbb{Z}G} \mathbb{C}G] = \chi(G \setminus X) \cdot [\mathbb{C}G]$ . Hence we get the following equality in  $R_{\mathbb{C}}(G)$ 

$$\chi(G \setminus X) \cdot [\mathbb{C}G] = [H_0(X;\mathbb{C})] + (-1)^d \cdot [H_d(X;\mathbb{C})].$$

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Obviously  $H_0(X; \mathbb{C})$  is  $\mathbb{C}G$ -isomorphic to the trivial 1-dimensional G-representation  $[\mathbb{C}]$ . Since  $H_d(X) \cong \mathbb{Z}$ , there is a group homomorphism  $w: G \to \{\pm 1\}$  such that  $H_d(X; \mathbb{C})$  is the 1-dimensional G-representation  $\mathbb{C}^w$  for which  $g \in G$  acts by multiplication by w(g). Thus we get in  $R_{\mathbb{C}}(G)$ 

$$\chi(G \setminus X) \cdot [\mathbb{C}G] = [\mathbb{C}] + (-1)^d \cdot [\mathbb{C}^w]$$

Computing the characters on both sides yields the following equalities for  $g \in G$ 

$$\chi(G \setminus X) \cdot |G| = 1 + (-1)^d;$$
  
$$0 = 1 + (-1)^d \cdot w(g) \quad \text{for } g \neq 1.$$

Since we assume  $|G| \ge 3$  and  $\chi(G \setminus X)$  is an integer, the first equality implies that d is odd. The second inequality implies that w(g) = 1 for all  $g \in G$ . Hence G acts trivially on  $H_m(X)$  for all  $m \ge 0$ .

(ii) Let  $C_*(X)$  and  $C_*(Y)$  be the free cellular  $\mathbb{Z}G$ -chain complexes. Choose finite projective  $\mathbb{Z}G$ -chain complexes  $P_*$  and  $Q_*$  together with  $\mathbb{Z}G$ -chain homotopy equivalences  $u_* \colon P_* \to C_*(X)$  and  $v_* \colon Q_* \to C_*(Y)$ . The map  $f \colon X \to Y$  induces a  $\mathbb{Z}G$ -chain map  $C_*(f): C_*(X) \to C_*(Y)$ . Choose a  $\mathbb{Z}G$ -chain map  $h_*: P_* \to Q_*$ satisfying  $v_* \circ h_* \simeq C_*(f) \circ u_*$ . Let cone<sub>\*</sub> = cone<sub>\*</sub>( $h_*$ ) be the mapping cone of  $h_*$ . It is a (d+1)-dimensional free  $\mathbb{Z}G$ -chain complex such that  $H_m(\text{cone}_*) = 0$  for  $m \neq d$ and  $H_d(\operatorname{cone}(C_*(f)))$  is  $\mathbb{Z}G$ -isomorphic to  $\mathbb{Z}/n$  with the trivial G-action. This follows from the long exact homology sequence associated to the short exact sequence of  $\mathbb{Z}G$ -chain complexes  $0 \to Q_* \to \operatorname{cone}(h_*) \to \Sigma P_* \to 0$  and assertion (i). Let  $D_*$  be the  $\mathbb{Z}G$ -chain subchain complex of cone<sub>\*</sub> such that  $D_{d+1} = \operatorname{cone}_{d+1}$ ,  $D_d$  is the kernel of the *d*-th differential of cone<sub>\*</sub> and  $D_k = 0$  for  $k \neq d, d + 1$ . Then  $D_*$  is a projective  $\mathbb{Z}G$ -chain complex and the inclusion  $D_* \to \operatorname{cone}_*$  induces an isomorphism on homology and hence is a  $\mathbb{Z}G$ -chain homotopy equivalence. In particular, we get a short exact sequence  $0 \to D_{d+1} \to D_d \to \mathbb{Z}/n \to 0$ . This excludes n = 0since the cohomological dimension of a non-trivial finite group is  $\infty$ . Suppose that (n, |G|) = 1 is not true. Then we can find a prime number p such that  $\mathbb{Z}/p$  is a subgroup of G and  $\mathbb{Z}/p^l$  is a direct summand in  $\mathbb{Z}/n$  for some  $l \ge 1$ . This implies that the cohomological dimension of the trivial  $\mathbb{Z}[\mathbb{Z}/p]$ -module  $\mathbb{Z}/p^l$  is bounded by 1. An easy computation shows that  $\operatorname{Ext}_{\mathbb{Z}[\mathbb{Z}/p]}^{n}(\mathbb{Z},\mathbb{Z}/p^{l})$  does not vanish for all  $n \geq 2$ , a contradiction. Hence (n, |G|) = 1.

We conclude from Lemma 2.36

$$(-1)^{d} \cdot [\mathbb{Z}/n] = (-1)^{d+1} \cdot [D_{d+1}] + (-1)^{d} \cdot [D_{d}] = o(D_{*}) = o(\operatorname{cone}_{*})$$
$$= [Q_{*}] - [P_{*}] = o(G \setminus Y) - o(G \setminus X).$$

Since d is odd by assertion (i), we conclude  $sw(\overline{n}) = o(G \setminus Y) - o(G \setminus X)$  from Remark 3.100.

**Exercise 3.103.** Let *X* be a free *d*-dimensional *G*-homotopy representation of the finite cyclic group *G*. Then  $G \setminus X$  is homotopy equivalent to a finite *CW*-complex.

#### 3.10.3 The Generalized Swan Homomorphism

In this subsection we briefly introduce the generalized Swan homomorphism. For proofs and more information we refer to [644, Chapter 19].

Fix a finite group G. Let m be its order |G|. We obtain a pullback of rings



Despite the fact that neither the right horizontal arrow nor the lower vertical arrow are surjective, one obtains a long exact sequence, which is an example of a localization sequence

$$(3.104) \quad K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}[1/m]G) \oplus K_1(\mathbb{Z}_{(m)}G) \to K_1(\mathbb{Q}G) \xrightarrow{\sigma} K_0(\mathbb{Z}G) \\ \to K_0(\mathbb{Z}[1/m]G) \oplus K_0(\mathbb{Z}_{(m)}G) \to K_0(\mathbb{Q}G).$$

In the sequel, we denote by  $K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$  the cokernel of the change of rings homomorphism  $K_1(\mathbb{Z}_{(m)}G) \to K_1(\mathbb{Q}G)$ .

**Definition 3.105 (Generalized Swan homomorphism).** The generalized Swan homomorphism

$$\overline{\mathrm{sw}}^G : \mathbb{Z}/m^{\times} \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$$

sends  $\overline{r}$  to the element in  $K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$  that is given by the element in  $K_1(\mathbb{Q}G)$ represented by the  $\mathbb{Q}G$ -automorphism  $r \cdot id \colon \mathbb{Q} \to \mathbb{Q}$  of the trivial  $\mathbb{Q}G$ -module  $\mathbb{Q}$ .

This is well-defined by the argument in [644, page 381]. The following result is taken from [644, Theorem 19.4 on page 381]

**Theorem 3.106 (The generalized Swan homomorphism).** *Let G be a finite group of order m.* 

(i) The composite of the generalized Swan homomorphism

$$\overline{\mathrm{sw}}^G : \mathbb{Z}/m^{\times} \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$$

introduced in Definition 3.105 with the homomorphism

$$\partial: K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G) \to K_0(\mathbb{Z}G)$$

induced by the boundary homomorphism of the localization sequence (3.104) is the classical Swan homomorphism sw<sup>G</sup>:  $\mathbb{Z}/m^{\times} \to K_0(\mathbb{Z}G)$  of Definition 3.98;

(ii) The generalized Swan homomorphism  $\overline{sw}^G : \mathbb{Z}/m^{\times} \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$  is injective.

3.10 Swan Homomorphisms

### 3.10.4 The Generalized Swan Homomorphism and Free Homotopy Representations

In this subsection we briefly discuss Reidemeister torsion for free homotopy representations. For proofs and more information we refer to [644, Chapter 20].

Let *G* be a finite group of order m = |G|. Let *X* be a free *d*-dimensional *G*-homotopy representation. Suppose that we have fixed an orientation, i.e., a generator of  $H_d(X;\mathbb{Z})$ . Then we can define a kind of Reidemeister torsion of *X* 

(3.107) 
$$\overline{\rho}^{G}(X) \in K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$$

as follows. The change of rings map  $\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Z}_{(m)}G)$  is trivial, see [937, Theorem 7.1 and Theorem 8.1]. Hence there is a finite free  $\mathbb{Z}_{(m)}G$ -chain complex  $F_*$ together with a  $\mathbb{Z}_{(m)}G$ -chain homotopy equivalence  $f_* \colon F_* \to C_*(X) \otimes_{\mathbb{Z}G} \mathbb{Z}_{(m)}G$ . Choose a  $\mathbb{Z}_{(m)}G$ -basis for  $F_*$ . Then  $F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G$  is a finite based free  $\mathbb{Q}G$ -chain complex. Note that we have preferred isomorphisms of abelian group  $H_0(X) \cong \mathbb{Z}$ and  $H_d(X) \cong \mathbb{Z}$  and G acts trivially on  $H_0(X)$  and  $H_d(X)$ . This induces preferred  $\mathbb{Q}G$ -isomorphisms  $H_i(F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G) \cong \mathbb{Q}$  for i = 0, d where we equip  $\mathbb{Q}$  with the trivial G-action. This enables us to define a torsion invariant  $\tau(F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G) \in$  $\widetilde{K}_1(\mathbb{Q}G)$  although  $F_* \otimes_{\mathbb{Z}_{(m)}G} \mathbb{Q}G$  is not acyclic. Define  $\overline{\rho}^G(X)$  to be its image under the projection  $\widetilde{K}_1(\mathbb{Q}G) \to K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G)$ . One easily checks that  $\overline{\rho}^G(X)$  is independent of the choice of  $F_*$ ,  $f_*$ , and the choice of the  $\mathbb{Z}_{(m)}G$ -basis for  $F_*$ . The proof of the following result is a special case of the results in [644, Theorem 20.37 on page 403 and Corollary 20.39 on page 404].

**Theorem 3.108 (Torsion and free homotopy representations).** Let G be a finite group of order  $m = |G| \ge 3$ . Let X and Y be free oriented G-homotopy representations.

- (i) The homomorphism  $\overline{\partial}$ :  $K_1(\mathbb{Q}G)/K_1(\mathbb{Z}_{(m)}G) \to K_0(\mathbb{Z}G)$  sends the torsion  $\overline{\rho}^G(X)$  to the finiteness obstruction  $o(G \setminus X)$ ;
- (ii) Let  $f: X \to Y$  be a G-map, which always exists. Then its degree  $\deg(f)$  is prime to m and

$$\overline{\mathrm{sw}}^G(\overline{\mathrm{deg}(f)}) = \overline{\rho}^G(Y) - \overline{\rho}^G(X);$$

(iii) The free *G*-homotopy representations *X* and *Y* are oriented *G*-homotopy equivalent if and only if  $\overline{\rho}^G(X) = \overline{\rho}^G(Y)$ .

Theorem 3.108 gives an interesting relation between torsion invariants and finite obstructions and generalizes the homotopy classification of lens spaces to free *G*-homotopy representations.

All this can be extended to not necessarily free G-homotopy representations, see [644, Section 20]. The theory of G-homotopy representations was initiated by tom Dieck-Petrie [955].

# **3.11** Variants of the Farrell-Jones Conjecture for $K_1(RG)$

In this section we state variants of the Farrell-Jones Conjecture for  $K_1(RG)$ . The Farrell-Jones Conjecture itself will give a complete answer for arbitrary rings but to formulate the full version some additional effort will be needed. If one assumes that *R* is regular and *G* is torsionfree, the conjecture reduces to an easy to formulate statement, which we will present next. Moreover, this special case is already very interesting.

**Conjecture 3.109 (Farrell-Jones Conjecture for**  $K_0(RG)$  **and**  $K_1(RG)$  **for regular** R **and torsionfree** G). Let G be a torsionfree group and let R be a regular ring. Then the maps defined in (3.26) and (3.27)

$$A_0 \colon K_0(R) \xrightarrow{\cong} K_0(RG);$$
  
$$A_1 \colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(RG),$$

are both isomorphisms. In particular the groups  $Wh_0^R(G)$  and  $Wh_1^R(G)$  introduced in Definition 3.28 vanish.

We mention the following important special case of Conjecture 3.109.

**Conjecture 3.110 (Farrell-Jones Conjecture for**  $\widetilde{K}_0(\mathbb{Z}G)$  and Wh(G) for torsionfree *G*). Let *G* be a torsionfree group. Then  $\widetilde{K}_0(\mathbb{Z}G)$  and Wh(G) vanish.

We have already discussed the  $K_0$ -part of the two conjectures above in Section 2.8. The following exercise shows that we cannot expect to have an analog for  $K_1(RG)$  of the Conjecture 2.67.

Exercise 3.111. Let G be a group and let R be a ring. Suppose that the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{T}\mathcal{N}}(G \times \mathbb{Z})} K_1(RH) \to K_1(R[G \times \mathbb{Z}])$$

is surjective. Show that then  $K_0(RG) = 0$  and hence  $K_0(R) = 0$ . In particular, *R* cannot be a commutative integral domain.

**Remark 3.112 (Relevance of Conjecture 3.110).** In view of Remark 3.13 Conjecture 3.110 predicts for a torsionfree group *G* that any matrix *A* in  $GL_n(\mathbb{Z}G)$  can be transformed by a sequence of the operations mentioned in Remark 3.13 to a (1, 1)-matrix of the form  $(\pm g)$  for some  $g \in G$ . This is the algebraic relevance of this conjecture. Its geometric meaning comes from the following conclusion of the *s*-Cobordism Theorem 2.39. Namely, if *G* is a finitely presented torsionfree group, and *n* an integer with  $n \ge 6$ , then it implies that every compact *n*-dimensional *h*-cobordism is trivial.

# **3.12** Survey on Computations of $K_1(\mathbb{Z}G)$ for Finite Groups

In contrast to  $K_0(\mathbb{Z}G)$  for finite groups *G*, the Whitehead group Wh(*G*) of a finite group is very well understood. The key source for the computation of Wh(*G*) for finite groups *G* is the book written by Oliver [776].

**Definition 3.113** ( $SK_1(\mathbb{Z}G)$ ) and Wh'(G)). Let G be a finite group. Define

$$SK_1(\mathbb{Z}G) := \ker((K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G));$$
  
Wh'(G) = Wh(G)/tors(Wh(G)).

**Remark 3.114** ( $SK_1(\mathbb{Z}G)$ ) and reduced norms). Let *G* be a finite group. The *reduced norm* on  $\mathbb{C}G$  is defined as the composite of isomorphisms of abelian groups

$$\operatorname{nr}_{\mathbb{C}G} \colon K_1(\mathbb{C}G) \xrightarrow{\phi_*} K_1\left(\prod_{i=1}^k M_{r_i}(\mathbb{C})\right) \xrightarrow{\cong} \prod_{i=1}^k K_1(M_{r_i}(\mathbb{C}))$$
$$\xrightarrow{\cong} \prod_{i=1}^k K_1(\mathbb{C}) \xrightarrow{\prod_{i=1}^k \det} \prod_{i=1}^k \mathbb{C}^{\times}$$

where the isomorphism of rings  $\phi \colon \mathbb{C}G \xrightarrow{\cong} \prod_{i=1}^{k} M_{r_i}(\mathbb{C})$  comes from Wedderburn's Theorem applied to the semisimple ring  $\mathbb{C}G$  and the remaining three isomorphisms come from Theorem 3.6, Lemma 3.8, and Lemma 3.9. The *reduced norm* on *RG* for  $R = \mathbb{Z}, \mathbb{Q}$  is defined as the composite

$$\operatorname{nr}_{RG} \colon K_1(RG) \xrightarrow{i_R} K_1(\mathbb{C}G) \xrightarrow{\operatorname{nr}_{\mathbb{C}G}} \prod_{i=1}^k \mathbb{C}^{\times}$$

where  $i_R$  is the obvious change of rings homomorphism. The map  $i_Q$  is injective, see [776, Theorem 2.5 on page 43]). Thus we can identify

$$SK_1(\mathbb{Z}G) = \ker\left(\operatorname{nr}_{\mathbb{Z}G} \colon K_1(\mathbb{Z}G) \to \prod_{i=1}^k \mathbb{C}^{\times}\right).$$

This identification is useful for investigating  $SK_1(\mathbb{Z}G)$  and Wh'(G). We conclude that for abelian groups the two definitions of  $SK_1(\mathbb{Z}G)$  appearing in Definition 3.15 and Definition 3.113 agree.

We denote by  $r_F(G)$  the number of isomorphism classes of irreducible representations of the finite *G* over the field *F*. Recall that  $r_F = |\operatorname{con}_F(G)|$  by Lemma 2.89. The proof of the next result can be found for instance in [776, Theorem 2.5 on page 48] and is based on the Dirichlet Unit Theorem 3.21.

**Theorem 3.115** ( $SK_1(\mathbb{Z}G)$  = tors(Wh(G))). Let G be a finite group. Then the abelian group  $SK_1(\mathbb{Z}G)$  is finite and agrees with the torsion subgroup tors(Wh(G)) of Wh(G). The group Wh'(G) = Wh(G)/tors(Wh(G)) is a finitely generated free abelian group of rank  $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$ .

Hence the next step is to compute  $SK_1(\mathbb{Z}G)$ . This is done using localization sequences, see [776, Theorem 1.17 on page 36 and Section 3c], which also involve the second algebraic *K*-group, see Chapter 5, and are consequences of the general result of Quillen stated in Theorem 6.49. Define

$$SK_1(\mathbb{Z}_p^{\widehat{G}}G) := \ker \left( K_1(\mathbb{Z}_p^{\widehat{G}}G) \to K_1(\mathbb{Q}_p^{\widehat{G}}G) \right).$$

Put

$$\operatorname{Cl}_1(\mathbb{Z}G) := \ker\left(SK_1(\mathbb{Z}G) \to \prod_{p \mid |G|} SK_1(\mathbb{Z}_p^{\widehat{G}}G)\right)$$

where p runs over all prime numbers dividing |G|. Then one obtains an exact sequence, see [776, (2) on page 7],

$$0 \to \operatorname{Cl}_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to \prod_{p \mid |G|} SK_1(\mathbb{Z}_p^{\widehat{}}G) \to 0.$$

The analysis of  $\operatorname{Cl}_1(\mathbb{Z}G)$  and  $SK_1(\mathbb{Z}_pG)$  is carried out independently and with different methods. Besides localization sequences *p*-adic logarithms play a key role. Details can be found in Oliver [776].

Given groups G and Q, the wreath product  $G \wr Q$  is defined to be the semidirect product  $\prod_Q G \rtimes Q$  where Q acts on  $\prod_Q G \rtimes Q$  permuting the factors.

**Theorem 3.116 (Finite groups with vanishing** Wh(G) **or**  $SK_1(\mathbb{Z}G)$ ). Let G be a *finite group*.

- (i) Let p be a prime number. If the p-Sylow subgroup S<sub>p</sub>G of G is isomorphic to Z/p<sup>n</sup> or Z/p<sup>n</sup> × Z/p for some n ≥ 0, then SK<sub>1</sub>(ZG)<sub>(p)</sub> = 0, i.e., the finite abelian group SK<sub>1</sub>(ZG) contains no p-torsion;
- (ii) Let G be a finite abelian group. Then  $SK_1(\mathbb{Z}G) = 0$  if and only if one of the following conditions hold:
  - (a) For every prime p the p-Sylow subgroup  $S_pG$  is isomorphic to  $\mathbb{Z}/p^n$  or  $\mathbb{Z}/p^n \times \mathbb{Z}/p$  for some  $n \ge 0$ ;
  - (b) We have  $G = (\mathbb{Z}/2)^n$  for some  $n \ge 1$ ;
- (iii) Let  $C_{Wh}$  be the smallest class of groups that is closed under finite products and wreath products with  $S_n$  for every  $n \ge 2$  and contains the trivial group. Let  $C_{SK_1}$  be the smallest class of groups that is closed under finite products and wreath products with  $S_n$  for every  $n \ge 2$  and contains the dihedral groups  $D_{2n}$ for  $n \ge 1$ .

Then Wh(G) = 0 for  $G \in C_{Wh}$  and  $SK_1(\mathbb{Z}G) = 0$  if  $G \in C_{SK_1}$ ;

3.12 Survey on Computations of  $K_1(\mathbb{Z}G)$  for Finite Groups

(iv) We have  $SK_1(\mathbb{Z}G) = 0$  if G is one of the following groups:

- (a) G is finite cyclic;
- (b)  $\mathbb{Z}/p^n \times \mathbb{Z}/p$  for some prime p and  $n \ge 1$ ;
- (c)  $(\mathbb{Z}/2)^n$  for  $n \ge 1$ ;
- (d) *G* is any symmetric group;
- (e) G is any dihedral group;
- (f) G is any semidihedral 2-group.

*Proof.* (i) See Oliver [776, Theorem 14.2 (i) on page 330].

- (ii) See Oliver [776, Theorem 14.2 (iii) on page 330].
- (iii) See Oliver [776, Theorem 14.1 on page 328].

(iv) This follows essentially from the other assertions. See Oliver [776, Examples 1 and 2 on page 14].

The group  $SK_1(\mathbb{Z}G)$  can be computed for many examples. We mention the following example taken from [776, Theorem 14.6 on page 336].

**Example 3.117** ( $SK_1(\mathbb{Z}[A_n])$ ). We have  $SK_1(\mathbb{Z}[A_n]) \cong \mathbb{Z}/3$  if we can write  $n = \sum_{i=1}^r 3^{m_i}$  such that  $m_1 > m_2 > \cdots > m_r > 0$  and  $\sum_{i=1}^r m_i$  is odd. Otherwise we get  $SK_1(\mathbb{Z}[A_n]) = \{0\}$ .

**Exercise 3.118.** Show that the Whitehead group  $Wh(\mathbb{Z}/m)$  of the finite cyclic group  $\mathbb{Z}/m$  of order *m* is a free abelian group of rank  $\lfloor m/2 \rfloor + 1 - \delta(m)$ , where  $\lfloor m/2 \rfloor$  is the greatest integer less or equal to m/2 and  $\delta(m)$  is the number of divisors of *m*.

Let p be a prime. Show that  $Wh(\mathbb{Z}/p)$  is isomorphic to  $\mathbb{Z}^{(p-1)/2-1}$  if p is odd and is trivial if p = 2.

**Exercise 3.119.** Find the finite abelian group of smallest order for which Wh(G) is finite and non-trivial.

The following result taken from [776, Theorem 14.5 on page 333] is rather puzzling. If p is any prime,  $k \ge 1$  is any natural number, and  $\mathbb{F}_{p^k}$  is the finite field with  $p^k$  elements, then

$$SK_1(\mathbb{Z}[SL_2(\mathbb{F}_{p^k})]) \cong \begin{cases} \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } p = 3, k \text{ is odd, and } k \ge 5; \\ \{0\} & \text{otherwise.} \end{cases}$$

The standard involution on  $\mathbb{Z}G$  sending  $\sum_{g \in G} \lambda_g \cdot g$  to  $\sum_{g \in G} \lambda_g \cdot g^{-1}$  induces an involution \*: Wh(*G*)  $\rightarrow$  Wh(*G*). If *G* is a finite group, then the induced involution on Wh'(*G*) is trivial by a result of Wall, see [776, Corollary 7.5 on page 182]. Computation of the induced involution on  $SK_1(\mathbb{Z}G)$  can be found in [776], e.g., the involution induced on Cl<sub>1</sub>( $\mathbb{Z}G$ )  $\subseteq SK_1(\mathbb{Z}G)$  is the identity, see [776, Theorem 5.12 on page 151] or [61]. Note that it is not true that the involution on Wh(*G*) is trivial for all finite groups *G*.

# 3.13 Survey on Computations of Algebraic $K_1(C_r^*(G))$ and $K_1(\mathcal{N}(G))$

Define  $SL(R) := \{A \in GL(R) \mid \det(A) = 1\}$ . Let *B* be a commutative Banach algebra. Then  $GL_n(B)$  inherits a topology, namely, the subspace topology for the obvious embedding  $GL_n(B) \subseteq M_n(B) = \prod_{i=1}^{n^2} B$ . Equip  $GL(B) = \bigcup_{n \ge 1} GL_n(B)$  with the weak topology, i.e., a subset  $A \subset GL(B)$  is closed if and only if  $A \cap GL_n(B)$  is a closed subset of  $GL_n(B)$  for all  $n \ge 1$ . Equip  $SL(B) \subseteq GL(B)$  with the subspace topology.

The following results are due to Milnor [727, Corollary 7.2 on page 57 and Corollary 7.3 on page 58].

**Theorem 3.120** ( $K_1(B)$  of a commutative Banach algebra). Let B be a commutative Banach algebra. Then there is a natural isomorphism

$$K_1(B) \xrightarrow{\cong} B^{\times} \times \pi_0(\mathrm{SL}(B)).$$

Define the *infinite special orthogonal group* SO =  $\bigcup_{n\geq 1}$  SO(*n*) and *infinite special unitary group* SU =  $\bigcup_{n\geq 1}$  SU(*n*) for SO(*n*) = { $A \in GL_n(\mathbb{R}) \mid AA^t = I$ , det(A) = 1} the special *n*-th orthogonal group and SU(*n*) = { $A \in GL_n(\mathbb{C}) \mid AA^* = I$ , det(A) = 1} the special *n*-th unitary group. Denote by [X, SO] and [X, SU] respectively the set of homotopy classes of maps from X to SO and SU respectively.

**Theorem 3.121** ( $K_1(C(X))$ ) of a commutative  $C^*$ -algebra C(X)). Let X be compact space. Then there are natural isomorphisms

$$K_1(C(X,\mathbb{R})) \xrightarrow{\cong} C(X,\mathbb{R})^{\times} \times [X, \mathrm{SO}];$$
  
$$K_1(C(X,\mathbb{C})) \xrightarrow{\cong} C(X,\mathbb{C})^{\times} \times [X, \mathrm{SU}].$$

The sets [X, SO] and [X, SU] are closely related to the topological K-groups  $KO^{-1}(X)$  and  $K^{-1}(X)$ .

If *B* is a group *C*<sup>\*</sup>-algebra  $C_r^*(G)$ , then not much is known about the algebraic *K*-group  $K_1(B)$  in general. At least we mention [345, Remark 1.3], where it is shown that for a simple infinite unital *C*<sup>\*</sup>-algebra *B* the canonical map from the algebraic to the topological  $K_1$ -group is bijective. An example for *B* is  $C_r^*(F_g)$  for the free group  $F_g$  of rank *g* for  $g \ge 2$ .

However, the algebraic  $K_1$ -group of a von Neumann algebra is fully understood, see [650, Section 9.3],[678]. We mention the special case, see [650, Example 9.34 on page 353], that for a finitely generated group *G* which is not virtually finitely generated abelian the Fuglede-Kadison determinant induces an isomorphism

(3.122) 
$$K_1(\mathcal{N}(G)) \xrightarrow{=} \mathcal{Z}(\mathcal{N}(G))^{+,\mathrm{inv}}$$

where  $\mathcal{Z}(\mathcal{N}(G))^{+,\text{inv}}$  consists of the elements of the center of  $\mathcal{N}(G)$  that are both positive and invertible.

### 3.14 Notes

The connection between the algebraic and the topological K-theory of a  $C^*$ -algebra will be discussed in Section 10.7.

# **3.14 Notes**

A universal property describing the Whitehead group and the Whitehead torsion similar to the description of the finiteness obstruction in Section 2.7 is presented in [644, Theorem 6.11].

Geometric versions or analogs of maps related to the Bass-Heller-Swan decomposition are described in [350], [380], [644, (7.34) on page 130], and [840, § 10].

Given two groups  $G_1$  and  $G_2$ , let  $G_1 * G_2$  by the amalgamated free product. Then the natural maps  $G_k \rightarrow G_0 * G_1$  for k = 1, 2 induce an isomorphism, see [924],

$$(3.123) \qquad \qquad \mathsf{Wh}(G_1) \oplus \mathsf{Wh}(G_2) \cong \mathsf{Wh}(G_1 * G_2).$$

Compare this with the analog for the reduced projective class groups stated in (2.126).

**Exercise 3.124.** Show that the projections  $pr_k: G_1 \times G_2 \rightarrow G_k$  for k = 1, 2 do *not* in general induce an isomorphism

$$\operatorname{Wh}(G_1 \times G_2) \xrightarrow{\cong} \operatorname{Wh}(G_1) \times \operatorname{Wh}(G_2).$$

There are also equivariant versions of the Whitehead torsion, see for instance [644, Chapter 4 and Chapter 12], where more references can be found.

Next we discuss the following conjecture.

**Conjecture 3.125 (Unit-Conjecture).** Let *R* be an integral domain and *G* be a torsionfree group. Then every unit in *RG* is trivial, i.e., of the form  $r \cdot g$  for some unit  $r \in R^{\times}$  and  $g \in G$ .

For more information about it we refer for instance to [610, page 95]. We have discussed its relations to some other conjectures already in Remark 2.85.

**Remark 3.126 (Status of the Unit Conjecture and its stable version).** Actually, Gardam found an explicit counterexample to the Unit Conjecture, see [417, Theorem A]. His group G is given by the presentation

$$\langle a, b \mid ba^2b^{-1} = a^{-2}, ab^2a^{-1} = b^{-2} \rangle.$$

It can be written as a non-split extension  $1 \to \mathbb{Z}^3 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$  and is a crystallographic group. The underlying coefficient ring is the field of two elements  $\mathbb{F}_2$ . Note that Gardam found his counterexample using computer algebra, but in his paper he presents a short human-readable proof. Counterexamples where the coefficient ring is a field of (arbitrary) prime characteristic were constructed by Murray [755]. Gardam [418, Theorem A] constructed counterexamples with coefficients in  $\mathbb{C}$  for the same group *G* as above.

Note that Conjecture 3.109 does not imply the Unit Conjecture 3.125. At least the bijectivity of the map  $A_1$  implies the stable version of the Unit Conjecture 3.125 that the class  $[x] \in K_1(RG)$  of any unit  $x \in RG^{\times}$  is represented by the class [u] of some trivial unit u, or, equivalently, by a sequence of elementary row and column operation and (de-)stabilization one can transform the (1, 1)-matrix (x) to the (1, 1)-matrix (u), see Remark 3.13, provided that  $\widetilde{K}_0(R)$  vanishes.

Note, that the map  $(\mathbb{Z}G)^{\times} \to K_1(\mathbb{Z}G)$  sending a unit to its class in the  $K_1$ -group is in general not injective and in general not every unit is a trivial unit, as the following example shows. If *G* is a finite group, then a result of Hartley-Pickel [468, Theorem 2] says that exactly one of the following cases occurs:

- *G* is abelian and  $(\mathbb{Z}G)^{\times}$  is abelian;
- *G* is a Hamiltonian 2-group and  $(\mathbb{Z}G)^{\times} = \{\pm g \mid g \in G\};$
- $(\mathbb{Z}G)^{\times}$  contains a free subgroup of rank 2.

Hence for the symmetric group  $S_n$  for  $n \ge 3$ , the group of units  $\mathbb{Z}[S_n]^{\times}$  is infinite, whereas  $Wh(S_n)$  vanishes, see Theorem 3.116 (iii), and hence  $K_1(\mathbb{Z}[S_n])$  and  $\{\pm g \mid g \in S_n\}$  are finite. This implies that the map  $(\mathbb{Z}[S_n])^{\times} \to K_1(\mathbb{Z}[S_n])$  has an infinite kernel for  $n \ge 3$  and that there are infinitely many elements in  $(\mathbb{Z}[S_n])^{\times}$  which are not trivial units.

# Chapter 4 Negative Algebraic *K*-Theory

# 4.1 Introduction

In this chapter we introduce *negative K-groups*. They are designed such that the *Bass-Heller-Swan decomposition* and the long exact sequence of a pullback of rings and of a two-sided ideal extend beyond  $K_0$ . We give a geometric interpretation of negative *K*-groups of group rings in terms of bounded *h*-cobordisms. We state variants of the *Farrell-Jones Conjecture for negative K-groups* and give a survey of computations for group rings of finite groups.

# 4.2 Definition and Basic Properties of Negative K-Groups

Recall that we get from Theorem 3.75 an isomorphism

$$K_0(R) = \operatorname{coker} \left( K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \right).$$

This motivates the following definition of negative K-groups due to Bass.

**Definition 4.1.** Given a ring *R*, define inductively for n = -1, -2, ...

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t,t^{-1}])\right).$$

Define for n = -1, -2, ...

$$NK_n(R) := \operatorname{coker} (K_n(R) \to K_n(R[t])).$$

Obviously a ring homomorphism  $f : R \to S$  induces for  $n \le -1$  a map of abelian groups

(4.2) 
$$K_n(f): K_n(R) \to K_n(S).$$

The Bass-Heller-Swan decomposition 3.72 for  $K_1(R[t, t^{-1}])$  extends to negative *K*-theory.

4 Negative Algebraic K-Theory

**Theorem 4.3 (Bass-Heller-Swan decomposition for middle and lower** *K***-theory).** *There are isomorphisms of abelian groups, natural in R, for* n = 1, 0, -1, -2, ...

$$\begin{split} NK_n(R) \oplus K_n(R) &\stackrel{\cong}{\to} K_n(R[t]);\\ K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) &\stackrel{\cong}{\to} K_n(R[t,t^{-1}]). \end{split}$$

There is a sequence which is natural in R and exact for n = 1, 0, -1, ...

$$0 \to K_n(R) \xrightarrow{K_n(k_+) \oplus -K_n(k_-)} K_n(R[t]) \oplus K_n(R[t^{-1}])$$
$$\xrightarrow{K_n(l_+) \oplus K_n(l_-)_*} K_n(R[t,t^{-1}]) \xrightarrow{C_n} K_{n-1}(R) \to 0$$

where  $k_+$ ,  $k_-$ ,  $l_+$ , and  $l_-$  are the obvious inclusions.

If we regard it as an acyclic  $\mathbb{Z}$ -chain complex, there exists a chain contraction, natural in R.

*Proof.* We give the proof only for n = 0, then an iteration of the argument proves the claim for all  $n \le 0$ . Take  $S = R[\mathbb{Z}] = R[x, x^{-1}]$ . We obtain a commutative diagram



where the right column is the exact sequence appearing in Theorem 3.75, the map C' is the canonical projection, the maps  $f_1$ ,  $f_2$ , and  $f_3$  come from the Bass-Heller-Swan decompositions for  $S = R[x, x^{-1}]$ ,  $S[t] = R[t][x, x^{-1}]$ ,  $S[t^{-1}] = R[t^{-1}][x, x^{-1}]$ , and  $S[t, t^{-1}] = R[t, t^{-1}][x, x^{-1}]$ , and the map  $f_0$  is the unique map that makes the diagram commutative. There are natural retractions  $r_k$  of  $f_k$  for k = 1, 2, 3 for which the diagram remains commutative, and a natural chain contraction  $\gamma = \{\gamma_k \mid k = 0, 1, 2\}$  of the right column, see Theorem 3.72. Let  $r_0: K_0(S) \to K_{-1}(R)$  be the unique map that satisfies  $r_0 \circ C = C' \circ r_1$ . An easy diagram shows that  $r_0$  is well-

defined, since  $C' \circ r_3$  sends the kernel of *C* to zero. One easily checks  $r_0 \circ f_0 = id$ . We obtain a chain contraction for the left column by considering the composites  $r_{k+1} \circ \gamma_k \circ f_k$  for k = 0, 1, 2.

**Remark 4.4 (Extending exact sequences to negative** *K***-theory).** The Mayer-Vietoris sequence of a pullback of rings, see Theorem 3.86, can be extended to negative *K*-theory and also to  $K_2$ , as we will explain in Theorem 5.9. Similarly, the long exact sequence of a two-sided ideal appearing in Theorem 3.89 can be extended to negative *K*-theory and also to  $K_2$ , as we will explain in Theorem 5.12.

**Exercise 4.5.** Let *R* and *S* be rings. Show for  $n \le 1$  that the projections induce an isomorphism

$$K_n(R \times S) \xrightarrow{=} K_n(R) \times K_n(S).$$

**Definition 4.6.** Define for  $n \le 1$  inductively for p = 0, 1, 2, ...

$$N^{0}K_{n}(R) := K_{n}(R);$$
  

$$N^{p+1}K_{n}(R) := \operatorname{coker} \left(N^{p}K_{n}(R) \to N^{p}K_{n}(R[t])\right).$$

Obviously  $N^1 K_n(R)$  agrees with  $NK_n(R)$ .

**Theorem 4.7 (Bass-Heller-Swan decomposition for lower and middle** *K***-theory for regular rings).** Suppose that *R* is regular. Then we get

$$K_n(R) = 0 \quad \text{for } n \le -1;$$
  

$$N^p K_n(R) = 0 \quad \text{for } n \le 1 \text{ and } p \ge 1,$$

and the Bass-Heller-Swan decomposition appearing in Theorem 4.3 reduces for  $n \leq 1$  to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]).$$

*Proof.* The Bass-Heller-Swan decomposition, see Theorem 4.3, applied to R and R[t] together with the obvious maps  $i: R \to R[t]$  and  $\epsilon: R[t] \to R$  satisfying  $\epsilon \circ i = id_R$  yield a natural Bass-Heller-Swan decomposition

$$(4.8) NK_n(R) \oplus NK_{n-1}(R) \oplus N^2K_n(R) \oplus N^2K_n(R) \xrightarrow{=} NK_n(R[\mathbb{Z}])$$

Hence  $NK_{n-1}(R) = 0$  if  $NK_n(R[\mathbb{Z}]) = 0$ . If *R* is regular, then  $R[\mathbb{Z}]$  is regular by Theorem 3.80 (i). Hence  $NK_{n-1}(R)$  vanishes for all regular rings *R* if  $NK_n(R)$ vanishes for all regular rings. We have shown in Theorem 3.81 that  $NK_1(R)$  vanishes for all regular rings *R*. We conclude by induction over *n* that  $NK_n(R)$  vanishes for all regular rings *R* and  $n \le 1$ . Obviously  $N^p K_n(R)$  is a direct summand in  $NK_n(R[t])$ and R[t] is regular by Theorem 3.80 (i). Hence  $N^p K_n(R)$  vanishes for  $p \ge 1$  and  $n \le 1$  if *R* is a regular ring. Next we show  $K_{-1}(R) = 0$  for every regular ring R. It suffices to show that the obvious map  $K_0(R[t]) \rightarrow K_0(R[t, t^{-1}])$  is surjective. The homomorphism

$$\alpha: G_0(R[t]) \to G_0(R[t, t^{-1}]), \quad [M] \to [M \otimes_{R[t]} R[t, t^{-1}]]$$

is well-defined, since  $R[t, t^{-1}]$  is a localization of R[t] and hence flat as an R[t]module. Since R by assumption and hence R[t] and  $R[t, t^{-1}]$  by Theorem 3.80 (i) are regular, we conclude from Lemma 3.85 that it remains to prove surjectivity of  $\alpha$ . Let M be a finitely generated  $R[t, t^{-1}]$ -module. Since  $R[t, t^{-1}]$  is Noetherian, we can find a matrix  $A \in M_{m,n}(R[t, t^{-1}])$  such that there exists an exact sequence of  $R[t, t^{-1}]$ -modules  $R[t, t^{-1}]^m \xrightarrow{A} R[t, t^{-1}]^n \to M \to 0$ . Since t is invertible in  $R[t, t^{-1}]$ , the sequence remains exact if we replace A by  $t^k A$  for some  $k \ge 1$ . Hence we can assume without loss of generality that  $A \in M_{m,n}(R[t])$ . Define the R[t]-module N to be the cokernel of  $R[t]^m \xrightarrow{A} R[t]^n$ . Then  $N \otimes_{R[t]} R[t, t^{-1}]$  is  $R[t, t^{-1}]$ -isomorphic to M and hence  $\alpha([N]) = [M]$ .

Now  $K_n(R) = 0$  follows inductively for  $n \le -1$  for every regular ring from Theorem 3.80 (i) and the Bass-Heller-Swan decomposition 4.3.

Finally apply Theorem 4.3.

Exercise 4.9. Let *R* be a regular ring. Prove

$$K_1(R[\mathbb{Z}^k]) = K_1(R) \oplus \bigoplus_{i=1}^k K_0(R);$$
  

$$K_0(R[\mathbb{Z}^k]) \cong K_0(R);$$
  

$$K_n(R[\mathbb{Z}^k]) \cong 0 \quad \text{for } n \le -1.$$

**Example 4.10**  $(K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  for  $n \leq 0$  and a prime p). Let p be a prime number. We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$$
 for  $n \leq -1$  and  $k \geq 0$ 

and that  $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is finitely generated for  $k \ge 0$ . Consider the pullback of rings appearing in the proof of Rim's Theorem in Section 3.8.



If we apply  $- \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^k]$ , we obtain the pullback of rings

4.2 Definition and Basic Properties of Negative K-Groups



The ring  $\mathbb{Z}[\exp(2\pi i/p)]$  is a Dedekind domain, see Theorem 2.23, and in particular regular. The rings  $\mathbb{Z}$  and  $\mathbb{F}_p$  are regular as well. Hence the rings  $\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]$ ,  $\mathbb{Z}[\mathbb{Z}^k]$ , and  $\mathbb{F}_p[\mathbb{Z}^k]$  are regular by Theorem 3.80 (i). The negative *K*-groups of  $\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]$ ,  $\mathbb{Z}[\mathbb{Z}^k]$  and  $\mathbb{F}_p[\mathbb{Z}^k]$  and  $\mathbb{F}_p[\mathbb{Z}^k]$  vanish by Theorem 4.7. The obvious maps

$$K_0(\mathbb{Z}) \xrightarrow{\cong} K_0(\mathbb{Z}[\mathbb{Z}^k]);$$
  

$$K_0(\mathbb{Z}[\exp(2\pi i/p)]) \xrightarrow{\cong} K_0(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]);$$
  

$$K_0(\mathbb{F}_p) \xrightarrow{\cong} K_0(\mathbb{F}_p[\mathbb{Z}^k]),$$

are bijective because of Theorem 4.7. Hence the associated long exact Mayer-Vietoris sequence, see Remark 4.4, implies that  $K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$  holds for  $n \leq -2$  and that we get the exact sequence

$$K_1(\mathbb{F}_p[\mathbb{Z}^k]) \to K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$$
  
 
$$\to K_0(\mathbb{Z}) \oplus K_0(\mathbb{Z}[\exp(2\pi i/p)]) \to K_0(\mathbb{F}_p) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to 0.$$

Since  $\mathbb{F}_p$  is a field and hence  $K_0(\mathbb{F}_p)$  is generated by  $[\mathbb{F}_p]$ , see Example 2.4, we conclude  $K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$ . Example 2.4, Theorem 3.17, and Theorem 4.7 imply  $K_1(\mathbb{F}_p[\mathbb{Z}^k]) \cong K_1(\mathbb{F}_p) \oplus K_0(\mathbb{F}_p)^k \cong (\mathbb{F}_p)^{\times} \oplus \mathbb{Z}^k$ . The abelian group  $K_0(\mathbb{Z}) \oplus K_0(\mathbb{Z}[\exp(2\pi i/p)])$  is finitely generated by Theorem 2.23. Hence  $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is finitely generated.

**Exercise 4.11.** Consider  $k \in \{0, 1, 2, ...\}$ . Show  $\widetilde{K}_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$  for  $n \le 0$ . Prove that  $N^p K_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$  holds for  $n \le -1$  and  $p \ge 0$  and for n = 0 and  $p \ge 1$ .

**Example 4.12** (Negative *K*-theory of  $\mathbb{Z}[\mathbb{Z}/6]$ ). We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z} & n = -1; \\ 0 & n \le -2. \end{cases}$$

Consider the pullback of rings



where  $i_1$  sends a + bt to a - b and  $i_2$  sends a + bt to a + b for  $t \in \mathbb{Z}/2$  the generator and the two maps from  $\mathbb{Z}$  to  $\mathbb{Z}/2$  are the canonical projections. Since  $\mathbb{Z}[\mathbb{Z}/3]$  is free as an abelian group, this remains to be a pullback of rings if we apply  $-\otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$ . We have isomorphisms of rings  $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] = \mathbb{Z}[\mathbb{Z}/6]$  and  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] = \mathbb{Z}[\mathbb{Z}/3]$ . From the pullback for p = 3 appearing in Example 4.10 we obtain an isomorphism of rings

$$\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times (\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\exp(2\pi i/3)]).$$

The ring  $\mathbb{Z}[\exp(2\pi i/3)]$  is as an abelian group free with two generators 1 and  $\omega = \exp(2\pi i/3)$  and the multiplication is uniquely determined by  $\omega^2 = -1 - \omega$ . Hence  $\mathbb{F}_2 \otimes_\mathbb{Z} \mathbb{Z}[\exp(2\pi i/3)]$  contains four elements, namely 0, 1, 1  $\otimes \omega$ , and the sum  $1 + 1 \otimes \omega$ . Since  $(1 \otimes \omega) \cdot (1 + 1 \otimes \omega) = 1$ , it is the field  $\mathbb{F}_4$  consisting of four elements. Hence we obtain a pullback of rings



Since  $K_n(\mathbb{F}_2 \times \mathbb{F}_4) \cong K_n(\mathbb{F}_2) \times K_n(\mathbb{F}_4)$  vanishes for  $n \leq -1$  and  $K_n(\mathbb{Z}[\mathbb{Z}/3])$  vanishes for  $n \leq -1$  by Example 4.10, the associated long exact Mayer-Vietoris sequence, see Remark 4.4, implies that  $K_n(\mathbb{Z}[\mathbb{Z}/6]) = 0$  holds for  $n \leq -2$  and there is an exact sequence

$$K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \to 0.$$

Since  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$  is trivial, see Example 2.107, and the projections induce an isomorphism  $K_0(\mathbb{F}_2 \times \mathbb{F}_4) \xrightarrow{\cong} K_0(\mathbb{F}_2) \times K_0(\mathbb{F}_4) \cong \mathbb{Z} \oplus \mathbb{Z}$ , we conclude  $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}$ .

**Exercise 4.13.** Consider  $k \in \{0, 1, 2, ...\}$ . Compute

$$K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z}^{k+1} & \text{for } n = 0; \\ \mathbb{Z} & \text{for } n = -1; \\ 0 & \text{for } n \le -2, \end{cases}$$

and prove  $N^p K_n(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) = 0$  for  $p \ge 1$  and  $n \le 0$ .

The Bass-Heller-Swan decomposition can be used to show that certain results about the *K*-groups in a fixed degree *m* have implications for all the *K*-groups in degree  $n \le m$ , as illustrated by the next result.

4.2 Definition and Basic Properties of Negative K-Groups

**Lemma 4.14.** Consider a ring R and  $m \in \mathbb{Z}$  with  $m \leq 1$ . Suppose that for every  $k \geq 1$  the map  $K_m(R) \to K_m(R[\mathbb{Z}^k])$  induced by the inclusion  $R \to R[\mathbb{Z}^k]$  is bijective.

Then  $K_n(R[\mathbb{Z}^l]) = 0$  for  $n \le m - 1$  and  $NK_n(R[\mathbb{Z}^l]) = 0$  for  $n \le m$  hold for all  $l \ge 0$ .

*Proof.* Since the bijectivity of  $K_m(R) \to K_m(R[\mathbb{Z}^k])$  for all  $k \ge 1$  implies the bijectivity of  $K_m(R[\mathbb{Z}^l]) \to K_m((R[\mathbb{Z}^l])[\mathbb{Z}^k])$  for all  $k, l \ge 0$  because of the identification  $(R[\mathbb{Z}^l])[\mathbb{Z}^k] = R[\mathbb{Z}^{k+l}]$ , it suffices to treat the case l = 0.

Consider any integer  $k \ge 1$ . The assumptions in Lemma 4.14 imply that the map  $K_m(R[\mathbb{Z}^{k-1}]) \to K_m(R[\mathbb{Z}^k])$  induced by the inclusion  $R[\mathbb{Z}^{k-1}] \to R[\mathbb{Z}^k]$  is bijective. Theorem 4.3 applied to the ring  $R[\mathbb{Z}^{k-1}]$  together with the identity  $R[\mathbb{Z}^k] = (R[\mathbb{Z}^{k-1}])[\mathbb{Z}]$  shows that  $K_{m-1}(R[\mathbb{Z}^{k-1}]) = 0$  and  $NK_m(R[\mathbb{Z}^{k-1}]) = 0$ . Using Theorem 4.3 and the Bass-Heller-Swan decomposition for *NK*, see (4.8), one shows inductively for  $i = 0, 1, \ldots, (k-1)$  that  $K_{m-1-j}(R[\mathbb{Z}^{k-i-1}]) = 0$  and  $NK_{m-j}(R[\mathbb{Z}^{k-i-1}]) = 0$  holds for  $j = 0, 1, \ldots, i$ . Then the case i = k - 1 shows that  $K_n(R) = 0$  for  $m - k \le n \le m - 1$  and  $NK_n(R) = 0$  for  $m - k + 1 \le n \le m$ . Since  $k \ge 1$  was arbitrary, Lemma 4.14 follows.

**Exercise 4.15.** Consider a ring R and  $m \in \mathbb{Z}$  with  $m \leq 1$ . Suppose that  $K_m(R[\mathbb{Z}^k]) = 0$  holds for every  $k \geq 1$ . Then  $K_i(R[\mathbb{Z}^l]) = NK_i(R[\mathbb{Z}^l]) = 0$  holds for  $i \leq m$  and  $l \geq 0$ .

**Theorem 4.16 (The middle and lower** *K***-theory of** *RG* **for finite** *G* **and Artinian** *R***).** *Let G be a finite group, and let R be an Artinian ring. Then:* 

(i) For every  $k \ge 0$  the map

$$K_0(RG) \xrightarrow{\cong} K_0(RG[\mathbb{Z}^k])$$

induced by the inclusion is bijective;

(ii) Given any  $k \ge 0$ , we have  $K_n(RG[\mathbb{Z}^k]) = 0$  for  $n \le -1$  and  $NK_n(RG[\mathbb{Z}^k]) = 0$  for  $n \le 0$ .

*Proof.* (i) Denote by  $J = \operatorname{rad}(RH) \subseteq RH$  the Jacobson radical of *RH*. Since *R* and hence *RH* are Artinian, there exists a natural number *l* with  $JJ^l = J^l$ . By Nakayama's Lemma, see [916, Proposition 8 in Chapter 2 on page 20],  $J^l$  is {0}, in other words, *J* is nilpotent. The ring *RH/J* is a semisimple Artinian ring, see [610, Definition 20.3 on page 311 and (20.3) on page 312], and in particular regular. Theorem 3.80 (ii) implies that  $(RH/J)[\mathbb{Z}^k]$  is regular for all  $k \ge 1$ . We derive from Theorem 4.7 that  $K_n((RH/J)[\mathbb{Z}^k]) = 0$  for  $n \le -1$  and  $NK_n((RH/J)[\mathbb{Z}^k])$  for  $n \le 0$  hold for all  $k \ge 0$ . We conclude from Theorem 4.7 by induction over k = 0, 1, 2, ... that the inclusion  $RH/J \to (RH/J)[\mathbb{Z}^k]$  induces an isomorphism

$$K_0(RH/J) \xrightarrow{=} K_0((RH/J)[\mathbb{Z}^k])$$

for all  $k \ge 0$ .

The following diagram



commutes. Since *J* is a nilpotent two-sided ideal of *RH*,  $J[\mathbb{Z}^k]$  is a nilpotent two-sided ideal of  $RH[\mathbb{Z}^k]$ . Obviously  $(RH/J)[\mathbb{Z}^k]$  can be identified with  $(RH[\mathbb{Z}^k])/(J[\mathbb{Z}^k])$ . Hence the vertical arrows in the diagram above are bijective by Lemma 2.125. Since the lower horizontal arrow is bijective for every  $k \ge 1$ , the upper horizontal arrow is bijective for every  $k \ge 1$ .

(ii) This follows from assertion (i) and Lemma 4.14 applied in the case m = 0 to the ring *RG*.

# 4.3 Geometric Interpretation of Negative K-Groups

One possible geometric interpretation of negative K-groups is in terms of bounded h-cobordisms.

We consider *manifolds* W parametrized over  $\mathbb{R}^k$ , i.e., manifolds that are equipped with a surjective proper map  $p: W \to \mathbb{R}^k$ . Recall that proper map means that preimages of compact subsets are compact again. We will always assume that the fundamental group(oid) is bounded, see [797, Definition 1.3]. A map  $f: W \to W'$ between two manifolds parametrized over  $\mathbb{R}^k$  is called *bounded* if  $\{p' \circ f(x) - p(x) \mid x \in W\}$  is a bounded subset of  $\mathbb{R}^k$ .

A bounded cobordism  $(W; M_0, f_0, M_1, f_1)$  is defined just as in Section 3.5 but compact manifolds are replaced by manifolds parametrized over  $\mathbb{R}^k$  and the parametrization for  $M_l$  is given by  $p_W \circ f_l$ . If we assume that the inclusions  $i_l: \partial_k W \to W$  are homotopy equivalences, then there exist deformations  $r_l: W \times I \to W$  such that  $r_l|_{W \times \{0\}} = \mathrm{id}_W$  and  $r_l(W \times \{1\}) \subset \partial_l W$ . A bounded cobordism is called a *bounded h-cobordism* if the inclusions  $i_l$  are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S_{l} = \{ p_{W}(r_{l}(x,t)) - p_{W}(r_{l}(x,1)) \mid x \in W, t \in [0,1] \}$$

are bounded subsets of  $\mathbb{R}^k$ .

The following theorem, see [797] and [1001, Appendix], contains the *s*-Cobordism Theorem 3.47 as a special case, gives another interpretation of elements in  $\tilde{K}_0(\mathbb{Z}\pi)$  and explains one aspect of the geometric relevance of negative *K*-groups.
4.4 Variants of the Farrell-Jones Conjecture for Negative K-Groups

**Theorem 4.17 (Bounded** *h*-**Cobordism Theorem).** Suppose that  $M_0$  is parametrized over  $\mathbb{R}^k$  and satisfies dim  $M_0 \ge 5$ . Let  $\pi$  be its fundamental group(oid). Equivalence classes of bounded *h*-cobordisms over  $M_0$  modulo bounded diffeomorphism relative  $M_0$  correspond bijectively to elements in  $\kappa_{1-k}(\pi)$  where

$$\kappa_{1-k}(\pi) = \begin{cases} Wh(\pi) & \text{if } k = 0; \\ \widetilde{K}_0(\mathbb{Z}\pi) & \text{if } k = 1; \\ K_{1-k}(\mathbb{Z}\pi) & \text{if } k \ge 2. \end{cases}$$

## 4.4 Variants of the Farrell-Jones Conjecture for Negative *K*-Groups

In this section we state variants of the Farrell-Jones Conjecture for negative *K*-theory. The Farrell-Jones Conjecture itself will give a complete answer for arbitrary rings but to formulate the full version some additional effort will be needed. If one assumes that *R* is regular and *G* torsionfree or that  $R = \mathbb{Z}$ , the conjecture reduces to an easy to formulate statement, which we will present next.

**Conjecture 4.18 (The Farrell-Jones Conjecture for negative** *K***-theory and regular coefficient rings).** Let *R* be a regular ring and *G* be a group such that for every finite subgroup  $H \subseteq G$  the element  $|H| \cdot 1_R$  of *R* is invertible in *R*. Then we get

 $K_n(RG) = 0$  for  $n \leq -1$ .

Exercise 4.19. Prove that Conjecture 4.18 is true if G is finite.

Conjecture 4.20 (The Farrell-Jones Conjecture for negative *K*-theory of the ring of integers in an algebraic number field). Let R be the ring of integers in an algebraic number field. Then, for every group G, we have

$$K_n(RG) = 0$$
 for  $n \le -2$ ,

and the canonical map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_{-1}(RH) \xrightarrow{\cong} K_{-1}(RG)$$

is an isomorphism.

Conjecture 4.21 (The Farrell-Jones Conjecture for negative K-theory and Artinian rings as coefficient rings). Let G be a group, and let R be an Artinian ring. Then we have

$$K_n(RG) = 0$$
 for  $n \leq -1$ .

# 4.5 Survey on Computations of Negative *K*-Groups for Finite Groups

The following result is due to Carter [217]. See also [102, Theorem 10.6 on page 695] and [626].

**Theorem 4.22 (Negative** *K*-theory of *RG* for a finite group *G* and a Dedekind domain of characteristic zero *R*). Let *R* be a Dedekind domain of characteristic zero. Let *k* be its fraction field. For any maximal ideal *P* of *R*, let  $k_P$  be the *P*-adic completion. Let *G* be a finite group of order n = |G|.

For a field F we denote by  $r_F(G)$  the number of isomorphism classes of irreducible representations of G over the field F. Then:

- (i)  $K_m(RG) = 0$  for  $m \le -2$ ;
- (ii)  $K_{-1}(RG)$  is a finitely generated group;
- (iii) Suppose that no prime divisor of n is invertible in R. Then the rank r of the finitely generated abelian group  $K_{-1}(RG)$  is given by

$$r = 1 - r_k(G) + \sum_{p|nR} r_{k_p}(G) - r_{R/P}(G)$$

where the sum runs over all maximal (= non-zero prime) ideals P dividing nR; (iv) Let R be the ring of integers in an algebraic number field k. Then

$$K_{-1}(RG) = \mathbb{Z}^r \oplus \mathbb{Z}/2^s$$

There is an explicit description of the integer s in terms of global and local Schur indices.

If G contains a normal abelian subgroup of odd index, then  $K_{-1}(RG)$  is torsionfree;

- (v) Let A be a finite abelian group. Then  $K_{-1}(\mathbb{Z}A)$  vanishes if and only if |A| is a prime power;
- (vi) The group  $K_{-1}(\mathbb{Z}G)$  is torsion if and only if every element of G has a prime power order;
- (vii) If the order of |G| is not divisible by 4 or if G contains a normal abelian subgroup of odd index, then the group  $K_{-1}(\mathbb{Z}G)$  is torsionfree;
- (viii) If the order of |G| is a p-power for some odd prime p, then the group  $K_{-1}(\mathbb{Z}G)$  vanishes.

If  $R = \mathbb{Z}$ , then  $r = 1 - r_{\mathbb{Q}}(G) + \sum_{p|n} r_{\mathbb{Q}\hat{p}}(G) - r_{\mathbb{F}_p}(G)$  where *p* runs through the prime numbers dividing *n*.

A computation of  $K_{-1}(\mathbb{Z}G)$  for all finite groups of order  $\leq 100$  can be found in Lehner [626]. In particular,  $K_{-1}(\mathbb{Z}G)$  is torsionfree for  $n \leq 15$  and the smallest group for which  $K_{-1}(\mathbb{Z}G)$  is not torsionfree is the generalized quaternion group  $Q_{16}$ of order 16. Actually we have  $K_{-1}(\mathbb{Z}[Q_{16}]) \cong \mathbb{Z}/2$ .

4.6 Notes

## 4.6 Notes

More information about  $NK_n(RG)$  for all  $n \in \mathbb{Z}$  will be given in Theorem 6.17, Theorem 6.18, Theorem 6.19, and Theorem 6.21.

More information about negative *K*-groups can be found for instance in [30, 102, 216, 217, 368, 530, 686, 700, 796, 797, 825, 840, 860, 998].

## Chapter 5 The Second Algebraic K-Group

## 5.1 Introduction

This chapter is devoted to the *second algebraic K-group*.

We give two equivalent definitions, namely, in terms of the Steinberg group and in terms of the universal central extension of E(R). We extend the long exact sequence associated to a pullback of rings and to a two-sided ideal beyond  $K_1$  to  $K_2$ . The long exact sequence associated to a pullback of rings cannot be extended to the left to higher algebraic *K*-groups, whereas this will be done for the long exact sequence associated to a two-sided ideal later.

We will introduce the *second Whitehead group* and state a variant of the *Farrell-Jones Conjecture* for it, namely, that it vanishes for torsionfree groups. Finally we give some information about computations of the second algebraic *K*-group.

## **5.2 Definition and Basic Properties of** $K_2(R)$

**Definition 5.1** (*n*-th Steinberg group). For  $n \ge 3$  and a ring *R*, define its *n*-th Steinberg group  $St_n(R)$  to be the group given by generators and relations as follows. The set of generators is

$$\{x_{i,j}^r \mid i, j \in \{1, 2, \dots, n\} \text{ and } r \in R\}.$$

The relations are

(i)  $x_{i,j}^r \cdot x_{i,j}^s = x_{i,j}^{r+s}$  for  $i, j \in \{1, 2, ..., n\}$  and  $r, s \in R$ ; (ii)  $[x_{i,j}^r, x_{j,k}^s] = x_{i,k}^{rs}$  for  $i, j, k \in \{1, 2, ..., n\}$  with  $i \neq k$  and  $r, s \in R$ ; (iii)  $[x_{i,j}^r, x_{k,l}^s] = 1$  for  $i, j, k, l \in \{1, 2, ..., n\}$  with  $i \neq l, j \neq k$ , and  $r, s \in R$ ,

where [a, b] denotes the commutator  $aba^{-1}b^{-1}$ .

The idea behind the Steinberg group is that for every ring *R* the corresponding relations hold in  $GL_n(R)$  if we replace  $x_{i,j}^r$  by the matrix  $I_n + r \cdot E_n(i, j)$  appearing in Section 3.2. Hence we get a canonical group homomorphism

$$\phi_n^R \colon \operatorname{St}_n(R) \to \operatorname{GL}_n(R), \quad x_{i,j}^r \mapsto I_n + r \cdot E_n(i,j).$$

The image of  $\phi_n^R$  is by definition the subgroup of  $GL_n(R)$  generated by all elements of the form  $I_n + r \cdot E_n(i, j)$  for  $i, j \in \{1, 2, ..., n\}$  and  $r \in R$ . There is an obvious inclusion  $St_n(R) \to St_{n+1}(R)$  sending a generator  $x_{i,j}^r$  to  $x_{i,j}^r$ . **Definition 5.2 (Steinberg group).** Define the *Steinberg group* St(R) to be the union of the groups  $St_n(R)$ .

The set of maps  $\{\phi_n^R \mid n \ge 3\}$  defines a homomorphism of groups

(5.3) 
$$\phi^R \colon \operatorname{St}(R) \to \operatorname{GL}(R).$$

The image of  $\phi^R$  is just the group E(R), which agrees with [GL(R), GL(R)], see Lemma 3.11.

**Definition 5.4** ( $K_2(R)$ ). Define the *algebraic*  $K_2$ -group  $K_2(R)$  of a ring R to be the kernel of the group homomorphism  $\phi^R \colon St(R) \to GL(R)$  of (5.3).

Obviously a ring homomorphism  $f: R \rightarrow S$  induces a map of abelian groups

(5.5) 
$$K_2(f) \colon K_2(R) \to K_2(S).$$

Exercise 5.6. Show that there is a natural exact sequence

$$0 \to K_2(R) \to \operatorname{St}(R) \to \operatorname{GL}(R) \to K_1(R) \to 0.$$

#### 5.3 The Steinberg Group as Universal Extension

A central extension of a group Q is a surjective group homomorphism  $\phi: G \to Q$ with Q as target such that the kernel of  $\phi$  is contained in the center  $\{g \in G \mid g \in G \}$ g'g = gg' for all  $g' \in G$  of G. A central extension  $\phi: U \to Q$  of a group Q is called *universal* if for every central extension  $\psi: G \to Q$  there is precisely one group homomorphism  $f: U \to G$  with  $\psi \circ f = \phi$ . If a group O admits a universal central extension, it is unique up to unique isomorphism. A group Q possesses a universal central extension if and only if it is *perfect*, i.e., it is equal to its commutator subgroup, see [727, Theorem 5.7 on page 44] or [860, Theorem 4.1.3 on page 163]. In this case the kernel of the universal central extension  $\phi: U \to O$  is isomorphic to the second homology  $H_2(Q;\mathbb{Z})$  of Q, see [727, Corollary 5.8 on page 46] or [860, Theorem 4.1.3 on page 163]. A central extension  $\phi: G \to Q$  of a group Q is universal if and only if G is perfect and every central extension  $\psi: H \to G$  of G splits, i.e., there is a homomorphism  $s: G \to H$  with  $\psi \circ s = id_G$ , see [727, Theorem 5.3 on page 43] or [860, Theorem 4.1.3 on page 163]. A central extension  $\phi: G \to Q$  of a perfect group Q is universal if and only if  $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$ , see [860, Corollary 4.1.18 on page 177]. The proof of the next result can be found in [727, Theorem 5.10 on page 47] or [860, Theorem 4.2.7 on page 190].

**Theorem 5.7** ( $K_2(R)$  and universal central extensions of E(R)). The canonical epimorphism  $\phi^R$ : St(R)  $\rightarrow E(R)$  coming from the map (5.3) is the universal central extension of E(R).

**Exercise 5.8.** Prove  $K_2(R) \cong H_2(\mathbb{E}(R); \mathbb{Z})$ .

## 5.4 Extending Exact Sequences of Pullbacks and Ideals

**Theorem 5.9** (Mayer-Vietoris sequence for *K*-theory in degree  $\leq 2$  of a pullback of rings). Consider a pullback of rings



such that both  $j_1$  and  $j_2$  are surjective. Then there exists a natural exact sequence, infinite to the right,

$$K_{2}(R) \xrightarrow{(i_{1})_{*} \oplus (i_{2})_{*}} K_{2}(R_{1}) \oplus K_{2}(R_{2}) \xrightarrow{(j_{1})_{*} - (j_{2})_{*}} K_{2}(R_{0})$$

$$\xrightarrow{\partial_{2}} K_{1}(R) \xrightarrow{(i_{1})_{*} \oplus (i_{2})_{*}} K_{1}(R_{1}) \oplus K_{1}(R_{2}) \xrightarrow{(j_{1})_{*} - (j_{2})_{*}} K_{1}(R_{0})$$

$$\xrightarrow{\partial_{1}} K_{0}(R) \xrightarrow{(i_{1})_{*} \oplus (i_{2})_{*}} K_{0}(R_{1}) \oplus K_{0}(R_{2}) \xrightarrow{(j_{1})_{*} - (j_{2})_{*}} K_{0}(R_{0})$$

$$\xrightarrow{\partial_{0}} K_{-1}(R) \xrightarrow{(i_{1})_{*} \oplus (i_{2})_{*}} K_{-1}(R_{1}) \oplus K_{-1}(R_{2})$$

$$\xrightarrow{(j_{1})_{*} - (j_{2})_{*}} K_{-1}(R_{0}) \xrightarrow{\partial_{-1}} \cdots$$

*Proof.* See [727, Theorem 6.4 on page 55] for the extension to  $K_2$ . The extension for negative *K*-theory follows for example from the fact that the passage going from *R* to  $R[\mathbb{Z}]$  sends a pullback of rings to a pullback of rings.

**Remark 5.10 (Surjectivity assumption is necessary).** Swan [942, Corollary 1.2] has shown that the assumption that both  $j_1$  and  $j_2$  are surjective in Theorem 5.9 is necessary. It is not enough that  $j_1$  or  $j_2$  is surjective, in contrast to the weaker Theorem 3.86.

**Remark 5.11** (No exact sequence for pullbacks in higher degrees). Swan [942, Corollary 6.9] has shown that it is not possible to define a functor  $K_3$  so that the natural exact sequence appearing in Theorem 5.9 can be extended to  $K_3$ .

**Theorem 5.12 (Exact sequence of a two-sided ideal** *K***-theory in degree**  $\leq$  2). *Given a two-sided ideal*  $I \subset R$ *, we obtain an exact sequence, natural in*  $I \subseteq R$  *and infinite to the right* 

$$K_{2}(R) \xrightarrow{\operatorname{pr}_{*}} K_{2}(R/I) \xrightarrow{\partial_{2}} K_{1}(R,I) \xrightarrow{j_{1}} K_{1}(R) \xrightarrow{\operatorname{pr}_{*}} K_{1}(R/I)$$
$$\xrightarrow{\partial_{1}} K_{0}(R,I) \xrightarrow{j_{0}} K_{0}(R) \xrightarrow{\operatorname{pr}_{*}} K_{0}(R/I) \xrightarrow{\partial_{0}} K_{-1}(R,I)$$
$$\xrightarrow{j_{-1}} K_{-1}(R) \xrightarrow{\operatorname{pr}_{*}} \cdots$$

where pr:  $R \rightarrow R/I$  is the projection.

*Proof.* See [727, Theorem 6.2 on page 54], [860, Theorem 3.3.4. on page 155 and Theorem 4.3.1 on page 200], or [998, Theorem 5.7.1 in Section III.5 on page 223].

**Remark 5.13 (Dependence of**  $K_n(R, I)$  **on** R**).** The group  $K_n(R, I)$  can be identified for  $n \le 0$  with  $K_n(I)$ , see Definition 3.90, and hence depends only on the structure of I as a ring without unit but not on the embedding  $I \subseteq R$ . But for  $n \ge 1$  the group  $K_n(R, I)$  does depend on the embedding  $I \subseteq R$ , see Example 3.94.

The sequence appearing in Theorem 5.12 is indeed an extension of the long exact sequence appearing in Theorem 3.89.

Often one wants to get information about  $K_2$  in order to compute  $K_1$ -groups using for instance Theorem 5.12. This is illustrated by the following example.

**Example 5.14.** Let R be the ring of integers in an algebraic number field, and let P be a non-zero prime ideal. Then the exact sequence appearing in Theorem 5.12 induces an exact sequence

$$K_2(R/P) \to SK_1(R,P) \to SK_1(R) \to SK_1(R/P)$$

where  $SK_1(R)$  has been defined in Definition 3.15 and we put:

$$SK_1(R, P) := (SL(R) \cap GL(R, P))/E(R, P)$$
  

$$\cong \ker(\det: GL(R, P) \to \{r \in R \mid r \equiv 1 \mod P\}).$$

Since R/P is a finite field,  $SK_1(R/P)$  and  $K_2(R/P)$  vanish by Theorem 3.17 and Theorem 5.18 (v). Hence we obtain an isomorphism

$$SK_1(R,P) \xrightarrow{\cong} SK_1(R).$$

The group  $SK_1(R)$  vanishes by [727, Corollary 16.3]. Hence also  $SK_1(R, P)$  vanishes.

**Example 5.15**  $(K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  for  $n \leq 1$  and a prime p). Let p be a prime number. We want to show

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) = 0$$
 for  $n \le -1$  and  $k \ge 0$ 

and that  $K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  and  $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  are finitely generated. All of these statements except the claim for  $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  have already been proved in Example 4.10. The same method of proof applies to this case, since Theorem 5.9 yields the exact sequence

$$K_2(\mathbb{F}_p[\mathbb{Z}^k]) \to K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to K_1(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k])$$

and  $K_2(\mathbb{F}_p[\mathbb{Z}^k])$ ,  $K_1(\mathbb{Z}[\mathbb{Z}^k])$ , and  $K_1(\mathbb{Z}[\exp(2\pi i/p)])$  are finitely generated abelian groups by Theorem 4.7, as  $K_m(\mathbb{F}_p)$  for  $m = 0, 1, 2, K_m(\mathbb{Z})$  for m = 0, 1, and  $K_m(\mathbb{Z}[\exp(2\pi i/p)])$  for m = 0, 1 are finitely generated and  $K_m(\mathbb{F}_p)$ ,  $K_m(\mathbb{Z})$ , and  $K_m(\mathbb{Z}[\exp(2\pi i/p)])$  vanish for  $m \le -1$  because of Example 2.4, Theorem 2.23, Theorem 3.17, Theorem 3.21, Theorem 3.80 (i) Theorem 4.7, and Theorem 5.18 (iv).

#### 5.5 Steinberg Symbols

Let *R* be a commutative ring and  $u, v \in R^{\times}$ . Consider the elements  $d_{1,2}(u), d_{1,3}(v) \in E(R)$  given by the invertible (3, 3)-matrices

1	u	0	0)		(v 0)	0)	
I	0	$u^{-1}$	0	and	01	0	
l	0	0	1)		00	$v^{-1}$	

Let  $\tilde{d}_{1,2}(u)$  and  $\tilde{d}_{1,3}(v)$  be any preimages of  $d_{1,2}(u)$  and  $d_{1,3}(v)$  under the canonical map  $\phi^R$ : St(R)  $\rightarrow$  E(R). Then the commutator [ $\tilde{d}_{1,2}(u), \tilde{d}_{1,3}(v)$ ] in St(R) defines an element in the kernel of  $\phi^R$ : St(R)  $\rightarrow$  E(R) and hence in  $K_2(R)$ . It depends only on u and v. The proof of the facts above can be found for instance in [860, page 192].

**Definition 5.16 (Steinberg symbol).** Let *R* be a commutative ring and  $u, v \in R^{\times}$ . The element in  $K_2(R)$  given by the construction above is called the *Steinberg symbol* of *u* and *v* and is denoted by  $\{u, v\}$ .

Exercise 5.17. Prove that the Steinberg symbol of Definition 5.16 is well-defined.

**Theorem 5.18 (Properties of the Steinberg symbol).** *Let R be a commutative ring. Then:* 

(i) *The Steinberg symbol defines a bilinear skew-symmetric pairing* 

$$R^{\times} \times R^{\times} \to K_2(R), \quad (u, v) \mapsto \{u, v\},\$$

*i.e.*,  $\{u_1 \cdot u_2, v\} = \{u_1, v\} + \{u_2, v\}$  and  $\{u, v\} = -\{v, u\}$  for all  $u_1, u_2, u, v$  in  $\mathbb{R}^{\times}$ ;

(ii) For  $u \in \mathbb{R}^{\times}$  we have  $\{u, -u\} = 0$ ;

- (iii) If for  $u \in \mathbb{R}^{\times}$  also  $1 u \in \mathbb{R}^{\times}$ , then  $\{u, 1 u\} = 0$ ;
- (iv) (*Matsumoto's Theorem*) If F is a field, then  $K_2(F)$  is isomorphic to the abelian group given by the generators  $\{u, v\}$  for  $u, v \in F^{\times}$  and the relations:
  - (a)  $\{u, 1 u\} = 0$  for  $u \in F$  with  $u \neq 0, 1$ ;
  - (b)  $\{u_1 \cdot u_2, v\} = \{u_1, v\} + \{u_2, v\}$  for  $u_1, u_2, v \in F^{\times}$ ;
  - (c)  $\{u, v_1 \cdot v_2\} = \{u, v_1\} + \{u, v_2\}$  for  $u, v_1, v_2 \in F^{\times}$ ;
- (v) If F is a finite field, then  $K_2(F) = 0$ ;
- (vi) We have  $K_2(\mathbb{Z}) = \mathbb{Z}/2$ . A generator is given by the Steinberg symbol  $\{-1, -1\}$ ;
- (vii) Let  $m \ge 2$  be an integer. If  $m \ne 0 \mod 4$ , then  $K_2(\mathbb{Z}/m) = \{0\}$ . If  $m = 0 \mod 4$ , then  $K_2(\mathbb{Z}/m) = \mathbb{Z}/2$  and a generator is given by the Steinberg symbol  $\{1, 1\}$ ;

- (viii) (Tate) We have  $K_2(\mathbb{Q}) = \mathbb{Z}/2 \times \prod_p \mathbb{F}_p^{\times}$  where p runs through the odd prime numbers;
- (ix) (Bass, Tate) Let R be a Dedekind domain with quotient field F. Then there is an exact sequence

$$\begin{split} K_2(F) &\to \bigoplus_P K_1(R/P) \to K_1(R) \to K_1(F) \\ &\to \bigoplus_P K_0(R/P) \to K_0(R) \to K_0(F) \to 0, \end{split}$$

where P runs through the maximal ideals of R.

*Proof.* (i) See [727, Theorem 8.2 on page 64] or [860, Lemma 4.2.14 on page 194].
(ii) and (iii) See [727, Theorem 9.8 on page 74] or [860, Theorem 4.2.17 on page 197].
(iv) See [727, Theorem 11.1 on page 93] or [860, Theorem 4.3.15 on page 214].

(v) See [727, Theorem 9.13 on page 78] or [860, Theorem 4.3.13 and Remark 4.3.14 on page 213].

(vi) See [727, Corollary 10.2 on page 81].

(vii) See [727, Corollary 10.8 on page 92], [307, Theorem 5.1], and [860, Exercise 4.3.19 on page 217].

(viii) See [727, Theorem 11.6 on page 101].

(ix) See [727, Corollary 13.1 on page 123] and [102, pages 702, 323]. □

## 5.6 The Second Whitehead Group

Let *R* be a ring. Consider  $u \in R^{\times}$  and integers  $i, j \ge 1$ . If  $x_{i,j}^{u}$  is the canonical generator of St(*R*), see Definition 5.1, then define

$$w_{i,j}^{u} := x_{i,j}^{u} x_{j,i}^{-u^{-1}} x_{ij}^{u} \in \operatorname{St}(R).$$

Let *G* be a group. Let  $W_G$  be the subgroup of  $St(\mathbb{Z}G)$  generated by all elements of the shape  $w_{i,j}^g$  for  $g \in G$  and integers  $i, j \ge 1$ . Recall that we can think of  $K_2(\mathbb{Z}G)$  as a subgroup of  $St(\mathbb{Z}G)$ .

**Definition 5.19 (The second Whitehead group).** Let G be a group. Define the *second Whitehead group of G* by

$$Wh_2(G) := K_2(\mathbb{Z}G)/(K_2(\mathbb{Z}G) \cap W_G).$$

**Exercise 5.20.** Show that the second Whitehead group of the trivial group vanishes using the fact, see [860, Example 4.2.19 on page 198], that  $w_{1,2}(1)^4 = \{-1, -1\}$  holds in St( $\mathbb{Z}$ ).

#### 5.7 A Variant of the Farrell-Jones Conjecture for the Second Whitehead Group

Let *I* denote the unit interval [0, 1]. Let *M* be a closed smooth manifold. A smooth *pseudoisotopy* of *M* is a diffeomorphism  $h: M \times I \to M \times I$  that restricted to  $M \times \{0\} \subseteq M \times I$  is the obvious inclusion. The group  $P^{\text{Diff}}(M)$  of smooth pseudoisotopies is the group of all such diffeomorphisms under composition. Pseudoisotopies play an important role if one tries to understand the homotopy type of the topological group Diff(*M*) of self-diffeomorphisms of *M*. Two self-diffeomorphisms  $f_0, f_1: M \to M$  are called *isotopic* if there is a smooth map  $h: M \times [0, 1] \to M$ , called an isotopy, such that  $h_t: M \to M, x \mapsto h(x, t)$  is a self-diffeomorphism for each  $t \in [0, 1]$  and  $h_k = f_k$  for k = 0, 1. They are called *pseudoisotopic* if there exists a diffeomorphism  $H: M \times [0, 1] \to M \times [0, 1]$  such that  $H(x, k) = (f_k(x), k)$  for all  $x \in M$  and k = 0, 1. If *h* is an isotopy, then we obtain a pseudoisotopy by H(x, k) = (h(x, k), k). Hence isotopic self-diffeomorphisms are pseudoisotopic. The converse is not true in general, there is no reason why a pseudoisotopy should be level preserving, i.e., it need not send  $M \times \{t\}$  to  $M \times \{t\}$  for every  $t \in [0, 1]$ .

In order to decide whether two self-diffeomorphisms are isotopic, it is often very useful to firstly decide whether they are pseudoisotopic, which is in general easier.

The set of path components  $\pi_0(\text{Diff}(M))$  of the space Diff(M) agrees with the set of isotopy classes of self-diffeomorphisms of M. The group  $P^{\text{DIFF}}(M)$  acts on Diff(M) by  $h \cdot f := h_1 \circ f$ . If  $P^{\text{DIFF}}(M)$  is path-connected, then two pseudoisotopic diffeomorphisms  $M \to M$  are isotopic, since the orbit of the identity  $\text{id}_M : M \to M$  under the  $P^{\text{DIFF}}(M)$ -action consists of the diffeomorphisms  $M \to M$  that are pseudoisotopic to the identity. If M is simply connected,  $P^{\text{DIFF}}(M)$  is known to be path connected by a result of Cerf [219, 220] if  $\dim(M) \ge 5$ .

The relevance of the second Whitehead group comes from the following result, see [469, 470].

**Theorem 5.21 (Pseudoisotopy and the second Whitehead group).** *Let* M *be a smooth closed manifold of dimension*  $\geq$  5. *Then there is an epimorphism* 

$$\pi_0(P^{\mathsf{DIFF}}(M)) \to \mathsf{Wh}_2(\pi_1(M)).$$

More information about pseudoisotopy and its relation to algebraic *K*-theory will be given in Chapter 7. The Farrell-Jones Conjecture for pseudoisotopy will be stated as Conjecture 15.63.

## 5.7 A Variant of the Farrell-Jones Conjecture for the Second Whitehead Group

**Conjecture 5.22 (Farrell-Jones Conjecture for**  $Wh_2(G)$  **for torsionfree** *G***).** Let *G* be a torsionfree group. Then  $Wh_2(G)$  vanishes.

5 The Second Algebraic K-Group

## 5.8 The Second Whitehead Group of Some Finite Groups

We give some information about  $K_2(\mathbb{Z}G)$  and  $Wh_2(G)$  for some finite groups.

The group  $K_2(RG)$  is finite for every finite group *G* and every ring of integer *R* in a number field, see [597, Theorem 1.1]. In particular  $K_2(\mathbb{Z}G)$  and  $Wh_2(G)$  are finite for any finite group *G*.

We have

$$Wh_2(G) = 0, \text{ for } G = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4;$$
  
 $|Wh_2(\mathbb{Z}/6)| \le 2;$   
 $Wh_2(D_6) \cong \mathbb{Z}/2,$ 

where  $D_6$  is the dihedral group of order six. The claim for the finite cyclic groups follow from [318, page 482] and [928, pages 218 and 221]. We get  $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$  from [928, Theorem 3.1]. This implies  $Wh_2(D_6) \cong \mathbb{Z}/2$  as explained in [683, Theorem 3.2.d.iii].

Given a prime *p*, the *p*-rank of an abelian group *A* is  $\dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{Z}} A)$ . The 2-rank of the finite abelian group  $Wh_2((\mathbb{Z}/2)^n)$  is at least  $(n-1) \cdot 2^n - \frac{(n+2)(n-1)}{2}$  by [302, Corollary 7]. If *p* is an odd prime, then the *p*-rank of the finite abelian group  $Wh_2((\mathbb{Z}/p)^n)$  is at least  $(n-1) \cdot (p^n-1) - {p+n-1 \choose p} - \frac{n(n-1)}{2}$  by [302, Corollary 8]. In particular  $Wh_2((\mathbb{Z}/p)^n)$  is non-trivial for a prime *p* and  $n \ge 2$ .

Some information about  $K_2(\mathbb{F}_p G)$  for finite groups can be found in [704].

**Exercise 5.23.** Determine all integers  $n \ge 1$  for which  $\widetilde{K}_i(\mathbb{Z}[\mathbb{Z}/n])$  for all  $i \le 0$ , Wh $(\mathbb{Z}/n)$ , and Wh<sub>2</sub> $(\mathbb{Z}/n)$  vanish.

### 5.9 Notes

We have already mentioned that often computations involving  $K_1$  use information about  $K_2$ , since there are various long exact sequences relating K-groups of different rings. Examples of such sequences have been given in Theorem 5.9, Theorem 5.12, and Theorem 5.18 (ix). Another important class of such exact sequences are given by localization sequences, see [776, Chapter 3].

The second algebraic *K*-group of fields also plays a role in number theory, as for instance explained in [727, Chapters 11, 15, 16], [922, Chapter 8] and [859, Chapter 4, Section 4]. Keywords are Hilbert symbols, Gauss' laws of quadratic reciprocity, Brauer groups, and the Mercurjev-Suslin Theorem. Relations to operator theory are discussed in [727, Chapter 7], and [859, Chapter 4, Section 4].

Further references to *K*<sub>2</sub> and the second Whitehead group are [23, 303, 304, 305, 306, 307, 470, 703, 929, 998].

## Chapter 6 Higher Algebraic *K*-Theory

## 6.1 Introduction

In this chapter we extend the definition of the *algebraic K-groups*  $K_n(R)$  to all integers  $n \in \mathbb{Z}$ .

We first present the plus-construction to define *higher algebraic K-theory* and record the basic properties. We introduce *algebraic K-theory with coefficients in*  $\mathbb{Z}/k$ . We discuss other constructions of *K*-theory that apply to more general situations such as to exact categories or Waldhausen categories. These constructions lead only to spaces and one can find deloopings which result in spectra whose homotopy groups are the algebraic *K*-groups also in negative degrees. We present the *K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings*. We introduce *Mayer-Vietoris sequences* for amalgamated free products and *Wang sequences* for HNN extensions for the algebraic *K*-theory of group rings. The appearance of Nil*terms* in these exact sequences is responsible for some complications concerning algebraic *K*-theory and the Farrell-Jones Conjecture that do not occur in the Baum-Connes setting. We discuss *homotopy K-theory*, a theory that is on the one hand close to algebraic *K*-theory and on the other hand is free of Nil-phenomena. We briefly explain relations between algebraic *K*-theory and cyclic homology.

## 6.2 The Plus-Construction

Let *R* be a ring. So far the algebraic *K*-groups  $K_n(R)$  for  $n \le 2$  have been described in a purely algebraic fashion by generators and relations. The definition of the higher algebraic *K*-groups  $K_n(R)$  for  $n \ge 3$  has been achieved topologically, namely, one assigns to a ring *R* a space K(R) and defines  $K_n(R)$  by the *n*-th homotopy group  $\pi_n(K(R))$  for  $n \ge 0$ . This will coincide with the previous definition for n = 0, 1, 2. There are various definitions of the space K(R) that extend to more general settings, as explained below, which are appropriate in different situations. We briefly recall the technically less demanding one, the plus-construction.

A space Z is called *acyclic* if it has the homology of a point, i.e., the singular homology with integer coefficients  $H_n(Z)$  vanishes for  $n \ge 1$  and is isomorphic to  $\mathbb{Z}$  for n = 0.

**Exercise 6.1.** Prove that an acyclic space is path connected and that its fundamental group  $\pi$  is perfect and satisfies  $H_2(\pi; \mathbb{Z}) = 0$ .

In the following we will suppress choices of and questions about base points. The *homotopy fiber* hofib(f) of a map  $f: X \to Y$  of path connected spaces has the property that it is the fiber of a fibration  $p_f: X \to E_f$  which comes with a homotopy equivalence  $h: E_f \to X$  satisfying  $p_f = f \circ h$ , see [1006, Theorem 7.30 in Chapter I.7 on page 42]. The *long exact homotopy sequence associated to* f, see [1006, Corollary 8.6 in Chapter IV.8 on page 187], looks like

(6.2) 
$$\cdots \xrightarrow{\partial_3} \pi_2(\operatorname{hofib}(f)) \xrightarrow{\pi_2(i)} \pi_2(X) \xrightarrow{\pi_2(f)} \pi_2(Y) \xrightarrow{\partial_2} \pi_1(\operatorname{hofib}(f))$$
  
 $\xrightarrow{\pi_1(i)} \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y) \xrightarrow{\partial_1} \pi_0(\operatorname{hofib}(f)) \to \{\{\bullet\}\}.$ 

**Definition 6.3 (Acyclic map).** Let *X* and *Y* be connected *CW*-complexes. A map  $f: X \rightarrow Y$  is called *acyclic* if its homotopy fiber hofib(*f*) is acyclic.

We conclude for an acyclic map  $f: X \to Y$  from the long exact homotopy sequence (6.2) that  $f_1: \pi_1(X) \to \pi_1(Y)$  is surjective and its kernel is a perfect subgroup P of  $\pi_1(X)$ , since P is a quotient of the perfect group  $\pi_1(\text{hofib}(f))$  and  $\pi_0(\text{hofib}(f))$  consists of one element. Obviously a space Z is acyclic if and only if the map  $Z \to \{\bullet\}$  is acyclic.

**Definition 6.4 (Plus-construction).** Let *X* be a connected *CW*-complex and  $P \subseteq \pi_1(X)$  be a perfect subgroup. A map  $f: X \to X^+$  to a *CW*-complex is called a *plus-construction of X relative to P* if *f* is acyclic and the kernel of  $f_1: \pi_1(X) \to \pi_1(X^+)$  is *P*.

The next result is due to Quillen. A proof can be found for instance in [860, Theorem 5.2.2 on page 266 and Proposition 5.2.4 on page 268].

**Theorem 6.5 (Properties of the plus-construction).** *Let* X *be a connected* CW-*complex and let*  $P \subseteq \pi_1(X)$  *be a perfect subgroup. Then:* 

- (i) There exists a plus-construction  $f: X \to X^+$  relative to P. (One can construct  $X^+$  by attaching 2- and 3-cells to X);
- (ii) Let f: X → X<sup>+</sup> be a plus-construction relative to P, and let g: X → Y be a map such that the kernel of π<sub>1</sub>(g): π<sub>1</sub>(X) → π<sub>1</sub>(Y) contains P. Then there is a map ḡ: X<sup>+</sup> → Y which is up to homotopy uniquely determined by the property that ḡ ∘ f is homotopic to g;
- (iii) If  $f_1: X \to X_1^+$  and  $f_2: X \to X_2^+$  are two plus-constructions for X relative to P, then there exists a homotopy equivalence  $g: X_1^+ \to X_2^+$  which is up to homotopy uniquely determined by the property  $g \circ f_1 \simeq f_2$ ;
- (iv) If  $f: X \to X^+$  is a plus-construction relative to P, then  $\pi_1(f): \pi_1(X) \to \pi_1(X^+)$  can be identified with the canonical projection  $\pi_1(X) \to \pi_1(X)/P$ ;
- (v) If  $f: X \to X^+$  is a plus-construction, then  $H_n(f; M): H_n(X; f^*M) \to H_n(X^+; M)$  is bijective for all  $n \ge 0$  and all local coefficient systems M on  $X^+$ .

6.2 The Plus-Construction

**Remark 6.6 (Perfect radical).** Every group G has a unique largest perfect subgroup  $P \subseteq G$ , called the *perfect radical* of G. In the following we will always use the perfect radical of G for P unless explicitly stated otherwise.

Exercise 6.7. Show that every group has a unique largest perfect subgroup.

**Exercise 6.8.** Show that E(R) = [GL(R), GL(R)] is the perfect radical of GL(R).

**Definition 6.9 (Higher algebraic** *K*-groups of a ring). Let  $BGL(R) \rightarrow BGL(R)^+$  be a plus-construction in the sense of Definition 6.4 for the classifying space BGL(R) of GL(R) (relative to the perfect radical of GL(R), which is E(R)). Define the *K*-theory space associated to *R* 

$$K(R) := K_0(R) \times BGL(R)^+$$

where we equip  $K_0(R)$  with the discrete topology. Define the *n*-th algebraic K-group

$$K_n(R) := \pi_n(K(R)) \quad \text{for } n \ge 0$$

This definition makes sense because of Theorem 6.5 (i) and (iii). Note that for  $n \ge 1$  we have  $K_n(R) = \pi_n(BGL(R)^+)$ .

**Exercise 6.10.** Show that the Definition 6.9 of  $K_n(R)$  for n = 0, 1 is compatible with the one of Definitions 2.1 and 3.1.

For n = 0, 1, 2, Definition 6.9 is compatible with the previous Definitions 2.1, 3.1, and 5.4, and we have  $K_3(R) \cong H_3(\text{St}(R))$  and  $K_n(R) = \pi_n(B\text{St}(R)^+)$  for  $n \ge 3$ , see [859, Corollary 5.2.8 on page 273], [423].

A ring homomorphism  $f: R \to S$  induces a group homomorphism  $GL(R) \to GL(S)$  and hence maps  $BGL(R) \to BGL(S)$  and  $BGL(R)^+ \to BGL(S)^+$ . We have already defined a map  $K_0(f): K_0(R) \to K_0(S)$  in (2.2). Therefore f induces a map  $K(f): K(R) \to K(S)$  and hence for every  $n \ge 0$  a map of abelian groups  $K_n(f): K_n(R) \to K_n(S)$ . This turns out to be compatible with the previous definitions for n = 0, 1, 2 in (2.2), (3.2), and (5.5). We have also defined  $K_n(f): K_n(R) \to K_n(S)$  for  $n \le -1$  in (4.2). Hence we get a covariant functor from the category of rings to the category of abelian groups by  $K_n(-)$  for  $n \in \mathbb{Z}$ .

**Definition 6.11 (Relative** *K***-groups).** Define for a two-sided ideal  $I \subseteq R$  and  $n \ge 0$ 

$$K_n(R, I) := \pi_n(\operatorname{hofib}(K(\operatorname{pr}): K(R) \to K(R/I)))$$

for pr:  $R \rightarrow R/I$  the projection.

The long exact homotopy sequence (6.2) associated to  $K(\text{pr}): K(R) \rightarrow K(R/I)$  together with Theorem 5.12 implies

**Theorem 6.12 (Long exact sequence of a two-sided ideal for higher algebraic** *K***-theory).** Let  $I \subseteq R$  be a two sided ideal. Then there is a long exact sequence, infinite to both sides,

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$$\cdots \xrightarrow{\partial_3} K_2(R,I) \xrightarrow{j_2} K_2(R) \xrightarrow{K_2(\mathrm{pr})} K_2(R/I) \xrightarrow{\partial_2} K_1(R,I) \xrightarrow{j_1} K_1(R)$$

$$\xrightarrow{K_1(\mathrm{pr})} K_1(R/I) \xrightarrow{\partial_1} K_0(R,I) \xrightarrow{j_0} K_0(R) \xrightarrow{K_0(\mathrm{pr})} K_0(R/I)$$

$$\xrightarrow{\partial_0} K_{-1}(R,I) \xrightarrow{j_{-1}} K_{-1}(R) \xrightarrow{K_{-1}(\mathrm{pr})} K_{-1}(R/I) \xrightarrow{\partial_{-1}} \cdots$$

The existence of the long exact sequence of a two-sided ideal of Theorem 6.12 has been an important requirement of an extension of middle and lower algebraic K-theory to higher degrees. It is indeed an extension of the long exact sequences appearing in Theorem 3.89 and Theorem 5.12.

For more information about the plus-construction we refer for instance to [130], [859, Chapter 5], [922, Chapter 2].

## 6.3 Survey on Main Properties of Algebraic K-Theory of Rings

#### 6.3.1 Compatibility with Finite Products

The basic idea of the proof of the following result for  $n \ge 1$  can be found in [821, (8) in §2 on page 20]. The case  $n \le 1$  follows from Lemma 2.12, Lemma 3.9, and by inspecting Definition 4.1, see also Exercise 4.5.

**Theorem 6.13 (Algebraic** *K*-theory and finite products). Let  $R_0$  and  $R_1$  be rings. Denote by  $pr_i: R_0 \times R_1 \rightarrow R_i$  for i = 0, 1 the projection. Then we obtain for  $n \in \mathbb{Z}$  isomorphisms

$$K_n(\mathrm{pr}_0) \times K_n(\mathrm{pr}_1) \colon K_n(R_0 \times R_1) \xrightarrow{=} K_n(R_0) \times K_n(R_1).$$

#### 6.3.2 Morita Equivalence

The idea of the proof of the next result is essentially the same as that of Theorem 2.10.

**Theorem 6.14 (Morita equivalence for algebraic** *K***-theory).** *For every ring R and integer*  $k \ge 1$  *there are for all*  $n \in \mathbb{Z}$  *natural isomorphisms* 

$$\mu_n \colon K_n(R) \xrightarrow{\cong} K_n(\mathbf{M}_k(R)).$$

#### 6.3.3 Compatibility with Colimits over Directed Sets

We conclude from [821, (12) in §2 on page 20], (at least in the connective setting) and [900, Lemma 6 in Section 7].

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**Theorem 6.15 (Algebraic** *K*-theory and colimits over directed sets). Let  $\{R_i \mid i \in I\}$  be a directed system of rings. Then the canonical map

$$\operatorname{colim}_{i \in I} K_n(R_i) \xrightarrow{\cong} K_n(\operatorname{colim}_{i \in I} R_i)$$

is bijective for  $n \in \mathbb{Z}$ .

Actually, one may consider more generally filtered colimits.

#### 6.3.4 The Bass-Heller-Swan Decomposition

We have already explained the following result for  $n \le 1$  in Theorem 3.72 and Theorem 4.3. Definition 3.68 of  $NK_n(R)$  makes sense for every  $n \in \mathbb{Z}$ . The proof for higher algebraic *K*-theory can be found in [922, Theorem 9.8 on page 207], see also [859, Theorem 5.3.30 on page 295]. More general versions where twistings are allowed and additive categories are considered are presented in [434, 436, 457, 531, 533, 609, 686].

#### Theorem 6.16 (Bass-Heller-Swan decomposition for algebraic K-theory).

(i) There are isomorphisms of abelian groups, natural in R, for  $n \in \mathbb{Z}$ 

$$NK_n(R) \oplus K_n(R) \xrightarrow{=} K_n(R[t]);$$
  
$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]).$$

*There is a sequence, which is natural in* R *and exact, for*  $n \in \mathbb{Z}$ 

$$0 \to K_n(R) \xrightarrow{K_n(k_+) \oplus -K_n(k_-)} K_n(R[t]) \oplus K_n(R[t^{-1}])$$
$$\xrightarrow{K_n(l_+) \oplus K_n(l_-)} K_n(R[t,t^{-1}]) \xrightarrow{C_n} K_{n-1}(R) \to 0$$

where  $k_+$ ,  $k_-$ ,  $l_+$ , and  $l_-$  are the obvious inclusions.

If we regard it as an acyclic  $\mathbb{Z}$ -chain complex, there exists a chain contraction, natural in R;

(ii) If R is regular, then

$$NK_n(R) = \{0\} \quad for \ n \in \mathbb{Z};$$
  
$$K_n(R) = \{0\} \quad for \ n \le -1.$$

#### 6.3.5 Some Information about NK-groups

The proof of the next result can be found in Weibel [995, Corollary 3.2].

**Theorem 6.17**  $(NK_n(R)[1/N]$  vanishes for characteristic *N*). Let *R* be a ring of finite characteristic *N*. Then we get for  $n \in \mathbb{Z}$ 

$$NK_n(R)[1/N] = 0.$$

**Theorem 6.18 (Vanishing criterion of**  $NK_n(RG)$  **for finite groups).** *Let* R *be a ring and let* G *be a finite group. Fix*  $n \in \mathbb{Z}$ *. Suppose*  $NK_n(R) = 0$ *. Then we get* 

$$NK_n(RG)[1/|G|] = 0.$$

*Proof.* This follows from Hambleton-Lück [457, Theorem A].

The following result is taken from Hambleton-Lück [457, Corollary B].

**Theorem 6.19** (*p*-elementary induction for  $NK_n(RG)$ ). Let *R* be a ring and let *G* be a finite group. For all  $n \in \mathbb{Z}$ , the sum of the induction maps

$$\bigoplus_E NK_n(RE)_{(p)} \to NK_n(RG)_{(p)}$$

is surjective, where E runs through all p-elementary subgroups.

The following theorem due to Prasolov [814] is an extension of a result due to Farrell [351] for n = 1 to  $n \ge 1$ .

**Theorem 6.20** ( $NK_n(R)$  is trivial or infinitely generated for  $n \ge 1$ ). Let R be a ring. Then  $NK_n(R)$  is either trivial or infinitely generated as abelian group for  $n \ge 1$ .

**Theorem 6.21 (Vanishing of**  $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k])$  for  $n \le 1$ ,  $k \ge 0$  and finite G of square-free order). Let G be a finite group whose order is square-free. Then  $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k]) = 0$  for  $n \le 1$  and  $k \ge 0$ .

*Proof.* Fix a prime p. We know from Example 5.15 that  $K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is finitely generated for every  $k \leq 0$ . We conclude from Theorem 6.16 that  $K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is finitely generated for every  $n \leq 1$  and  $k \geq 0$  and hence that  $NK_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is finitely generated for every  $n \leq 1$  and  $k \geq 0$ . Theorem 6.20 implies that  $NK_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k])$  is trivial every  $n \leq 1$  and  $k \geq 0$ .

We conclude from [457, Theorem A] that for any ring R, any finite group G, and any prime number p, there is a surjection

$$\bigoplus_{P} NK_n(RP)_{(p)} \to NK_n(RG)_{(p)},$$

where *P* runs through the *p*-subgroups of *G*. This implies that  $NK_n(RG)$  vanishes if  $NK_n(RP)_{(p)}$  vanishes for every prime *p* and every *p*-subgroup *P* of *G*. In particular,  $NK_n(RG)$  vanishes for a finite group *G* of square-free order if  $NK_n(R[\mathbb{Z}/p])_{(p)}$  vanishes for every prime number *p*. Put  $R = \mathbb{Z}[\mathbb{Z}^k]$ . Then  $R[\mathbb{Z}/p] = \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]$  and  $RG = \mathbb{Z}[G \times \mathbb{Z}^k]$ , and we know already that  $NK_n(R[\mathbb{Z}/p])_{(p)}$  vanishes for every prime number *p*,  $n \le 1$  and  $k \ge 0$ . Hence  $NK_n(\mathbb{Z}[G \times \mathbb{Z}^k]) = NK_n(RG)$  vanishes for  $n \le 1$  and  $k \ge 0$  if *G* is a finite group of square-free order.

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Theorem 6.21 has been proved in the case k = 0 by Harmon [467].

**Exercise 6.22.** Let *G* be a finite group of square-free order. Show for all  $k \ge 1$ 

$$K_{n}(\mathbb{Z}[G \times \mathbb{Z}^{k}]) = \begin{cases} K_{1}(\mathbb{Z}G) \oplus K_{0}(\mathbb{Z}G)^{k} \oplus K_{-1}(\mathbb{Z}G)^{k(k-1)/2} & \text{if } n = 1; \\ K_{0}(\mathbb{Z}G) \oplus K_{-1}(\mathbb{Z}G)^{k} & \text{if } n = 0; \\ K_{-1}(\mathbb{Z}G) & \text{if } n = -1; \\ \{0\} & \text{if } n \leq -2. \end{cases}$$

#### 6.3.6 Algebraic K-Theory of Finite Fields

The following result has been proved by Quillen [820].

**Theorem 6.23 (Algebraic K-theory of finite fields).** Let  $\mathbb{F}_q$  be a finite field of order q. Then  $K_n(\mathbb{F}_q)$  vanishes if n = 2k for some integer  $k \ge 1$ , and is a finite cyclic group of order  $q^k - 1$  if n = 2k - 1 for some integer  $k \ge 1$ .

Recall that  $K_0(F) \cong \mathbb{Z}$  and  $K_n(F) = \{0\}$  for  $n \le -1$  if *F* is a field, see Example 2.4 and Theorem 4.7.

#### 6.3.7 Algebraic K-Theory of the Ring of Integers in a Number Field

The computation of the higher algebraic *K*-groups of  $\mathbb{Z}$  or, more generally, of the ring of integers *R* in an algebraic number field *F*, is very hard. Quillen [820] showed that these are finitely generated as abelian groups. Their ranks as abelian groups have been determined by Borel [152].

**Theorem 6.24 (Rational Algebraic K-theory of ring of integers of number fields).** Let R be a ring of integers in an algebraic number field. Let  $r_1$  be the number of distinct embeddings of F into  $\mathbb{R}$  and let  $r_2$  be the number of distinct conjugate pairs of embeddings of F into  $\mathbb{C}$  with image not contained in  $\mathbb{R}$ . Then

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \{0\} & n \leq -1; \\ \mathbb{Q} & n = 0; \\ \mathbb{Q}^{r_1 + r_2 - 1} & n = 1; \\ \mathbb{Q}^{r_1 + r_2} & n \geq 2 \quad and \quad n = 1 \mod 4; \\ \mathbb{Q}^{r_2} & n \geq 2 \quad and \quad n = 3 \mod 4; \\ \{0\} & n \geq 2 \quad and \quad n = 0 \mod 2. \end{cases}$$

We have  $K_n(\mathbb{Z}) = \{0\}$  for  $n \le -1$  and the first values of  $K_n(\mathbb{Z})$  for n = 0, 1, 2, 3, 4, 5, 6, 7 are given by  $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48, \{0\}, \mathbb{Z}, \{0\}, \mathbb{Z}/240.$ 

The Lichtenbaum-Quillen Conjecture makes a prediction about the torsion, see [630, 631], relating the algebraic *K*-groups to number theory via the zeta-function. We refer to the survey article of Weibel [994], where a complete picture about the algebraic *K*-theory of ring of integers in algebraic number fields and in particular of  $K_*(\mathbb{Z})$  is given and a list of relevant references can be found. See also Weibel [998, Section VI.10 on pages 527ff].

An outline of how the next corollary follows from Theorem 6.49 can be found in [821, page 29] and [859, page 294]. It is a basic tool for computations.

**Corollary 6.25.** *Let R be a Dedekind domain with quotient field F. Then there is an exact sequence* 

$$\dots \to K_{n+1}(F) \to \bigoplus_{P} K_n(R/P) \to K_n(R) \to K_n(F) \to \bigoplus_{P} K_{n-1}(R/P)$$
$$\dots \to K_1(F) \to \bigoplus_{P} K_0(R/P) \to K_0(R) \to K_0(F) \to 0$$

where P runs through the maximal ideals of R.

Exercise 6.26. Consider the part of the sequence

$$K_1(\mathbb{Z}) \to K_1(\mathbb{Q}) \xrightarrow{\partial_1} \bigoplus_p K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Q}) \to 0$$

of Corollary 6.25 for  $R = \mathbb{Z}$ . Compute the five terms appearing in it. Guess what the map  $\partial_1$  is and determine the others.

**Exercise 6.27.** Show that the map  $K_n(\mathbb{Z}) \to K_n(\mathbb{Q})$  is injective if n = 2k for  $k \ge 1$ , is surjective if n = 2k - 1 for  $k \ge 2$ , and rationally bijective for  $n \ge 2$ .

## 6.4 Algebraic K-Theory with Coefficients

By invoking the Moore space associated to  $\mathbb{Z}/k$ , one can introduce *K*-theory mod *k*, denoted by  $K_n(R; \mathbb{Z}/k)$ , for any integer  $k \ge 2$  and every  $n \in \mathbb{Z}$ . Its main feature is that there exists a long exact sequence

(6.28) 
$$\cdots \to K_{n+1}(R; \mathbb{Z}/k) \to K_n(R) \xrightarrow{k \cdot \mathrm{id}} K_n(R) \to K_n(R; \mathbb{Z}/k)$$
  
 $\to K_{n-1}(R) \xrightarrow{k \cdot \mathrm{id}} K_{n-1}(R) \to K_{n-1}(R; \mathbb{Z}/k) \to \cdots$ 

The next theorem is due to Suslin [934].

**Theorem 6.29** (Algebraic *K*-theory mod *k* of algebraically closed fields). The inclusion of algebraically closed fields induces isomorphisms on  $K_*(-; \mathbb{Z}/k)$ .

Let *p* be a prime number. Quillen [820] has computed the algebraic *K*-groups for any algebraic extension of the field  $\mathbb{F}_p$  of *p* elements for every prime *p*. One can determine  $K_n(\overline{\mathbb{F}_p}; \mathbb{Z}/k)$  for the algebraic closure  $\overline{\mathbb{F}_p}$  of  $\mathbb{F}_p$  from (6.28). Hence one obtains  $K_n(F; \mathbb{Z}/k)$  for any algebraically closed field of prime characteristic *p* by Suslin's Theorem 6.29.

The next theorem is due to Suslin [935]. We will explain the topological *K*-groups  $K_n^{\text{TOP}}(\mathbb{R})$  and  $K_n^{\text{TOP}}(\mathbb{C})$  of the *C*<sup>\*</sup>-algebras  $\mathbb{R}$  and  $\mathbb{C}$  in Subsection 10.3.2. There are mod *k* versions  $K_n^{\text{TOP}}(\mathbb{R}; \mathbb{Z}/k)$  and  $K_n^{\text{TOP}}(\mathbb{C}; \mathbb{Z}/k)$ , for which a long exact sequence analogous to that of (6.28) exists.

**Theorem 6.30 (Algebraic and topological** *K*-theory mod k for  $\mathbb{R}$  and  $\mathbb{C}$ ). The comparison map from algebraic to topological K-theory induces for all integers  $k \ge 2$  and all  $n \ge 0$  isomorphisms

$$K_n(\mathbb{R};\mathbb{Z}/k) \xrightarrow{\cong} K_n^{\text{TOP}}(\mathbb{R};\mathbb{Z}/k);$$
  
$$K_n(\mathbb{C};\mathbb{Z}/k) \xrightarrow{\cong} K_n^{\text{TOP}}(\mathbb{C};\mathbb{Z}/k).$$

Generalizations of Theorem 6.30 to  $C^*$ -algebras will be discussed in Section 10.7.

Since  $K_n^{\text{TOP}}(\mathbb{C})$  is  $\mathbb{Z}$  for *n* even and vanishes for *n* odd and for every algebraically closed field *F* of characteristic 0 we have an injection  $\overline{\mathbb{Q}} \to F$  for the algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ , Theorem 6.29 and Theorem 6.30 imply for every algebraically closed field *F* of characteristic zero

$$K_n(F; \mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \ge 0, n \text{ even}; \\ \{0\} & n \ge 1, n \text{ odd.} \\ \{0\} & n \le -1. \end{cases}$$

**Exercise 6.31.** Using the fact that  $K_n^{\text{TOP}}(\mathbb{R})$  is 8-periodic and its values for n = 0, 1, 2, 3, 4, 5, 6, 7 are given by  $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \{0\}, \mathbb{Z}, \{0\}, \{0\}, \{0\}, \{0\}, compute <math>K_n(\mathbb{R}; \mathbb{Z}/k)$  and  $K_n^{\text{TOP}}(\mathbb{R}; \mathbb{Z}/k)$  for  $n \in \mathbb{Z}$  and  $k \ge 3$  an odd natural number.

## 6.5 Other Constructions of Connective Algebraic K-Theory

The plus-construction works for rings and finitely generated free or projective modules. However, it turns out that it is important to consider more general situations where one can feed in categories with certain extra structures. The main examples are Quillen's *Q*-construction, see [821, §2], [859, Chapter 5], [922, Chapter 4], designed for exact categories, the group completion construction, see [434, 906], designed for symmetric monoidal categories, and Waldhausen's  $wS_{\bullet}$ -construction, see [979] and Subsection 7.3.2, designed for categories with cofibrations and weak equivalences. Given a ring *R*, the category of finitely generated projective *R*-modules yields examples of the type of categories above and the appropriate construction always yields the same, namely, the *K*-groups as defined by the plus-construction above. The

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*Q*-construction and exact categories can be used to define *K*-theory for the category of finitely generated *R*-modules (dropping projective) or the category of locally free  $O_X$ -modules of finite rank over a scheme *X*. One important feature is that the notion of exact sequences can be different from the one given by split exact sequences, or, equivalently, by direct sums. Whereas in Quillen's setting one needs exact structures in an algebraic sense, Waldhausen's  $wS_{\bullet}$ -construction is also suitable for categories where the input are spaces and one can replace isomorphisms by weak equivalences.

We briefly recall the setup of exact categories beginning with additive categories. A category *C* is called *small* if its objects form a set. An *additive category*  $\mathcal{A}$  is a small category  $\mathcal{A}$  such that for two objects *A* and *B* the morphism set mor<sub> $\mathcal{A}$ </sub>(*A*, *B*) has the structure of an abelian group, there exists a zero-object, i.e., an object which is both initial and terminal, the direct sum  $A \oplus B$  of two objects *A* and *B* exists, and the obvious compatibility conditions hold, e.g., composition of morphisms is bilinear. A *functor of additive categories*  $F : \mathcal{A}_0 \to \mathcal{A}_1$  is a functor respecting the zero-objects such that for two objects *A* and *B* in  $\mathcal{A}_0$  the map mor<sub> $\mathcal{A}_0$ </sub>(*A*, *B*)  $\to$  mor<sub> $\mathcal{A}_1$ </sub>(*F*(*A*), *F*(*B*)) sending *f* to *F*(*f*) respects the abelian group structures and *F*( $A \oplus B$ ) is a model for *F*(A)  $\oplus$  *F*(B).

A *skeleton*  $\mathcal{D}$  of a category *C* is a full subcategory such that  $\mathcal{D}$  is small and the inclusion  $\mathcal{D} \to C$  is an equivalence of categories, or, equivalently, for every object  $C \in C$  there is an object *D* in  $\mathcal{D}$  together with an isomorphism  $C \xrightarrow{\cong} D$  in *C*.

**Definition 6.32 (Exact category).** An *exact category*  $\mathcal{P}$  is a full additive subcategory of some abelian category  $\mathcal{A}$  with the following properties:

- $\mathcal{P}$  is closed under extensions in  $\mathcal{A}$ , i.e., for any exact sequence  $0 \to P_0 \to P_1 \to P_2 \to 0$  in  $\mathcal{A}$  with  $P_0, P_2$  in  $\mathcal{P}$  we have  $P_1 \in \mathcal{P}$ ;
- $\mathcal{P}$  has a small skeleton.

An *exact functor*  $F : \mathcal{P}_0 \to \mathcal{P}_1$  is a functor of additive categories that sends exact sequences to exact sequences.

Examples of exact categories are abelian categories possessing a small skeleton, the category of finitely generated projective R-modules, the category of finitely generated R-modules, the category of vector bundles over a compact space, the category of algebraic vector bundles over a projective algebraic variety, and the category of locally free sheaves of finite rank on a scheme.

An additive category becomes an exact category in the sense of Quillen with respect to split exact sequences. On the other hand there are interesting exact categories where the exact sequences are not necessarily split exact sequences.

The *Q*-construction, see [821, §2], [859, Chapter 5], [922, Chapter 4], assigns to any exact category  $\mathcal{P}$  its *K*-theory space  $K(\mathcal{P})$  and one defines  $K_n(\mathcal{P}) := \pi_n(K(\mathcal{P}))$ for  $n \ge 0$ . If  $\mathcal{P}$  is the category of finitely generated projective *R*-modules, this definition coincides with the Definition 6.9 of  $K_n(R)$  coming from the plus-construction.

The Q-construction allows us to define algebraic K-theory for objects naturally appearing in algebraic geometry, arithmetic geometry, and number theory, since these give exact categories as described above.

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**Example 6.33 (The category of nilpotent endomorphism).** Let NIL(*R*) be the exact category whose objects are pairs (P, f) of finitely generated projective *R*-modules together with nilpotent endomorphisms  $f: P \rightarrow P$ . Its *K*-theory Nil<sub>n</sub>(*R*) :=  $K_n(\text{NIL}(R))$  splits as  $K_n(R) \oplus \widetilde{\text{Nil}}_n(R)$  for  $n \ge 0$  where  $\widetilde{\text{Nil}}_n(R)$  is the cokernel of the homomorphism  $K_n(R) \rightarrow K_n(\text{NIL}(R))$  induced by the obvious functor sending a finitely generated projective *R*-module *P* to 0:  $P \rightarrow P$ . We get for  $n \ge 1$ 

$$NK_n(R) = \widetilde{Nil}_{n-1}(R).$$

This has been considered for n = 1 already in Theorem 3.72. A proof, which works also for the more general context of non-connective *K*-theory of additive categories where a twist with an automorphism is allowed, can be found in [686, Theorem 0.4], see also [436].

## 6.6 Non-Connective Algebraic K-Theory of Additive Categories

The approaches mentioned in Section 6.5 will always yield spaces K(R) such that the algebraic *K*-groups are defined to be its homotopy groups. Since a space has no negative homotopy groups, this definition will not encompass the negative algebraic *K*-groups. In order to take these into account, one has to find appropriate deloopings.

So the task is to replace the space K(R) by a (non-connective) spectrum  $\mathbf{K}(R)$  such that one can define  $K_n(R)$  by  $\pi_n(\mathbf{K}(R))$  for  $n \in \mathbb{Z}$  and this definition coincides with the other definitions for all  $n \in \mathbb{Z}$ . For rings this has been achieved by Gersten [422] and Wagoner [973].

We would like to feed in additive categories.

The category of spectra SPECTRA will be introduced in Section 12.4. Denote by ADDCAT the category of additive categories. There is an obvious notion of the direct sum of two additive categories. We will use a construction of Pedersen-Weibel [800], see also Schlichting [209] or Lück-Steimle [684], of a functor

(6.34) **K**: ADDCAT 
$$\rightarrow$$
 SPECTRA,  $\mathcal{A} \mapsto \mathbf{K}(\mathcal{A})$ .

**Definition 6.35 (Algebraic** *K*-groups of additive categories). We call  $\mathbf{K}(\mathcal{A})$  the *non-connective K*-theory spectrum associated to an additive category. Define for  $n \in \mathbb{Z}$  the *n*-th algebraic *K*-group of an additive category  $\mathcal{A}$  by

$$K_n(\mathcal{A}) := \pi_n(\mathbf{K}(\mathcal{A}))$$

**Definition 6.36 (Flasque and Eilenberg swindle).** An additive category  $\mathcal{A}$  is called *flasque* if there exists a functor of additive categories  $S: \mathcal{A} \to \mathcal{A}$  together with a natural equivalence  $T: \operatorname{id}_{\mathcal{A}} \oplus S \xrightarrow{\cong} S$ . Sometimes the pair (S,T) is called an *Eilenberg swindle*.

The next result follows from Pedersen-Weibel [800], see also Cardenas-Pedersen [209] or Lück-Steimle [684].

#### **Theorem 6.37 (Properties of K** $(\mathcal{A})$ **).**

- (i) If *R* is a ring and  $\mathcal{A}$  is the additive category of finitely generated projective *R*-modules, then  $K_n(\mathcal{A})$  coincides with  $K_n(R)$  for  $n \in \mathbb{Z}$ ;
- (ii) Let  $F_1$  and  $F_2$  be functors of additive categories. If there exists a natural equivalence of such functors from  $F_1$  to  $F_2$ , then the maps of spectra  $\mathbf{K}(F_1)$  and  $\mathbf{K}(F_2)$ are homotopic;

In particular, a functor F: A → A' of additive categories which is an equivalence of categories induces a homotopy equivalence K(F): K(A) → K(A');
(iii) If A is flasque, then K(A) is weakly contractible.

**Exercise 6.38.** Give a definition of  $K_0(\mathcal{A})$  and  $K_1(\mathcal{A})$  as abelian groups in terms of generators and relations such that in the case where *R* is a ring and  $\mathcal{A}$  is the category of finitely generated projective *R*-modules, this definition coincides with the ones appearing in Definitions 2.1 and 3.1. Show that  $K_0(\mathcal{A})$  and  $K_1(\mathcal{A})$  are trivial if  $\mathcal{A}$  is flasque.

**Exercise 6.39.** Let  $\mathcal{A}$  be the category of countably generated projective *R*-modules. Show that  $K_n(\mathcal{A}) = 0$  for all  $n \in \mathbb{Z}$ .

**Remark 6.40** (Non-connective *K*-theory spectra for exact categories). Schlichting [900] has constructed for an exact category  $\mathcal{P}$  a delooping of the space  $K(\mathcal{P})$ . Thus he can assign to an exact category  $\mathcal{P}$  a (non-connective) spectrum  $\mathbf{K}(\mathcal{P})$  and define  $K_n(\mathcal{P}) := \pi_n(\mathbf{K}(\mathcal{P}))$  for  $n \in \mathbb{Z}$ . If  $\mathcal{P}$  is the category of finitely generated projective *R*-modules, this definition coincides with our previous definition of  $K_n(R)$ . If the exact sequences in  $\mathcal{P}$  are given by split exact sequences, this definition agrees with the one of Definition 6.35 when we consider  $\mathcal{P}$  as an additive category.

Later we will use the following construction for additive categories.

Given an additive category  $\mathcal{A}$ , its *idempotent completion* Idem( $\mathcal{A}$ ) is defined to be the following additive category. Objects are morphisms  $p: A \to A$  in  $\mathcal{A}$  satisfying  $p \circ p = p$ . A morphism f from  $p_1: A_1 \to A_1$  to  $p_2: A_2 \to A_2$  is a morphism  $f: A_1 \to A_2$  in  $\mathcal{A}$  satisfying  $p_2 \circ f \circ p_1 = f$ . The identity of an object (A, p) is given by the morphism  $p: (A, p) \to (A, p)$ . The structure of an additive category on A induces the structure of an additive category on Idem( $\mathcal{A}$ ) in the obvious way. A functor of additive categories  $F: \mathcal{A} \to \mathcal{A}'$  induces a functor Idem( $\mathcal{F}$ ): Idem( $\mathcal{A}$ )  $\to$  Idem( $\mathcal{A}'$ ) of additive categories by sending (A, p) to (F(A), F(p)).

There is an obvious embedding

$$\eta(\mathcal{A}): \mathcal{A} \to \text{Idem}(\mathcal{A})$$

sending an object A to  $\operatorname{id}_A: A \to A$  and a morphism  $f: A \to B$  to the morphism given by f again. An additive category  $\mathcal{A}$  is called *idempotent complete* if  $\eta(\mathcal{A}): \mathcal{A} \to \operatorname{Idem}(\mathcal{A})$  is an equivalence of additive categories, or, equivalently, if for every idempotent  $p: A \to A$  in A there exists objects B and C and an isomorphism  $f: A \stackrel{\cong}{\to} B \oplus C$  in  $\mathcal{A}$  such that  $f \circ p \circ f^{-1}: B \oplus C \to B \oplus C$  is given

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by  $\begin{pmatrix} id_B & 0 \\ 0 & 0 \end{pmatrix}$ . The idempotent completion Idem( $\mathcal{A}$ ) of an additive category  $\mathcal{A}$  is idempotent complete.

**Theorem 6.41.** The map  $\eta$  induces an equivalence

$$\mathbf{K}(\eta) \colon \mathbf{K}(\mathcal{A}) \xrightarrow{\simeq} \mathbf{K}(\mathrm{Idem}(\mathcal{A}))$$

on the non-connective K-theory spectra.

*Proof.* This follows from [949, Theorem A.9.1] and [684, Corollary 3.7].

Note that Theorem 6.41 is not true for the standard construction of the connective *K*-theory of an additive category. Therefore in the construction of the connective *K*-theory spectrum we always replace  $\mathcal{A}$  by its idempotent completion Idem( $\mathcal{A}$ ). This passage does not change  $K_n(\mathcal{A})$  for  $n \ge 1$ , but it does change  $K_0(\mathcal{A})$ , see [949, Theorem A.9.1]. This is analogous to the fact that in previous constructions of the connective *K*-theory of a ring we had to take the cross product with  $K_0(\mathcal{A})$ , see Definition 6.9.

Let *R* be a ring. Let *R*-MOD<sub>fgf</sub> and *R*-MOD<sub>fgp</sub> respectively be the additive category of finitely generated free *R*-modules and of finitely generated projective *R*-modules. We obtain an equivalence of additive categories  $\text{Idem}(R-\text{MOD}_{\text{fgf}}) \xrightarrow{\simeq} R-\text{MOD}_{\text{fgp}}$  by sending an object (F, p) to im(p). Let  $\underline{R}_{\oplus}$  be the additive category which has as objects the natural numbers  $0, 1, 2, \ldots$  and morphisms from *m* to *n* are given by (m, n)-matrices over *R*. The composition is given by multiplication of matrices, more precisely, given morphisms  $A: l \to m$  and  $B: m \to n$ , their composite is  $AB: l \to m$ . The direct sum of two objects *m* and *n* is the object m + n and the direct sum of morphisms is given by the block sum of matrices. We have the obvious equivalence of additive categories

$$(6.42) \qquad \underline{R}_{\oplus} \xrightarrow{\simeq} R\text{-}\mathrm{MOD}_{\mathrm{fgf}}$$

which sends an object *m* to  $R^m$  and a morphism  $A: m \to n$  to the *R*-linear homomorphism  $r_A: R^m \to R^n$ ,  $(s_1, \ldots, s_m) \mapsto (s_1, \ldots, s_m)A$  given by right multiplication with *A*. Thus we obtain an equivalence of additive categories, natural in the ring *R*,

(6.43) 
$$\Theta_R : \operatorname{Idem}(R_{\oplus}) \xrightarrow{\simeq} R \operatorname{-MOD}_{\operatorname{fgp}}$$

Note that  $\operatorname{Idem}(\underline{R}_{\oplus})$  is small, in contrast to R-MOD<sub>fgp</sub>. The non-connective K-theory spectrum of a ring  $\mathbf{K}(R)$  is defined to be  $\mathbf{K}(\underline{R}_{\oplus})$  for  $\mathbf{K}$  defined in (6.34). Then  $\pi_n(\mathbf{K}(R))$  can be identified with all other definitions of  $K_n(R)$  above for every  $n \in \mathbb{Z}$ .

## 6.7 Survey on Main Properties of Algebraic K-Theory of Exact Categories

Next we state some basic and important general results about the algebraic *K*-theory of exact categories.

## 6.7.1 Additivity

For a proof of the next result we refer for instance to [821, Corollary 1 in §3 on page 22], [922, Corollary 4.3 on page 41], [998, Theorem 1.2 in Section V.I on page 366] (at least in the connective setting), and [900, Corollary 4 in Section 7].

**Theorem 6.44 (Additivity Theorem for exact categories).** Let  $0 \to F_0 \xrightarrow{i} F_1 \xrightarrow{p} F_2 \to 0$  be an exact sequence of functors  $F_k : \mathcal{P}_1 \to \mathcal{P}_2$  of exact categories  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , i.e., *i* and *p* are natural transformations such that for each object *P* the sequence  $0 \to F_0(P) \xrightarrow{i(P)} F_1(P) \xrightarrow{p(P)} F_2(P) \to 0$  is exact.

Then we get for the induced morphisms  $K_n(F_k)$ :  $K_n(\mathcal{P}_1) \to K_n(\mathcal{P}_2)$  for every  $n \in \mathbb{Z}$ 

$$K_n(F_1) = K_n(F_0) + K_n(F_2).$$

## 6.7.2 Resolution Theorem

Let  $\mathcal{M}$  and  $\mathcal{P}$  be exact categories which are contained in the same abelian category  $\mathcal{A}$ . Suppose that  $\mathcal{P}$  is a full subcategory of  $\mathcal{M}$ . A *finite resolution* of an object M of  $\mathcal{M}$  by objects in  $\mathcal{P}$  is an exact sequence  $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$  for some natural number n. We say that  $\mathcal{P}$  is *closed under extensions in*  $\mathcal{M}$  if for any exact sequence  $0 \to M_0 \to M_1 \to M_2 \to 0$  in  $\mathcal{M}$  with  $M_0, M_2$  in  $\mathcal{P}$  we have  $M_1 \in \mathcal{P}$ . For a proof of the next theorem we refer for instance to [821, Corollary 1 in §4 on page 25] or [922, Theorem 4.6 on page 41], [998, Theorem 3.1 in Section V.3 on page 385] (at least in the connective setting), and [900].

**Theorem 6.45 (Resolution Theorem).** Let  $\mathcal{M}$  and  $\mathcal{P}$  be exact categories which are contained in the same abelian category  $\mathcal{A}$ . Suppose that  $\mathcal{P}$  is a full subcategory of  $\mathcal{M}$  and is closed under extensions in  $\mathcal{M}$ . Suppose that every object in  $\mathcal{M}$  has a finite resolution by objects in  $\mathcal{P}$ .

Then the inclusion  $\mathcal{P} \to \mathcal{M}$  induces for every  $n \in \mathbb{Z}$  an isomorphism

$$K_n(\mathcal{P}) \xrightarrow{\cong} K_n(\mathcal{M}).$$

**Exercise 6.46.** Let *R* be a regular ring. Show that for every  $n \in \mathbb{Z}$  the canonical map  $K_n(R) \rightarrow G_n(R)$  is bijective.

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#### 6.7.3 Devissage

For a proof of the next result we refer for instance to [821, Theorem 4 in §5 on page 28], [922, Theorem 4.8 on page 42], or [998, Theorem 4.1 in Section V.4 on page 400].

**Theorem 6.47 (Devissage).** Let  $\mathcal{A}$  be an abelian category. Let  $\mathcal{B}$  be a full abelian subcategory of  $\mathcal{A}$  which is closed under taking subobjects, quotients, and finite products in  $\mathcal{A}$ . Suppose that each object  $\mathcal{A}$  in  $\mathcal{A}$  has a finite filtration in  $\mathcal{A}$ 

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$$

such that  $A_i/A_{i-1}$  is isomorphic to an object in  $\mathcal{B}$  for i = 1, 2, ..., n.

Then the inclusion of exact categories  $i: \mathcal{B} \to \mathcal{A}$  induces an isomorphism

$$K_n(i): K_n(\mathcal{B}) \xrightarrow{\cong} K_n(\mathcal{A})$$

for  $n \ge 0$ .

Note that in Theorem 6.47 the condition  $n \ge 0$  appears. To the author's knowledge it is not known whether Theorem 6.47 also holds for  $n \le -1$ . If  $\mathcal{A}$  is a Noetherian abelian category, then its negative *K*-groups vanish and Theorem 6.47 also holds for negative *K*-groups for trivial reasons, see [900, Theorem 7].

An object N in an abelian category is called *simple* if  $N \neq 0$  and any monomorphism  $M \rightarrow N$  is the zero-homomorphism or an isomorphism. For a simple object M its ring of automorphisms  $\operatorname{end}_{\mathcal{R}}(M)$  is a skew-field (Schur's Lemma). An object N in an abelian category is called *semisimple* if it is isomorphic to a finite direct sum of simple objects. A zero object is called an object of length 0. Call the simple objects of an abelian category objects of length  $\leq 1$ . We define inductively for  $l \geq 2$  an object M to be of length  $\leq l$  if there exists an exact sequence  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  for an object  $M_1$  of length  $\leq 1$  and an object  $M_2$  of length  $\leq (l-1)$ . An object is of finite length if it has length  $\leq l$  for some natural number l. For a proof of the following corollary of Theorem 6.47 we refer to [821, Corollary 1 in §5 on page 28].

**Corollary 6.48.** Let  $\mathcal{A}$  be an abelian category. Suppose that there is a subset S of the set of objects of  $\mathcal{A}$  with the property that any simple object in  $\mathcal{A}$  is isomorphic to precisely one object in S. Let  $\mathcal{A}_{ss}$  be the full subcategory of  $\mathcal{A}$  consisting of semisimple objects and let  $\mathcal{A}_{fl}$  be the full subcategory consisting of objects of finite length. Then we obtain for every  $n \in \mathbb{Z}$ ,  $n \geq 0$  isomorphisms

$$\bigoplus_{M \in S} K_n(\operatorname{end}_{\mathcal{A}}(M)) \xrightarrow{\cong} K_n(\mathcal{A}_{ss}) \xrightarrow{\cong} K_n(\mathcal{A}_{fl}).$$

In particular we get in the situation of Corollary 6.48 from Example 2.4 and Theorem 3.6

$$K_0(\mathcal{A}_{fl}) \cong \bigoplus_{S} \mathbb{Z};$$
  
$$K_1(\mathcal{A}_{fl}) \cong \prod_{S} \operatorname{end}_{\mathcal{A}}(S)^{\times} / [\operatorname{end}_{\mathcal{A}}(S)^{\times}, \operatorname{end}_{\mathcal{A}}(S)^{\times}].$$

#### 6.7.4 Localization

**Theorem 6.49 (Localization).** Let  $\mathcal{A}$  be a small abelian category and let  $\mathcal{B}$  be an additive subcategory such that for any exact sequence  $0 \to M_0 \to M_1 \to M_2 \to 0$  in  $\mathcal{A}$  the object  $M_1$  belongs to  $\mathcal{B}$  if and only if both  $M_0$  and  $M_2$  belong to  $\mathcal{B}$ . Then there exists a well-defined quotient abelian category  $\mathcal{A}/\mathcal{B}$ . It has the same objects as  $\mathcal{A}$ , and its morphisms are obtained from those in  $\mathcal{A}$  by formally inverting morphisms whose kernel and cokernel belong to  $\mathcal{B}$ .

Then there are obvious functors  $\mathcal{B} \to \mathcal{A}$  and  $\mathcal{A} \to \mathcal{A}/\mathcal{B}$  that induce a long exact sequence

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to \cdots$$

The full description of  $\mathcal{A}/\mathcal{B}$  can be found in [922, Appendix B.3] or [998, Section II.6 on page 119]. A proof of the last theorem is given in [821, Theorem 5 in §5 on page 29], [922, Theorem 4.9 on page 42], [998, Theorem 5.1 in Section V.5 on page 402] (at least in the connective setting), and [900, Theorem 1].

The next example is taken from [998, Application 6.1 in Section V.6 on page 406]

**Example 6.50.** Let R be a Noetherian ring and s be an element in the center of R which is different from 0. Then one can consider the subcategory of finitely generated s-torsion modules of the abelian category of finitely generated R-modules and the localization sequence of Theorem 6.49 reduces to a long exact sequence

$$\cdots \to G_{n+1}(R[s^{-1}]) \to G_n(R/(s)) \to G_n(R) \to G_n(R[s^{-1}])$$
$$\to G_{n-1}(R/(s)) \to G_{n-1}(R) \to \cdots$$

where, roughly speaking,  $R[s^{-1}]$  is obtained from R by inverting s.

Exercise 6.51. Let p be a prime number. Then we obtain a long exact sequence

$$\cdots \to K_{n+1}(\mathbb{Z}[p^{-1}]) \to K_n(\mathbb{F}_p) \to K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[p^{-1}])$$
$$\cdots \to K_1(\mathbb{Z}[p^{-1}]) \to K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[p^{-1}]) \to 0$$

#### 6.7.5 Filtered Colimits

For a proof of the next theorem we refer for instance to [821, (9) in §2 on page 20] or [922, Lemma 3.8 on page 35], [998, (6.4) in Section IV.6 on page 321] (at least in the connective setting), and [900, Corollary 5].

**Theorem 6.52** (*K*-theory and directed colimits). Let  $\mathcal{A}$  be an exact category. Let  $\{\mathcal{A}_i \mid i \in I\}$  be a directed set of exact subcategories of  $\mathcal{A}$ , directed by inclusion such that  $\mathcal{A}$  is the union of the categories  $\mathcal{A}$  in the sense that for every object A in  $\mathcal{A}$  and every morphism  $f : A \to A'$  there is an  $i \in I$  with  $A \in \mathcal{A}$  and  $f \in \mathcal{A}_i$ . Then the canonical map

$$\operatorname{colim}_{i\in I} K_n(\mathcal{A}_i) \to K_n(\mathcal{A})$$

is bijective for  $n \in \mathbb{Z}$ .

Theorem 6.52 holds more generally for filtered colimits.

## 6.8 The *K*-Theoretic Farrell-Jones Conjecture for Torsionfree Groups and Regular Rings

The Farrell-Jones Conjecture for algebraic *K*-theory, which we will formulate in full generality in Conjecture 13.1, reduces for a torsionfree group and a regular ring to the following conjecture. Under the additional assumption that there is a finite model for *BG* it appears already in [503].

**Conjecture 6.53 (Farrell-Jones Conjecture for** *K***-theory for torsionfree groups and regular rings).** Let *G* be a torsionfree group. Let *R* be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for  $n \in \mathbb{Z}$ .

Here  $H_*(-; \mathbf{K}(R))$  denotes the homology theory that is associated to the (nonconnective) *K*-spectrum  $\mathbf{K}(R)$ . Recall that  $H_n(\{\bullet\}; \mathbf{K}(R))$  is  $K_n(R)$  for  $n \in \mathbb{Z}$ , where here and elsewhere  $\{\bullet\}$  denotes the space consisting of one point. The space *BG* is the *classifying space of the group G*, which is up to homotopy characterized by the property that it is a *CW*-complex with  $\pi_1(BG) \cong G$  whose universal covering is contractible. The technical details of the construction of  $H_n(-; \mathbf{K}(R))$  and the assembly map will be explained in a more general setting in Sections 12.4 and 12.5.

The point of Conjecture 6.53 is that on the right-hand side of the assembly map we have the group  $K_n(RG)$  we are interested in, whereas the left-hand side is a homology theory and hence much easier to compute. A basic tool for the computation of a homology theory is the Atiyah-Hirzebruch spectral sequence, which in our case has as  $E^2$ -term  $E_{p,q}^2 = H_p(BG; K_q(R))$  and converges to  $H_{p+q}(BG; \mathbf{K}(R))$ . **Remark 6.54 (The conditions appearing in Conjecture 6.53 are necessary).** The condition that G is torsionfree and that R is regular are necessary in Conjecture 6.53. If one drops one of these conditions, one obtains counterexamples as follows.

If G is a finite group, then we obtain an isomorphism

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(\{\bullet\}; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{=} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Hence Conjecture 6.53 would predict for a finite group that the change of rings homomorphism  $K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$  is bijective. This contradicts for instance Lemma 2.89.

In view of the Bass-Heller-Swan decomposition 6.16, Conjecture 6.53 is true for  $G = \mathbb{Z}$  in degree *n* only if  $NK_n(R)$  vanishes.

**Exercise 6.55.** Let *R* be a regular ring. Let  $G = G_1 *_{G_0} G_2$  be an amalgamated free product of torsionfree groups, where  $G_0$  is a common subgroup of  $G_1$  and  $G_2$ . Suppose that Conjecture 6.53 is true for  $G_0$ ,  $G_1$ ,  $G_2$ , and *G* with coefficients in the ring *R*. Show that then there exists a long exact Mayer-Vietoris sequence

$$\cdots \to K_n(RG_0) \to K_n(RG_1) \oplus K_n(RG_2) \to K_n(RG)$$
$$\to K_{n-1}(RG_0) \to K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \to \cdots$$

**Exercise 6.56.** Let *R* be a regular ring. Let  $\phi: G \to G$  be an automorphism of the torsionfree group *G*. Suppose that Conjecture 6.53 is true for *G* and the semidirect product  $G \rtimes_{\phi} \mathbb{Z}$  with coefficients in the ring *R*. Show that then there exists a long exact Wang sequence

$$\cdots \to K_n(RG) \xrightarrow{\operatorname{id} - K_n(\phi)} K_n(RG) \to K_n(R[G \rtimes_{\phi} \mathbb{Z}])$$
$$\to K_{n-1}(RG) \xrightarrow{\operatorname{id} - K_{n-1}(\phi)} K_{n-1}(RG) \to \cdots$$

**Remark 6.57** ( $K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$  for torsionfree *G*). Rationally the Atiyah-Hirzebruch spectral sequence always collapses and the homological Chern character gives an isomorphism

ch: 
$$\bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah-Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Subsection 12.6.1 and Section 12.7.

Because of Theorem 6.24 the left-hand side of the isomorphism described in Remark 6.57 specializes for  $R = \mathbb{Z}$  to  $H_n(BG; \mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG; \mathbb{Q})$ . Hence Conjecture 6.53 predicts for a torsionfree group *G* 

(6.58) 
$$K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(BG; \mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG; \mathbb{Q}).$$

6.9 Mayer-Vietoris Sequences

**Conjecture 6.59** (Nil-groups for regular rings and torsionfree groups). Let *G* be a torsionfree group and let *R* be a regular ring. Then we get

$$NK_n(RG) = 0$$
 for all  $n \in \mathbb{Z}$ .

**Exercise 6.60.** Show that a torsionfree group G satisfies Conjecture 6.59 for all regular rings R if it satisfies Conjecture 6.53 for all regular rings R.

## 6.9 Mayer-Vietoris Sequences for Amalgamated Free Products and Wang Sequences for HNN-Extensions

We have seen in the introduction that for the topological *K*-theory of reduced group  $C^*$ -algebras there exist Mayer-Vietoris sequences associated to amalgamated free products, see (1.4), and long exact Wang sequences for semidirect products of the shape  $G = H \rtimes_{\phi} \mathbb{Z}$ , see (1.5). These lead to the final formulation of the Baum-Connes Conjecture 1.1. Because of Exercises 6.55 and 6.56 one can expect similar long exact sequences to exist for the algebraic *K*-theory of group rings for torsionfree groups and regular rings, but not in general, as one can derive for instance from the Bass-Heller-Swan decomposition 6.16.

We want to explain the more complicated general answer for the algebraic K-theory of group rings, which is given by Waldhausen [975] and [976].

A ring *R* is called *regular coherent* if every finitely presented *R*-module possesses a finite projective resolution. A ring *R* is regular if and only if it is regular coherent and Noetherian. A group *G* is called *regular* or *regular coherent* respectively if for any regular ring *R* the group ring *RG* is regular or regular coherent. If  $G = G_1 *_{G_0} G_2$ for regular coherent groups  $G_1$  and  $G_2$  and a regular group  $G_0$  or if  $G = H \rtimes_{\phi} \mathbb{Z}$  for a regular group *H*, then *G* is regular coherent. In particular,  $\mathbb{Z}^n$  is regular and regular coherent, whereas a non-abelian finitely generated free group is regular coherent but not regular. For proofs of the claims above and for more information about regular coherent groups we refer to [976, Theorem 19.1].

The maps of spectra appearing in the theorem below are all induced by obvious functors between categories.

**Theorem 6.61 (Waldhausen's cartesian squares for non-connective algebraic** *K***-theory).** Let  $G = G_1 *_{G_0} G_2$  be an amalgamated free product and let *R* be a ring.

(i) There exists a homotopy cartesian square of spectra



where Nil( $RG_0$ ;  $RG_1$ ,  $RG_2$ ) is a certain non-connective Nil-spectrum associated to  $G = G_1 *_{G_0} G_2$  and R and K is the (non-connective) K-theory spectrum;

(ii) There is a map  $\mathbf{f} : \mathbf{K}(RG_0) \lor \mathbf{K}(RG_0) \to \mathbf{Nil}(RG_0; RG_1, RG_2)$  and for k = 1, 2a map  $\mathbf{g}_k : \mathbf{Nil}(RG_0; RG_1, RG_2) \to \mathbf{K}(RG_0)$  with the following properties. The composite  $\mathbf{g}_k \circ \mathbf{f} : \mathbf{K}(RG_0) \lor \mathbf{K}(RG_0) \to \mathbf{K}(RG_0)$  is the projection to the k-th summand, the composite

$$\mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \xrightarrow{\mathbf{I}} \mathbf{Nil}(RG_0; RG_1, RG_2) \xrightarrow{\mathbf{I}} \mathbf{K}(RG_1) \vee \mathbf{K}(RG_2)$$

is homotopic to  $\mathbf{K}(j_1) \vee \mathbf{K}(j_2)$  for  $j_k \colon G_0 \to G_k$  the canonical inclusion, and  $\mathbf{i} \circ \mathbf{f}$  is homotopic to  $\mathbf{id} \vee \mathbf{id} \colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \to \mathbf{K}(RG_0)$ ;

- (iii) If R is regular and  $G_0$  is regular coherent, then  $\mathbf{f} \colon \mathbf{K}(RG_0) \vee \mathbf{K}(RG_0) \rightarrow \mathbf{Nil}(RG_0; RG_1, RG_2)$  is a weak homotopy equivalence;
- (iv) The composite of the map  $\Omega \mathbf{K}(RG) \rightarrow \mathbf{Nil}(RG_0; RG_1, RG_2)$  associated to the homotopy cartesian square of assertion (i) with the canonical map from  $\mathbf{Nil}(RG_0; RG_1, RG_2)$  to the homotopy cofiber of the map **f** induces a split surjection on homotopy groups.

*Proof.* All these claims are proved for connective *K*-theory in Waldhausen [976, 11.2, 11.3, 11.6]. In [75, Section 9 and 10] the definitions and assertions are extended to the non-connective version except for assertion (iv). Assertion (iv) can be derived from the connective version by using the Bass-Heller-Swan decomposition 6.16.

**Theorem 6.62 (Mayer-Vietoris sequence of an amalgamated free product for algebraic** *K*-theory). Let  $G = G_1 *_{G_0} G_2$  be an amalgamated free product and let R be a ring. Denote by  $i_k : G_0 \to G_k$  and  $j_k : G_k \to G$  the obvious inclusions. Define  $NK_n(RG_0; RG_1, RG_2)$  to be the (n - 1)-homotopy group of the homotopy cofiber of the map **f** appearing in Theorem 6.61 (ii). Let  $p_n : K_n(RG) \to NK_n(RG_0; RG_1, RG_2)$  be the split surjection coming from Theorem 6.61 (iv). Then:

(i) We obtain a splitting

$$K_n(RG) \cong \ker(p_n) \oplus NK_n(RG_0; RG_1, RG_2);$$

(ii) There exists a long exact Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) \xrightarrow{K_n(i_1) \oplus K_n(i_2)} K_n(RG_1) \oplus K_n(RG_2)$$

$$\xrightarrow{K_n(j_1) - K_n(j_2)} \ker(p_n) \xrightarrow{\partial_n} K_{n-1}(RG_0)$$

$$\xrightarrow{K_{n-1}(i_1) \oplus K_{n-1}(i_2)} K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \xrightarrow{K_{n-1}(j_1) - K_{n-1}(j_2)} \cdots$$

(iii) If  $G_0$  is regular coherent and R is regular, then

 $NK_n(RG_0; RG_1, RG_2) = 0$  for  $n \in \mathbb{Z}$ 

and the sequence of assertion (ii) reduces to the long exact sequence

6.9 Mayer-Vietoris Sequences

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) \xrightarrow{K_n(i_1) \oplus K_n(i_2)} K_n(RG_1) \oplus K_n(RG_2)$$

$$\xrightarrow{K_n(j_1) - K_n(j_2)} K_n(RG) \xrightarrow{\partial_n} K_{n-1}(RG_0)$$

$$\xrightarrow{K_{n-1}(i_1) \oplus K_{n-1}(i_2)} K_{n-1}(RG_1) \oplus K_{n-1}(RG_2) \xrightarrow{K_{n-1}(j_1) - K_{n-1}(j_2)} \cdots$$

Exercise 6.63. Show that Theorem 6.61 implies Theorem 6.62.

Analogously one gets the next theorem from Waldhausen [975] and [976] using [75, Section 9 and 10].

**Theorem 6.64 (Wang sequence associated to an HNN-extension for algebraic** *K*-theory). Let  $\alpha, \beta: H \to K$  be two injective group homomorphisms. Let G be the *associated HNN-extension and let*  $j: K \to G$  be the canonical inclusion. Then there are certain Nil-groups  $NK_n(RH, RK, \alpha, \beta)$  and homomorphisms  $p_n: K_n(RG) \to$  $NK_n(RH, RK, \alpha, \beta)$  such that the following holds:

(i) There is a long exact Wang sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RH) \xrightarrow{K_n(\alpha) - K_n(\beta)} K_n(RK) \xrightarrow{K_n(j)} \ker(p_n)$$
$$\xrightarrow{\partial_n} K_{n-1}(RH) \xrightarrow{K_{n-1}(\alpha) - K_{n-1}(\beta)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots ;$$

(ii) The map  $p_n: K_n(RG) \to NK_n(RH, RK, \alpha, \beta)$  is split surjective;

(iii) If *R* is regular and *H* is regular coherent, then  $NK_n(RH, RK, \alpha, \beta)$  vanishes for all  $n \in \mathbb{Z}$ . In this case the Wang sequence reduces to

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RH) \xrightarrow{K_n(\alpha) - K_n(\beta)} K_n(RK) \xrightarrow{K_n(j)} K_n(RG)$$
$$\xrightarrow{\partial_n} K_{n-1}(RH) \xrightarrow{K_{n-1}(\alpha) - K_{n-1}(\beta)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots$$

**Remark 6.65 (Wang sequence of a semidirect product**  $G = K \rtimes_{\phi} \mathbb{Z}$  for algebraic *K*-theory). A semidirect product  $G = K \rtimes_{\phi} \mathbb{Z}$  for a group automorphism  $\phi : K \to K$  is a special case of an HNN-extension, namely take H = K,  $\alpha = id$ , and  $\beta = \phi$ . In this case the Wang sequence appearing in Theorem 6.64 (i) takes the form

$$\cdots \xrightarrow{\partial_{n+1}} K_n(RK) \xrightarrow{\operatorname{id} - K_n(\phi)} K_n(RK) \xrightarrow{K_n(j)} \operatorname{ker}(p_n)$$
$$\xrightarrow{\partial_n} K_{n-1}(RK) \xrightarrow{\operatorname{id} - K_{n-1}(\phi)} K_{n-1}(RK) \xrightarrow{K_{n-1}(j)} \cdots$$

and we get an isomorphism

$$N_+K_n(RK,\phi) \oplus N_-K_n(RK,\phi) \xrightarrow{=} NK_n(RK,RK,\mathrm{id},\phi).$$

Here  $N_{\pm}K_n(RK, \phi)$  is the kernel of the split surjection  $K_n(RK_{\phi}[t^{\pm 1}]) \rightarrow K_n(RK)$  that is induced by the homomorphism  $RK_{\phi}[t^{\pm 1}] \rightarrow RK$  obtained by evaluation at t = 0.

Such a Wang sequence is established more generally for additive categories in [686, Theorem 0.1].

We mention the following computation from [666, Corollary 1.14].

**Theorem 6.66 (Vanishing of**  $NK_n(RK, \phi)$ ). Let *R* be a regular ring. Let  $\phi : K \xrightarrow{\cong} K$  be an automorphism of the finite group *K*. Let  $\mathcal{P}(K, R)$  be the set of primes which divide the order of *K* and are not invertible in *R*.

Then for every  $n \in \mathbb{Z}$  the abelian group  $N_{\pm}K_n(RK, \phi)$  vanishes after inverting all primes in  $\mathcal{P}(K, R)$ . In particular, we get  $N_{\pm}K_n(RK, \phi) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$  for all  $n \in \mathbb{Z}$ .

## 6.10 Homotopy Algebraic K-Theory

*Homotopy algebraic K-theory* has been introduced for rings by Weibel [996]. He constructs for a ring R a spectrum **KH**(R) and defines

(6.67) 
$$KH_n(R) := \pi_n(\mathbf{KH}(R)) \quad \text{for } n \in \mathbb{Z}.$$

The main feature of homotopy *K*-theory is that it is *homotopy invariant*, i.e., for every ring *R* and every  $n \in \mathbb{Z}$  the canonical inclusion induces an isomorphism [996, Theorem 1.2 (i)]

Note that homotopy invariance does not hold for algebraic *K*-theory unless *R* is regular, see Theorem 6.16.

A consequence of homotopy invariance is that we get for every ring *R* and  $n \in \mathbb{Z}$  isomorphisms, see [996, Theorem 1.2 (iii)],

(6.69) 
$$KH_n(R) \oplus KH_{n-1}(R) \xrightarrow{=} KH_n(R\mathbb{Z}).$$

Hence the are no Nil-terms appearing for the trivial HNN-extension  $G \times \mathbb{Z}$ . It turns out that there are no Nil-phenomena concerning amalgamated free products and HNN-extensions in general, as illustrated by the next result, which follows from [75, Theorem 11.3].

**Theorem 6.70** (Mayer-Vietoris sequence of an amalgamated free product for homotopy *K*-theory). Let  $G = G_1 *_{G_0} G_2$  be an amalgamated free product and let *R* be a ring. Denote by  $i_k : G_0 \to G_k$  and  $j_k : G_k \to G$  the obvious inclusions.

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Then there exists a Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial_{n+1}} KH_n(RG_0) \xrightarrow{KH_n(i_1) \oplus KH_n(i_2)} KH_n(RG_1) \oplus KH_n(RG_2) \xrightarrow{KH_n(j_1) - KH_n(j_2)} KH_n(RG) \xrightarrow{\partial_n} KH_{n-1}(RG_0) \xrightarrow{KH_{n-1}(i_1) \oplus KH_{n-1}(i_2)} KH_{n-1}(RG_1) \oplus KH_{n-1}(RG_2) \xrightarrow{KH_{n-1}(j_1) - KH_{n-1}(j_2)} \cdots$$

**Theorem 6.71 (Wang sequence associated to an HNN-extension for homotopy** *K***-theory).** Let  $\alpha, \beta: H \to K$  be two injective group homomorphisms. Let *G* be the associated HNN-extension and let  $j: K \to G$  be the canonical inclusion. Then there is a long exact Wang sequence

$$\cdots \xrightarrow{\partial_{n+1}} KH_n(RH) \xrightarrow{KH_n(\alpha) - KH_n(\beta)} KH_n(RK) \xrightarrow{KH_n(j)} KH_n(RG)$$
$$\xrightarrow{\partial_n} KH_{n-1}(RH) \xrightarrow{KH_{n-1}(\alpha) - KH_{n-1}(\beta)} KH_{n-1}(RK) \xrightarrow{KH_{n-1}(j)} \cdots$$

There is a natural map of (non-connective) spectra  $\mathbf{K}(R) \rightarrow \mathbf{KH}(R)$  and hence one obtains natural homomorphisms

(6.72) 
$$K_n(R) \to KH_n(R) \text{ for } n \in \mathbb{Z}.$$

This map is in general neither injective nor surjective. It is bijective if R is regular by Theorem 6.16. In some sense homotopy algebraic K-theory is the best approximation of algebraic K-theory by a homotopy invariant functor.

**Exercise 6.73.** Let  $R = R_0 \oplus R_1 \oplus R_2 \oplus ...$  be a graded ring. Show that the inclusion  $i: R_0 \to R$  induces isomorphisms  $KH_n(R_0) \xrightarrow{\cong} KH_n(R)$  for  $n \in \mathbb{Z}$ .

The same discussion as for the Baum Conjecture in Subsection 1.3.3 leads to the following conjecture.

**Conjecture 6.74 (Farrell-Jones Conjecture for homotopy** *K***-theory for torsion-free groups).** Let *G* be a torsionfree group. Then the assembly map

$$H_n(BG; \mathbf{KH}(R)) \to KH_n(RG)$$

is an isomorphism for every  $n \in \mathbb{Z}$  and every ring *R*.

**Lemma 6.75.** (i) Let R be a ring of finite characteristic N. Then the canonical map from algebraic K-theory to homotopy K-theory induces an isomorphism

$$K_n(R)[1/N] \xrightarrow{=} KH_n(R)[1/N]$$

for all  $n \in \mathbb{Z}$ ;

(ii) Let *H* be a finite group and let *R* be a regular ring. Then for all  $n \in \mathbb{Z}$  the canonical map from algebraic *K*-theory to homotopy *K*-theory

$$K_n(RH) \xrightarrow{\cong} KH_n(RH)$$

becomes an isomorphism after inverting all the primes p which divide the order of H and are not invertible in R.

*Proof.* The proof can be found [88, Lemma 3.11], where in assertion (ii) all primes have to be inverted and  $R = \mathbb{Z}$ . The proof in [88, Lemma 3.11] carries over to the case where *R* is any regular ring and one has only to invert all the primes *p* which divide the order of *H* and are not invertible in *R* because of Theorem 6.66.

**Conjecture 6.76 (Comparison of algebraic K-theory and homotopy** *K***-theory for torsionfree groups).** Let *R* be a regular ring and let *G* be a torsionfree group. Then the canonical map

$$K_n(RG) \rightarrow KH_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

Note that Conjecture 6.76 follows from Conjecture 6.53 and Conjecture 6.74.

## 6.11 Algebraic K-Theory and Cyclic Homology

Fix a commutative ring k, referred to as the ground ring. Let R be a k-algebra. We denote by  $HH_*^{\otimes k}(R)$  the Hochschild homology of R relative to the ground ring k, and similarly by  $HC_*^{\otimes k}(R)$ ,  $HP_*^{\otimes k}(R)$ , and  $HN_*^{\otimes k}(R)$  the cyclic, the periodic cyclic, and the negative cyclic homology of R relative to k. Hochschild homology receives a map from the algebraic K-theory, which is known as the Dennis trace map. There are variants of the Dennis trace taking values in cyclic, periodic cyclic, and negative cyclic homology (sometimes called Chern characters), as displayed in the following commutative diagram.

(6.77)  
$$HN_{*}^{\otimes_{k}}(R) \longrightarrow HP_{*}^{\otimes_{k}}(R)$$
$$\downarrow h \qquad \qquad \downarrow h$$
$$K_{*}(R) \xrightarrow{\operatorname{dtr}} HH_{*}^{\otimes_{k}}(R) \longrightarrow HC_{*}^{\otimes_{k}}(R).$$

For the definition of these maps, see [636, Chapters 8 and 11] and [674, Section 5]. The article [674] investigates which parts of  $K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$  can be detected by using the linear traces above. Here is an example, see [674, Theorem 0.7].
6.12 Notes

**Theorem 6.78 (Detection Result for**  $\mathbb{Q}$  and  $\mathbb{C}$  as coefficients). For every group *G* and every integer  $n \ge 0$ , there exist injective homomorphisms

$$\bigoplus_{(C)\in(\mathcal{FC}\mathcal{Y})} H_*(BN_GC;\mathbb{Q}) \to K_*(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q};$$
$$\bigoplus_{(g)\in\operatorname{con}(G),|g|<\infty} H_*(BC_G\langle g\rangle;\mathbb{C}) \to K_*(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C},$$

where we denote by  $(\mathcal{FCY})$  the conjugacy classes of finite cyclic subgroups of G, by  $\operatorname{con}(G)$  the set of conjugacy classes (g) of elements  $g \in G$ , by  $N_GC$  the normalizer of  $C \subseteq G$ , and by  $C_G\langle g \rangle$  the centralizer of  $g \in G$ .

**Remark 6.79.** In [150], Bökstedt, Hsiang, and Madsen define the *cyclotomic trace*, a map out of *K*-theory, which takes values in *topological cyclic homology*. The cyclotomic trace map can be thought of as an even more elaborate refinement of the Dennis trace map. In contrast to the Dennis trace, the cyclotomic trace has the potential to detect almost all of the rationalized *K*-theory of an integral group ring. This question is investigated in detail by Lück-Rognes-Reich-Varisco [675, 676]. More information will be given in Subsection 15.11.2.

# **6.12** Notes

A good source of survey articles about algebraic *K*-theory is the handbook of *K*-theory, edited by Friedlander and Grayson [406]. There the relevance of higher algebraic *K*-theory for algebra, topology, arithmetic geometry, and number theory is explained. Other good sources are the books by Rosenberg [860], Srinivas [922], and Weibel [998].

The relation of the exact sequences for amalgamated free products and HNNextensions appearing in Sections 6.9 and 6.10 to the Farrell-Jones Conjecture is explained in Section 15.7.

The exact sequences for amalgamated free products and HNN-extensions appearing in Sections 6.9 and 6.10 are the main ingredients in the proof that Conjecture 6.53 holds for a certain class of groups  $C\mathcal{L}$ , see [976, Theorem 19.4 on page 249] in the connective case and [75, Corollary 0.12] in general. The class  $C\mathcal{L}$  is described and analyzed in [976, Definition 19.2 on page 248 and Theorem 17.5 on page 250] and [75, Definition 0.10]. It is closed under taking subgroups and contains for instance all torsionfree one-relator groups.

We remark that algebraic K-theory does commute with infinite products for additive categories, see [212] and also [573, Theorem 1.2], but not with infinite products of rings.

The question of finding conditions under which the long exact sequence associated to a pullback of rings (see Remark 4.4 and Remark 5.11) can be extended to higher algebraic K-theory is investigated by Land-Tamme [616], actually for ring spectra.

The group  $K_{2n}(RG)$  is finite for every finite group G, every ring of integer R in a number field, and every  $n \ge 1$ , see [597, Theorem 1.1].

# Chapter 7 Algebraic *K*-Theory of Spaces

# 7.1 Introduction

We give a brief introduction to the *K*-theory of spaces called *A*-theory. This theory was initialized by Waldhausen. Its benefit is that it allows us to study interesting spaces of geometric structures such as groups of diffeomorphisms or homeomorphism of manifolds, pseudoisotopy spaces, spaces of *h*-cobordisms, and Whitehead spaces. It is an instance of a very successful strategy in topology to extend algebraic notions to spaces. Other examples of this type are topological Hochschild homology and topological cyclic homology. We will see in Section 9.21 how the results of this chapter combined with surgery theory lead to quite explicit results about the homotopy groups of the space Top(M) of self-homeomorphisms and the space Diff(M) of self-diffeomorphisms for an aspherical closed (smooth) manifold *M*.

## 7.2 Pseudoisotopy

Let *I* denote the unit interval [0, 1]. A topological *pseudoisotopy* of a compact manifold *M* is a homeomorphism  $h: M \times I \to M \times I$  that restricted to  $M \times \{0\} \cup \partial M \times I$  is the obvious inclusion. The space P(M) of pseudoisotopies is the group of all such homeomorphisms, where the group structure comes from composition. If we allow *M* to be non-compact, we will demand that *h* has compact support, i.e., there is a compact subset  $C \subseteq M$  such that h(x, t) = (x, t) for all  $x \in M - C$  and  $t \in [0, 1]$  holds.

There is a stabilization map  $P(M) \rightarrow P(M \times I)$  given by crossing a pseudoisotopy with the identity on the interval *I* and the stable pseudoisotopy space is defined as  $\mathcal{P}(M) = \operatorname{hocolim}_{j \to \infty} P(M \times I^j)$ . There also exist smooth versions  $P^{\text{DIFF}}(M)$  and  $\mathcal{P}^{\text{DIFF}}(M) = \operatorname{hocolim}_{j \to \infty} P^{\text{DIFF}}(M \times I^j)$ . For closed manifolds of dimension  $\geq 6$ , the PL-version agrees with the topological version, see [191].

The natural maps  $P^{\text{Diff}}(M) \to \mathcal{P}^{\text{Diff}}(M)$  induce isomorphisms on the *i*-th homotopy group if the dimension *n* of *M* is large compared to the dimension *n* by work of Igusa, building on earlier work of Hatcher, roughly for  $i \le n/3$ , see [471, 512]. This implies that the same also holds for the map  $P(M) \to \mathcal{P}(M)$  for smoothable manifolds *M*, see the argument in [981, Corollary 1.4.2]. Meanwhile more information on this connectivity range is known due to work of Goodwillie, Krannich, Kupers, and Randal-Williams, see [430, 583, 584]. This is most conveniently stated in terms of the following quantities on the connectivity of the single stabilization map  $s_M : P^{\text{Diff}}(M) \to P^{\text{Diff}}(M \times I)$ 

 $\phi(M) := \min\{k \ge 0 \mid s_{M \times I^m} \text{ is } k \text{-connected for all } m \ge 0\};$  $\phi(M)_{\mathbb{Q}} := \min\{k \ge 0 \mid s_{M \times I^m} \text{ is rationally } k \text{-connected for all } m \ge 0\}.$ 

Note that lower bounds for  $\phi(M)$  imply lower bounds for  $\phi(M)_{\mathbb{Q}}$  and upper bounds for  $\phi(M)_{\mathbb{Q}}$  imply upper bounds for  $\phi(M)$ . Igusa's work shows  $\phi(M) \ge n/3$  for all M (precisely:  $\phi(M) \ge \min(\frac{n-4}{3}, \frac{n-7}{2})$ ) and the following can be extracted from the papers by Goodwillie, Krannich, Kupers, and Randal-Williams mentioned above:

(i) φ(M)<sub>Q</sub> ≤ n - 4 for spin-manifolds M of dimension n ≥ 5;
(ii) φ(M)<sub>Q</sub> = n - 4 for simply connected spin-manifolds M of dimension n ≥ 10.

Next we want to define a delooping of P(M). Let  $p: M \times \mathbb{R}^k \times I \to \mathbb{R}^k$  denote the natural projection. For a manifold M the space  $P_b(M; \mathbb{R}^k)$  of bounded pseudoisotopies is the space of all self-homeomorphisms  $h: M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$  satisfying: (i) The restriction of h to  $M \times \mathbb{R}^k \times \{0\} \cup \partial M \times \mathbb{R}^k \times [0, 1]$  is the inclusion, (ii) the map h is bounded in the  $\mathbb{R}^k$ -direction, i.e., the set  $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$  is a bounded subset of  $\mathbb{R}^k$ , and (iii) the map h has compact support in the M-direction, i.e., there is a compact subset  $C \subseteq M$  such that h(x, y, t) = (x, y, t) for all  $x \in M - C$ ,  $y \in \mathbb{R}^k$  and  $t \in [0, 1]$ . There is an obvious stabilization map  $P_b(M; \mathbb{R}^k) \to P_b(M \times I; \mathbb{R}^k)$  and a stable bounded pseudoisotopy space  $\mathcal{P}_b(M; \mathbb{R}^k) \to \Omega \mathcal{P}_b(M; \mathbb{R}^{k+1})$ , see [472, Appendix II]. Hence the sequences of spaces  $\mathcal{P}_b(M; \mathbb{R}^k)$  for  $k = 0, 1, 2, \ldots$  and  $\Omega^{-i} \mathcal{P}_b(M)$  for  $i = 0, -1, -2, \ldots$  define an  $\Omega$ -spectrum  $\mathbf{P}(M)$ . Analogously one defines the differentiable bounded pseudoisotopy space

**Definition 7.1 ((Non-connective) pseudoisotopy spectrum).** We call the  $\Omega$ -spectra  $\mathbf{P}(X)$  and  $\mathbf{P}^{\text{DIFF}}(X)$  associated to a topological space *X* the (non-connective) pseudoisotopy spectrum and the smooth (non-connective) pseudoisotopy spectrum of *X*.

**Remark 7.2 (Strict Functoriality).** A priori the pseudoisotopy space and its nonconnective version are only homotopy functors in the following sense. They assign to a map between manifolds only a homotopy class of maps between the pseudoisotopy spaces and not a specific map. At least the homotopy class of maps between the pseudoisotopy spaces depends only on the homotopy class of the map between manifolds we started with. The homotopy class of the identity is sent to the homotopy class of the identity and the construction is compatible with composition up to homotopy. Moreover, it is a priori not clear what the values of the pseudoisotopy space on general topological spaces are.

There are several places in the literature where a construction as a strict functor from the category of topological spaces to the category of non-connective spectra is indicated, but it seems to be the case that the only places where all the details of this non-trivial extensions are carried out in the smooth, topological, and PL category are the PhD theses of Enkelmann [343] and Pieper [810]. This is important

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for the construction of the assembly map appearing in the Farrell-Jones Conjecture for pseudoisotopy spaces 15.63, since we want the pseudoisotopy functor to digest for instance classifying spaces of groups and groupoids, which obviously are not compact manifolds in general, and to construct the assembly map we need strict functoriality.

**Theorem 7.3 (Pseudoisotopy is a homotopy-invariant functor).** Let  $f: X \to Y$  be a weak homotopy equivalence. Then the induced maps

$$\mathbf{P}(f) \colon \mathbf{P}(X) \to \mathbf{P}(Y);$$
$$\mathbf{P}^{\mathsf{DIFF}}(f) \colon \mathbf{P}^{\mathsf{DIFF}}(X) \to \mathbf{P}^{\mathsf{DIFF}}(Y),$$

are weak homotopy equivalences.

Proof. See [472, Proposition 1.3].

**Remark 7.4.** There is also a PL-version  $\mathbf{P}^{PL}(X)$  of  $\mathbf{P}(X)$ . Since the canonical map  $\mathbf{P}^{PL}(X) \rightarrow \mathbf{P}(X)$  is a weak homotopy equivalence, we do not consider it further.

# 7.3 Whitehead Spaces and A-Theory

#### 7.3.1 Categories with Cofibrations and Weak Equivalences

The following definition is a generalization of the notion of an exact category of Definition 6.32 in the sense of Quillen. It allows us to deal with spaces instead of algebraic objects such as modules. It is due to Waldhausen.

A category *C* is called *pointed* if it comes with a distinguished *zero-object*, i.e., an object that is both initial and terminal.

**Definition 7.5 (Category with cofibrations and weak equivalences).** A *category with cofibrations and weak equivalences* is a small pointed category with a subcategory *coC*, called the *category of cofibrations*, in *C* and a subcategory *wC*, called the *category of weak equivalences*, in *C* such that the following axioms are satisfied:

- (i) The isomorphisms in C are cofibrations, i.e., belong to coC;
- (ii) For every object C the map  $* \to C$  is a cofibration, where \* is the distinguished zero-object;
- (iii) If in the diagram  $A \xleftarrow{i} B \xrightarrow{f} C$  the left arrow is a cofibration, the pushout



exists and  $\overline{i}$  is a cofibration;

(iv) The isomorphisms in C are contained in wC;

(v) If in the commutative diagram



the horizontal arrows on the left are cofibrations, and all vertical arrows are weak equivalences, then the induced map on the pushout of the upper row to the pushout of the lower row is a weak homotopy equivalence.

**Example 7.6 (Exact categories are categories with cofibrations and weak equivalences).** Let  $\mathcal{P} \subseteq \mathcal{A}$  be an exact category in the sense of Definition 6.32. The zero-object is just a zero-object in the abelian category  $\mathcal{A}$ . A cofibration in  $\mathcal{P}$  is a morphism  $i: A \to B$  that occurs in an exact sequence  $0 \to A \to B \to C \to 0$  of  $\mathcal{P}$ . The weak equivalences are given by the isomorphisms.

**Exercise 7.7.** Let *C* be the category of finite projective *R*-chain complexes. Define cofibrations to be chain maps  $i_* : C_* \to D_*$  such that  $i_n : C_n \to D_n$  is split injective for all  $n \ge 0$ . Define weak equivalences to be homology equivalences. Show that *C* is a category with cofibrations and weak equivalences in the sense of Definition 7.5, ignoring the fact that *C* is not small.

**Example 7.8 (The category**  $\mathcal{R}(X)$  **of retractive spaces).** Let *X* be a space. A retractive space over *X* is a triple (Y, r, s) consisting of a space *Y* and maps  $s: X \to Y$  and  $r: Y \to X$  such that *s* is a cofibration and  $r \circ s = id_X$ . A morphism from (Y, r, s) to (Y', r', s') is a map  $f: Y \to Y'$  satisfying  $r' \circ f = r$  and  $f \circ s = s'$ . The zero-object is  $(X, id_X, id_X)$ . A morphism  $f: (Y, r, s) \to (Y', r', s')$  is declared to be a cofibration if the underlying map of spaces  $f: Y \to Y'$  is a cofibration. Now there are several possibilities to define weak equivalences. One may require that  $f: Y \to Y'$  is a homeomorphism, a homotopy equivalence, weak homotopy equivalence, or a homology equivalence with respect to some fixed homology theory. Then one obtains a category  $\mathcal{R}(X)$  with cofibrations and weak equivalences in the sense of Definition 7.5 except that  $\mathcal{R}(X)$  is not small.

To achieve that  $\mathcal{R}(X)$  is small and later to get interesting *K*-theory, one may for instance require that (Y, X) is a relative *CW*-complex which is relatively finite,  $s: X \to Y$  is the inclusion, and morphisms are cellular maps. Denote this category with cofibrations and weak equivalences by  $\mathcal{R}^f(X)$ , where we choose all weak homotopy equivalences as weak equivalences and inclusion of relative *CW*-complexes as cofibrations.

#### 7.3.2 The $wS_{\bullet}$ -Construction

Let *C* be a category with cofibrations and weak equivalences. Next we briefly recall Waldhausen's  $wS_{\bullet}$ -construction, see [979, Section 1.3].

For an integer  $n \ge 0$  let [n] be the ordered set  $\{0, 1, 2, ..., n\}$ . Let  $\Delta$  be the category whose set of objects is  $\{[n] \mid n = 0, 1, 2, ...\}$  and whose set of morphisms from [m]to [n] consists of the order preserving maps. A simplicial category is a contravariant functor from  $\Delta$  to the category CAT of categories. Analogously, a simplicial category with cofibrations and weak equivalences is a contravariant functor from  $\Delta$  to the category CAT<sub>cof,weq</sub> of categories with cofibrations and weak equivalences. Now we assign to *C* a simplicial category with cofibrations and weak equivalences *S*•*C* as follows. Define  $S_nC$  to be the category for which an object is a sequence of cofibrations  $A_{0,1} \xrightarrow{k_{0,1}} A_{0,2} \xrightarrow{k_{0,2}} \cdots \xrightarrow{k_{0,n-1}} A_{0,n}$  together with explicit choices of quotient objects  $\operatorname{pr}_{i,j} : A_{0,j} \to A_{i,j} = A_{0,j}/A_{0,i}$  for  $i, j \in \{1, 2, ..., n\}, i < j$ , i.e., we fix pushouts



Morphisms are given by a collection of morphisms  $\{f_{i,j}\}$  which make the obvious diagrams commute.

With these explicit choices of quotient objects, it is easy to define the relevant face and degeneracy maps. For instance the face map  $d_i: S_n C \to S_{n-1}C$  is given for  $i \ge 1$  by dropping  $A_{0,i}$  and for i = 0 by passing to  $A_{0,2}/A_{0,1} \to A_{0,3}/A_{0,1} \to \cdots \to A_{0,n}/A_{0,1}$ . An arrow in  $S_n C$  is declared to be a cofibration if each arrow  $A_{i,j} \to A'_{i,j}$  is a cofibration and analogously for weak equivalences.

We obtain a simplicial category  $wS \cdot C$  by considering the category of weak equivalences of  $S \cdot C$ . Let  $|wS \cdot C|$  be the geometric realization of the simplicial category  $wS \cdot C$  which is the geometric realization of the bisimplicial set obtained by the composite of the functor nerve of a category with  $wS \cdot C$ .

**Definition 7.9 (Algebraic** *K*-theory space of a category with cofibrations and weak equivalences). Let *C* be a category with cofibrations and weak equivalences. Its *algebraic K*-theory space K(C) is defined by

$$K(C) := \Omega |wS_{\bullet}C|.$$

The 1-skeleton in the  $S_{\bullet}$  direction of  $|wS_{\bullet}C|$  is obtained from  $|wSC| \times [0, 1] = |wS_1C| \times \Delta_1$  by collapsing  $\{*\} \times [0, 1] \cup |wSC| \times \{0\}$  to a point because of  $|wS_0| = \{\bullet\}$ . Hence there is a canonical map  $|wC| \rightarrow \Omega |wS_{\bullet}C|$  that is the adjoint of the obvious identification of the 1-skeleton in the  $S_{\bullet}$ -direction of  $|wS_{\bullet}C|$  with the reduced suspension  $|wC| \wedge S^1$ . If we apply the construction to  $S_nC$ , we obtain a map of spaces  $|wS_nC| \rightarrow \Omega |wS_{\bullet}S_nC|$ . The collection of these maps for  $n \ge 0$  yields a map of simplicial spaces and hence by geometric realization a map of

spaces  $|wS_{\bullet}C| \rightarrow \Omega |wS_{\bullet}S_{\bullet}C|$ . By iterating this construction, we obtain a sequence of maps

$$|wC| \rightarrow \Omega |wS_{\bullet}C| \rightarrow \Omega \Omega |wS_{\bullet}S_{\bullet}C| \rightarrow \Omega \Omega \Omega |wS_{\bullet}S_{\bullet}S_{\bullet}C| \rightarrow \cdots$$

such that all maps except the first one are weak homotopy equivalences. So K(C) is an infinite loop space beyond the first term.

### 7.3.3 A-Theory

Next we recall Waldhausen's definition of the *A*-theory of a topological space, see [979, Chapter 2].

**Definition 7.10 (Connective** *A***-theory).** Let *X* be a topological space. Let  $\mathcal{R}^f(X)$  be the category with cofibrations and weak equivalences defined in Example 7.8. Define the *A*-theory space *A*(*X*) associated to *X* to be the algebraic *K*-theory space *K*( $\mathcal{R}^f(X)$ ) in the sense of Definition 7.9.

**Remark 7.11 (The**  $wS_{\bullet}$ -construction encompasses the *Q*-construction). Waldhausen's construction encompasses the *Q*-construction of Quillen, see [979, Section 1.9].

As in the case of the algebraic *K*-theory of rings or pseudoisotopy, it will be crucial for us to consider a non-connective version. Vogell [967] has defined a delooping of A(X) yielding a non-connective  $\Omega$ -spectrum  $\mathbf{A}(X)$  for a topological space. The idea is similar to the construction of the (non-connective) pseudoisotopy spectrum in Section 7.2, where one considers parametrizations over  $\mathbb{R}^k$  and imposes control conditions. This construction actually yields a covariant functor from the category of topological spaces to the category of  $\Omega$ -spectra

(7.12) 
$$A: \text{TOP} \rightarrow \Omega\text{-SPECTRA}.$$

**Definition 7.13 (Non-connective** *A***-theory).** We call  $\mathbf{A}(X)$  the (non-connective) *A*-theory spectrum associated to the topological space *X*. We write for  $n \in \mathbb{Z}$ 

$$A_n(X) := \pi_n(\mathbf{A}(X)).$$

Note that  $A_n(X)$  agrees with  $\pi_n(A(X))$  for  $n \ge 1$  if A(X) is the space appearing in Definition 7.10. Actually there is a map of spectra, natural in X,

(7.14) 
$$\mathbf{i}(X): A(X) \to \mathbf{A}(X)$$

which induces isomorphisms  $\pi_n(\mathbf{i}(X)): \pi_n(A(X)) \xrightarrow{\cong} \pi_n(\mathbf{A}(X))$  for  $n \ge 1$ .

**Remark 7.15**  $(\pi_0(A(X)))$ . If X is path connected, then  $A_0(X) \cong \mathbb{Z}$ . The isomorphism comes from taking the Euler characteristic of a relatively finite relative *CW*-complex (Y, X).

#### 7.3 Whitehead Spaces and A-Theory

One may replace in the definition of A(X) the category  $\mathcal{R}^f(X)$  by the full subcategory of  $\mathcal{R}(X)$  of those triples (Y, r, s) such that (Y, X) is a relative *CW*complex consisting of countably many cells,  $s: X \to Y$  is the inclusion and the object (Y, r, s) is up to homotopy the retract of an object (Y', r', s') such that (Y', X)is a relatively finite relative *CW*-complex. Then  $\pi_n(A(X))$  is unchanged for  $n \ge 1$ , whereas  $\pi_0(A(X))$  can now be identified with  $K_0(\mathbb{Z}[\pi_1(X)])$  if X is path connected. The identification comes from taking an appropriate finiteness obstruction. With this new definition the map  $\pi_0(\mathbf{i}): \pi_0(A(X)) \to \pi_0(\mathbf{A}(X))$  is bijective.

For the proof of the next result see [979, Proposition 2.1.7].

**Theorem 7.16** (*A*-theory is a homotopy-invariant functor). Let  $f: X \to Y$  be a weak homotopy equivalence. Then the induced maps

$$A(f): A(X) \to A(Y);$$
  
$$\mathbf{A}(f): \mathbf{A}(X) \to \mathbf{A}(Y),$$

are weak homotopy equivalences.

Let X be a connected space with fundamental group  $\pi = \pi_1(X)$  which admits a universal covering  $p_X: \widetilde{X} \to X$ . Consider an object in  $\mathcal{R}^f(X)$ . Recall that it is given by a relatively finite relative *CW*-complex (Y, X) together with a map  $r: Y \to X$ satisfying  $r|_X = \operatorname{id}_X$ . Let  $\widetilde{Y} \to Y$  be the  $\pi$ -covering obtained from  $p_X: \widetilde{X} \to X$  by the pullback construction applied to  $r: Y \to X$ . The cellular  $\mathbb{Z}\pi$ -chain complex  $C_*(\widetilde{Y}, \widetilde{X})$ of the relative free  $\pi$ -*CW*-complex  $(\widetilde{Y}, \widetilde{X})$  is a finite free  $\mathbb{Z}\pi$ -chain complex. This yields a functor of categories with cofibrations and weak equivalences from  $\mathcal{R}^f(X)$ to the category of finite free  $\mathbb{Z}\pi$ -chain complexes. The algebraic *K*-theory of the category of finite free  $\mathbb{Z}\pi$ -chain complexes agrees with that of the finitely generated free  $\mathbb{Z}\pi$ -modules. Hence we get a natural map of spectra called the *linearization map* 

(7.17) 
$$\mathbf{L}(X): \mathbf{A}(X) \to \mathbf{K}(\mathbb{Z}\pi_1(X)).$$

The next result follows by combining [968, Section 4] and [978, Proposition 2.2 and Proposition 2.3].

**Theorem 7.18 (Connectivity of the linearization map).** Let X be a connected CW-complex. Then:

(i) The linearization map L(X) of (7.17) is 2-connected, i.e., the map

$$L_n(X) := \pi_n(\mathbf{L}(X)) \colon A_n(X) \to K_n(\mathbb{Z}\pi_1(X))$$

*is bijective for*  $n \le 1$  *and surjective for* n = 2*;* 

(ii) Rationally the map  $L_n(X)$  is bijective for all  $n \in \mathbb{Z}$  provided that X is aspherical.

**Exercise 7.19.** Show that the canonical map of spectra  $A(\{\bullet\}) \rightarrow A(\{\bullet\})$  is a weak homotopy equivalence.

**Remark 7.20.** We obtain from the transformation L of (7.17) for every group G and every  $n \in \mathbb{Z}$  a commutative diagram

whose horizontal arrows are assembly maps. We conclude from Theorem 7.18 that its vertical arrows are bijective for  $n \le 1$ , surjective for n = 2, and rationally bijective for all  $n \in \mathbb{Z}$ . Hence the upper horizontal arrow is rationally bijective if and only if the lower horizontal arrow is rationally bijective. Recall that Conjecture 6.53 says that the lower horizontal map is bijective. So one may wonder whether the upper horizontal is always bijective. The answer is no, already for  $G = \mathbb{Z}$  the assembly map

$$H_n(B\mathbb{Z}; \mathbf{A}(\{\bullet\})) = A_{n-1}(\{\bullet\}) \oplus A_n(\{\bullet\}) \to A_n(B\mathbb{Z}) = \mathbf{A}(S^1)$$

is known to be not surjective by the following consideration.

Let  $NA_n(\{\bullet\})$  be the Nil-term occurring in the Bass-Heller-Swan-isomorphisms for non-connective *A*-theory, see [510, 511],

(7.22) 
$$A_n(S^1) = A_n(\{\bullet\}) \oplus A_{n-1}(\{\bullet\}) \oplus NA_n(\{\bullet\}) \oplus NA_n(\{\bullet\}).$$

We conclude  $NA_n(\{\bullet\}) = \{0\}$  for  $n \leq 1$  and  $NA_n(\{\bullet\}) \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$  for  $n \in \mathbb{Z}$  from (7.21) and Theorem 13.51. On the other hand,  $NA_n(\{\bullet\})$  for n = 2, 3 is an infinite-dimensional  $\mathbb{F}_2$ -vector space. For more information about  $NA_n(\{\bullet\})$  we refer to Grunewald-Klein-Macko [445] and Hesselholt [478].

**Exercise 7.23.** Show that the linearization map

$$L_2(S^1): A_2(S^1) \to K_2(\mathbb{Z}\pi_1(S^1))$$

is not injective using the fact that  $Wh_2(\mathbb{Z})$  vanishes.

#### 7.3.4 Whitehead Spaces

Waldhausen [978, 979] defines the functor Wh(X) from spaces to infinite loop spaces, which can be viewed as connective  $\Omega$ -spectra, and a fibration sequence

(7.24) 
$$X_+ \wedge A(\{\bullet\}) \to A(X) \to Wh(X).$$

Here  $X_+ \wedge A(\{\bullet\}) \to A(X)$  is an assembly map. After taking homotopy groups, it can be compared with the algebraic *K*-theory assembly map that appears in Conjecture 6.53 via a commutative diagram

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(7.25) 
$$\begin{aligned} \pi_n(X_+ \wedge A(\{\bullet\})) &\longrightarrow \pi_n(A(X)) \\ &\cong \bigvee & & \downarrow \\ \pi_n(X_+ \wedge \mathbf{A}(\{\bullet\})) &= H_n(X; \mathbf{A}(\{\bullet\})) &\longrightarrow \pi_n(\mathbf{A}(X)) \\ & & \downarrow \\ H_n(X; \mathbf{L}) & & \downarrow \\ H_n(B\pi_1(X); \mathbf{K}(\mathbb{Z})) &\longrightarrow K_n(\mathbb{Z}\pi_1(X)) \end{aligned}$$

Here the vertical arrows from the first row to the second row come from the map **i** of (7.14). The left one of these is bijective for  $n \in \mathbb{Z}$  by Exercise 7.19 and the right one is bijective for  $n \ge 1$ . As already discussed in Remark 7.20, the lower vertical arrows from the second row to the third row come from the linearization map **L** of (7.17) and because of Theorem 7.18 the left lower vertical arrow is bijective for  $n \le 1$  and rationally bijective for  $n \in \mathbb{Z}$ . In the case where *X* is aspherical, the lower right vertical map  $\pi_n(L)$  is bijective for  $n \le 1$  and rationally bijective for n < 1 and rationally bijective for n < 1 and rationally bijective for n < 1 and rationally bijective for all  $n \in \mathbb{Z}$  because of Theorem 7.18. Because of (7.24) and the fact that

(7.26) 
$$\Omega^2 \operatorname{Wh}(X) \simeq \mathcal{P}(X),$$

see [322, Section 9] and [981], Conjecture 6.53 implies rational vanishing results for the groups  $\pi_n(\mathcal{P}(M))$  if *M* is an aspherical closed manifold.

**Theorem 7.27 (Homotopy groups of** Wh(*BG*) and  $\mathcal{P}(BG)$  rationally for torsionfree *G*). Let *G* be a torsionfree group. Suppose that Conjecture 6.53 holds for *G* and  $R = \mathbb{Z}$ . Then we get for all  $n \ge 0$ 

$$\pi_n(\operatorname{Wh}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0;$$
  
$$\pi_n(\mathcal{P}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

**Exercise 7.28.** Show that  $\pi_1(Wh(BG))$  is Wh(G).

There is also a smooth version of the Whitehead space  $Wh^{DIFF}(X)$  defined as the homotopy cofiber

(7.29) 
$$\Sigma^{\infty}(X_{+}) \to A(X) \to \operatorname{Wh}^{\mathsf{DIFF}}(X)$$

where  $\Sigma^{\infty}(X_+) \to A(X)$  factors as the unit map  $\Sigma^{\infty}(X_+) = X_+ \wedge \mathbf{S} \to \Sigma^{\infty}(X_+) \wedge A(\{\bullet\})$  and the assembly map  $\Sigma^{\infty}(X_+) \wedge A(\{\bullet\}) \to A(X)$ . We have

(7.30) 
$$\Omega^2 \operatorname{Wh}^{\operatorname{DIFF}}(X) \simeq \mathcal{P}^{\operatorname{DIFF}}(X).$$

Again there is a close relation to *A*-theory via the natural splitting of connective spectra due to Waldhausen [978, 980, 981]

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(7.31) 
$$A(X) \simeq \Sigma^{\infty}(X_{+}) \lor \mathrm{Wh}^{\mathrm{DIFF}}(X).$$

Here  $\Sigma^{\infty}(X_+)$  denotes the suspension spectrum associated to  $X_+$ . Since for every space  $\pi_n(\Sigma^{\infty}(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X; \mathbb{Q})$ , Conjecture 6.53 combined with Remark 6.57 and Theorem 7.18 yields the following result.

**Theorem 7.32 (Homotopy groups of** Wh<sup>DIFF</sup>(*BG*) and  $\mathcal{P}^{DIFF}(BG)$  rationally for torsionfree *G*). Let *G* be a torsionfree group. Suppose that Conjecture 6.53 holds for  $R = \mathbb{Z}$  and *G*. Then we get for all  $n \ge 0$ 

$$\pi_{n}(\operatorname{Wh}^{\operatorname{DIFF}}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(BG; \mathbb{Q});$$
$$\pi_{n}(\mathcal{P}^{\operatorname{DIFF}}(BG)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(BG; \mathbb{Q}).$$

Note that Theorem 7.27 and Theorem 7.32 is a key ingredient in the computation of the homotopy groups of Top(M) and Diff(M) for a closed (smooth) manifold M, as they appear in Theorem 9.195 and Theorem 9.196.

**Exercise 7.33.** Show that there is no connected closed manifold M with the property that the homomorphism induced by the forgetful map  $\pi_n(Wh^{D1FF}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Wh(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$  is bijective for all  $n \ge 0$ . Use the fact that the composite of the obvious inclusion of  $Wh^{D1FF}(X)$  into  $\Sigma^{\infty}(X_+) \vee Wh^{D1FF}(X)$  with the inverse of the splitting (7.31) and the map  $A(X) \to Wh(X)$  of (7.24) is up to homotopy the obvious forgetful map  $Wh^{D1FF}(M) \to Wh(M)$ .

**Remark 7.34.** There are also non-connective versions **Wh** of the Whitehead space Wh defined by the homotopy fibration sequence of non-connective spectra

(7.35) 
$$X_+ \wedge \mathbf{A}(\{\bullet\}) \to \mathbf{A}(X) \to \mathbf{Wh}(X)$$

and  $\mathbf{Wh}^{\mathsf{DIFF}}(X)$  of the smooth Whitehead space  $\mathsf{Wh}^{\mathsf{DIFF}}(X)$  defined to be the homotopy cofiber in the sequence of non-connective spectra

(7.36) 
$$\Sigma^{\infty}(X_{+}) \to \mathbf{A}(X) \to \mathbf{W}\mathbf{h}^{\mathsf{DIFF}}(X)$$

such that the results above have non-connective versions working for all  $n \in \mathbb{Z}$ .

Integral computations of the homotopy groups of Whitehead spaces are much harder. We at least state one example, which follows directly from [344, Theorem 1.3].

**Theorem 7.37 (Homotopy groups of Wh**(*BG*) **of a torsionfree hyperbolic group** *G*). Let *G* be a torsionfree hyperbolic group. Then we get for  $n \in \mathbb{Z}$  an isomorphism

$$\pi_n(\mathbf{Wh}(BG)) \cong \bigoplus_{(C)} NA_n(\{\bullet\}) \oplus NA_n(\{\bullet\})$$

where (C) ranges over the conjugacy classes of maximal infinite cyclic subgroups C of G and  $NA_n(\{\bullet\})$  has been introduced in (7.22).

In particular,  $\pi_n(\mathbf{Wh}(BG)) = 0$  for  $n \leq 1$ .

# 7.4 Notes

One of the basic tools to investigate the algebraic *K*-theory of spaces is the Additivity Theorem, see [712] and [979, Theorem 1.4.2]. If *C* is a category with cofibrations and weak equivalences, we can assign to it a category with cofibrations and weak equivalences E(C) whose objects are exact sequences  $A \xrightarrow{i} B \xrightarrow{p} C$ , where exact means that the map *i* is a cofibration and the following diagram is a pushout

$$\begin{array}{c} A \xrightarrow{i} B \\ \downarrow \\ \downarrow \\ \ast \xrightarrow{} C \end{array}$$

**Theorem 7.38 (Additivity Theorem for categories with cofibrations and weak equivalences).** Let  $F_1$  and  $F_3$  respectively be the functors  $E(C) \rightarrow C$  of categories with cofibrations and weak equivalences sending an object  $A \xrightarrow{i} B \xrightarrow{p} C$  to A and C respectively. Then we obtain a weak homotopy equivalence

$$K(F_1) \times K(F_3) \colon K(E(C)) \xrightarrow{-} K(C) \times K(C).$$

Further useful tools are the Approximation Theorem, see [979, Theorem 1.6.7], the Fibration Theorem, see [979, Theorem 1.6.4], and the Cofinality Theorem, see [979, Theorem 1.5.9], which give criteria to decide when a functor of categories with cofibrations and weak equivalences induces a weak homotopy equivalence on the K-theory spaces.

To the author's knowledge, it is not known how to define a non-connective *K*-theory spectrum for an arbitrary Waldhausen category. If we restrict ourselves to homotopical Waldhausen category, a non-connective *K*-theory spectrum has been defined by Bunke-Kasprowski-Winges [186, Definition 2.37].

There is also a *space of parametrized h-cobordisms* H(M) for a closed topological manifold M. Roughly speaking, the space is designed such that a map  $N \to H(M)$  is the same as a bundle over N whose fibers are *h*-cobordisms over M. The set of path component  $\pi_0(H(M))$  agrees with the isomorphism classes of *h*-cobordisms over M. In particular the *s*-Cobordism Theorem 3.47 is equivalent to the statement that for dim $(M) \ge 5$  we obtain a bijection  $\pi_0(H(M)) \xrightarrow{\cong} Wh(\pi_1(M))$  coming from taking the Whitehead torsion, or, equivalently, that we obtain a bijection  $\pi_0(H(M)) \xrightarrow{\cong} \pi_0(\Omega Wh(M))$ . There is also a stable version, *the space of stable parametrized h*-cobordisms  $\mathcal{K}(M) = hocolim_{i\to\infty} H(M \times I^j)$ .

**Theorem 7.39 (The stable parametrized** *h***-cobordism theorem).** *If M is a closed topological manifold, then there is a homotopy equivalence* 

$$\mathcal{K}(M) \xrightarrow{\sim} \Omega \operatorname{Wh}(M).$$

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There is also a smooth version of the result above. For the proof and more information about the stable parametrized h-cobordism theorem we refer to [981].

# Chapter 8 Algebraic *K*-Theory of Higher Categories

# 8.1 Introduction

The development of higher category theory over the last few decades has further promoted a shift in perspective on mathematical concepts which prioritizes universal properties over explicit constructions. As a result of the work of Barwick [99] and Blumberg, Gepner, and Tabuada [143], this includes algebraic *K*-theory, which can be characterized in terms of a universal property once the requisite  $\infty$ -categorical machinery has been set up.

In Section 8.3, we describe connective and non-connective algebraic *K*-theory as the *universal additive* and *universal localizing invariant* under the groupoid core on stable  $\infty$ -categories, and extend their domain of definition slightly to cover *right-exact*  $\infty$ -categories.

Section 8.4 compares this definition of algebraic *K*-theory to the algebraic *K*-theory functors from previous chapters, thus explaining in which sense this version of algebraic *K*-theory subsumes (almost) all instances of algebraic *K*-theory encountered previously.

In Section 8.5 we explain how to produce from right-exact G- $\infty$ -categories the necessary data, namely an equivariant homology theory, that will allow for the formulation of the Farrell-Jones Conjecture in the setting of higher categories in Subsection 13.3.1. This encompasses the construction of an appropriate equivariant homology theory and the Farrell-Jones Conjecture with coefficients in additive G-categories appearing in Section 13.3.

A reader who is only interested in the Farrell-Jones Conjecture for group rings may just skip this chapter.

Throughout this chapter, as well as Chapter 24, we will assume familiarity with the basic notions of higher category theory as laid out for example in [242, 613, 691, 851]. This includes the notions of  $\infty$ -category, functors between  $\infty$ -categories, adjoints, limits and colimits, Kan extensions, and the concept of cofinality. Parts of the discussion will also use the theory of (symmetric) monoidal  $\infty$ -categories and (commutative) algebra objects in such  $\infty$ -categories, which is developed in [689].

# 8.2 Notational and Terminological Conventions

We adopt the following notational and terminological conventions:

• We denote by  $CAT_{\infty}$  the  $\infty$ -category of small  $\infty$ -categories;

- We denote by Spc the full subcategory of CAT<sub>∞</sub> spanned by the ∞-groupoids, synonymously "spaces", "weak homotopy types", or "anima". Moreover, Spc<sub>\*</sub> := Spc<sub>\*</sub>/ is the ∞-category of pointed spaces;
- The groupoid core functor  $\iota: CAT_{\infty} \to Spc$  takes an  $\infty$ -category to its subcategory of equivalences. This functor is right adjoint to the inclusion functor  $Spc \to CAT_{\infty}$ ;
- The inclusion functor Spc → CAT<sub>∞</sub> also admits a left adjoint, sometimes called groupoidification, which we denote by |-|: CAT<sub>∞</sub> → Spc;
- Given an  $\infty$ -category *C* and a collection of morphisms *S* in *C*, the *Dwyer-Kan localization* of *C* at *S* is a functor  $\ell: C \to C[S^{-1}]$  such that the restriction functor

$$\ell^*$$
: Fun( $C[S^{-1}], \mathcal{D}$ )  $\rightarrow$  Fun( $C, \mathcal{D}$ )

is fully faithful with essential image given by those functors which send every morphism in S to an invertible morphism in  $\mathcal{D}$ ;

• There exists a functor

$$(8.1) L: \text{ Top} \to \text{Spc}.$$

which exhibits the  $\infty$ -category Spc as the Dwyer-Kan localization of the category of topological spaces at the weak homotopy equivalences;

- In order to make the notation consistent, we deviate from standard usage and write  $\operatorname{mor}_C \colon C^{\operatorname{op}} \times C \to \operatorname{Spc}$  for the functor which sends a pair of objects (x, y) in an  $\infty$ -category *C* to its space of morphisms (and therefore gives rise to the *Yoneda embedding*  $C \to \operatorname{Fun}(C^{\operatorname{op}}, \operatorname{Spc})$ );
- We denote the ∞-category of functors between two ∞-categories *C* and *D* by Fun(*C*, *D*). Mapping spaces in functor categories will typically be denoted by nat instead of the unwieldy mor<sub>Fun(C, D</sub>);
- We denote by Sp the ∞-category of spectra. Recall that this ∞-category can be characterized uniquely by a number of universal properties, for example as the stabilization of the finitely complete ∞-category Spc<sub>\*</sub> [689, Section 1.4.3], or as the stable, presentable ∞-category freely generated by a single object **S**, *the sphere spectrum*, see [689, Corollary 1.4.4.5]. We will recall the notion of stability in Definition 8.3 below.

The  $\infty$ -category Sp is also the  $\infty$ -category obtained from SPECTRA using the Bousfield-Friedlander model structure [158]. In particular, any functor valued in SPECTRA gives rise to an Sp-valued functor by composition with the localization functor

$$(8.2) L: SPECTRA \to Sp.$$

which sends weak homotopy equivalences to equivalences. The notions of homotopy equivalence, homotopy fiber sequence, ... in SPECTRA correspond to the intrinsic notions of equivalence, fiber sequence, ... in the  $\infty$ -category Sp.

There exists an adjunction  $\Sigma^{\infty}$ : Spc<sub>\*</sub>  $\rightleftharpoons$  Sp :  $\Omega^{\infty}$  in which  $\Sigma^{\infty}$  sends a pointed space to its suspension spectrum and  $\Omega^{\infty}$  sends a spectrum to its underlying infinite loop space.

One recovers the homotopy groups of a spectrum X by setting  $\pi_n(X) := \pi_0(\operatorname{mor}_{\operatorname{Sp}}(\Sigma^n \mathbf{S}, X))$ . As usual, a spectrum X is *connective* if  $\pi_n(X) = 0$  for n < 0. The inclusion of the full subcategory  $\operatorname{Sp}_{\geq 0}$  of connective spectra admits a right adjoint  $\tau_{\geq 0} \colon \operatorname{Sp} \to \operatorname{Sp}_{\geq 0}$  which produces the connective cover of a given spectrum;

• The word "unique" will be used in favor of "essentially unique" to indicate that an object or map is determined up to a contractible space of choices, as happened already above when we referred for example to "the" Yoneda embedding.

# 8.3 The Universal Property of Algebraic K-Theory

The works of Barwick [99] and Blumberg, Gepner, and Tabuada [143] provide two different routes to formulating a universal property for algebraic *K*-theory. Since we will ultimately be interested in non-connective algebraic *K*-theory, we focus on the approach of [143] and understand algebraic *K*-theory as an invariant of small stable  $\infty$ -categories. Let us begin by introducing the relevant notions.

**Definition 8.3.** An  $\infty$ -category *C* is

- (i) *pointed* if it admits a zero object, i.e., there exist both an initial object Ø and a final object \*, and the unique morphism Ø → \* is an equivalence;
- (ii) *stable* if it is pointed, admits all finite limits and all finite colimits, and a commutative square in *C* is a pushout if and only if it is a pullback.

An *exact functor*  $F: C \to \mathcal{D}$  between stable  $\infty$ -categories is a functor which preserves both finite limits and finite colimits. Denote by  $\operatorname{Fun}^{\operatorname{ex}}(C, \mathcal{D}) \subseteq \operatorname{Fun}(C, \mathcal{D})$  the full subcategory of exact functors.

The  $\infty$ -category CATST is the subcategory of CAT $_{\infty}$  given by small stable  $\infty$ -categories and exact functors. In particular, the mapping spaces in CATST are given by

$$\operatorname{mor}_{\operatorname{CATST}}(\mathcal{C},\mathcal{D}) \simeq \iota \operatorname{Fun}^{\operatorname{ex}}(\mathcal{C},\mathcal{D}),$$

the groupoid core of the  $\infty$ -category of exact functors from *C* to  $\mathcal{D}$ .

**Remark 8.4.** The homotopy category of a stable  $\infty$ -category *C* carries an induced triangulated structure, in which the shift functor is induced by the suspension  $\Sigma: C \to C$  and the distinguished triangles correspond to commutative diagrams



with both squares bicartesian [689, Theorem 1.1.2.14].

#### Example 8.5.

- (i) The ∞-category Sp of spectra is a stable ∞-category, as is its full subcategory Sp<sup>ω</sup> of compact spectra. The inclusion functor Sp<sup>ω</sup> → Sp is exact;
- (ii) If R is an E<sub>1</sub>-ring spectrum, the ∞-category R-MOD of left R-modules is stable, and so is its full subcategory PERF(R) of compact objects. The inclusion functor PERF(R) → R-MOD is exact. Since S-MOD ≈ Sp, this generalizes the first example;
- (iii) For a small additive category  $\mathcal{A}$ , consider the category  $Ch(\mathcal{A})$  of bounded chain complexes over  $\mathcal{A}$ . Localizing at the subcategory  $hCh(\mathcal{A})$  of chain homotopy equivalences, one obtains a stable  $\infty$ -category

(8.6) 
$$\mathcal{K}^{\mathfrak{b}}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[h\mathrm{Ch}(\mathcal{A})^{-1}].$$

This can be deduced from Proposition 8.29 below, using the criterion that a pointed, finitely cocomplete  $\infty$ -category is stable if the suspension functor is an equivalence [689, Corollary 1.4.2.27].

Comparing universal properties, one finds that the homotopy category of  $\mathcal{K}^{b}(\mathcal{A})$  is equivalent to the classical homotopy category of bounded chain complexes over  $\mathcal{A}$ , cf. [997, Section 10.1]. This equivalence refines to an equivalence of triangulated categories;

(iv) If  $\mathcal{A}$  is a small abelian category, define

(8.7) 
$$\mathcal{D}^{\mathsf{b}}(\mathcal{A}) := \mathsf{Ch}(\mathcal{A})[q\mathsf{Ch}(\mathcal{A})^{-1}],$$

as the localization at the subcategory  $qCh(\mathcal{A})$  of quasi-isomorphisms. As in the case of  $\mathcal{K}^{b}(\mathcal{A})$ , Proposition 8.29 below implies that this is a stable  $\infty$ -category. The homotopy category of  $\mathcal{D}^{b}(\mathcal{A})$  is equivalent to the classical bounded derived category of  $\mathcal{A}$ , also as a triangulated category.

Since every chain homotopy equivalence is a quasi-isomorphism, the identity functor on  $\mathcal{A}$  induces a functor  $\mathcal{K}^{b}(\mathcal{A}) \to \mathcal{D}^{b}(\mathcal{A})$  which is exact. This functor vanishes on the full subcategory of  $\mathcal{K}^{b}(\mathcal{A})$  spanned by the acyclic complexes over  $\mathcal{A}$ , and it is in fact universal among exact functors with this property in the sense that restriction along this functor induces for every stable  $\infty$ -category *C* a fully faithful functor

$$\operatorname{Fun}^{\operatorname{ex}}(\mathcal{D}^{\operatorname{b}}(\mathcal{A}), C) \to \operatorname{Fun}^{\operatorname{ex}}(\mathcal{K}^{\operatorname{b}}(\mathcal{A}), C)$$

whose essential image is given by those functors which vanish on the full subcategory of acyclic complexes. This exhibits the bounded derived category as a *Verdier quotient* of  $\mathcal{K}^{b}(\mathcal{A})$ , which is the next concept we introduce.

**Definition 8.8.** Let *C* be a small stable  $\infty$ -category.

(i) A *full stable subcategory* C is a full subcategory  $\mathcal{U} \subseteq C$  which contains the zero object and is closed under finite limits and finite colimits in C;

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- (ii) Let  $\mathcal{U}$  be a full stable subcategory of C. A *Verdier quotient* of C by  $\mathcal{U}$  is an exact functor  $p: C \to C'$  to a stable  $\infty$ -category C' such that the restriction functor

$$p^*$$
: Fun<sup>ex</sup>( $\mathcal{C}', \mathcal{D}$ )  $\rightarrow$  Fun<sup>ex</sup>( $\mathcal{C}, \mathcal{D}$ )

is fully faithful with essential image given by those exact functors which vanish on  $\mathcal{U}$ .

By definition, a full stable subcategory of a stable  $\infty$ -category is also stable. By passing to groupoid cores, their universal property implies that Verdier quotients are cofibers in the  $\infty$ -category CATST.

By stability, an exact functor  $F: C \to \mathcal{D}$  vanishes on a full stable subcategory  $\mathcal{U}$  of *C* if and only if *F* inverts all morphisms in *C* whose cofiber is equivalent to an object in  $\mathcal{U}$ . In particular, a Verdier quotient  $p: C \to C'$  is always a localization at the collection of morphisms whose cofiber lies in ker $(p) := \{x \in C \mid p(x) \simeq 0\}$ . In fact, the converse is also true and proves the existence of arbitrary Verdier quotients.

**Proposition 8.9** ([193, Proposition A.1.5 and Lemma A.1.8]). Let *C* be a small stable  $\infty$ -category and let  $\mathcal{U}$  be a full stable subcategory of *C*. Then the localization  $C/\mathcal{U}$  of *C* at the subcategory of morphisms whose cofiber is equivalent to an object in  $\mathcal{U}$  is stable, and the localization functor  $p: C \to C/\mathcal{U}$  is exact. In particular, *p* is a Verdier quotient of *C* by  $\mathcal{U}$ .

Moreover, the kernel of p is exactly the full stable subcategory of objects in C which are retracts of objects in U.

**Example 8.10.** As remarked above, the canonical functor  $\mathcal{K}^{b}(\mathcal{A}) \to \mathcal{D}^{b}(\mathcal{A})$  is a Verdier quotient of  $\mathcal{K}^{b}(\mathcal{A})$  by the full subcategory of acyclic complexes for every abelian category  $\mathcal{A}$ .

A particular class of Verdier quotients is given by those which are actually Bousfield localization, i.e., those Verdier quotients which admit a right adjoint. Such an adjoint is automatically fully faithful, since Verdier quotients are localizations [193, Lemma A.2.1].

### Definition 8.11 ([193, Definition A.2.4]).

- (i) A *split Verdier quotient* is a Verdier quotient  $p: C \rightarrow C'$  which admits both a left and a right adjoint;
- (ii) A split Verdier square is a pullback square



of stable  $\infty$ -categories in which p and q are split Verdier quotients.

**Example 8.12.** Let *C* and  $\mathcal{D}$  be stable  $\infty$ -categories.

(i) The commutative square



in which all arrows are the obvious projection functors is a split Verdier square; (ii) The commutative square



in which f sends  $x \in C$  to  $x \to 0$  and c sends  $x \to y$  to cofib $(x \to y)$  is a split Verdier square.

**Definition 8.13.** Let X be an  $\infty$ -category. A functor  $F: CATST \rightarrow X$  is

- (i) *finitary* if it preserves filtered colimits;
- (ii) an *additive invariant* if it preserves zero objects and sends split Verdier squares to pullback squares.

Denote by  $\operatorname{Fun}^{\operatorname{fadd}}(\operatorname{CATST}, X)$  the full subcategory of  $\operatorname{Fun}(\operatorname{CATST}, X)$  spanned by the finitary additive invariants.

**Example 8.14.** A key example of a finitary additive invariant is the groupoid core functor  $\iota$ : CATST  $\rightarrow$  Spc. In fact, this functor preserves all limits because the inclusion functor CATST  $\rightarrow$  CAT<sub> $\infty$ </sub> preserves limits [689, Theorem 1.1.4.4], and  $\iota$  is a right adjoint. Similarly, the inclusion functor preserves filtered colimits by [689, Proposition 1.1.4.6], and  $\iota$  does so because it is corepresented by a compact object, namely the category with a single object and a single morphism.

Note that taking infinite loop spaces induces a functor  $\Omega^{\infty} \colon Sp_{\geq 0} \to Spc$  which preserves pullbacks and filtered colimits.

Theorem 8.15 (Group completion). The functor

 $\Omega^{\infty} \circ -: \operatorname{Fun}^{fadd}(CATST, Sp_{>0}) \to \operatorname{Fun}^{add}(CATST, Spc)$ 

is fully faithful and admits a left adjoint

 $(-)^{\text{grp}}$ : Fun<sup>add</sup>(CATST, Spc)  $\rightarrow$  Fun<sup>fadd</sup>(CATST, Sp<sub>>0</sub>)

called group completion.

*Proof.* This is a consequence of an appropriate version of the Additivity Theorem [193, Theorem 2.7.4], see also [193, Remark 3.4.9]. For the special case that the groupoid core functor admits a group completion in the sense of the theorem, see [474, Section 5].

**Definition 8.16.** Define the *connective algebraic K-theory functor* by

$$K := \iota^{\operatorname{grp}} \colon \operatorname{CATST} \to \operatorname{Sp}_{>0}$$

where the groupoid core functor  $\iota$  and  $(-)^{grp}$  have been defined in Example 8.14 and Theorem 8.15.

Unwinding what left adjointness of  $(-)^{grp}$  means, we obtain for every finitary additive invariant  $F: CATST \rightarrow Sp_{\geq 0}$  an equivalence

$$\operatorname{nat}(K, F) \simeq \operatorname{nat}(\iota, \Omega^{\infty} \circ F).$$

An important point is that the group completion functor admits a more concrete description in terms of the Segal-Waldhausen  $S_{\bullet}$ -construction. To this end, observe that  $S_{\bullet}(C)$  makes perfectly good sense for a stable  $\infty$ -category C: the  $\infty$ -category  $S_n(C)$  is the full subcategory of Fun(Ar[n], C) spanned by those functors x: Ar[n]  $\rightarrow C$  such that x(i, i) is a zero object for all i and each square

$$\begin{array}{c} x(i,j) \longrightarrow x(i,k) \\ \downarrow \\ 0 \simeq x(i,i) \longrightarrow x(j,k) \end{array}$$

is a pushout for all  $i \le j \le k$ . One observes that  $S_{n+1}(C)$  is equivalent to Fun([n], C), but as usual the additional data encoded in  $S_{n+1}(C)$  are needed to define a simplicial  $\infty$ -category  $S_{\bullet}(C)$ .

For every finitary additive invariant F, we have natural equivalences

$$\Omega^{\infty} F^{\rm grp}(\mathcal{C}) \simeq \Omega |F(S_{\bullet}(\mathcal{C}))|.$$

This follows from the discussion in [193, Section 2.7], using that the *Q*-construction is the edgewise subdivision of the  $S_{\bullet}$ -construction. This is explained for example in [98]. Since taking groupoid cores defines a finitary additive invariant, we obtain in particular

$$\Omega^{\infty}\iota^{\operatorname{grp}}(C) \simeq \Omega |\iota S_{\bullet}(C)|.$$

This matches the formula for algebraic K-theory given in Section 7.3.2, even though the type of input category is different. We will make the comparison more precise in Section 8.4 below.

It turns out that algebraic K-theory preserves more fiber sequences than just the split Verdier sequences, as witnessed by the following analog of Waldhausen's Fibration Theorem taken from [474, Theorem 6.1].

**Theorem 8.17** (*K*-theory and Verdier quotients). Let  $p: C \to D$  be a Verdier quotient. Then the sequence

$$K(\ker(p)) \to K(\mathcal{C}) \to K(\mathcal{D})$$

## is a cofiber sequence of connective spectra.

This theorem does not apply to every sequence  $\mathcal{U} \to C \to C/\mathcal{U}$  obtained from a stable  $\infty$ -category *C* and a full stable subcategory  $\mathcal{U}$ , since  $\mathcal{U}$  need not be closed under retracts in *C*.

**Definition 8.18.** A stable  $\infty$ -category *C* is *idempotent complete* if every idempotent in *C* splits.

There exists an endofunctor Idem: CATST  $\rightarrow$  CATST which is a Bousfield localization onto the full subcategory of idempotent complete stable  $\infty$ -categories [689, Corollary 1.1.3.7].

Theorem 8.17 then yields for every stable  $\infty$ -category *C* with a full stable subcategory  $\mathcal{U}$  a cofiber sequence of connective spectra

 $K(\operatorname{Idem}(\mathcal{U})) \to K(\operatorname{Idem}(C)) \to K(\operatorname{Idem}(C)/\operatorname{Idem}(\mathcal{U})).$ 

In general, there is no reason to expect that the induced functor

$$\operatorname{Idem}(\mathcal{C})/\operatorname{Idem}(\mathcal{U}) \to \operatorname{Idem}(\mathcal{C}/\mathcal{U})$$

is an equivalence, but it is an idempotent completion. By virtue of the Cofinality Theorem [99, Theorem 10.19], the induced map on *K*-theory is an injection on  $\pi_0$  and an isomorphism on all higher homotopy groups, so

$$K(\operatorname{Idem}(\mathcal{U})) \to K(\operatorname{Idem}(C)) \to K(\operatorname{Idem}(C/\mathcal{U}))$$

is still a fiber sequence of connective spectra. This leads to the following definition.

**Definition 8.19.** Let X be an  $\infty$ -category.

(i) A functor  $F: CATST \rightarrow X$  is a *localizing invariant* if

$$F(\mathcal{U}) \to F(\mathcal{C}) \to F(\mathcal{C}/\mathcal{U})$$

is a fiber sequence in X for every stable  $\infty$ -category C and full stable subcategory  $\mathcal{U}$  of C:

(ii) Denote by Fun<sup>floc</sup> (CATST, X) the  $\infty$ -category of finitary localizing invariants.

The next theorem is taken from [243, Theorem 4.3.3].

**Theorem 8.20 (Comparing functors from** CATST to Sp and  $Sp_{\geq 0}$ ). *The functor* 

 $\tau_{\geq 0}$ : Fun<sup>floc</sup>(CATST, Sp)  $\rightarrow$  Fun<sup>floc</sup>(CATST, Sp<sub>>0</sub>)

is an equivalence.

Definition 8.21. The non-connective algebraic K-theory functor

$$\mathbf{K}: CATST \rightarrow Sp$$

is the unique finitary localizing invariant satisfying  $\tau_{\geq 0} \mathbf{K} \simeq K \circ \text{Idem}$ .

From the preceding discussion, one immediately deduces that non-connective algebraic *K*-theory also enjoys a universal property.

**Corollary 8.22.** Let  $F: CATST \to Sp$  be a finitary localizing invariant. Then restriction along the natural transformation  $K \circ Idem \simeq \tau_{\geq 0} \mathbf{K} \to \mathbf{K}$  induces an equivalence

$$\operatorname{nat}(\mathbf{K}, F) \to \operatorname{nat}(K \circ \operatorname{Idem}, F).$$

*Proof.* Since  $K \circ$  Idem takes values in connective spectra, we have

$$\operatorname{nat}(K \circ \operatorname{Idem}, F) \simeq \operatorname{nat}(K \circ \operatorname{Idem}, \tau_{>0}F),$$

so the statement follows from Theorem 8.20.

For later applications, it is useful to extend the definition of algebraic K-theory to a slightly larger class of  $\infty$ -categories.

**Definition 8.23.** An  $\infty$ -category *C* is *right-exact* if it is pointed and admits all finite colimits. Denote by CATREX the subcategory of CAT<sub> $\infty$ </sub> given by the small right-exact  $\infty$ -categories and functors which preserve finite colimits.

There is an easy characterization of those right-exact  $\infty$ -categories which are stable: a right-exact  $\infty$ -category *C* is stable if and only if the suspension functor  $\Sigma$  is an equivalence [689, Corollary 1.4.2.27]. In addition, a functor  $F: C \to D$  between stable  $\infty$ -categories is exact if and only if it preserves finite colimits [689, Proposition 1.1.4.1], so CATST is a full subcategory of CATREX.

**Proposition 8.24** ([690, Proposition C.1.17]). *The fully faithful functor* CATST  $\rightarrow$  CATREX *admits a left adjoint* 

SW: CATREX 
$$\rightarrow$$
 CATST

called Spanier-Whitehead stabilization. This functor has the property

$$\mathrm{SW}(C) \simeq \operatorname{colim}\left(C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} C \to \ldots\right).$$

We extend the algebraic *K*-theory functors to CATREX by precomposing with the Spanier-Whitehead stabilization. The resulting functors

$$K: CATREX \rightarrow Sp_{>0}, \qquad C \mapsto K(SW(C))$$

and

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**K**: CATREX  $\rightarrow$  Sp,  $C \mapsto$  **K**(SW(C))

preserves filtered colimits. The property of being a localizing invariant also generalizes to this context.

**Proposition 8.25.** Let C be a small right-exact  $\infty$ -category and let  $\mathcal{U}$  be a full subcategory of C which is closed under finite colimits. Define  $W_{\mathcal{U}}$  as the collection of morphisms whose cofiber is equivalent to an object in  $\mathcal{U}$ .

- (i) The Dwyer-Kan localization  $C[W_{\mathcal{U}}^{-1}]$  is right-exact and the localization functor  $C \to C[W_{\mathcal{U}}^{-1}]$  preserves finite colimits;
- (ii) The induced functor  $SW(C)/SW(\mathcal{U}) \to SW(C[W_{\mathcal{U}}^{-1}])$  is an equivalence. In particular, the induced sequence

$$\mathbf{K}(\mathcal{U}) \to \mathbf{K}(\mathcal{C}) \to \mathbf{K}(\mathcal{C}[W_{q_I}^{-1}])$$

is a cofiber sequence of spectra.

Proof. See for example [180, Lemma 2.4.6].

**Remark 8.26.** It is not quite true that  $C[W_{\mathcal{U}}^{-1}]$  is the cofiber of the inclusion  $\mathcal{U} \to C$  in CATREX: there may be functors  $C \to \mathcal{D}$  which preserve finite colimits and vanish on  $\mathcal{U}$ , but do not invert every morphism in  $W_{\mathcal{U}}$ .

# 8.4 Relating the Different Definitions of Algebraic K-Theory

The goal of this section is to indicate a comparison of the algebraic K-theory functors introduced in 8.3 with the previous definitions of algebraic K-theory.

**Theorem 8.27 (Identification of non-connective** *K*-theory for additive categories in the classical setting and in the setting of higher categories). Let  $\mathcal{A}$  be an additive category. Let  $\mathbf{K}(\mathcal{A})$  in SPECTRA be the non-connective *K*-theory spectrum of (6.34). Let  $\mathbf{L}(\mathbf{K}(\mathcal{A}))$  in Sp be its image under the functor  $\mathbf{L}$  of (8.2). Let  $\mathcal{K}^{\mathbf{b}}(\mathcal{A})$ be the stable  $\infty$ -category associated to  $\mathcal{A}$  in (8.6). Denote by  $\mathbf{K}(\mathcal{K}^{\mathbf{b}}(\mathcal{A}))$  in Sp the associated non-connective *K*-theory spectrum defined in (8.21).

Then there is a weak equivalence of spectra in Sp, natural in A,

$$\mathbf{L}(\mathbf{K}(\mathcal{A})) \xrightarrow{\simeq} \mathbf{K}(\mathcal{K}^{\mathbf{b}}(\mathcal{A})).$$

We begin by discussing connective algebraic *K*-theory. As explained in the previous section, the connective algebraic *K*-theory K(C) of a right-exact  $\infty$ -category is given in terms of the *S*<sub>•</sub>-construction. This allows us to identify the algebraic *K*-theory of certain categories with cofibrations and weak equivalences with the algebraic *K*-theory of a right-exact  $\infty$ -category.

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#### Definition 8.28.

- (i) A subcategory wC of some category C satisfies the *two-out-of-six property*: if for any three composable morphisms  $C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3$  in C both  $f_2 \circ f_1$ and  $f_3 \circ f_2$  are weak equivalences, then  $f_1, f_2, f_3$ , and  $f_3 \circ f_2 \circ f_1$  are also weak equivalences;
- (ii) The category with cofibrations and weak equivalences (C, coC, wC) admits factorizations if every morphism in C can be factorized into a cofibration followed by a weak equivalence; no functoriality of this factorization is assumed;
- (iii) A homotopical Waldhausen category (C, coC, wC) is a category with cofibrations and weak equivalences which admits factorizations and whose subcategory of weak equivalences satisfies the two-out-of-six-property.

**Proposition 8.29.** Let (C, coC, wC) be a category with cofibrations and weak equivalences which admits factorizations. Let  $\ell: C \to C[wC^{-1}]$  be the Dwyer-Kan localization of C at wC.

- (i) The  $\infty$ -category  $C[wC^{-1}]$  is right-exact, and the localization functor  $\ell$  preserves zero objects and pushouts along cofibrations;
- (ii) If (C, coC, wC) is a homotopical Waldhausen category, the comparison map

$$K(C) \to \Omega^{\infty} K(C[wC^{-1}])$$

induced by  $\ell$  is an equivalence in Spc.

*Proof.* The first assertion is [242, Proposition 7.5.6]. In particular, the  $S_{\bullet}$ -construction is defined for  $C[wC^{-1}]$ , since its definition involves only finite colimits.

Since *wC* is assumed to satisfy the two-out-of-six property, a morphism in *C* is inverted by the localization functor  $\ell$  if and only if it is a weak equivalence: this follows either using the methods of [242, Corollary 7.5.19], or directly by appealing to [145, Theorem 6.4]. Therefore, [242, Corollary 7.6.18] implies that

$$|wC| \rightarrow \iota C[wC^{-1}]$$

is an equivalence in Spc.

The category  $S_nC$  inherits the structure of a homotopical Waldhausen category from *C*. In particular,

$$|wS_nC| \rightarrow \iota \left(S_n(C)[wS_n(C)^{-1}]\right)$$

is an equivalence for all *n*. Moreover, there is a natural comparison map

$$S_n(C)[wS_n(C)^{-1}] \to S_n(C[wC^{-1}])$$

which is an equivalence for all n by [242, Corollary 7.6.18]. In particular, we obtain a natural equivalence

$$\Omega|wS_{\bullet}C| \to \Omega|\iota S_{\bullet}(C[wC^{-1}])|.$$

Finally, the natural map

$$\Omega[\iota S_{\bullet}(C[wC^{-1}])] \to \Omega[\iota S_{\bullet}(SW(C[wC^{-1}]))] \simeq \Omega^{\infty} K(C[wC^{-1}])$$

is an equivalence by [99, Corollary 8.2.1].

**Corollary 8.30.** Let  $\mathcal{E}$  be an exact category. Let  $K(\mathcal{E})$  in SPECTRA be the classical *K*-theory of an exact category a la Quillen. Let  $L(K(\mathcal{E}))$  in Sp be its image under the functor L of (8.2). Let  $K(\mathcal{D}^{b}(\mathcal{E}))$  in Sp be the connective *K*-theory spectrum of  $\mathcal{D}^{b}(\mathcal{E})$  in terms of higher categories, see (8.7) and Definition 8.16.

Then there is a weak equivalence of connective spectra in Sp, natural in  $\mathcal{E}$ ,

$$L(K(\mathcal{E})) \xrightarrow{\sim} K(\mathcal{D}^{b}(\mathcal{E})).$$

In particular, there is a natural equivalence of connective spectra in Sp  $L(K(\mathcal{A})) \simeq K(\mathcal{K}^{b}(\mathcal{A}))$  for every additive category  $\mathcal{A}$ .

*Proof.* The Cofinality Theorem (see e.g., [923, Theorem 2.1] for exact categories and [949, 1.10.1] for the category of chain complexes) allows us to assume without loss of generality that  $\mathcal{E}$  is weakly idempotent complete. The category  $Ch(\mathcal{E})$  of bounded chain complexes over  $\mathcal{E}$  carries a homotopical Waldhausen structure whose weak equivalences are the quasi-isomorphisms. The Gillet-Waldhausen theorem [949, 1.11.7] identifies  $K(\mathcal{E})$  with the algebraic K-theory of the homotopical Waldhausen category  $Ch(\mathcal{E})$ . Since  $\mathcal{D}^{b}(\mathcal{E})$  is precisely the Dwyer-Kan localization of  $Ch(\mathcal{E})$  at the quasi-isomorphisms, the statement follows from Proposition 8.29.

Next we deal with non-connective algebraic *K*-theory for additive categories. In order to make this comparison, recall that an *additive*  $\infty$ -*category*  $\mathcal{A}$ , just as in the case of ordinary categories, is an  $\infty$ -category with a zero object, finite products, and finite coproducts such that the obvious comparison map  $a \sqcup b \rightarrow a \times b$  is an equivalence for all objects *a* and *b* of  $\mathcal{A}$ , and such that the shear map

$$s_a : \begin{pmatrix} \text{id id} \\ 0 & \text{id} \end{pmatrix} : a \oplus a \to a \oplus a$$

is an equivalence for every object *a*. The small additive  $\infty$ -categories and additive functors form an  $\infty$ -category ADD<sub> $\infty$ </sub>. Using [691, Proposition 5.5.8.15], one can show that the forgetful functor CATST  $\rightarrow$  ADD<sub> $\infty$ </sub> admits a left adjoint

$$\mathcal{P}_{\Sigma}^{\mathrm{st},f}:\mathrm{ADD}_{\infty}\to\mathrm{CATST}.$$

Explicitly, this functor sends an additive  $\infty$ -category  $\mathcal{A}$  to the Spanier-Whitehead stabilization of the finite colimit closure of the essential image of the Yoneda embedding in the  $\infty$ -category of additive presheaves.

The existence of this left adjoint directly implies a localization theorem for additive  $\infty$ -categories.

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8.4 Relating the Different Definitions of Algebraic K-Theory

**Proposition 8.31.** Let  $\mathcal{A}$  be an additive  $\infty$ -category and let  $\mathcal{U} \subseteq \mathcal{A}$  be a full additive subcategory. If  $\mathcal{A} \to \mathcal{A} /\!\!/ \mathcal{U}$  is a cofiber of the inclusion map  $\mathcal{U} \to \mathcal{A}$  in the  $\infty$ -category  $ADD_{\infty}$ , then

$$\mathbf{K}(\mathcal{P}^{\mathrm{st},f}_{\Sigma}(\mathcal{U})) \to \mathbf{K}(\mathcal{P}^{\mathrm{st},f}_{\Sigma}(\mathcal{A})) \to \mathbf{K}(\mathcal{P}^{\mathrm{st},f}_{\Sigma}(\mathcal{A}/\!\!/\mathcal{U}))$$

is a cofiber sequence in Sp.

*Proof.* If  $\mathcal{U} \subseteq \mathcal{A}$  is a full additive subcategory, one observes by unpacking the construction of  $\mathcal{P}_{\Sigma}^{\text{st},f}$  that  $\mathcal{P}_{\Sigma}^{\text{st},f}(\mathcal{U}) \to \mathcal{P}_{\Sigma}^{\text{st},f}(\mathcal{U})$  is also fully faithful. Since  $\mathcal{P}_{\Sigma}^{\text{st},f}$ , being a left adjoint, preserves cofiber sequences, the statement follows immediately from the fact that **K** is a localizing invariant.

The catch of Proposition 8.31 is that, even if  $\mathcal{U}$  and  $\mathcal{A}$  are additive categories (in the classical sense), the cofiber  $\mathcal{A}/\!\!/\mathcal{U}$  is not necessarily an additive category, it can a priori only be defined as an  $\infty$ -category. To obtain a better understanding of the situation, let us give an explicit construction of the cofiber in ADD<sub> $\infty$ </sub>.

Given an additive  $\infty$ -category  $\mathcal{A}$  and a full additive subcategory  $\mathcal{U}$ , one can consider the wide subcategory  $\mathcal{A}_{\mathcal{U}}$  of  $\mathcal{A}$  given by the projections onto direct summands whose complement is an object of  $\mathcal{U}$ .

**Lemma 8.32.** The Dwyer-Kan localization  $\mathcal{A} \to \mathcal{A}[\mathcal{A}_{\mathcal{U}}^{-1}]$  is a cofiber of the inclusion functor  $\mathcal{U} \to \mathcal{A}$  in ADD<sub> $\infty$ </sub>.

*Proof.* The subcategory  $\mathcal{A}_{\mathcal{U}}$  is closed under direct sums, so it follows from [242, Proposition 7.1.7 and Corollary 7.1.16] that the Dwyer-Kan localization  $\mathcal{A}[\mathcal{A}_{\mathcal{U}}^{-1}]$  is also an additive  $\infty$ -category and that the localization functor  $\mathcal{A} \to \mathcal{A}[\mathcal{A}_{\mathcal{U}}^{-1}]$  is additive. Observing that an additive functor  $\mathcal{A} \to \mathcal{B}$  vanishes on  $\mathcal{U}$  if and only if it inverts  $\mathcal{A}_{\mathcal{U}}$ , the universal property of the localization implies the lemma.

**Proposition 8.33.** Let  $\mathcal{A}$  be an additive category and let  $\mathcal{U} \subseteq \mathcal{A}$  be a full additive subcategory. Suppose that the following condition is satisfied:

(\*) Every morphism  $u \to a$  from an object u in  $\mathcal{U}$  to an object a of  $\mathcal{A}$  factors in the form  $u \xrightarrow{f} v \xrightarrow{i} a$  for some morphism f in  $\mathcal{U}$  and a direct summand inclusion i.

Then  $\mathcal{A} \parallel \mathcal{U}$  is an (ordinary) category and in particular an additive category (in the classical sense).

*Proof.* Denote by  $\mathcal{A}_{\mathcal{U}}(a)$  the full subcategory of  $a \downarrow \mathcal{A}$  on the objects  $a \to a'$  which lie in  $\mathcal{A}_{\mathcal{U}}$ . We claim that this category is filtered. It is non-empty since  $\mathrm{id}_a$  is an object in  $\mathcal{A}_{\mathcal{U}}(a)$ . Given two objects  $p_1: a \to a_1$  and  $p: a \to a_2$ , we can apply condition  $(\star)$  to the induced map  $\ker(p_1) \oplus \ker(p_2) \to a$  to obtain a summand inclusion  $i: u \to a$  such that the projection  $p: a \to \mathrm{coker}(i)$  factors over both  $p_1$  and  $p_2$ . If two morphisms  $f, g: a_1 \to a_2$  define a morphism between  $p_1: a \to a_1$  and  $p_2: a \to a_2$  in  $\mathcal{A}_{\mathcal{U}}(a)$ , then g = f because  $p_1$  is an epimorphism.

Consider now the functor

$$M_a := \operatornamewithlimits{colim}_{a \to a' \in \mathcal{A}_{\mathcal{U}}(a)} \operatorname{mor}_{\mathcal{A}}(-, a') \colon \mathcal{A}^{\operatorname{op}} \to \operatorname{Spc}$$

Since  $\mathcal{A}_{\mathcal{U}}(a)$  is filtered and  $\mathcal{A}$  is a category, this colimit may equivalently be computed in the category of sets. Next, we show that for every morphism  $q: b_1 \to b_2$  in  $\mathcal{A}_{\mathcal{U}}$ , the induced map  $q^*: M_a(b_2) \to M_a(b_1)$  is a bijection.

We can think of elements in  $M_a(b_1)$  as zig-zags  $b_1 \xrightarrow{f} a' \xleftarrow{p} a$  with f an arbitrary morphism in  $\mathcal{A}$  and p in  $\mathcal{A}_{\mathcal{U}}$ . Condition ( $\star$ ) allows us to factor the composite morphism ker $(q) \rightarrow b_1 \xrightarrow{f} a'$  into a morphism ker $(q) \rightarrow u$  in  $\mathcal{U}$  followed by a summand inclusion  $i: u \rightarrow a'$ . Then the zig-zag  $b_1 \operatorname{coker}(i) \leftarrow a$  represents the same element as the original zig-zag, and  $b_1 \rightarrow \operatorname{coker}(i)$  factors over q. This shows that  $q^*$  is surjective.

For injectivity, note that a zig-zag  $b_2 \xrightarrow{g} a' \xleftarrow{p} a$  in  $M_a(b_2)$  maps to 0 under  $q^*$  precisely if  $gq: b_1 \to a'$  becomes trivial after postcomposition with a morphism  $p': a' \to a''$  in  $\mathcal{A}_{\mathcal{U}}(a)$ . Since q is an epimorphism, this implies p'g = 0 as required.

Since  $M_a$  inverts  $\mathcal{A}_{\mathcal{U}}$ , it follows from general facts that  $M_a \simeq \operatorname{mor}_{\mathcal{A}/\mathcal{U}}(-, a)$ , see [474, Step 1 in the proof of Theorem 6.9]. We have already observed that all values of  $M_a$  are discrete, so this finishes the proof.

**Corollary 8.34.** Let  $\mathcal{A}$  be an additive category and let  $\mathcal{U} \subseteq \mathcal{A}$  be a full additive subcategory satisfying the condition  $(\star)$  appearing in Proposition 8.33. Denote by  $\mathcal{A}/\mathcal{U}$  the usual quotient of additive categories. Then

$$\mathbf{K}(\mathcal{K}^{\mathsf{b}}(\mathcal{U})) \to \mathbf{K}(\mathcal{K}^{\mathsf{b}}(\mathcal{A})) \to \mathbf{K}(\mathcal{K}^{\mathsf{b}}(\mathcal{A}/\mathcal{U}))$$

is a cofiber sequence in Sp.

*Proof.* The additive functor  $\mathcal{B} \to \mathcal{K}^{b}(\mathcal{B})$  induces an equivalence  $\mathcal{P}_{\Sigma}^{\text{st},f}(\mathcal{B}) \xrightarrow{\sim} \mathcal{K}^{b}(\mathcal{B})$  for every additive category  $\mathcal{B}$ , see e.g., [180, Theorem 7.4.9] for a proof. By Proposition 8.33, we have  $\mathcal{R}/\!\!/\mathcal{U} \simeq \mathcal{R}/\mathcal{U}$ , so the corollary follows from Proposition 8.31.

After these preparations, we can now show that the usual constructions of nonconnective algebraic *K*-theory for additive categories produce spectra which are equivalent to  $\mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{A}))$ . There exist various constructions of functors

$$\mathcal{F}$$
: ADDCAT  $\rightarrow$  ADDCAT

equipped with a natural transformation id  $\Rightarrow \mathcal{F}$  such that  $\mathcal{A} \to \mathcal{F}\mathcal{A}$  is fully faithful and satisfies condition ( $\star$ ) appearing in Proposition 8.33 for every additive category, and  $\mathbf{K}(\mathcal{K}^{b}(\mathcal{F}\mathcal{A})) \simeq 0$ . In particular,

$$\mathbf{K}(\mathcal{K}^{\mathsf{b}}(\mathcal{F}\mathcal{A}/\mathcal{A})) \simeq \Sigma \mathbf{K}(\mathcal{K}^{\mathsf{b}}(\mathcal{A})).$$

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Setting  $S\mathcal{A} := \mathcal{F}\mathcal{A}$ , iterating this construction, and taking connective covers, we obtain natural equivalences

$$\begin{split} K(\mathrm{Idem}(\mathcal{K}^{\mathrm{b}}(\mathcal{S}^{n}\mathcal{A}))) &\simeq \tau_{\geq 0} \mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{S}^{n}\mathcal{A})) \\ &\simeq \tau_{\geq 0} \Sigma^{n} \mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{A})) \\ &\simeq \Sigma^{n} \tau_{\geq -n} \mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{A})), \end{split}$$

where  $\tau_{\geq -n}$  is the (-n)-connective cover of a spectrum. Using the cofinality theorem, we find that

$$\mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{A})) \simeq \operatorname{colim}\left(K(\mathcal{K}^{\mathrm{b}}(\mathcal{A})) \to \Omega K(\mathcal{K}^{\mathrm{b}}(\mathcal{S}\mathcal{A})) \to \Omega^{2} K(\mathcal{K}^{\mathrm{b}}(\mathcal{S}^{2}\mathcal{A})) \to \ldots\right).$$

The typical construction of a non-connective algebraic *K*-theory spectrum for additive categories takes such a functor  $\mathcal{F}$  and defines

$$\mathbf{K}(\mathcal{A}) := \operatorname{colim} \left( K(\mathcal{A}) \to \Omega K(\mathcal{S}\mathcal{A}) \to \Omega^2 K(\mathcal{S}^2 \mathcal{A}) \to \ldots \right),$$

where the structure maps are induced by the squares

using that  $K(\mathcal{FS}^n\mathcal{A}) \simeq 0$ . In light of Corollary 8.30, this induces the required weak homotopy equivalence

$$\mathbf{L}(\mathbf{K}(\mathcal{A})) \xrightarrow{\simeq} \mathbf{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{A})).$$

and hence Theorem 8.27 is proven.

# 8.5 Spectra over Groupoids and Equivariant Homology Theories

In this section, we explain how to produce from right-exact G- $\infty$ -categories the necessary data, namely an equivariant homology theory, that will allow for the formulation of the Farrell-Jones Conjecture in the setting of higher categories in Subsection 13.3.1. We will also show that this encompasses the construction of an appropriate equivariant homology theory and the Farrell-Jones Conjecture with coefficients in additive *G*-categories, see Section 13.3. We will also deal with the analogous statement for the Farrell-Jones Conjecture for Waldhausen A-theory, see Section 15.10.

Recall that for a group G, the category I(G) is the category with one object and group of automorphisms G. As an  $\infty$ -category, the standard notation for this category would be BG, but we retain the name I(G) for notational consistency. **Definition 8.35.** A *right-exact* G- $\infty$ -*category* is a functor

$$C: I(G) \to CATREX.$$

The name G- $\infty$ -category is chosen to provide consistent terminology. Note that many authors use the same term to describe contravariant functors from the orbit category of G to CAT $_{\infty}$ .

The main result of this section will be the following result.

**Theorem 8.36 (Equivariant homology theories associated to right exact** G- $\infty$ -categories). Let G be a group.

(i) Let C be right-exact G- $\infty$ -category C. Then one can assign to C a covariant functor, see (8.38),

$$\mathbf{K}_{\mathcal{C}}$$
: GROUPOIDS  $\downarrow I(G) \rightarrow \operatorname{Sp}$ 

and an equivariant homology theory  $\mathcal{H}^{?\downarrow G}_{*}(-; \mathbf{K}_{C})$  over G in the sense of Definition 12.91 such that for every group  $(H, \xi)$  over G and subgroup  $L \subseteq H$  we have a natural identification

$$\mathcal{H}_{n}^{L,\xi|L}(\{\bullet\};\mathbf{K}_{C}) = \mathcal{H}_{n}^{H,\xi}(H/L,\mathbf{K}_{C}) = \pi_{n}(\mathbf{K}_{C}(L,\xi|_{L}));$$

(ii) Let A be an additive G-category. Then we can construct a right-exact G-∞-category K<sup>b</sup>(A), see (8.40), such that equivariant homology theories over G given by H<sup>?↓G</sup><sub>\*</sub>(-; K<sub>A</sub>) of assertion (i) and H<sup>↓G</sup>?<sub>\*</sub>(-; K<sub>A</sub>) coming from (13.10) and Theorem 12.93 are naturally equivalent.

Most of the remainder of this section is occupied with the proof of Theorem 8.36.

### 8.5.1 From Right-Exact G-∞-Categories to Spectra over Groupoids

Recall that  $\mathcal{G}^G(S)$  denotes the transport groupoid of a *G*-set. Note that there is an obvious identification of  $\mathcal{G}^G(G/G)$  with the groupoid I(G).

**Proposition 8.37.** Let C be a right-exact G- $\infty$ -category. Then there exists a functor

$$E_C$$
: GROUPOIDS  $\downarrow I(G) \rightarrow CATREX$ 

with the following properties:

(i)  $E_C$  sends an object  $\operatorname{pr}: \mathcal{G} \to I(G)$  to  $\operatorname{colim}_{\mathcal{G}} C \circ \operatorname{pr}$ ; (ii) The restriction of  $E_C$  along the functor

$$O: \operatorname{Or}(G) \to \operatorname{GROUPOIDS} \downarrow I(G)$$
  
 $G/H \mapsto \left( \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/G) \cong I(G) \right)$ 

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is a left Kan extension of the functor  $C: I(G) \to CATREX$  along the fully faithful functor  $j: I(G) \to Or(G)$  which sends the unique object in I(G) to Gand a morphism g to G-map  $r_{g^{-1}}: G \to G$  given by right multiplication by  $g^{-1}$ . Moreover,  $E_C(\mathcal{G}^G(G/H) \to I(G)) \simeq \operatorname{colim}_{I(H)} C$ .

*Proof.* The proof relies on the (un)straightening equivalence [691, Theorem 3.2.0.1]. Denoting by  $\widehat{CAT}_{\infty}$  the very large  $\infty$ -category of large  $\infty$ -categories, this equivalence allows us to identify the functor category

Fun( GROUPOIDS 
$$\downarrow I(G)^{\text{op}}, \overline{C}A\overline{T}_{\infty})$$

with the subcategory of  $\widehat{CAT}_{\infty} \downarrow (GROUPOIDS \downarrow I(G))$  given by the cartesian fibrations and morphisms being those functors over  $GROUPOIDS \downarrow I(G)$  which preserve cartesian morphisms. Given such a functor *E*, we denote its unstraightening by  $Un(F) \rightarrow GROUPOIDS \downarrow I(G)$ .

Consider now the functor

A: GROUPOIDS 
$$\downarrow I(G)^{\text{op}} \rightarrow \widehat{CAT}_{\infty}$$

which sends a groupoid  $\mathcal{G} \to I(G)$  over I(G) to the functor category Fun( $\mathcal{G}$ , CATREX) as well as the constant functor

$$C: \text{ GROUPOIDS} \downarrow I(G)^{\text{op}} \rightarrow \text{CAT}_{\infty}$$

with value CATREX. There exists a natural transformation  $t: C \Rightarrow A$  such that the component of t at  $\mathcal{G} \rightarrow I(G)$  is the functor which sends a right-exact  $\infty$ -category C to the constant diagram with value C in Fun( $\mathcal{G}$ , CATREX). Via unstraightening, this natural transformation corresponds to a commutative diagram



in which the functor Un(t) preserves cartesian morphisms. Since CATREX admits all colimits, the functor CATREX  $\rightarrow$  Fun( $\mathcal{G}$ , CATREX) taking constant diagrams has a left adjoint for all  $\mathcal{G}$ . Hence [689, Proposition 7.3.2.6] applies to show that Un(t) has a left adjoint *L* over GROUPOIDS  $\downarrow I(G)$ , which on each fiber is given by the colimit functor colim<sub> $\mathcal{G}$ </sub>: Fun( $\mathcal{G}$ , CATREX)  $\rightarrow$  CATREX.

We are now going to construct a functor GROUPOIDS  $\downarrow I(G) \rightarrow$  CATREX by precomposing *L* with a certain section to *p*. One consequence of the unstraightening equivalence is the fact that limits in  $\widehat{CAT}_{\infty}$  are given by the  $\infty$ -category of cartesian sections in the cartesian fibration associated to a diagram [691, Corollary 3.3.3.2]. In particular, the  $\infty$ -category of cartesian sections to *p* is a model for the limit of *A*. Noting that the indexing category GROUPOIDS  $\downarrow I(G)^{\text{op}}$  has an initial object, this limit is equivalent to  $A(\operatorname{id}_{I(G)}) = \operatorname{Fun}(I(G), \operatorname{CATREX})$ . The given right-exact G- $\infty$ -category C specifies an object in this  $\infty$ -category, and therefore induces a cartesian section s: GROUPOIDS  $\downarrow I(G) \rightarrow Un(A)$  of p. Composing this section with L and projecting to CATREX now provides a functor

$$E_C$$
: GROUPOIDS  $\downarrow I(G) \rightarrow CATREX.$ 

Since s is a cartesian section, s sends an object pr:  $\mathcal{G} \to I(G)$  to the image  $C \circ pr$ of C under the restriction functor along pr. Since L is given fiberwise by the colimit functor, it follows that  $E_{\mathcal{C}}(pr) \simeq \operatorname{colim}_{\mathcal{G}} \mathcal{C} \circ pr$  as claimed.

Consider now the restriction  $E_C \circ O$ . Then observe that O is naturally isomorphic to the functor

$$J: \operatorname{Or}(G) \to \operatorname{GROUPOIDS} \downarrow I(G)$$
$$G/H \mapsto (j \downarrow G/H \to j \downarrow G/G \cong I(G)),$$

where the isomorphism  $I(G) \cong j \downarrow G/G$  sends a morphism  $g: G/G \to G/G$  in I(G) to the morphism  $r_{g^{-1}}: G \to G$  in  $j \downarrow G/G$ . Generalizing this identification, such a natural isomorphism is induced by the isomorphism of categories

$$\begin{aligned} \mathcal{G}^{G}(G/H) &\xrightarrow{\cong} j \downarrow G/H \\ gH \mapsto \left( G \xrightarrow{e \mapsto gH} G/H \right), \\ \left( gH \xrightarrow{\gamma} g'H \right) \mapsto & G \xrightarrow{r_{\gamma^{-1}}} G \\ e \mapsto gH \xrightarrow{q \mapsto g'H} G/H. \end{aligned}$$

From the preceding description of  $E_C$ , it now follows that

$$E_C(G/H) \simeq \operatorname{colim}_{i \mid G/H} C,$$

which is precisely the pointwise formula for the left Kan extension along j. Since the functor  $I(H) \to j \downarrow G/H$  which sends the unique object in I(H) to  $G \xrightarrow{e \mapsto eH} G/H$ and an element  $h \in G$  to the multiplication map with h is an equivalence of categories, it also follows that  $E_C(G/H) \simeq \operatorname{colim}_{I(H)} C$ . 

This finishes the proof of Proposition 8.37.

By composing  $E_C$  with the algebraic K-theory functor, one obtains for every right-exact G- $\infty$ -category C a functor

(8.38) 
$$\mathbf{K}_C \colon \mathsf{GROUPOIDS} \downarrow I(G) \to \mathsf{Sp}.$$

By construction, the functor  $\mathbf{K}_C$  inverts equivalences of groupoids over I(G).

#### 8.5.2 From Spectra over Groupoids to Equivariant Homology Theories

Let us explain how such a functor gives rise to a *G*-homology theory on the  $\infty$ -category of *G*-spaces. By Elmendorf's theorem [334], the  $\infty$ -category obtained by localizing the category of *G*-*CW*-complexes at the collection of equivariant homotopy equivalences is equivalent to the  $\infty$ -category Fun(Or(*G*)<sup>op</sup>, Spc) of presheaves on the orbit category of *G*. Given a functor **E**: Or(*G*)  $\rightarrow$  Sp, the universal property of the Yoneda embedding implies that **E** extends uniquely to a colimit-preserving functor

$$\mathbf{H}^{G}(-; \mathbf{E})$$
: Fun(Or(G)<sup>op</sup>, Spc)  $\rightarrow$  Sp

such that the composition of  $\mathbf{H}^G(-; \mathbf{E})$  with the Yoneda embedding  $Or(G) \rightarrow Fun(Or(G)^{op}, Spc)$  is identified with  $\mathbf{E}$ .

Starting from a functor  $\mathbf{E} \colon \operatorname{Or}(G) \to \operatorname{SPECTRA}$ , Chapter 12.10 provides an alternative approach to the construction of such a homology theory. Here is an indication in which sense the two constructions lead to isomorphic outcomes. In Chapter 12.10, a *G*-homology theory is obtained from  $\mathbf{E}$  by considering the SPECTRA-valued functor on *G*-*CW*-complexes which sends *X* to map<sub>*G*</sub>( $-, X_+$ )  $\wedge_{\operatorname{Or}(G)}$   $\mathbf{E}$  and taking homotopy groups. The proof of this statement amounts to showing that the Sp-valued functor taking *X* to  $\mathbf{L}$  (map<sub>*G*</sub>( $-, X_+$ )  $\wedge_{\operatorname{Or}(G)}$   $\mathbf{E}$ ) has the following properties:

- It inverts *G*-equivariant homotopy equivalences. In particular, it factors uniquely over the localization Fun(Or(*G*)<sup>op</sup>, Spc) of the category of *G*-*CW*-complexes;
- The induced functor  $\mathbf{h}_{\mathbf{E}}$ : Fun $(Or(G)^{op}, Spc) \rightarrow Sp$  preserves colimits. To see this, it is enough to check that the induced functor preserves initial objects, pushouts, and filtered colimits. The preservation of initial objects is obvious. For pushouts, one can, for example, rely on Proposition 8.29 again to see that every pushout in Fun $(Or(G)^{op}, Spc)$  can be realized by a pushout of *G-CW*-complexes along the inclusion of a subcomplex. Then the preservation of pushouts follows from excision as formulated in Definition 12.1. By a similar argument, the preservation of filtered colimits follows from the disjoint union axiom for *G*-homology theories.

As a colimit-preserving functor on Fun( $Or(G)^{op}$ , Spc), the functor  $\mathbf{h}_{\mathbf{E}}$  is uniquely determined by its restriction along the Yoneda embedding. Essentially by construction, this restriction is given by  $\mathbf{L} \circ \mathbf{E}$ , so  $\mathbf{h}_{\mathbf{E}}$  is equivalent to the functor  $\mathbf{H}^{G}(-; \mathbf{L} \circ \mathbf{E})$  considered above.

More generally, consider now a functor **E**: GROUPOIDS  $\downarrow I(G) \rightarrow$  Sp. If  $\xi: H \rightarrow G$  is a group over *G*, then we can restrict **E** along the functor

$$T_{\xi} \colon \operatorname{Or}(H) \to \operatorname{GROUPOIDS} \downarrow I(G)$$
$$H/L \mapsto \left( \mathcal{G}^{H}(H/L) \to \mathcal{G}^{H}(H/H) \cong I(H) \xrightarrow{I(\xi)} I(G) \right)$$

to obtain the functor  $\mathbf{E} \circ T_{\boldsymbol{\xi}} : \operatorname{Or}(G) \to \operatorname{Sp.}$  This functor in turn induces a colimitpreserving functor

$$\mathbf{H}^{H,\xi}(-;\mathbf{E})$$
: Fun(Or( $H$ )<sup>op</sup>, Spc)  $\rightarrow$  Sp.

Suppose that  $\alpha : (H, \xi) \to (K, \nu)$  is a morphism of groups over *G*. Then the induction functor  $Or(H) \to Or(K), G/H \mapsto \alpha_*G/H$  induces a restriction functor

$$\operatorname{Fun}(\operatorname{Or}(K)^{\operatorname{op}},\operatorname{Spc}) \to \operatorname{Fun}(\operatorname{Or}(H)^{\operatorname{op}},\operatorname{Spc}).$$

Since Spc is cocomplete, this restriction functor has a left adjoint

$$\alpha_*$$
: Fun(Or( $H$ )<sup>op</sup>, Spc)  $\rightarrow$  Fun(Or( $K$ )<sup>op</sup>, Spc)

which corresponds to the analogous induction functor on the level of *G-CW*complexes. Being a left adjoint,  $\alpha_*$  preserves colimits, so  $\mathbf{H}^{K,\nu}(\alpha_*-; \mathbf{E})$  is a colimitpreserving functor Fun( $Or(H)^{op}$ , Spc)  $\rightarrow$  Sp. Using the universal property of the Yoneda embedding once more, natural transformations between  $\mathbf{H}^{H,\xi}(-, \mathbf{E})$  and  $\mathbf{H}^{K,\nu}(\alpha_*-; \mathbf{E})$  are identified with natural transformations of the restricted functors  $Or(H) \rightarrow$  Sp. Since  $\alpha$  induces a natural transformation  $T_{\xi} \Rightarrow T_{\nu}$ , this observation shows that  $\alpha$  gives rise to a natural transformation

$$\operatorname{ind}_{\alpha} \colon \mathbf{H}^{H,\xi}(-,\mathbf{E}) \Rightarrow \mathbf{H}^{K,\nu}(\alpha_*-;\mathbf{E}).$$

One can check that this produces (by passing to homotopy groups) an equivariant homology theory over G in the sense of Definition 12.91.

Now to finish the proof of assertion (i) of Theorem 8.36, just apply the construction above for  $\mathbf{E} = \mathbf{K}_C$  defined in (8.38).

**Remark 8.39.** If we start with a functor  $\mathbf{E}$ : GROUPOIDS  $\downarrow I(G) \rightarrow$  SPECTRA instead, one observes that the proof of Theorem 12.93 actually produces natural transformations ind<sub> $\alpha$ </sub> of functors  $Or(H) \rightarrow$  SPECTRA for every morphism  $\alpha$  of groups over *G*, and that these transformations agree with the transformations for the functor  $\mathbf{L} \circ \mathbf{E}$  after applying the localization functor  $\mathbf{L}$ : SPECTRA  $\rightarrow$  Sp. Hence, the equivariant homology theory over *G* giving by considering the homotopy groups of  $\mathbf{H}^{?}_{*}(-; \mathbf{L} \circ \mathbf{E})$  and considering a *G*-*CW*-complex as *G*- $\infty$ -space defined above and the equivariant homology theory  $H^{?}_{*}(-; \mathbf{E})$  over *G* constructed in Theorem 12.30 are naturally equivalent.

We refer to Section 12.10 and Chapter 15 for more details and explanations of how such equivariant homology theories allow for the formulation of Isomorphism Conjectures.

8.5 Spectra over Groupoids and Equivariant Homology Theories

#### 8.5.3 The Special Case of Additive G-Categories

In this subsection, we will compare the output of Proposition 8.37 to analogous constructions for the algebraic *K*-theory of additive categories, see Subsection 13.3.1.

As explained in Example 8.5, one obtains from any additive category  $\mathcal{A}$  a stable  $\infty$ -category  $\mathcal{K}^{b}(\mathcal{A})$ , see (8.6), by localizing the category of bounded chain complexes over  $\mathcal{A}$  at the chain homotopy equivalences. If  $\mathcal{A}$  carries a *G*-action, the functoriality of the localization implies that we obtain a right-exact *G*- $\infty$ -category, again denoted by  $\mathcal{K}^{b}(\mathcal{A})$ ,

(8.40) 
$$\mathcal{K}^{b}(\mathcal{A}) \colon I(G) \to \mathsf{CATREX}.$$

Through Proposition 8.37, we obtain a functor

$$E_{\mathcal{K}^{\mathsf{b}}(\mathcal{A})} \circ O \colon \mathsf{Or}(G) \to \mathsf{CATREX}$$

which is a left Kan extension of  $\mathcal{K}^{b}(\mathcal{A})$  along the inclusion  $I(G) \to Or(G)$ . Now [180, Corollary 7.4.16] identifies this functor with another functor

$$E_{\mathcal{A}}: \operatorname{Or}(G) \to \operatorname{CATREX}$$

which is given as follows. By localizing the category of additive categories at the equivalences, one obtains an  $\infty$ -category ADDCAT $_{\infty}$ . Since sending  $\mathcal{A}$  to  $\mathcal{K}^{b}(\mathcal{A})$  respects equivalences, we obtain an induced functor  $\mathcal{K}^{b}$ : ADDCAT $_{\infty} \rightarrow$  CATREX. Regarding  $\mathcal{A}$  as an object with *G*-action in ADDCAT $_{\infty}$ , one can take the left Kan extension  $j_{!}\mathcal{A}$  of  $\mathcal{A}$  along  $j: I(G) \rightarrow Or(G)$ . Postcomposing with  $\mathcal{K}^{b}$  yields the functor  $E_{\mathcal{A}} := \mathcal{K}^{b} \circ j_{!}\mathcal{A}$ .

After applying K-theory, the discussion from the preceding section yields an identification

$$\mathbf{K}_{\mathcal{K}^{\mathfrak{b}}(\mathcal{A})} \circ O \simeq \mathbf{K} \circ j_{!} \mathcal{A}.$$

As explained in [182, Section 3.3], the latter functor is equivalent to the composite of the functor  $\mathbf{K}_{\mathcal{R}}$ :  $Or(G) \rightarrow SPECTRA$  of (13.10), which sends G/H to  $\mathbf{K}\left(\int_{\mathcal{G}^{G}(G/H)} \mathcal{R} \circ \mathrm{pr}\right)$ , with the functor **L** of (8.2). This together with Remark 8.39 finishes the proof of assertion (ii) of Theorem 8.36 and hence of Theorem 8.36.

## 8.5.4 The Special Case of Waldhausen's A-Theory

In this subsection, we will compare the output of Proposition 8.37 to the analogous constructions for *A*-theory, see Conjecture 15.61.

Recall the following notation:

- We denote by  $\mathcal{R}^f$ : SPACES  $\to$  Wald<sup>ho</sup> the functor which sends a space X to the Waldhausen category of finite retractive spaces over X, whose subcategory of weak equivalences we denote by  $h\mathcal{R}^f(X)$ . In order to shorten notation, let us denote the localization  $\mathcal{R}^f(X)[h\mathcal{R}^f(X)^{-1}]$  simply by  $\mathcal{R}^f(X)[h^{-1}]$ ;
- We denote by  $\mathbf{A} := \mathbf{K} \circ \mathcal{R}^f : SPACES \to Sp$  the non-connective A-theory functor.

**Proposition 8.41.** Let Z be a free G-CW-complex. Denoting by L(Z):  $I(G) \rightarrow \text{Spc}$  the underlying homotopy type with G-action, we obtain a right-exact G- $\infty$ -category  $L(Z) \otimes \text{Spc}^{\omega}$  because CATREX is cocomplete.

Then the functors

$$\mathbf{A}_Z^G \colon \operatorname{Or}(G) \to \operatorname{Sp}, \quad G/H \mapsto \mathbf{A}(Z \times_G G/H)$$

and

$$\mathbf{K}_{L(Z)\otimes \operatorname{Spc}^{\omega}}\circ O\colon \operatorname{Or}(G)\to \operatorname{Sp}$$

are equivalent.

*Proof.* Proposition 8.37 implies that the second functor is obtained by left Kan extending  $L(Z) \otimes \operatorname{Spc}^{\omega}$  along the inclusion functor  $I(G) \to \operatorname{Or}(G)$ . Observing that free *G*-CW-complexes are principal *G*-bundles, the proposition is now a direct consequence of [180, Corollary 7.5.6].

### 8.5.5 Ring Spectra as Coefficients

Finally, let us mention that this framework also incorporates the *K*-theory of group rings over ring spectra. If **R** is any ring spectrum, we can equip the stable  $\infty$ -category PERF(**R**) of perfect **R**-modules with the trivial *G*-action. It is known that colim<sub>*I*(*G*)</sub> PERF(**R**)  $\simeq$  PERF(**R**[*G*]), where **R**[*G*] :=  $\Sigma^{\infty}_{+}G \otimes \mathbf{R}$  denotes the group ring of *G* over **R**, see for example [121, Corollary 3.6]. In light of Proposition 8.37, the functor **K**<sub>PERF(**R**)</sup> induces a functor  $Or(G) \rightarrow Sp$  which sends *G/H* to a spectrum equivalent to **K**(**R**[*H*]).</sub>

Note that this reduces to Waldhausen's A-theory if we take **R** to be the sphere spectrum **S**, as the non-connective K-theory of S[G] is the non-connective Waldhausen theory A(BG), and also encompasses the case of group rings RG for (ordinary) rings R.
# 8.6 Karoubi Filtrations and the Associated Weak Homotopy Fibration Sequences

This section is devoted to the notion of a Karoubi filtration, which is given by a full additive subcategory  $\mathcal{U}$  of an additive category  $\mathcal{A}$  satisfying certain conditions, and the existence of the associated weak homotopy fibration sequences

$$\begin{split} \mathbf{K}(\mathcal{U}) &\to \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U});\\ \mathbf{L}^{\langle -\infty \rangle}(\mathcal{U}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}) \to \mathbf{L}^{\langle -\infty \rangle}(\mathcal{A}/\mathcal{U}), \end{split}$$

which induce long exact sequences of K- and L-groups. This will be a basic tool in Chapter 21, where we will define G-homology theories in terms of controlled topology and need to check the axioms of a G-homology theory such as the long exact sequence of a pair or excision.

#### 8.6.1 Karoubi Filtration and Quotient Categories

If  $\mathcal{U}$  is a full additive subcategory of  $\mathcal{A}$ , then one can define the quotient category  $\mathcal{A}/\mathcal{U}$  as follows. The set of objects of  $\mathcal{A}/\mathcal{U}$  agrees with the set of objects of  $\mathcal{A}$ . The set of morphism  $\operatorname{mor}_{\mathcal{A}/\mathcal{U}}(A, A')$  for objects A and A' in  $\mathcal{A}/\mathcal{U}$  is defined to be  $\operatorname{mor}_{\mathcal{A}}(A, A')/\sim$  for the equivalence relation  $\sim$  where we call two morphisms  $f, f': A \to A'$  in  $\mathcal{A}$  equivalent if their difference  $f - f': A \to A'$  factorizes in  $\mathcal{A}$  as a composite  $A \to U \to A'$  for some object U in  $\mathcal{U}$ . We leave to the reader the elementary proof of the fact that  $\mathcal{A}/\mathcal{U}$  inherits from  $\mathcal{A}$  the structure of an additive categories. For a morphism  $f: A \to A'$  in  $\mathcal{A}$ , we denote by  $[f]: A \to A'$  the morphism in  $\mathcal{A}/\mathcal{U}$  represented by f.

**Definition 8.42 (Quotients for additive categories).** We call the additive category  $\mathcal{A}/\mathcal{U}$  the *quotient category* of  $\mathcal{A}$  by  $\mathcal{U}$ .

**Definition 8.43** (*U*-filtered). We say that  $\mathcal{A}$  is *right U*-filtered or, equivalently, that the inclusion  $\mathcal{U} \to \mathcal{A}$  is a *right Karoubi filtration*, if the following holds:

The additive subcategory  $\mathcal{U} \subseteq \mathcal{A}$  is full. Moreover, given an object A in  $\mathcal{A}$ , an objects  $U \in \mathcal{U}$ , and a morphism  $f: A \to U$ , there are objects  $A^{\mathcal{U}}$  in  $\mathcal{U}$  and  $A^{\perp}$  in  $\mathcal{A}$  and morphisms  $i^{\mathcal{U}}: A^{\mathcal{U}} \to A$  and  $i^{\perp}: A^{\perp} \to A$  satisfying:

- $i^{\mathcal{U}} \oplus i^{\perp} : A^{\mathcal{U}} \oplus A^{\perp} \xrightarrow{\cong} A$  is an isomorphism in  $\mathcal{R}$ ;
- There exists a morphism  $f^{\mathcal{U}}: A^{\mathcal{U}} \to \hat{U}$  such that the following diagram commutes



where  $\operatorname{pr}_{A^{\mathcal{U}}} : A^{\mathcal{U}} \oplus A^{\perp} \to A^{\mathcal{U}}$  is the canonical projection.

We say that  $\mathcal{A}$  is *left U-filtered* or, equivalently, that the inclusion  $\mathcal{U} \to \mathcal{A}$  is a *left Karoubi filtration*, if the following holds:

The additive subcategory  $\mathcal{U} \subseteq \mathcal{A}$  is full. Moreover, given an object A in  $\mathcal{A}$ , an object  $V \in \mathcal{U}$ , and a morphism  $g: V \to A$  in  $\mathcal{A}$ , there are objects  $A^{\mathcal{U}}$  in  $\mathcal{U}$  and  $A^{\perp}$  in  $\mathcal{A}$  and morphisms  $i^{\mathcal{U}}: A^{\mathcal{U}} \to A$  and  $i^{\perp}: A^{\perp} \to A$  satisfying:

- $i^{\mathcal{U}} \oplus i^{\perp} : A^{\mathcal{U}} \oplus A^{\perp} \xrightarrow{\cong} A$  is an isomorphism in  $\mathcal{R}$ ;
- There exists a morphism  $g^{\mathcal{U}}: V \to A^{\mathcal{U}}$  such that the following diagram commutes



where  $i_{A^{\mathcal{U}}} \colon A^{\mathcal{U}} \to A^{\mathcal{U}} \oplus A^{\perp}$  is the canonical inclusion.

We say that  $\mathcal{A}$  is  $\mathcal{U}$ -filtered or, equivalently, that the inclusion  $\mathcal{U} \to \mathcal{A}$  is a *Karoubi filtration*, if it is both left and right  $\mathcal{U}$ -filtered.

**Remark 8.44.** The morphisms  $f^{\mathcal{U}}$  and  $g^{\mathcal{U}}$  appearing in Definition 8.43 are uniquely determined by the desired properties. Namely, if  $f^{\mathcal{U}}$  and  $g^{\mathcal{U}}$  exist, then  $f^{\mathcal{U}} = f \circ i^{\mathcal{U}}$  and  $g^{\mathcal{U}} = \operatorname{pr}^{\mathcal{U}} \circ (i^{\mathcal{U}} \oplus i^{\perp})^{-1} \circ g$ .

**Remark 8.45 (Relation to the classical definition of a Karoubi filtration).** If one requires in the definition of  $\mathcal{U}$ -filtered appearing in Definition 8.43 additionally that U = V, then it reduces to [566, Definition 5.4]. One easily checks that Definition 8.43 and [566, Definition 5.4] are equivalent, the special case U = V in [566, Definition 5.4] implies the general case of Definition 8.43 by considering  $U \oplus V$ . Note that [566, Definition 5.4] agrees with the more complicated notion of a  $\mathcal{U}$ -filtration due to Karoubi [547], see [566, Lemma 5.6].

#### 8.6.2 The Weak Homotopy Fibration Sequence of a Karoubi Filtration

The main result of this section is stated next.

8.6 Karoubi Filtrations and the Associated Weak Homotopy Fibration Sequences

**Theorem 8.46 (The weak homotopy fibration sequence of a Karoubi filtration).** Let  $\mathcal{A}$  be an additive category and  $i: \mathcal{U} \to \mathcal{A}$  be the inclusion of a full additive subcategory. Let  $p: \mathcal{A} \to \mathcal{A}/\mathcal{U}$  be the canonical projection. Suppose that  $\mathcal{A}$  is left  $\mathcal{U}$ -filtered or right  $\mathcal{U}$ -filtered.

(i) The sequence of spectra

$$\mathbf{K}(\mathcal{U}) \xrightarrow{\mathbf{K}(i)} \mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(p)} \mathbf{K}(\mathcal{A}/\mathcal{U})$$

is a weak homotopy fibration sequence of non-connective spectra, i.e., the composite  $\mathbf{K}(p) \circ \mathbf{K}(i)$  admits a preferred nullhomotopy, since there is a preferred natural transformation from  $p \circ i$  to the trivial functor, and the induced map

$$\mathbf{K}(\mathcal{U}) \to \mathrm{hofib}(\mathbf{K}(p) \colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U}))$$

is a weak homotopy equivalence.

In particular we get a long exact sequence, infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{U}) \xrightarrow{K_n(i)} K_n(\mathcal{A}) \xrightarrow{K_n(p)} K_n(\mathcal{A}/\mathcal{U}) \xrightarrow{\partial_n} K_{n-1}(\mathcal{U})$$
$$\xrightarrow{K_{n-1}(i)} K_{n-1}(\mathcal{A}) \xrightarrow{K_{n-1}(p)} K_{n-1}(\mathcal{A}/\mathcal{U}) \xrightarrow{\partial_{n-1}} \cdots$$

(ii) Suppose additionally that A is an additive category with involution such that the involution induces the structure of an additive category with involution on U. Then A/U inherits the structure of an additive category with involution and the sequence of spectra

$$\mathbf{L}^{\langle -\infty\rangle}(\mathcal{U}) \xrightarrow{\mathbf{L}^{\langle -\infty\rangle}(i)} \mathbf{L}^{\langle -\infty\rangle}(\mathcal{A}) \xrightarrow{\mathbf{L}^{\langle -\infty\rangle}(p)} \mathbf{L}^{\langle -\infty\rangle}(\mathcal{A}/\mathcal{U})$$

is a weak homotopy fibration sequence of non-connective spectra. In particular we get a long exact sequence, infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} L_n^{\langle -\infty \rangle}(\mathcal{U}) \xrightarrow{L_n^{\langle -\infty \rangle}(i)} L_n^{\langle -\infty \rangle}(\mathcal{R}) \xrightarrow{L_n^{\langle -\infty \rangle}(p)} L_n^{\langle -\infty \rangle}(\mathcal{R}/\mathcal{U})$$

$$\xrightarrow{\partial_n} L_{n-1}^{\langle -\infty \rangle}(\mathcal{U}) \xrightarrow{L_{n-1}^{\langle -\infty \rangle}(i)} L_{n-1}^{\langle -\infty \rangle}(\mathcal{R})$$

$$\xrightarrow{L_{n-1}^{\langle -\infty \rangle}(p)} L_{n-1}^{\langle -\infty \rangle}(\mathcal{R}/\mathcal{U}) \xrightarrow{\partial_{n-1}} \cdots$$

*Proof.* (i) Corollary 8.34 takes care of the case when  $\mathcal{A}$  is left  $\mathcal{U}$ -filtered. Since in an additive category finite sums and finite products agree, the category  $\mathcal{A}^{op}$  is right  $\mathcal{U}$ -filtered if  $\mathcal{A}$  is left  $\mathcal{U}$ -filtered. Since *K*-theory does not see the difference between  $\mathcal{A}$  and  $\mathcal{A}^{op}$ , assertion (i) follows from Corollary 8.34.

If one assumes that  $\mathcal{A}$  is  $\mathcal{U}$ -filtered, more classical proofs can be found for instance in [209, 211, 801], based on the work of Karoubi [547].

(ii) Note that for an additive category  $\mathcal{A}$  with involution left  $\mathcal{U}$  filtered is equivalent to right  $\mathcal{U}$  filtered and hence to  $\mathcal{U}$ -filtered. Now apply [214, Theorem 4.2].

**Example 8.47.** Suppose that  $\mathcal{U} \to \mathcal{A}$  is a left or right Karoubi filtration and  $\mathcal{A}$  is flasque. Then there is weak homotopy equivalence  $\mathbf{K}(\mathcal{U}) \xrightarrow{\simeq} \Omega \mathbf{K}(\mathcal{A}/\mathcal{U})$  by the following argument.

We get a weak homotopy equivalence

$$\mathbf{K}(\mathcal{U}) \xrightarrow{-} \operatorname{hofib}(\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U}))$$

from Theorem 8.46 (i). The projection  $\mathbf{K}(\mathcal{A}) \rightarrow *$  to the trivial spectrum \* is a weak homotopy equivalence by Theorem 6.37 (iii) and hence induces a weak homotopy equivalence

$$\operatorname{hofib}(\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}/\mathcal{U})) \xrightarrow{\simeq} \operatorname{hofib}(* \to \mathbf{K}(\mathcal{A}/\mathcal{U})) = \Omega \mathbf{K}(\mathcal{A}/\mathcal{U})$$

## 8.7 Notes

Without question, higher categories have become a very important tool in many branches of mathematics in the last years, including *K*-theory. Here we want briefly to discuss its relevance in connection with the Isomorphism Conjectures.

If one goes through the list of applications of the Farrell-Jones Conjecture in Section 13.12, one sees that for all of them except the applications to the computations of homotopy groups of automorphisms of aspherical closed manifolds it suffices to consider rings as coefficients. For the computations of homotopy groups of automorphisms of aspherical closed manifolds, it is enough to know the Farrell-Jones Conjecture for *A*-theory, which has already been treated in detail in [344]. The passage from rings to additive categories is not directly relevant for applications, but is clearly motivated by the fact that it also allows us to handle twisted group rings and orientation characters and ensures all the useful inheritance properties. At the time of writing, we know no application of the Farrell-Jones Conjecture to prominent problems in algebra, geometry, group theory, or topology where it does not suffice to deal with rings as coefficients or with the *A*-theory version and one is forced to consider higher categories. We also think that computations using cyclotomic traces have come to their limit concerning the detection of the algebraic *K*-theory of integral group rings.

Nevertheless, we expect that in the future the version of the Farrell-Jones Conjecture for higher categories will become important. Actually there are already instances where the passage to higher categories was necessary to get information about the classical setting. The constructions of natural transformations from the topological *K*-theory to the algebraic *L*-theory of  $C^*$ -algebra have been poorly (and incorrectly) treated in the classical setting, but in a very satisfactory way using higher category theory in [614, 615] and thus open the door to a link between the Farrell-Jones Conjecture and the Baum-Connes Conjecture as explained in Subsection 15.14.4. Even

#### 8.7 Notes

for computations of the topological *K*-theory of  $C^*$ -algebras methods from higher category theory are useful and actually needed, see for instance [615, Theorem B].

Another related topic is Hermitian *K*-theory in the setting of higher categories and all its applications, see for instance Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus, and Steimle [193, 194, 195, 196].

So far the *L*-theoretic version of the Farrell-Jones Conjecture has only been established for additive categories with involution. Christoph Winges is at the time of writing working on a generalization to the setting of higher categories for all Dress-Farrell-Hsiang-Jones groups.

# Chapter 9 Algebraic *L*-Theory

# 9.1 Introduction

In Remark 3.53 we have briefly discussed the *Surgery Program*. Starting with a map of degree one of connected closed manifolds  $f: M \to N$ , the goal is to modify it by surgery steps so that it becomes a homotopy equivalence. This will change the source but not the target, and can only be carried out if the map f is covered by bundle data. With the bundle data, one is able to make the map highly connected, but in the last step towards a homotopy equivalence an obstruction, the *surgery obstruction*, occurs, whose appearance is among other things due to *Poincaré duality*. This surgery obstruction takes values in the *algebraic L-groups*  $L_n(\mathbb{Z}G)$  for  $G = \pi_1(N)$ . An introduction to the surgery obstruction and the algebraic *L*-groups will be given in this chapter. These are the key tools for the classification of manifolds besides the *s*-Cobordism Theorem 3.47. All this will be carried out in Sections 9.2 to 9.5 in the even-dimensional case and in Sections 9.6 to 9.8 in the odd-dimensional case.

We will also consider normal maps between compact manifolds with boundary that induce homotopy equivalences on the boundary. Here we want to achieve a homotopy equivalence by surgery on the interior, see Section 9.9.

Since the Whitehead torsion appears in the *s*-Cobordism Theorem 3.47, it will be important to achieve a simple homotopy equivalence and not only a homotopy equivalence by surgery. This leads to the *simple surgery obstruction* and *decorated L*-groups, see Section 9.10. The various decorated *L*-groups are linked by *Rothen*-*berg sequences*. The *L*-theoretic analog of the Bass-Heller-Swan decomposition for *K*-theory is the *Shaneson splitting*.

We will present the *L*-theoretic Farrell-Jones Conjecture for torsionfree groups 9.114, which relates the algebraic *L*-groups  $L_n(\mathbb{Z}G)$  to the homology of *BG* with coefficient in the *L*-theory spectrum, analogous to the Farrell-Jones Conjecture for torsionfree groups and regular rings for *K*-theory 6.53. This together with the *Surgery Exact Sequence* of Section 9.12 opens the door to many applications. We will discuss the *Novikov Conjecture* 9.137 about the homotopy invariance of higher signatures and the *Borel Conjecture* 9.163 about the topologically rigidity of aspherical closed manifolds. Moreover, we will deal with the problems of whether a given finitely presented *Poincaré duality group* occurs as the fundamental group of an aspherical closed manifold, see Section 9.18, the *stable Cannon Conjecture*, see Section 9.19, and when a *product decomposition* of the fundamental group of an aspherical closed manifold already implies a product decomposition of the manifold itself, see Section 9.20. *Automorphism groups* of aspherical closed manifolds are

treated in Section 9.21. A brief survey on computations of *L*-theory of group rings of finite groups is presented in Section 9.22.

This chapter is an extract of the book [667] by Lück and Macko.

# 9.2 Symmetric and Quadratic Forms

## 9.2.1 Symmetric Forms

**Definition 9.1 (Ring with involution).** A *ring with involution* R is an associative ring R with unit together with an *involution of rings* 

$$-: R \to R, \quad r \mapsto \overline{r},$$

i.e., a map satisfying  $\overline{\overline{r}} = r$ ,  $\overline{r+s} = \overline{r} + \overline{s}$ ,  $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$ , and  $\overline{1} = 1$  for  $r, s \in R$ .

If *R* is commutative, we can equip it with the trivial involution  $\overline{r} = r$ .

Below we fix a ring R with involution. Module is to be understood as left module unless explicitly stated otherwise.

**Example 9.2 (Involutions on group rings).** Let  $w: G \to \{\pm 1\}$  be a group homomorphism. Then the group ring *RG* inherits an involution, the so-called *w*-twisted involution, that sends  $\sum_{g \in G} r_g \cdot g$  to  $\sum_{g \in G} w(g) \cdot \overline{r_g} \cdot g^{-1}$ .

**Remark 9.3 (Dual modules).** The main purpose of the involution is to ensure that the dual of a left *R*-module can be viewed as a left *R*-module again. Namely, let *M* be a left *R*-module. Then  $M^* := \hom_R(M, R)$  carries a canonical right *R*-module structure given by  $(fr)(m) = f(m) \cdot r$  for a homomorphism of left *R*-modules  $f: M \to R$  and  $m \in M$ . The involution allows us to view  $M^* = \hom_R(M, R)$  as a left *R*-module, namely, define rf for  $r \in R$  and  $f \in M^*$  by  $(rf)(m) := f(m) \cdot \overline{r}$  for  $m \in M$ .

**Notation 9.4.** Given a finitely generated projective *R*-module *P*, we denote by  $e(P) : P \xrightarrow{\cong} (P^*)^*$  the canonical isomorphism of (left) *R*-modules that sends  $p \in P$  to the element in  $(P^*)^*$  given by  $P^* \to R$ ,  $f \mapsto \overline{f(p)}$ .

We will often use the following elementary fact. Let  $f: P \rightarrow Q$  be a homomorphism of finitely generated projective *R*-modules. Then the following diagram commutes

(9.5)  $P \xrightarrow{f} Q$  $e(P) \bigvee_{\cong} \bigoplus_{\cong} \bigvee_{\cong} e(Q)$  $(P^*)^* \xrightarrow{(f^*)^*} (Q^*)^*.$ 

9.2 Symmetric and Quadratic Forms

**Exercise 9.6.** Show that the map  $e(P): P \rightarrow (P^*)^*$  of Notation (9.4) is a well-defined isomorphism of finitely generated projective *R*-modules, is compatible with direct sums, and is natural, i.e., the diagram (9.5) commutes.

**Definition 9.7 (Non-singular**  $\epsilon$ -symmetric form). Let  $\epsilon \in \{\pm 1\}$ . An  $\epsilon$ -symmetric form  $(P, \phi)$  over an associative ring R with involution is a finitely generated projective (left) R-module P together with an R-map  $\phi \colon P \to P^*$  such that the composite  $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{\phi^*} P^*$  agrees with  $\epsilon \cdot \phi$ .

A morphism  $f: (P, \phi) \to (P', \phi)$  of  $\epsilon$ -symmetric forms is an *R*-homomorphism  $f: P \to P'$  satisfying  $f^* \circ \phi' \circ f = \phi$ .

We call an  $\epsilon$ -symmetric form  $(P, \phi)$  non-singular if  $\phi$  is an isomorphism.

If  $\epsilon$  is 1 or -1 respectively, we often replace  $\epsilon$ -symmetric by symmetric or skewsymmetric respectively. The direct sum of two  $\epsilon$ -symmetric forms is defined in the obvious way. The direct sum of two non-singular  $\epsilon$ -symmetric forms is again a non-singular  $\epsilon$ -symmetric form.

**Remark 9.8** ( $\epsilon$ -symmetric forms as pairings). We can rewrite an  $\epsilon$ -symmetric form  $(P, \phi)$  as a pairing

$$\lambda: P \times P \to R, \qquad (p,q) \mapsto \phi(p)(q).$$

The conditions that  $\phi$  is *R*-linear and that  $\phi(p)$  is *R*-linear for all  $p \in P$  translates to

$$\lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2) = r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2);$$
  
$$\lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) = \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}.$$

The condition  $\phi = \epsilon \cdot \phi^* \circ e(P)$  translates to  $\lambda(q, p) = \epsilon \cdot \overline{\lambda(p, q)}$ .

If we consider the real numbers  $\mathbb{R}$  as a ring with involution by the trivial involution, then a non-singular 1-symmetric form  $\phi$  on a finite-dimensional  $\mathbb{R}$ -vector space Vsuch that  $\phi(x)(x) \ge 0$  holds for all  $x \in \mathbb{R}^n$  is the same as a scalar product on V. If we consider the complex numbers  $\mathbb{C}$  as a ring with involution by taking complex conjugation, then the corresponding statement holds for a finite-dimensional complex vector space.

**Definition 9.9 (The standard hyperbolic**  $\epsilon$ **-symmetric form).** Let *P* be a finitely generated projective *R*-module. The *standard hyperbolic*  $\epsilon$ *-symmetric form*  $H^{\epsilon}(P)$  is given by the *R*-module  $P \oplus P^*$  and the *R*-isomorphism

$$\phi \colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\operatorname{id} \oplus e(P)} P^* \oplus (P^*)^* \xrightarrow{\gamma} (P \oplus P^*)^*$$

where  $\gamma$  is the obvious *R*-isomorphism.

If we write the standard hyperbolic  $\epsilon$ -symmetric form  $H^{\epsilon}(P)$  as a pairing, see Remark 9.8, we obtain

$$(P \oplus P^*) \times (P \oplus P^*) \to R,$$
  $((p, \alpha), (p', \alpha')) \mapsto \alpha'(p) + \epsilon \cdot \alpha(p').$ 

#### 9.2.2 The Signature

Consider a non-singular symmetric bilinear pairing  $s: V \times V \to \mathbb{R}$  for a finitedimensional real vector space V, or, equivalently, a non-singular symmetric form of finitely generated free  $\mathbb{R}$ -modules. Choose a basis for V and let A be the square matrix describing s with respect to this basis. Since s is symmetric and non-singular, A is symmetric and invertible. Hence A can be diagonalized by an orthogonal matrix U to a diagonal matrix whose entries on the diagonal are non-zero real numbers. Let  $n_+$  be the number of positive entries and  $n_-$  be the number of negative entries on the diagonal. These two numbers are independent of the choice of the basis and the orthogonal matrix U. Namely  $n_+$  is the maximum of the dimensions of subvector spaces  $W \subset V$  on which s is positive-definite, i.e.,  $s(w, w) \ge 0$  for  $w \in W$ , and analogous for  $n_-$ . Obviously  $n_+ + n_- = \dim_{\mathbb{R}}(V)$ .

**Definition 9.10 (Signature).** Define the *signature* of the non-singular symmetric bilinear pairing  $s: V \times V \to \mathbb{R}$  for a finite-dimensional real vector space V to be the integer

$$\operatorname{sign}(s) := n_+ - n_-$$

Define the signature of a non-singular symmetric form over  $\mathbb{Z}$  to be the signature of the associated non-singular symmetric form over  $\mathbb{R}$ .

**Lemma 9.11.** Let  $s: V \times V \to \mathbb{R}$  be a non-singular symmetric bilinear pairing for a finite-dimensional real vector space V. Then  $\operatorname{sign}(s) = 0$  if and only if there exists a subvector space  $L \subset V$  such that  $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{R}}(L)$  and s(a, b) = 0 for  $a, b \in L$ .

*Proof.* Suppose that sign(s) = 0. Then one can find an orthogonal (with respect to *s*) basis  $\{b_1, b_2, \ldots, b_{n_+}, c_1, c_2, \ldots, c_{n_-}\}$  such that  $s(b_i, b_i) = 1$  and  $s(c_j, c_j) = -1$  holds. Since  $0 = sign(s) = n_+ - n_-$ , we can define *L* to be the subvector space generated by  $\{b_i + c_i \mid i = 1, 2, \ldots, n_+\}$ . One easily checks that *L* has the desired properties.

Suppose such an  $L \subset V$  exists. Choose subvector spaces  $V_+$  and  $V_-$  of V such that s is positive-definite on  $V_+$  and negative-definite on  $V_-$  and that  $V_+$  and  $V_-$  are maximal with respect to this property. Then  $V_+ \cap V_- = \{0\}$  and  $V = V_+ \oplus V_-$ . Obviously  $V_+ \cap L = V_- \cap L = \{0\}$ . From

$$\dim_{\mathbb{R}}(V_{\pm}) + \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(V_{\pm} \cap L) \le \dim_{\mathbb{R}}(V),$$

we conclude  $\dim_{\mathbb{R}}(V_{\pm}) \leq \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(L)$ . Since  $2 \cdot \dim_{\mathbb{R}}(L) = \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V_{\pm}) + \dim_{\mathbb{R}}(V_{-})$  holds, we get  $\dim_{\mathbb{R}}(V_{\pm}) = \dim_{\mathbb{R}}(L)$ . This implies

$$\operatorname{sign}(s) = \dim_{\mathbb{R}}(V_{+}) - \dim_{\mathbb{R}}(V_{-}) = \dim_{\mathbb{R}}(L) - \dim_{\mathbb{R}}(L) = 0.$$

#### 9.2 Symmetric and Quadratic Forms

If *M* is an orientable connected closed manifold of dimension *d*, then  $H_d(M)$  is infinite cyclic. An *orientation* on *M* is equivalent to a choice of generator  $[M] \in H_d(M)$  called a *fundamental class*. This definition extends to a (not necessarily connected) orientable closed manifold *M* of dimension *d* by defining  $[M] \in H_d(M)$  to be the image of  $\{[C] \mid C \in \pi_0(M)\}$  under the canonical isomorphism  $\bigoplus_{C \in \pi_0(M)} H_{\dim(M)}(C) \xrightarrow{\cong} H_{\dim(M)}(M)$ .

**Example 9.12 (Intersection pairing).** Let *M* be a closed oriented manifold of even dimension 2*n*. Then we obtain a  $(-1)^n$ -symmetric form on the finitely generated free  $\mathbb{R}$ -module  $H^n(M; \mathbb{R})$ 

$$i: H^n(M; \mathbb{R}) \times H^n(M; \mathbb{R}) \to \mathbb{R}$$

by sending ([x], [y]) for  $x, y \in H^n(M; \mathbb{R})$  to  $\langle x \cup y, [M]_{\mathbb{R}} \rangle$  where  $\langle u, v \rangle$  denotes the *Kronecker pairing* and  $[M]_{\mathbb{R}}$  is the image of the fundamental class [M] under the change of rings homomorphism  $H_n(M; \mathbb{Z}) \to H_n(M; \mathbb{R})$ . It is non-singular by Poincaré duality.

Next we define a fundamental invariant of a closed oriented manifold, namely, its signature. This is the first kind of surgery obstruction we will encounter.

**Definition 9.13 (Signature of a closed oriented manifold).** Let M be a closed oriented manifold of dimension n. If n is divisible by four, then the signature sign(M) of M is defined to be the signature of its intersection pairing. If n is not divisible by four, define sign(M) = 0.

One easily checks  $sign(M) = \sum_{C \in \pi_0(M)} sign(C)$ .

**Exercise 9.14.** Let *M* be an oriented closed 4*k*-dimensional manifold. Let  $\chi(M)$  be its Euler characteristic. Show sign(*M*)  $\equiv \chi(M) \mod 2$ .

The signature can also be defined for oriented compact manifolds with possibly non-empty boundary, see for instance [667, Definition 5.84 on page 135], and has the following properties.

## Theorem 9.15 (Properties of the signature of oriented compact manifolds).

(i) The signature is an oriented bordism invariant, i.e., if M is a (4k + 1)-dimensional oriented compact manifold with boundary  $\partial M$ , then

$$sign(\partial M) = 0;$$

(ii) Let M and N be oriented compact manifolds and  $f: \partial M \to \partial N$  be an orientation reversing diffeomorphism. Then  $M \cup_f N$  inherits an orientation from M and N and

$$\operatorname{sign}(M \cup_f N) = \operatorname{sign}(M) + \operatorname{sign}(N);$$

(iii) Let M and N be oriented compact manifolds. Then we get

$$\operatorname{sign}(M \times N, \partial(M \times N)) = \operatorname{sign}(M, \partial M) \cdot \operatorname{sign}(N, \partial N);$$

(iv) Let  $p: \overline{M} \to M$  be a finite covering with d sheets of oriented closed manifolds. Then

$$\operatorname{sign}(M) = d \cdot \operatorname{sign}(M);$$

(v) *If the oriented closed manifolds M and N are oriented homotopy equivalent, then* 

$$\operatorname{sign}(M) = \operatorname{sign}(N);$$

(vi) If M is an oriented closed manifold and  $M^-$  is obtained from M by reversing the orientation, then

$$\operatorname{sign}(M^{-}) = -\operatorname{sign}(M).$$

*Proof.* (i) Let  $i: \partial M \to M$  be the inclusion. Then the following diagram commutes

$$\begin{array}{c|c} H^{2k}(M;\mathbb{R}) \xrightarrow{H^{2k}(i)} & H^{2k}(\partial M;\mathbb{R}) \xrightarrow{\delta^{2k}} & H^{2k+1}(M,\partial M;\mathbb{R}) \\ \hline & -\cap [M,\partial M]_{\mathbb{R}} \middle| \cong & -\cap \partial_{4k+1}([M,\partial M]_{\mathbb{R}}) \middle| \cong & -\cap [M,\partial M]_{\mathbb{R}} \middle| \cong \\ & H_{2k+1}(M,\partial M;\mathbb{R}) \xrightarrow{\partial_{2k+1}} & H_{2k}(\partial M;\mathbb{R}) \xrightarrow{H_{2k}(i)} & H_{2k}(M;\mathbb{R}). \end{array}$$

This implies dim<sub>R</sub>(ker( $H_{2k}(i)$ )) = dim<sub>R</sub>(im( $H^{2k}(i)$ )). Since  $\mathbb{R}$  is a field, we get from the Kronecker pairing an isomorphism  $H^{2k}(M; \mathbb{R}) \cong (H_{2k}(M; \mathbb{R}))^*$  and analogously for  $\partial M$ . Under these identifications  $H^{2k}(i)$  becomes  $(H_{2k}(i))^*$ . Hence dim<sub>R</sub>(im( $H_{2k}(i)$ )) = dim<sub>R</sub>(im( $H^{2k}(i)$ )). From

$$\dim_{\mathbb{R}}(H_{2k}(\partial M;\mathbb{R})) = \dim_{\mathbb{R}}(\ker(H_{2k}(i))) + \dim_{\mathbb{R}}(\operatorname{im}(H_{2k}(i)))$$

we conclude

$$\dim_{\mathbb{R}}(H^{2k}(\partial M;\mathbb{R})) = 2 \cdot \dim_{\mathbb{R}}(\operatorname{im}(H^{2k}(i))).$$

We have for  $x, y \in H^{2k}(M; \mathbb{R})$ 

$$\langle H^{2k}(i)(x) \cup H^{2k}(i)(y), \partial_{4k+1}([M, \partial M]_{\mathbb{R}}) \rangle$$
  
=  $\langle H^{4k}(i)(x \cup y), \partial_{4k+1}([M, \partial M]_{\mathbb{R}}) \rangle$   
=  $\langle x \cup y, H_{4k}(i) \circ \partial_{4k+1}([M, \partial M]_{\mathbb{R}}) \rangle$   
=  $\langle x \cup y, 0 \rangle$   
= 0.

If we apply Lemma 9.11 to the non-singular symmetric bilinear pairing

$$H^{2k}(\partial M;\mathbb{R}) \otimes_{\mathbb{R}} H^{2k}(\partial M;\mathbb{R}) \xrightarrow{\cup} H^{4k}(\partial M;\mathbb{R}) \xrightarrow{\langle -,\partial_{4k+1}([M,\partial M]_{\mathbb{R}})\rangle} \mathbb{R}$$

with *L* the image of  $H^{2k}(i): H^{2k}(M;\mathbb{R}) \to H^{2k}(\partial M;\mathbb{R})$ , we see that the signature of this pairing is zero.

(ii) This is due to Novikov. For a proof see for instance [53, Proposition 7.1 on page 588].

- (iii) See for instance [667, Lemma 5.85 (ii) on page 136].
- (iv) For a smooth manifold M this follows from Atiyah's  $L^2$ -index theorem [46, (1.1)]. Topological closed manifolds are treated in [892, Theorem 8].
- (v) The two intersection pairings are isomorphic and hence have the same signatures.

(vi) This follows from  $[M^{-}] = -[M]$ .

**Exercise 9.16.** Compute for  $n \ge 1$  the signature of:

- (i) the complex projective space  $\mathbb{CP}^n$ ;
- (ii) the total space *STM* of the sphere tangent bundle of an oriented closed *n*-dimensional manifold *M*;
- (iii) an oriented closed *n*-dimensional manifold *M* admitting an orientation reversing self-diffeomorphism.

#### 9.2.3 Quadratic Forms

Next we introduce quadratic forms, which are refinements of symmetric forms.

**Notation 9.17.** For a finitely generated projective *R*-module *P* define an involution of *R*-modules

$$T = T(P): \operatorname{hom}_{R}(P, P^{*}) \to \operatorname{hom}_{R}(P, P^{*}), \quad u \mapsto u^{*} \circ e(P).$$

Notation 9.18. Let P be a finitely generated projective R-module. Define abelian groups

$$Q^{\epsilon}(P) := \ker \left( (1 - \epsilon \cdot T) \colon \hom_{R}(P, P^{*}) \to \hom_{R}(P, P^{*}) \right);$$
  
$$Q_{\epsilon}(P) := \operatorname{coker} \left( (1 - \epsilon \cdot T) \colon \hom_{R}(P, P^{*}) \to \hom_{R}(P, P^{*}) \right).$$

An *R*-homomorphism  $f: P \rightarrow Q$  induces a homomorphism of abelian groups

$$\begin{aligned} & Q^{\epsilon}(f) \colon Q^{\epsilon}(Q) \to Q^{\epsilon}(P), \quad u \mapsto f^{*} \circ u \circ f; \\ & Q_{\epsilon}(f) \colon Q_{\epsilon}(Q) \to Q_{\epsilon}(P), \quad [u] \mapsto [f^{*} \circ u \circ f]. \end{aligned}$$

Let

$$(1 + \epsilon \cdot T) \colon Q_{\epsilon}(P) \to Q^{\epsilon}(P)$$

be the homomorphism that sends the class [u] represented by  $u: P \to P^*$  to the element  $u + \epsilon \cdot T(u)$ .

**Definition 9.19 (Non-singular**  $\epsilon$ -quadratic form). Let  $\epsilon \in \{\pm 1\}$ . An  $\epsilon$ -quadratic form  $(P, \psi)$  is a finitely generated projective *R*-module *P* together with an element  $\psi \in Q_{\epsilon}(P)$ . It is called *non-singular* if the associated  $\epsilon$ -symmetric form  $(P, (1 + \epsilon \cdot T)(\psi))$  is non-singular, i.e.  $(1 + \epsilon \cdot T)(\psi)$ :  $P \rightarrow P^*$  is bijective.

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A morphism  $f: (P, \psi) \to (P', \psi')$  of two  $\epsilon$ -quadratic forms is an *R*-homomorphism  $f: P \xrightarrow{\cong} P'$  such that the induced map  $Q_{\epsilon}(f): Q_{\epsilon}(P') \to Q_{\epsilon}(P)$  sends  $\psi'$  to  $\psi$ .

Given a non-singular  $\epsilon$ -symmetric form  $(P, \phi)$ , a *quadratic refinement* is a non-singular  $\epsilon$ -quadratic form  $(P, \psi)$  with  $\phi = (1 + \epsilon \cdot T)(\psi)$ .

There is an obvious notion of a direct sum of two  $\epsilon$ -quadratic forms. The direct sum of two non-singular  $\epsilon$ -quadratic forms is a non-singular  $\epsilon$ -quadratic form.

Consider the pairing

(9.20) 
$$\rho \colon R \times Q_{(-1)^k}(R) \to Q_{(-1)^k}(R), \quad (r, [s]) \mapsto [rs\overline{r}].$$

It is well defined, since for  $r, s, t \in R$  we get if we put  $t' = rt\overline{r}$ 

$$r(s + (t - (-1)^k \cdot \overline{t}))\overline{r} = rs\overline{r} + (t' - (-1)^k \cdot \overline{t'}).$$

It is additive in the second variable, i.e.,  $\rho(r, [s_1] - [s_2]) = \rho(r, [s_1]) - \rho(r, [s_2])$ , but it is not additive in the first variable and in particular  $\rho$  does *not* give the structure of a left *R*-module on  $Q_{(-1)^k}(R)$ . Nevertheless, sometimes in the literature  $\rho(r, [s])$ is denoted by  $r[s]\overline{r}$ .

**Remark 9.21 (Writing a quadratic form as a triple**  $(P, \lambda, \mu)$ ). An  $\epsilon$ -quadratic form  $(P, \psi)$  is equivalent to a triple  $(P, \lambda, \mu)$  consisting of a pairing

$$\lambda\colon P\times P\to R$$

satisfying

$$\begin{split} \lambda(p,r_1 \cdot q_1 + r_2 \cdot q_2) &= r_1 \cdot \lambda(p,q_1) + r_2 \cdot \lambda(p,q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2,q) &= \lambda(p_1,q) \cdot \overline{r_1} + \lambda(p_2,q) \cdot \overline{r_2}; \\ \lambda(q,p) &= \epsilon \cdot \overline{\lambda(p,q)}, \end{split}$$

and a map

$$\mu \colon P \to Q_{\epsilon}(R) = R/\{r - \epsilon \cdot \overline{r} \mid r \in R\}$$

satisfying

$$\mu(rp) = \rho(r, \mu(p));$$
  

$$\mu(p+q) - \mu(p) - \mu(q) = \operatorname{pr}(\lambda(p,q));$$
  

$$\lambda(p,p) = (1 + \epsilon \cdot T)(\mu(p)),$$

where the pairing  $\rho$  was introduced in (9.20), pr:  $R \to Q_{\epsilon}(R)$  is the projection, and  $(1 + \epsilon \cdot T): Q_{\epsilon}(R) \to R$  the map sending the class of r to  $r + \epsilon \cdot \overline{r}$ . Namely, put

$$\lambda(p,q) = ((1 + \epsilon \cdot T)(\psi)(p))(q);$$
  
$$\mu(p) = \operatorname{pr}(\psi(p)(p)).$$

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These two descriptions of an  $\epsilon$ -quadratic form are equivalent, see [986, Theorem 1].

**Definition 9.22 (The standard hyperbolic**  $\epsilon$ **-quadratic form).** Let *P* be a finitely generated projective *R*-module. The *standard hyperbolic*  $\epsilon$ *-quadratic form*  $H_{\epsilon}(P)$  is given by the *R*-module  $P \oplus P^*$  and the class in  $Q_{\epsilon}(P \oplus P^*)$  of the *R*-homomorphism

$$\phi \colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\operatorname{id} \oplus e(P)} P^* \oplus (P^*)^* \xrightarrow{\gamma} (P \oplus P^*)^*$$

where  $\gamma$  is the obvious *R*-isomorphism.

If we write the standard hyperbolic  $\epsilon$ -quadratic form  $H_{\epsilon}(P)$  as a pairing, see Remark 9.21, we obtain

$$\begin{split} \lambda \colon (P \oplus P^*) \times (P \oplus P^*) \to R, \quad ((p, \alpha), (p', \alpha')) \mapsto \alpha'(p) + \epsilon \cdot \alpha(p'); \\ \mu \colon P \oplus P^* \to Q_{\epsilon}(R), \qquad (p, \alpha) \mapsto \operatorname{pr}(\alpha(p)). \end{split}$$

In particular, the  $\epsilon$ -symmetric form associated to the standard  $\epsilon$ -quadratic form  $H_{\epsilon}(P)$  is just the standard  $\epsilon$ -symmetric form  $H^{\epsilon}(P)$ .

**Exercise 9.23.** Let  $\lambda: P \times P \to \mathbb{Z}$  be a non-singular symmetric  $\mathbb{Z}$ -bilinear pairing on the finitely generated free  $\mathbb{Z}$ -module *P*. Show that it has, when considered as a non-singular symmetric form, a quadratic refinement if and only if  $\lambda(x, x)$  is even for all  $x \in P$ .

**Remark 9.24.** Suppose that  $1/2 \in R$ . Then the homomorphism

$$(1 + \epsilon \cdot T): Q_{\epsilon}(P) \to Q^{\epsilon}(P), \quad [u] \mapsto [u + \epsilon \cdot T(u)]$$

is bijective. The inverse sends v to [v/2]. Hence any  $\epsilon$ -symmetric form carries a unique  $\epsilon$ -quadratic structure. Therefore there is no difference between the symmetric and the quadratic setting if 2 is invertible in *R*.

## **9.3 Even-Dimensional** *L*-groups

Next we define even-dimensional L-groups. Below R is an associative ring with involution.

**Definition 9.25** (*L*-groups in even dimensions). For an even integer n = 2k define the abelian group  $L_n(R)$ , called the *n*-th quadratic *L*-group, of *R* to be the abelian group of equivalence classes  $[P, \psi]$  of non-singular  $(-1)^k$ -quadratic forms  $(P, \psi)$  whose underlying *R*-module *P* is a finitely generated free *R*-module, with respect to the following equivalence relation: We call  $(P, \psi)$  and  $(P', \psi')$  equivalent or

stably isomorphic if and only if there exist integers  $u, u' \ge 0$  and an isomorphism of non-singular  $(-1)^k$ -quadratic forms

$$(P,\psi) \oplus H_{(-1)^k}(R)^u \cong (P',\psi') \oplus H_{(-1)^k}(R)^{u'}.$$

Addition is given by the direct sum of two  $(-1)^k$ -quadratic forms. The zero element is represented by  $[H_{(-1)^k}(R)^u]$  for any integer  $u \ge 0$ . The inverse of  $[P, \psi]$  is given by  $[P, -\psi]$ .

A morphism  $u: R \to S$  of rings with involution induces homomorphisms  $u_*: L_n(R) \to L_n(S)$  and  $u_*: L^n(R) \to L^n(S)$  for even  $n \in \mathbb{Z}$  by induction satisfying  $(u \circ v)_* = u_* \circ v_*$  and  $(\operatorname{id}_R)_* = \operatorname{id}_{L_k(R)}$  for k = 0, 2.

Next we will present a criterion for an  $\epsilon$ -quadratic form  $(P, \psi)$  to represent zero in  $L_{1-\epsilon}(R)$ . Let  $(P, \psi)$  be an  $\epsilon$ -quadratic form. A *sublagrangian*  $L \subset P$  is an *R*-submodule such that the inclusion  $i: L \to P$  is split injective, the image of  $\psi$  under the map  $Q_{\epsilon}(i): Q_{\epsilon}(P) \to Q_{\epsilon}(L)$  is zero, and *L* is contained in its *annihilator*  $L^{\perp}$ , that is by definition the kernel of

$$P \xrightarrow{(1+\epsilon \cdot T)(\psi)} P^* \xrightarrow{i^*} L^*.$$

A sublagrangian  $L \subset P$  is called *lagrangian* if  $L = L^{\perp}$ . Equivalently, a lagrangian  $L \subset P$  is an *R*-submodule *L* with inclusion  $i: L \to P$  such that the sequence

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1+\epsilon \cdot T)(\psi)} L^* \to 0.$$

is exact.

**Lemma 9.26.** Let  $(P, \psi)$  be an  $\epsilon$ -quadratic form. Let  $L \subset P$  be a sublagrangian. Then L is a direct summand in  $L^{\perp}$  and  $\psi$  induces the structure of a non-singular  $\epsilon$ -quadratic form  $(L^{\perp}/L, \psi^{\perp}/\psi)$ . Moreover, the inclusion  $i: L \to P$  extends to an isomorphism of  $\epsilon$ -quadratic forms

$$H_{\epsilon}(L) \oplus (L^{\perp}/L, \psi^{\perp}/\psi) \xrightarrow{\cong} (P, \psi).$$

In particular, a non-singular  $\epsilon$ -quadratic form  $(P, \psi)$  is isomorphic to  $H_{\epsilon}(Q)$  if and only if it contains a lagrangian  $L \subset P$  which is isomorphic as an *R*-module to *Q*.

*Proof.* See for instance [667, Lemma 8.95 on page 261].

**Exercise 9.27.** Show for a non-singular  $\epsilon$ -quadratic form  $(P, \psi)$  that  $(P, \psi) \oplus (P, -\psi)$  is isomorphic to  $H_{\epsilon}(P)$  and hence an inverse of  $[P, \psi]$  in  $L_{1-\epsilon}(R)$  is given by  $[P, -\psi]$ .

**Exercise 9.28.** Show that the signature defines an isomorphism  $L_0(\mathbb{R}) \xrightarrow{=} \mathbb{Z}$ .

Finally we state the computation of the even-dimensional *L*-groups of the ring of integers  $\mathbb{Z}$ . Consider an element  $(P, \psi)$  in  $L_0(\mathbb{Z})$ . By tensoring over  $\mathbb{Z}$  with  $\mathbb{R}$  and only taking the symmetric structure into account, we obtain a non-singular

#### 9.3 Even-Dimensional L-groups

symmetric  $\mathbb{R}$ -bilinear pairing  $\lambda: \mathbb{R} \otimes_{\mathbb{Z}} P \times \mathbb{R} \otimes_{\mathbb{Z}} P \to \mathbb{R}$ . It turns out that its signature is always divisible by eight. A proof the following classical result can be found for instance in [667, Subsection 8.5.2], see also [728].

**Theorem 9.29** (*L*-groups of the ring of integers in dimension 4n). The signature defines for  $n \in \mathbb{Z}$  an isomorphism

$$\frac{1}{8} \cdot \operatorname{sign} \colon L_{4n}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \qquad [P, \psi] \mapsto \frac{1}{8} \cdot \operatorname{sign}(\mathbb{R} \otimes_{\mathbb{Z}} P, \lambda).$$

Consider a non-singular quadratic form  $(P, \psi)$  over the field  $\mathbb{F}_2$  of two elements. Write  $(P, \psi)$  as a triple  $(P, \lambda, \mu)$  as explained in Remark 9.21. Choose any symplectic basis  $\{b_1, b_2, \ldots, b_{2m}\}$  for P, where symplectic means that  $\lambda(b_i, b_j)$  is 1 if i - j = mand 0 otherwise. Such a symplectic basis always exists. Define the *Arf invariant* of  $(P, \psi)$  by

(9.30) 
$$\operatorname{Arf}(P,\psi) := \sum_{i=1}^{m} \mu(b_i) \cdot \mu(b_{i+m}) \quad \in \mathbb{Z}/2.$$

It turns out that the Arf invariant of  $(P, \psi)$  is 1 if and only if  $\mu$  sends a (strict) majority of the elements of *P* to 1, see [169, Corollary III.1.9 on page 55]. (Because of this property sometimes the Arf invariant is called the democratic invariant.) This description shows that (9.30) is independent of the choice of symplectic basis.

**Exercise 9.31.** Let *V* be a two-dimensional  $\mathbb{F}_2$ -vector space. Classify all non-singular quadratic forms on *V* up to isomorphism and compute their Arf invariants.

The Arf invariant defines an isomorphism

Arf: 
$$L_{2n}(\mathbb{F}_2) \xrightarrow{=} \mathbb{Z}/2$$
,

essentially, since two non-singular quadratic forms over  $\mathbb{F}_2$  on the same finite dimensional  $\mathbb{F}_2$ -vector space are isomorphic if and only if they have the same Arf invariant, see [169, Theorem III.1.12 on page 55]. The change of rings homomorphism  $\mathbb{Z} \to \mathbb{F}_2$  induces an isomorphism,

$$L_{4n+2}(\mathbb{Z}) \xrightarrow{\cong} L_{4n+2}(\mathbb{F}_2).$$

This implies, see for instance [667, Subsection 8.5.3],

**Theorem 9.32** (*L*-groups of the ring of integers in dimension 4n + 2). The Arf invariant defines for  $n \in \mathbb{Z}$  an isomorphism

Arf: 
$$L_{4n+2}(\mathbb{Z}) \xrightarrow{=} \mathbb{Z}/2$$
,  $[P, \psi] \mapsto \operatorname{Arf}(\mathbb{F}_2 \otimes_{\mathbb{Z}} (P, \psi))$ .

For more information about forms over the integers and the Arf invariant we refer for instance to [169, 728]. Implicitly the computation of  $L_n(\mathbb{Z})$  is already in [576].

## 9.4 Intersection and Self-Intersection Pairings

The notions of an  $\epsilon$ -symmetric form as presented in Remark 9.8 and of an  $\epsilon$ -quadratic form as presented in Remark 9.21 are best motivated by considering intersections and self-intersection pairings. When trying to solve a surgery problem in even dimensions, one faces in the final step, namely, when dealing with the middle dimension, the problem to decide whether we can change an immersion  $f: S^k \to M$  within its regular homotopy class to an embedding where M is a compact manifold of dimension n = 2k. This problem leads in a natural way to self-intersection pairings and  $\epsilon$ -quadratic forms, as explained next.

## 9.4.1 Intersections of Immersions

Let  $k \ge 2$  be a natural number, and let M be a connected compact smooth manifold of dimension n = 2k. We fix base points  $s \in S^k$  and  $b \in M$ . We will consider *pointed immersions* (f, w), i.e., an immersion  $f: S^k \to M$  together with a path wfrom b to f(s) in M. A *regular homotopy*  $h: M \times [0, 1] \to N$  from an immersion  $q_0: M \to N$  to an immersion  $q_1: M \to N$  is a (continuous, but not necessarily smooth) homotopy  $h: M \times [0, 1] \to N$  such that  $h_0 = q_0, h_1 = q_1, h_t: M \to N$  is a (smooth) immersion for each  $t \in [0, 1]$ , and the derivatives  $Th_t: TM \to TN$  of  $h_t$ fit together to define a (continuous) homotopy of bundle monomorphisms

$$TM \times [0,1] \to TN, \quad (v,t) \mapsto Th_t(v)$$

between  $Tq_0$  and  $Tq_1$ . A pointed regular homotopy from  $(f_0, w_0)$  to  $(f_1, w_1)$  is a regular homotopy  $h: S^k \times [0, 1] \to M$  from  $h_0 = f_0$  to  $h_1 = f_1$  such that  $w_0 * h(s, -)$ and  $w_1$  are homotopic paths relative end points. Here h(s, -) is the path from  $f_0(s)$ to  $f_1(s)$  given by restricting h to  $\{s\} \times [0, 1]$ . Denote by  $I_k(M)$  the set of pointed regular homotopy classes [f, w] of pointed immersions (f, w) from  $S^k$  to M. We need the paths to define the structure of an abelian group on  $I_k(M)$ . The sum of  $[f_0, w_0]$  and  $[f_1, w_1]$  is given by the connected sum along the path  $w_0^- * w_1$  from  $f_0(s)$  to  $f_1(s)$ . The zero element is given by the composite of the standard embedding  $S^k \to \mathbb{R}^{k+1} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{k-1} = \mathbb{R}^n$  with some embedding  $\mathbb{R}^n \subset M$  and any path wfrom b to the image of s. The inverse of the class of (f, w) is the class of  $(f \circ a, w)$ for any base point preserving diffeomorphism  $a: S^k \to S^k$  of degree -1.

The fundamental group  $\pi = \pi_1(M, b)$  operates on  $I_k(M)$  by composing the path *w* with a loop at *b*. Thus  $I_k(M)$  inherits the structure of a  $\mathbb{Z}\pi$ -module.

Next we want to define the *intersection pairing* 

(9.33) 
$$\lambda: I_k(M) \times I_k(M) \to \mathbb{Z}\pi.$$

For this purpose we will have to fix an orientation of  $T_bM$  at *b*. Consider  $\alpha_0 = [f_0, w_0]$  and  $\alpha_1 = [f_1, w_1]$  in  $I_k(M)$ . Choose representatives  $(f_0, w_0)$  and  $(f_1, w_1)$ . We can arrange without changing the pointed regular homotopy classes that  $D = \operatorname{im}(f_0) \cap \operatorname{im}(f_1)$  is finite, for any  $y \in D$  both the preimage  $f_0^{-1}(y)$  and the preimage  $f_1^{-1}(y)$  consists of precisely one point, and, for any two points  $x_0$  and  $x_1$  in  $S^k$  with  $f_0(x_0) = f_1(x_1)$ , we have  $T_{x_0}f_0(T_{x_0}S^k) + T_{x_1}f_1(T_{x_1}S^k) = T_{f_0(x_0)}M$ . Consider  $d \in D$ . Let  $x_0$  and  $x_1$  in  $S^k$  be the points uniquely determined by  $f_0(x_0) = f_1(x_1) = d$ . Let  $u_i$  be a path in  $S^k$  from *s* to concatenation  $w_1 * f_1(u_1) * f_0(u_0)^- * w_0^-$ . Recall that we have fixed an orientation of  $T_bM$ . The fiber transport along the path  $w_0 * f(u_0)$  yields an isomorphism  $T_bM \xrightarrow{\cong} T_dM$  that is unique up to isotopy. Hence  $T_dM$  inherits an orientation on  $T_{x_0}S^k$  and  $T_{x_1}S^k$ . We have the isomorphism of oriented vector spaces

$$T_{x_0}f_0 \oplus T_{x_1}f_1 \colon T_{x_0}S^k \oplus T_{x_1}S^k \xrightarrow{=} T_dM.$$

Define  $\epsilon(d) = 1$  if this isomorphism respects the orientations and  $\epsilon(d) = -1$  otherwise. The elements  $g(d) \in \pi$  and  $\epsilon(d) \in \{\pm 1\}$  are independent of the choices of  $u_0$  and  $u_1$ , since  $S^k$  is simply connected for  $k \ge 2$ . Define

$$\lambda(\alpha_0, \alpha_1) := \sum_{d \in D} \epsilon(d) \cdot g(d).$$

Lift  $b \in M$  to a base point  $\tilde{b} \in \tilde{M}$ . Let  $\tilde{f}_i : S^k \to \tilde{M}$  be the unique lift of  $f_i$ determined by  $w_i$  and  $\tilde{b}$  for i = 0, 1. Let  $\lambda_{\mathbb{Z}}(\tilde{f}_0, \tilde{f}_1)$  be the  $\mathbb{Z}$ -valued intersection number of  $\tilde{f}_0$  and  $\tilde{f}_1$ . This is the same as the algebraic intersection number of the classes in the k-th homology with compact support defined by  $\tilde{f}_0$  and  $\tilde{f}_1$ , which obviously depends only on the homotopy classes of  $\tilde{f}_0$  and  $\tilde{f}_1$ ; the proof in [163, Theorem 11.9 in Chapter VI on page 372] can be extended to our setting. Then

(9.34) 
$$\lambda(\alpha_0, \alpha_1) = \sum_{g \in \pi} \lambda_{\mathbb{Z}}(\widetilde{f}_0, l_{g^{-1}} \circ \widetilde{f}_1) \cdot g,$$

where  $l_{g^{-1}}$  denotes left multiplication by  $g^{-1}$ . This shows that  $\lambda(\alpha_0, \alpha_1)$  depends only on the pointed regular homotopy classes of  $(f_0, w_0)$  and  $(f_1, w_1)$ .

Below we use the  $w_1(M)$ -twisted involution on  $\mathbb{Z}\pi$  that sends  $\sum_{g \in \pi} a_g \cdot g$  to  $\sum_{g \in \pi} w_1(M)(g) \cdot a_g \cdot g^{-1}$ , where  $w_1(M) \colon \pi \to \{\pm 1\}$  is the first Stiefel-Whitney class of M. The elementary proof of the next lemma is left to the reader.

**Lemma 9.35.** For  $\alpha, \beta, \beta_1, \beta_2 \in I_k(M)$  and  $u_1, u_2 \in \mathbb{Z}\pi$  we have

$$\lambda(\alpha,\beta) = (-1)^k \cdot \overline{\lambda(\beta,\alpha)};$$
  
$$\lambda(\alpha, u_1 \cdot \beta_1 + u_2 \cdot \beta_2) = u_1 \cdot \lambda(\alpha,\beta_1) + u_2 \cdot \lambda(\alpha,\beta_2).$$

**Remark 9.36 (Intersection pairing and**  $(-1)^k$ -symmetric forms). Lemma 9.35 says that the pair  $(I_k(M), \lambda)$  satisfies all the requirements appearing in Remark 9.8 except that  $I_k(M)$  may not be finitely generated projective over  $\mathbb{Z}\pi$ .

**Remark 9.37 (The intersection pairing as necessary obstruction for finding an embedding).** Suppose that the normal bundle of the immersion  $f: S^k \to M$  has a nowhere vanishing section. (In the typical situation that appears in surgery theory it actually will be trivial.) Suppose that f is regular homotopic to an embedding g. Then the normal bundle of g has a nowhere vanishing section  $\sigma$ . Let g' be the embedding obtained by moving g a little bit in the direction of this normal vector field  $\sigma$ . Choose a path  $w_f$  from f(s) to b. Then for appropriate paths  $w_g$  and  $w_{g'}$  we get pointed embeddings  $(g, w_g)$  and  $(g', w_{g'})$  such that the pointed regular homotopy classes of  $(f, w), (g, w_g)$  and  $(g', w_{g'})$  agree. Since g and g' have disjoint images, we conclude

$$\lambda([f,w],[f,w]) = 0.$$

Hence the vanishing of  $\lambda([f, w], [f, w])$  is a necessary condition for finding an embedding in the regular homotopy class of f, provided that the normal bundle of f has a nowhere vanishing section. It is not a sufficient condition. To get a sufficient condition we have to consider self-intersections, which we will do next.

#### 9.4.2 Self-Intersections of Immersions

Let  $\alpha \in I_k(M)$  be an element. Let (f, w) be a pointed immersion representing  $\alpha$ . Recall that we have fixed base points  $s \in S^k$ ,  $b \in M$ , and an orientation of  $T_bM$ . Since we can find arbitrarily close to f an immersion which is in general position, we can assume without loss of generality that f itself is in general position. This means that there is a finite subset D of im(f) such that  $f^{-1}(y)$  consists of precisely two points for  $y \in D$  and of precisely one point for  $y \in im(f) - D$  and that for two points  $x_0$  and  $x_1$  in  $S^k$  with  $x_0 \neq x_1$  and  $f(x_0) = f(x_1)$  we have  $T_{x_0}f(T_{x_0}S^k) + T_{x_1}f(T_{x_1}S^k) = T_{f_0(x_0)}M$ . Now fix for any  $d \in D$  an ordering  $x_0(d), x_1(d)$  of  $f^{-1}(d)$ . Analogously to the construction above one defines  $\epsilon(x_0(d), x_1(d)) \in \{\pm 1\}$  and  $g(x_0(d), x_1(d)) \in \mathbb{Z}\pi$ . It does not only depend on f, but also on the choice of the ordering of  $f^{-1}(d)$  for  $d \in D$ . One easily checks that the change of ordering of  $f^{-1}(d)$  has the following effect for  $w = w_1(M)$ :  $\pi \to \{\pm 1\}$ 

$$g(x_1(d), x_0(d)) = g(x_0(d), x_1(d))^{-1};$$
  

$$w(g(x_1(d), x_0(d))) = w(g(x_0(d), x_1(d)));$$
  

$$\epsilon(x_1(d), x_0(d)) = (-1)^k \cdot w(g(x_0(d), x_1(d))) \cdot \epsilon(x_0(d), x_1(d));$$
  

$$\epsilon(x_1(d), x_0(d)) \cdot g(x_1(d), x_0(d)) = (-1)^k \cdot \epsilon(x_0(d), x_1(d)) \cdot \overline{g(x_0(d), x_1(d))}.$$

We have defined the abelian group  $Q_{(-1)^k}(\mathbb{Z}\pi, w)$  in Notation 9.18. Define the *self-intersection element* for  $\alpha \in I_k(M)$ 

(9.38) 
$$\mu(\alpha) := \left[ \sum_{d \in D} \epsilon(x_0(d), x_1(d)) \cdot g(x_0(d), x_1(d)) \right] \in Q_{(-1)^k}(\mathbb{Z}\pi, w).$$

The passage from  $\mathbb{Z}\pi$  to  $Q_{(-1)^k}(\mathbb{Z}\pi, w)$  ensures that the definition is independent of the choice of the order on  $f^{-1}(d)$  for  $d \in D$ . It remains to show that it depends only on the pointed regular homotopy class of (f, w). Let *h* be a pointed regular homotopy from (f, w) to (g, v). We can arrange that *h* is in general position. In particular, each immersion  $h_t$  is in general position and comes with a set  $D_t$ . The collection of the  $D_t$ -s yields a bordism *W* from the finite set  $D_0$  to the finite set  $D_1$ . Since *W* is a compact one-dimensional manifold, it consists of circles and arcs joining points in  $D_0 \cup D_1$  to points in  $D_0 \cup D_1$ . Suppose that the point *e* and the point *e'* in  $D_0 \cup D_1$  are joined by an arc. Then one easily checks that their contributions to

$$\mu(f, w) - \mu(g, w) := \left[ \sum_{d_0 \in D_0} \epsilon(x_0(d_0), x_1(d_0)) \cdot g(x_0(d_0), x_1(d_0)) - \sum_{d_1 \in D_1} \epsilon(x_0(d_1), x_1(d_1)) \cdot g(x_0(d_1), x_1(d_1)) \right]$$

cancel out. This implies  $\mu(f, w) = \mu(g, w)$ .

Consider the pairing which is a special case of the pairing (9.20)

$$(9.39) \qquad \rho \colon \mathbb{Z}\pi \times Q_{(-1)^k}(\mathbb{Z}\pi, w) \to Q_{(-1)^k}(\mathbb{Z}\pi, w), \quad (u, [v]) \mapsto [uv\overline{u}].$$

Recall that it is additive in the second variable, i.e.,  $\rho(x, [y_1] - [y_2]) = \rho(x, [y_1]) - \rho(x, [y_2])$ , but it is not additive in the first variable, and in particular  $\rho$  does *not* give the structure of a left  $\mathbb{Z}\pi$ -module on  $Q_{(-1)^k}(\mathbb{Z}\pi, w)$ . Sometimes in the literature  $\rho(x, [y])$  is denoted by  $x[y]\overline{x}$ , but this is a little bit misleading since it might lead to the wrong impression that  $Q_{(-1)^k}(\mathbb{Z}\pi, w)$  is a left or right  $\mathbb{Z}\pi$ -module.

**Lemma 9.40.** Let  $\mu: I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$  be the map given by the selfintersection element, see (9.38), and let  $\lambda: I_k(M) \times I_k(M) \to \mathbb{Z}\pi$  be the intersection pairing, see (9.33). Then:

(i) Let  $(1 + (-1)^k \cdot T) : Q_{(-1)^k}(\mathbb{Z}\pi, w) \to \mathbb{Z}\pi$  be the homomorphism of abelian groups that sends [u] to  $u + (-1)^k \cdot \overline{u}$ . For  $\alpha \in I_k(M)$  denote by  $\chi(\alpha) \in \mathbb{Z}$  the Euler number of the normal bundle v(f) for any representative (f, w) of  $\alpha$  with respect to the orientation of v(f) given by the standard orientation on  $S^k$  and the orientation on  $f^*TM$  given by the fixed orientation on  $T_bM$  and w. Then:

$$\lambda(\alpha, \alpha) = (1 + (-1)^k \cdot T)(\mu(\alpha)) + \chi(\alpha) \cdot 1;$$

(ii) We get for pr:  $\mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi, w)$  the canonical projection and  $\alpha, \beta \in I_k(M)$ 

$$\mu(\alpha + \beta) - \mu(\alpha) - \mu(\beta) = \operatorname{pr}(\lambda(\alpha, \beta));$$

(iii) For  $\alpha \in I_k(M)$  and  $u \in \mathbb{Z}\pi$ , we get, where  $\rho$  is defined in (9.39),

$$\mu(x \cdot \alpha) = \rho(x, \mu(\alpha)).$$

*Proof.* (i) Represent  $\alpha \in I_k(M)$  by a pointed immersion (f, w) which is in general position. Choose a section  $\sigma$  of v(f) which meets the zero section transversally. Note that then the Euler number satisfies

$$\chi(f) = \sum_{y \in N(\sigma)} \epsilon(y)$$

where  $N(\sigma)$  is the (finite) set of zero points of  $\sigma$  and  $\epsilon(y)$  is a sign that depends on the local orientations. We can arrange that no zero of  $\sigma$  is the preimage of an element in the set of double points  $D_f$  of f. Now move f a little bit in the direction of this normal field  $\sigma$ . We obtain a new immersion  $g: S^k \to M$  with a path v from b to g(s) such that (f, w) and (g, v) are pointed regularly homotopic.

We want to compute  $\lambda(\alpha, \alpha)$  using the representatives (f, w) and (g, v). Note that any point in  $d \in D_f$  corresponds to two distinct points  $x_0(d)$  and  $x_1(d)$  in the set  $D = im(f) \cap im(g)$  and any element  $n \in N(\sigma)$  corresponds to one point x(n) in D. Moreover any point in D occurs as  $x_i(d)$  or x(n) in a unique way. Now the contribution of d to  $\lambda([f, w], [g, v])$  is  $\epsilon(d) \cdot g(d) + (-1)^k \cdot \epsilon(d) \cdot \overline{g(d)}$  and the contribution of  $n \in N(\sigma)$  is  $\epsilon(n) \cdot 1$ . Now assertion (i) follows.

(ii) and (iii) The proof of these assertions are left to the reader.

**Remark 9.41 (Self-intersection pairing and**  $(-1)^k$ -quadratic forms). Lemma 9.40 says that the triple  $(I_k(M), \lambda, \mu)$  satisfies all the requirements appearing in Remark 9.21 except that  $I_k(M)$  may not be finitely generated projective over  $\mathbb{Z}\pi$  and we have to require  $\chi(\alpha) = 0$ , which will be satisfied in the cases of interest.

The following theorem of Wall is taken from [987, Theorem 5.2 on page 45].

**Theorem 9.42 (Self-intersections and embeddings).** Let M be a connected compact manifold of even dimension n = 2k. Fix base points  $s \in S^k$  and  $b \in M$  and an orientation of  $T_bM$ . Let (f, w) be a pointed immersion of  $S^k$  in M. Suppose that  $k \ge 3$ . Then (f, w) is pointed homotopic to a pointed immersion (g, v) for which  $g: S^k \to M$  is an embedding if and only  $\mu(f, w) = 0$ .

*Proof.* If *f* is represented by an embedding, then  $\mu(f, w) = 0$  by definition. Suppose that  $\mu(f, w) = 0$ . We can assume without loss of generality that *f* is in general position. Since  $\mu(f, w) = 0$ , we can find *d* and *e* in the set of double points  $D_f$  of *f* and a numeration  $x_0(d), x_1(d)$  of  $f^{-1}(d)$  and  $x_0(e), x_1(e)$  of  $f^{-1}(e)$  satisfying

$$\epsilon(x_0(d), x_1(d)) = -\epsilon(x_0(e), x_1(e));$$
  
$$g(x_0(d), x_1(d)) = g(x_0(e), x_1(e)).$$

Therefore we can find arcs  $u_0$  and  $u_1$  in  $S^k$  such that  $u_0(0) = x_0(d)$ ,  $u_0(1) = x_0(e)$ ,  $u_1(0) = x_1(d)$ , and  $u_1(1) = x_1(e)$  hold, the paths  $u_0$  and  $u_1$  are disjoint from one another,  $f(u_0((0, 1)))$  and  $f(u_1((0, 1)))$  do not meet  $D_f$ , and  $f(u_0)$  and  $f(u_1)$ are homotopic relative endpoints. We can change  $u_0$  and  $u_1$  without destroying the properties above and find a smooth map  $U: D^2 \to M$  whose restriction to  $S^1$  is an embedding (ignoring corners on the boundary) and is given on the upper hemisphere  $S_{\perp}^{1}$  by  $u_{0}$  and on the lower hemisphere  $S_{\perp}^{1}$  by  $u_{1}$  and which meets  $\operatorname{im}(f)$  transversally. There is a compact neighborhood K of  $S^{1}$  such that  $U|_{K}$  is an embedding. Since  $k \geq 3$  we can find arbitrarily close to U an embedding which agrees with U on K. Hence we can assume without loss of generality that U itself is an embedding. The Whitney trick, see [725, Theorem 6.6 on page 71], [1012], allows us to change f within its pointed regular homotopy class to a new pointed immersion (g, v) such that  $D_{g} = D_{f} - \{d, e\}$  and  $\mu(g, v) = 0$ . By iterating this process we achieve  $D_{f} = \emptyset$ .

**Remark 9.43 (The dimension assumption** dim $(M) \ge 5$ ). The condition dim $(M) \ge 5$ , which arises in high-dimensional manifold theory, ensures in the proof of Theorem 9.42 that  $k \ge 3$  and hence we can arrange U to be an embedding. If k = 2, one can achieve that U is an immersion but not necessarily an embedding. This is the technical reason why surgery in dimension 4 is much more complicated than in dimensions  $\ge 5$ .

**Exercise 9.44.** Let  $f: S^k \to M$  be an immersion into a compact 2k-dimensional manifold. Suppose that it is in general position and the set of double points consists of precisely one element. Show that f is not regular homotopic to an embedding.

**Exercise 9.45.** Construct an immersion  $f: M \to N$  of connected closed manifolds which is homotopic but not regularly homotopic to an embedding.

# 9.5 The Surgery Obstruction in Even Dimensions

We give a brief introduction to the surgery obstruction in even dimension to motivate the relevance of the L-groups for topology. We will use the sign conventions for chain complexes as they appear in [667, Section 14.4].

### 9.5.1 Poincaré Duality Spaces

Consider a connected finite *CW*-complex *X* with fundamental group  $\pi$  and a group homomorphism  $w: \pi \to \{\pm 1\}$ . Below we use the *w*-twisted involution on  $\mathbb{Z}\pi$ . Denote by  $C_*(\widetilde{X})$  the cellular  $\mathbb{Z}\pi$ -chain complex of the universal covering. It is a finite free  $\mathbb{Z}\pi$ -chain complex. The product  $\widetilde{X} \times \widetilde{X}$  equipped with the diagonal  $\pi$ -action is again a  $\pi$ -*CW*-complex. The diagonal map  $D: \widetilde{X} \to \widetilde{X} \times \widetilde{X}$  sending  $\widetilde{x}$  to  $(\widetilde{x}, \widetilde{x})$  is  $\pi$ -equivariant but not cellular. By the Equivariant Cellular Approximation Theorem, see for instance [644, Theorem 2.1 on page 32], there is up to cellular  $\pi$ -homotopy precisely one cellular  $\pi$ -map  $\overline{D}: \widetilde{X} \to \widetilde{X} \times \widetilde{X}$  which is  $\pi$ -homotopic to D. It induces a  $\mathbb{Z}\pi$ -chain map unique up to  $\mathbb{Z}\pi$ -chain homotopy

$$(9.46) C_*(D): C_*(X) \to C_*(X \times X).$$

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There is a natural isomorphism of  $\mathbb{Z}\pi$ -chain complexes

**Definition 9.48 (Dual chain complex).** Given an *R*-chain complex of left *R*-modules  $C_*$  and  $n \in \mathbb{Z}$ , we define its *dual chain complex*  $C^{n-*}$  to be the chain complex of left *R*-modules whose *p*-th chain module is  $\hom_R(C_{n-p}, R)$  and whose *p*-th differential is given by

$$(-1)^{n-p+1} \cdot \hom_R(c_{n-p+1}, \operatorname{id}) \colon (C^{n-*})_p = \hom_R(C_{n-p}, R)$$
  
 $\to (C^{n-*})_{p-1} = \hom_R(C_{n-p+1}, R).$ 

Denote by  $\mathbb{Z}^w$  the  $\mathbb{Z}\pi$ -module whose underlying abelian group is  $\mathbb{Z}$  and on which  $g \in \pi$  acts by  $w(g) \cdot id$ . Given two projective  $\mathbb{Z}\pi$ -chain complexes  $C_*$  and  $D_*$ , we obtain a natural  $\mathbb{Z}$ -chain map unique up to  $\mathbb{Z}$ -chain homotopy

$$(9.49) s: \mathbb{Z}^w \otimes_{\mathbb{Z}\pi} (C_* \otimes_{\mathbb{Z}} D_*) \to \hom_{\mathbb{Z}\pi} (C^{-*}, D_*)$$

by sending  $1 \otimes x \otimes y \in \mathbb{Z} \otimes C_p \otimes D_q$  to

$$s(1 \otimes x \otimes y) \colon \hom_{\mathbb{Z}\pi}(C_p, \mathbb{Z}\pi) \to D_q, \quad (\phi \colon C_p \to \mathbb{Z}\pi) \mapsto (-1)^{|x| \cdot |y| + |x|} \cdot \overline{\phi(x)} \cdot y.$$

The composite of the chain map (9.49) for  $C_* = D_* = C_*(\widetilde{X})$ , the inverse of the chain map (9.47) tensored with  $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} -$ , and the chain map (9.46) tensored with  $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} -$  yield a  $\mathbb{Z}$ -chain map

$$\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) \to \hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X})).$$

Note that the *n*-th homology of  $\hom_{\mathbb{Z}\pi}(C^{-*}(\widetilde{X}), C_*(\widetilde{X}))$  is the set of  $\mathbb{Z}\pi$ -chain homotopy classes  $[C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}\pi}$  of  $\mathbb{Z}\pi$ -chain maps from  $C^{n-*}(\widetilde{X})$  to  $C_*(\widetilde{X})$ . Define  $H_n(X; \mathbb{Z}^w) := H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$ . Taking the *n*-th homology group yields a well-defined  $\mathbb{Z}$ -homomorphism

(9.50) 
$$\cap: H_n(X; \mathbb{Z}^w) \to [C^{n-*}(\widetilde{X}), C_*(\widetilde{X})]_{\mathbb{Z}^n}$$

that sends a class  $x \in H_n(X; \mathbb{Z}^w) = H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}))$  to the  $\mathbb{Z}\pi$ -chain homotopy class of a  $\mathbb{Z}\pi$ -chain map denoted by  $- \cap x \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ .

**Definition 9.51 (Poincaré complex).** A connected finite *n*-dimensional Poincaré complex is a connected finite CW-complex of dimension *n* together with a group homomorphism  $w = w_1(X) : \pi_1(X) \to \{\pm 1\}$ , called an *orientation homomorphism*, if there exists an element  $[X] \in H_n(X; \mathbb{Z}^w)$ , called a *fundamental class*, such that the  $\mathbb{Z}\pi$ -chain map  $- \cap [X] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$  is a  $\mathbb{Z}\pi$ -chain homotopy equivalence. We will call it the *Poincaré*  $\mathbb{Z}\pi$ -chain homotopy equivalence.

**Exercise 9.52.** Show that the orientation homomorphism  $w: \pi_1(X) \to \{\pm 1\}$  is uniquely determined by the homotopy type of the finite *n*-dimensional Poincaré complex *X*.

Obviously there are two possible choices for [X], since it has to be a generator of the infinite cyclic group  $H_n(X, \mathbb{Z}^w) \cong H^0(X; \mathbb{Z}) \cong \mathbb{Z}$ . A choice of [X] is called a *w*-orientation on X. We call X *w*-oriented if we have chosen a *w*-orientation.

A map  $f: Y_1 \to Y_2$  of *w*-oriented connected Poincaré complexes has degree one if  $w_1(Y_2) \circ \pi_1(f) = w_1(Y_2)$  and the map  $H_n(Y_1, \mathbb{Z}^{w_1(Y_1)}) \to H_n(Y_2, \mathbb{Z}^{w_1(Y_2)})$  induced by *f* sends  $[Y_1]$  to  $[Y_2]$ .

**Theorem 9.53.** *Let M* be a connected closed manifold of dimension n. Then *M carries the structure of a connected finite n-dimensional Poincaré complex.* 

For a proof we refer for instance to [987, Theorem 2.1 on page 23].

Below a *w*-orientation of a connected closed manifold M of dimension n is a choice of a generator [M] of the infinite cyclic group  $H_n(M; \mathbb{Z}^{w_1(M)})$ . We call M *w*-oriented if we have chosen a *w*-orientation. Note that *w*-oriented does not necessarily mean that  $w_1(M)$  is trivial. Following the standard conventions, we say that M is orientable if  $w_1(M)$  is trivial, and we call M oriented if  $w_1(M)$  is trivial and we have chosen a fundamental class  $[M] \in H_n(M; \mathbb{Z})$ .

Remark 9.54 (Poincaré duality as obstruction for being homotopy equivalent to a closed manifold). Theorem 9.53 gives us the first obstruction for a topological space X to be homotopy equivalent to a connected closed n-dimensional manifold. Namely, X must be homotopy equivalent to a connected finite n-dimensional Poincaré complex.

### 9.5.2 Normal Maps and the Surgery Step

**Definition 9.55 (Normal map of degree one).** Let *X* be a *w*-oriented connected finite *n*-dimensional Poincaré complex together with an *m*-dimensional vector bundle  $\xi: E \to X$ . A normal  $\xi$ -map or briefly normal map  $(M, i, f, \overline{f})$  with  $(X, \xi)$  as target consists of

- a *w*-oriented connected closed manifold *M* of dimension *n*;
- an embedding  $i: M \to \mathbb{R}^{n+m}$ ;
- a bundle map  $(f, f): v(i) \to \xi$ ,

where v(i) denotes the normal bundle of the embedding  $i: M \to \mathbb{R}^{m+n}$ . xs A normal map of degree one is a normal map such that the degree of  $f: M \to X$  is one.

**Remark 9.56.** We are being somewhat sloppy here since we have ignored the problem that the choices of the fundamental classes and the bundle data have to be consistent with one another. This is an issue that has been overlooked in many places. It is explained in detail and fixed in [667, Section 7.4, Example 7.44 on page 215 and Remark 7.45 on page 215]. However, to keep this exposition comprehensible, we ignore this issue and also will not treat the notion of an intrinsic fundamental class of [667, Section 5.5]. Given a normal map  $(M, i, f, \overline{f})$  with  $(X, \xi)$  as target, we obtain for  $k \ge 1$  a normal map  $(M, i, f, \overline{f'})$  with  $(X, \xi \oplus \mathbb{R}^k)$  as target as follows. Let  $i' : M \to \mathbb{R}^{n+m+k}$ be the composite of the embedding  $i: M \to \mathbb{R}^{n+m}$  with the standard inclusion  $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m+k}$ . Then v(i') is the Whitney sum  $v(i) \oplus \mathbb{R}^k$ , where  $\mathbb{R}^k$  is the trivial *k*-dimensional bundle. Let  $\overline{f'}: v(i') \to \xi \oplus \mathbb{R}^k$  be the stabilization of  $\overline{f}$ . We call  $(M, i', f, \overline{f'})$  a stabilization of  $(M, i, f, \overline{f})$ .

The next result is due to Whitney [1011, 1012].

**Theorem 9.57 (Whitney's Approximation Theorem).** Let M and N be closed manifolds of dimensions m and n. Then any map  $f: M \to N$  is arbitrarily close and in particular homotopic to an immersion, provided that  $2m \le n$ , and arbitrarily close and in particular homotopic to an embedding, provided that 2m < n.

Remark 9.58 (Existence of a normal map of degree one as obstruction for being homotopy equivalent to a closed manifold). Given a connected finite *n*-dimensional Poincaré complex *X*, the existence of a normal map of degree one with  $(X, \xi)$  as target for some vector bundle  $\xi$  over *X* (for some appropriate choice of *w*-orientations) is necessary for *X* to be homotopy equivalent to a closed manifold. Namely, if  $f: M \to X$  is such a homotopy equivalence, choose a homotopy inverse  $g: X \to M$ and put  $\xi = g^*v(i)$  for some embedding  $i: M \subseteq \mathbb{R}^{n+m}$ . Such an embedding always exists for n < m by Theorem 9.57. Obviously *f* can be covered by a bundle map  $\overline{f}: v(M) \to \xi$  and *f* has degree one (for some appropriate choice of *w*-orientations).

Note that an orientation of a compact manifold W induces an orientation of its boundary  $\partial W$ , see for instance [667, Remark 5.37 on page 119]. In the special case  $W = M \times [0, 1]$  for closed M, the induced orientations on  $M = M \times \{0\}$  and  $M = M \times \{1\}$  are inverse to one another.

**Definition 9.59 (Normal bordism).** Consider two normal maps of degree one  $(M_k, i_k, f_k, \overline{f_k})$  with the same target  $(X, \xi)$  for k = 0, 1. A *normal bordism* from  $(\overline{f_0}, f_0)$  to  $(\overline{f_1}, f_1)$  consists of

- a *w*-oriented connected compact manifold *W* with boundary  $\partial W$ ;
- an embedding  $j: (W, \partial W) \to (\mathbb{R}^{n+m} \times [0, 1], \mathbb{R}^{n+m} \times \{0, 1\});$
- a map  $(F, \partial F)$ :  $(W, \partial W) \rightarrow (X \times [0, 1], X \times \{0, 1\})$  of degree one;
- a bundle map  $\overline{F}$ :  $\nu(j) \rightarrow \xi$  covering f;
- an orientation preserving diffeomorphism  $u: \partial W \xrightarrow{\cong} M_0 \amalg M_1$ ,

such that the obvious compatibility conditions are satisfied.

We call  $(M_0, i_0, f_0, f_0)$  and  $(M_1, i_1, f_1, f_1)$  normally bordant if after stabilization there exists a normal bordism between them.

Note Definition 9.59 corresponds in [667] to the notion of a normal bordism with cylindrical target, see [667, Definition 7.16 on page 203].

**Exercise 9.60.** Let  $(M, i_0, f, \overline{f}_0)$  be a normal map of degree one with target  $(X, \xi)$ . Let  $i_1: M \to \mathbb{R}^{n+k}$  be an embedding. Show that there exists a normal map of degree one  $(M, i_1, f, \overline{f}_1)$  with target  $(X, \xi)$  which is normally bordant to  $(M, i_0, f, \overline{f}_0)$ . In the sequel we will often suppress the embedding  $i: M \to \mathbb{R}^{n+m}$  in the notation and write  $\nu(M)$  instead of  $\nu(i)$ .

### 9.5.3 The Surgery Step

So the question is whether we can modify a normal map of degree one with  $(X, \xi)$  as target (without changing the target) so that the underlying map f is a homotopy equivalence. There is a procedure in the world of *CW*-complexes to turn a map into a weak homotopy equivalence, namely, by attaching cells. If  $f: Y_1 \to Y_2$  is already k-connected, we can attach (k + 1) cells to  $Y_1$  to obtain an extension  $f': Y'_1 \to Y_2$  of f which is (k + 1)-connected. In principle we want to do the same for a normal map of degree one with target  $(X, \xi)$ . However, there are two fundamental difficulties. First of all we have to keep the manifold structure on the source and cannot therefore just attach cells. Moreover, by Poincaré duality any modification in dimension k will cause a dual modification in dimension n - k if n is the dimension of X so that one encounters at any rate problems when n happens to be 2k.

Consider a normal map (f, f):  $v(M) \rightarrow \xi$  such that  $f: M \rightarrow X$  is a *k*-connected map. Consider an element  $\omega \in \pi_{k+1}(f)$  represented by a diagram



We cannot attach a single cell to M without destroying the manifold structure. But one can glue two manifolds together along a common boundary such that the result is a manifold. Suppose that the map  $q: S^k \to M$  extends to an embedding  $\overline{q}: S^k \times D^{n-k} \to M$ . (This assumption will be justified later.) Let  $\operatorname{int}(\operatorname{im}(\overline{q}))$  be the interior of the image of  $\overline{q}$ . Then  $M - \operatorname{int}(\operatorname{im}(\overline{q}))$  is a manifold with boundary  $\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})$ . We can get rid of the boundary by attaching  $D^{k+1} \times S^{n-k-1}$  along  $\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})$ . Call the result

$$M' := D^{k+1} \times S^{n-k-1} \cup_{\operatorname{im}(\overline{q}|_{S^k \times S^{n-k-1}})} (M - \operatorname{int}(\operatorname{im}(\overline{q}))) .$$

Here and elsewhere we apply without further mention the technique of straightening the angle in order to get a well-defined smooth structure, see [167, Definition 13.11 on page 145 and (13.12) on page 148] and [497, Chapter 8, Section 2]. Choose a map  $\overline{Q}: D^{k+1} \times D^{n-k} \to X$  which extends Q and  $f \circ \overline{q}$ . The restriction of f to  $M - \operatorname{int}(\operatorname{im}(\overline{q}))$  extends to a map  $f': M' \to X$  using  $\overline{Q}|_{D^{k+1} \times S^{n-k}}$ . Note that the inclusion  $M - \operatorname{int}(\operatorname{im}(\overline{q})) \to M$  is (n - k - 1)-connected, since  $S^k \times S^{n-k-1} \to$  $S^k \times D^{n-k}$  is (n - k - 1)-connected. So the passage from M to  $M - \operatorname{int}(\operatorname{im}(\overline{q}))$ will not affect  $\pi_j(f)$  for j < n - k - 1. All in all we see that  $\pi_l(f) = \pi_l(f')$  for  $l \le k$  and that there is an epimorphism  $\pi_{k+1}(f) \to \pi_{k+1}(f')$  whose kernel contains  $\omega$ , provided that  $2(k + 1) \leq n$ . The condition  $2(k + 1) \leq n$  can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality forces a change in dimension (n - l). This phenomenon will cause surgery obstructions to appear.

Note that  $f: M \to X$  and  $f': M' \to X$  are bordant. The relevant bordism is given by  $W = D^{k+1} \times D^{n-k} \cup_{\overline{q}} M \times [0, 1]$ , where we think of  $\overline{q}$  as an embedding  $S^k \times D^{n-k} \to M \times \{1\}$ . In other words, W is obtained from  $M \times [0, 1]$  by attaching a handle  $D^{k+1} \times D^{n-k}$  to  $M \times \{1\}$ . Then M appears in W as  $M \times \{0\}$  and M' as another part of the boundary of W. Define  $F: W \to X$  by  $f \times id_{[0,1]}$  and  $\overline{Q}$ . Then F restricted to M and M' is f and f'.

Why can we assume that the map  $q: S^k \to M$  extends to an embedding  $\overline{q}: S^k \times D^{n-k} \to M$ ? This will be ensured by the bundle data in the case 2k + 1 < n by the following argument.

Because of Theorem 9.57 we can arrange that q is an embedding. The extension  $\overline{q}$  exists if and only if the normal bundle v(q) of the embedding  $q: S^k \to M$  is trivial. Since  $D^{k+1}$  is contractible, every vector bundle over  $D^{k+1}$  is trivial. Hence  $Q^*\xi$  is a trivial vector bundle over  $D^{k+1}$ . Recall that  $i: M \to \mathbb{R}^{m+n}$  is a fixed embedding and v(M) is defined to be the normal bundle v(i) of *i*. Pullbacks of trivial vector bundles are trivial again. This implies that  $q^*v(M) \cong q^*f^*\xi \cong j^*Q^*\xi$  is a trivial vector bundle over  $S^k$ . Since  $v(q) \oplus q^*v(M) \cong v(i: S^k \to \mathbb{R}^{n+m})$  is trivial, v(q) is a stably trivial (n - k)-dimensional vector bundle over  $S^k$ . Since  $2k + 1 \le n$ , this implies that v(q) itself is trivial.

So we see that the bundle data are needed to carry out the desired surgery step. Note that the construction yields a map  $f': M' \to X$  of degree one and a bundle map  $\overline{f'}: v(M') \to \xi$  covering f' so that we end up with a normal map of degree one with target X again. Hence we are able to repeat this surgery step over and over again in dimensions  $2k - 1 \le n$ . Actually, the bordism W together with the map  $F: W \to X$  also come with a bundle map  $\overline{F}: v(W) \to \xi$  covering F and is therefore a normal bordism in the sense of Definition 9.59. In particular, surgery does not change the normal bordism class.

For the proof of the next lemma we refer for instance to [667, Theorem 7.41 on page 214].

**Lemma 9.61.** Consider a normal map of degree one  $(\overline{f}, f)$ :  $v(M) \to \xi$  covering  $f: M \to X$ , where M is a w-oriented connected closed manifold of dimension n and X is a connected finite Poincaré complex of dimension n. Let k be the natural number given by n = 2k or n = 2k + 1.

Then we can always arrange by finitely many surgery steps that for the resulting normal map of degree one  $(\overline{f'}, f')$ :  $v(M') \to \xi$  its underlying map  $f' : M' \to X$  is *k*-connected.

Now assume that *n* is even, let us say n = 2k. As mentioned above, we can arrange that *f* is *k*-connected. If we can achieve that *f* is (k + 1)-connected, then by Poincaré duality the map *f* is a homotopy equivalence.

But in this last step we encounter a problem which actually leads to the surgery obstruction in the even-dimensional case. Namely, in the argument above we used at one point that the map  $q: S^k \to M$  can be arranged to be an embedding by general position if  $2k + 1 \le n$  and that certain normal bundle are trivial. In the situation n = 2k we can arrange q to be an immersion by Theorem 9.57 and simultaneously ensure that the bundle data carry over to the desired normal bordism, essentially, because of a systematic use of Theorem 9.63 below. However, the latter fixes the regular homotopy class of the immersion q. Hence one open problem is to ensure that we can change q to an embedding within its regular homotopy class. We have already introduced the main obstruction for that, the self-intersection element in (9.38). We also encounter the problem that by Poincaré duality any change in the homology of the middle dimension comes with a dual change and one has to ensure that these two have the desired effect and do not disturb one another. Next we explain how this leads to the so-called surgery obstruction in  $L_{2k}(\mathbb{Z}\pi_1(X))$  with respect to the  $w_1(X)$ -twisted involution on  $\mathbb{Z}\pi$ .

#### 9.5.4 The Even-Dimensional Surgery Obstruction

For the rest of this subsection we fix a normal map  $(\overline{f}, f): v(M) \to \xi$  of degree one covering  $f: M \to X$ , where M is a w-oriented connected closed manifold of dimension n and X is a w-oriented connected finite Poincaré complex of dimension n. Suppose that f induces an isomorphism on the fundamental groups. Fix a base point  $b \in M$  together with lifts  $\widetilde{b} \in \widetilde{M}$  of b and  $\widetilde{f(b)} \in \widetilde{X}$  of f(b). We identify  $\pi = \pi_1(M, b) = \pi(X, f(b))$  by  $\pi_1(f, b)$ . The choices of  $\widetilde{b}$  and  $\widetilde{f(b)}$  determine  $\pi$ -operations on  $\widetilde{M}$  and on  $\widetilde{X}$  and a lift  $\widetilde{f}: \widetilde{M} \to \widetilde{X}$  which is  $\pi$ -equivariant.

**Definition 9.62 (Surgery kernels).** Let  $K_k(\widetilde{M})$  be the kernel of the  $\mathbb{Z}\pi$ -map  $H_k(\widetilde{f}): H_k(\widetilde{M}) \to H_k(\widetilde{X})$ . Denote by  $K^k(\widetilde{M})$  the cokernel of the  $\mathbb{Z}\pi$ -map  $H^k(\widetilde{f}): H^k(\widetilde{X}) \to H^k(\widetilde{M})$  which is the  $\mathbb{Z}\pi$ -map induced by  $C^*(\widetilde{f}): C^*(\widetilde{X}) \to C^*(\widetilde{M})$ . We call  $K_k(\widetilde{M})$  the surgery kernel.

Given two vector bundles  $\xi: E \to M$  and  $\eta: F \to N$ , we have so far only considered bundle maps  $(\overline{f}, f): \xi \to \eta$  which are fiberwise isomorphisms. We need to consider, at least for the next theorem, more generally bundle monomorphisms, i.e., we will only require that the map is fiberwise injective. Consider two bundle monomorphism  $(\overline{f}_0, f_0), (\overline{f}_1, f_1): \xi \to \eta$ . Let  $\xi \times [0, 1]$  be the vector bundle  $\xi \times \text{id}: E \times [0, 1] \to M \times [0, 1]$ . A homotopy of bundle monomorphisms  $(\overline{h}, h)$  from  $(\overline{f}_0, f_0)$  to  $(\overline{f}_1, f_1)$  is a bundle monomorphism  $(\overline{h}, h): \xi \times [0, 1] \to \eta$  whose restriction to  $X \times \{j\}$  is  $(\overline{f}_j, f_j)$  for j = 0, 1. Denote by  $\pi_0(\text{Mono}(\xi, \eta))$  the set of homotopy classes of bundle monomorphisms.

For a proof of the following result we refer to Haefliger-Poenaru [451], Hirsch [496], and Smale [920]. Denote by  $\pi_0(\text{Imm}(M, N))$  the set of regular homotopy classes of immersions from *M* to *N*.

**Theorem 9.63 (Immersions and Bundle Monomorphisms).** Let *M* be an *m*-dimensional and *N* be an *n*-dimensional manifold.

(i) Suppose that  $1 \le m < n$ . Then taking the differential of an immersion yields a bijection

$$T: \pi_0(\operatorname{Imm}(M, N)) \xrightarrow{=} \pi_0(\operatorname{Mono}(TM, TN));$$

(ii) Suppose that  $1 \le m \le n$  and that M has a handlebody decomposition consisting of q-handles for  $q \le n - 2$ . Then taking the differential of an immersion yields a bijection

$$T \colon \pi_0(\operatorname{Imm}(M, N)) \xrightarrow{\cong} \operatorname{colim}_{a \to \infty} \pi_0(\operatorname{Mono}(TM \oplus \underline{\mathbb{R}^a}, TN \oplus \underline{\mathbb{R}^a}))$$

where the colimit is given by stabilization.

**Lemma 9.64.** (i) The cap product with [M] induces isomorphisms

$$-\cap [M]: K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M});$$

(ii) Suppose that f is k-connected. Then there is the composite of natural  $\mathbb{Z}\pi$ -isomorphisms

$$h_k \colon \pi_{k+1}(f) \xrightarrow{\cong} \pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f}) \xrightarrow{\cong} K_k(\widetilde{M});$$

(iii) Suppose that f is k-connected and n = 2k. Then there is a natural  $\mathbb{Z}\pi$ -homomorphism

$$t_k \colon \pi_{k+1}(f) \to I_k(M).$$

*Proof.* (i) The following diagram commutes and has isomorphisms as vertical arrows

(9.65) 
$$H^{n-k}(\widetilde{M}) \xleftarrow{H^{n-k}(\widetilde{f})} H^{n-k}(\widetilde{X})$$
$$-\cap [M] \downarrow \cong \cong \left( -\cap [X] \right)$$
$$H_k(\widetilde{M}) \xrightarrow{H_k(\widetilde{f})} H_k(\widetilde{X}).$$

Hence the composite  $K_k(\widetilde{M}) \to H_k(\widetilde{M}) \xrightarrow{(-\cap [M])^{-1}} H^{n-k}(\widetilde{M}) \to K^{n-k}(\widetilde{M})$  is bijective.

(ii) The commutative square (9.65) above implies that  $H_l(\tilde{f}) : H_l(\tilde{M}) \to H_l(\tilde{X})$  is split surjective for all *l*. We conclude from the long exact sequence of  $C_*(\tilde{f})$  that the boundary map

$$\partial : H_{k+1}(\widetilde{f}) := H_{k+1}(\operatorname{cone}(C_*(\widetilde{f}))) \to H_k(\widetilde{M})$$

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induces an isomorphism

$$\partial_{k+1} \colon H_{k+1}(\widetilde{f}) \xrightarrow{\equiv} K_k(\widetilde{M}).$$

Since f and hence  $\tilde{f}$  is k-connected, the Hurewicz homomorphism

$$\pi_{k+1}(\widetilde{f}) \xrightarrow{\cong} H_{k+1}(\widetilde{f})$$

is bijective [1006, Corollary IV.7.10 on page 181]. The canonical map

$$\pi_{k+1}(\widetilde{f}) \to \pi_{k+1}(f)$$

is bijective. The composite of the maps above or their inverses yields a natural isomorphism  $h_k: \pi_{k+1}(f) \to K_k(\widetilde{M})$ .

(iii) Note that an element in  $\pi_{k+1}(f, b)$  is given by a commutative diagram

together with a path w from b to f(s) for a fixed base point  $s \in S^k$ . We leave the details of the rest of the proof, which is based on Theorem 9.63 (ii), to the reader. The necessary bundle monomorphisms come from the bundle data of  $(\overline{f}, f)$ , the stable triviality of  $TS^k$ , and the fact that any vector bundle over  $D^{k+1}$  is trivial.  $\Box$ 

Suppose that n = 2k. The Kronecker pairing  $\langle , \rangle : H^k(\widetilde{M}) \times H_k(\widetilde{M}) \to \mathbb{Z}\pi$  is induced by the evaluation pairing  $\hom_{\mathbb{Z}\pi}(C_p(\widetilde{M}),\mathbb{Z}\pi) \times C_p(\widetilde{M}) \to \mathbb{Z}\pi$  which sends  $(\beta, x)$  to  $\beta(x)$ . It induces a pairing

$$\langle , \rangle \colon K^k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

Together with the isomorphism

$$-\cap [M]: K^{n-k}(\widetilde{M}) \xrightarrow{\cong} K_k(\widetilde{M})$$

of Theorem 9.64 (i) it yields the intersection pairing

(9.66) 
$$s: K_k(\widetilde{M}) \times K_k(\widetilde{M}) \to \mathbb{Z}\pi.$$

We get from Lemma 9.64 (ii) and (iii) a  $\mathbb{Z}\pi$ -homomorphism

(9.67) 
$$\alpha \colon K_k(\widetilde{M}) \to I_k(\widetilde{M}).$$

The elementary proof of the next lemma is left to the reader.

**Lemma 9.68.** *The following diagram commutes* 



where the upper pairing is defined in (9.66), the lower pairing in (9.33), and the left vertical arrows in (9.67).

**Exercise 9.69.** Let  $f: X \to Y$  be a map of connected finite Poincaré complexes of dimension  $n \ge 4$ . Suppose that f has degree one and that f is (k + 1)-connected where k is given by n = 2k if n is even, and by n = 2k + 1 if n is odd. Show that then f is a homotopy equivalence.

Recall that an *R*-module *V* is called *stably finitely generated free* if for some non-negative integer *l* the *R*-module  $V \oplus R^l$  is a finitely generated free *R*-module.

**Lemma 9.70.** If  $f: X \to Y$  is k-connected for n = 2k or n = 2k + 1, then  $K_k(\tilde{M})$  is stably finitely generated free.

*Proof.* See for instance [667, Lemma 8.55 (ii) on page 248].

**Example 9.71 (Effect of trivial surgery).** Consider the normal map  $(\overline{f}, f) : v(M) \to \xi$  covering the *k*-connected map of degree one  $f : M \to X$  for a *w*-oriented connected closed *n*-dimensional manifolds *M* for n = 2k. If we do surgery on the zero element in  $\pi_{k+1}(f)$ , then the effect on *M* is that *M* is replaced by the connected sum  $M' = M \# (S^k \times S^k)$ . The effect on  $K_k(\widetilde{M})$  is that it is replaced by  $K_k(\widetilde{M}') = K_k(\widetilde{M}) \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)$ . The intersection pairing on this new kernel is the sum of the given intersection pairing on  $K_k(\widetilde{M})$  together with the standard hyperbolic symmetric form  $H^{(-1)^k}(\mathbb{Z}\pi)$ . Moreover, taking the self-intersections into account, the non-singular  $(-1)^k$ -quadratic form on the new kernel is the direct sum of the one of the old kernel and the standard hyperbolic  $(-1)^k$ -quadratic form  $H_{(-1)^k}(\mathbb{Z}\pi)$ . In particular, we can arrange by finitely many surgery steps on the trivial element in  $\pi_{k+1}(f)$  that  $K_k(\widetilde{M})$  is a finitely generated free  $\mathbb{Z}\pi$ -module.

**Remark 9.72.** Let  $(\overline{f}, f): v(M) \to \xi$  be a normal map of degree one covering  $f: M \to X$ , where *M* is a *w*-oriented connected closed manifold of dimension *n* and *X* is a *w*-oriented connected finite Poincaré complex of dimension *n*. Suppose that n = 2k and *f* is *k*-connected.

By Lemma 9.70 and Example 9.71, we can do finitely many trivial surgery steps to achieve that the kernel  $K_k(\widetilde{M})$  is a finitely generated free  $\mathbb{Z}\pi$ -module. By the intersection pairing *s* of (9.66), we obtain a non-singular  $(-1)^k$ -symmetric form  $(K_k(\widetilde{M}), s)$ , see Remark 9.8.

So far we have not used the bundle data. They now come into play, when we want to refine  $(K_k(\widetilde{M}), s)$  to a non-singular  $(-1)^k$ -quadratic form. Because of Remark 9.21 we have to specify a map  $t: K_k(\widetilde{M}) \to Q_{(-1)^k}(\mathbb{Z}\pi)$ . We will take the composite

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$$K_k(\widetilde{M}) \xrightarrow{h_k^{-1}} \pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi)$$

where  $\mu$  has been defined (9.38) and the isomorphism  $h_k$  and the map  $t_k$  have been introduced in Lemma 9.64. This is indeed a quadratic refinement by Lemma 9.40 and Lemma 9.68.

**Definition 9.73 (Even-dimensional surgery obstruction).** Consider a normal map of degree one  $(\overline{f}, f)$ :  $v(M) \to \xi$  covering  $f: M \to X$ , where M is a w-oriented connected closed manifold of even dimension n = k and X is a connected finite Poincaré complex of dimension n with fundamental group  $\pi$ . Perform surgery below the middle dimension and trivial surgery in the middle dimension so that we obtain a k-connected normal map of degree one  $(\overline{f'}, f'): v(M) \to \xi$  such that  $K_k(\widetilde{M'})$  is a finitely generated free  $\mathbb{Z}\pi$ -module. Define the *surgery obstruction* of  $(\overline{f}, f): v(M) \to \xi$ 

$$\sigma(\overline{f}, f) \in L_{2k}(\mathbb{Z}\pi, w_1(X))$$

to be the class of the non-singular  $(-1)^k$ -quadratic form  $(K_k(\widetilde{M'}), s, t)$  of Remark 9.72.

We omit the proof that this element is well-defined, e.g., independent of the previous surgery steps; details of the proof can be found in [667, Section 8.6.3]

**Theorem 9.74 (Surgery obstruction in even dimensions).** Consider a normal map of degree one  $(\overline{f}, f): v(M) \rightarrow \xi$  covering  $f: M \rightarrow X$ , where M is a w-oriented connected closed manifold of even dimension n = 2k and X is a w-oriented connected finite Poincaré complex of dimension n with fundamental group  $\pi$ . Then:

- (i) Suppose  $k \ge 3$ . Then  $\sigma(\overline{f}, f) = 0$  in  $L_n(\mathbb{Z}\pi, w_1(X))$  if and only if we can do a finite number of surgery steps to obtain a normal map  $(\overline{f'}, f'): v(M') \to \xi$  which covers a homotopy equivalence  $f': M' \to X$ ;
- (ii) The surgery obstruction  $\sigma(\overline{f}, f)$  depends only on the normal bordism class of  $(\overline{f}, f)$ .

*Proof.* We only give the proof of assertion (i). More details can be found in [667, Theorem 8.112 on page 270] or [987, Chapter 5]. By Lemma 9.61, Example 9.71, and the definition of  $L_{2k}(\mathbb{Z}\pi, w)$ , we can arrange by finitely many surgery steps that the non-singular  $(-1)^k$ -quadratic form  $(K_k(\widetilde{M}), s, t)$  is isomorphic to  $H_{(-1)^k}(\mathbb{Z}\pi^v)$ . Thus we can choose for some natural number v a  $\mathbb{Z}\pi$ -basis  $\{b_1, b_2, \ldots, b_v, c_1, c_2, \ldots, c_v\}$  for  $K_k(\widetilde{M})$  such that

$s(b_i, c_i) = 1$	$i \in \{1, 2, \ldots, v\};$
$s(b_i, c_j) = 0$	$i, j \in \{1, 2, \dots, v\}, i \neq j$
$s(b_i, b_j) = 0$	$i, j \in \{1, 2, \dots, v\};$
$s(c_i, c_j) = 0$	$i, j \in \{1, 2, \dots, v\};$
$t(b_i) = 0$	$i \in \{1, 2, \ldots, v\}.$

Note that *f* is a homotopy equivalence if and only if the number *v* is zero. Hence it suffices to explain how we can lower the number *v* to (v - 1) by a surgery step on an element in  $\pi_{k+1}(f)$ . Of course our candidate is the element  $\omega$  in  $\pi_{k+1}(f)$  which corresponds under the isomorphism  $h: \pi_{k+1}(f) \to K_k(\widetilde{M})$ , see Lemma 9.64 (ii), to the element  $b_v$ . By construction the composite

$$\pi_{k+1}(f) \xrightarrow{t_k} I_k(M) \xrightarrow{\mu} Q_{(-1)^k}(\mathbb{Z}\pi, w)$$

of the maps defined in (9.38) and Lemma 9.64 (iii) sends  $\omega$  to zero. Now Theorem 9.42 ensures that we can perform surgery on  $\omega$ . Note that the assumption  $k \ge 3$  and the quadratic structure on the kernel become relevant exactly at this point. Finally it remains to check whether the effect on  $K_k(\tilde{M})$  is the desired one, namely, that we get rid of one of the hyperbolic summands  $H_{\epsilon}(\mathbb{Z}\pi)$ , or equivalently, v is lowered to v - 1.

We have explained earlier that doing surgery yields not only a new manifold M', but also a bordism from M to M'. Namely, take  $W = M \times [0, 1] \cup_{S^k \times D^{n-k}} D^{k+1} \times D^{n-k}$ , where we attach  $D^{k+1} \times D^{n-k}$  by an embedding  $S^k \times D^{n-k} \to M \times \{1\}$ , and  $M' := \partial W - M$  using the identification  $M = M \times \{0\}$ . The manifold W comes with a map  $F: W \to X \times [0, 1]$  whose restriction to M is the given map f: M = $M \times \{0\} \to X = X \times \{0\}$  and whose restriction to M' is a map  $f': M' \to X \times \{1\}$ . The definition of the kernels also makes sense for pair of maps. We obtain an exact braid



which combines the various long exact sequences of pairs.

The (k + 1)-handle  $D^{k+1} \times D^{n-k}$  defines an element  $\phi^{k+1}$  in  $K_{k+1}(\widetilde{W}, \widetilde{M})$  and the associated dual k-handle defines an element  $\psi^k \in K_k(\widetilde{W}, \widetilde{M'})$ . These elements constitute a  $\mathbb{Z}\pi$ -basis for  $K_{k+1}(\widetilde{W}, \widetilde{M}) \cong \mathbb{Z}\pi$  and  $K_k(\widetilde{W}, \widetilde{M'}) \cong \mathbb{Z}\pi$ . The  $\mathbb{Z}\pi$ homomorphism  $K_{k+1}(\widetilde{W}, \widetilde{M}) \to K_k(\widetilde{M})$  maps  $\phi$  to  $b_v$ . The  $\mathbb{Z}\pi$ -homomorphism  $K_k(\widetilde{M}) \to K_k(\widetilde{W}, \widetilde{M'})$  sends x to  $s(b_v, x) \cdot \psi^k$ . Hence we can find elements  $b'_1, b'_2,$  $\ldots, b'_v$  and  $c'_1, c'_2, \ldots, c'_{v-1}$  in  $K_{k+1}(\widetilde{W}, \partial \widetilde{W})$  uniquely determined by the property that  $b'_i$  is mapped to  $b_i$  and  $c'_i$  to  $c_i$  under the  $\mathbb{Z}\pi$ -homomorphism  $K_{k+1}(\widetilde{W}, \partial \widetilde{W}) \to$  $K_k(\widetilde{M})$ . Moreover, these elements form a  $\mathbb{Z}\pi$ -basis for  $K_{k+1}(\widetilde{W}, \partial \widetilde{W})$ , and the element  $\phi^{k+1}$  is mapped to  $b'_v$  under the  $\mathbb{Z}\pi$ -homomorphism  $K_{k+1}(\widetilde{W}, \partial \widetilde{W})$ .

#### 9.6 Formations

Define  $b''_i$  and  $c''_i$  for i = 1, 2, ..., (v - 1) to be the image of  $b'_i$  and  $c'_i$  under the  $\mathbb{Z}\pi$ -homomorphism  $K_{k+1}(\widetilde{W}, \widetilde{\partial W}) \to K_k(\widetilde{M'})$ . Then

$$\{b_i'' \mid i = 1, 2, \dots, (v-1)\} \bigsqcup \{c_i'' \mid i = 1, 2, \dots, (v-1)\}$$

is a  $\mathbb{Z}\pi$ -basis for  $K_k(\widetilde{M'})$ . One easily checks for the quadratic structure (s', t') on  $K_k(\widetilde{M'})$ 

```
 \begin{split} s'(b''_i,c''_i) &= s(b_i,c_i) = 1 & i \in \{1,2,\ldots,(v-1)\}; \\ s'(b''_i,c''_j) &= s(b_i,c_j) = 0 & i,j \in \{1,2,\ldots,(v-1)\}, i \neq j; \\ s'(b''_i,b''_j) &= s(b_i,b_j) = 0 & i,j \in \{1,2,\ldots,(v-1)\}; \\ s'(c''_i,c''_j) &= s(c_i,c_j) = 0 & i,j \in \{1,2,\ldots,(v-1)\}; \\ t'(b''_i) &= t(b_i) = 0 & i \in \{1,2,\ldots,(v-1)\}. \end{split}
```

This finishes the proof of assertion (i) of Theorem 9.74.

**Exercise 9.75.** Let *M* be a stably framed manifold of dimension (4k+2), i.e., a closed (4k + 2)-dimensional manifold together with a choice of a stable trivialization of its tangent bundle. Assign to it an element  $\alpha(M) \in \mathbb{Z}/2$  such that  $\alpha(M) = \alpha(N)$  depends only on the stably framed bordism class of *M*. (The easy solution that  $\alpha$  is constant is not what we have in mind.)

# 9.6 Formations

In this subsection we explain the algebraic objects, so-called formations, which describe the surgery obstruction and which will be the typical elements in the surgery obstruction group in odd dimensions. Throughout this section *R* will be an associative ring with involution and  $\epsilon \in \{\pm 1\}$ .

**Definition 9.76 (Formation).** An  $\epsilon$ -quadratic formation  $(P, \psi; F, G)$  is a nonsingular  $\epsilon$ -quadratic form  $(P, \psi)$  together with two lagrangians F and G.

An isomorphism  $f: (P, \psi; F, G) \to (P', \psi'; F, G')$  of  $\epsilon$ -quadratic formations is an isomorphism  $f: (P, \psi) \to (P', \psi')$  of non-singular  $\epsilon$ -quadratic forms such that f(F) = F' and f(G) = G' holds.

**Definition 9.77 (Trivial formation).** The *trivial*  $\epsilon$ -quadratic formation associated to a finitely generated projective *R*-module *P* is the formation  $(H_{\epsilon}(P); P, P^*)$ . A formation  $(P, \psi; F, G)$  is called *trivial* if it isomorphic to the trivial  $\epsilon$ -quadratic formation associated to some finitely generated projective *R*-module. Two formations are *stably isomorphic* if they become isomorphic after taking the direct sum with trivial formations.

**Remark 9.78 (Formations and automorphisms).** We conclude from Lemma 9.26 that any  $\epsilon$ -quadratic formation is isomorphic to an  $\epsilon$ -quadratic formation of the type  $(H_{\epsilon}(P); P, F)$  for some lagrangian  $F \subset P \oplus P^*$ . Given an automorphism

 $v: H_{\epsilon}(P) \xrightarrow{=} H_{\epsilon}(P)$  of the standard hyperbolic  $\epsilon$ -quadratic form  $H_{\epsilon}(P)$  for some finitely generated projective *R*-module *P*, we get a formation by  $(H_{\epsilon}(P); P, v(P))$ .

Consider an  $\epsilon$ -quadratic formation  $(P, \psi; F, G)$  such that P, F, and G are finitely generated free and suppose that R has the property that  $R^n$  and  $R^m$  are R-isomorphic if and only if n = m. Then  $(P, \psi; F, G)$  is stably isomorphic to  $(H_{\epsilon}(Q); Q, v(Q))$ for some finitely generated free R-module Q and automorphism v of  $H_{\epsilon}(Q)$  by the following argument. Because of Lemma 9.26 we can choose isomorphisms of non-singular  $\epsilon$ -quadratic forms  $f: H_{\epsilon}(F) \xrightarrow{\cong} (P, \psi)$  and  $g: H_{\epsilon}(G) \xrightarrow{\cong} (P, \psi)$  such that f(F) = F and g(G) = G. Since  $F \cong R^a$  and  $G \cong R^b$  by assumption and  $R^{2a} \cong F \oplus F^* \cong P \cong G \oplus G^* \cong R^{2b}$ , we conclude a = b. Hence we can choose an R-isomorphism  $u: F \to G$ . Then we obtain an automorphism of non-singular  $\epsilon$ -quadratic forms by the composite

$$v \colon H_{\epsilon}(F) \xrightarrow{H_{\epsilon}(u)} H_{\epsilon}(G) \xrightarrow{g} (P, \psi) \xrightarrow{f^{-1}} H_{\epsilon}(F)$$

and an isomorphism of  $\epsilon$ -quadratic formations

$$f: (H_{\epsilon}(F); F, v(F)) \xrightarrow{=} (P, \psi; F, G).$$

Recall that  $K_1(R)$  is defined in terms of automorphisms of finitely generated free *R*-modules. Hence it is plausible that the odd-dimensional *L*-groups will be defined in terms of formations, which is essentially the same as in terms of automorphisms of the standard hyperbolic form over a finitely generated free *R*-module.

**Definition 9.79 (Boundary formation).** Let  $(P, \psi)$  be a (not necessarily nonsingular)  $(-\epsilon)$ -quadratic form. Define its *boundary*  $\partial(P, \psi)$  to be the  $\epsilon$ -quadratic formation  $(H_{\epsilon}(P); P, \Gamma_{\psi})$  where  $\Gamma_{\psi}$  is the lagrangian given by the image of the *R*-homomorphism

$$P \to P \oplus P^*$$
,  $x \mapsto (x, (1 - \epsilon \cdot T)(\psi)(x))$ .

One easily checks that  $\Gamma_{\psi}$  appearing in Definition 9.79 is indeed a lagrangian. Two lagrangians *F*, *G* of a non-singular  $\epsilon$ -quadratic form  $(P, \psi)$  are called *complementary* if  $F \cap G = \{0\}$  and F + G = P.

**Lemma 9.80.** Let  $(P, \psi; F, G)$  be an  $\epsilon$ -quadratic formation. Then:

- (i)  $(P, \psi; F, G)$  is trivial if and only F and G are complementary to one another;
- (ii)  $(P, \psi; F, G)$  is isomorphic to a boundary if and only if there is a lagrangian  $L \subset P$  such that L is a complement of both F and G;
- (iii) There is an  $\epsilon$ -quadratic formation  $(P', \psi'; F', G')$  such that  $(P, \psi; F, G) \oplus (P', \psi'; F', G')$  is a boundary;
- (iv) An  $(-\epsilon)$ -quadratic form  $(Q, \mu)$  is non-singular if and only if its boundary is trivial.

*Proof.* See for instance [667, Lemma 9.13 on page 331].
9.8 The Surgery Obstruction in Odd Dimensions

# 9.7 Odd-Dimensional *L*-groups

Now we can define the odd-dimensional L-groups.

**Definition 9.81 (Odd-dimensional** *L*-groups). Let *R* be an associative ring with involution. For an odd integer n = 2k + 1 define the abelian group  $L_n(R)$ , called the *n*-th quadratic *L*-group, of *R* to be the abelian group of equivalence classes  $[P, \psi; F, G]$  of  $(-1)^k$ -quadratic formations  $(P, \psi; F, G)$  such that *P*, *F*, and *G* are finitely generated free *R*-modules with respect to the following equivalence relation. We call  $(P, \psi; F, G)$  and  $(P', \psi'; F', G')$  equivalent if and only if there exist  $(-(-1)^k)$ -quadratic forms  $(Q, \mu)$  and  $(Q', \mu')$  for finitely generated free *R*-modules *Q* and *Q'* and finitely generated free *R*-modules *S* and *S'* together with an isomorphism of  $(-1)^k$ -quadratic formations

$$(P,\psi;F,G) \oplus \partial(Q,\mu) \oplus (H_{\epsilon}(S);S,S^{*})$$
  
$$\cong (P',\psi';F',G') \oplus \partial(Q',\mu') \oplus (H_{\epsilon}(S');S',(S')^{*}).$$

Addition is given by the sum of two  $(-1)^k$ -quadratic formations. The zero element is represented by  $\partial(Q, \mu) \oplus (H_{(-1)^k}(S); S, S^*)$  for any  $(-(-1)^k)$ -quadratic form  $(Q, \mu)$ for any finitely generated free *R*-module *Q* and any finitely generated free *R*-module *S*. The inverse of  $[P, \psi; F, G]$  is represented by  $(P, -\psi; F', G')$  for any choice of lagrangians *F'* and *G'* in  $H_{\epsilon}(P)$  such that *F* and *F'* are complementary and *G* and *G'* are complementary.

A morphism  $u: R \to S$  of rings with involution induces homomorphisms  $u_*: L_k(R) \to L_k(S)$  for k = 1, 3 by induction satisfying  $(u \circ v)_* = u_* \circ v_*$  and  $(\mathrm{id}_R)_* = \mathrm{id}_{L_k(R)}$  for k = 1, 3. xs

**Theorem 9.82 (Vanishing of the odd-dimensional** *L*-groups of the ring of integers). We have  $L_{2k+1}(\mathbb{Z}) = 0$  for all  $k \in \mathbb{Z}$ .

*Proof.* See for instance [667, Subsection 9.2.4].

**Remark 9.83 (Four-periodicity of the** *L***-groups).** Obviously the *L*-groups are fourperiodic, i.e.,  $L_n(R) = L_{n+4k}(R)$  holds for all  $k, n \in \mathbb{Z}$ .

# 9.8 The Surgery Obstruction in Odd Dimensions

Next we very briefly treat the odd-dimensional surgery obstruction. Consider a normal map of degree one  $(\overline{f}, f): v(M) \to \xi$  covering  $f: M \to X$ , where *M* is a *w*-oriented closed manifold of dimension *n* and *X* is a *w*-oriented connected finite Poincaré complex of dimension *n* for odd n = 2k + 1. Put  $\pi = \pi_1(X)$ . To these data one can assign the *surgery obstruction* of  $(\overline{f}, f)$ 

(9.84) 
$$\sigma(\overline{f}, f) \in L_{2k+1}(\mathbb{Z}\pi, w).$$

Its construction and the proof of the following result can be found in [667, Section 9.3] or [987, Chapter 6].

**Theorem 9.85 (Surgery obstruction in odd dimensions).** We get under the conditions above:

- (i) Suppose k ≥ 2. Then σ(f, f) = 0 in L<sub>n</sub>(Zπ, w) if and only if we can do a finite number of surgery steps to obtain a normal map (f', f'): v(M') → ξ covering a homotopy equivalence f': M' → X;
- (ii) The surgery obstruction  $\sigma(\overline{f}, f)$  depends only on the normal bordism class of  $(\overline{f}, f)$ .

**Example 9.86 (The surgery obstruction in the simply connected case).** Consider a normal map of degree one  $(\overline{f}, f)$ :  $v(M) \rightarrow \xi$  covering  $f: M \rightarrow X$ , where *M* is a *w*-oriented connected closed manifold of dimension *n* and *X* is a *w*-oriented connected finite Poincaré complex of dimension *n*. Suppose that *X* is simply connected.

If *n* is odd,  $L_n(\mathbb{Z})$  is trivial and hence  $\sigma(f, f) = 0$ . In particular, we can arrange by finitely many surgery steps that the underlying map is a homotopy equivalence, provided  $n \ge 5$ .

If *n* is divisible by four, we obtain an isomorphism  $L_n(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$  by sending a quadratic form to its signature divided by eight, see Theorem 9.29. It turns out that under this isomorphism we get

$$\sigma(\overline{f}, f) = \frac{\operatorname{sign}(X) - \operatorname{sign}(M)}{8}.$$

Note that in this case the surgery obstruction depends only on M and X, but not on f and  $\overline{f}$ . This is not true in general.

If *n* is even, but not divisible by four, then the Arf invariant yields an isomorphism  $L_n(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$ . It turns out that  $\sigma(\overline{f}, f)$  depends not only on *f* but also on the bundle data  $\overline{f}$ . For instance, for different framings of  $T^2$  one obtains different invariants  $\alpha(T^2)$  in Exercise 9.75.

More details can be found in [667, Subsection 8.7.6].

# 9.9 Surgery Obstructions for Manifolds with Boundary

Next we deal with manifolds with boundary.

**Definition 9.87 (Poincaré pairs).** The notion of a Poincaré complex can be extended to pairs as follows. Let X be a connected finite *n*-dimensional CW-complex with fundamental group  $\pi$  together with a subcomplex  $A \subset X$  of dimension (n-1). Denote by  $\widetilde{A} \subset \widetilde{X}$  the preimage of A under the universal covering  $\widetilde{X} \to X$ . We call (X, A)a finite *n*-dimensional Poincaré pair with respect to the orientation homomorphism  $w: \pi_1(X) \to \{\pm 1\}$  if there is a fundamental class  $[X, A] \in H_n(X, A; \mathbb{Z}^w)$  such that the  $\mathbb{Z}\pi$ -chain maps  $-\cap [X, A] : C^{n-*}(\widetilde{X}, \widetilde{A}) \to C_*(\widetilde{X})$  and  $-\cap [X, A] : C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{A})$  are  $\mathbb{Z}\pi$ -chain homotopy equivalences.

We call (X, A) simple if the Whitehead torsions of these  $\mathbb{Z}\pi$ -chain homotopy equivalences vanish.

If  $A = \emptyset$ , we speak of a *simple Poincaré complex* 

If *M* is a connected compact manifold of dimension *n* with boundary  $\partial M$ , then  $(M, \partial M)$  is a simple finite *n*-dimensional Poincaré pair.

We want to extend the notion of a normal map from closed manifolds to manifolds with boundary. The underlying map f is a map of pairs  $(f, \partial f): (M, \partial M) \rightarrow (X, \partial X)$ , where M is a *w*-oriented compact manifold with boundary  $\partial M$  and  $(X, \partial X)$ is a *w*-oriented finite Poincaré pair, the degree of f is one and  $\partial f: \partial M \rightarrow \partial X$  is required to be a homotopy equivalence. The bundle data are unchanged, they consist of a vector bundle  $\xi$  over X and a bundle map  $\overline{f}: \nu(M) \rightarrow \xi$  covering f.

The notion of a normal bordism for manifolds with boundaries is rather complicated, but also obvious. We will at least explain what happens for the underlying spaces and maps. More details can be found in [667, Subsection 8.8.2].

Consider two normal maps in dimension *n* whose underlying maps are  $(f_m, \partial f_m)$ :  $(M_m, \partial M_m) \rightarrow (X_m, \partial X_m)$  such that  $\partial f_m$  is a homotopy equivalence. A normal bordism between them is defined a follows. As in the closed case *W* is a *w*-oriented compact (n + 1)-dimensional manifold with boundary  $\partial W$ , but now the boundary is the union of three pieces

$$\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$$

where  $\partial_m W$  is a codimension zero submanifold of  $\partial W$  possibly with non-empty boundary  $\partial \partial_m W$  for m = 0, 1, 2 satisfying

$$\partial_0 W \cap \partial_1 W = \emptyset;$$
  

$$\partial_2 W \cap \partial_m W = \partial \partial_m W \quad \text{for } m = 0, 1;$$
  

$$\partial \partial_2 W = \partial \partial_0 W \amalg \partial \partial_1 W.$$

We have an (n + 1)-dimensional finite Poincaré pair  $(Y, \partial Y)$  with a decomposition of  $\partial Y$  into three *n*-dimensional finite *CW*-subcomplexes

$$\partial Y = \partial_0 Y \cup \partial_1 Y \cup \partial_2 Y,$$

such that for appropriate (n-1)-dimensional finite *CW*-subcomplexes  $\partial \partial_m Y \subseteq \partial_m Y$  for m = 0, 1, 2 we have

$$\partial_0 Y \cap \partial_1 Y = \emptyset;$$
  

$$\partial_2 Y \cap \partial_m Y = \partial \partial_m Y \quad \text{for } m = 0, 1;$$
  

$$\partial \partial_2 Y = \partial \partial_0 Y \amalg \partial \partial_1 Y.$$

The map  $F: W \to Y$  is required to induce maps  $\partial_m F: \partial_m W \to \partial_m Y$  for m = 0, 1, 2and  $\partial_2 F: \partial_2 W \to \partial_2 Y$  is required to be a homotopy equivalence. The various identifications  $M_m \xrightarrow{\cong} \partial_m W$  and  $X_m \to \partial_m Y$  for m = 0, 1 in the closed case are now required to be identifications  $(M_m, \partial M_m) \xrightarrow{\cong} (\partial_m W, \partial \partial_m W)$  and  $(X_m, \partial X_m) \xrightarrow{\cong} (\partial_m Y, \partial Y_m)$  for m = 0, 1.

The definition and the main properties of the surgery obstruction carry over from normal maps for closed manifolds to normal maps for compact manifolds with boundary. The main reason is that we require  $\partial f : \partial M \rightarrow \partial X$  to be a homotopy equivalence so that the surgery kernels "do not feel the boundary". All arguments such as making a map highly connected by surgery steps and intersection pairings and self-intersection can be carried out in the interior of M without affecting the boundary. Thus we get the following, see [667, Theorem 8.189 on page 311 and Theorem 9.113 on page 389].

**Theorem 9.88.** (Surgery Obstruction for Manifolds with Boundary). Let  $(\overline{f}, f)$  be a normal map of degree one with underlying map  $(f, \partial f)$ :  $(M, \partial M) \rightarrow (X, \partial X)$  such that  $\partial f$  is a homotopy equivalence. Put  $n = \dim(M)$  and  $\pi = \pi_1(X)$ . Then:

(i) We can associate to it its surgery obstruction

$$\sigma(\overline{f}, f) \in L_n(\mathbb{Z}\pi, w);$$

- (ii) The surgery obstruction depends only on the normal bordism class of  $(\overline{f}, f)$ ;
- (iii) Suppose  $n \ge 5$ . Then  $\sigma(\overline{f}, f) = 0$  in  $L_n(\mathbb{Z}\pi, w)$  if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map  $(\overline{f'}, f')$  which covers a homotopy equivalence of pairs  $(f', \partial f'): (M', \partial M') \to (X, \partial X)$  with  $\partial M' = \partial M$  and  $\partial f' = \partial f$ .

More details can be found in [667, Sections 8.8 and 9.5].

## 9.10 Decorations

Next we want to modify the *L*-groups and the surgery obstruction so that the surgery obstruction is the obstruction to achieve a simple homotopy equivalence. This will force us to study *L*-groups with decorations.

### 9.10.1 *L*-groups with *K*<sub>1</sub>-Decorations

We begin with the *L*-groups. It is clear that this requires us to take equivalence classes of bases into account. Suppose that we have specified a subgroup  $U \subset K_1(R)$  such that *U* is closed under the involution on  $K_1(R)$  coming from the involution of *R* and contains the image of the change of rings homomorphism  $K_1(\mathbb{Z}) \to K_1(R)$ .

#### 9.10 Decorations

Two bases *B* and *B'* for the same finitely generated free *R*-module *V* are called *U*-equivalent if the change of basis matrix defines an element in  $K_1(R)$  which belongs to *U*. Note that the *U*-equivalence class of a basis *B* is unchanged if we permute the order of elements of *B*. We call an *R*-module *V U*-based if *V* is finitely generated free and we have chosen a *U*-equivalence class of bases.

Let V be a stably finitely generated free R-module. A *stable basis* for V is a basis B for  $V \oplus R^u$  for some integer  $u \ge 0$ . Denote for any integer v the direct sum of the basis B and the standard basis  $S^a$  for  $R^a$  by  $B \coprod S^a$ , which is a basis for  $V \oplus R^{u+a}$ . Let C be a basis for  $V \oplus R^v$ . We call the stable basis B and C *stably U-equivalent* if and only if there is an integer  $w \ge u, v$  such that  $B \coprod S^{w-u}$  and  $C \coprod S^{w-v}$  are U-equivalent basis. We call an R-module V *stably U-based* if V is stably finitely generated free and we have specified a stable U-equivalence class of stable basis for V.

Let *V* and *W* be stably *U*-based *R*-modules. Let  $f: V \oplus R^a \xrightarrow{\cong} W \oplus R^b$  be an *R*-isomorphism. Choose a non-negative integer *c* together with basis for  $V \oplus R^{a+c}$  and  $W \oplus R^{b+c}$  which represent the given stable *U*-equivalence classes of basis for *V* and *W*. Let *A* be the matrix of  $f \oplus id_{R^c}: V \oplus R^{a+c} \xrightarrow{\cong} W \oplus R^{b+c}$  with respect to these bases. It defines an element [A] in  $K_1(R)$ . Define the *U*-torsion

(9.89) 
$$\tau^U(f) \in K_1(R)/U$$

by the class represented by [A]. It is easy to prove that  $\tau^U(f)$  is independent of the choices of *c* and the basis and depends only on *f* and the stable *U*-basis for *V* and *W*. Moreover, one easily checks

$$\begin{aligned} \tau^U(g \circ f) &= \tau^U(g) + \tau^U(f); \\ \tau^U\begin{pmatrix} f & 0 \\ u & v \end{pmatrix} &= \tau^U(f) + \tau^U(v); \\ \tau^U(\mathrm{id}_V) &= 0, \end{aligned}$$

for *R*-isomorphisms  $f: V_0 \xrightarrow{\cong} V_1$ ,  $g: V_1 \xrightarrow{\cong} V_2$ , and  $v: V_3 \xrightarrow{\cong} V_4$  and an *R*-homomorphism  $u: V_0 \to V_4$  of stably *U*-based *R*-modules  $V_i$ . Let  $C_*$  be a contractible stably *U*-based finite *R*-chain complex, i.e., a contractible *R*-chain complex  $C_*$  of stably *U*-based *R*-modules which satisfies  $C_i = 0$  for |i| > N for some integer *N*. The definition of Whitehead torsion in (3.32) carries over to the definition of the *U*-torsion

(9.90) 
$$\tau^U(C_*) = [A] \in K_1(R)/U.$$

Analogously we can associate to an *R*-chain homotopy equivalence  $f: C_* \to D_*$  of stably *U*-based finite *R*-chain complexes its *U*-torsion, cf. (3.33),

(9.91) 
$$\tau^{U}(f_{*}) := \tau(\operatorname{cone}_{*}(f_{*})) \in K_{1}(R)/U.$$

We will consider stably U-based  $\epsilon$ -quadratic forms  $(P, \psi)$ , i.e., non-singular  $\epsilon$ -quadratic forms, whose underlying *R*-module *P* is a stably U-based *R*-module such that the U-torsion of the isomorphism  $(1 + \epsilon \cdot T)(\psi)$ :  $P \xrightarrow{\cong} P^*$  is zero in  $K_1(R)/U$ . An isomorphism  $f: (P, \psi) \to (P', \psi')$  of stably U-based  $\epsilon$ -quadratic forms is U-simple if the U-torsion of  $f: P \to P'$  vanishes in  $K_1(R)/U$ . Note that for a stably U-based *R*-module *P* the  $\epsilon$ -quadratic form  $H_{\epsilon}(P)$  is a stably U-based  $\epsilon$ -quadratic form. The sum of two stably U-based  $\epsilon$ -quadratic forms is again a stably U-based  $\epsilon$ -quadratic form. It is worthwhile to mention the following U-simple version of Lemma 9.26.

**Lemma 9.92.** Let  $(P, \psi)$  be a stably U-based  $\epsilon$ -quadratic form. Let  $L \subset P$  be a lagrangian such that L is a stably U-based R-module and the U-torsion of the following 2-dimensional stably U-based finite R-chain complex

$$0 \to L \xrightarrow{i} P \xrightarrow{i^* \circ (1 + \epsilon \cdot T)(\psi)} L^* \to 0$$

vanishes in  $K_1(R)/U$ . Then the inclusion  $i: L \to P$  extends to a U-simple isomorphism of stably U-based  $\epsilon$ -quadratic forms

$$H_{\epsilon}(L) \xrightarrow{\cong} (P, \psi).$$

Next we give the simple version of the even-dimensional *L*-groups.

**Definition 9.93 (Even-dimensional** *U*-decorated quadratic *L*-groups). Let *R* be an associative ring with involution. For  $\epsilon \in \{\pm 1\}$  define  $L_{1-\epsilon}^U(R)$  to be the abelian group of equivalence classes  $[P, \psi]$  of stably *U*-based non-singular  $\epsilon$ -quadratic forms  $(P, \psi)$  with respect to the following equivalence relation. We call  $(P, \psi)$  and  $(P', \psi')$ equivalent if and only if there exist integers  $u, u' \ge 0$  and a *U*-simple isomorphism of stably *U*-based non-singular  $\epsilon$ -quadratic forms

$$(P,\psi) \oplus H_{\epsilon}(R^{u}) \cong (P',\psi') \oplus H_{\epsilon}(R^{u'}).$$

Addition is given by the sum of two  $\epsilon$ -quadratic forms. The zero element is represented by  $[H_{\epsilon}(R^{u})]$  for any integer  $u \ge 0$ . The inverse of  $[P, \psi]$  is given by  $[P, -\psi]$ .

For an even integer *n* define the abelian group  $L_n^U(R)$ , called the *n*-th U-decorated quadratic L-group, of R by

$$L_n^U(R) := \begin{cases} L_0^U(R) & \text{if } n \equiv 0 \mod 4; \\ L_2^U(R) & \text{if } n \equiv 2 \mod 4. \end{cases}$$

A stably U-based  $\epsilon$ -quadratic formation  $(P, \psi; F, G)$  consists of an  $\epsilon$ -quadratic formation  $(P, \psi; F, G)$  such that  $(P, \psi)$  is a stably U-based  $\epsilon$ -quadratic form, the lagrangians F and G are stably U-based R-modules, and the U-torsion of the following two contractible stably U-based finite R-chain complexes

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$$0 \to F \xrightarrow{i} P \xrightarrow{i^* \circ (1 + \epsilon \cdot T)(\psi)} F^* \to 0$$

and

$$0 \to G \xrightarrow{j} P \xrightarrow{j^* \circ (1 + \epsilon \cdot T)(\psi)} G^* \to 0$$

vanish in  $K_1(R)/U$  where  $i: F \to P$  and  $j: G \to P$  denote the inclusions. An isomorphism  $f: (P, \psi; F, G) \to (P', \psi'; F', G')$  of U-based  $\epsilon$ -quadratic formations is U-simple if the U-torsion of the induced R-isomorphisms  $P \xrightarrow{\cong} P', F \xrightarrow{\cong} F'$ and  $G \xrightarrow{\cong} G'$  vanishes in  $K_1(R)/U$ . Note that for a U-stably based R-module P the trivial  $\epsilon$ -quadratic formation  $(H_{\epsilon}(P); P, (P)^*)$  has the structure of a stably based  $\epsilon$ -quadratic formation. Given a stably U-based  $(-\epsilon)$ -quadratic form  $(Q, \psi)$ , its boundary  $\partial(Q, \psi)$  is a stably U-based  $\epsilon$ -quadratic formation. Obviously the sum of two stably U-based  $\epsilon$ -quadratic formations is again a stably U-based  $\epsilon$ -quadratic formation. Next we give the simple version of the odd-dimensional L-groups.

**Definition 9.94 (Odd-dimensional** *U*-decorated quadratic *L*-groups). Let *R* be an associative ring with involution. For  $\epsilon \in \{\pm 1\}$  define  $L_{2-\epsilon}^U(R)$  to be the abelian group of equivalence classes  $[P, \psi; F, G]$  of stably *U*-based  $\epsilon$ -quadratic formations  $(P, \psi; F, G)$  with respect to the following equivalence relation. We call two stably *U*-based  $\epsilon$ -quadratic formations  $(P, \psi; F, G)$  and  $(P', \psi'; F', G')$  equivalent if and only if there exist stably *U*-based  $(-\epsilon)$ -quadratic forms  $(Q, \mu)$  and  $(Q', \mu')$  and nonnegative integers *u* and *u'* together with a *U*-simple isomorphism of stably *U*-based  $\epsilon$ -quadratic formations

$$\begin{aligned} (P,\psi;F,G) &\oplus \partial(Q,\mu) \oplus (H_{\epsilon}(R^{u});R^{u},(R^{u})^{*}) \\ &\cong (P',\psi';F',G') \oplus \partial(Q',\mu') \oplus (H_{\epsilon}(R^{u'});R^{u'},(R^{u'})^{*}). \end{aligned}$$

Addition is given by the sum of two stably *U*-based  $\epsilon$ -quadratic forms. The zero element is represented by  $\partial(Q, \mu) \oplus (H_{\epsilon}(R^{u}); R^{u}, (R^{u})^{*})$  for any stably *U*-based  $(-\epsilon)$ -quadratic form  $(Q, \mu)$  and non-negative integer *u*. The inverse of  $[P, \psi; F, G]$  is represented by  $(P, -\psi; F', G')$  for any choice of stably *U*-based lagrangians *F'* and *G'* in  $H_{\epsilon}(P)$  such that *F* and *F'* are complementary and *G* and *G'* are complementary and the *U*-torsion of the obvious isomorphism  $F \oplus F' \xrightarrow{\cong} P$  and  $G \oplus G' \xrightarrow{\cong} P$  vanishes in  $K_1(R)/U$ .

For an odd integer *n* define the abelian group  $L_n^U(R)$  called the *n*-th U-decorated quadratic L-group of R

$$L_n^U(R) := \begin{cases} L_1^U(R) & \text{if } n \equiv 1 \mod 4; \\ L_3^U(R) & \text{if } n \equiv 3 \mod 4. \end{cases}$$

**Notation 9.95.** Consider the case of a group ring  $R\pi$  with the *w*-twisted involution. For a group *G* denote by  $Wh_n^R(G)$  the *n*-th Whitehead group of *RG*, which is the (n - 1)-th homotopy group of the homotopy fiber of the assembly map  $BG_+ \wedge \mathbf{K}(R) \rightarrow \mathbf{K}(RG)$ . Then we define  $L_n^s(R\pi, w)$  by  $L_n^U(R\pi)$  for *U* the ker-

nel of the map  $K_1(R\pi) \to \operatorname{Wh}_1^R(\pi)$ . Observe that  $L_n^s(R\pi)$  depends on the pair  $(R,\pi)$ . Sometimes one denotes  $L_n^s(R\pi,w)$  also by  $L_n^{\langle 2 \rangle}(R\pi,w)$ .

If  $R = \mathbb{Z}\pi$  with the *w*-twisted involution, then  $U \subseteq K_1(\mathbb{Z}\pi)$  reduces to the abelian group  $V \subseteq K_1(\mathbb{Z}G)$  of elements of the shape  $(\pm g)$  for  $g \in \pi$ . So we get the *simple quadratic L*-groups

$$L_n^s(\mathbb{Z}\pi, w) = L_n^{\langle 2 \rangle}(\mathbb{Z}\pi, w) = L_n^V(\mathbb{Z}\pi, w).$$

### 9.10.2 The Simple Surgery Obstruction

Let  $(\overline{f}, f)$  be a normal map of degree one with  $(f, \partial f): (M, \partial M) \to (X, \partial X)$ as underlying map such that  $(X, \partial X)$  is a simple finite Poincaré complex and  $\partial f$ is a simple homotopy equivalence. Then the definition of the surgery obstruction appearing in Theorem 9.88 (i) can be modified to the simple setting. Note that the difference between the *L*-groups  $L_n^h(\mathbb{Z}\pi, w)$  and the simple *L*-groups  $L_n^s(\mathbb{Z}\pi, w)$  is the additional structure of a *U*-basis. The definition of the simple surgery obstruction

(9.96) 
$$\sigma(\overline{f}, f) \in L_n^s(\mathbb{Z}\pi, w)$$

is the same as the one appearing in Theorem 9.88 (i) except that we must explain how the various surgery kernels inherit a stable *U*-basis.

The elementary proof of the following lemma is left to the reader. Note that for any stably *U*-based *R*-module *V* and element  $x \in K_1(R)/U$  we can find another stable *U*-basis *C* for *V* such that the *U*-torsion  $\tau^U(\text{id}: (V, B) \to (V, C))$  is *x*. This is not true in the unstable setting. For instance, there exists a ring *R* with an element  $x \in K_1(R)/U$  for *U* the image of  $K_1(\mathbb{Z}) \to K_1(R)$  such that *x* cannot be represented by a unit in *R*, in other words *x* is not the *U*-torsion of any *R*-automorphism of *R*.

**Lemma 9.97.** Let  $C_*$  be a contractible finite stably free *R*-chain complex and *r* be an integer. Suppose that each chain module  $C_i$  with  $i \neq r$  comes with a stable *U*-basis. Then  $C_r$  inherits a preferred stable *U*-basis which is uniquely defined by the property that the *U*-torsion of  $C_*$  vanishes in  $K_1(R)/U$ .

We have the following version of Lemma 9.70

**Lemma 9.98.** If  $f: X \to Y$  is k-connected for n = 2k or n = 2k + 1, then  $K_k(M)$  is stably finitely generated free and inherits a preferred stable U-basis.

*Proof.* See [667, Lemma 10.27 (i) on page 403].

Next we can give the simple version of the surgery obstruction theorem. For its proof see for instance [667, Theorem 10.30 on page 404]. Note that simple normal bordism class means that in the definition of normal nullbordisms the pairs  $(Y, \partial Y)$ ,  $(\partial_0 Y, \partial_0 Y \cap \partial_1 Y)$ , and  $(\partial_1 Y, \partial_0 Y \cap \partial_1 Y)$  are required to be simple finite Poincaré pairs and the map  $\partial_2 F : \partial_2 M \rightarrow \partial_2 Y$  is required to be a simple homotopy equivalence.

**Theorem 9.99.** (Simple surgery obstruction for manifolds with boundary) Let  $(\overline{f}, f)$  be a normal map of degree one, whose underlying map is  $(f, \partial f)$ :  $(M, \partial M) \rightarrow (X, \partial X)$  such that  $(X, \partial X)$  is a simple finite Poincaré complex and  $\partial f$  is a simple homotopy equivalence. Put  $n = \dim(M)$  and  $\pi = \pi_1(X)$ . Then:

- (i) The simple surgery obstruction depends only on the simple normal bordism class of  $(\overline{f}, f)$ ;
- (ii) Suppose  $n \ge 5$ . Then  $\sigma(\overline{f}, f) = 0$  in  $L_n^s(\mathbb{Z}\pi, w)$  if and only if we can do a finite number of surgery steps on the interior of M leaving the boundary fixed to obtain a normal map  $(\overline{f'}, f'): vM' \to \xi$  which covers a simple homotopy equivalence of pairs  $(f', \partial f'): (M', \partial M') \to (X, \partial X)$  with  $\partial M' = \partial M$  and  $\partial f' = \partial f$ .

**Exercise 9.100.** Let *W* be a compact manifold of dimension *n* whose boundary is the disjoint union *M*  $\amalg N$ . Let  $(\overline{f}, f)$  be a normal map such that the underlying map of pairs is of the shape  $f: (W, \partial W) \rightarrow (X \times [0, 1], X \times \{0, 1\})$  for some closed manifold *X* and induces a simple homotopy equivalence  $\partial W \rightarrow X \times \{0, 1\}$ . Show that *M* and *N* are diffeomorphic provided that the simple surgery obstruction  $\sigma(\overline{f}, f)$  of (9.96) vanishes and  $n \ge 6$ .

## 9.10.3 Decorated L-Groups

*L*-groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in.

The *L*-group  $L_n(R)$  of Definitions 9.25 and 9.81 are also denoted by  $L_n^{(1)}(R)$  or by  $L_n^h(R)$ . If one works with finitely generated projective modules instead of finitely generated free *R*-modules in Definitions 9.25 and 9.81, one obtains *projec*tive quadratic *L*-groups  $L_n^p(R)$ , which are also denoted by  $L_n^{(0)}(R)$ . The negative decorations  $L_n^{(j)}(R)$  for  $j \in \mathbb{Z}$ ,  $j \leq -1$  can be obtained using suitable categories of modules parametrized over  $\mathbb{R}^k$ . There are forgetful maps  $L_n^{(j+1)}(R) \to L_n^{(j)}(R)$  for  $j \in \mathbb{Z}$ ,  $j \leq 0$ . The group  $L_n^{(-\infty)}(R)$  is defined as the colimit over these maps. For details the reader can consult [833, 840].

Let us summarize the decorations for integral group rings. We have already introduced  $L_n^s(\mathbb{Z}\pi, w) = L_n^{(2)}(\mathbb{Z}\pi, w)$  in Notation 9.95. We get

$$L_n^h(\mathbb{Z}\pi, w) = L_n^{\langle 1 \rangle}(\mathbb{Z}\pi, w) = L_n(\mathbb{Z}\pi, w);$$
$$L_n^p(\mathbb{Z}\pi, w) = L_n^{\langle 0 \rangle}(\mathbb{Z}\pi, w),$$

and have furthermore  $L_n^{\langle j \rangle}(\mathbb{Z}\pi)$  for  $j \in \mathbb{Z}, j \leq -1$  and  $L_n^{\langle -\infty \rangle}(\mathbb{Z}\pi)$ .

For the Farrell-Jones Conjecture we will have to take the decoration  $\langle -\infty \rangle$  where for applications the decorations *h* and *s* will be relevant. So we have to understand how one can compare them.

### 9.10.4 The Rothenberg Sequence

Next we explain how decorated *L*-groups can be computed from one another for a ring with involution. We have the long exact *Rothenberg sequence* [837, Proposition 1.10.1 on page 104], [840, 17.2] for  $j \in \{0, -1, -2, ...\}$  II  $\{-\infty\}$  and  $n \in \mathbb{Z}$ 

$$(9.101) \quad \dots \to L_n^{\langle j+1 \rangle}(R) \to L_n^{\langle j \rangle}(R) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R)) \\ \to L_{n-1}^{\langle j+1 \rangle}(R) \to L_{n-1}^{\langle j \rangle}(R) \to \dots .$$

Here  $\widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R))$  is the Tate-cohomology of the group  $\mathbb{Z}/2$  with coefficients in the  $\mathbb{Z}[\mathbb{Z}/2]$ -module  $\widetilde{K}_j(R)$ . The involution on  $\widetilde{K}_j(R)$  comes from the involution on R.

For a group ring  $R\pi$  with the *w*-twisted involution and elements *j* in  $\{1, 0, -1, ...\} \amalg \{-\infty\}$  and *n* in  $\mathbb{Z}$ , we get the long exact sequence

$$(9.102) \quad \dots \to L_n^{\langle j+1 \rangle}(R\pi, w) \to L_n^{\langle j \rangle}(R\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j^R(\pi)) \\ \to L_{n-1}^{\langle j+1 \rangle}(R\pi, w) \to L_{n-1}^{\langle j \rangle}(R\pi, w) \to \cdots.$$

Over the integral group ring  $Wh_1^{\mathbb{Z}}(\pi)$  agrees with  $Wh(\pi)$  and  $Wh_j^{\mathbb{Z}}(\pi)$  agrees with  $\widetilde{K}_j(\mathbb{Z}\pi)$  for  $j \leq 0$ . Hence (9.102) reduces for  $R = \mathbb{Z}G$  and  $j \leq 0$  to

$$(9.103) \quad \dots \to L_n^{\langle j+1 \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle j \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathbb{Z}\pi)) \\ \to L_{n-1}^{\langle j+1 \rangle}(\mathbb{Z}\pi, w) \to L_{n-1}^{\langle j \rangle}(\mathbb{Z}\pi, w) \to \cdots .$$

In particular, we get the long exact sequences

$$(9.104) \quad \dots \to L_n^{\langle h \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle p \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}\pi)) \\ \to L_{n-1}^{\langle h \rangle}(\mathbb{Z}\pi, w) \to L_{n-1}^{\langle p \rangle}(\mathbb{Z}\pi, w) \to \cdots.$$

Moreover, we have the long exact sequence

$$(9.105) \quad \dots \to L_n^{\langle s \rangle}(\mathbb{Z}\pi, w) \to L_n^{\langle h \rangle}(\mathbb{Z}\pi, w) \to \widehat{H}^n(\mathbb{Z}/2; Wh(\pi)) \\ \to L_{n-1}^{\langle s \rangle}(R) \to L_{n-1}^{\langle h \rangle}(R) \to \cdots .$$

**Theorem 9.106 (Independence of decorations).** Let G be a group such that Wh(G),  $\widetilde{K}_0(\mathbb{Z}G)$ , and  $K_n(\mathbb{Z}G)$  for all  $n \in \mathbb{Z}$ ,  $n \leq -1$  vanish. Then for every  $j \in \mathbb{Z}$ ,  $j \leq -1$  and every  $n \in \mathbb{Z}$  the forgetful maps induce isomorphisms

$$L_n^s(\mathbb{Z}G) \xrightarrow{\cong} L_n^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^p(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

Proof. This follows from the various Rothenberg sequences.

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**Exercise 9.107.** Show that for every group G, every  $j \in \mathbb{Z}$ ,  $j \leq -1$ , and every  $n \in \mathbb{Z}$  the forgetful maps induces isomorphisms after inverting 2

$$\begin{split} L^{s}(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^{h}(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^{p}(\mathbb{Z}G)[1/2] \\ \xrightarrow{\cong} L^{\langle j \rangle}(\mathbb{Z}G)[1/2] \xrightarrow{\cong} L^{\langle -\infty \rangle}(\mathbb{Z}G)[1/2]. \end{split}$$

### 9.10.5 The Shaneson Splitting

The Bass-Heller-Swan decomposition in K-theory, see Theorem 6.16, has the following analog for the algebraic L-groups.

**Theorem 9.108 (Shaneson splitting).** For every group G, every ring with involution R, every  $j \in \mathbb{Z}$ ,  $j \leq 2$ , and  $n \in \mathbb{Z}$ , there is a natural isomorphism

$$L_n^{\langle j \rangle}(RG) \oplus L_{n-1}^{\langle j-1 \rangle}(RG) \xrightarrow{\cong} L_n^{\langle j \rangle}(R[G \times \mathbb{Z}])$$

and we have the natural isomorphism

$$(9.109) L_n^{\langle -\infty \rangle}(RG) \oplus L_{n-1}^{\langle -\infty \rangle}(RG) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(R[G \times \mathbb{Z}]).$$

The map appearing in the theorem above comes from the map  $L_n^{\langle j \rangle}(RG) \rightarrow L_n^{\langle j \rangle}(R[G \times \mathbb{Z}])$  induced by the inclusion  $G \rightarrow G \times \mathbb{Z}$  and a map  $L_{n-1}^{\langle j-1 \rangle}(RG) \rightarrow L_n^{\langle j \rangle}(R[G \times \mathbb{Z}])$  which is essentially given by the cartesian product with  $S^1$ . The latter explains the raise from (n-1) to n. But why does the decoration raise from (j-1) to j? The reason is the product formula for Whitehead torsion, see Theorem 3.37 (iv). It predicts for any (not necessarily simple) homotopy equivalence  $f: X \rightarrow Y$  of finite *CW*-complexes that the homotopy equivalence  $f \times id_{S^1}: X \times S^1 \rightarrow Y \times S^1$  is a simple homotopy equivalence. There is also a product formula for the finiteness obstruction which predicts for a finitely dominated (not necessarily up to homotopy finite) *CW*-complex *X* that  $X \times S^1$  is homotopy equivalent to a finite *CW*-complex. The original proof of the Shaneson splitting for the case j = 2 and  $R = \mathbb{Z}$  i.e., for the isomorphism

$$L_n^s(\mathbb{Z}G) \oplus L_{n-1}^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^s(\mathbb{Z}[G \times \mathbb{Z}])$$

can be found in [913]. The proof for arbitrary j and R is given in [840, 17.2]. Note that for j = 1 we obtain an isomorphism

(9.110) 
$$L_n^h(RG) \oplus L_{n-1}^p(RG) \xrightarrow{\cong} L_n^h(R[G \times \mathbb{Z}])$$

**Remark 9.111 (UNil-groups).** Even though in the Shaneson splitting (9.109) above there are no terms analogous to the Nil-terms in the Bass-Heller-Swan decomposition in *K*-theory, see Theorem 6.16, such Nil-phenomena do also occur in *L*-theory, as

soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They are (not necessarily finitely generated) 2-primary abelian groups see [204]. For more information about the UNil-groups we refer for instance to [63, 201, 202, 254, 255, 258, 352, 841]. How the Farrell-Jones Conjecture predicts exact Mayer-Vietoris sequences for amalgamated free products after inverting 2 is explained in Section 15.7.

**Exercise 9.112.** Compute  $L_n^{\langle j \rangle}(\mathbb{Z}[\mathbb{Z}])$ .

# 9.11 The Farrell-Jones Conjecture for Algebraic *L*-Theory for Torsionfree Groups

The Farrell-Jones Conjecture for algebraic *L*-theory, which will later be formulated in full generality in Chapter 13, reduces for a torsionfree group to the following conjecture. Given a ring with involution *R*, there exists an *L*-spectrum associated to *R* with decoration  $\langle -\infty \rangle$ 

$$(9.113) L^{\langle -\infty \rangle}(R)$$

with the property that  $\pi_n(\mathbf{L}^{\langle -\infty \rangle}(R)) = L_n^{\langle -\infty \rangle}(R)$  holds for  $n \in \mathbb{Z}$ .

### Conjecture 9.114 (Farrell-Jones Conjecture for L-theory for torsionfree groups).

Let G be a torsionfree group. Let R be any ring with involution.

Then the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for all  $n \in \mathbb{Z}$ .

We get for every  $j \in \{1, 0, -1, ...\} \amalg \{-\infty\}$ 

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j \rangle}(R)) \cong H_n(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(R)) \oplus H_{n-1}(\{\bullet\}; \mathbf{L}^{\langle j \rangle}(R))$$
$$\cong L_n^{\langle j \rangle}(R) \oplus L_{n-1}^{\langle j \rangle}(R).$$

In view of the Shaneson splitting of Theorem 9.108 it is now obvious why we have passed to the decoration  $j = -\infty$  in Conjecture 9.114.

**Exercise 9.115.** Let  $F_g$  be the closed orientable surface of genus g. Compute  $L_n^{\langle j \rangle}(\mathbb{Z}[\pi_1(F_g)])$  for all  $j \in \mathbb{Z}, j \leq 2$ , and  $n \in \mathbb{Z}$  using the fact that Conjecture 9.114 holds for  $G = \pi_1(F_g)$ .

Lemma 9.116. Let X be a CW-complex.

(i) If X is finite and we localize at the prime 2, we obtain a natural isomorphism

$$H_n(X; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))_{(2)} \cong \prod_{j \in \mathbb{Z}} (H_{n+4j}(X; \mathbb{Z}_{(2)}) \times H_{n+4j-2}(X; \mathbb{Z}/2));$$

(ii) If we invert 2, we obtain a natural isomorphism

$$H_n(X; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))[1/2] \cong KO_n(X)[1/2].$$

*Proof.* (i) The *L*-theory spectrum  $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})_{(2)}$  localized at (2) is an infinite product of Eilenberg-MacLane spectra by [946, Theorem A].

(ii) This follows from the more general case discussed in Subsection 15.14.4, which is based on [614, 615].  $\Box$ 

# 9.12 The Surgery Exact Sequence

In this section we introduce the Surgery Exact Sequence. It is the realization of the Surgery Program, which we have explained in Remark 3.53. The Surgery Exact Sequence is the main theoretical tool in solving the classification problem of manifolds of dimensions greater than or equal to five.

### 9.12.1 The Structure Set

**Definition 9.117 (Simple structure set).** Let *X* be a closed manifold of dimension *n*. We call two simple homotopy equivalences  $f_i: M_i \to X$  from closed manifolds  $M_i$  of dimension *n* to *X* for i = 0, 1 equivalent if there exists a diffeomorphism  $g: M_0 \to M_1$  such that  $f_1 \circ g$  is homotopic to  $f_0$ . The *simple structure set*  $S_n^s(X)$  of *X* is the set of equivalence classes of simple homotopy equivalences  $M \to X$  from closed manifolds of dimension *n* to *X*. This set has a preferred base point, namely, the class of the identity id:  $X \to X$ .

The simple structure set  $S_n^s(X)$  is the basic object in the study of manifolds which are diffeomorphic to X. Note that a simple homotopy equivalence  $f: M \to X$  is homotopic to a diffeomorphism if and only if it represents the base point in  $S_n^s(X)$ . A manifold M is diffeomorphic to X if and only if for some simple homotopy equivalence  $f: M \to X$  the class of [f] agrees with the preferred base point. Some care is necessary since it may be possible that a given simple homotopy equivalence  $f: M \to X$  is not homotopic to a diffeomorphism, although M and X are diffeomorphic. Hence it does not suffice to compute  $S_n^s(X)$ , one also has to understand the operation of the group of homotopy classes of simple self-equivalences of X on  $S_n^s(X)$ . This can be rather complicated in general. But it will be no problem in the case  $X = S^n$  because any self-homotopy equivalence  $S^n \to S^n$  is homotopic to a self-diffeomorphism.

There is also a version of the structure set which does not take Whitehead torsion into account.

**Definition 9.118 (Structure set).** Let *X* be a closed manifold of dimension *n*. We call two homotopy equivalences  $f_i: M_i \to X$  from closed manifolds  $M_i$  of dimension *n* to *X* for i = 0, 1 equivalent if there is a manifold triad  $(W; \partial_0 W, \partial_1 W)$  with  $\partial_0 W \cap$  $\partial_1 W = \emptyset$  and a homotopy equivalence of triads  $(F; \partial_0 F, \partial_1 F): (W; \partial_0 W, \partial_1 W) \to$  $(X \times [0, 1]; X \times \{0\}, X \times \{1\})$  together with diffeomorphisms  $g_0: M_0 \to \partial_0 W$  and  $g_1: M_1 \to \partial_1 W$  satisfying  $\partial_i F \circ g_i = f_i$  for i = 0, 1. The *structure set*  $S_n^h(X)$  of *X* is the set of equivalence classes of homotopy equivalences  $M \to X$  from a closed manifold *M* of dimension *n* to *X*. This set has a preferred base point, namely, the class of the identity id:  $X \to X$ .

**Remark 9.119** (The simple structure set and *s*-cobordisms). If we require in Definition 9.118 the homotopy equivalences F,  $f_0$ , and  $f_1$  to be simple homotopy equivalences, we get the simple structure set  $S_n^s(X)$  of Definition 9.117, provided that  $n \ge 5$ . We have to show that the two equivalence relations are the same. This follows from the *s*-Cobordism Theorem 3.47. Namely, W appearing in Definition 9.118 is an *h*-cobordism and is even an *s*-cobordism if we require F,  $f_0$ , and  $f_1$  to be simple homotopy equivalences, see Theorem 3.37. Hence there is a diffeomorphism  $\Phi: \partial_0 W \times [0, 1] \to W$  inducing the obvious identification  $\partial_0 W \times \{0\} \to \partial_0 W$  and some diffeomorphism  $\phi_1: (\partial_0 W) = (\partial_0 W \times \{1\}) \to \partial_1 W$ . Then  $\phi: M_0 \to M_1$  given by  $g_1^{-1} \circ \phi_1 \circ g_0$  is a diffeomorphism such that  $f_1 \circ \phi$  is homotopic to  $f_0$ . The other implication is obvious.

### 9.12.2 Realizability of Surgery Obstructions

In this section we explain that any element in the *L*-group  $L_n(\mathbb{Z}\pi, w)$  for  $n \ge 5$  can be realized as the surgery obstruction of a normal map  $(\overline{f}, f)$  covering a map  $(f, \partial f): (M, \partial M) \to (X, \partial X)$  of compact manifolds if we require that X has nonempty boundary  $\partial X$  and that  $\partial f$  is a (simple) homotopy equivalence.

**Theorem 9.120 (Realizability of the surgery obstruction).** Suppose  $n \ge 5$ . Consider a w-oriented connected compact manifold X with non-empty boundary  $\partial X$ . Let  $\pi$  be its fundamental group and let  $w : \pi \to \{\pm 1\}$  be its orientation homomorphism. Consider an element  $x \in L_n(\mathbb{Z}\pi, w)$ .

Then we can find a normal map of degree one  $(\overline{f}, f)$  covering a map of triads

$$f = (f; \partial_0 f, \partial_1 f) \colon (M; \partial_0 M, \partial_1 M) \to (X \times [0, 1]; X \times \{0\} \cup \partial X \times [0, 1], X \times \{1\})$$

with the following properties:

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- (i)  $\partial_0 f$  is a diffeomorphism and  $f|_{\partial_0 M}$  is a stabilization of  $T(\partial f_0)$ ;
- (ii)  $\partial_1 f$  is a homotopy equivalence;
- (iii) The surgery obstruction  $\sigma(\overline{f}, f)$  in  $L_n(\mathbb{Z}\pi, w)$ , see (9.84), is the given element x.

The analogous statement holds for  $x \in L_n^s(\mathbb{Z}\pi, w)$  if we require  $\partial_1 f$  to be a simple homotopy equivalence and we consider the simple surgery obstruction, see (9.96).

*Proof.* See [667, Theorem 8.195 on page 317 and Theorem 9.115 on page 390]. □

**Remark 9.121 (Surgery obstructions of closed manifolds).** It is not true that for any *w*-oriented closed manifold *N* of dimension *n* with fundamental group  $\pi$  and orientation homomorphism  $w: \pi \to \{\pm 1\}$  and any element  $x \in L_n(\mathbb{Z}\pi, w)$  there is a normal map  $(\overline{f}, f)$  covering a map of *w*-oriented closed manifolds  $f: M \to N$  of degree one such that  $\sigma(\overline{f}, f) = x$ . Note that in Theorem 9.120 the target manifold  $X \times [0, 1]$  is not closed. The same remark holds for  $L_n^s(\mathbb{Z}\pi, w)$ . These questions are discussed in in [454, 459, 720, 721], see also [667, Remark 8.199 on page 321 and Remark 9.117 on page 392].

### 9.12.3 The Surgery Exact Sequence

Now we can establish one of the main tools in the classification of manifolds, the Surgery Exact Sequence. We have already extended the notion of a normal map for closed manifolds to manifolds with boundary and explained the notion of a normal bordism for normal maps of pairs in Section 9.9. In this Subsection 9.12.3, we will consider only normal maps with the same target  $(X, \partial X)$ , whose underlying maps are diffeomorphisms on the boundary. We call two of them with the same target normally bordant if there is a normal bordism between them in the sense of Definition 9.59, whose underlying map induces a diffeomorphism  $\partial_1 W \rightarrow \partial X \times [0, 1]$ .

**Definition 9.122.** Let  $(X, \partial X)$  be a *w*-oriented compact manifold of dimension *n* with boundary  $\partial X$ . Define the *set of normal maps* to  $(X, \partial X)$ 

$$\mathcal{N}_n(X,\partial X)$$

to be the set of normal bordism classes of normal maps of degree one  $(\overline{f}, f)$  with underlying map  $(f, \partial f): (M, \partial M) \to (X, \partial X)$  for which  $\partial f: \partial M \to \partial X$  is a diffeomorphism.

Let *X* be a closed *w*-oriented connected manifold of dimension  $n \ge 5$ . Denote by  $\pi$  its fundamental group and by  $w: \pi \to \{\pm 1\}$  its orientation homomorphism. Let  $\mathcal{N}_{n+1}(X \times [0, 1], X \times \{0, 1\})$  and  $\mathcal{N}_n(X)$  be the set of normal maps of degree one as introduced in Definition 9.122. Let  $\mathcal{S}_n^s(X)$  be the structure set of Definition 9.117. Denote by  $L_n^s(\mathbb{Z}\pi, w)$  the simple surgery obstruction group, see Notation 9.95. Denote by

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(9.123) 
$$\sigma_{n+1}^s \colon \mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \to L^s_{n+1}(\mathbb{Z}\pi, w);$$

(9.124) 
$$\sigma_n^s \colon \mathcal{N}_n(X) \to L_n^s(\mathbb{Z}\pi, w),$$

the maps that assign to the normal bordism class of a normal map of degree one its simple surgery obstruction, see (9.96). This is well-defined by Theorem 9.88 (ii). Let

(9.125) 
$$\eta_n^s \colon \mathcal{S}_n^s(X) \to \mathcal{N}_n(X)$$

be the map that sends the class  $[f] \in S_n^s(X)$  represented by a simple homotopy equivalence  $f: M \to X$  to the normal bordism class of the following normal map of degree one. Choose a homotopy inverse  $f^{-1}: X \to M$  and a homotopy  $h: \operatorname{id}_M \simeq f^{-1} \circ f$ . Put  $\xi = (f^{-1})^*TM$ . Up to isotopy of bundle maps there is precisely one bundle map  $(\overline{h}, h): TM \times [0, 1] \to TM$  covering  $h: M \times [0, 1] \to M$ whose restriction to  $TM \times \{0\}$  is the identity map  $TM \times \{0\} \to TM$ . The restriction of  $\overline{h}$  to  $X \times \{1\}$  induces a bundle map  $\overline{f}: TM \to \xi$  covering  $f: M \to X$ . Put  $\eta([f]) := [(\overline{f}, f)]$ . One easily checks that the normal bordism class of  $(\overline{f}, f)$ depends only on  $[f] \in S_n^s(X)$  and hence that  $\eta$  is well-defined.

Next we define an action of the abelian group  $L^s_{n+1}(\mathbb{Z}\pi, w)$  on the structure set  $S^s_n(X)$ 

(9.126) 
$$\rho_n^s \colon L^s_{n+1}(\mathbb{Z}\pi, w) \times \mathcal{S}^s_n(X) \to \mathcal{S}^s_n(X).$$

Fix  $x \in L_{n+1}^{s}(\mathbb{Z}\pi, w)$  and  $[f] \in S_{n}^{s}(X)$  represented by a simple homotopy equivalence  $f: M \to X$ . By Theorem 9.120 we can find a normal map  $(\overline{F}, F)$  covering a map of triads  $(F; \partial_{0}F, \partial_{1}F): (W; \partial_{0}W, \partial_{1}W) \to (M \times [0, 1]; M \times \{0\}, M \times \{1\})$  such that  $\partial_{0}F$  is a diffeomorphism,  $\partial_{1}F$  is a simple homotopy equivalence, and  $\sigma(\overline{F}, F) = x$ . Then define  $\rho_{n}^{s}(x, [f])$  by the class  $[f \circ \partial_{1}F: \partial_{1}W \to X]$ . We have to show that this is independent of the choice of  $(\overline{F}, F)$ . Let  $(\overline{F'}, F')$  be a second choice. We can glue W' and  $W^{-}$  together along the diffeomorphism  $(\partial_{0}F)^{-1} \circ \partial_{0}F': \partial_{0}W' \to \partial_{0}W$  and obtain a normal bordism from  $(\overline{F}|_{\partial_{1}W}, \partial_{1}F)$  to  $(\overline{F'}|_{\partial_{1}W'}, \partial_{1}F')$ . The simple obstruction of this normal bordism is

$$\sigma(\overline{F'}, F') - \sigma(\overline{F}, F) = x - x = 0.$$

Because of Theorem 9.99 (ii) we can perform surgery relative boundary on this normal bordism to arrange that the reference map from it to  $X \times [0, 1]$  is a simple homotopy equivalence. In view of Remark 9.119 this shows that  $f \circ \partial_1 F$  and  $f \circ \partial_1 F'$  define the same element in  $S_n^s(X)$ . One easily checks that this defines a group action, since the surgery obstruction is additive under stacking normal bordisms together. The next result is the main result of this chapter and follows from the definitions and Theorem 9.99 (ii).

**Theorem 9.127 (The Surgery Exact Sequence).** Let X be a closed w-oriented connected manifold of dimension  $n \ge 5$ . Then, in the notation above, the so-called Surgery Exact Sequence

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$$\mathcal{N}_{n+1}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^s} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^s} \mathcal{S}_n^s(X) \xrightarrow{\eta_n^s} \mathcal{N}_n(X) \xrightarrow{\sigma_n^s} L_n^s(\mathbb{Z}\pi, w)$$

is exact for  $n \ge 5$  in the following sense. An element  $z \in N_n(X)$  lies in the image of  $\eta_n^s$ if and only if  $\sigma_n^s(z) = 0$ . Two elements  $y_1, y_2 \in S_n^s(X)$  have the same image under  $\eta_n^s$ if and only if there exists an element  $x \in L_{n+1}^s(\mathbb{Z}\pi, w)$  with  $\rho_n^s(x, y_1) = y_2$ . For two elements  $x_1, x_2$  in  $L_{n+1}^s(\mathbb{Z}\pi)$  we have  $\rho_n^s(x_1, [\text{id}: X \to X]) = \rho_n^s(x_2, [\text{id}: X \to X])$ if and only if there is a  $u \in N_{n+1}(X \times [0, 1], X \times \{0, 1\})$  with  $\sigma_{n+1}^s(u) = x_1 - x_2$ .

There is an analogous Surgery Exact Sequence

$$\mathcal{N}_{n+1}^h(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^h} L_{n+1}^h(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^h} \mathcal{S}^h(X) \xrightarrow{\eta_n^h} \mathcal{N}_n(X) \xrightarrow{\sigma_n^h} L_n^h(\mathbb{Z}\pi, w),$$

where  $S^h(X)$  is the structure set of Definition 9.118 and  $L_n^h(\mathbb{Z}\pi, w) := L_n(\mathbb{Z}\pi, w)$  has been introduced in Definitions 9.25 and 9.81.

**Remark 9.128** (Extending the Surgery Exact Sequence to the left). The Surgery Exact Sequence of Theorem 9.127 can be extended to infinity to the left. In the range far enough to the left it is a sequence of abelian groups.

# 9.13 Surgery Theory in the PL and in the Topological Category

One can also develop surgery theory in the PL (=piecewise linear) category or in the topological category [579]. This requires us to carry over the notions of vector bundles and tangent bundles to these categories. There are analogs of the sets of normal invariants  $\mathcal{N}_n^{\text{PL}}(X)$  and  $\mathcal{N}_n^{\text{TOP}}(X)$  and the structure sets  $\mathcal{S}_n^{\text{PL},h}(X)$ ,  $\mathcal{S}_n^{\text{PL},s}(X)$ ,  $\mathcal{S}_n^{\text{TOP},h}(X)$ , and  $\mathcal{S}_n^{\text{TOP},s}(X)$ . There are analogs PL and TOP of the group O = colim\_{n\to\infty} O\_n. The topological group TOP is the limit of the groups TOP(k)that are the groups of homeomorphisms of  $\mathbb{R}^k$  fixing the origin:

$$TOP = \operatorname{colim}_{k \to \infty} TOP(k).$$

The definition of PL is more elaborate and therefore omitted. Let  $G = \operatorname{colim}_{n \to \infty} G(n)$ where G(n) is the monoid of self-homotopy equivalences of  $S^n$ . There are classifying spaces BPL (resp. BTOP), which classify stable isomorphism classes of PL (resp. TOP)  $\mathbb{R}^k$  bundles and which are infinite loop spaces with multiplication corresponding to the Whitney sum of bundles. The space BG is the classifying space for spherical fibrations. There are also canonical maps BPL  $\rightarrow$  BG (resp. BTOP  $\rightarrow$  BG) which classify the passage to strong fiber homotopy equivalence classes of stable spherical fibrations. The homotopy fibers of these maps are denoted G/PL (resp. G/TOP) and have infinite loop space structures so that the canonical maps G/PL  $\rightarrow$  BPL and  $G/TOP \rightarrow BTOP$  are maps of infinite loop spaces. Define G/O as the homotopy fiber of the map  $BO \rightarrow BG$ .

**Theorem 9.129 (The set of normal maps and** G/O, G/PL, and G/TOP). *Let* X *be* a connected compact n-dimensional manifold with (possibly empty) boundary  $\partial X$ . (*In the sequel, if*  $\partial X = \emptyset$ , by  $X/\partial X$  we mean the space X itself.)

Then there exists a canonical group structure on the set  $[X/\partial X, G/O]$ ,  $[X/\partial X, G/PL]$ , or  $[X/\partial X, G/TOP]$  respectively, and a transitive free operation of this group on  $\mathcal{N}_n(X, \partial X)$ ,  $\mathcal{N}_n^{PL}(X, \partial X)$ , or  $\mathcal{N}_n^{TOP}(X, \partial X)$  respectively. In particular, we get bijections

$$\begin{split} & [X/\partial X, \mathsf{G}/\mathsf{O}] \xrightarrow{\cong} \mathcal{N}_n(X, \partial X); \\ & [X/\partial X, \mathsf{G}/\mathsf{PL}] \xrightarrow{\cong} \mathcal{N}_n^{\mathsf{PL}}(X, \partial X); \\ & [X/\partial X, \mathsf{G}/\mathsf{TOP}] \xrightarrow{\cong} \mathcal{N}_n^{\mathsf{TOP}}(X, \partial X), \end{split}$$

respectively.

There are analogs of the Surgery Exact Sequence, see Theorem 9.127, for the PL category and the topological category.

**Theorem 9.130 (The Surgery Exact Sequence for the** PL and the topological category). Let X be a closed w-oriented connected PL manifold of dimension  $n \ge 5$ . Then there is a Surgery Exact Sequence

$$\mathcal{N}_{n+1}^{\mathsf{PL}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^s} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^s} \mathcal{S}_n^{\mathsf{PL},s}(X) \xrightarrow{\eta_n^s} \mathcal{N}_n^{\mathsf{PL}}(X) \xrightarrow{\sigma_n^s} L_n^s(\mathbb{Z}\pi, w)$$

which is exact for  $n \ge 5$  in the sense of Theorem 9.127. There is an analogous Surgery Exact Sequence

$$\mathcal{N}_{n+1}^{\mathsf{PL}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^h} L_{n+1}^h(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^h} \mathcal{S}_n^{\mathsf{PL},h}(X) \xrightarrow{\eta_n^h} \mathcal{N}_n^{\mathsf{PL}}(X) \xrightarrow{\sigma_n^h} L_n^h(\mathbb{Z}\pi, w)$$

The analogous sequences exists in the topological category. Namely, for a closed w-oriented connected topological manifold X of dimension  $n \ge 5$ , there is a Surgery Exact Sequence

$$\mathcal{N}_{n+1}^{\mathsf{TOP}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^s} L_{n+1}^s(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^s} \mathcal{S}_n^{\mathsf{TOP},s}(X) \\ \xrightarrow{\eta_n^s} \mathcal{N}_n^{\mathsf{TOP}}(X) \xrightarrow{\sigma_n^s} L_n^s(\mathbb{Z}\pi, w)$$

which is exact for  $n \ge 5$  in the sense of Theorem 9.127, and an analogous Surgery *Exact Sequence* 

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$$\mathcal{N}_{n+1}^{\mathsf{TOP}}(X \times [0,1], X \times \{0,1\}) \xrightarrow{\sigma_{n+1}^h} L_{n+1}^h(\mathbb{Z}\pi, w) \xrightarrow{\partial_{n+1}^h} \mathcal{S}_n^{\mathsf{TOP},h}(X) \xrightarrow{\eta_n^h} \mathcal{N}_n^{\mathsf{TOP}}(X) \xrightarrow{\sigma_n^h} L_n^h(\mathbb{Z}\pi, w)$$

Note that the surgery obstruction groups are the same in the smooth category, PL category, and topological category. Only the set of normal invariants and the structure sets are different. The set of normal invariants in the smooth category, PL category, or topological category do not depend on the decoration h and s, whereas the structure sets and the surgery obstruction groups depend on the decorations h and s. In particular, the structure set depends on both the choice of category and choice of decoration.

As in the smooth setting, the Surgery Exact Sequence above can be extended to infinity to the left.

Some interesting constructions can be carried out in the topological category, which do not have smooth counterparts.

**Remark 9.131 (The total surgery obstruction).** Given a finite Poincaré complex *X* of dimension  $\geq$  5, a single obstruction, the so-called *total surgery obstruction*, is constructed in [839, §17], see also [596]. It vanishes if and only if *X* is homotopy equivalent to a closed topological manifold. It combines the two stages of the classical obstructions, namely, the problem whether the Spivak normal fibration has a reduction to a TOP-bundle (which is equivalent to the condition that  $N^{\text{TOP}}(X)$  is non-empty) and whether the surgery obstruction of the associated normal map is trivial.

**Remark 9.132 (Group structures on the Surgery Exact Sequence).** An algebraic Surgery Exact Sequence is constructed in [839, § 14, § 18] and identified with the geometric Surgery Exact Sequence above in the topological category. Moreover, in the topological situation one can find abelian group structures on  $S_n^{\text{TOP},s}(X)$ ,  $S_n^{\text{TOP},h}(X)$  and  $\mathcal{N}_n^{\text{TOP}}(X)$  such that the surgery sequence becomes a sequence of abelian groups. The main point is to find the right infinite loop space structure on G/TOP.

There cannot be a group structure in the smooth category for  $S_n^h(X)$  and  $\mathcal{N}_n(X)$  such that  $S_n^h(X) \xrightarrow{\eta} \mathcal{N}_n(X) \xrightarrow{\sigma} L_n^h(\mathbb{Z}\pi, w)$  is a sequence of groups (and analogous for the simple version), see [263]. Note that the composite, see Theorem 9.129,

$$[X, G/O] \cong \mathcal{N}_n(X) \xrightarrow{\sigma_n^s} L_n^s(\mathbb{Z}\pi, w)$$

is a map whose source and target come with canonical group structures but it is not a homomorphism of abelian groups in general, see [987, page 114]. The same problem arises with the decoration h. More information about this topic can be found for instance in [667, Sections 11.8 and 17.6].

**Remark 9.133** (The homotopy type of G/TOP and TOP/PL). The computation of the homotopy type of the space G/TOP (and also of G/PL) due to Sullivan [933] is explained in detail in [697, Chapter 4]. One obtains homotopy equivalences

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$$\begin{array}{l} \text{G/TOP}\left[\frac{1}{2}\right] \simeq BO\left[\frac{1}{2}\right];\\\\ \text{G/TOP}_{(2)} \simeq \prod_{j\geq 1} K(\mathbb{Z}_{(2)}, 4j) \times \prod_{j\geq 1} K(\mathbb{Z}/2, 4j-2). \end{array}$$

where K(A, i) denotes the *Eilenberg-MacLane space* of type (A, i), i.e., a *CW*-complex such that  $\pi_n(K(A, i))$  is trivial for  $n \neq i$  and is isomorphic to A if n = i, the subscript  $_{(2)}$  stands for localizing at (2), i.e., all primes except 2 are inverted, and  $\lfloor \frac{1}{2} \rfloor$  stands for localization of 2, i.e. 2 is inverted. In particular, we get for a space X isomorphisms

$$\begin{split} & [X, \mathsf{G}/\mathsf{TOP}] \left[ \frac{1}{2} \right] \cong \widetilde{KO}^0(X) \left[ \frac{1}{2} \right]; \\ & [X, \mathsf{G}/\mathsf{TOP}]_{(2)} \cong \prod_{j \ge 1} H^{4j}(X; \mathbb{Z}_{(2)}) \times \prod_{j \ge 1} H^{4j-2}(M; \mathbb{Z}/2), \end{split}$$

where  $KO^*$  is K-theory of real vector bundles, see Subsection 10.2.2.

The various groups G, TOP, and PL, and their (homotopy-theoretic) quotients G/PL, PL/O, and G/PL fit into a braid by inspecting long exact sequences of fibrations. This braid can be interpreted geometrically in terms of *L*-groups, bordism groups, and homotopy groups of exotic spheres in dimensions  $\geq 5$ , see for instance [667, Chapter 12].

Kirby and Siebenmann [579, Theorem 5.5 in Essay V on page 251], see also [883], have proved

#### **Theorem 9.134.** The space TOP/PL is an Eilenberg MacLane space of type $(\mathbb{Z}/2, 3)$ .

More information about the homotopy type of G/O, G/PL, and G/TOP can be found for instance in [667, Chapter 17].

# 9.14 The Novikov Conjecture

In this section we introduce the Novikov Conjecture in its original form in terms of higher signatures and make a first link to surgery theory. It follows from both the Baum-Connes Conjecture and the Farrell-Jones Conjecture and has been an important interface between topology and non-commutative geometry.

### 9.14.1 The Original Formulation of the Novikov Conjecture

Let G be a (discrete) group. Let  $u: M \to BG$  be a map from an oriented closed smooth manifold M to BG. Let

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(9.135) 
$$\mathcal{L}(M) \in \bigoplus_{k \in \mathbb{Z}, k \ge 0} H^{4k}(M; \mathbb{Q})$$

be the *L*-class of *M*. Its *k*-th entry  $\mathcal{L}(M)_k \in H^{4k}(M; \mathbb{Q})$  is a certain homogeneous polynomial of degree *k* in the rational Pontrjagin classes  $p_i(M; \mathbb{Q}) \in H^{4i}(M; \mathbb{Q})$  for i = 1, 2, ..., k such that the coefficient  $s_k$  of the monomial  $p_k(M; \mathbb{Q})$  is different from zero. It is defined in terms of multiplicative sequences, see for instance [729, § 19]. We mention at least the first values

$$\mathcal{L}(M)_1 = \frac{1}{3} \cdot p_1(M; \mathbb{Q});$$
  

$$\mathcal{L}(M)_2 = \frac{1}{45} \cdot \left(7 \cdot p_2(M; \mathbb{Q}) - p_1(M; \mathbb{Q})^2\right);$$
  

$$\mathcal{L}(M)_3 = \frac{1}{945} \cdot \left(62 \cdot p_3(M; \mathbb{Q}) - 13 \cdot p_1(M; \mathbb{Q}) \cup p_2(M; \mathbb{Q}) + 2 \cdot p_1(M; \mathbb{Q})^3\right)$$

The *L*-class  $\mathcal{L}(M)$  is determined by all the rational Pontrjagin classes and vice versa. Recall that the *k*-th rational Pontrjagin class  $p_k(M, \mathbb{Q}) \in H^{4k}(M; \mathbb{Q})$  is defined as the image of *k*-th Pontrjagin class  $p_k(M)$  under the obvious change of coefficients map  $H^{4k}(M; \mathbb{Z}) \to H^{4k}(M; \mathbb{Q})$ . The *L*-class depends on the tangent bundle and thus on the differentiable structure of *M*. For  $x \in \prod_{k\geq 0} H^k(BG; \mathbb{Q})$  define the higher signature of *M* associated to *x* and *u* to be

(9.136) 
$$\operatorname{sign}_{x}(M, u) := \langle \mathcal{L}(M) \cup u^{*}x, [M]_{\mathbb{Q}} \rangle \in \mathbb{Q}$$

where  $[M]_{\mathbb{Q}}$  denotes the image of the fundamental class [M] of an oriented closed *d*-dimensional manifold *M* under the change of coefficients map  $H_d(M; \mathbb{Z}) \rightarrow H_d(M; \mathbb{Q})$ . We say that sign<sub>x</sub> for  $x \in H^*(BG; \mathbb{Q})$  is *homotopy invariant* if for two oriented closed smooth manifolds *M* and *N* with reference maps  $u: M \rightarrow BG$  and  $v: N \rightarrow BG$  we have

$$\operatorname{sign}_{x}(M, u) = \operatorname{sign}_{x}(N, v),$$

whenever there is an orientation preserving homotopy equivalence  $f: M \to N$  such that  $v \circ f$  and u are homotopic.

**Conjecture 9.137** (Novikov Conjecture). The group *G* satisfies the *Novikov Conjecture* if sign<sub>x</sub> is homotopy invariant for all elements *x* of  $\prod_{k \in \mathbb{Z}, k \ge 0} H^k(BG; \mathbb{Q})$ .

This conjecture appears for the first time in the paper by Novikov [765, § 11]. A survey about its history can be found in [383].

### 9.14.2 Invariance Properties of the L-Class

One motivation for the Novikov Conjecture comes from the Signature Theorem due to Hirzebruch [498, 499]. Recall that for  $\dim(M) = 4n$  the *signature*  $\operatorname{sign}(M)$  of M is the signature of the non-singular bilinear symmetric pairing on the middle

cohomology  $H^{2n}(M; \mathbb{R})$  given by the intersection pairing  $(a, b) \mapsto \langle a \cup b, [M]_{\mathbb{R}} \rangle$ . Obviously sign(*M*) depends only on the oriented homotopy type of *M*.ö

**Theorem 9.138 (Signature Theorem).** Let M be an oriented closed manifold of dimension n. Then the higher signature  $\operatorname{sign}_1(M, u) = \langle \mathcal{L}(M), [M]_{\mathbb{Q}} \rangle$  associated to  $1 \in H_0(M)$  and some map  $u: M \to BG$  coincides with the signature  $\operatorname{sign}(M)$  of M if  $\dim(M) = 4n$ , and is zero if  $\dim(M)$  is not divisible by four.

The Signature Theorem 9.138 leads to the question whether the Pontrjagin classes or the *L*-classes are homotopy invariants or homeomorphism invariants. They are obviously invariants of the diffeomorphism type. However, the Pontrjagin classes  $p_k(M) \in H^{4k}(M;\mathbb{Z})$  for  $k \ge 2$  are not homeomorphism invariants, see for instance [587, Theorem 4.8 on page 31]. On the other hand, there is the following deep result due to Novikov [762, 763, 764].

**Theorem 9.139 (Topological invariance of rational Pontrjagin classes).** The rational Pontrjagin classes  $p_k(M, \mathbb{Q}) \in H^{4k}(M; \mathbb{Q})$  are topological invariants, i.e., for a homeomorphism  $f: M \to N$  of closed smooth manifolds we have

$$H^{4k}(f;\mathbb{Q})(p_k(N;\mathbb{Q})) = p_k(M;\mathbb{Q})$$

for all  $k \ge 0$  and in particular  $H^*(f; \mathbb{Q})(\mathcal{L}(N)) = \mathcal{L}(M)$ .

**Example 9.140 (The** *L***-class is not a homotopy invariant).** The rational Pontrjagin classes and the *L*-class are not homotopy invariants, as the following example shows. There exists for  $k \ge 1$  and large enough  $j \ge 0$  a (j + 1)-dimensional vector bundle  $\xi: E \to S^{4k}$  with Riemannian metric whose *k*-th Pontrjagin class  $p_k(\xi)$  is not zero and which is trivial as a fibration. The total space SE of the associated sphere bundle is a closed (4k + j)-dimensional manifold which is homotopy equivalent to  $S^{4k} \times S^j$  and satisfies

$$p_k(SE) = -p_k(\xi) \neq 0;$$
  
$$\mathcal{L}(SE)_k = s_k \cdot p_k(SE) \neq 0.$$

where  $s_k \neq 0$  is the coefficient of  $p_k$  in the polynomial defining the *L*-class. But  $p_k(S^{4k} \times S^j)$  and  $\mathcal{L}(S^{4k} \times S^j)_k$  vanish since the tangent bundle of  $S^{4k} \times S^j$  is stably trivial. In particular, *SE* and  $S^{4k} \times S^j$  are simply connected homotopy equivalent closed manifolds which are not homeomorphic. This example is taken from [841, Proposition 2.9] and attributed to Dold and Milnor there. See also [841, Proposition 2.10] or [729, Section 20].

**Remark 9.141 (The homological version of the Novikov Conjecture).** One may understand the Novikov Conjecture as an attempt to figure out how much of the *L*-class is a homotopy invariant of *M*. If one considers the oriented homotopy type and the simply connected case, it is just the expression  $\langle \mathcal{L}(M), [M]_Q \rangle$  or, equivalently, the top component of  $\mathcal{L}(M)$ . In the Novikov Conjecture one asks the

#### 9.14 The Novikov Conjecture

same question, but now taking the fundamental group into account by remembering the classifying map  $u_M: M \to B\pi_1(M)$ , or, more generally, a reference map  $u: M \to BG$ . The Novikov Conjecture can also be rephrased by saying that for any group G and any pair (M, u) consisting of an oriented closed manifold M of dimension n together with a reference map  $u: M \to BG$  the term

$$u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) \in \bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q})$$

depends only on the oriented homotopy type of the pair (M, u). This follows from the elementary computation for  $x \in H^*(BG; \mathbb{Q})$ 

$$\langle \mathcal{L}(M) \cup u^* x, [M]_{\mathbb{Q}} \rangle = \langle u^* x, \mathcal{L}(M) \cap [M]_{\mathbb{Q}} \rangle = \langle x, u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) \rangle$$

and the fact that the Kronecker pairing  $\langle -, - \rangle$  for rational coefficients is non-singular. Note that  $- \cap [M]_{\mathbb{Q}} : H^{n-i}(M; \mathbb{Q}) \to H_i(M; \mathbb{Q})$  is an isomorphism for all  $i \ge 0$  by Poincaré duality. Hence  $\mathcal{L}(M) \cap [M]_{\mathbb{Q}}$  carries the same information as  $\mathcal{L}(M)$ .

**Exercise 9.142.** Let  $f: M \to N$  be an orientation preserving homotopy equivalence of oriented closed manifolds which are aspherical. Assume that the Novikov Conjecture 9.137 holds for  $G = \pi_1(M)$ . Show that then  $\mathcal{L}(M) = f^*\mathcal{L}(N)$  must be true.

# 9.14.3 The Novikov Conjecture and Surgery Theory

**Remark 9.143 (The Novikov Conjecture and assembly map).** There exists an assembly map

(9.144) 
$$\operatorname{asmb}_{n}^{G} \colon \bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q}) \to L_{n}^{h}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which fits into the following commutative diagram

$$S_{n}^{\text{TOP},h}(M) \xrightarrow{\eta_{n}^{h}} \mathcal{N}_{n}^{\text{TOP}}(M) \xrightarrow{\sigma_{n}^{s}} L_{n}^{h}(\mathbb{Z}\pi_{1}(M))$$

$$\downarrow b \qquad \qquad \downarrow u_{*}$$

$$[M, G/\text{TOP}] \qquad L_{n}^{h}(\mathbb{Z}G)$$

$$\downarrow c \qquad \qquad \downarrow i$$

$$\bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q}) \xrightarrow{\text{asmb}_{n}^{G}} L_{n}^{h}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The map *i* is the obvious map and  $u_*$  is the homomorphism coming from  $\pi_1(u): \pi_1(M) \to \pi_1(BG) = G$ . The bijection *b* is taken from Theorem 9.129. The map *c* comes from the rational version of the homotopy equivalences describing G/TOP appearing in Remark 9.133, and Poincaré duality. The composite  $c \circ b$  sends the class of a normal map  $(\overline{f}, f)$  with underlying map  $f: N \to M$  of degree one to  $(u \circ f)_*(\mathcal{L}(N) \cap [N]_{\mathbb{Q}}) - u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}})$ . This fact is for instance explained in [627, page 728]. The map *s* is defined analogously, it sends the class [f] of a homotopy equivalence  $f: N \to M$  to the difference  $(u \circ f)u_*(\mathcal{L}(N) \cap [N]_{\mathbb{Q}}) - u_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}})$ , where we choose [N] such that the map *f* has degree one. We conclude from Remark 9.141 that the Novikov Conjecture 9.137 is equivalent to the statement that *s* is trivial. The upper row is part of the Surgery Exact Sequence of Theorem 9.130. This implies that the composite

$$S_n^{\operatorname{TOP},h}(M) \xrightarrow{s} \bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q}) \xrightarrow{\operatorname{asmb}_n^G} L_n^h(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is trivial.

Thus we can conclude that the group *G* satisfies the Novikov Conjecture 9.137 if the map  $\operatorname{asmb}_n^G$ :  $\bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q}) \to L_n^h(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$  is injective. See also Kaminker-Miller [541] or [587, Proposition 15.4 on page 112]. Note that the last map involves only *G*. This conclusion shows that the *L*-theoretic Novikov Conjecture 13.64 implies the Novikov Conjecture 9.137 using the fact under the Chern character the assembly map (9.144) can be identified with the assembly map appearing in the *L*-theoretic Novikov Conjecture 13.64 for *G*.

Moreover, the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring  $\mathbb{Z}$  implies the *L*-theoretic Novikov Conjecture 13.64, see Theorem 13.65 (xi).

**Remark 9.145 (The converse of the Novikov Conjecture).** A kind of converse to the Novikov Conjecture 9.137 is the following result. Let N be an oriented connected closed smooth manifold of dimension  $n \ge 5$ . Let  $u: N \to BG$  be a map inducing an isomorphism on the fundamental groups. Consider any element  $l \in \prod_{i\ge 0} H^{4i}(N;\mathbb{Q})$  such that  $u_*(l \cap [N]_{\mathbb{Q}}) = 0$  holds in  $H_*(BG;\mathbb{Q})$ . Then there exists a non-negative integer K such that for any multiple k of K there is a homotopy equivalence  $f: M \to N$  of oriented closed smooth manifolds satisfying

$$f^*(\mathcal{L}(N) + k \cdot l) = \mathcal{L}(M).$$

A proof can be found for instance in [278, Theorem 6.5]. This shows that the top dimension part of the *L*-class  $\mathcal{L}(M)$  is essentially the only homotopy invariant rational characteristic class for simply connected closed 4k-dimensional manifolds.

More information about the Novikov Conjecture can be found for instance in [384, 385, 587, 865, 1029]. An algebraic geometric and an equivariant version of the Novikov Conjecture is introduced in [863] and [870].

# 9.15 Topologically Rigidity and the Borel Conjecture

In this section we deal with the Borel Conjecture and how it follows from the Farrell-Jones Conjecture in dimensions  $\geq 5$ .

### 9.15.1 Aspherical Spaces

**Definition 9.146 (Aspherical).** A space *X* is called *aspherical* if it is path connected and all its higher homotopy groups vanish, i.e.,  $\pi_n(X)$  is trivial for  $n \ge 2$ .

**Remark 9.147 (Homotopy classification of aspherical** *CW*-complexes). A *CW*-complex is aspherical if and only if it is connected and its universal covering is contractible. Given two aspherical *CW*-complexes *X* and *Y*, the map from the set of homotopy classes of maps  $X \to Y$  to the set of group homomorphisms  $\pi_1(X) \to \pi_1(Y)$  modulo inner automorphisms of  $\pi_1(Y)$  given by the map induced on the fundamental groups is a bijection. In particular, two aspherical *CW*-complexes are homotopy equivalent if and only if they have isomorphic fundamental groups and every isomorphism between their fundamental groups comes from a homotopy equivalence.

**Remark 9.148 (Classifying space of a group).** An aspherical *CW*-complex *X* with fundamental group  $\pi$  is the same as an Eilenberg-MacLane space  $K(\pi, 1)$  of type  $(\pi, 1)$  and the same as the classifying space  $B\pi$  for the group  $\pi$ .

**Exercise 9.149.** Let  $F \to E \to B$  be a fibration. Suppose that *F* and *B* are aspherical. Show that then *E* is aspherical.

**Exercise 9.150.** Let *X* be an aspherical *CW*-complex of finite dimension. Show that  $\pi_1(X)$  is torsionfree.

### **Example 9.151 (Examples of aspherical manifolds).**

- (i) A connected closed 1-dimensional manifold is homeomorphic to S<sup>1</sup> and hence aspherical;
- (ii) Let *M* be a connected closed 2-dimensional manifold. Then *M* is either aspherical or homeomorphic to S<sup>2</sup> or ℝP<sup>2</sup>;
- (iii) A connected closed 3-manifold M is called *prime* if for any decomposition as a connected sum  $M \cong M_0 \# M_1$  one of the summands  $M_0$  or  $M_1$  is homeomorphic to  $S^3$ . It is called *irreducible* if any embedded sphere  $S^2$  bounds a disk  $D^3$ . Every irreducible closed 3-manifold is prime. A prime closed 3-manifold is either irreducible or an  $S^2$ -bundle over  $S^1$ . The following statements are equivalent for a closed 3-manifold M:
  - *M* is aspherical;
  - *M* is irreducible and its fundamental group is infinite and contains no element of order 2;

- The fundamental group  $\pi_1(M)$  cannot be written in a non-trivial way as a free product of two groups, is infinite, different from  $\mathbb{Z}$ , and contains no element of order 2;
- The universal covering of *M* is homeomorphic to  $\mathbb{R}^3$ .
- (iv) Let *L* be a Lie group with finitely many path components. Let  $K \subseteq L$  be a maximal compact subgroup. Let  $G \subseteq L$  be a discrete torsionfree subgroup. Then  $M = G \setminus L/K$  is an aspherical closed manifold with fundamental group *G*, since its universal covering L/K is diffeomorphic to  $\mathbb{R}^n$  for appropriate *n*;
- (v) Every closed Riemannian (smooth) manifold with non-positive sectional curvature has a universal covering which is diffeomorphic to  $\mathbb{R}^n$  and is in particular aspherical.

Exercise 9.152. Classify all simply connected aspherical closed manifolds.

**Exercise 9.153.** Suppose that M is a connected sum  $M_1 \# M_2$  of two closed manifolds  $M_1$  and  $M_2$  of dimension  $n \ge 3$ , which are not homotopy equivalent to a sphere. Show that M is not aspherical.

There exist exotic aspherical manifolds, as the following results illustrate. The following theorem is due to Davis-Januszkiewicz [291, Theorem 5a.4].

**Theorem 9.154 (Non-PL-example).** For every  $n \ge 4$  there exists an aspherical closed topological *n*-manifold that is not homotopy equivalent to a PL-manifold

The following result is proved by Davis-Fowler-Lafont [290] using the work of Manolescu [706, 705].

**Theorem 9.155** (Non-triangulable aspherical closed manifolds). There exists for each  $n \ge 6$  an n-dimensional aspherical closed topological manifold that cannot be triangulated. One can arrange that the fundamental group is hyperbolic.

The proof of the following theorem can be found in [288], [291, Theorem 5b.1].

**Theorem 9.156 (Exotic universal covering of aspherical closed manifolds).** For each  $n \ge 4$  there exists an aspherical closed n-dimensional manifold such that its universal covering is not homeomorphic to  $\mathbb{R}^n$ .

By the Hadamard-Cartan Theorem, see [414, 3.87 on page 134], the manifold appearing in Theorem 9.156 above cannot be homeomorphic to a smooth manifold with Riemannian metric with non-positive sectional curvature.

The following theorem is proved in [291, Theorem 5c.1 and Remark on page 386].

**Theorem 9.157 (Exotic aspherical closed manifolds with hyperbolic fundamental group).** For every  $n \ge 5$ , there exists an aspherical closed smooth n-dimensional manifold M that is homeomorphic to a strictly negatively curved polyhedron and has in particular a hyperbolic fundamental group such that the universal covering is homeomorphic to  $\mathbb{R}^n$ , but M is not homeomorphic to a smooth manifold with Riemannian metric with negative sectional curvature.

9.15 Topologically Rigidity and the Borel Conjecture

The next results are due to Belegradek [120, Corollary 5.1], Mess [715], Osajda [783, Corollary 3.5], and Weinberger, see [286, Section 13].

### Theorem 9.158 (Aspherical closed manifolds with exotic fundamental groups).

- (i) For every n ≥ 4, there is an aspherical closed topological manifold of dimension n whose fundamental group contains an infinite divisible abelian group;
- (ii) For every  $n \ge 4$ , there is an aspherical closed PL manifold of dimension n whose fundamental group has an unsolvable word problem and whose simplicial volume in the sense of Gromov [438] is non-zero;
- (iii) For every n ≥ 4, there is an aspherical closed manifold of dimension n whose fundamental group contains coarsely embedded expanders.

**Theorem 9.159 (Closed aspherical manifolds whose fundamental groups contain coarsely embedded expanders).** There exist closed aspherical manifolds of dimension 4 and higher whose fundamental groups contain coarsely embedded expanders.

More information about fundamental groups of aspherical closed manifolds with unusual properties can be found for instance in [888].

The question of when the isometry group of the universal covering of an aspherical closed manifold is non-discrete is studied by Farb-Weinberger [347].

**Remark 9.160** ( $S^1$ -actions on aspherical closed manifolds). If  $S^1$  acts on an aspherical closed manifold, then the orbit circle is a non-trivial element in the center by a result of Borel, see for instance [249, Lemma 5.1 on page 242]. Conner-Raymond [249, page 229] conjecture that the converse is true, namely, if the fundamental group of an aspherical closed manifold has nontrivial center, then the manifold has a circle action, such that the orbit circle is a nontrivial central element of the fundamental group. A counterexample in dimensions  $\geq 6$  was constructed by Cappell-Weinberger-Yan [208].

It is an open question whether the conjecture of Conner-Raymond above is true if one allows the passage to a finite covering.

**Remark 9.161.** Another interesting open question is whether the center of the fundamental group of an aspherical closed manifold is finitely generated.

For more information about aspherical closed manifolds we refer for instance to [660].

# 9.15.2 Formulation and Relevance of the Borel Conjecture

**Definition 9.162 (Topologically rigid).** We call a closed topological manifold *N* topologically rigid if any homotopy equivalence  $M \rightarrow N$  with a closed topological manifold *M* as source is homotopic to a homeomorphism.

**Conjecture 9.163 (Borel Conjecture (for a group** *G* **in dimension** *n*)). The *Borel Conjecture for a group G in dimension n* predicts, for two aspherical closed topological manifolds *M* and *N* of dimensions *n* with  $\pi_1(M) \cong \pi_1(N) \cong G$ , that *M* and *N* are homeomorphic and any homotopy equivalence  $M \to N$  is homotopic to a homeomorphism.

The *Borel Conjecture* says that every aspherical closed topological manifold is topologically rigid.

**Remark 9.164 (The Borel Conjecture in low dimensions).** The Borel Conjecture is true in dimension  $\leq 2$ . It is true in dimension 3 if Thurston's Geometrization Conjecture is true. This follows from results of Waldhausen, see Hempel [477, Lemma 10.1 and Corollary 13.7], and Turaev, see [957], as explained for instance in [588, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [580, 751] following ideas of Perelman. Some information in dimension 4 can be found in Davis [279].

**Remark 9.165 (Topological rigidity for non-aspherical manifolds).** Topological rigidity phenomena also hold for some non-aspherical closed manifolds. For instance the sphere  $S^n$  is topologically rigid by the Poincaré Conjecture. The Poincaré Conjecture is known to be true in all dimensions. This follows in high dimensions from the *h*-cobordism theorem, in dimension four from the work of Freedman [401, 402], in dimension three from the work of Perelman as explained in [580, 750], and in dimension two from the classification of surfaces.

Many more examples of classes of manifolds which are topologically rigid are given and analyzed in Kreck-Lück [588]. For instance, the connected sum of closed manifolds of dimension  $\geq 5$  which are topologically rigid and whose fundamental groups do not contain elements of order two is again topologically rigid. The product  $S^k \times S^n$  is topologically rigid if and only if k and n are odd. An integral homology sphere of dimension  $n \geq 5$  is topologically rigid if and only if the inclusion  $\mathbb{Z} \to \mathbb{Z}[\pi_1(M)]$  induces an isomorphism of simple *L*-groups  $L_{n+1}^s(\mathbb{Z}) \to L_{n+1}^s(\mathbb{Z}[\pi_1(M)])$ . Every 3-manifold with torsionfree fundamental group is topologically rigid.

**Exercise 9.166.** Give an example of a closed orientable 3-manifold with finite fundamental group that is not topologically rigid.

**Exercise 9.167.** Give an example of two topologically rigid orientable closed smooth manifolds whose cartesian product is not topologically rigid.

**Remark 9.168 (The smooth Borel Conjecture holds in dimension** n **if and only if**  $n \leq 3$ ). The Borel Conjecture 9.163 is false in the smooth category, i.e., if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The torus  $T^n$  for  $n \geq 5$  is an example, see [987, 15A]. The smooth Borel Conjecture is also false in dimension 4, see Davis-Hayden-Huang-Ruberman-Sunukjian [287]. So the smooth Borel Conjecture is true in dimension n if and only if  $n \leq 3$  since in dimension  $\leq 3$  there is no difference between the smooth and the

topological category, see Moise [747, 748], and the Borel Conjecture 9.163 holds in dimension  $\leq$  3, see Remark 9.164.

Other counterexamples involving negatively curved manifolds are constructed by Farrell-Jones [362, Theorem 0.1].

**Remark 9.169 (The Borel Conjecture versus Mostow rigidity).** The examples of Farrell-Jones [362, Theorem 0.1] give actually more. Namely, they yield for given  $\epsilon > 0$  a closed Riemannian manifold  $M_0$  whose sectional curvature lies in the interval  $[1 - \epsilon, -1 + \epsilon]$  and a closed hyperbolic manifold  $M_1$  such that  $M_0$  and  $M_1$  are homeomorphic but not diffeomorphic. The idea of the construction is essentially to take the connected sum of  $M_1$  with exotic spheres. Note that by definition  $M_0$  would be hyperbolic if we could take  $\epsilon = 0$ . Hence this example is remarkable in view of *Mostow rigidity*, which predicts for two closed hyperbolic manifolds  $N_0$  and  $N_1$  that they are isometrically diffeomorphic if and only if  $\pi_1(N_0) \cong \pi_1(N_1)$  and any homotopy equivalence  $N_0 \rightarrow N_1$  is homotopic to an isometric diffeomorphism.

One may view the Borel Conjecture as the topological version of Mostow rigidity. The conclusion in the Borel Conjecture is weaker, one gets only homeomorphisms and not isometric diffeomorphisms, but the assumption is also weaker, since there are many more aspherical closed topological manifolds than hyperbolic closed manifolds.

**Remark 9.170 (The work of Farrell-Jones).** Farrell-Jones have made deep contributions to the Borel Conjecture. They have proved it in dimension  $\geq 5$  for non-positively curved closed Riemannian manifolds, for compact complete affine flat manifolds, and for aspherical closed manifolds whose fundamental group is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold that is A-regular. Relevant references are [363, 364, 367, 369, 370].

The Borel Conjecture for higher-dimensional graph manifolds is studied by Frigerio-Lafont-Sisto [407].

More information about the Borel Conjecture can be found in [667, Chapter 19] and [999].

### 9.15.3 The Farrell-Jones and the Borel Conjecture

**Theorem 9.171 (The Farrell-Jones and the Borel Conjecture).** Let *G* be a finitely presented torsionfree group. Suppose that it satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ .

Then every aspherical closed manifold of dimension  $\geq 5$  with G as fundamental group is topologically rigid, in other words, the Borel Conjecture 9.163 holds for G in dimensions  $\geq 5$ .

For its proof we need the following lemma.

**Lemma 9.172.** Let M be a closed topological manifold with  $Wh(\pi_1(M)) = 0$ . Then M is topologically rigid if and only if the simple topological structure set  $S^{TOP,s}(M)$  consists of precisely on element, namely the class of  $id_M$ .

*Proof.* Suppose that M is topologically rigid. Consider any element in  $\eta \in S^{\text{TOP},s}(M)$ . Choose a simple homotopy equivalence  $f: N \to M$  representing  $\eta$ . Since M is topologically rigid, f is homotopic to a homeomorphism  $h: N \to M$ . Hence  $\text{id}_M \circ h \simeq f$ . This implies that  $\eta$  is represented by  $\text{id}_M$ .

Suppose that  $S^{\text{TOP},s}(M)$  consists only of one class, the one represented by  $\mathrm{id}_M$ . Consider any homotopy equivalence  $f: N \to M$ . Since  $\mathrm{Wh}(\pi_1(M)) = 0$  holds by assumption, f is a simple homotopy equivalence and thus represents an element in  $S^{\text{TOP},s}(M)$ . Since it represents the same class as  $\mathrm{id}_M$  by assumption, there exists a homeomorphism  $h: N \to M$  such that  $h = \mathrm{id}_M \circ h$  is homotopic to f.

**Lemma 9.173.** Let M be a closed topological manifold of dimension  $n \ge 5$ . Let  $w: \pi := \pi_1(M) \to \{\pm 1\}$  be given by its first Stiefel-Whitney class. Suppose  $Wh(\pi_1(M)) = 0$ . Assume that the homomorphism of abelian groups  $\sigma_{n+1}^s : \mathcal{N}_{n+1}^{\mathsf{TOP}}(M \times [0, 1], M \times \{0, 1\}) \to L_{n+1}^s(\mathbb{Z}\pi, w)$  of (9.123) is surjective and that the preimage of 0 under the map  $\sigma_n^s : \mathcal{N}_n^{\mathsf{TOP}}(X) \to L_n^s(\mathbb{Z}\pi, w)$  of (9.124) consists of one point.

Then M is topologically rigid.

*Proof.* This follows from the simple topological Surgery Exact Sequence of Theorem 9.130 and Lemma 9.172.

Now we can give a sketch of the proof of Theorem 9.171.

Sketch of the proof of Theorem 9.171. We deal for simplicity with the orientable case, i.e.,  $w_1 = 0$ , only. Let  $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$  be the *L*-theory spectrum appearing in the version of the *L*-theoretic Farrell-Jones Conjecture 9.114. Since it holds by assumption, the so-called assembly map

$$\operatorname{asmb}_{k}^{\langle -\infty \rangle} \colon H_{k}(B\pi; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_{k}^{\langle -\infty \rangle}(\mathbb{Z}\pi)$$

is bijective for all k. Let  $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle$  be the 1-connected cover of  $\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$ . This spectrum comes with a map of spectra  $\mathbf{i} \colon \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle \to \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})$  such that  $\pi_k(\mathbf{i})$  is bijective for  $k \geq 1$  and  $\pi_k(\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})\langle 1 \rangle) = 0$  for  $k \leq 0$ . For  $k \geq 1$  there is a connective version of the assembly map  $\operatorname{asmb}_k$  above

$$\operatorname{asmb}_{k}^{\langle -\infty \rangle} \langle 1 \rangle \colon H_{k}(B\pi; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}) \langle 1 \rangle) \to L_{k}^{\langle -\infty \rangle}(\mathbb{Z}\pi)$$

such that  $\operatorname{asmb}_{k}^{\langle -\infty \rangle} \langle 1 \rangle = \operatorname{asmb}_{k}^{\langle -\infty \rangle} \circ H_{k}(\operatorname{id}_{B\pi}; \mathbf{i})$  holds. A comparison argument of the Atiyah-Hirzebruch spectral sequence shows that the bijectivity of  $\operatorname{asmb}_{k}^{\langle -\infty \rangle}$  for k = n, n + 1 implies that  $\operatorname{asmb}_{n+1}^{\langle -\infty \rangle} \langle 1 \rangle$  is bijective and in particular surjective and that  $\operatorname{asmb}_{n}^{\langle -\infty \rangle} \langle 1 \rangle$  is injective if *n* is the dimension of the aspherical closed manifold under consideration. Because by assumption Conjectures 3.110 and 4.20 hold for

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 $\pi$ , we conclude from Theorem 9.106 that the simple versions of the 1-connective assembly maps

 $\operatorname{asmb}_{k}^{s}\langle 1 \rangle \colon H_{k}(B\pi; \mathbf{L}^{s}(\mathbb{Z})\langle 1 \rangle) \to L_{k}^{s}(\mathbb{Z}\pi)$ 

agree with the maps  $\operatorname{asmb}_{k}^{\langle -\infty \rangle} \langle 1 \rangle$ . One can identify the map  $\operatorname{asmb}_{n+1}^{s} \langle 1 \rangle$  with the map  $\sigma_{n+1}^s \colon \mathcal{N}_{n+1}^{\text{TOP}}(M \times [0,1], M \times \{0,1\}) \to L_{n+1}^s(\mathbb{Z}\pi)$  of (9.123) and the map  $\operatorname{asmb}_n^s(1)$  with the map  $\sigma_n^s \colon \mathcal{N}_n^{\text{PL}}(X) \to L_n^s(\mathbb{Z}\pi)$  of (9.124), see [839, Theorem 18.5] on page 198], [834], [596] using Remark 18.18, Remark 18.19, and Example 18.23. 

Now Theorem 9.171 follows from Lemma 9.173.

Remark 9.174 (Dimension 4). The conclusion of Theorem 9.171 also holds in dimension 4, provided that the fundamental group is good in the sense of Freedman, see [401, 402]. Groups of subexponential growth are good, see [404, 594]. Every elementary amenable group is good, see [118, Example 19.6 on page 280]. In [118, page 282] it is mentioned that it is likely that amenable groups are good. All groups are good if and only if  $\mathbb{Z} * \mathbb{Z}$  is good, see [118, Proposition 19.7 on page 281]. No satisfactory characterization of the class of good groups seems to be known. Our personal belief is that a group is good if and only if it is amenable.

Remark 9.175 (The Novikov Conjecture implies a stable version of the Borel **Conjecture**). For a group G that satisfies the Novikov Conjecture 9.137, the following stable version of the Borel Conjecture holds: For any homotopy equivalence  $f: M \rightarrow$ N of aspherical closed manifolds of dimension  $\geq 5$  whose fundamental groups are isomorphic to G, the map  $f \times id_{\mathbb{R}^3} \colon M \times \mathbb{R}^3 \to N \times \mathbb{R}^3$  is homotopic to a homeomorphism. See [521, Proposition 2.8], where the proof is attributed to Shmuel Weinberger, see also [356, Proof of Corollary B on page 207].

Remark 9.176 (Homology-ANR-manifolds). If one works in the category of homology ANR-manifolds, one does not have to pass to the 1-connective cover, see [174, Main Theorem].

# 9.16 Homotopy Spheres

An oriented closed smooth manifold is called a *homotopy sphere* if it is homotopy equivalent to the standard sphere. By the Poincaré Conjecture a homotopy sphere is always homeomorphic to a standard sphere and actually topologically rigid. However, it may not be diffeomorphic to a standard sphere, and in this case it is called an *exotic* sphere.

The classification of homotopy spheres due to Kervaire-Milnor [576] marks the beginning of surgery theory. In order to understand the surgery machinery and in particular the long Surgery Exact Sequence, we recommend to the reader to study the classification of homotopy spheres, which boils down to computing  $S_n^s(S^n)$ . Moreover, there are some beautiful constructions of exotic spheres and results about the curvature properties of Riemannian metrics on an exotic sphere. We refer for

instance to the following survey articles [526], [611], [628], and [648, Chapter 6], and to [667, Chapter 12].

## 9.17 Poincaré Duality Groups

The following definition is due to Johnson-Wall [528].

**Definition 9.177 (Poincaré duality group).** A group *G* is called a *Poincaré duality group of dimension n* if the following conditions holds:

- (i) The group G is of type FP, i.e., the trivial ZG-module Z possesses a finitedimensional projective ZG-resolution by finitely generated projective ZGmodules;
- (ii) We get an isomorphism of abelian groups

$$H^{i}(G; \mathbb{Z}G) \cong \begin{cases} \{0\} \text{ for } i \neq n; \\ \mathbb{Z} \text{ for } i = n. \end{cases}$$

Recall that a *CW*-complex *X* is called *finitely dominated* if there exists a finite *CW*-complex *Y* and maps  $i: X \to Y$  and  $r: Y \to X$  with  $r \circ i \simeq id_X$ .

A metric space X is called an *absolute neighborhood retract* or briefly ANR if, for every embedding  $i: X \to Y$  as a closed subspace into a metric space Y, there is an open neighbourhood U of im(i) together with a retraction  $r: U \to im(i)$ , or, equivalently, for every metric space Z, every closed subset  $Y \subseteq Z$ , and every (continuous) map  $f: Y \to X$ , there exists an open neighborhood U of Y in Z together with an extension  $F: U \to X$  of f to U. Every ANR is locally contractible, see [505, Theorem 7.1 in Chapter III on page 96]. A metrizable space of finite dimension is an ANR if and only if it is locally contractible, see [505, Theorem 7.1 in Chapter V on page 168]. Being an ANR is a local property, see [505, Theorem 8.1 in Chapter III on page 98]. Every finite CW-complex and every topological manifold is an ANR. Another good source about ANR-s is the book by Borsuk [154].

A compact n-dimensional homology ANR-manifold X is a compact absolute neighborhood retract such that it has a countable basis for its topology, has finite topological dimension, see Definition 22.35, and for every  $x \in X$  the abelian group  $H_i(X, X - \{x\})$  is trivial for  $i \neq n$  and infinite cyclic for i = n. A closed n-dimensional topological manifold is an example of a compact n-dimensional homology ANRmanifold, see [275, Corollary 1A in V.26 page 191].

**Exercise 9.178.** Show that the product of two Poincaré duality groups is again a Poincaré duality group.

**Theorem 9.179 (Homology** ANR-manifolds and finite Poincaré complexes). Let *M* be a closed topological manifold, or more generally, a compact homology ANR-manifold of dimension n. Then *M* is homotopy equivalent to a finite n-dimensional Poincaré complex.

9.17 Poincaré Duality Groups

*Proof.* A closed topological manifold, and more generally a compact ANR, has the homotopy type of a finite *CW*-complex, see [579, Theorem 2.2], [1004]. The usual proof of Poincaré duality for closed manifolds carries over to homology ANR-manifolds.

**Theorem 9.180 (Poincaré duality groups).** Let G be a group and  $n \ge 1$  be an integer. Then:

- (i) The following assertions are equivalent:
  - (a) G is finitely presented and a Poincaré duality group of dimension n;
  - (b) There exists a finitely dominated n-dimensional aspherical Poincaré complex with G as fundamental group;
- (ii) Suppose that  $\widetilde{K}_0(\mathbb{Z}G) = 0$ . Then the following assertions are equivalent:
  - (a) *G* is finitely presented and a Poincaré duality group of dimension *n*;
  - (b) There exists a finite n-dimensional aspherical Poincaré complex with G as fundamental group;
- (iii) A group G is a Poincaré duality group of dimension 1 if and only if  $G \cong \mathbb{Z}$ ;
- (iv) A group G is a Poincaré duality group of dimension 2 if and only if G is isomorphic to the fundamental group of an aspherical closed surface.

*Proof.* (i) Every finitely dominated *CW*-complex has a finitely presented fundamental group, since every finite *CW*-complex has a finitely presented fundamental group and a group that is a retract of a finitely presented group is again finitely presented, see [983, Lemma 1.3]. If there exists a *CW*-model for *BG* of dimension *n*, then the cohomological dimension of *G* satisfies  $cd(G) \le n$  and the converse is true, provided that  $n \ge 3$ , see [171, Theorem 7.1 in Chapter VIII.7 on page 205], [330], [983], and [984]. This implies that the implication (ib)  $\implies$  (ia) holds for all  $n \ge 1$  and that the implication (ia)  $\implies$  (ib) holds for  $n \ge 3$ . For more details we refer to [528, Theorem 1]. The remaining part, to show the implication (ia)  $\implies$  (ib) for n = 1, 2, follows from assertions (iii) and (iv).

(ii) This follows in dimension  $n \ge 3$  from assertion (i) and Wall's results about the finiteness obstruction, which decides whether a finitely dominated *CW*-complex is homotopy equivalent to a finite *CW*-complex, and takes values in  $\tilde{K}_0(\mathbb{Z}\pi)$ , see [382, 740, 983, 984] or Section 2.5. The implication (iib)  $\implies$  (iia) holds for all  $n \ge 1$ . The remaining part, to show the implication (iia)  $\implies$  (iib) holds, follows from assertions (iii) and (iv).

(iii) Since  $S^1 = B\mathbb{Z}$  is a 1-dimensional closed manifold,  $\mathbb{Z}$  is a finite Poincare duality group of dimension 1 by Theorem 9.179. We conclude from the (easy) implication (ib)  $\implies$  (ia) appearing in assertion (i) that  $\mathbb{Z}$  is a Poincaré duality group of dimension 1. Suppose that *G* is a Poincaré duality group of dimension 1. Since the cohomological dimension of *G* is 1, it has to be a free group, see [925, 941]. Since the homology group of a group of type FP is finitely generated, *G* is isomorphic to a finitely generated free group  $F_r$  of rank *r*. Since  $H^1(BF_r) \cong \mathbb{Z}^r$  and  $H_0(BF_r) \cong \mathbb{Z}$ , Poincaré duality can only hold for r = 1, i.e., *G* is  $\mathbb{Z}$ .

(iv) This is proved in [328, Theorem 2]. See also [138, 139, 326, 329]. □

**Conjecture 9.181 (Manifold structures on aspherical Poincaré complexes).** Every finitely dominated aspherical Poincaré complex is homotopy equivalent to a closed topological manifold.

**Remark 9.182 (Existence and uniqueness part of the Borel Conjecture).** Conjecture 9.181 can be viewed as the existence part of the Borel Conjecture 9.163, namely, the question whether an aspherical finite Poincaré complex carries up to homotopy the structure of a closed topological manifold. The Borel Conjecture 9.163 as stated above is the uniqueness part.

**Conjecture 9.183 (Poincaré duality groups).** A finitely presented group is an *n*-dimensional Poincaré duality group if and only if it is the fundamental group of an aspherical closed *n*-dimensional topological manifold.

The *disjoint disk property* says that for any  $\epsilon > 0$  and maps  $f, g: D^2 \to M$  there are maps  $f', g': D^2 \to M$  so that the distance between f and f' and the distance between g and g' are bounded by  $\epsilon$  and  $f'(D^2) \cap g'(D^2) = \emptyset$ .

**Theorem 9.184 (Poincaré duality groups and aspherical compact homology** ANR-manifolds). Suppose that the torsionfree group G is a finitely presented Poincaré duality group of dimension  $n \ge 6$  and satisfies the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ . Let X be some aspherical finite Poincaré complex with  $\pi_1(X) \cong G$ . (It exists because of Theorem 9.180 (ii).) Suppose that the Spivak normal fibration of X admits a TOP-reduction.

Then BG is homotopy equivalent to an aspherical compact homology ANRmanifold satisfying the disjoint disk property.

*Proof.* See [839, Remark 25.13 on page 297], [174, Main Theorem on page 439 and Section 8] and [175, Theorem A and Theorem B].

**Remark 9.185.** Note that in Theorem 9.184 the condition appears that for some aspherical finite Poincaré complex *X* with  $\pi_1(X) \cong G$  the Spivak normal fibration of *X* admits a TOP-reduction. This condition does not appear in earlier versions. The reason is that there seems to be a mistake in [174], as explained in the Erratum [176]. The problem was pointed out by Hebestreit-Land-Weiss-Winges, see [476]. The problem is that the proof that any compact homology ANR-manifold has a TOP-reduction of its Spivak normal fibration is not correct. In the applications of [174] to results appearing in this book one has either to assume that the TOP-reduction exists or to prove its existence. This is the reason why this extra assumption in Theorem 9.184 appears.

As pointed out in [176], Theorem 9.188 and 9.192 remain true without adding any further hypothesis. This is also true for Theorem 9.194 by the following argument. Let  $X_1$  and  $X_2$  be connected finite Poincare complexes. Let  $p_1: E_1 \rightarrow X_1$  and

 $p_2: E_2 \to X_2$  be spherical fibrations representing their Spivak normal fibration. Then the fibration  $p_1 * p_2: E_1 * E_2 \to X_1 \times X_2$  is a representative of the Spivak normal fibration of  $X_1 \times X_2$ , where the fiber over  $(x_1, x_2) \in X_1 \times X_2$  is the join  $p_1^{-1}(x_1) * p_2^{-1}(x_2)$  of the fibers of  $p_1$  over  $x_1$  and  $p_2$  over  $x_2$ . Now suppose that  $p_1 * p_2$  has a TOP reduction after possibly stabilization. Then  $i^*(p_1 * p_2)$  also has a TOP-reduction for the inclusion  $i: X_1 \to X_1 \times X_2$  sending  $x_1$  to  $(x_1, x_2)$  for some fixed  $x_2 \in X_2$ . But  $i^*(p_1 * p_2)$  is a stabilization of  $p_1$ . Hence the Spivak normal fibration of  $X_1$  has a TOP-reduction.

Remark 9.186 (Compact homology ANR-manifolds versus closed topological **manifolds**). In the following all manifolds have dimension  $\geq 6$ . One would prefer that in the conclusion of Theorem 9.184 one could replace "compact homology ANRmanifold" by "closed topological manifold". The problem is that in the geometric Surgery Exact Sequence one has to work with the 1-connective cover L(1) of the L-theory spectrum L, whereas in the assembly map appearing in the Farrell-Jones setting one uses the L-theory spectrum L. The L-theory spectrum L is 4-periodic, i.e.,  $\pi_n(\mathbf{L}) \cong \pi_{n+4}(\mathbf{L})$  for  $n \in \mathbb{Z}$ . The 1-connective cover  $\mathbf{L}(1)$  comes with a map of spectra **f**:  $\mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$  such that  $\pi_n(\mathbf{f})$  is an isomorphism for  $n \geq 1$  and  $\pi_n(\mathbf{L}\langle 1 \rangle) = 0$ for  $n \leq 0$ . Since  $\pi_0(\mathbf{L}) \cong \mathbb{Z}$ , one misses a part involving  $L_0(\mathbb{Z})$  of the so-called total surgery obstruction due to Ranicki, i.e., the obstruction for a finite Poincaré complex to be homotopy equivalent to a closed topological manifold. If one deals with the periodic L-theory spectrum L, one picks up only the obstruction for a finite Poincaré complex to be homotopy equivalent to a compact homology ANR-manifold, the so-called *four-periodic total surgery obstruction*. The difference of these two obstructions is related to the *resolution obstruction* of Quinn, which takes values in  $L_0(\mathbb{Z})$ . Any element of  $L_0(\mathbb{Z})$  can be realized by an appropriate compact homology ANR-manifold as its resolution obstruction. There are compact homology ANRmanifolds that are not homotopy equivalent to closed manifolds. But no example of an aspherical compact homology ANR-manifold that is not homotopy equivalent to a closed topological manifold is known. For an aspherical compact homology ANR-manifold M, the total surgery obstruction and the resolution obstruction carry the same information. So we could replace in the conclusion of Theorem 9.184 "compact homology ANR-manifold" by "closed topological manifold" if and only if every aspherical compact homology ANR-manifold with the disjoint disk property admits a resolution.

We refer for instance to [174, 381, 826, 827, 839] for more information about this topic.

**Question 9.187 (Vanishing of the resolution obstruction in the aspherical case).** Is every aspherical compact homology ANR-manifold homotopy equivalent to a closed manifold?

# 9.18 Boundaries of Hyperbolic Groups

If G is the fundamental group of an *n*-dimensional closed Riemannian (smooth) manifold with negative sectional curvature, then G is a hyperbolic group in the sense of Gromov, see for instance [159], [165], [424], and [440]. Moreover, such a group is torsionfree and its boundary  $\partial G$  is homeomorphic to a sphere. This leads to the natural question whether a torsionfree hyperbolic group with a sphere as boundary occurs as the fundamental group of an aspherical closed manifold, see Gromov [441, page 192]. In high dimensions this question is answered by the following two theorems taken from Bartels-Lück-Weinberger [90]. For the notion of and information about the boundary of a hyperbolic group and its main properties we refer for instance to [545].

**Theorem 9.188 (Hyperbolic groups with spheres as boundary).** *Let G be a torsionfree hyperbolic group and let n be an integer*  $\ge$  6. *Then:* 

- (i) The following statements are equivalent:
  - (a) The boundary  $\partial G$  is homeomorphic to  $S^{n-1}$ ;
  - (b) There is an aspherical closed topological manifold M such that G ≅ π<sub>1</sub>(M), its universal covering M̃ is homeomorphic to ℝ<sup>n</sup> and the compactification of M̃ by ∂G is homeomorphic to D<sup>n</sup>;
- (ii) *The aspherical closed topological manifold M appearing in the assertion above is unique up to homeomorphism.*

**Theorem 9.189** (Hyperbolic groups with Čech-homology spheres as boundary). Let *G* be a torsionfree hyperbolic group and let *n* be an integer  $\geq 6$ . Then

- (i) The following statements are equivalent:
  - (a) The boundary  $\partial G$  has the integral Čech cohomology of  $S^{n-1}$ ;
  - (b) *G* is a Poincaré duality group of dimension *n*;
  - (c) There exists a compact homology ANR-manifold M homotopy equivalent to BG. In particular, M is aspherical and  $\pi_1(M) \cong G$ ;
- (ii) If the statements in assertion (i) hold, then the compact homology ANR-manifold M appearing there is unique up to s-cobordism of compact ANR-homology manifolds.

One of the main ingredients in the proof of the two theorems above is the fact that both the K-theoretic and the L-theoretic Farrell-Jones Conjecture hold for hyperbolic groups, see [78] and [87].
# 9.19 The Stable Cannon Conjecture

Tremendous progress in the theory of 3-manifolds has been made during the last decade. A proof of Thurston's Geometrization Conjecture is given in [580], [751] following ideas of Perelman. The Virtually Fibering Conjecture was settled by the work of Agol, Liu, Przytycki-Wise, and Wise [20, 21, 633, 816, 817, 1016, 1017].

However, the following famous conjecture, taken from [197, Conjecture 5.1], is still open at the time of writing.

**Conjecture 9.190 (Cannon Conjecture).** Let *G* be a hyperbolic group. Suppose that its boundary is homeomorphic to  $S^2$ .

Then G acts properly cocompactly and isometrically on the 3-dimensional hyperbolic space.

In the torsionfree case it boils down to

**Conjecture 9.191 (Cannon Conjecture in the torsionfree case).** Let G be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to  $S^2$ .

Then G is the fundamental group of a closed hyperbolic 3-manifold.

More information about Conjecture 9.190 and its status can be found for instance in [377, Section 2] and [151].

The following theorem is taken from [377, Theorem 2]. It is a stable version of the Conjecture 9.191 above. Its proof is based on high-dimensional surgery theory and the theory of homology ANR-manifolds.

**Theorem 9.192 (Stable Cannon Conjecture).** Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL, or topological manifold respectively, that is closed and whose dimension is  $\geq 2$ .

Then there is a closed smooth, PL, or topological manifold M and a normal map of degree one

satisfying

- (i) *The map f is a simple homotopy equivalence;*
- (ii) Let  $\widehat{M} \to M$  be the *G*-covering associated to the composite of the isomorphism  $\pi_1(f): \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$  with the projection  $G \times \pi_1(N) \to G$ . Suppose additionally that *N* is aspherical, dim $(N) \ge 3$ , and  $\pi_1(N)$  satisfies the Full Farrell-Jones Conjecture 13.30. (Its status is discussed in Theorem 16.1.) Then  $\widehat{M}$  is homeomorphic to  $\mathbb{R}^3 \times N$ . Moreover, there is a compact topological manifold  $\overline{\widehat{M}}$  whose interior is homeomorphic to  $\widehat{M}$  and for which there exists a homeomorphism of pairs  $(\overline{\widehat{M}}, \partial \overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$ .

If we could choose  $N = \{\bullet\}$  in Theorem 9.192, it would imply Conjecture 9.191.

**Exercise 9.193.** Show that the manifold *M* appearing in Theorem 9.192 is unique up to homeomorphism if *N* is aspherical and  $\pi_1(N)$  satisfies the Full Farrell-Jones Conjecture 13.30.

# 9.20 Product Decompositions

In this section we show that, roughly speaking, an aspherical closed topological manifold M is a product  $M_1 \times M_2$  if and only if its fundamental group is a product  $\pi_1(M) = G_1 \times G_2$  and that such a decomposition is unique up to homeomorphism.

**Theorem 9.194 (Product decompositions of aspherical closed manifolds).** Let M be an aspherical closed topological manifold of dimension n with fundamental group  $G = \pi_1(M)$ . Suppose we have a product decomposition

$$p_1 \times p_2 \colon G \xrightarrow{=} G_1 \times G_2.$$

Suppose that G, G<sub>1</sub>, and G<sub>2</sub> satisfy the versions of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ .

Then G,  $G_1$  and  $G_2$  are Poincaré duality groups whose cohomological dimensions satisfy

$$n = \operatorname{cd}(G) = \operatorname{cd}(G_1) + \operatorname{cd}(G_2).$$

Suppose in the following:

• the cohomological dimension  $cd(G_i)$  is different from 3, 4, and 5 for i = 1, 2;

•  $n \neq 4$  or (n = 4 and G is good in the sense of Freedman).

Then:

(i) There are aspherical closed topological manifolds  $M_1$  and  $M_2$  together with isomorphisms

$$v_i \colon \pi_1(M_i) \xrightarrow{=} G_i$$

and maps

$$f_i: M \to M_i$$

for i = 1, 2 such that

$$f = f_1 \times f_2 \colon M \to M_1 \times M_2$$

is a homeomorphism and  $v_i \circ \pi_1(f_i) = p_i$  (up to inner automorphisms) for i = 1, 2;

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- (ii) Suppose we have another such choice of aspherical closed topological manifolds M'<sub>1</sub> and M'<sub>2</sub> together with isomorphisms

$$v_i' \colon \pi_1(M_i') \xrightarrow{\cong} G_i$$

and maps

$$f'_i \colon M \to M'_i$$

for i = 1, 2 such that the map  $f' = f'_1 \times f'_2$  is a homotopy equivalence and  $v'_i \circ \pi_1(f'_i) = p_i$  (up to inner automorphisms) for i = 1, 2. Then there are for i = 1, 2 homeomorphisms  $h_i: M_i \to M'_i$  such that  $h_i \circ f_i \simeq f'_i$  and  $v_i \circ \pi_1(h_i) = v'_i$  holds for i = 1, 2.

*Proof.* The case  $n \neq 3$  is proved in [660, Theorem 6.1]. The case n = 3 is done as follows. We conclude from [477, Theorem 11.1 on page 100] that  $G_1 \cong \mathbb{Z} \cong \pi_1(S^1)$  and  $G_2$  is the fundamental group  $\pi_1(F)$  of a closed surface or the other way around. Now use the fact that the Borel Conjecture is true in dimensions  $\leq 3$ .

# 9.21 Automorphisms of Manifolds

We record the following two results that deduce information about the homotopy groups of the automorphism group of an aspherical closed manifold from the Farrell-Jones Conjecture and the material from Chapter 7 about pseudoisotopy spaces.

**Theorem 9.195 (Homotopy Groups of** Top(M) **rationally for closed aspherical** M). Let M be a aspherical closed topological manifold with fundamental group  $\pi$ . Suppose that M is smoothable, the L-theory assembly map

$$H_n(B\pi; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}\pi)$$

is an isomorphism for  $n \in \mathbb{Z}$  and suppose the K-theory assembly map

$$H_n(B\pi; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}\pi)$$

is an isomorphism for  $n \le 1$  and a rational isomorphism for  $n \ge 2$ . Then for  $1 \le i \le (\dim M - 7)/3$  one has

$$\pi_i(\operatorname{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \operatorname{center}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ 0 & \text{if } i > 1. \end{cases}$$

In the differentiable case one additionally needs to study involutions on the higher *K*-theory groups.

**Theorem 9.196 (Homotopy Groups of** Diff(M) **rationally for closed aspherical** *M*). Let *M* be an aspherical orientable closed smooth manifold with fundamental group  $\pi$ .

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Then under the same assumptions as in Theorem 9.195 we have for  $1 \le i \le (\dim M - 7)/3$ 

$$\pi_i(\operatorname{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \operatorname{center}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even.} \end{cases}$$

For a proof see for instance [354], [364, Section 2], and [353, Lecture 5]. For a survey on automorphisms of manifolds we refer to [1003].

**Remark 9.197** (Homotopy Groups of G(M) for closed aspherical M). Let M be an aspherical closed topological manifold with fundamental group  $\pi$ . Let G(M) be the monoid of self-homotopy equivalences of M. Choose  $id_M$  as its base point. Then there are isomorphisms

$$\begin{aligned} \alpha_0 \colon \pi_0(G(M)) &\xrightarrow{=} \operatorname{Out}(\pi); \\ \alpha_1 \colon \pi_1(G(M)) &\xrightarrow{\cong} \operatorname{center}(\pi); \\ \pi_n(G(M)) &\cong \{0\} \quad \text{for } n \ge 2, \end{aligned}$$

where  $\alpha_0$  comes from taking the map induced on the fundamental group, see Remark 9.147, and  $\alpha_1$  comes from the evaluation map  $G(M) \to M$ ,  $f \mapsto f(x)$ for a base point  $x \in M$ , see [432, Theorem III.2]. Define maps  $\beta_n$  for n = 0, 1 to be the composites

(9.198) 
$$\beta_0: \pi_0(\operatorname{Top}(M)) \to \pi_0(G(M)) \xrightarrow{\alpha_0} \operatorname{Out}(\pi);$$

(9.199) 
$$\beta_1 : \pi_1(\operatorname{Top}(M)) \to \pi_1(G(M)) \xrightarrow{\alpha} \operatorname{center}(\pi).$$

The maps  $\beta_0$  and  $\beta_1$  are rationally bijective if the assumptions appearing in Theorem 9.195 are satisfied.

**Exercise 9.200.** Show that for a topologically rigid aspherical closed manifold *M* the map  $\beta_0: \pi_0(\text{Top}(M)) \to \text{Out}(\pi)$  of (9.198) is surjective.

**Remark 9.201.** Let *M* be an aspherical closed topological manifold such that the assumptions appearing in Theorem 9.195 are satisfied. Then we also get some information about the (co)homology of  $BTop(M)^\circ$ , where  $Top(M)^\circ$  denotes the component of the identity of Top(M). We get from  $\beta_1$  defined in (9.199) and Theorem 9.195 a map

$$BTop(M)^{\circ} \rightarrow K(center(\pi), 2)$$

of simply connected spaces inducing isomorphism on the rationalized homotopy groups in dimensions  $\leq (\dim M - 7)/3 + 1$ . This implies that we get isomorphisms

$$H_n(\operatorname{BTop}(M)^\circ; \mathbb{Q}) \xrightarrow{\cong} H_n(K(\operatorname{center}(\pi), 2); \mathbb{Q});$$
$$H^n(K(\operatorname{center}(\pi), 2); \mathbb{Q}) \xrightarrow{\cong} H^n(\operatorname{BTop}(M)^\circ; \mathbb{Q}),$$

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for  $n \le (\dim M - 7)/3 + 1$ .

**Remark 9.202.** Let *M* be an aspherical closed topological manifold with fundamental group  $\pi$  such that the assumptions appearing in Theorem 9.195 are satisfied.

Then the following assertions are equivalent by Theorem 9.195 and Remark 9.201:

- (i) The abelian group center( $\pi$ ) of  $\pi$  is finitely generated;
- (ii) The  $\mathbb{Q}$ -module  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{center}(\pi)$  is finitely generated;
- (iii) The  $\mathbb{Q}$ -module  $H^2(\text{BTop}(M)^\circ; \mathbb{Q})$  is finitely generated;
- (iv) The  $\mathbb{Q}$ -module  $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_1(\text{Top}(M))$  is finitely generated;
- (v) The Q-module  $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_1(\text{Top}(M))$  is finitely generated and the Q-module  $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_i(\text{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$  is trivial for  $2 \le i \le (\dim M 7)/3 + 1$ ;
- (vi) The Q-module  $H^{i}(\operatorname{BTop}(M)^{\circ}; \mathbb{Q})$  is finitely generated for  $1 \leq i \leq (\dim M 7)/3 + 1$ ;

Recall the open question whether center( $\pi_1(M)$ ) is finitely generated for a closed aspherical manifold *M*.

In this context we mention the result of Budney-Gabai [177, Theorem 1.3] that, for  $n \ge 4$  and an *n*-dimensional hyperbolic closed manifold, both  $\pi_{n-4}(\text{Top}^0(M))$ and  $\pi_{n-4}(\text{Diff}^0(M))$  are not finitely generated. Recall that  $\pi_1(M)$  satisfies the Farrell-Jones Conjecture, center( $\pi_1(M)$ ) is trivial, and  $\pi_i(\text{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $H^i(\text{BTop}(M)^\circ; \mathbb{Q})$  vanishes for  $1 \le i \le (\dim M - 7)/3 + 1$ .

Integral computations of the homotopy groups of automorphisms are much harder. We mention at least the following result taken from [344, Theorem 1.3].

**Theorem 9.203 (Homotopy groups of** Top(M) **for closed aspherical** M **with hyperbolic fundamental group).** *Let* M *be a smoothable aspherical closed topological manifold whose fundamental group*  $\pi$  *is hyperbolic.* 

Then there is a  $\mathbb{Z}/2$ -action on  $\mathbf{Wh}^{\text{TOP}}(B\pi)$  such that we obtain for every *n* satisfying  $1 \le n \le \min\{(\dim M - 7)/2, (\dim M - 4)/3\}$  isomorphisms

$$\pi_n(\operatorname{TOP}(M)) \cong \pi_{n+2} \Big( E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big( \bigvee_C \operatorname{Wh}^{\operatorname{TOP}}(BC) \big) \Big)$$

and an exact sequence

$$1 \to \pi_2 \Big( E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big( \bigvee_{(C)} \mathbf{Wh}^{\mathrm{TOP}}(BC) \big) \Big) \to \pi_0(\mathrm{TOP}(M)) \to \mathrm{Out}(\pi) \to 1$$

where (C) ranges over the conjugacy classes (C) of maximal infinite cyclic subgroups C of  $\pi$ .

It is conceivable that one can replace the range  $(\dim M - 7)/3$  appearing in Theorem 9.195, Theorem 9.196, and Remark 9.202 with the range min{ $(\dim M - 7)/2, (\dim M - 4)/3$ } appearing in Theorem 9.203.

The methods described in this book about automorphism groups of closed manifolds apply only to aspherical closed manifolds. It is of course also essential to study automorphism groups of disks. The techniques used to analyze them are quite different. There has been tremendous progress on this topic during recent years. We refer to the survey article by Randal-Williams [832], where also further references in the literature about this topic are given.

Moreover, there has recently been tremendous progress on the moduli spaces of manifolds, which gives information about the cohomology of the classifying space BDiff(M) of closed smooth manifolds M such as the connected sum of several copies of a product of spheres. We refer to the survey article by Galatius and Randal-Williams [413], where also further references in the literature about this topic are given.

# **9.22** Survey on Computations of *L*-Theory of Group Rings of Finite Groups

**Theorem 9.204** (Algebraic *L*-theory of  $\mathbb{Z}G$  for finite groups). Let *G* be a finite group. Then:

- (i) The groups  $L_n^{\langle j \rangle}(\mathbb{Z})$  are independent of the decoration j and given by  $\mathbb{Z}$ ,  $\{0\}$ ,  $\mathbb{Z}/2$ ,  $\{0\}$  for  $n \equiv 0, 1, 2, 3 \mod (4)$ ;
- (ii) For every  $n \in \mathbb{Z}$ , the groups  $L_n^{\langle s \rangle}(\mathbb{Z}G)$ ,  $L_n^{\langle h \rangle}(\mathbb{Z}G)$ ,  $L_n^{\langle p \rangle}(\mathbb{Z}G)$ ,  $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$ , and  $L_n^{\langle j \rangle}(\mathbb{Z}G)$  for every  $j \leq 1$  are finitely generated as abelian groups and contain no p-torsion for odd primes p. Moreover, they all are finite for odd n;
- (iii) Let r(G) be the number of isomorphisms classes of irreducible real *G*-representations. Let  $r_{\mathbb{C}}(G)$  be the number of isomorphisms classes of irreducible real *G*-representations *V* that are of complex type. For every decoration  $\langle j \rangle$  we have

$$\begin{split} L_n^{\langle j \rangle}(\mathbb{Z}G)[1/2] &\cong L_n^{\langle j \rangle}(\mathbb{Q}G)[1/2] \cong L_n^{\langle j \rangle}(\mathbb{R}G)[1/2] \\ &\cong \begin{cases} \mathbb{Z}[1/2]^{r(G)} & n \equiv 0 \ (4); \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \ (4); \\ 0 & n \equiv 1, 3 \ (4); \end{cases} \end{split}$$

- (iv) If G has odd order and n is odd, then  $L_n^{\varepsilon}(\mathbb{Z}G) = 0$  for  $\varepsilon = p, h, s$  and  $L_n^{(j)}(\mathbb{Z}G) \cong \mathbb{Z}/2^r$  for  $j \in \{-1, -2, ...,\} \amalg \{-\infty\}$ , where r is the number appearing in Theorem 4.22 (iii);
- (v) If G is a cyclic group of odd order, then the kernel of the split epimorphism  $L_n^s(\mathbb{Z}G) \to L_n^s(\mathbb{Z})$  is torsionfree. In particular,  $\operatorname{tors}(L_n^s(\mathbb{Z}G))$  is  $\mathbb{Z}/2$  if  $n \equiv 2 \mod 4$  and trivial otherwise.

*Proof.* (i) See for instance [667, Theorem 16.11 (i) on page 768].

(ii) See [987, Theorem 13.A.4 (i) on page 177], [465] for the decoration *s*. Now the claim follows for all decorations from the Rothenberg sequences, see Subsection 9.10.4, since the relevant *K*-groups of  $\mathbb{Z}G$  are all finitely generated abelian groups.

#### 9.23 Notes

(iii) See [839, Proposition 22.34 on page 253].

(iv) See [59], [465, Theorem 10.1] for  $\epsilon \in \{s, p, h\}$ . Note that  $K_n(\mathbb{Z}G) = 0$  for  $n \leq -2$  and  $K_{-1}(\mathbb{Z}G) = \mathbb{Z}^r$  by Theorem 4.22. The involution on  $K_{-1}(\mathbb{Z}G) = \mathbb{Z}^r$  is given by – id. Hence  $\widehat{H}^0(Z/2, K_{-1}(ZG)) = 0$  and  $H^1(Z/2, K_{-1}(ZG)) = (Z/2)^r$ . Since  $L_n^p(ZG) = 0$  for odd *n* and  $L_n^p(ZG)$  is known to be torsionfree for even *n*, the claim follows from the Rothenberg sequence (9.101). See also [458, Section 3].

(v) See [987, Theorem 13.A.4 (ii) on page 177], [465, Section 10].

# **9.23** Notes

The next problem is meanwhile solved and triggered surgery theory for non-simply connected manifolds. It is a kind of generalization of the *Space Form Problem* asking which finite groups occur as fundamental groups of closed Riemannian manifolds with constant positive sectional curvature.

**Problem 9.205 (Spherical Space Form Problem).** Which finite groups can act freely (topologically or smoothly) on a standard sphere, or, equivalently, occur as fundamental groups of closed manifolds whose universal covering is (homeomorphic or diffeomorphic to) a standard sphere.

More information about this interesting problem and its solution can be found in [284] and [694].

For a survey of the classification of fake spaces such as fake products of spheres, fake projective spaces, fake lens spaces, and fake tori, and the literature about them, we refer to [667, Chapter 18].

Our definition of the *L*-groups follows the original approach due to Wall. A much more satisfactory and elegant approach via chain complexes is due to Mishchenko and Ranicki and is of fundamental importance for many applications and generalizations, see for instance [667, 734, 735, 736, 835, 836, 837, 839].

We mention that a different approach to surgery has been developed by Kreck. A survey about his approach is given in [586]. Its advantage is that one does not have to get a complete homotopy classification first. The price to pay is that the *L*-groups are much more complicated, they are not necessarily abelian groups any more. This approach is in particular successful when the manifolds under consideration are already highly connected. See for instance [589, 590, 930].

More information about surgery theory can be found for instance in [169, 199, 200, 226, 667, 648, 843, 987].

We will relate the algebraic *L*-theory of  $C^*$ -algebras to their topological *K*-theory in Theorem 10.78. In particular, we get for all  $n \in \mathbb{Z}$  natural isomorphisms

$$L_n(C_r^*(G, \mathbb{R}))[1/2] \cong K_n^{\text{TOP}}(C_r^*(G; \mathbb{R}))[1/2];$$
  
$$L_n(C_r^*(G, \mathbb{C}))[1/2] \cong K_n^{\text{TOP}}(C_r^*(G; \mathbb{C}))[1/2].$$

We mention already here Conjecture 15.89, which deals with the passage for *L*-theory from  $\mathbb{Q}G$  to  $\mathbb{R}G$  to  $C_r^*(G;\mathbb{R})$ . Its connection to the Baum-Connes Conjecture and the Farrell-Jones Conjecture is analyzed in Lemma 15.90. For more information about the algebraic *L*-theory of  $C^*$ -algebras we refer to [615].

There is also a version of the Borel Conjecture for manifolds with boundary, which is implied by the Farrell-Jones Conjecture, see for instance [383, page 17 and page 31].

Another survey article about topological rigidity is [555].

There is an equivariant version of the Borel Conjecture where one replaces EG with the classifying space for proper *G*-actions  $\underline{E}G$ , see Definition 11.18. One may ask whether there is a compact closed *G*-manifold which is a model for  $\underline{E}G$  and whether, for two compact proper topological *G*-manifolds *M* and *N* that both are models for  $\underline{E}G$ , any *G*-homotopy equivalence between them is *G*-homotopic to a *G*-homeomorphism. This version is not true in general and investigated for instance in [251, 252, 253, 257, 277, 665, 999].

The vanishing of  $\kappa$ -classes for aspherical closed manifolds is analyzed in [475] using the Farrell-Jones Conjecture.

# Chapter 10 Topological *K*-Theory

# **10.1 Introduction**

In this chapter we deal with the *topological K-theory* of reduced group  $C^*$ -algebras, which is the target of the *Baum-Connes Conjecture*, in contrast to the algebraic *K*- and *L*-theory of group rings, which is the target of the Farrell-Jones Conjecture. We begin by reviewing the topological *K*-theory of spaces and its equivariant version for proper actions of possibly infinite discrete groups. Then we pass to its generalization to  $C^*$ -algebras. We discuss the *Baum-Connes Conjecture for torsionfree groups* 10.44 and present two applications, namely, to the *Trace Conjecture* about the integrality of the trace map and to the *Kadison Conjecture* about idempotents in reduced group  $C^*$ -algebras of torsionfree groups. Then we briefly state the main properties of *Kasparov's KK-theory* and its equivariant version (without explaining its construction). This will later be needed in Chapter 14 to explain the *analytic Baum-Connes assembly* map and state the *Baum-Connes Conjecture for arbitrary groups and with coefficients* in a G- $C^*$ -algebra.

# **10.2** Topological K-Theory of Spaces

# 10.2.1 Complex Topological K-Theory of Spaces

The complex topological K-theory of spaces, sometimes also called the complex topological K-cohomology of spaces, is a generalized cohomology theory, i.e., it assigns to a pair of CW-complexes  $(X, A) \ a \mathbb{Z}$ -graded abelian group  $K^*(X, A)$  and a homomorphism of degree one  $\delta^* : K^*(A) \to K^{*+1}(X, A)$  and to a map  $f : (X, A) \to (Y, B)$  of such pairs a homomorphism  $K^*(f) : K^*(Y, B) \to K^*(X, A)$  of  $\mathbb{Z}$ -graded abelian groups such that the Eilenberg-Steenrod axioms of a cohomology theory are satisfied, i.e., one has naturality, homotopy invariance, the long exact sequence of a pair, and excision. Moreover, the disjoint union axiom holds, see Definition 12.1. In contrast to singular cohomology the dimension axiom is not satisfied, actually  $K^n(\{\bullet\})$  is  $\mathbb{Z}$  if *n* is even and is trivial if *n* is odd. A very important feature is that topological complex *K*-theory satisfies *Bott periodicity*, i.e., there is a natural isomorphism of degree two compatible with the boundary map in the long exact sequence of pairs

$$\beta^*(X,A) \colon K^*(X,A) \xrightarrow{=} K^{*+2}(X,A).$$

Topological complex K-theory comes with a multiplicative structure.

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It can be constructed by the so-called *complex topological K-theory spectrum*  $\mathbf{K}_{\mathbb{C}}^{\text{TOP}}$  that is the following  $\Omega$ -spectrum. (Spectra will be defined in Section 12.4.) The *n*-th space is  $\mathbb{Z} \times BU$  for even *n* and  $\Omega(\mathbb{Z} \times BU)$  for odd *n*. The *n*-th structure map is given by the identity id:  $\Omega(\mathbb{Z} \times BU) \rightarrow \Omega(\mathbb{Z} \times BU)$  for odd *n* and by an explicit homotopy equivalence due to Bott  $\mathbb{Z} \times BU \stackrel{\sim}{\to} \Omega^2(\mathbb{Z} \times BU)$  for even *n*. As usual, associated to this spectrum is also a generalized homology theory  $K_*(X, A)$ , called the *topological complex K-homology of spaces*, such that  $K_n(\{\bullet\})$  is  $\mathbb{Z}$  if *n* is even and is trivial if *n* is odd. A proof of a universal coefficient theorem for complex *K*-theory can be found in [27] and [1023, (3.1)], the homological version then follows from [12, Note 9 and 15].

Rationally one can compute complex topological *K*-theory by Chern characters. (Equivariant versions will be explained in Section 12.7.) Namely, we get for any pair of *CW*-complexes (X, A) a natural Q-isomorphism

(10.1) 
$$\bigoplus_{p \in \mathbb{Z}, p \equiv n(2)} H_p(X, A; \mathbb{Q}) \xrightarrow{\cong} K_n(X, A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

and for any pair of finite CW-complexes (X, A) a natural Q-isomorphism

(10.2) 
$$K^n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{p \in \mathbb{Z}, p \equiv n(2)} H^p(X,A;\mathbb{Q}).$$

The condition that (X, A) is finite is needed in (10.2). The cohomological Chern character (10.2) is compatible with the multiplicative structures.

For integral computations one has to use the Atiyah-Hirzebruch spectral sequence, which does not collapse in general.

Exercise 10.3. Let X be a finite CW-complex. Show for its Euler characteristic

$$\chi(X) = \operatorname{rk}_{\mathbb{Z}}(K^0(X)) - \operatorname{rk}_{\mathbb{Z}}(K^1(X)) = \operatorname{rk}_{\mathbb{Z}}(K_0(X)) - \operatorname{rk}_{\mathbb{Z}}(K_1(X)).$$

The groups  $K^*(BG)$  can be computed explicitly for all finite groups G using the

Completion Theorem due to Atiyah-Segal [43, 51], see for instance [656, Theorem 0.3]. Namely, if for a prime p we denote by r(p) the number of conjugacy classes (g) of elements  $g \in G$  whose order |g| is  $p^d$  for some integer  $d \ge 1$ , and by  $\mathbb{Z}_p$  the *p*-adic integers, then there are isomorphisms of abelian groups

(10.4) 
$$K^{0}(BG) \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\mathbb{Z}_{p})^{r(p)};$$

(10.5) 
$$K^1(BG) \cong 0$$

One can also figure out the multiplicative structure on  $K^0(BG)$  in (10.4). This shows how accessible topological *K*-theory is. For instance, one does *not* know the group cohomology  $H^*(BG)$  of all finite groups *G*.

If X is a finite CW-complex,  $K^*(X)$  can be described in terms of vector bundles. For instance,  $K^0(X)$  is the Grothendieck group associated to the abelian monoid of isomorphism classes of (finite-dimensional complex) vector bundles over X under the Whitney sum. Naturality comes from the pullback construction, the multiplicative structure from the tensor product of vector bundles.

There are a *Thom isomorphism* and a *Künneth Theorem* for finite *CW*-complexes for topological complex *K*-cohomology, see [48, Corollary 2.7.12 on page 111 and Corollary 2.7.15 on page 113].

Using exterior powers one can construct the so-called *Adams operations* on topological complex *K*-cohomology. They were a key ingredient in the work of Adams on the Hopf invariant one problem, see [3, 14], and on linear independent vector fields on spheres, see [4, 5, 6]. Atiyah [44] introduced the groups J(X) where vector bundles are considered up to fiber homotopy equivalence. They were studied by Adams [8, 9, 10, 11].

Complex topological *K*-theory is one of the first generalized cohomology theories. There are other generalized (co)homology theories such as *bordism*, see for instance [982], *complex bordism*, see for instance [844], *Morava K-theory*, see for instance [1021], *elliptic cohomology*, see for instance [692, 947], and *topological modular forms* tmf, see for instance [501, 502, 692], which have been of great interest in algebraic topology over the last decades.

The connection between topological K-theory and spaces of Fredholm operators was explained by Jänich [519]. Namely, there exists a natural bijection of abelian groups for finite *CW*-complexes X

(10.6) 
$$[X, \operatorname{Fred}] \cong K^0(X)$$

where Fred is the space of *Fredholm operators*, i.e., bounded operators with finitedimensional kernel and cokernel. This shows that there is a relation between topological *K*-theory and index theory. For instance, we get from (10.6) applied to  $X = \{\bullet\}$  an isomorphism  $\pi_0(\text{Fred}) = K^0(\{\bullet\}) \cong \mathbb{Z}$  that sends a Fredholm operator to its classical index which is the difference of the dimension of its kernel and the dimension of its cokernel. The bijection of (10.6) assigns to a map  $X \to \text{Fred}$ which can be interpreted as a family of Fredholm operators parametrized by *X*, its family index which is essentially the difference of the class of the vector bundle over *X* whose fiber over *x* is the kernel of the Fredholm operator associated to  $x \in X$ and the vector bundle over *X* whose fiber over *x* is the cokernel of the Fredholm operator associated to  $x \in X$ . Good introductions to index theory are the seminal papers [50, 52, 53, 55, 56]. Other references about index theory are [142, 853, 1022].

#### **10.2.2 Real Topological K-Theory of Spaces**

There is also the *real topological K-theory of spaces*, sometimes also called the *real topological KO-cohomology of spaces*,  $KO^*(X, A)$  and *real topological K-homology*  $KO_*(X, A)$ , where one considers real vector bundles instead of complex vector bundles and *BO* instead of *BU*. One uses a specific homotopy equivalence  $\mathbb{Z} \times BO \xrightarrow{\simeq} \Omega^8(\mathbb{Z} \times BO)$  to construct the so-called *real K-theory spectrum*  $\mathbf{K}_{\mathbb{R}}^{\text{TOP}}$ . A much more sophisticated and structured symmetric spectrum representing real *K*-theory in terms of Fredholm operators was constructed by Joachim [523, 524] and Mitchener [744] based on ideas of Atiyah-Singer [54].

The main difference between the real and the complex version is that  $KO_*$  is 8-periodic and  $KO_n(\{\bullet\}) = KO^{-n}(\{\bullet\})$  is given by  $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0$  for n = 0, 1, 2, 3, 4, 5, 6, 7. There are natural transformations  $i^*: KO^*(X) \to K^*(X)$ and  $r^*: K^*(X) \to KO^*(X)$  that corresponds to assigning to a real vector bundle its complexification and to a complex vector bundle its restriction to a real vector bundle. They satisfy  $r \circ i = 2 \cdot id$ . They also exist on K-homology. It is sometimes useful to consider the real topological K-theory instead of the complex version, since one loses information when passing to the complex topological version. On the other hand computations for the real topological K-theory are harder than for the complex topological K-theory, since the real version is 8-periodic and its value at  $\{\bullet\}$  contains 2-torsion, whereas the complex version is 2-periodic and its evaluation at  $\{\bullet\}$  is much simpler than for the real version.

Rationally we get again a Chern character, namely, for any pair of *CW*-complexes (X, A) a natural  $\mathbb{Q}$ -isomorphism

(10.7) 
$$\bigoplus_{p \in \mathbb{Z}, p \equiv n(4)} H_p(X, A; \mathbb{Q}) \xrightarrow{\cong} KO_n(X, A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

and for any pair of finite CW-complexes (X, A) a natural Q-isomorphism

a .

(10.8) 
$$KO^n(X,A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{p \in \mathbb{Z}, p \equiv n(4)} H^p(X,A;\mathbb{Q}).$$

There is a natural transformation of homology theories called *KO-orientation of Spin bordism* due to Atiyah-Bott-Shapiro [49], which can be interpreted by sending a Spin manifold to the index class of the associated Dirac operator

(10.9) 
$$D: \Omega_n^{\text{spin}}(X) \to KO_n(X).$$

It plays an important role for the question when a closed spin manifold admits a Riemannian metric of positive sectional curvature, see Subsection 14.8.4.

A relation of *KO*-theory to surgery theory has already been explained in Remark 9.133.

10.2 Topological K-Theory of Spaces

Another variant of topological *K*-theory denoted by  $KR^*(X, A)$  was defined by Atiyah [45]. *Twisted topological K-theory* has been studied intensively, see instance [41, 42, 400, 552].

More information about the topological *K*-theory of spaces can be found for instance in [7, 40, 47, 48, 508, 509, 551, 620].

#### 10.2.3 Equivariant Topological K-Theory of Spaces

Equivariant topological *K*-theory has been considered for compact topological groups acting on compact spaces, see for instance [48, 905]. For our purposes it will be important to treat the more general case of a proper action of a not necessarily compact group. It suffices for us to consider discrete groups *G* and proper *G*-*CW*-complexes, or, equivalently, *CW*-complexes with a *G*-action such that all isotropy groups are finite and for every open cell *e* of *X* with  $g \cdot e \cap e \neq \emptyset$  we have gx = x for all  $x \in e$ . This is difficult enough, but not as hard as the much less understood case of a topological group acting properly on a locally compact Hausdorff space.

If *G* is a discrete group, *G*-cohomology theories  $K_G^*$  and  $KO_G^*$  are constructed by Lück-Oliver [670] for pairs of proper *G*-*CW*-complexes (*X*, *A*) using classifying spaces for *G*-vector bundles. More precisely, for every pair of proper *G*-*CW*complexes (*X*, *A*) one obtains  $\mathbb{Z}$ -graded abelian groups  $K_G^*(X, A)$  and  $KO_G^*(X, A)$ such that one has naturality, *G*-homotopy invariance, a long exact sequence of pairs, excision, and the disjoint union axiom holds, see Definition 12.1. The complex version  $K_G^*$  is 2-periodic, the real version is 8-periodic.

Let  $H \subseteq G$  be a finite subgroup. Then

(10.10) 
$$K_G^n(G/H) = \begin{cases} \operatorname{Rep}_{\mathbb{C}}(H) & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

There is a decomposition of the real group ring  $\mathbb{R}H$  as a direct product  $\prod_{i=0}^{r} M_{n_i,n_i}(D_i)$  of matrix algebras over skew-fields  $D_i$  where  $D_i$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . Then one obtains a decomposition for each  $n \in \mathbb{Z}$ 

(10.11) 
$$KO_G^{-n}(G/H) = \prod_{i=1}^r KO_G^{-n}(G/H)_i$$

where

$$KO_G^{-n}(G/H)_i = \begin{cases} KO_n(\{\bullet\}) & \text{if } D_i = \mathbb{R}; \\ K_n(\{\bullet\}) & \text{if } D_i = \mathbb{C}; \\ KO_{n+4}(\{\bullet\}) & \text{if } D_i = \mathbb{H}. \end{cases}$$

There is a natural *external multiplicative structure*, i.e., there is a natural pairing

(10.12) 
$$K_G^m(X,A) \otimes_{\mathbb{Z}} K_H^n(X,B) \to K_{G \times H}^{m+n}((X,A) \times (Y,B))$$

for discrete groups G and H and a pair (X, A) of proper G-CW-complexes and a pair (Y, B) of proper H-CW-complexes. There exists a natural *restriction homomorphism* for any inclusion  $i: H \to G$  of discrete groups

(10.13) 
$$i^*: K^*_G(X, A) \to K^*_H(i^*(X, A)),$$

where (X, A) is a pair of proper *G*-*CW*-complexes and  $i^*(X, A)$  is its restriction to *H*. Applying this to the diagonal map  $G \to G \times G$  and the external product and using the diagonal embedding  $X \to X \times X$ , one obtains a natural *internal multiplicative structure*, i.e., natural pairings

(10.14) 
$$K_G^m(X,A) \otimes_{\mathbb{Z}} K_G^n(X,B) \to K_G^{m+n}(X,A \cup B)$$

for a discrete group G and a proper G-CW-complex X with G-CW-subcomplexes A and B. In particular,  $K_G^*(X)$  becomes a  $\mathbb{Z}$ -graded algebra for any proper G-CW-complex X. Given a group homomorphism  $\alpha : H \to G$ , there is an *induction homomorphism* 

(10.15) 
$$\operatorname{ind}_{\alpha} \colon K^*_H(X, A) \to K^*_G(\operatorname{ind}_{\alpha}(X, A)),$$

where (X, A) is a proper *H*-*CW*-complex and  $\operatorname{ind}_{\alpha}(X, A)$  is the proper *G*-*CW*-complex  $G \times_{\alpha} (X, A)$ . If ker $(\alpha)$  acts freely on (X, A), the map  $\operatorname{ind}_{\alpha}$  is bijective.

All the constructions and results above are carried out in [670], and the corresponding statements also hold for the real version  $KO_G^*$ . If G is finite, they all reduce to the classical constructions and results.

One can give a description for pairs (X, A) of finite proper *G*-*CW*-complexes for a discrete group *G* in terms of *G*-vector bundles such that for instance  $K_G^0(X)$  and  $KO_G^0(X)$  respectively agree with the Grothendieck groups of isomorphism classes of *G*-equivariant complex and real respectively vector bundles over the finite proper *G*-*CW*-complex *X*. This follows from [671, Theorem 3.2 and Theorem 3.15] and [670, Proposition 1.5]. (A *C*\*-theoretic analog of this result is discussed in [115, Section 6].) However, the interpretation of  $K_G^0(X)$  in terms of vector bundles does not hold if *G* is a Lie group, as explained in [671, Section 5]. A description in terms of infinite-dimensional *G*-vector bundles is discussed by Phillips [806]. The question of what the Grothendieck group of isomorphism classes of *G*-vector bundles over a classifying space *BG* of a compact Lie group *G* looks like and how it is related to  $K^0(BG)$  is treated in [518]. (Note that this is a non-trivial question already for finite groups, since *BG* does not have a finite-dimensional *CW*-model for non-trivial finite groups.)

Let *G* be a discrete group. For any cyclic group  $C \subseteq G$  of order  $n < \infty$  we denote by  $\mathbb{Z}[\zeta_C] \subseteq \mathbb{Q}(\zeta_C)$  the cyclotomic ring and field generated by the *n*-th roots of unity. We regard them as quotient rings of the group rings  $\mathbb{Z}[\hom(C, \mathbb{C}^*)] \subseteq$ 

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 $\mathbb{Q}[\hom(C, \mathbb{C}^*)]$ . In other words, we fix an identification of the *n*-th roots of unity in  $\mathbb{Q}(\zeta_C)$  with the irreducible characters of *C*. Let C(G) be a set of conjugacy class representatives for the cyclic subgroups  $C \subseteq G$  of finite order. Denote by  $C_G C$  the centralizer and by  $N_G C$  the normalizer of *C* in *G*. Then for any pair of finite proper *G*-complexes (*X*, *A*), there is the following version of an *equivariant Chern character*, namely, a natural isomorphism of rings

(10.16) 
$$K_G^*(X;A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{C \in C(G)} \left( H^*((X,A)^C / C_G C; \mathbb{Q}(\zeta_C)) \right)^{N_G C / C_G C}$$

where  $N_G C/C_G C$  acts via the conjugation action on  $\mathbb{Q}(\zeta_C)$  and on  $X^C/C_G C$  in terms of the given *G*-action on *X*.

Equivariant Chern characters can be used to compute  $K^*(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$  for infinite groups possessing a finite *G*-*CW*-model for its classifying space for proper *G*-actions, i.e., for instance for hyperbolic groups *G* or compact lattices *G* in connected Lie groups, see [656] and also [15, 16]. More information about  $K^*(BG)$  for infinite groups can be found in [525, Theorem 0.1], and about cohomological Chern characters in [653].

**Exercise 10.17.** Let *G* be an abelian group. Let *X* be a finite proper *G*-*CW*-complex. Show that there is a  $\mathbb{Q}$ -isomorphism

$$K^*_G(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{C \in C(G)} H^*(X^C/G; \mathbb{Q})^{\varphi(|C|)}$$

for the Euler Phi-function  $\varphi$ .

We will construct in Section 10.6 the equivariant *K*-homology  $K_*^G$  that is a *G*-homology theory defined for pairs of proper *G*-*CW*-complexes for discrete groups *G* and satisfies the disjoint union axiom.

An equivariant Universal Coefficient Theorem for equivariant complex K-theory for discrete groups G and finite proper G-CW-complexes X is given in [525, Theorem 0.3], namely, there are short exact sequences, natural in X,

(10.18) 
$$0 \to \operatorname{Ext}_{\mathbb{Z}}(K^G_{*-1}(X), \mathbb{Z}) \to K^*_G(X) \to \hom_{\mathbb{Z}}(K^G_*(X), \mathbb{Z}) \to 0;$$

(10.19) 
$$0 \to \operatorname{Ext}_{\mathbb{Z}}(K_G^{*+1}(X), \mathbb{Z}) \to K_*^G(X) \to \hom_{\mathbb{Z}}(K_G^*(X), \mathbb{Z}) \to 0.$$

It reduces for a finite group G to the one of Bökstedt [146], as explained in [525, Remark 5.21],

An external Künneth Theorem for complex K-theory relating  $K^*_{G \times H}(X \times Y)$  to  $K^*_G(X)$  and  $K^*_H(Y)$  is given in [730] for compact Lie groups G and H and finite G-CW-complexes X and Y, namely, there is a short exact sequence

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$$(10.20) \quad 0 \to \bigoplus_{i+j=n} K_G^i(X) \otimes_{\mathbb{Z}} K_H^j(Y) \to K_{G \times H}^n(X \times Y)$$
$$\to \bigoplus_{i+j=n+1} \operatorname{Tor}_{\mathbb{Z}}(K_G^i(X), K_H^j(Y)) \to 0.$$

The situation is much more complicated and much less understood if one wants to relate  $K_G^*(X \times Y)$  to  $K_G^*(X)$  and  $K_G^*(Y)$  for a finite group *G* and finite *G*-*CW*-complexes *X* and *Y*, see [500, 864]. This complication is not surprising, since it is related to the difficult question of computing  $K_G^*(X \times Y)$  for the diagonal *G*-action on  $X \times Y$  from  $K_{G \times G}^*(X \times Y)$  for a finite group *G* and finite *G*-*CW*-complexes *X* and *Y*.

**Exercise 10.21.** Let *G* and *H* be discrete groups. Let (X, A) be a pair of finite proper *G*-*CW*-complexes, and let (Y, B) be a pair of finite proper *H*-*CW*-complexes. Suppose that either  $K_G^i(X)$  is torsionfree for  $i \in \mathbb{Z}$  or that  $K_H^j(Y)$  is torsionfree for all  $j \in \mathbb{Z}$ .

Then the external multiplicative structure induces for every  $n \in \mathbb{Z}$  an isomorphism

$$\bigoplus_{i+j=n} K_G^i(X,A) \otimes_{\mathbb{Z}} K_H^j(Y,B) \xrightarrow{\cong} K_{G\times H}^n((X,A) \times (Y,B)).$$

Consider a discrete group G and a complex G-vector bundle  $p: E \to X$  with Hermitian metric over a finite proper G-CW-complex. Let  $p_{DE}: DE \to X$  be the disk bundle and  $p_{SE}: SE \to X$  be the sphere bundle associated to p whose fiber over  $x \in X$  is the disk and sphere in  $p^{-1}(x)$ . Then there exists a Thom class  $\lambda_E \in K_G^0(DE, SE)$  and the composite

(10.22) 
$$T_E \colon K_G^*(X) \xrightarrow{K_G^*(p_{DE})} K_G^*(DE) \xrightarrow{-\cup \lambda_E} K_G^*(DE, SE)$$

is an isomorphism of  $\mathbb{Z}$ -graded abelian groups called the *Thom isomorphism*, see [671, Theorem 3.14].

**Exercise 10.23.** For a discrete group *G* and a complex *G*-vector bundle  $p: E \to X$  over a finite proper *G*-*CW*-complex define its *Euler class*  $e(p) \in K_0^G(X)$  to be the image of the Thom class under the composite  $K_G^0(DE, SE) \xrightarrow{K_G^0(j)} K_G^0(DE) \xrightarrow{K_G^0(p_{DE})^{-1}} K_G^0(X)$  for  $j: DE \to (DE, SE)$  the inclusion. Show that there exists a long exact *Gysin sequence* 

$$(10.24) \quad \cdots \xrightarrow{\delta^{n-1}} K_G^n(X) \xrightarrow{-\cup e(p)} K_G^n(X) \xrightarrow{K_G^n(p_{SE})} K_G^n(SE)$$
$$\xrightarrow{\delta^n} K_G^{n+1}(X) \xrightarrow{-\cup e(p)} K_G^{n+1}(X) \xrightarrow{K_G^{n+1}(p_{SE})} \cdots$$

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A *Completion Theorem* for complex and real topological K-theory allowing families of subgroups is proved in [670, Theorem 6.5] for a discrete group G and a finite proper G-CW-complex X in terms of isomorphisms of pro-systems, see also [671, Theorem 4.3]. Let  $p: EG \rightarrow BG$  be the universal covering of BG, or, equivalently, the universal principal G-bundle. Up to G-homotopy EG is uniquely characterized by the property that it is a free G-CW-complex which is (after forgetting the group action) contractible. A consequence of the Completion Theorem is that the inverse system

$$\left[K^*\left((EG\times_G X)^{(n)}\right)\right]_{n>0}$$

satisfies the Mittag-Leffler condition and we obtain isomorphisms

(10.25) 
$$K_G^*(X)_I \cong K^*(EG \times_G X) \cong \operatorname{invlim}_{n \to \infty} K^*((EG \times_G X)^{(n)})$$

Here  $K_G^*(X)_I$  is the completion of  $K_G^*(X)$  with respect to the so-called augmentation ideal *I* that is the kernel of the dimension map  $K_G^0(\underline{E}G) \to \mathbb{Z}$  for  $\underline{E}G$  the classifying space for proper *G*-actions, and we have to assume that there is a finite-dimensional model for  $\underline{E}G$ . If *G* is finite and we take  $X = \{\bullet\}$ , this reduces to the classical Atiyah-Segal Completion Theorem predicting an isomorphism

$$K^{n}(BG) = \begin{cases} \operatorname{Rep}_{\mathbb{C}}(G)_{\widehat{I}} & n \text{ even;} \\ 0 & n \text{ odd,} \end{cases}$$

where *I* is the augmentation ideal, i.e., kernel of the map given by taking complex dimension  $\operatorname{Rep}_{\mathbb{C}}(G) \to \mathbb{Z}$ . There is also a version for the real topological *K*-theory.

A Cocompletion Theorem for the topological complex K-homology for discrete groups and finite proper G-CW-complexes is proved in [525, Theorem 0.2]. It assigns to a finite proper G-CW-complex X a short exact sequence

(10.26) 
$$0 \to \operatorname{colim}_{n \ge 1} \operatorname{Ext}_{\mathbb{Z}}^{1}(K_{G}^{*+1}(X)/I^{n} \cdot K_{G}^{*+1}(X), \mathbb{Z}) \to K_{*}(EG \times_{G} X)$$
$$\to \operatorname{colim}_{n \ge 1} \hom_{\mathbb{Z}}(K_{G}^{*}(X)/I^{n} \cdot K_{G}^{*}(X), \mathbb{Z}) \to 0.$$

The Completion and Cocompletion Theorems are not only interesting in their own right, they are needed in the computation of the topological *K*-theory of certain group  $C^*$ -algebras, see for instance [282, 283, 619, 988].

Another important tool for equivariant *K*-theory over compact Lie groups is the *Localization Theorem for equivariant topological complex K-theory* of Segal [905, Proposition 4.1]. Given a prime ideal  $\mathcal{P}$  of  $\operatorname{Rep}_{\mathbb{C}}(G) = K^0_G(\{\bullet\})$ , there is a topologically cyclic group *S* associated to  $\mathcal{P}$ , its so-called support. If *X* is a finite *G-CW*-complex, let  $X^{(S)}$  be the *G-CW*-subcomplex  $G \cdot X^S$ . Then after localization at  $\mathcal{P}$  the inclusion  $X^{(S)} \to X$  induces an isomorphism

(10.27) 
$$K_G^*(X)_{(\mathcal{P})} \xrightarrow{\cong} K_G^*(X^{(S)})_{(\mathcal{P})}.$$

Localization for equivariant cohomology theories for compact Lie groups is treated in general in [952, Chapter 7] and [953, III.3 and III.4].

Equivariant topological K-theory was designed for and is a key ingredient when one considers indices of equivariant operators. See for instance [50, 52, 53] where also applications such as Lefschetz Theorems, Riemann-Roch Theorems, and G-Signature Theorems are treated for compact Lie groups.

The  $K_G$ -degree of G-maps between spheres of unitary G-representations for a compact Lie group G is an important tool, see [953, II.5].

A discussion about equivariant *K*-theory and orbifold *K*-theory can be found in [18, Chapter 3].

A geometric description of equivariant *K*-homology for proper actions in term cycles built by proper cocompact G-Spin<sup>*c*</sup>-manifolds and smooth complex *G*-vector bundles over them is given in [115], extending the non-equivariant versions of [110, 114].

# **10.3** Topological *K*-Theory of *C*\*-Algebras

#### **10.3.1 Basics about** *C*\*-Algebras

For this section let F be  $\mathbb{R}$  or  $\mathbb{C}$ . For  $\lambda \in F$ , denote by  $\overline{\lambda}$  the complex conjugate of  $\lambda$ . A *Banach algebra* over F is an associative F-algebra  $A = (A, +, \cdot)$  together with a norm  $|| \quad ||$  for the underlying F-vector space such that the underlying F-vector space is complete with respect to the given norm and we have the inequality  $||a \cdot b|| \le ||a|| \cdot ||b||$  for all elements  $a, b \in A$ .

A Banach \*-algebra is a Banach algebra together with an involution \*:  $A \to A$ ,  $a \mapsto a^*$  satisfying  $(a^*)^* = a$ ,  $(a \cdot b)^* = b^* \cdot a^*$ ,  $(\lambda \cdot a + \mu \cdot b)^* = \overline{\lambda} \cdot a^* + \overline{\mu} \cdot b^*$ , and  $||a^*|| = ||a||$  for  $a, b \in A$  and  $\lambda, \mu \in F$ . If G is a discrete group,  $L^1(G, F)$  carries the structure of a Banach \*-algebra coming from the convolution product, the  $L^1$ -norm, and the involution sending  $\sum_{g \in G} \lambda_g \cdot g$  to  $\sum_{g \in G} \overline{\lambda_g} \cdot g^{-1}$ .

A  $C^*$ -algebra is a Banach \*-algebra A that additionally satisfies the  $C^*$ -identity  $||a^*a|| = ||a||^2$  for all  $a \in A$ . A homomorphism of  $C^*$ -algebras  $f: A \to B$  is a homomorphism of F-algebras in the algebraic sense that respects the involutions. A consequence of the C<sup>\*</sup>-identity is that a homomorphism of C<sup>\*</sup>-algebras  $f: A \to B$ automatically satisfies  $||f(a)|| \le ||a||$  for all  $a \in A$  and is in particular continuous. Moreover, any injective homomorphism of  $C^*$ -algebras  $f: A \to B$  is automatically isometric, i.e., satisfies ||f(a)|| = ||a|| for all  $a \in A$ , and two C<sup>\*</sup>-algebras which are isomorphic as F-algebras with involutions in the purely algebraic sense are automatically isomorphic as  $C^*$ -algebras. Two homomorphisms  $f, g: A \to B$  are *homotopic* if there is a path  $\{\gamma_t \mid t \in [0,1]\}$  of homomorphisms of C<sup>\*</sup>-algebras  $\gamma_t: A \to B$  such that  $\gamma_0 = f$  and  $\gamma_1 = g$  and for every a the evaluation map  $[0,1] \rightarrow B, t \mapsto \gamma_t(a)$  is continuous with respect to the C<sup>\*</sup>-norm on B. Equivalently, there is a homomorphism of C<sup>\*</sup>-algebras  $\gamma: A \to C([0, 1], B)$  to the C<sup>\*</sup>-algebra of continuous functions from [0, 1] to B under the supremum norm such that its composite with the evaluation maps at t = 0 and t = 1 from C([0, 1], B) to B are f and g.

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If *H* is a Hilbert *F*-space, then the algebra of bounded operators  $\mathcal{B}(H)$  with the involution given by taking adjoint operators and the operator norm is a  $C^*$ -algebra. Any subalgebra  $A \subseteq \mathcal{B}(H)$  which is closed in the norm topology and closed under taking adjoints inherits the structure of a  $C^*$ -algebra, and any  $C^*$ -algebra is isomorphic as a  $C^*$ -algebra to such an *A*.

We are not requiring a multiplicative unit. If the Banach algebra or  $C^*$ -algebra A has a multiplicative unit, we call A a *unital Banach algebra* or *unital C\*-algebra*.

Given a  $C^*$ -algebra A, an *ideal* in A is a two-sided ideal of the underlying F-algebra that is closed in the norm topology. It is automatically closed under the involution and hence inherits the structure of a  $C^*$ -algebra. The quotient A/I inherits the structure of a  $C^*$ -algebra by the obvious F-algebra structure and the norm  $||a + I||_{A/I} := \inf\{||a + i||_A \mid i \in I\}$ . Kernels of  $C^*$ -homomorphisms  $f : A \to B$  are ideals A and each ideal in A is the kernel of some homomorphism of  $C^*$ -algebras with A as source, namely, of the projection  $A \to A/I$ .

Fix an infinite-dimensional separable *F* Hilbert space *H*. Let  $\mathcal{B}$  be the unital *C*<sup>\*</sup>-algebra of bounded operators  $H \to H$ . An element  $T \in \mathcal{B}(H)$  is *compact* if for any bounded subset  $B \subseteq H$  the closure of T(B) is a compact subset of *H*. The compact operators form an ideal  $\mathcal{K}$  in  $\mathcal{B}$ . The *Calkin algebra* is the unital *C*<sup>\*</sup>-algebra  $\mathcal{B}/\mathcal{K}$ 

Let *X* be a locally compact Hausdorff space. Denote by  $C_0(X, F)$  the  $C^*$ -algebra of continuous functions  $f: X \to F$  that vanish at infinity, i.e., for every  $\epsilon > 0$  there exists a compact subset  $C \subseteq X$  such that  $|f(x)| \le \epsilon$  holds for all  $x \in X \setminus C$ . If *F* is clear from the context, we often abbreviate  $C_0(X) = C_0(X, F)$ . Define an involution  $*: C_0(X, F) \to C_0(X, F)$  by sending *f* to the function mapping  $x \in X$  to  $\overline{f(x)}$ . Equip  $C_0(X, F)$  with the supremum norm. Then  $C_0(X, F)$  is a  $C^*$ -algebra. If *X* is compact, the constant function on *X* with value 1 is a unit. Moreover,  $C_0(X, F)$  is unital if and only if *X* is compact.

**Example 10.28 (One-point and Stone-Čech compactification).** If X is a locally compact Hausdorff space, then we can assign to it two compactifications, the one-point compactification  $X_+$  and the Stone-Čech compactification  $\beta X$ , see [754, page 183 and Section 5.3]. Then  $C_0(X_+, F)$  agrees with  $C_0(X, F)_+$  and  $C(\beta X, F)$  agrees with  $C_b(X, F)$ , the C\*-algebra of bounded continuous functions  $X \to F$ . (Actually  $C_b(X, F)$  is the so-called multiplier algebra of  $C_0(X, F)$ .) See for instance [991, Example 2.1.2 on page 28 and Example 2.2.4 on page 32].

Let  $L^2(G, F)$  be the Hilbert *F*-space whose orthonormal basis is *G*. If *F* is clear from the context, we often abbreviate  $L^2(G) = L^2(G, F)$ . Let  $\mathcal{B}(L^2(G, F))$  denote the bounded linear operators on the Hilbert *F*-space  $L^2(G, F)$ . The *reduced group*  $C^*$ -algebra  $C_r^*(G, F)$  is the closure in the norm topology of the image of the regular representation  $FG \to \mathcal{B}(L^2(G, F))$  that sends an element  $u \in FG$  to the (left) *G*-equivariant bounded operator  $L^2(G, F) \to L^2(G, F)$  given by right multiplication by *u*. Let  $L^1(G, F)$  be the Banach \*-algebra of formal sums  $\sum_{g \in G} \lambda_g \cdot g$  with coefficients in *F* such that  $\sum_{g \in G} |\lambda_g| < \infty$ . If *F* is clear from the context, we often abbreviate  $L^1(G) = L^1(G, F)$ . There are natural inclusions

$$FG \subseteq L^1(G,F) \subseteq C_r^*(G,F) \subseteq \mathcal{B}(L^2(G,F))^G \subseteq \mathcal{B}(L^2(G,F)).$$

**Exercise 10.29.** Show for a discrete group *G* that  $L^1(G, F)$  is a *C*<sup>\*</sup>-algebra if and only if *G* is trivial or (*G* has order 2 and  $F = \mathbb{R}$ ).

For a group G let  $C_m^*(G, F)$  be its maximal group  $C^*$ -algebra, that is, the norm closure of the image of the so-called universal representation  $FG \rightarrow \mathcal{B}(H_u)$ , compare [802, 7.1.5 on page 229]. The maximal group  $C^*$ -algebra has the advantage that every homomorphism of groups  $\phi: G \rightarrow H$  induces a homomorphism  $C_m^*(G, F) \rightarrow C_m^*(H, F)$  of  $C^*$ -algebras. This is not true for the reduced group  $C^*$ -algebra  $C_r^*(G, F)$ . Here is a counterexample. Since  $C_r^*(G, F)$  is a simple algebra if G is a non-abelian free group [813], there is no unital algebra homomorphism  $C_r^*(G, F) \rightarrow C_r^*(\{1\}, F) = F$ . There is a canonical homomorphism of  $C^*$ -algebras  $C_m^*(G, F) \rightarrow C_r^*(G, F)$ , which is an isomorphism of  $C^*$ -algebra if and only if G is amenable, see [802, Theorem 7.3.9 on page 243].

If F is clear from the context, we often abbreviate  $C_r^*(G) = C_r^*(G, F)$  and  $C_m^*(G) = C_m^*(G, F)$ .

Given a discrete group G, a G-C\*-algebra A is a C\*-algebra together with a G-action  $\rho: G \rightarrow \operatorname{aut}(A)$  by C\*-automorphisms. One can associate to a G-C\*-algebra A two new C\*-algebras, its reduced crossed product of C\*-algebras A  $\rtimes_r G$  and its maximal crossed product of C\*-algebras A  $\rtimes_m G$ , see [802, 7.6.5 on page 257 and 7.7.4 on page 262]. There is a canonical homomorphism from the maximal crossed product to the reduced crossed product, which is an isomorphism if G is amenable, see [802, Theorem 7.7.7. on page 263]. If we take A = F with the trivial G-action, then  $F \rtimes_r G$  and  $F \rtimes_m G$  are just  $C_r^*(G, F)$  and  $C_m^*(G, F)$ .

Let  $\{A_i \mid i \in I\}$  be a directed system of  $C^*$ -algebras. Then its *colimit*, often also called the *inductive limit*, or *direct limit*, is a  $C^*$ -algebra denoted by  $\operatorname{colim}_{i \in I} A_i$ , together with homomorphisms of  $C^*$ -algebras  $\psi_j : A_i \to \operatorname{colim}_{i \in I} A_i$  for every  $j \in I$ such that  $\psi_j \circ \phi_{i,j} = \psi_i$  holds for  $i, j \in I$  with  $i \leq j$  and the following universal property is satisfied: For every  $C^*$ -algebra B and every system of homomorphisms of  $C^*$ -algebras  $\{\mu_i : A_i \to B \mid i \in I\}$  such that  $\mu_j \circ \phi_{i,j} = \mu_i$  holds for  $i, j \in I$  with  $i \leq j$ , there is precisely one homomorphism of  $C^*$ -algebras  $\mu$ :  $\operatorname{colim}_{i \in I} A_i \to B$ satisfying  $\mu \circ \psi_i = \mu_i$  for every  $i \in I$ . The colimit exists and is unique up to isomorphism of  $C^*$ -algebras.

An extensive discussions about tensor products  $A \otimes B$  of  $C^*$ -algebras can be found in [991, Appendix T]. There are various ways for two  $C^*$ -algebras A and B to complete their algebraic tensor product  $A \otimes_F B$  to a new  $C^*$ -algebra  $A \otimes B$ . One is the spatial norm, which turns out to be the minimal norm and leads to the *spatial tensor product*, sometimes also called *the minimal tensor product*. A second is the maximal norm, which leads to the *maximal tensor product*. Any  $C^*$ -norm on the algebraic tensor product lies between the minimal and the maximal norm. The favorite situation is the case where A is a so-called *nuclear*  $C^*$ -algebra, i.e., the minimal and the maximal norm on the algebraic tensor product  $A \otimes_F B$  agree for any  $C^*$ -algebra B. Then for any  $C^*$ -algebra B there exists only one  $C^*$ -norm on the

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algebraic tensor product  $A \otimes_F B$  and hence there is a unique tensor product  $C^*$ -algebra  $A \widehat{\otimes} B$ . Commutative  $C^*$ -algebras and finite-dimensional  $C^*$ -algebras are nuclear. The class of nuclear  $C^*$ -algebras is closed under taking colimits over directed systems and extensions. In particular, the  $C^*$ -algebra of compact operators  $\mathcal{K}$  is nuclear. Ideals in and quotients of nuclear  $C^*$ -algebras are again nuclear. The reduced group  $C^*$ -algebra of G is nuclear if and only if G is amenable.

Given a  $C^*$ -algebra A, define  $M_n(A) = A \otimes M_n(F)$ , which is well-defined since  $M_n(F) = \mathcal{B}(F^n)$  is nuclear. Actually, the underlying F-algebra of  $M_n(A)$  is the algebraic tensor product  $A \otimes_F M_n(F)$  itself, one does not have to complete.

The *C*\*-algebra  $\mathcal{K}$  of compact operators on an infinite-dimensional separable Hilbert *F*-space is the colimit of the directed system  $M_1(F) \to M_2(F) \to M_3(F) \to \cdots$ where the structure maps are given by taking the block sum with the (1, 1)-zero matrix (0). Given a *C*\*-algebra *A*, the tensor product  $A \otimes \mathcal{K}$  is the colimit of the directed system  $M_1(A) \to M_2(A) \to M_3(A) \to \cdots$ .

A  $C^*$ -algebra is called *separable* if its underlying topological space is separable, i.e., contains a dense countable subset.

A  $C^*$ -algebra A is called *stable* if A is isomorphic as a  $C^*$ -algebra to  $A \otimes \mathcal{K}$ . Since  $\mathcal{K} \otimes \mathcal{K}$  is isomorphic to  $\mathcal{K}$ , the tensor product  $A \otimes \mathcal{K}$  is a stable  $C^*$ -algebra for every  $C^*$ -algebra A.

More information about  $C^*$ -algebras can be found for instance in [38, 141, 250, 276, 310, 388, 539, 540, 802].

#### 10.3.2 Basic Properties of the Topological K-Theory of C\*-Algebras

*Topological K-theory* assigns to any (not necessarily unital)  $C^*$ -algebra A a  $\mathbb{Z}$ -graded abelian group  $K_*(A)$  such that the following properties hold:

(i) Functoriality

A homomorphism  $f: A \to B$  of  $C^*$ -algebras induces a map of  $\mathbb{Z}$ -graded abelian groups  $K_*(f): K_*(A) \to K_*(B)$ . If  $g: B \to C$  is another homomorphism of  $C^*$ -algebras, we have  $K_*(g \circ f) = K_*(g) \circ K_*(f)$ . Moreover  $K_*(\mathrm{id}_A) = \mathrm{id}_{K_*(A)}$ ;

(ii) *Homotopy invariance* Homotopic homomorphisms of C\*-algebras induce the same map on the topological K-theory;

(iii) Finite direct products

If *A* and *B* are  $C^*$ -algebras, their direct product  $A \times B$  inherits the structure of a  $C^*$ -algebra by  $||(a, b)|| = \max\{||a||, ||b||\}$ . The projections to the factors are homomorphisms of  $C^*$ -algebras and induce a natural isomorphism of  $\mathbb{Z}$ -graded abelian groups

$$K_*(A \times B) \xrightarrow{=} K_*(A) \times K_*(B);$$

(iv) Compatibility with colimits over directed systems

Let  $\{A_i \mid i \in I\}$  be a directed system of  $C^*$ -algebras. Then the canonical map of  $\mathbb{Z}$ -graded abelian groups is an isomorphism

$$\operatorname{colim}_{i \in I} K_*(A_i) \xrightarrow{=} K_*(\operatorname{colim}_{i \to I} A_i);$$

(v) *Morita equivalence* 

There are canonical isomorphisms  $K_*(A) \rightarrow K_*(M_n(A))$ ;

(vi) Stabilization

The canonical inclusion  $F = M_1(F) \rightarrow \mathcal{K}$  yields an inclusion  $i_A \colon A \rightarrow A \widehat{\otimes} \mathcal{K}$ . The induced map of  $\mathbb{Z}$ -graded abelian groups  $K_*(i_A) \colon K_*(A) \rightarrow K_*(A \widehat{\otimes} \mathcal{K})$  is an isomorphism;

(vii) Long exact sequence of an ideal

Let *I* be a (two-sided closed) ideal in the  $C^*$ -algebra *A*. Denote by  $i: I \to A$  the inclusion and by  $p: A \to A/I$  the projection. Then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

$$\cdots \xrightarrow{\partial_{n+1}} K_n(I) \xrightarrow{K_n(i)} K_n(A) \xrightarrow{K_n(p)} K_n(A/I) \xrightarrow{\partial_n} K_{n-1}(I)$$
$$\xrightarrow{K_{n-1}(i)} K_{n-1}(A) \xrightarrow{K_{n-1}(p)} K_{n-1}(A/I) \xrightarrow{\partial_{n-1}} \cdots ;$$

#### (viii) Bott periodicity

For any  $C^*$ -algebra A over F there exists an isomorphism of degree b(F)

$$\beta_*(A) \colon K_*(A) \xrightarrow{\cong} K_{*+b(F)}(A),$$

which is natural in *A*, and compatible with the boundary operator  $\partial_*$  of the long exact sequence of an ideal, where b(F) = 2 if  $F = \mathbb{C}$  and b(F) = 8 if  $F = \mathbb{R}$ ;

(ix) Commutative C\*-algebras

Let *X* be a finite *CW*-complex (or more generally, compact Hausdorff space). Then there are isomorphisms of  $\mathbb{Z}$ -graded abelian groups, natural in *X*,

$$K^*(X) \xrightarrow{=} K_*(C(X, \mathbb{C}));$$
  
$$KO^*(X) \xrightarrow{\cong} K_*(C(X, \mathbb{R})),$$

from the topological complex or real *K*-theory of *X* to the topological *K*-theory of the unital  $C^*$ -algebra C(X, F) of continuous functions  $X \to F$ .

Of course the last property about commutative  $C^*$ -algebras is closely related to the material in Section 2.4 about Swan's Theorem 2.27.

Notation 10.30 (*K* and *KO*). If one considers a real  $C^*$ -algebra, one often writes  $KO_*(A)$  instead of  $K_*(A)$  to indicate that the  $C^*$ -algebra under consideration lives over  $\mathbb{R}$ .

#### 10.3 Topological K-Theory of C\*-Algebras

The 0-th topological *K*-group  $K_0(A)$  of a  $C^*$ -algebra *A* agrees with the projective class group  $K_0(A)$  of the underlying ring (possibly without unit) in the sense of Definition 3.90. In contrast to  $K_0(A)$  the topology of *A* enters in the definition of  $K_1(A)$  as explained next.

If *A* is a *C*<sup>\*</sup>-algebra (with or without unit), then we define the unital *C*<sup>\*</sup>-algebra  $A_+$ as follows. The underlying unital *F*-algebra is  $A \oplus F$  with the addition  $(a, \lambda) + (b, \mu) =$  $(a + b, \lambda + \mu)$ , multiplication  $(a, \lambda) \cdot (b, \mu) = (a \cdot b + \lambda \cdot b + \mu \cdot a, \lambda \cdot \mu)$ , and unit (0, 1). The involutions sends  $(a, \lambda)$  to  $(a^*, \overline{\lambda})$ . The *C*<sup>\*</sup>-norm is explained for instance in [802, 1.1.3 on page 1] or [991, Proposition 2.1.7 on page 30]. Let  $p: A_+ \to F$ be the canonical projection sending  $(a, \lambda)$  to  $\lambda$ . It induces maps  $M_n(A_+) \to M_n(F)$ and  $GL_n(A_+) \to GL_n(F)$ , denoted again by *p*. Define

(10.31) 
$$\operatorname{GL}_{n}^{+}(A) := \{B \in \operatorname{GL}_{n}(A_{+}) \mid p(B) = 1\}.$$

This becomes a topological group by the subspace topology with respect to the inclusion  $GL_n^+(A) \subseteq M_n(A_+)$ . There is an obvious directed system of topological groups

$$\operatorname{GL}_1^+(A) \subseteq \operatorname{GL}_2^+(A) \subseteq \operatorname{GL}_3^+(A) \subseteq \cdots$$

coming from embedding  $M_n(A_+)$  into  $M_{n+1}(A_+)$  by taking the block sum with the (1, 1)-identity matrix (1). Its colimit is a topological group denoted by  $GL^+(A)$ . Let  $GL^+(A)^0$  be the path component of the unit element in  $GL^+(A)$ . Then we get

(10.32) 
$$K_1(A) = GL^+(A)/GL^+(A)_0 = \pi_0(GL^+(A)).$$

More generally, we have

(10.33) 
$$K_n(A) = \pi_{n-1}(\operatorname{GL}^+(A)) \text{ for } n \ge 1.$$

If *A* is unital, then one defines the topological group  $GL(A) = \operatorname{colim}_{n \to \infty} GL_n(A)$ and obtains a canonical isomorphism

(10.34) 
$$K_n(A) \cong \pi_{n-1}(\operatorname{GL}(A)) \quad \text{for } n \ge 1$$

**Exercise 10.35.** Compute for  $F = \mathbb{C}$  the topological *K*-theory of  $\mathcal{K}$ .

**Remark 10.36 (Six term sequence of an ideal).** Let  $F = \mathbb{C}$  in this Remark 10.36. Since  $K_*$  is two-periodic, one often thinks about it as a  $\mathbb{Z}/2$ -graded theory. The long exact sequence of an extension  $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$  becomes the *six-term exact sequence of an ideal* 

**Remark 10.37 (Topological** *K*-theory in terms of unitary groups). Let  $F = \mathbb{C}$  in this Remark 10.37. Let  $U_n(A)$  be the group of unitary (n, n)-matrices over *A*, i.e., (n, n)-matrices *U* that are invertible and satisfy  $U^{-1} = U^*$ , where  $U^*$  is defined by transposing and applying to each entry the involution on *A*. Define  $U_n^+(A) := \{U \in U_n(A_+) \mid p(U) = 1\}$ . Put  $U(A) = \operatorname{colim}_{n \to \infty} U_n(A)$  and  $U^+(A) := \operatorname{colim}_{n \to \infty} U_n^+(A)$ . Then then we have isomorphisms of groups, see [991, Proposition 4.2.6 on page 77],

$$K_1(A) = \mathrm{GL}^+(A)/\mathrm{GL}^+(A)_0 \cong \mathrm{GL}(A^+)/\mathrm{GL}(A^+)_0$$
$$\cong U^+(A)/U^+(A)_0 \cong U(A^+)/U(A^+)_0.$$

**Example 10.38 (On the boundary map and indices).** Let  $F = \mathbb{C}$  in this Example 10.38. Let *A* be a unital *C*<sup>\*</sup>-algebra,  $I \subseteq A$  be an ideal, and  $p: A \to A/I$  be the projection. Let *u* be a unitary element in A/I. Let  $a \in A$  be any element in *A* with p(a) = u and ||a|| = 1. Consider the (2, 2)-matrices over *A* 

$$\begin{split} P &:= \begin{pmatrix} aa^* & a(1a^*a)^{1/2} \\ a^*(1-aa^*)^{1/2} & 1-a^*a \end{pmatrix}; \\ Q &:= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{split}$$

where  $(1 - aa^*)^{1/2}$  is uniquely determined by the properties that it is positive, i.e., of the form  $b^*b$  for some  $b \in A$ , and satisfies  $(1 - aa^*)^{1/2} \cdot (1 - aa^*)^{1/2} = 1 - aa^*$ , and analogously for  $(1 - a^*a)^{1/2}$ . Then *P* is a projection, i.e.,  $P^2 = P$  and  $P^* = P$ , and *Q* is a projection. Moreover, P - Q lies in M<sub>2</sub>(*I*). Define matrices in M<sub>2</sub>(*I*<sub>+</sub>) by

$$P_{+} := \begin{pmatrix} (aa^{*} - 1, 1) & (a(1a^{*}a)^{1/2}, 0) \\ (a^{*}(1 - aa^{*})^{1/2}, 0) & (1 - a^{*}a, 0) \end{pmatrix};$$
$$Q := \begin{pmatrix} (0, 1) & (0, 0) \\ (0, 0) & (0, 0) \end{pmatrix}.$$

One easily checks  $P_+^2 = P_+$  and  $Q_+^2 = Q_+$  and  $P_+ - Q_+ \in I$ . Hence  $P_+$  and  $Q_+$  determine elements  $[P_+], [Q_+] \in K_0(I_+)$  such that the difference  $[P_+] - [Q_+]$  is mapped under the canonical projection  $K_0(I_+) \to K_0(\mathbb{C})$  to zero. Hence  $[P_+] - [Q_+]$  defines an element in  $K_0(I)$ . It turns out that the image  $\partial_1([u])$  of the class  $[u] \in K_1(A)$  under the boundary homomorphism  $\partial_1 \colon K_1(A/I) \to K_0(I)$  is the class  $[P_+] - [Q_+]$ , see [491, Proposition 4.8.10 (a) on page 109].

If we can additionally arrange that *a* is a partial isometry, i.e.,  $a^*a$  is a projection, then  $1 - a^*a$  and  $1 - aa^*$  lie in *I* and are projections, and we obtain an element  $[1 - a^*a] - [1 - aa^*]$  in  $K_0(I)$  which agrees with  $\partial_1([u])$ , see [491, Proposition 4.8.10 (b) on page 109].

Now we apply this to  $A = \mathcal{B} = \mathcal{B}(H)$  and  $I = \mathcal{K} = \mathcal{K}(H)$  for an infinitedimensional separable Hilbert space *H*. Let  $a \in \mathcal{B}$  be a Fredholm operator such that *a* is a partial isometry. Then  $1 - a^*a$  is the orthogonal projection onto the kernel of *a* and  $1 - aa^*$  is the orthogonal projection onto the cokernel of *a*. Hence the element  $[1-a^*a]-[1-aa^*] \in K_0(\mathcal{K})$  becomes under the standard identification  $K_0(\mathcal{K}) \cong \mathbb{Z}$  the difference of the dimension of the kernel of *a* and the dimension of the cokernel of *a*, which is by definition the classical index of the Fredholm operator *a*. This shows that  $\partial_1 : K_1(\mathcal{B}/\mathcal{K}) \to K_0(\mathcal{K}) \cong \mathbb{Z}$  sends the class of [a] to the classical index of *a*. Since  $K_n(\mathcal{B}) = 0$  holds for  $n \in \mathbb{Z}$ , see [991, 6.5 on page 123], the map  $\partial_1 : K_1(\mathcal{B}/\mathcal{K}) \xrightarrow{\cong} K_0(\mathcal{K}) \cong \mathbb{Z}$  is actually an isomorphisms and  $K_0(\mathcal{B}/\mathcal{K}) = \{0\}$ .

It will often occur in many more general and important situations that  $\partial_1$  can be viewed as an index map.

**Example 10.39 (Suspensions and cones).** The suspension of a  $C^*$ -algebra A is the  $C^*$ -algebra  $\Sigma A$  of continuous functions  $f: [0,1] \to A$  with f(0) = f(1) = 1 equipped with the obvious algebra structure and involution and the supremum norm inherited from A. Denote by  $\Sigma^n(A)$  the *n*-fold suspension. It can be identified with the tensor product of  $C^*$ -algebras  $A \otimes C_0(\mathbb{R}^n)$ . The *cone* is defined analogously as the  $C^*$ -algebra cone(A) of continuous functions  $f: [0,1] \to A$  with f(0) = 0. It can be identified with the tensor product of  $C^*$ -algebras  $A \otimes C_0(\mathbb{R}^n)$ . The *cone* is defined analogously as the  $C^*$ -algebra cone(A) of continuous functions  $f: [0,1] \to A$  with f(0) = 0. It can be identified with the tensor product of  $C^*$ -algebras  $A \otimes C_0((0,1])$ . There is an obvious exact sequence of  $C^*$ -algebras  $0 \to \Sigma A \to \text{cone}(A) \to A \to 0$ . Moreover, the  $C^*$ -algebra cone(A) is *contractible*, i.e., the zero and the identity endomorphism are homotopic. The desired homotopy is given by the formula  $f_t(s) := f(ts)$ , see [991, Proposition 6.4.7 on page 123]. Hence  $K_*(\text{cone}(A))$  is trivial and the boundary operator in the associated long exact sequence induces isomorphisms

$$\partial_n \colon K_n(A) \xrightarrow{\cong} K_{n-1}(\Sigma A).$$

For complex  $C^*$ -algebras A and B for which A lies in the so-called bootstrap category N, a *Künneth Theorem*, i.e., an exact sequence  $0 \to K_*(A) \otimes K_*(B) \to K_*(A \widehat{\otimes} B) \to \text{Tor}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$ , is established in [903]. The case of real  $C^*$ -algebras is treated in [147].

**Remark 10.40 (Topological** *K***-theory and the classification of**  $C^*$ **-algebras).** One prominent feature is that for certain classes of  $C^*$ -algebras their isomorphism type is determined by their topological *K*-theory, sometimes taking the order structure on  $K_0(A)$  coming from the positive cone of those elements that are represented by finitely generated projective modules into account. If one considers the topological *K*-theory of spaces, such nice classification results are not available.

One example is the class of *AF-algebras*, i.e.,  $C^*$ - algebras that occur as a colimit of a sequence of finite-dimensional  $C^*$ -algebras, due to Elliot, see [333], [856, Chapter 7], [991, 12.1]. The index *n* of the *Cuntz*  $C^*$ -algebra  $O_n$  is determined by the topological *K*-theory since  $K_0(O_n) \cong \mathbb{Z}/n$  and  $K_1(O_n) = 0$ , see [266], [991, 12.2]. A very important result about the classification of so-called *Kirchberg*  $C^*$ -algebras in terms of their topological *K*-theory is due to Kirchberg, see for instance [855, Chapter 8].

**Remark 10.41 (Topological** *K***-theory and generalized index theory).** One important motivation to study the topological *K*-theory of  $C^*$ -algebras is index theory and its generalizations. A first introduction to how one can assign to a Fredholm operator

over a  $C^*$ -algebra A an element in  $K_0(A)$  is given in [991, Chapter 17], following Mingo [733]. There are many other index theorems taking values in the topological K-theory of  $C^*$ -algebras. Often they are generalizations of the classical family index theorem for families of operators parametrized over a closed manifold M, which take values in  $K^*(M) = K_*(C(M))$ .

One can attach to geometric or topological situations new  $C^*$ -algebras and consider their topological *K*-theory and indices of appropriate operators, where it is no longer possible to work with topological spaces. Examples are foliations and coarse geometry. There are also plenty other generalizations of the classical index theorems using topological *K*-theory of  $C^*$ -algebras. For information about these topics we refer for instance to [250, 270, 491, 738].

More information about the topological *K*-theory of  $C^*$ -algebras can be found for instance in [140, 250, 270, 491, 856, 991].

# 10.4 The Baum-Connes Conjecture for Torsionfree Groups

Let *G* be a group. Then there exist for all  $n \in \mathbb{Z}$  assembly maps

(10.42) 
$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_n \colon K_n(BG) \to K_n(C_r^*(G,\mathbb{C}));$$

(10.43)  $\operatorname{asmb}^{G,\mathbb{R}}(BG)_n \colon KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})).$ 

**Conjecture 10.44 (Baum-Connes Conjecture for torsionfree groups).** The assembly maps appearing in (10.42) and (10.43) are isomorphisms for all  $n \in \mathbb{Z}$ , provided that *G* is torsionfree.

It is crucial for the Baum-Connes Conjecture to work with the reduced group  $C^*$ -algebra, it is definitely not true for the maximal group  $C^*$  algebra in general. Moreover, Conjecture 10.44 in general fails for groups with torsion. The general version, which makes sense for all groups, will be discussed in Chapter 14.

**Exercise 10.45.** Show for a finite group *G* that the following statements are equivalent:

(i)  $K_0(BG)$  and  $K_0(C_r^*(G))$  are rationally isomorphic;

(ii)  $KO_0(BG)$  and  $KO_0(C_r^*(G))$  are rationally isomorphic;

(iii) *G* is trivial.

One benefit of Conjecture 10.44 is that the right side is of great interest because of index theory but hard to compute, whereas the left side is accessible by standard methods from algebraic topology.

**Example 10.46 (Three-dimensional Heisenberg group).** Let  $\text{Hei}(\mathbb{R})$  be the *three-dimensional Heisenberg group*. It is the subgroup of  $\text{GL}_3(\mathbb{R})$  consisting of upper triangular matrices whose diagonal entries are all equal to 1. The *three-dimensional* 

*discrete Heisenberg group* Hei is the intersection of  $\text{Hei}(\mathbb{R})$  with  $\text{GL}_3(\mathbb{Z})$ . Obviously Hei is a torsionfree discrete subgroup of the contractible Lie group  $\text{Hei}(\mathbb{R})$ . Hence  $\text{Hei} \setminus \text{Hei}(\mathbb{R})$  is a model for *B*Hei, which is an orientable closed 3-manifold.

Define elements in Hei

$$u := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad v := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad w := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we get the presentation

Hei = 
$$\langle u, v, w | [u, w] = v, [u, v] = 1, [w, v] = 1 \rangle$$
.

Therefore we have a central extension  $1 \to \mathbb{Z} \xrightarrow{i} \text{Hei} \xrightarrow{p} \mathbb{Z}^2 \to 1$ , where *i* sends the generator of  $\mathbb{Z}$  to *v* and *p* sends *v* to (0, 0), *u* to (1, 0) and *w* to (0, 1). Hence the map  $H_1(B\text{Hei}) \to H_1(B\mathbb{Z}^2)$  is an isomorphism. Using Poincaré duality we conclude

$$H_n(B\text{Hei}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 3; \\ \mathbb{Z}^2 & \text{if } n = 1, 2. \end{cases}$$

We conclude from the Chern character (10.1) for every  $n \in \mathbb{Z}$ .

$$K_n(B\text{Hei}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^3.$$

Next we consider the Atiyah-Hirzebruch spectral sequence converging to  $K_{p+q}(B\text{Hei})$ whose  $E^2$ -term is  $E^2_{p,q} = H_p(B\text{Hei}; K_q(\{\bullet\}))$ . Its  $E^2$ -page looks as follows

÷	÷	÷	÷	÷	÷	
Z	$\mathbb{Z}^2$	$\mathbb{Z}^2$	Z	0	0	
0	0	0	0	0		
$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	Z	0	0	
0	0	0	0	0		
:	÷	:	:	:	:	

Each entry is a finitely generated free  $\mathbb{Z}$ -module and we have for every  $n \in \mathbb{Z}$ 

$$\sum_{p+q=n} \dim_{\mathbb{Q}}(E_{p,q}^2) \otimes_{\mathbb{Z}} \mathbb{Q} = 3 = K_n(B\text{Hei}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This implies that all differentials must vanish and we get for every  $n \in \mathbb{Z}$ 

$$K_n(B\text{Hei}) \cong \mathbb{Z}^3.$$

Conjecture 10.44 is known to be true for Hei and hence we conclude for every  $n \in \mathbb{Z}$ 

$$K_n(C_r^*(\text{Hei})) \cong \mathbb{Z}^3$$
.

**Exercise 10.47.** Let *G* be the semidirect product  $\mathbb{Z} \rtimes \mathbb{Z}$ , where the generator of  $\mathbb{Z}$  acts on  $\mathbb{Z}$  by – id. Compute  $K_*(C_r^*(G))$  using the fact that Conjecture 10.44 is known to be true for *G*.

Next we discuss some consequences of the Baum-Connes Conjecture for torsionfree groups 10.44.

#### 10.4.1 The Trace Conjecture in the Torsionfree Case

The assembly map appearing in the Baum-Connes Conjecture has an interpretation in terms of index theory. Namely, an element  $\eta \in K_0(BG)$  can be represented by a pair  $(M, P^*)$  consisting of a cocompact free proper smooth *G*-manifold *M* with a *G*-invariant Riemannian metric together with an elliptic *G*-complex  $P^*$  of differential operators of order 1 on *M*, see [110]. To such a pair one can assign an index  $\operatorname{ind}_{C_r^*(G)}(M, P^*)$  in  $K_0(C_r^*(G))$ , see [738], that is the image of  $\eta$  under the assembly map  $K_0(BG) \to K_0(C_r^*(G))$  appearing in Conjecture 10.44. With this interpretation the surjectivity of the assembly map for a torsionfree group says that any element in  $K_0(C_r^*(G))$  can be realized as an index. This allows us to apply index theorems to get interesting information. It is of the same significance as the interpretation of the *L*-theoretic assembly map as the map  $\sigma$  appearing in the Surgery Exact Sequence discussed in the proof of Theorem 9.171.

Here is a prototype of such an argument. The standard trace

(10.48) 
$$\operatorname{tr}_{C_r^*(G)} \colon C_r^*(G) \to \mathbb{C}$$

sends an element  $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$  to  $\langle f(1), 1 \rangle_{l^2(G)}$ . Applying the trace to idempotent matrices yields a homomorphism

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}.$$

Let pr:  $BG \rightarrow \{\bullet\}$  be the projection. For a group G the following diagram commutes

10.4 The Baum-Connes Conjecture for Torsionfree Groups



where  $i: \mathbb{Z} \to \mathbb{R}$  is the inclusion. This non-trivial statement follows from Atiyah's  $L^2$ -index theorem [46]. Atiyah's theorem says that the  $L^2$ -index  $\operatorname{tr}_{C_r^*(G)} \circ \operatorname{asmb}_*(\eta)$  of an element  $\eta \in K_0(BG)$ , which is represented by a pair  $(M, P^*)$ , agrees with the ordinary index of  $(G \setminus M; G \setminus P^*)$ , which is given by  $\operatorname{tr}_{\mathbb{C}} \circ K_0(\operatorname{pr})(\eta) \in \mathbb{Z}$ .

The following conjecture is taken from [108, page 21].

**Conjecture 10.50 (Trace Conjecture for torsionfree groups).** For a torsionfree group *G* the image of

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

The commutativity of diagram (10.49) shows

**Lemma 10.51.** If the Baum-Connes assembly map  $K_0(BG) \rightarrow K_0(C_r^*(G))$  of (10.42) is surjective, then the Trace Conjecture for Torsionfree Groups 10.50 holds for G.

A Modified Trace Conjecture for not necessarily torsionfree groups is discussed in Subsection 14.8.3.

#### 10.4.2 The Kadison Conjecture

**Conjecture 10.52 (Kadison Conjecture).** If *G* is a torsionfree group, then the only idempotent elements in  $C_r^*(G)$  are 0 and 1.

**Lemma 10.53.** *The Trace Conjecture for Torsionfree Groups* 10.50 *implies the Kadison Conjecture* 10.52.

*Proof.* Assume that  $p \in C_r^*(G)$  is an idempotent different from 0 or 1. From p one can construct a non-trivial projection  $q \in C_r^*(G)$ , i.e.  $q^2 = q$ ,  $q^* = q$ , with  $\operatorname{im}(p) = \operatorname{im}(q)$  and hence with 0 < q < 1. Since the standard trace  $\operatorname{tr}_{C_r^*(G)}(q)$  is faithful, we conclude  $\operatorname{tr}_{C_r^*(G)}(q) \in \mathbb{R}$  with  $0 < \operatorname{tr}_{C_r^*(G)}(q) < 1$ . Since  $\operatorname{tr}_{C_r^*(G)}(q)$  is by definition the image of the element  $[\operatorname{im}(q)] \in K_0(C_r^*(G))$  under  $\operatorname{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \to \mathbb{R}$ , we get a contradiction to the assumption  $\operatorname{im}(\operatorname{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$ .

Remark 10.54 (The Kadison Conjecture 10.52 and Kaplansky's Idempotent Conjecture 2.73). Obviously the Kadison Conjecture 10.52 implies Kaplansky's Idempotent Conjecture 2.73 in the case that *R* can be embedded in  $\mathbb{C}$ . Because of Remark 2.84 the Kadison Conjecture 10.52 implies Kaplansky's Idempotent Conjecture 2.73 if *R* is any field of characteristic zero. The Bost Conjecture 14.23 implies that there are no non-trivial idempotents in  $L^1(G)$  and hence the Kaplansky's Idempotent Conjecture 2.73 for fields of characteristic zero, see [131, Corollary 1.6].

#### 10.4.3 The Zero-in-the-Spectrum Conjecture

The following conjecture is due to Gromov [439, page 120].

**Conjecture 10.55 (Zero-in-the-Spectrum Conjecture).** Suppose that  $\widetilde{M}$  is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure

$$(\Delta_p)_{\min} \colon L^2 \Omega^p(M) \supset \operatorname{dom}(\Delta_p)_{\min} \to L^2 \Omega^p(M)$$

for some  $p \in \{0, 1, ..., \dim M\}$ , where  $\Delta_p$  denotes the Laplacian acting on smooth *p*-forms on  $\widetilde{M}$ .

**Theorem 10.56 (The strong Novikov Conjecture implies the Zero-in-the-Spectrum Conjecture).** Suppose that M is an aspherical closed Riemannian manifold with fundamental group G, then the injectivity of the assembly map

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-Spectrum Conjecture 10.55 for  $\widetilde{M}$ .

*Proof.* We give a sketch of the proof. More details can be found in [638, Corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on  $\widetilde{M}$  yields a contradiction in the case that  $n = \dim(M)$  is even. Namely, this assumption implies that the  $C_r^*(G)$ -valued index of the signature operator twisted with the flat bundle  $\widetilde{M} \times_G C_r^*(G) \to M$  in  $K_0(C_r^*(G))$  is zero where  $G = \pi_1(M)$ . This index is the image of the class [S] defined by the signature operator in  $K_0(BG)$  under the assembly map  $K_0(BG) \to K_0(C_r^*(G))$ . Since by assumption the assembly map is rationally injective, this implies [S] = 0 in  $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Note that M is aspherical by assumption and hence M = BG. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \ge 0} H_{2p}(M; \mathbb{Q})$$

that sends [S] to the Poincaré dual  $\mathcal{L}(M) \cap [M]_{\mathbb{Q}}$  of the Hirzebruch *L*-class  $\mathcal{L}(M) \in \bigoplus_{p \ge 0} H^{2p}(M; \mathbb{Q})$ . This implies that  $\mathcal{L}(M) \cap [M]_{\mathbb{Q}} = 0$  and hence  $\mathcal{L}(M) = 0$ . This contradicts the fact that the component of  $\mathcal{L}(M)$  in  $H^0(M; \mathbb{Q})$  is 1.  $\Box$ 

More information about the Zero-in-the-Spectrum Conjecture 10.55 can be found for instance in [638] and [650, Section 12].

# 10.5 Kasparov's KK-Theory

Kasparov introduced the bivariant *KK-theory* that assigns to two separable  $C^*$ -algebras A and B a  $\mathbb{Z}$ -graded abelian group  $KK_*(A, B)$ . We give a very brief summary of it. In the sequel all  $C^*$ -algebras are assumed to be separable and the ground ring F is  $\mathbb{C}$  or  $\mathbb{R}$ .

#### 10.5.1 Basic Properties of KK-theory for C\*-Algebras

(i) *Bi-functoriality* 

A homomorphism  $f: A \to B$  of  $C^*$ -algebras induces homomorphisms of  $\mathbb{Z}$ -graded abelian groups

$$KK_*(f, \mathrm{id}_D) \colon KK^*(B, D) \to KK_*(A, D);$$
  
$$KK_*(\mathrm{id}_D, f) \colon KK_*(D, A) \to KK_*(D, B).$$

If  $g: B \to C$  is another homomorphism of  $C^*$ -algebras, we have

$$KK_*(g \circ f, \mathrm{id}_D) = KK_*(f, \mathrm{id}_D) \circ K_*(g, \mathrm{id}_D);$$
  
$$KK_*(\mathrm{id}_D, g \circ f) = KK_*(\mathrm{id}_D, g) \circ K_*(\mathrm{id}_D, f).$$

Moreover  $K_*(id_A, id_B) = id_{KK_*(A,B)}$ . In particular,  $KK_*(-, D)$  is a contravariant and  $KK_*(D, -)$  is a covariant functor from the category of *C*\*-algebras to the category of  $\mathbb{Z}$ -graded abelian groups;

#### (ii) Homotopy invariance

If  $f,g: A \to B$  are homotopic homomorphisms of  $C^*$ -algebras, then  $KK_*(f, \mathrm{id}_D) = KK_*(g, \mathrm{id}_D)$  and  $KK_*(\mathrm{id}_D, f) = KK_*(\mathrm{id}_D, g)$ ;

(iii) Finite direct products

If *A* and *B* are *C*<sup>\*</sup>-algebras, there are natural isomorphisms of  $\mathbb{Z}$ -graded abelian groups

$$\begin{split} & \mathit{KK}_*(A \times B, C) \xrightarrow{\cong} \mathit{KK}_*(A, C) \times \mathit{KK}_*(B, C); \\ & \mathit{KK}_*(C, A \times B) \xrightarrow{\cong} \mathit{KK}_*(C, A) \times \mathit{KK}_*(C, B); \end{split}$$

#### (iv) Countable direct sums in the first variable

If  $A = \bigoplus_{i=0}^{\infty} A_i$  is a countable direct sum of  $C^*$ -algebras, then there is a natural isomorphism

$$KK_n\left(\bigoplus_{i=0}^{\infty} A_i, B\right) \xrightarrow{\cong} \prod_{i=0}^{\infty} K_n(A_i, B);$$

(v) Morita equivalence

For any integers  $m, n \ge 1$  there are natural isomorphisms of  $\mathbb{Z}$ -graded abelian groups  $KK_*(A, B) \xrightarrow{\cong} KK_*(M_m(A), M_n(B));$ 

#### (vi) Stabilization

There are natural isomorphisms of Z-graded abelian groups

$$KK_*(A, B) \xrightarrow{\cong} KK_*(A\widehat{\otimes}\mathcal{K}, B);$$
  
$$KK_*(A, B) \xrightarrow{\cong} KK_*(A, B\widehat{\otimes}\mathcal{K});$$

# (vii) Long exact sequence of an ideal

Let  $0 \to I \xrightarrow{i} A \xrightarrow{p} A/I \to 0$  be an extension of (separable)  $C^*$ -algebras. If it is semisplit in the sense of [140, Definition 19.5.1. on page 195], (which is automatically true if A is nuclear,) then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

$$\cdots \xrightarrow{\delta_{n-1}} KK_n(A/I, B) \xrightarrow{KK_n(p, \mathrm{id}_B)} KK_n(A, B) \xrightarrow{KK_n(i, \mathrm{id}_B)} KK_n(I, B)$$

$$\xrightarrow{\delta_n} KK_{n+1}(A/I, B) \xrightarrow{KK_{n+1}(p, \mathrm{id}_B)} KK_{n+1}(A, B)$$

$$\xrightarrow{KK_{n+1}(i, \mathrm{id}_B)} KK_{n+1}(I, B, ) \xrightarrow{\delta_{n+1}} \cdots$$

If the extension is semisplit or if B is nuclear, then there exists a long exact sequence, natural in (A, I) and infinite to both sides,

(viii) Bott periodicity

There exists an isomorphism of degree b(F)

$$\beta_*(A) \colon K\!K_*(A,B) \xrightarrow{\cong} K\!K_{*+b(F)}(A,B),$$

which is natural in *A* and *B*, where  $b(\mathbb{C}) = 2$  and  $b(\mathbb{R}) = 8$ ; (ix) *Connection to topological K-theory* 

There are natural isomorphisms of  $\mathbb{Z}$ -graded abelian groups

$$K_*(A) \xrightarrow{\cong} KK_*(F, A) \quad \text{if } F = \mathbb{C};$$
  
$$KO_*(A) \xrightarrow{\cong} KK_*(F, A) \quad \text{if } F = \mathbb{R};$$

(x) Homomorphisms of  $C^*$ -algebras

A homomorphism  $f: A \to B$  of  $C^*$ -algebras defines an element [f] in  $KK_*(A, B)$ .

10.5 Kasparov's KK-Theory

**Remark 10.57 (Some failures).** The second variable is in general not compatible with countable direct sums and in particular not with colimits over directed sets. However, in the special case  $A = \mathbb{C}$ , this is the case, since then  $KK_*(\mathbb{C}, B)$  is just the topological *K*-theory of *B*.

Conditions about the existence of a long exact sequence of an ideal such as semisplit or B being nuclear are needed.

# 10.5.2 The Kasparov's Intersection Product

One of the basic features of *KK*-theory is *Kasparov's intersection product*, which is a bilinear pairing of  $\mathbb{Z}$ -graded abelian groups

(10.58) 
$$\widehat{\otimes}_B \colon KK_*(A, B) \otimes KK_*(B, C) \to KK_*(A, C).$$

It has the following properties

```
(i) Naturality
```

It is natural in *A*, *B*, and *C*;

(ii) Associativity

It is associative;

(iii) Composition of homomorphisms

If  $f: A \to B$  and  $g: B \to C$  are homomorphisms of  $C^*$ -algebras, then we get for the associated elements  $[f] \in KK_0(A, B), [g] \in KK_0(B, C)$  and  $[g \circ f] \in KK_0(A, C)$ 

 $[g \circ f] = [f]\widehat{\otimes}_B[g];$ 

(iv) Units

There is a unit  $1_A := [id_A]$  in  $KK_0(A, A)$  for the intersection product.

**Remark 10.59** (*KK*-equivalence). We consider in this Remark 10.59 only  $F = \mathbb{C}$ , the case  $F = \mathbb{R}$  is analogous.

One of the basic features of the product is that an element *x* in  $KK_0(A, B)$  induces a homomorphism

$$-\bigotimes_B x: K_n(A) = KK_n(F, A) \to K_n(B) = KK_n(F, B).$$

Of course  $-\widehat{\otimes}_B[f]$  agrees with  $K_n(f)$  if  $f: A \to B$  is a homomorphism of  $C^*$ -algebras. An element  $x \in KK_0(A, B)$  is called a *KK*-equivalence if there exists an element  $y \in KK_0(B, A)$  satisfying  $x \widehat{\otimes}_B y = 1_A$  and  $y \widehat{\otimes}_A x = 1_B$ . The basic feature of a *KK*-equivalence is that

$$-\otimes_B x \colon K_n(A) = KK_n(F, A) \to K_n(B) = KK_n(F, B)$$

is automatically an isomorphism, the inverse is  $-\widehat{\otimes}_B y$ .

**Remark 10.60 (K-homology of**  $C^*$ -algebras). We consider in this Remark 10.60 only  $F = \mathbb{C}$ , the case  $F = \mathbb{R}$  is analogous.

One can define the *topological K-homology of a*  $C^*$ -algebra  $K^*(A)$  by  $K^n(A) := KK_{-n}(A, F)$ . It is in some sense dual to the topological K-theory  $K_*(A)$ . Moreover, the intersection product yields the *index pairing* 

$$K_n(A) \otimes_{\mathbb{Z}} K^n(A) \to KK_0(F,F) = \mathbb{Z}, \quad (x,y) \mapsto \langle x,y \rangle := x \widehat{\otimes}_A y.$$

If we take n = 0 and A = C(M) for a smooth closed Riemannian manifold M, then an appropriate elliptic operator P over M defines an element in [P] in  $K^0(C(M)) = K_0(M)$ , a vector bundle  $\xi$  over M defines an element in  $K_0(C(M)) = K^0(M)$ , and the pairing  $\langle [\xi], [P] \rangle$  is the classical index of the elliptic operator obtained from Pby twisting with  $\xi$ .

There are Universal Coefficient Theorems and Künneth Theorems for KK-theory, see for instance [147, 148, 868, 903]. The Pimsner-Voiculescu sequences associated to an automorphisms of a  $C^*$ -algebra are explained for KK-theory in [140, Theorem 19.6.1 on page 198].

More information about *KK*-theory, for instance about its construction in terms of Kasparov modules or quasi-homomorphisms, other bivariant theories such as Ext for extensions of  $C^*$ -algebras, kk-theory, E-theory, and their relation to *KK*-theory, generalizations of these theories to more general operator algebras than  $C^*$ -algebras, universal properties of these theories, applications to index theory, and the relevant literature can be found for instance in [140, 270, 481, 483, 491, 520], or in the papers of Kasparov [560, 561, 562, 563].

# **10.6 Equivariant Topological K-Theory and KK-Theory**

In the sequel groups are assumed to be discrete. In the sequel all  $C^*$ -algebras are assumed to be separable and the ground ring F is  $\mathbb{C}$  or  $\mathbb{R}$ . Given a group G, there exists an equivariant version of KK-theory. It assigns to two G- $C^*$ -algebras A and B a  $\mathbb{Z}$ -graded abelian group  $KK^G_*(A, B)$  and has essentially the same basic properties as non-equivariant KK-theory. Namely, it is a bi-functor, contravariant in the first and covariant in the second variable, is G-homotopy invariant, satisfies Morita equivalence and stabilization, is split exact, i.e., has long exact sequences for appropriate ideals, satisfies Bott periodicity, is compatible with finite direct products in both variables and countable direct sums in the first variable, and a homomorphism of G- $C^*$ -algebras  $f : A \to B$  defines an element  $[f] \in KK^G_0(A, B)$ . There is also an *equivariant version of Kasparov's intersection product* 

$$\widehat{\otimes}_B \colon K\!K_i^G(A, B) \otimes K\!K_i^G(B, C) \to K\!K_{i+i}^G(A, C),$$

which has all the expected properties as in the non-equivariant case.

In particular, we get on  $KK_0^G(F, F)$  an interesting structure of a commutative ring with unit, sometimes called the *representation ring* of *G*. If *G* is finite,  $KK_0^G(F, F)$  is indeed isomorphic as a ring to  $\text{Rep}_F(G)$ .

There exist certain additional structures in the equivariant setting. Given a homomorphism  $\alpha: H \to G$ , there are natural *restriction homomorphisms* 

(10.61) 
$$\alpha^* \colon K\!K^G_*(A,B) \to K\!K^H(\alpha^*A,\alpha^*B),$$

where  $\alpha^*A$  and  $\alpha^*B$  are the *H*-*C*<sup>\*</sup>-algebras obtained from the *G*-*C*<sup>\*</sup>-algebras *A* and *B* by restring the *G*-action to an *H*-action using  $\alpha$ . It is compatible with the equivariant Kasparov product.

Let  $i: H \to G$  be the inclusion of groups. Given an H- $C^*$ -algebra A, we define its *induction*  $i_*A$ , to be the G- $C^*$ -algebra of bounded functions  $f: G \to A$  which satisfy  $f(gh) = h^{-1} \cdot f(g)$  and vanish at infinity, i.e., for every  $\epsilon > 0$  there exists a finite subset  $S \subseteq G/H$  such that for every  $g \in G$  with  $gH \notin S$  we have  $||f(g)|| \le \epsilon$ . The norm is the supremum norm. Given  $g \in G$  and such a function f, define  $g \cdot f$ to be the function sending  $g' \in G$  to  $f(g^{-1}g')$ .

Note that the left *FG*-module *FG*  $\otimes_{FH} A$ , which is the algebraic induction of *A* viewed as *FH*-module, embeds as a dense *FG*-submodule into  $i_*A$  by sending  $g \otimes a$  to the function that maps gh to  $h^{-1}a$  for  $h \in H$  and  $g' \in G$  with  $g'H \neq gH$  to zero. In other words, we can think of *FG*  $\otimes_{FH} A$  as the set of elements  $f \in i_*A$  such that  $\{gH \in G/H \mid f(g) \neq 0\}$  is finite. In contrast to modules over group rings, induction  $i_*$  and restriction  $i^*$  do *not* form an adjoint pair  $(i_*, i^*)$  for equivariant  $C^*$ -algebras, as the following exercise illustrates.

**Exercise 10.62.** Let  $i: \{1\} \to G$  be the inclusion of the trivial group into an infinite discrete group *G*. Show that hom<sub>*G*</sub>( $i_*F, F$ ) and hom<sub>{1}</sub>( $F, i^*F$ ) are not isomorphic, where  $F = \mathbb{R}, \mathbb{C}$  denotes both the obvious  $\{1\}$ -*C*\*-algebra and the obvious *G*-*C*\*-algebra with trivial *G*-action.

If X is a proper *H*-*CW*-complex, then  $G \times_H X$  is a proper *G*-*CW*-complex, and we obtain an isomorphism of G- $C^*$ -algebras  $i_*C_0(X) \xrightarrow{\cong} C_0(G \times_H X)$  that sends  $f \in i_*C_0(X)$  to the function  $G \times_H X \to F$ ,  $(g, x) \mapsto f(g)(x)$ . Given an *H*-*C*\*-algebra A and an *H*-*C*\*-algebra B, there is a natural *induction homomorphism* 

(10.63) 
$$i_* \colon KK^H_*(A, B) \to KK^G(i_*A, i_*B)$$

It is compatible with the equivariant Kasparov's intersection product respecting the units. If  $j: G \to K$  is an inclusion, we get  $(j \circ i)_* = j_* \circ i_*$ .

There are *descent homomorphisms* 

(10.64)  $j_r^G \colon KK^G_*(A, B) \to KK_*(A \rtimes_r G, B \rtimes_r G);$ 

(10.65) 
$$j_r^{\mathsf{G}} \colon K\!K^{\mathsf{G}}_*(A,\mathbb{C}) \to K\!K_*(A \rtimes_r G,\mathbb{C});$$

(10.66)  $j_m^G \colon KK^G_*(A, B) \to KK_*(A \rtimes_m G, B \rtimes_m G).$ 

The dual of the Green-Julg Theorem says that (10.65) is an isomorphism. The descent homomorphisms are natural and compatible with Kasparov's intersection products respecting the units.

In the sequel we assume  $F = \mathbb{C}$ . Define the *equivariant complex K-homology* of a pair of proper *G-CW*-complexes (X, A) with coefficients in the complex *G-C*<sup>\*</sup>-algebra *B* by

(10.67) 
$$K_{n}^{G}(X, A; B) := \operatorname{colim}_{C \subseteq X} KK_{n}^{G}(C_{0}(C, C \cap A); B),$$

where the colimit is taken over the directed system of cocompact proper *G-CW*subcomplexes  $C \subseteq X$ , directed by inclusion, and  $C_0(C, C \cap A)$  is the *G-C*<sup>\*</sup>-algebra of continuous functions  $C \to \mathbb{C}$  that vanish on  $C \cap A$  and at infinity. This group is often denoted by  $RK_n(X, A; B)$  in the literature and called *equivariant K-homology with compact support*, but from a topologist's point of view it is better to call it equivariant *K*-homology in view of its description in terms of spectra, see Section 12.4. If *B* is  $\mathbb{C}$  with the trivial *G*-action, we just write  $K^G_*(X, A)$  for  $K^G_*(X, A; \mathbb{C})$ , and this is precisely the  $\mathbb{Z}$ -graded abelian group that we have mentioned already in Subsection 10.2.3 and will be constructed in terms of spectra in Section 12.4.

Next we explain the equivariant Chern character for equivariant complex *K*-homology. Denote for a proper *G*-*CW*-complex *X* by  $\mathcal{F}(X)$  the set of all subgroups  $H \subset G$  for which  $X^H \neq \emptyset$ , and by

(10.68) 
$$\Lambda^G(X) := \mathbb{Z}\left[\frac{1}{\mathcal{F}(X)}\right]$$

the ring  $\mathbb{Z} \subset \Lambda^G(X) \subset \Lambda^G$  obtained from  $\mathbb{Z}$  by inverting the orders of all subgroups  $H \in \mathcal{F}(X)$ . Denote by  $J^G(X)$  the set of conjugacy classes (*C*) of finite cyclic subgroups  $C \subset G$  for which  $X^C$  is non-empty. Let  $C \subset G$  be a finite cyclic subgroup. Let  $C_G C$  be the centralizer and  $N_G C$  be the normalizer of  $C \subset G$ . Then the quotient  $N_G C/C_G C = N_G C/C \cdot C_G C$  is a finite group. For a specific idempotent  $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$  defined in [651, Section 3] the cokernel of

$$\bigoplus_{D \subset C, D \neq C} \operatorname{ind}_{D}^{C} \colon \bigoplus_{D \subset C, D \neq C} \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(D) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$$

is isomorphic to the image of the idempotent endomorphism

$$\theta_C \colon \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Z}\left[\frac{1}{|C|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C).$$

The element  $\theta_C \in \Lambda^C \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$  is uniquely determined by the property that its character sends a generator of *C* to 1 and all other elements in *C* to 0.

The next theorem is taken from [651, Theorem 0.7].
10.6 Equivariant Topological K-Theory and KK-Theory

**Theorem 10.69 (Equivariant Chern character for equivariant complex** *K*-homology). Let X be a proper G-CW-complex. Put  $\Lambda = \Lambda^G(X)$  and  $J = J^G(X)$ . Let  $\operatorname{im}(\theta_C) \subseteq \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$  be the image of  $\theta_C : \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \Lambda \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C)$ . Then there is for  $n \in \mathbb{Z}$  a natural isomorphism

$$\operatorname{ch}_p^G(X) \colon \bigoplus_{(C) \in J} \Lambda \otimes_{\mathbb{Z}} K_n(C_G C \setminus X^C) \otimes_{\Lambda[N_G C/C_G C]} \operatorname{im}(\theta_C) \xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_n^G(X).$$

Note that the isomorphism appearing in Theorem 10.69 exists already over  $\Lambda$ , one does not have to pass to  $\mathbb{Q}$  or  $\mathbb{C}$ . This will be important when we deal with the Modified Trace Conjecture in Subsection 14.8.3.

**Example 10.70.** In the special case where *G* is finite, *X* is the one-point-space  $\{*\}$  and n = 0, the equivariant Chern character appearing in Theorem 10.69 reduces to an isomorphism

$$\bigoplus_{(C)\in J^G} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}\left[\frac{1}{|G|}\right][N_G C/C_G C]} \operatorname{im}(\theta_C) \xrightarrow{\cong} \mathbb{Z}\left[\frac{1}{|G|}\right] \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G)$$

where  $J^G$  is the set of conjugacy classes (*C*) of cyclic subgroups  $C \subset G$ . This is a strong version of the well-known theorem of Artin that the map induced by induction

$$\bigoplus_{C)\in J^G} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{C}}(G)$$

is surjective for any finite group G.

**Exercise 10.71.** Let *p* be an odd prime and let *V* be an orthogonal  $\mathbb{Z}/p$ -representation of dimension *d* such that  $V^{\mathbb{Z}/p} \neq \{0\}$ . Denote by *SV* the  $\mathbb{Z}/p$ -*CW*-complex consisting of elements  $v \in V$  of norm 1. Show

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV) \cong_{\mathbb{Z}[1/p]} \begin{cases} \mathbb{Z}[1/p]^p & \text{if } d \text{ is even;} \\ \mathbb{Z}[1/p]^{2p} & \text{if } d \text{ is odd and } n \text{ is even;} \\ 0 & \text{if } d \text{ is odd and } n \text{ is odd.} \end{cases}$$

Analogously to the complex case one defines the *equivariant real K-homology*  $KO^G_*(X, A; B)$  of a pair of proper *G-CW*-complexes (X, A) with coefficients in the real *G-C\**-algebra *B*. We will abbreviate  $KO^G_*(X, A) := KO^G_n(X, A; \mathbb{R})$  where  $\mathbb{R}$  carries the trivial *G*-action, This is precisely the  $\mathbb{Z}$ -graded abelian group that we will be constructed in terms of spectra in Section 12.4.

For discussions of universal coefficient theorems for equivariant *K*-theory see [698, 867, 868].

Further information about equivariant *KK*-theory can be found for instance in [140, Section 20], [564], and [950].

## 10.7 Comparing Algebraic and Topological *K*-theory of *C*\*-Algebras

Let *A* be a  $C^*$ -algebra. Then  $K_n(A)$  denotes in most cases topological *K*-theory, but it can also mean the algebraic *K*-theory of *A* considered just as a ring. To avoid this ambiguity, we will use in this Section 10.7 the superscripts TOP and ALG to make clear what we mean.

There is for any  $C^*$ -algebra over  $\mathbb{R}$  or  $\mathbb{C}$  a canonical map of spectra

(10.72) 
$$\mathbf{t}(A): \mathbf{K}^{\mathrm{ALG}}(A) \to \mathbf{K}^{\mathrm{TOP}}(A)$$

from the non-connective algebraic *K*-theory spectrum of *A* just considered as a ring to the topological *K*-theory spectrum associated to the *C*\*-algebra *A*, see [862, Theorem 4 on page 851]. It induces homomorphisms of abelian groups for all  $n \in \mathbb{Z}$ 

(10.73) 
$$t_n(A) = K_n(\mathbf{t}(A)) \colon K_n^{\mathrm{ALG}}(A) \to K_n^{\mathrm{TOP}}(A).$$

It is always an isomorphism for n = 0, but in general far from being a bijection, as illustrated by the following exercise.

**Exercise 10.74.** Let *X* be a finite *CW*-complex of dimension  $\geq 1$ . Prove that the comparison map  $K_1^{ALG}(C(X, F)) \rightarrow K_1^{TOP}(C(X, F))$  is never injective.

The situation is different if A is stable or if one uses coefficients in  $\mathbb{Z}/k$ . Namely, the following result is proved in [936, Theorem 10.9] over  $\mathbb{C}$  and  $n \ge 1$ , but holds in the more general form below by [862, Theorem 19 on page 863], see also Higson [482].

**Theorem 10.75 (Karoubi's Conjecture).** Karoubi's Conjecture is true, i.e., for any stable  $C^*$ -algebra A over  $\mathbb{R}$  or  $\mathbb{C}$  the canonical map  $\mathbf{t}$  of (10.72) is weak homotopy equivalence i.e., the maps  $t_n$  of (10.73) are bijective for  $n \in \mathbb{Z}$ .

Given an integer  $k \ge 2$ , we have introduced  $K_n^{ALG}(A; \mathbb{Z}/k)$  in Section 6.4. The analogous construction works for topological *K*-theory and there is the analog of (10.73), a natural homomorphism

(10.76) 
$$t_n(A;\mathbb{Z}/k)\colon K_n^{\mathrm{ALG}}(A;\mathbb{Z}/k)\to K_n^{\mathrm{TOP}}(A;\mathbb{Z}/k).$$

We mention the following conjecture of Rosenberg [858, Conjecture 4.1] or [862, Conjecture 26 on page 869].

**Conjecture 10.77 (Comparing algebraic and topological** *K***-theory with coefficients for**  $C^*$ **-algebras).** If *A* is a real or complex  $C^*$ -algebra and  $k \ge 2$  an integer, then the comparison map

$$K_n^{\mathrm{ALG}}(A; \mathbb{Z}/k) \to K_n^{\mathrm{TOP}}(A; \mathbb{Z}/k)$$

is bijective for  $n \ge 0$ .

The map  $K_n^{\text{ALG}}(A; \mathbb{Z}/k) \to K_n^{\text{TOP}}(A; \mathbb{Z}/k)$  appearing in Conjecture 10.77 is known to be bijective for n = 1 and to be surjective for  $n \ge 1$  by [550, Theorem 2.5]. Conjecture 10.77 is true if A is stable by Theorem 10.75, or if A is commutative, see [392],[815], [858, Theorem 4.2], and [862, Theorem 27 on page 870].

A discussion about  $K_i$ -regularity and the homotopy invariance of  $K_n^{ALG}(A)$  for  $n \le -1$  is discussed for  $C^*$ -algebras in [862, Section 3.3.4 on page 865ff].

More information about the relation between algebraic and topological K-theory can be found in [260].

# **10.8** Comparing Algebraic *L*-Theory and Topological *K*-theory of *C*\*-Algebras

Whereas the algebraic and the topological K-theory of a  $C^*$ -algebra are very different in general, the topological K-theory of a  $C^*$  algebra is closely related to the L-theory of the  $C^*$ -algebra just considered as a ring with involution. This is illustrated by the following result.

#### Theorem 10.78 (*L*-theory and topological *K*-theory of *C*\*-algebras).

(i) A generalized signature defines for any unital C\*-algebra over ℝ or ℂ a natural isomorphism

$$L_0^p(A) \xrightarrow{=} K_0(A);$$

(ii) Let A be a unital C<sup>\*</sup>-algebra over  $\mathbb{C}$ . Then there is for all  $n \in \mathbb{Z}$  a natural isomorphism

$$K_n(A) \xrightarrow{\cong} L_n^p(A);$$

(iii) Let A be a unital  $C^*$ -algebra over  $\mathbb{R}$ . Then there is a natural homomorphism

$$K_1(A) \xrightarrow{\cong} L_1^h(A)$$

which is surjective and whose kernel has at most order two; (iv) For any unital  $C^*$ -algebra over  $\mathbb{R}$  or  $\mathbb{C}$  there are natural isomorphisms

$$K_n(A)[1/2] \xrightarrow{\cong} L_n^p(A)[1/2] \xrightarrow{\cong} L_n^h(A)[1/2];$$

(v) Let A be a real  $C^*$ -algebra. There are natural isomorphisms

(a) 
$$L_1^p(A) \cong \operatorname{coker}(K_0(A) \xrightarrow{\cdot \eta} K_1(A)),$$
  
(b)  $L_2^p(A) \cong \ker(K_6(A) \xrightarrow{\cdot \eta} K_7(A));$   
(c)  $L_3^p(A) \cong K_7(A),$ 

where  $\eta$  is the non-trivial element in  $K_1(\mathbb{R}) \cong \mathbb{Z}/2$ .

*Proof.* (i) See [861, Theorem 1.6].

(ii) See [722, Theorem 0.2], [737], [861, Theorem 1.8].

(iii) See [861, Theorem 1.9].

(iv) See [861, Theorem 1.11], where this result is described as a consequence of Karoubi [548, 549].

(**v**) See [615, Theorem B].

## **10.9** Topological K-Theory for Finite Groups

Note that  $\mathbb{C}G = l^1(G) = C_r^*(G) = C_{\max}^*(G)$  holds for a finite group, and analogously for the real versions.

**Theorem 10.79** (Topological *K*-theory of the *C*\*-algebra of finite groups). Let *G* be a finite group.

(i) We have

$$K_n(C_r^*(G)) \cong \begin{cases} \operatorname{Rep}_{\mathbb{C}}(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \text{for } n \text{ even}; \\ 0 & \text{for } n \text{ odd}, \end{cases}$$

where  $r_{\mathbb{C}}(G)$  is the number of irreducible complex *G*-representations; (ii) There is an isomorphism of topological *K*-groups

$$KO_n(C_r^*(G,\mathbb{R})) \cong KO_n(\mathbb{R})^{r_{\mathbb{R}}(G;\mathbb{R})} \times KO_n(\mathbb{C})^{r_{\mathbb{R}}(G;\mathbb{C})} \times KO_{n+4}(\mathbb{H})^{r_{\mathbb{R}}(G;\mathbb{H})}$$

where  $r_{\mathbb{R}}(G;\mathbb{R})$ ,  $r_{\mathbb{R}}(G;\mathbb{C})$ , or  $r_{\mathbb{R}}(G;\mathbb{H})$  is the number of irreducible real *G*-representations of real, complex, or quaternionic type.

Moreover,  $KO_n(\mathbb{C}) = K_n(\mathbb{C})$  is 2-periodic with values  $\mathbb{Z}$ , 0 for n = 0, 1,  $KO_n(\mathbb{R}) = K(\mathbb{R})$  is 8-periodic with values  $\mathbb{Z}$ ,  $\mathbb{Z}/2$ ,  $\mathbb{Z}/2$ , 0,  $\mathbb{Z}$ , 0, 0, 0 for  $n = 0, 1, \ldots, 7$  and  $KO_n(\mathbb{H}) = KO_{n+4}(\mathbb{R})$  for  $n \in \mathbb{Z}$ .

*Proof.* One gets isomorphisms of  $C^*$ -algebras

$$C_r^*(G) \cong \prod_{j=1}^{r_{\mathbb{C}}(G)} \mathrm{M}_{n_i}(\mathbb{C})$$

and

$$C_r^*(G,\mathbb{R}) \cong \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{R})} \mathcal{M}_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{C})} \mathcal{M}_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{H})} \mathcal{M}_{p_i}(\mathbb{H})$$

from [908, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page 106]. Now the claim follows from Morita invariance and the well-known values for  $K_n(\mathbb{R})$ ,  $K_n(\mathbb{C})$ , and  $K_n(\mathbb{H})$ , see for instance [943, page 216].

10.10 Notes

## 10.10 Notes

Bivariant algebraic *K*-theory is investigated in [262, 419]. More information about index theory and non-commutative geometry can be found for instance in [250, 431].

## Chapter 11 Classifying Spaces for Families

## **11.1 Introduction**

This chapter is devoted to *classifying spaces for families of subgroups*. They are a key input in the general formulations of the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

If one wants to understand these conjectures, one only needs to know the following facts.

- A *family of subgroups*  $\mathcal{F}$  is a set of subgroups of *G*, closed under conjugation and passing to subgroups;
- A *G-CW*-model for the classifying space  $E_{\mathcal{F}}(G)$  is a *G-CW*-complex whose isotropy groups belong to  $\mathcal{F}$  and whose *H*-fixed point set is weakly contractible for every  $H \in \mathcal{F}$ ;
- Such a *G*-*CW*-model always exists, and two such *G*-*CW*-models are *G*-homotopy equivalent;
- For every *G*-*CW*-complex *X* whose isotropy groups belong to  $\mathcal{F}$ , there is up to *G*-homotopy precisely one *G*-map from *X* to  $E_{\mathcal{F}}(G)$ .

If one is interested in concrete computations, it is very useful to know situations where one can find small *G-CW*-models for specific *G* and  $\mathcal{F}$ .

We give much more information about classifying spaces for families, since they are interesting in their own right and are important tools for investigating groups. After we have explained the basic *G*-homotopy theoretic aspects, we pass to the classifying space  $\underline{E}G = E_{COM}(G)$  for proper actions, which is the same as the classifying space for the family *COM* of compact subgroups. If *G* is discrete,  $\underline{E}G$  reduces to  $E_{\mathcal{FIN}}(G)$ , where  $\mathcal{FIN}$  is the family of finite subgroups. There are many prominent groups for which there are nice geometric models for  $\underline{E}G$ . The *G*-*CW*-complex  $\underline{E}G$  is relevant for the Baum-Connes Conjecture. For the Farrell-Jones Conjecture we also have to deal with  $\underline{E}G = E_{\mathcal{VCY}}(G)$  for the family  $\mathcal{VCY}$  of virtually cyclic subgroups, which is much harder to analyze. We systematically address the question whether there are *finite or finite-dimensional G-CWmodels* and what the minimal dimension of such *G-CW*-models for  $E_{\mathcal{F}}(G)$  are for  $\mathcal{F} = \mathcal{FIN}, \mathcal{VCY}$ .

#### 11.2 Definition and Basic Properties of G-CW-Complexes

**Remark 11.1 (Compactly generated spaces).** In the sequel we will work in the *category of compactly generated spaces*. This convenient category is explained in detail in [687, Appendix A], [927] and [1006, I.4]. A reader may ignore this technical point without harm, but we nevertheless give a short explanation.

A topological space X is called *compactly generated* if X is a Hausdorff space and a subset  $A \subseteq X$  is closed if and only if  $A \cap K$  is closed for every compact subset  $K \subseteq X$ . Given a topological Hausdorff space X, let k(X) be the compactly generated Hausdorff space with the same underlying set as X and the topology for which a subset  $A \subseteq X$  is closed if and only if for every compact subset  $K \subseteq X$  the intersection  $A \cap K$  is closed in the given topology on X. The identity induces a continuous map  $k(X) \to X$  which is a homeomorphism if and only if X is compactly generated. The spaces X and k(X) have the same compact subsets. Locally compact Hausdorff spaces and every Hausdorff space which satisfies the first axiom of countability are compactly generated. In particular, metrizable spaces are compactly generated.

Working in the category of compactly generated spaces means that one only considers compactly generated spaces and whenever a topological construction such as the cartesian product or the mapping space leads out of this category, one retopologizes the result as described above to get a compactly generated space. The advantage is for example that in the category of compactly generated spaces the exponential map map( $X \times Y, Z$ )  $\rightarrow$  map(X, map(Y, Z)) is always a homeomorphism, for an identification  $p: X \rightarrow Y$  the map  $p \times \text{id}_Z: X \times Z \rightarrow Y \times Z$  is always an identification, and, for a filtration by closed subspaces  $X_1 \subset X_2 \subseteq ... \subseteq X$  such that X is the colimit colim<sub> $n \rightarrow \infty$ </sub>  $X_n$ , we always get  $X \times Y = \text{colim}_{n \rightarrow \infty}(X_n \times Y)$ .

One may also work in the category of compactly generated weak Hausdorff spaces, see for instance Strickland [932].

In the sequel G is a topologically group (which is compactly generated). Subgroups are understood to be closed.

**Definition 11.2** (*G-CW-complex*). A *G-CW-complex X* is a *G*-space together with a *G*-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subset X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq \bigcup_{n \ge 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration (i.e., a set  $C \subseteq X$  is closed if and only if  $C \cap X_n$  is closed in  $X_n$  for all  $n \ge 0$ ) and  $X_n$  is obtained from  $X_{n-1}$  for each  $n \ge 0$  by attaching equivariant *n*-dimensional cells, i.e., there exists a *G*-pushout

$$\begin{array}{c|c} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} X_{n-1} \\ & \downarrow & \downarrow \\ & \downarrow & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} X_n. \end{array}$$

#### 11.2 Definition and Basic Properties of G-CW-Complexes

The space  $X_n$  is called the *n*-skeleton of X. Note that only the filtration by skeletons belongs to the *G*-*CW*-structure but not the *G*-pushouts, only their existence is required. An equivariant open *n*-dimensional cell is a *G*-component of  $X_n \setminus X_{n-1}$ , i.e., the preimage under the projection  $X_n \setminus X_{n-1} \to G \setminus (X_n \setminus X_{n-1})$  of a path component of  $G \setminus (X_n \setminus X_{n-1})$ . The closure of an equivariant open *n*-dimensional cell is called an equivariant closed *n*-dimensional cell. If one has chosen the *G*-pushouts in Definition 11.2, then the equivariant open *n*-dimensional cells are the *G*-subspaces  $Q_i(G/H_i \times (D^n \setminus S^{n-1}))$  and the equivariant closed *n*-dimensional cells are the *G*-subspaces  $Q_i(G/H_i \times D^n)$ .

It is obvious what a pair of G-CW-complexes is.

**Remark 11.3** (*G-CW*-complexes and *CW*-complexes with *G*-action). Suppose that *G* is discrete. A *G-CW*-complex *X* is the same as a *CW*-complex *X* with *G*-action such that, for each open cell  $e \subseteq X$  and each  $g \in G$  with  $ge \cap e \neq \emptyset$ , we have gx = x for every  $x \in e$ .

The definition of a G-CW-complex appearing in Definition 11.2 has the advantage that it also makes sense for topological groups.

**Example 11.4 (Simplicial actions).** Let *X* be a simplicial complex on which the group *G* acts by simplicial automorphisms. Then *G* acts also on the barycentric subdivision X' by simplicial automorphisms. The filtration of the barycentric subdivision X' by the simplicial *n*-skeletons yields the structure of a *G*-*CW*-complex, which is not necessarily true for *X*. This becomes clear if one considers the standard 2-simplex with the obvious actions of the symmetric group  $S_3$  given by permuting the three vertices.

A map  $f: X \to Y$  between *G*-*CW*-complexes is called *cellular* if  $f(X_n) \subseteq Y_n$  holds for all  $n \ge 0$ .

For a subgroup  $H \subseteq G$  denote by  $N_G H = \{g \in G \mid gHg^{-1} = H\}$  its *normalizer* and by  $W_G H = N_G H/H$  its *Weyl group*.

#### Lemma 11.5.

- (i) Let X be a G-CW-complex and let Y be an H-CW-complex. Then X × Y with the product G × H-action is a G × H-CW-complex;
- (ii) Let X be a G-CW-complex and let  $H \subseteq G$  be a subgroup. Suppose that G is discrete or that H is open and closed in G. Then X viewed as an H-space by restriction inherits the structure of an H-CW-complex;
- (iii) Consider a G-pushout



Suppose that  $X_i$  for i = 0, 1, 2 is a G-CW-complex and that  $i_1$  is cellular and  $i_2$  is the inclusion of a pair of G-CW-complexes. Then X inherits the structure of a G-CW-complex;

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- (iv) Let X be a G-CW-complex and let  $H \subseteq G$  be a subgroup. Then  $X^H$  viewed as an  $W_GH$ -space inherits the structure of a  $W_GH$ -CW-complex provided that G is discrete, or that  $K \subseteq G$  is open and closed, or that G is a Lie group and  $H \subseteq G$  is compact;
- (v) Let X be a G-CW-complex and let  $H \subseteq G$  be a normal subgroup. Then X/H viewed as an G/H-space inherits the structure of a G/H-CW-complex.

*Proof.* (11.5) The skeletal filtration on  $X \times Y$  is given by

$$(X \times Y)_n = \bigcup_{p+q=n} X_p \times Y_q.$$

Then  $X \times Y$  is the colimit colim<sub> $n\to\infty$ </sub>  $(X \times Y)_n$ , as we work in the category of compactly generated spaces.

(ii) Use the same filtration on X viewed as an *H*-space as for the *G*-*CW*-complex X. (iii) Define the filtration on  $X^H$  given by

$$X_n = j_1((X_1)_n) \cup j_2((X_2)_n).$$

(iv) The *G*-action on *X* induces a  $N_GH$ -action on  $X^H$ , which in turn passes to a  $W_GH$ -action on  $X^H$ . Take the *n*-skeleton of  $X^H$  to be  $(X_n)^H$ . Use the fact that for every  $K \subseteq G$  the space  $(G/K)^H$  is a disjoint union of  $W_GH$ -orbits, which is obvious if *G* is discrete, or if  $K \subseteq G$  is open and closed, and follows for a Lie group *G* and compact  $K \subseteq G$  for instance from [644, Theorem 1.33 on page 23].

(v) The *n*-skeleton of X/H is the image of  $X_n$  under the canonical projection  $X \to X/H$ .

**Exercise 11.6.** Let  $p: \widetilde{X} \to X$  be the universal covering of the connected *CW*-complex *X* with fundamental group  $\pi$ . Show that the  $\pi$ -space  $\widetilde{X}$  inherits the structure of a  $\pi$ -*CW*-complex.

**Exercise 11.7.** Let *p* be a prime number and let *X* be a compact  $\mathbb{Z}/p$ -*CW*-complex. Show that *X* and  $X^{\mathbb{Z}/p}$  are compact *CW*-complexes and their Euler characteristics satisfy

$$\chi(X) \equiv \chi(X^{\mathbb{Z}/p}) \mod p.$$

**Definition 11.8 (Type of a** *G-CW*-complex). A *G-CW*-complex is called *finite* if it is built by finitely many equivariant cells.

A G-CW-complex is called *of finite type* if each *n*-skeleton is a finite G-CW-complex.

A *G*-*CW*-complex is called *of dimension*  $\leq n$  if  $X = X_n$ . It is called *n*-*dimensional* or of *dimension n* if  $X = X_n$  and  $X \neq X_{n-1}$  holds. It is called *finite-dimensional* if it is of dimension  $\leq n$  for some integer *n*.

**Remark 11.9 (Slice Theorem).** A *Slice Theorem* for *G-CW*-complexes is proved in [687, Theorem 7.1]. It says, roughly, that for a *G-CW*-complex X we can find for any  $x \in X$  an arbitrary small  $G_x$ -subspace  $S_x$  and an arbitrary small open

*G*-invariant neighborhood *U* of *x* such that the closure of  $S_x$  is contained in *U*, the inclusion  $\{x\} \rightarrow S_x$  is a  $G_x$ -homotopy equivalence and the canonical *G*-map

$$G \times_{G_x} S_x \to U, \quad (g, s) \mapsto g \cdot s$$

is a G-homeomorphism.

## **11.3 Proper G-Spaces**

**Definition 11.10 (Proper** *G*-space). A *G*-space *X* is called *proper* if for each pair of points *x* and *y* in *X* there are open neighbourhoods  $V_x$  of *x* and  $W_y$  of *y* in *X* such that the closure of the subset  $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$  of *G* is compact.

**Lemma 11.11.** A G-CW-complex X is proper if and only if all its isotropy groups are compact.

Proof.	This is shown	in [644, Theorem	1.23 on page 18].	
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In particular, a free *G*-*CW*-complex is always proper. However, not every free *G*-space is proper.

**Exercise 11.12.** Find a free compact  $\mathbb{Z}$ -space that is not proper.

**Remark 11.13 (Lie Groups acting properly and smoothly on manifolds).** Let G be a Lie group. If M is a proper smooth G-manifold, then an equivariant smooth triangulation induces a G-CW-structure on M. For the proof and for *equivariant smooth triangulations* we refer to [513, Theorem I and II].

**Exercise 11.14.** Let *p* be an odd prime. Show that there is no smooth free  $\mathbb{Z}/p$ -action on an even-dimensional sphere.

### **11.4 Maps between** *G-CW***-Complexes**

**Theorem 11.15 (Equivariant Cellular Approximation Theorem).** Let (X, A) be a pair of *G*-*CW*-complexes and let *Y* be a *G*-*CW*-complex. Let  $f: X \to Y$  be a *G*-map such that  $f|_A: A \to Z$  is cellular.

Then there exists a cellular G-map  $f': X \to Y$  such that  $f|_A = f'|_A$  and f and f' are G-homotopic relative A.

*Proof.* Since  $X = \text{colim}_{n\to\infty} X_n$  by definition, it suffices to construct inductively for n = -1, 0, 1, 2, ... G-maps

$$h_n: X_n \times [0,1] \cup X \times \{0\} \rightarrow Y$$

such that  $h_n(x, 0) = f(x)$  for every  $x \in X_n$  and  $h_n(x, t) = h_{n-1}(x, t)$  for every  $x \in X_{n-1}$  and  $t \in [0, 1]$  hold and the map  $f'_n: X \to Y$  sending  $x \in X_n$  to  $h_n(x, 1)$  is cellular. The induction beginning n = -1 is trivial, define  $h_{-1}: A \times [0, 1] \cup X \times \{0\} \to Y$  by sending (x, t) to f(x). The induction step from (n - 1) to n is done as follows. Choose a *G*-pushout

$$\begin{array}{c|c} \bigsqcup_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i} X_{n-1} \cup A \\ & & \downarrow \\ & & \downarrow \\ & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i} X_n \cup A. \end{array}$$

It yields the G-pushout

Because of the G-pushout property it suffices to explain for every  $i \in I_n$  how to extend the composite

$$\phi_i \colon G/H_i \times \left( S^{n-1} \times [0,1] \cup D^n \times \{0\} \right) \xrightarrow{q'_i} X_{n-1} \times [0,1] \cup X \times \{0\} \xrightarrow{h_{n-1}} Y$$

to a G-map

$$\Phi_i: G/H_i \times D^n \times [0,1] \to Y$$

satisfying  $\Phi_i(G/H_i \times D^n \times \{1\}) \subseteq Y_n$ . This is the same as the non-equivariant problem to extend the map

$$\phi_i' \colon S^{n-1} \times [0,1] \cup D^n \times \{0\} \to Y^H$$

obtained from  $\phi_i$  by restriction to  $\{eH_i\} \times (S^{n-1} \times [0, 1] \cup D^n \times \{0\})$  to a map

$$\Phi_i': D^n \times [0,1] \to Y^H$$

such that  $\Phi'_i(D^n \times \{1\}) \subseteq Y_n$ , since we can define  $\Phi_i(gH, x, t) := g \cdot \Phi'_i(x, t)$ . It is not hard to check that this non-equivariant problem can be solved if the inclusion  $Y_{m-1}^H \to Y_m^H$  is *m*-connected for every  $m \ge 0$ . Since we have the pushout of spaces

$$\begin{array}{c} \coprod_{i \in I_m} G/H_i^H \times S^{m-1} \xrightarrow{\coprod_{i \in I_m} q_i^m} Y_{m-1} \\ \downarrow \\ \downarrow \\ \coprod_{i \in I_n} G/H_i^H \times D^m \xrightarrow{\coprod_{i \in I_n} Q_i^m} Y_m \end{array}$$

the inclusion  $G/H_i^H \times S^{m-1} \to G/H_i^H \times D^m$  is *m*-connected and  $G/H_i^H \times S^{m-1}$  is a deformation retract of an open neighborhood in  $G/H_i^H \times D^m$ , this follows from the Blakers-Massey Excision Theorem, see [954, Proposition 6.4.2 on page 133].

A map  $f: X \to Y$  of spaces is called a *weak homotopy equivalence* if f induces a bijection  $\pi_0(f): \pi_0(X) \to \pi_0(Y)$  and for every  $x \in X$  and  $n \ge 1$  an isomorphism  $\pi_n(f, x): \pi_n(X, x) \to \pi_n(Y, f(x))$ . A *G*-map  $f: X \to Y$  of *G*-spaces is called a *weak G-homotopy equivalence* if  $f^H: X^H \to Y^H$  is a weak equivalence of spaces for all subgroups  $H \subseteq G$ .

#### Theorem 11.16 (Equivariant Whitehead Theorem).

(i) Let  $f: Y \to Z$  be a G-map between G-spaces. Then f is a weak G-homotopy equivalence if for every G-CW-complex X the map induced by f between the G-homotopy classes of G-maps

$$f_* \colon [X,Y]^G \to [X,Z]^G, \quad [h] \mapsto [f \circ h]$$

is bijective;

- (ii) Let  $f: Y \rightarrow Z$  be a *G*-map between *G*-CW-complexes. Then the following assertions are equivalent:
  - (a) *f* is a *G*-homotopy equivalence;
  - (b) *f* is a weak *G*-homotopy equivalence;
  - (c) For every  $H \subseteq G$  that occurs as an isotropy group of some point in X or Y, the map  $f^H : X^H \to Y^H$  is a weak homotopy equivalence of spaces.

*Proof.* See [953, II.2.6], [644, Theorem 2.4 on page 36]. □

**Exercise 11.17.** Let Y be a G-space. A G-CW-approximation of Y is a G-CW-complex X together with a weak G-homotopy equivalence  $f: X \to Y$ . Show that two G-CW-approximations of Y are G-homotopy equivalent.

## 11.5 Definition and Basic Properties of Classifying Spaces for Families

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, it is a set of subgroups of G that is closed under conjugation with elements of G and under passing to subgroups, and we listed some examples in Notation 2.63, for instance the family  $\mathcal{TR}$  consisting of the trivial subgroup, the family  $\mathcal{FIN}$  of finite subgroups, the family  $\mathcal{VCY}$  of virtually cyclic subgroups, and the family  $\mathcal{RLL}$  of all subgroups. Actually one could replace the condition that  $\mathcal{F}$  is closed under taking subgroups by the weaker condition that the intersection of finitely many elements of  $\mathcal{F}$  belongs to  $\mathcal{F}$ . Then the set of compact open subgroups is a family, too.

**Definition 11.18 (Classifying** *G*-*CW*-complex for a family of subgroups). Let  $\mathcal{F}$  be a family of subgroups of *G*. A model  $E_{\mathcal{F}}(G)$  for the *classifying G*-*CW*-complex for the family  $\mathcal{F}$  of subgroups of *G*, sometimes also called the *classifying space for* the family  $\mathcal{F}$  of subgroups of *G*, is a *G*-*CW*-complex  $E_{\mathcal{F}}(G)$  that has the following properties:

- (i) All isotropy groups of  $E_{\mathcal{F}}(G)$  belong to  $\mathcal{F}$ ;
- (ii) For any *G*-*CW*-complex *Y* whose isotropy groups belong to  $\mathcal{F}$ , there is up to *G*-homotopy precisely one *G*-map  $Y \to X$ .

We abbreviate  $\underline{E}G := E_{COM}(G)$  and call it the *universal G-CW-complex for* proper *G*-actions.

If G is discrete, we have  $\underline{E}G := E_{\mathcal{FIN}}(G)$ .

In other words,  $E_{\mathcal{F}}(G)$  is a terminal object in the *G*-homotopy category of *G*-*CW*-complexes whose isotropy groups belong to  $\mathcal{F}$ . In particular, two models for  $E_{\mathcal{F}}(G)$  are *G*-homotopy equivalent and for two families  $\mathcal{F}_0 \subseteq \mathcal{F}_1$  there is up to *G*-homotopy precisely one *G*-map  $E_{\mathcal{F}_0}(G) \to E_{\mathcal{F}_1}(G)$ .

**Theorem 11.19 (Homotopy characterization of**  $E_{\mathcal{F}}(G)$ ). Let  $\mathcal{F}$  be a family of subgroups.

- (i) There exists a model for  $E_{\mathcal{F}}(G)$  for any family  $\mathcal{F}$ ;
- (ii) A G-CW-complex X is a model for  $E_{\mathcal{F}}(G)$  if and only if all its isotropy groups belong to  $\mathcal{F}$  and for each  $H \in \mathcal{F}$  the H-fixed point set  $X^H$  is weakly contractible, i.e.,  $X^H$  is non-empty and path connected and  $\pi_n(X^H, y)$  vanishes for all  $n \ge 1$ and one (and hence all) basepoints  $y \in X^H$ .

*Proof.* (i) A model can be obtained by attaching equivariant cells  $G/H \times D^n$  for all  $H \in \mathcal{F}$  to make the *H*-fixed point sets weakly contractible. See for instance [644, Proposition 2.3 on page 35]. There are also functorial constructions for discrete *G* generalizing the bar construction, see [280, Section 3 and Section 7].

(ii) Suppose that the *G*-*CW*-complex *X* is a model for  $E_{\mathcal{F}}(G)$ . Let *Y* be any *CW*-complex and let  $H \in \mathcal{F}$ . Then there is up to *G*-homotopy precisely one *G*-map  $G/H \times Y \to X$ . Hence there is up to homotopy precisely one map  $Y \to X^H$ . This is equivalent to the condition that  $X^H$  is weakly contractible.

Suppose that  $X^H$  is weakly contractible for every  $H \in \mathcal{F}$ . Let (Y, B) be a G-CW-pair such that the isotropy group of any point in  $Y \setminus B$  belongs to  $\mathcal{F}$ , and let  $f_{-1}: B \to X$  be any G-map. We next show the existence of a G-map  $f: Y \to X$  extending  $f_{-1}$ . Obviously this implies that X is a model for  $E_{\mathcal{F}}(G)$ . Since Y is the colimit over the skeletons  $Y_n$  for  $n \ge -1$  and  $Y_{-1} = B$ , it suffices to prove for  $n \ge 0$  that, for a given G-map  $f_{n-1}: Y_{n-1} \to X$ , there exists a G-map  $f_n: Y_n \to X$  with  $f_n|_{Y_{n-1}} = f_{n-1}$ . Recall that by definition there exists a G-pushout

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such that each  $H_i$  belong to  $\mathcal{F}$ . Because of the universal property of a *G*-pushout it remains to show that for every  $H \in \mathcal{F}$  and every G-map  $u: G/H \times S^{n-1} \to X$  can be extended to a *G*-map  $v: G/H \times D^n \to X$ . This is equivalent to showing that every map  $u': S^{n-1} \to X^H$  can be extended to a map  $v': D^n \to X^H$ . This follows from the assumption that  $X^H$  is weakly contractible.

A model for  $E_{\mathcal{RLL}}(G)$  is G/G. A model for  $E_{\mathcal{TR}}(G)$  is the same as a model for EG, i.e, the total space of the universal *G*-principal bundle  $EG \rightarrow BG = G \setminus EG$ . In Section 11.6 we will give many interesting geometric models for classifying spaces  $\underline{E}G = E_{\mathcal{FIN}}(G)$ .

**Exercise 11.20.** Show for a discrete group *G* that there exists a zero-dimensional model for  $E_{\mathcal{F}}(G)$  if and only if  $G \in \mathcal{F}$ . Is there a non-trivial connected Lie group *L* with a 0-dimensional model for *EL*?

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In this section we present some interesting geometric models for the classifying space for proper actions  $\underline{E}G$  for some discrete groups. These models will often be small in the sense that they are finite, of finite type, or finite-dimensional. We will restrict ourselves to discrete groups G in this section. More information, also for non-discrete groups, can be found for instance in [109, 655].

#### 11.6.1 Simplicial Model

Let  $P_{\infty}(G)$  be the geometric realization of the abstract simplicial complex whose *k*-simplices consist of subsets of *G* of cardinality (k + 1). There is an obvious simplicial *G*-action of *G* on  $P_{\infty}(G)$  coming from the group structure. We get for instance from [1, Example 2.6].

**Theorem 11.21 (Simplicial model).**  $P_{\infty}(G)$  is a model for <u>E</u>G.

#### 11.6.2 Operator Theoretic Model

Let  $PC_0(G)$  be the metric space of functions  $f: G \to \{r \in \mathbb{R} \mid r \ge 0\}$  such that f is not identically zero and has finite support, where the metric comes from the supremum norm. The group G acts isometrically on  $PC_0(G)$  by  $(g \cdot f)(x) := f(g^{-1}x)$  for  $f \in PC_0(G)$  and  $g, x \in G$ . Obviously  $PC_0(G)$  is a subspace of the Banach space  $C_0(G)$ .

Let  $X_G$  be the metric space

$$X_G = \left\{ f \colon G \to [0,1] \; \middle| \; f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

with the metric coming from the supremum norm. The group *G* acts isometrically on  $X_G$  by  $(g \cdot f)(x) := f(g^{-1}x)$  for  $f \in X_G$  and  $g, x \in G$ .

**Theorem 11.22 (Operator theoretic model).** Both  $PC_0(G)$  and  $X_G$  are *G*-homotopy equivalent to a *G*-CW-model for EG.

*Proof.* See [1, Theorem 2.4] and [109, Section 2].  $\Box$ 

**Remark 11.23 (Comparing**  $P_{\infty}(G)$  **and**  $X_G$ ). The simplicial *G*-complex  $P_{\infty}(G)$  of Theorem 11.21 and the *G*-space  $X_G$  of Theorem 11.22 have the same underlying sets but in general they have different topologies. The identity map induces a (continuous) *G*-map  $P_{\infty}(G) \rightarrow X_G$  which is a *G*-homotopy equivalence, but in general not a *G*-homeomorphism, see also [963, A.2].

#### 11.6.3 Discrete Subgroups of Almost Connected Lie Groups

The next result is a special case of a much more general result due to Abels [1, Corollary 4.14]. Recall that a topological group *L* is called almost connected if  $\pi_0(L)$  is finite.

**Theorem 11.24** (Discrete subgroups of almost connected Lie groups). Let L be an almost connected Lie group. Let  $G \subseteq L$  be a discrete subgroup.

Then L contains a maximal compact subgroup K, which is unique up to conjugation, and the G-space L/K is a model for <u>E</u>G.

#### 11.6.4 Actions on Simply Connected Non-Positively Curved Manifolds

**Theorem 11.25 (Actions on simply connected non-positively curved manifolds).** Suppose that G acts properly and isometrically on the simply connected complete Riemannian manifold M with non-positive sectional curvature. Then M is a model for  $\underline{E}G$ .

*Proof.* See [1, Theorem 4.15].

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#### 11.6.5 Actions on Trees and Graphs of Groups

A tree is a 1-dimensional CW-complex that is contractible.

**Theorem 11.26 (Actions on trees).** Suppose that G acts on a tree T such that for each element  $g \in G$  and each open cell e with  $g \cdot e \cap e \neq \emptyset$  we have gx = x for any  $x \in e$ . Assume that the isotropy group of each  $x \in T$  is finite.

Then T is a model for  $\underline{E}G$ .

*Proof.* Obviously *T* is a *G*-*CW*-complex, see Remark 11.3. Let  $H \subseteq G$  be finite. If  $e_0$  is a zero-cell in *T*, then  $H \cdot e_0$  is finite. In the sequel we equip *T* with the obvious path length metric, for which each edge has length 1. Let *T'* be the union of all geodesics with extremities in  $H \cdot e$ . This is an *H*-invariant subtree of *T* of finite diameter. One shows now inductively over the diameter of *T'* that *T'* has a vertex that is fixed under the *H*-action, see [911, page 20] or [308, Proposition 4.7 on page 17]. Hence  $T^H$  is non-empty. If *e* and *f* are vertices in  $T^H$ , the geodesic in *T* from *e* to *f* must be *H*-invariant. Hence  $T^H$  is a connected *CW*-subcomplex of the tree *T* and hence is itself a tree. This shows that  $T^H$  is contractible. Now apply Theorem 11.19 (ii).

#### 11.6.6 Actions on CAT(0)-Spaces

For the notion of a CAT(0)-space we refer for instance to [165, Definition 1.1 in Chapter II.1 on page 158].

**Theorem 11.27 (Actions on CAT(0)-spaces).** Let X be a proper G-CW-complex. Suppose that X has the structure of a complete CAT(0)-space on which G acts by isometries. Then X is a model for EG.

*Proof.* By [165, Corollary 2.8 in Chapter II.2 on page 179] the *K*-fixed point set of *X* is a non-empty convex subset of *X* and hence contractible for any compact subgroup  $K \subset G$ .

This result contains as special cases Theorem 11.25 and Theorem 11.26, since simply connected complete Riemannian manifolds with non-positive sectional curvature and trees are complete CAT(0)-spaces.

#### 11.6.7 The Rips Complex of a Hyperbolic Group

A metric space X = (X, d) is called  $\delta$ -hyperbolic for a given real number  $\delta \ge 0$  if for any four points x, y, z, t the following inequality holds

$$(11.28) \quad d(x,y) + d(z,t) \le \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\} + 2\delta.$$

A group G with a finite set S of generators is called  $\delta$ -hyperbolic if the metric space  $(G, d_S)$  given by G and the word-metric  $d_S$  with respect to the finite set of generators S is  $\delta$ -hyperbolic.

The *Rips complex*  $P_d(G, S)$  of a group G with a finite set S of generators for a natural number d is the geometric realization of the abstract simplicial complex whose set of k-simplices consists of subsets S' of S of cardinality (k + 1) such that  $d_S(g,g') \leq d$  holds for all  $g,g \in S'$ . The obvious G-action by simplicial automorphisms on  $P_d(G,S)$  induces a G-action by simplicial automorphisms on the barycentric subdivision  $P_d(G,S)'$ , see Example 11.4.

**Theorem 11.29 (Rips complex).** Let G be a group with a finite set S of generators. Suppose that (G, S) is  $\delta$ -hyperbolic for the real number  $\delta \ge 0$ . Let d be a natural number with  $d \ge 16\delta + 8$ . Then the barycentric subdivision of the Rips complex  $P_d(G, S)'$  is a finite G-CW-model for EG.

#### *Proof.* See [713], [714].

A metric space is called *hyperbolic* if it is  $\delta$ -hyperbolic for some real number  $\delta \ge 0$ . A finitely generated group *G* is called *hyperbolic* if for one (and hence all) finite set *S* of generators the metric space  $(G, d_S)$  is a hyperbolic metric space. Since for metric spaces the property hyperbolic is invariant under quasiisometry and for two finite sets  $S_1$  and  $S_2$  of generators of *G* the metric spaces  $(G, d_{S_1})$  and  $(G, d_{S_2})$  are quasiisometric, the choice of *S* does not matter. Theorem 11.29 implies that for a hyperbolic group there is a finite *G*-*CW*-model for <u>*E*</u>*G*.

The notion of a hyperbolic group is due to Gromov and has intensively been studied, see for example [165, 424, 440]. The prototype is the fundamental group of a closed hyperbolic manifold.

#### 11.6.8 Arithmetic Groups

An arithmetic group *A* in a semisimple connected linear  $\mathbb{Q}$ -algebraic group possesses a finite *A*-*CW*-model for <u>E</u>*A*. Namely, let  $G(\mathbb{R})$  be the  $\mathbb{R}$ -points of a semisimple  $\mathbb{Q}$ -group  $G(\mathbb{Q})$ , and let  $K \subseteq G(\mathbb{R})$  be a maximal compact subgroup. If  $A \subseteq G(\mathbb{Q})$  is an arithmetic group, then  $G(\mathbb{R})/K$  with the left *A*-action is a model for <u>E</u>*A*, as already explained in Theorem 11.24. The *A*-space  $G(\mathbb{R})/K$  is not necessarily cocompact. The *Borel-Serre completion* of  $G(\mathbb{R})/K$ , see [153], [909], is a finite *A*-*CW*-model for <u>E</u>*G*, as pointed out in [19, Remark 5.8], where a private communication with Borel and Prasad is mentioned. 11.6 Models for the Classifying Space for Proper Actions

#### 11.6.9 Mapping Class Groups

Let  $\Gamma_{g,r}^s$  be the *mapping class group* of an orientable compact surface  $F_{g,r}^s$  of genus g with s punctures and r boundary components. This is the group of isotopy classes of orientation preserving self-diffeomorphisms  $F_{g,r}^s \to F_{g,r}^s$  that preserve the punctures individually and restrict to the identity on the boundary. We require that the isotopies leave the boundary pointwise fixed. We will always assume that 2g + s + r > 2, or, equivalently, that the Euler characteristic of the punctured surface  $F_{g,r}^s$  is negative. It is well-known that the associated *Teichmüller space*  $\mathcal{T}_{g,r}^s$  is a contractible space on which  $\Gamma_{g,r}^s$  acts properly.

**Theorem 11.30** (Mapping class group). The Teichmüler space  $\mathcal{T}_{g,r}^s$  is a model for  $\underline{E}\Gamma_{g,r}^s$ 

Proof. This follows from [574].

**Remark 11.31 (Finite model for**  $\underline{E}\Gamma_{g,r}^{s}$ **).** There exist a finite  $\Gamma_{g,r}^{s}$ -*CW*-model for  $\underline{E}\Gamma_{g,r}^{s}$ , see [741].

#### 11.6.10 Outer Automorphism Groups of Finitely Generated Free Groups

Let  $F_n$  be the free group of rank n. Denote by  $Out(F_n)$  the group of outer automorphisms of  $F_n$ , i.e., the quotient of the group of all automorphisms of  $F_n$  by the normal subgroup of inner automorphisms. Culler and Vogtmann [265, 970] have constructed a space  $X_n$ , called the *outer space*, on which  $Out(F_n)$  acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface. Fix a graph  $R_n$  with one vertex v and *n*-edges and identify  $F_n$  with  $\pi_1(R_n, v)$ . A marked metric graph  $(g, \Gamma)$  consists of a graph  $\Gamma$  with all vertices of valence at least three, a homotopy equivalence g:  $R_n \to \Gamma$ , called a marking, and an assignment of a positive length to each edge of  $\Gamma$ . This turns  $\Gamma$  into a metric space with the path metric. We call two marked metric graphs  $(g, \Gamma)$  and  $(g', \Gamma')$  equivalent of there is a homothety  $h: \Gamma \to \Gamma'$ such that  $g \circ h$  and h' are homotopic. Homothety means that there is a constant  $\lambda > 0$  with  $d(h(x), h(y)) = \lambda \cdot d(x, y)$  for all x, y. Elements in the outer space  $X_n$  are equivalence classes of marked graphs. The main result in [265] is that X is contractible. Actually, for each finite subgroup  $H \subseteq Out(F_n)$  the *H*-fixed point set  $X_n^H$  is contractible [593, Proposition 3.3 and Theorem 8.1], [1005, Theorem 5.1].

The space  $X_n$  contains a *spine*  $K_n$ , which is an  $Out(F_n)$ -equivariant deformation retraction. This space  $K_n$  is a simplicial complex of dimension (2n-3) on which the  $Out(F_n)$ -action is by simplicial automorphisms and cocompact. Actually the group of simplicial automorphisms of  $K_n$  is  $Out(F_n)$ , see [166]. Hence the barycentric subdivision  $K'_n$  is a finite (2n-3)-dimensional model of  $\underline{E}$   $Out(F_n)$ .

#### **11.6.11** Special Linear Groups of (2,2)-Matrices

In order to illustrate some of the general statements above, we consider the special example  $SL_2(\mathbb{Z})$ .

Let  $\mathbb{H}^2$  be the 2-dimensional hyperbolic space. We will use either the upper half-plane model or the Poincaré disk model. The group  $SL_2(\mathbb{R})$  acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations, i.e., a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acts by sending a complex number *z* with positive imaginary part to  $\frac{az+b}{cz+d}$ .

 $\begin{pmatrix} c & d \end{pmatrix}$  acts by sending a complex number z with positive imaginary part to  $\frac{dz+b}{cz+d}$ . This action is proper and transitive. The isotropy group of z = i is SO(2). Since  $\mathbb{H}^2$ 

is a simply connected Riemannian manifold whose sectional curvature is constant -1, the SL<sub>2</sub>( $\mathbb{Z}$ )-space  $\mathbb{H}^2$  is a model for <u>E</u> SL<sub>2</sub>( $\mathbb{Z}$ ) by Theorem 11.25.

One easily checks that  $SL_2(\mathbb{R})$  is a connected Lie group and  $SO(2) \subseteq SL_2(\mathbb{R})$  is a maximal compact subgroup. Since the  $SL_2(\mathbb{R})$ -action on  $\mathbb{H}^2$  is transitive and SO(2) is the isotropy group at  $i \in \mathbb{H}^2$ , we see that the  $SL_2(\mathbb{R})$ -manifolds  $SL_2(\mathbb{R})/SO(2)$  and  $\mathbb{H}^2$  are  $SL_2(\mathbb{R})$ -diffeomorphic.

As  $SL_2(\mathbb{Z})$  is a discrete subgroup of  $SL_2(\mathbb{R})$ , the space  $\mathbb{H}^2 = SL_2(\mathbb{R})/SO(2)$  with the obvious  $SL_2(\mathbb{Z})$ -action is a model for  $\underline{E} SL_2(\mathbb{Z})$  by Theorem 11.24.

The group  $SL_2(\mathbb{Z})$  is isomorphic to the amalgamated free product  $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ . This implies that  $SL_2(\mathbb{Z})$  acts cell preserving with finite stabilizers on a tree *T*, which has alternately two and three edges emanating from each vertex, see [911, Theorem 7 in I.4.1 on page 32 and Example 4.2 (c) in I.4.2 on page 35]. This tree is a model for  $E SL_2(\mathbb{Z})$  by Theorem 11.26.

The two models given by  $\mathbb{H}^2$  and *T* must be  $SL_2(\mathbb{Z})$ -homotopy equivalent. They can explicitly be related by the following construction.

Divide the Poincaré disk or the half plane model  $\mathbb{H}^2$  into fundamental domains for the SL<sub>2</sub>( $\mathbb{Z}$ )-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges whose end points lie in the interior of the Poincaré disk is a tree *T* with SL<sub>2</sub>( $\mathbb{Z}$ )-action. This is the tree model above. The tree is an SL<sub>2</sub>( $\mathbb{Z}$ )-equivariant deformation retraction of  $\mathbb{H}^2$ . A retraction is given by moving a point *p* in  $\mathbb{H}^2$  along a geodesic starting at the vertex at infinity that belongs to the triangle containing *p*, through *p* to the first intersection point of this geodesic with *T*, see for instance [911, Example 4.2 (c) in I.4.2 on page 35].

#### **11.6.12** Groups with Appropriate Maximal Finite Subgroups

Let G be a discrete group. Let MFIN be the subset of FIN consisting of elements in FIN that are maximal with respect to inclusion in FIN. Consider the following assertions concerning G:

- ( $\underline{M}$ ) Every non-trivial finite subgroup of *G* is contained in a unique maximal finite subgroup;
- $(\underline{NM}) \qquad M \in \mathcal{MFIN}, M \neq \{1\} \implies N_G M = M;$

For such a group there is a nice model for  $\underline{E}G$  with as few non-free cells as possible. Let  $\{M_i \mid i \in I\}$  be a complete set of representatives for the conjugacy classes of maximal finite subgroups of G, i.e., each  $M_i$  is a maximal finite subgroup of G and any maximal finite subgroup of G is conjugate to  $M_i$  for precisely one element  $i \in I$ . By attaching free G-cells, we get an inclusion of G-CW-complexes  $j_1: \prod_{i \in I} G \times_{M_i} EM_i \to EG$  where EG is the same as  $E_{\mathcal{TR}}(G)$ , i.e., a contractible free G-CW-complex.

**Theorem 11.32 (Passage from** EG to  $\underline{E}G$ ). Suppose that G satisfies ( $\underline{M}$ ) and ( $\underline{NM}$ ). Let X be the G-CW-complex defined by the G-pushout

$$\begin{array}{c|c} & \coprod_{i \in I} G \times_{M_i} EM_i \xrightarrow{J_1} EG \\ & u_1 & \downarrow \\ & \downarrow \\ & \coprod_{i \in I} G/M_i \xrightarrow{k_1} X \end{array}$$

where  $u_1$  is the obvious *G*-map obtained by collapsing each  $EM_i$  to a point. Then X is a model for EG.

*Proof.* We have to explain why  $\underline{E}G$  is a model for the classifying space for proper actions of *G*. Obviously it is a *G*-*CW*-complex. Its isotropy groups are all finite. We have to show for  $H \subseteq G$  finite that  $X^H$  weakly contractible. We begin with the case  $H \neq \{1\}$ . Because of conditions (M) and (NM) there is precisely one index  $i_0 \in I$  such that *H* is subconjugate to  $M_{i_0}$  and is not subconjugate to  $M_i$  for  $i \neq i_0$  and we get

$$\left(\prod_{i\in I} G/M_i\right)^H = \left(G/M_{i_0}\right)^H = \{\bullet\}.$$

Hence  $X^H = \{\bullet\}$ . It remains to treat  $H = \{1\}$ . Since  $u_1$  is a non-equivariant homotopy equivalence and  $j_1$  is a cofibration,  $f_1$  is a non-equivariant homotopy equivalence and hence <u>*E*</u>*G* is contractible (after forgetting the group action).

**Example 11.33.** Here are some examples of groups G that satisfy conditions (<u>M</u>) and (<u>NM</u>):

• Extensions  $1 \to \mathbb{Z}^n \to G \to F \to 1$  for finite *F* such that the conjugation action of *F* on  $\mathbb{Z}^n$  is free outside  $0 \in \mathbb{Z}^n$ .

The conditions (M) and (NM) are satisfied by [683, Lemma 6.3];

• Fuchsian groups

The conditions ( $\underline{M}$ ) and ( $\underline{NM}$ ) are satisfied by [683, Lemma 4.5]. In [683] the larger class of cocompact planar groups (sometimes also called cocompact NEC-groups) is treated;

• One-relator groups

Let *G* be a one-relator group. Let  $G = \langle (q_i)_{i \in I} | r \rangle$  be a presentation with one relation. We only have to consider the case where *G* contains torsion. Let *F* be the free group with basis  $\{q_i | i \in I\}$ . Then *r* is an element in *F*. There exists

an element  $s \in F$  and an integer  $m \ge 2$  such that  $r = s^m$ , the cyclic subgroup *C* generated by the class  $\overline{s} \in G$  represented by *s* has order *m*, any finite subgroup of *G* is subconjugate to *C*, and for any  $g \in G$  the implication  $g^{-1}Cg \cap C \ne 1 \Rightarrow g \in C$  holds. These claims follows from [693, Propositions 5.17, 5.18, and 5.19 in II.5 on pages 107 and 108]. Hence *G* satisfies conditions (M) and (NM).

**Remark 11.34 (Passing to larger families).** Theorem 11.32 is a special case of a general recipe to construct for two families  $\mathcal{F} \subseteq \mathcal{G}$  an efficient model for  $E_{\mathcal{G}}(G)$  from  $E_{\mathcal{F}}(G)$  in [688, Section 2]. These models are important for concrete calculations of the left-hand side appearing in the Baum-Conjecture or the Farrell-Jones Conjecture, see Chapter 17.

#### 11.6.13 One-Relator Groups

Let *G* be a one-relator group. Let  $G = \langle (q_i)_{i \in I} | r \rangle$  be a presentation with one relation. There is up to conjugacy one maximal finite subgroup *C*, which turns out to be cyclic. Let  $p: *_{i \in I} \mathbb{Z} \to G$  be the epimorphism from the free group generated by the set *I* to *G* that sends the generator  $i \in I$  to  $q_i$ . Let  $Y \to \bigvee_{i \in I} S^1$  be the *G*-covering associated to the epimorphism *p*. There is a 1-dimensional unitary *C*-representation *V* and a *C*-map  $f: SV \to \operatorname{res}_G^C Y$  such that the induced action on the unit sphere *SV* is free and the following is true: If we equip *SV* with the *C*-*CW*-structure with precisely one equivariant 0-cell and precisely one equivariant 1-cell and *DV* with the *C*-*CW*-complex structures coming from the fact that *DV* is the cone over *SV*, then the *C*-map *f* can be chosen to be cellular and we obtain a *G*-*CW*-model for <u>E</u>*G* by the *G*-pushout



where  $\overline{f}$  sends (g, x) to gf(x). Thus we get a 2-dimensional *G*-*CW*-model for  $\underline{E}G$  such that  $\underline{E}G$  is obtained from G/C for a maximal finite cyclic subgroup  $C \subseteq G$  by attaching free cells of dimensions  $\leq 2$ . The *CW*-*CW*-complex structure on  $\underline{E}G$  has precisely one 0-cell  $G/C \times D^0$ , one 0-cell  $G \times D^0$ ,  $(2 \cdot |I| \text{ many 1-cells } G \times D^1$  and |I| many 2-cells  $G \times D^2$ . All these claims follow from [171, Exercise 2 (c) II. 5 on page 44].

If *G* is torsionfree, the 2-dimensional complex associated to a presentation with one relation is a model for BG, see [693, Chapter III §§9-11].

**Exercise 11.35.** Let *G* be a one-relator group. Let  $M \subseteq G$  be a maximal cyclic subgroup. Show that the inclusion induces for  $n \geq 3$  an isomorphism  $H_n(BM) \xrightarrow{\cong} H_n(BG)$ .

**Exercise 11.36.** Let *G* be a finitely generated group. Suppose that for every integer *d* there is a  $k \ge d$  with  $H_k(BG; \mathbb{Q}) \ne 0$ . Show that *G* cannot be a hyperbolic group, an arithmetic group, a mapping class group,  $Out(F_n)$ , or a one-relator group.

## 11.7 Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

In general the *G*-*CW*-models for  $\underline{E}G$  are not as nice and small as the ones for  $\underline{E}G$ . We illustrate this in the case  $G = \mathbb{Z}^{\overline{n}}$  for  $n \ge 2$ . Then a  $\mathbb{Z}^n$ -*CW*-model for  $\underline{E}\mathbb{Z}^n = E\mathbb{Z}^n$  is  $\mathbb{R}^n$  with the standard translation action of  $\mathbb{Z}^n$ .

An explicit  $\mathbb{Z}^n$ -*CW*-model for  $\underline{E}\mathbb{Z}^n$  can be constructed as follows. Choose an enumeration  $\{C_i \mid i \in \mathbb{Z}\}$  of the infinite cyclic subgroups of  $\mathbb{Z}^n$ . Consider the space  $\mathbb{R}^n \times \mathbb{R}$ . For each  $i \in \mathbb{Z}$  we identify in  $\mathbb{R}^n \times \{i\}$  the subspace given by the  $\mathbb{R}$ -span of  $C_i \subseteq \mathbb{Z}^n \subseteq \mathbb{R}^n$  to a point. Then we obtain a  $\mathbb{Z}^n$ -*CW*-complex *X*. Since the  $C_i$ -fixed point set of *X* consists of precisely one point, the underlying topological space *X* is contractible, and all isotropy groups of the  $\mathbb{Z}^n$ -action are infinite cyclic or trivial, *X* is a  $\mathbb{Z}^n$ -*CW*-model for  $\underline{E}\mathbb{Z}^n$ . Note that the dimension of *X* is (n + 1). One can actually show that any  $\mathbb{Z}^n$ -*CW*-model for  $\underline{E}\mathbb{Z}^n$  has dimension greater than or equal to (n + 1), see [688, Example 5.21].

#### 11.7.1 Groups with Appropriate Maximal Virtually Cyclic Subgroups

Let G be a discrete group. Let  $\mathcal{MVCY}$  be the subset of  $\mathcal{VCY}$  consisting of elements in  $\mathcal{VCY}$  that are maximal with respect to inclusion in  $\mathcal{VCY}$ . Consider the following assertions concerning G:

- $(\underline{\underline{M}})$  Every infinite virtually cyclic subgroup of G is contained in a unique maximal virtually cyclic subgroup;
- (NM)  $V \in \mathcal{MVCY}, |V| = \infty \implies N_G V = V.$

For such a group there is a nice model for  $\underline{\underline{E}}G$  with as few cells of type G/V with infinite virtually cyclic V as possible. Let  $\{V_i \mid i \in I\}$  be a complete set of representatives for the conjugacy classes of maximal infinite virtually cyclic subgroups of G. By attaching G-cells of the type G/H for finite subgroups  $H \subseteq G$ , we get an inclusion of G-CW-complexes  $j_1: \coprod_{i \in I} G \times_{V_i} \underline{\underline{E}}V_i \to \underline{\underline{E}}G$ .

The next result is proved in [688, Corollary 2.11].

**Theorem 11.37 (Passage from**  $\underline{E}G$  **to**  $\underline{E}G$ ). Suppose that G satisfies ( $\underline{\underline{M}}$ ) and ( $\underline{\underline{NM}}$ ). Let X be the G-CW-complex defined by the G-pushout



where  $u_1$  is the obvious *G*-map obtained by collapsing each  $\underline{E}V_i$  to a point. Then *X* is a model for *EG*.

A useful criterion for a group G to satisfy both ( $\underline{M}$ ) and ( $\underline{NM}$ ) can be found in [688, Theorem 3.1]. It implies that any hyperbolic group satisfies both ( $\underline{M}$ ) and ( $\underline{NM}$ ), see [688, Example 3.6]. On the other hand the Klein bottle group  $\mathbb{Z} \rtimes \overline{\mathbb{Z}}$  does not satisfy ( $\underline{M}$ ), see [688, Example 3.7]. This is one of the few instances where  $\underline{EG}$ behaves more nicely than  $\underline{EG}$ , since the class of groups for which both ( $\underline{M}$ ) and ( $\underline{NM}$ ) hold is much richer than the class for which both ( $\underline{M}$ ) and ( $\underline{NM}$ ) hold.

Theorem 11.37 will be very helpful for computing the left-hand side appearing in the Farrell-Jones Conjecture, see Section 17.5.

## **11.8 Finiteness Conditions**

It has been very fruitful in group theory to investigate the question whether one can find small models for BG, for instance a finite CW-model, a CW-model of finite type, or a finite-dimensional CW-model, or equivalently, small G-CW-models for EG. The same question can be asked for  $\underline{E}G$  and  $\underline{E}G$ . For torsionfree groups there is no difference between EG and  $\underline{E}G$ , but for groups with torsion the space  $\underline{E}G$  seems to carry much more information than EG. In this section we collect some information about finiteness conditions on EG,  $\underline{E}G$ , and  $\underline{E}G$ . Having small models is also important for computations of the left-hand sides appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture, see Chapter 17.

Throughout this section G will be a discrete group.

#### 11.8.1 Review of Finiteness Conditions on BG

As an illustration we review what is known about finiteness properties of G-CW-models for EG for a discrete group G. This is equivalent to the same question about BG.

#### 11.8 Finiteness Conditions

We introduce the following notation. Let *R* be a commutative associative ring with unit. The trivial *RG*-module is *R* viewed as an *RG*-module by the trivial *G*-action. The *cohomological dimension*  $cd_R(M)$  of a *RG*-module *M* is  $\infty$  if there is no finite-dimensional projective *RG*-resolution of *M* and is equal to the integer *n* if there exists a projective resolution of *M* of dimension  $\leq n$  but not of dimension  $\leq n - 1$ . Note that *M* possesses a projective *RG*-resolution of dimension  $\leq n$  if and only if for any *RG*-module *N* we have  $Ext^i_{RG}(M, N) = 0$  for  $i \geq n + 1$ . The *cohomological dimension over R* of a group *G*, which is denoted by  $cd_R(G)$ , is the cohomological dimension of trivial *RG*-module *R*. If  $R = \mathbb{Z}$ , we abbreviate  $cd(G) := cd_{\mathbb{Z}}(G)$ .

An *RG*-module *M* is of *type* FP<sub>n</sub> if it admits a projective *RG*-resolution  $P_*$  such that  $P_i$  is finitely generated for  $i \le n$ , and of type FP<sub>∞</sub> if it admits a projective *RG*-resolution  $P_*$  such that  $P_i$  is finitely generated for all *i*. A group *G* is of type FP<sub>n</sub> or FP<sub>∞</sub> respectively if the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  is of type  $FP_n$  or  $FP_\infty$  respectively.

Here is a summary of well-known statements about finiteness conditions on BG.

#### **Theorem 11.38** (Finiteness conditions for *BG*). Let *G* be a discrete group.

- (i) If there exists a finite-dimensional model for BG, then G is torsionfree;
- (ii) (a) There exists a CW-model for BG with finite 1-skeleton if and only if G is finitely generated;
  - (b) *There exists a CW-model for BG with finite 2-skeleton if and only if G is finitely presented;*
  - (c) For  $n \ge 3$  there exists a CW-model for BG with finite n-skeleton if and only if G is finitely presented and of type FP<sub>n</sub>;
  - (d) There exists a CW-model for BG of finite type, i.e., all skeleta are finite if and only if G is finitely presented and of type  $FP_{\infty}$ ;
  - (e) *There exists groups G that are of type* FP<sub>2</sub> *and not finitely presented;*
- (iii) There is a finite CW-model for BG if and only if G is finitely presented and there is a finite free  $\mathbb{Z}G$ -resolution  $F_*$  for the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$ ;
- (iv) The following assertions are equivalent:
  - (a) The cohomological dimension over  $\mathbb{Z}$  of G is  $\leq 1$ ;
  - (b) *There is a model for BG of dimension*  $\leq 1$ ;
  - (c) *G* is free;
- (v) The following assertions are equivalent for  $d \ge 3$ :
  - (a) There exists a CW-model for BG of dimension d;
  - (b) The cohomological dimension over  $\mathbb{Z}$  of G is d;
- (vi) For Thompson's group F there is a CW-model of finite type for BG but no finite-dimensional model for BG.

*Proof.* (i) Suppose we can choose a finite-dimensional model for *BG*. Let  $C \subseteq G$  be a finite cyclic subgroup. Then  $C \setminus \widetilde{BG} = C \setminus EG$  is a finite-dimensional model for *BC*. Hence there is an integer *d* such that we have  $H_i(BC) = 0$  for  $i \ge d$ . This implies

that C is trivial [171, (2.1) in II.3 on page 35]. Hence G is torsionfree.

- (ii) See [135] and [171, Theorem 7.1 in VIII.7 on page 205].
- (iii) See [171, Theorem 7.1 in VIII.7 on page 205].
- (iv) See [925] and [941].
- (v) See [171, Theorem 7.1 in VIII.7 on page 205].
- (vi) See [172].

## 11.8.2 Cohomological Criteria for Finiteness Properties in Terms of Bredon Cohomology

We have seen that we can read off finiteness properties of *BG* or *EG* from the group cohomology of *G*. If one wants to investigate the same question for  $E_{\mathcal{F}}(G)$  analogous statements are true if one considers modules over the  $\mathcal{F}$ -restricted orbit category  $\operatorname{Or}_{\mathcal{F}}(G)$  in the sense of Definition 2.64. This is explained in [655, Subsection 5.2]. For instance, if  $d \ge 3$  is a natural number, then there is a *G*-*CW*-model of dimension  $\le d$  for  $E_{\mathcal{F}}(G)$  if and only if the trivial  $\mathbb{Z}\operatorname{Or}_{\mathcal{F}}(G)$ -module  $\mathbb{Z}$  has a projective  $\mathbb{Z}\operatorname{Or}_{\mathcal{F}}(G)$ -resolution of dimension  $\le d$ , see [655, Theorem 5.2 (i)]. The role of the cohomology of a group is now played by the Bredon cohomology of  $E_{\mathcal{F}}(G)$ . We will deal with Bredon cohomology in Example 12.2.

Other papers related to the topic of connecting geometric dimension or other finiteness properties for classifying spaces for families to algebraic analogues are [161, 395, 397, 766, 768].

#### **11.8.3** Finite Models for the Classifying Space for Proper Actions

The specific constructions of Sections 11.6 show that there is a finite *G*-*CW*-model for the classifying space for proper actions  $\underline{E}G$  if *G* is a cocompact discrete subgroups of an almost connected Lie group, a hyperbolic group, an arithmetic group, the outer automorphism group of a finitely generated free groups, a mapping class group, or a finitely generated one-relator group. This is also the case for an elementary amenable group of type FP<sub>∞</sub>, see [591, Theorem 1.1].

If  $1 \to K \to G \to Q \to 1$  is an extension of groups and there are finite models for  $\underline{E}K$  and  $\underline{E}Q$ , one may ask whether there is a finite model for  $\underline{E}G$ . Some sufficient conditions for this question are given in [647, Theorem 3.2 and Theorem 3.3], for instance that *K* is hyperbolic or virtually poly-cyclic. However, even in the case that *Q* is finite and *K* is torsionfree with a finite model for *BK*, it can happen that there is no finite model for  $\underline{E}G$ , see [623, Example 3 on page 149 in Section 7].

#### 11.8.4 Models of Finite Type for the Classifying Space for Proper Actions

The following result is proved in [647, Theorem 4.2].

#### **Theorem 11.39 (Models for** *EG* **of finite type).**

*The following statements are equivalent for the group G.* 

- (i) There is a G-CW-model for  $\underline{E}G$  of finite type;
- (ii) There are only finitely many conjugacy classes of finite subgroups of G and for any finite subgroup H ⊂ G there is a CW-model for BW<sub>G</sub>H of finite type where W<sub>G</sub>H := N<sub>G</sub>H/H;
- (iii) There are only finitely many conjugacy classes of finite subgroups of G and for any finite subgroup  $H \subset G$  the Weyl group  $W_GH$  is finitely presented and is of type  $FP_{\infty}$ .

The comments about extensions in Subsection 11.8.3 for finite models carry over to models of finite type.

#### 11.8.5 Finite-Dimensional Models for the Classifying Space for Proper Actions

The following result follows from Dunwoody [319, Theorem 1.1].

**Theorem 11.40 (A criterion for** 1-dimensional models for  $\underline{E}G$ ). Let G be a discrete group. Then there exists a 1-dimensional model for  $\underline{E}G$  if and only if the cohomological dimension of G over  $\mathbb{Q}$  is less or equal to one.

If *G* is finitely generated, then there is a 1-dimensional model for  $\underline{E}G$  if and only if *G* contains a finitely generated free subgroup of finite index [554, Theorem 1]. If *G* is torsionfree, we rediscover the results due to Swan and Stallings stated in Theorem 11.38 (iv) from Theorem 11.40.

If *G* is virtually torsionfree, one defines its *virtual cohomological dimension* vcd(G) by the cohomological dimension cd(H) of any torsionfree subgroup  $H \subseteq G$  of finite index. Since for any other torsionfree subgroup  $K \subseteq G$  of finite index we have cd(H) = cd(K), this notion is well-defined.

**Definition 11.41 (Homotopy dimension).** Given a *G*-space *X*, the *homotopy dimension* hdim<sup>*G*</sup>(*X*)  $\in$  {0, 1, . . .}  $\amalg$  {∞} of *X* is defined to be the infimum over the dimensions of all *G*-*CW*-complexes *Y* that are *G*-homotopy equivalent to *X*.

Notation 11.42. Put for a group G

$$\underline{gd}(G) := \mathrm{hdim}^G(\underline{E}G);$$
  
$$\underline{gd}(G) := \mathrm{hdim}^G(\underline{\underline{E}}G).$$

**Lemma 11.43.** Suppose that G is virtually torsionfree. Then

$$\operatorname{vcd}(G) \leq \operatorname{gd}(G).$$

*Proof.* Choose a torsionfree subgroup  $H \subseteq G$  of finite index. Then the restriction of  $\underline{E}G$  to H is a model for EH. This implies  $cd(H) \leq dim(\underline{E}G)$  and hence  $vcd(G) \leq gd(G)$ .

The next result is taken from [655, Theorem 5.24]

**Theorem 11.44 (Dimension of** <u>E</u>G **for a discrete subgroup** G **of an almost connected Lie group).** Let L be a Lie group with finitely many path components. Then L contains a maximal compact subgroup K, which is unique up to conjugation. Let  $G \subseteq L$  be a discrete subgroup of L. Then L/K with the left G-action is a model for EG.

Suppose additionally that G is virtually torsionfree. Then we have

 $\operatorname{vcd}(G) \leq \dim(L/K)$ 

and equality holds if and only if  $G \setminus L$  is compact.

The next result follows from [394, Theorem 1 and inequalities (1) and (2) on page 7] where also the notion of the Hirsch length for elementary amenable groups due to Hillman [495] is recalled. In the special case that there is a finite sequence  $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = \{1\}$  of subgroups such that  $G_{i+1}$  is normal in  $G_i$  and  $G_i/G_{i+1}$  is finitely generated abelian for  $i = 0, 1, \ldots, (n-1)$ , the Hirsch length h(G) is  $\sum_{i=0}^{n-1} \operatorname{rk}_{\mathbb{Z}}(G_i/G_{i+1})$ .

**Theorem 11.45 (Dimension of** <u>*E*</u>*G* **for countable elementary amenable groups of finite Hirsch length).** If *G* is an elementary amenable group, then its Hirsch length satisfies

$$h(G) \leq \operatorname{gd}(G).$$

If G is a countable elementary amenable group, then

 $gd(G) \le \max\{3, h(G) + 1\}.$ 

If *F* is a virtually poly-cyclic group *G*, then *G* is virtually torsionfree, and vcd(*G*) is finite and satisfies vcd(*G*) = h(G) = gd(G), see [655, Example 5.26].

If  $H \subseteq G$  is a subgroup of finite index [G : H] and there is a H-CW-model for  $\underline{E}H$  of dimension  $\leq d$ , then there is a G-CW-model for  $\underline{E}G$  of dimension  $\leq d \cdot [G : H]$ , see [647, Theorem 2.4]. In particular  $gd(G) \geq [G : H] \cdot gd(H)$ .

**Theorem 11.46 (Dimension of** <u>EG</u> and extension). Let  $1 \to K \to G \to Q \to 1$ be an exact sequence of groups. Suppose that there exists a positive integer d that is an upper bound on the orders of finite subgroups of Q. Then

$$gd(G) \le d \cdot gd(K) + gd(Q).$$

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**Remark 11.47** ( $\underline{\text{gd}}(G)$  for locally finite groups). For a locally finite group of cardinality  $\aleph_n$  the inequality  $\underline{\text{gd}}(G) \le n + 1$  is proved in [309, Theorem 2.6] and [688, Theorem 5.31]. The equality  $\underline{\text{gd}}(G) = n + 1$  is explained in [688, Example 5.32].

**Exercise 11.48.** Let *F* be a non-trivial finite group. Put  $H = \bigoplus_{\mathbb{Z}} F$ . Let  $H \rtimes \mathbb{Z}$  be the semidirect product with respect to the shift automorphism of *H*. Show  $\underline{gd}(H) = 1$  and  $\underline{gd}(H \rtimes \mathbb{Z}) = 2$ .

## **11.8.6** Brown's Problem about Virtual Cohomological Dimension and the Dimension of the Classifying Space for Proper Actions

The following problem, whether the converse of Lemma 11.43 is true, is stated by Brown [170, page 32].

**Problem 11.49 (Brown's problem about**  $vcd(G) = dim(\underline{E}G)$ ). For which virtually torsionfree groups G does the equality

$$\operatorname{vcd}(G) = \operatorname{gd}(G)$$

hold?

The *length*  $l(H) \in \{0, 1, ...\}$  of a finite group H is the supremum over all l for which there is a nested sequence  $H_0 \subset H_1 \subset \cdots \subset H_l$  of subgroups  $H_i$  of H with  $H_i \neq H_{i+1}$ . The following result is proved in [647, Theorem 6.4] and was motivated by Brown's Problem 11.49.

**Theorem 11.50 (Estimate on** dim( $\underline{E}G$ ) in terms of vcd(G)). Let G be a group with virtual cohomological dimension vcd(G)  $\leq d$ . Let  $l \geq 0$  be an integer such that the length l(H) of any finite subgroup  $H \subset G$  is bounded by l.

Then there is a G-CW-model for  $\underline{E}G$  such that for any finite subgroup  $H \subset G$ 

$$\dim(\underline{E}G^H) = \max\{3, d\} + l - l(H)$$

holds. In particular  $gd(G) \le max\{3, d\} + l$ .

However, we obtain the following from Leary-Petroysan [624, Corollary 1.2], see also Leary-Nucinkis [623, Example 12 on page 153 in Section 7].

**Theorem 11.51 (Brown's Problem 11.49 has a negative answer in general).** Given a natural number m, there exists a group G such that there is a finite model for  $\underline{E}G$  and we have vcd(G) = 2m and  $gd(G) \ge 3m$ .

Moreover, Leary-Petroysan [624, page 732] show that the estimate in Theorem 11.50 cannot be improved, even if one considers only finite models for EG.

#### 11.8.7 Finite-Dimensional Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

The following problem has triggered a lot of activity.

**Problem 11.52 (Relating the dimension of**  $\underline{E}G$  and  $\underline{E}G$ ). For which countable groups *G* do the inequalities

$$\underline{\mathrm{gd}}(G) - 1 \leq \underline{\mathrm{gd}}(G) \leq \underline{\mathrm{gd}}(G) + 1$$

hold?

The inequality appearing in Problem 11.52 holds for countable elementary amenable groups, see [299, Corollary 4.4]. There are groups of type  $FP_{\infty}$  for which the difference gd(G) - gd(G) is arbitrary large, see [299, Example 6.5].

All possible cases of the inequality appearing in Problem 11.52 can occur, in particular there are examples of finitely presented groups G with  $\underline{gd}(G) < \underline{gd}(G)$ , see Remark 11.56.

The next result is proved in [299, Theorem A].

**Theorem 11.53 (Dimension of**  $\underline{E}G$  **for elementary amenable groups of finite Hirsch length).** If G is an elementary amenable group of cardinality  $\aleph_n$  such that the Hirsch length h(G) of G is finite, then

$$\underline{\mathrm{gd}}(G) \le h(G) + n + 2.$$

Theorem 11.54 (The dimension of  $\underline{E}G$ ).

(i) We have for any group G

$$\underline{\mathrm{gd}}(G) \leq 1 + \underline{\mathrm{gd}}(G);$$

(ii) We have

$$\operatorname{gd}(G \times H) \le \operatorname{gd}(G) + \operatorname{gd}(H),$$

and

$$\underline{\underline{\mathrm{gd}}}(G \times H) \leq \underline{\underline{\mathrm{gd}}}(G) + \underline{\underline{\mathrm{gd}}}(H) + 3,$$

and these inequalities cannot be improved in general; (iii) If G satisfies condition ( $\underline{M}$ ) and ( $\underline{NM}$ ), then

$$\underbrace{\operatorname{gd}(G)}_{\leq =} \begin{cases} = \operatorname{\underline{gd}}(G) & \text{if } \operatorname{\underline{gd}}(G) \ge 2; \\ \le 2 & \text{if } \operatorname{\underline{gd}}(G) \le 1; \end{cases}$$

11.8 Finiteness Conditions

(iv) If  $H \subseteq G$  is a subgroup of finite index [G : H] then

$$\underline{\underline{gd}}(G) \le [G:H] \cdot \underline{\underline{gd}}(H)$$

*Proof.* (i) See [688, Corollary 5.4 (1)].

(ii) This is obvious for  $\underline{gd}(G \times H)$  and proved for  $\underline{gd}(G \times H)$  in [688, Corollary 5.6 and Remark 5.7].

(iii) See [688, Theorem 5.8 (2)].

(iv) This is proved in [647, Theorem 2.4].

**Exercise 11.55.** If G is the fundamental group of a hyperbolic closed Riemannian manifold M, then

$$\operatorname{cd}(G) = \dim(N) = \underline{\operatorname{gd}}(G) = \underline{\operatorname{gd}}(G)$$

**Remark 11.56 (Virtually-poly-cyclic-groups).** In [688, Theorem 5.13] a complete computation of  $\underline{gd}(G)$  is presented for virtually poly- $\mathbb{Z}$  groups. The answer is much more complicated than the one for  $\underline{gd}(G)$ , which is equal to both vcd(G) and the Hirsch length h(G), see [655, Example 5.26]. This leads to some interesting examples in [688, Subsection 5.4]. For instance, one can construct, for k = -1, 0, 1, automorphisms  $f_k$ : Hei  $\rightarrow$  Hei of the three-dimensional Heisenberg group Hei such that

$$\operatorname{gd}(\operatorname{Hei} \rtimes_{f_k} \mathbb{Z}) = 4 + k.$$

Note that  $\underline{gd}(\text{Hei} \rtimes_f \mathbb{Z}) = cd(\text{Hei} \rtimes_f \mathbb{Z}) = 4$  holds for every automorphism  $f: \text{Hei} \rightarrow \text{Hei}$ .

The following result is taken from [658, Theorem 1.1].

**Theorem 11.57 (Dimensions of** <u>EG</u> and <u>EG</u> for groups acting on CAT(0)spaces). Let G be a discrete group that acts properly and isometrically on a complete proper CAT(0)-space X. Let dim(X) be the topological dimension of X, see Definition 22.35.

(i) We have

$$\operatorname{gd}(G) \leq \dim(X);$$

 (ii) Suppose that G acts by semisimple isometries. (This is the case if we additionally assume that the G-action is cocompact.) Then

$$\underline{\mathrm{gd}}(G) \le \dim(X) + 1.$$

**Remark 11.58** ( $\underline{\text{gd}}(G)$  for locally virtually cyclic groups). For a locally virtually cyclic group of cardinality  $\aleph_n$  the inequality  $\underline{\text{gd}}(G) \le n+1$  is a special case of [688, Theorem 5.31].

The next result is taken from [296, Theorem A].

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**Theorem 11.59 (Finite-dimensional models for**  $\underline{E}G$  **for discrete subgroups of**  $\operatorname{GL}_n(\mathbb{R})$ ). Every discrete subgroup G of  $\operatorname{GL}_n(\mathbb{R})$  admits a finite-dimensional model for  $\underline{E}G$ . More precisely, if m is the dimension of the Zariski closure of G in  $\operatorname{GL}_n(\mathbb{R})$ , then

$$\underline{\mathrm{gd}}(G) \le m+1.$$

For information about  $\underline{gd}(G)$  we refer for (relatively) hyperbolic groups to [532, 606], for mapping class groups of finite type surfaces to [535, 767], for mapping class groups of punctured spheres to [36], for systolic groups to [784], for braid groups to [393], for normally poly-free groups to [522], for orientable 3-manifold groups to [527], and for  $Out(F_n)$  to [450].

#### 11.8.8 Low Dimensions

Besides Theorem 11.40 we have the following result proved in [688, Theorem 5.33].

### Theorem 11.60 (Low-dimensional models for $\underline{E}G$ and $\underline{E}G$ ).

(i) Let G be a countable group that is locally virtually cyclic. Then

$$\underline{gd}(G) = \begin{cases} 0 & \text{if } G \text{ is finite;} \\ 1 & \text{if } G \text{ is infinite and either locally finite} \\ & \text{or virtually cyclic;} \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\underbrace{\operatorname{gd}}_{=}(G) = \begin{cases} 0 & \text{if } G \text{ is virtually cyclic;} \\ 1 & \text{otherwise;} \end{cases}$$

(ii) Let G be a countable group satisfying  $gd(G) \leq 1$ . Then

$$\underline{\underline{gd}}(G) = \begin{cases} 0 & \text{if } G \text{ is virtually cyclic;} \\ 1 & \text{if } G \text{ is locally virtually cyclic but} \\ & \text{not virtually cyclic;} \\ 2 & \text{otherwise.} \end{cases}$$

**Exercise 11.61.** Let *G* be a countable group. Show that *G* is infinite locally finite if and only if  $\underline{gd}(G) = \underline{gd}(G) = 1$  holds.

## **11.8.9** Finite Models for the Classifying Space for the Family of Virtually Cyclic Subgroups

If G is virtually cyclic, a model for  $\underline{E}G$  is  $\{\bullet\} = G/G$ , which is in particular finite. There is no group known such that  $\underline{E}G$  has a finite G-CW-model and G is not virtually cyclic. This leads to the following conjecture of Juan-Pineda and Leary [532, Conjecture 1].

**Conjecture 11.62 (Finite Models for**  $\underline{\underline{E}}G$ **).** If a group *G* has a finite *G*-*CW*-model for  $\underline{\underline{E}}G$ , then *G* is virtually cyclic.

Conjecture 11.62 is known to be true in many cases, since the existence of a finite *G*-*CW*-model for  $\underline{E}G$  implies that there is a finite *G*-*CW*-model for  $\underline{E}G$ , see [688, Corollary 5.4 (2)], and that there are only finitely many conjugacy classes of infinite virtually cyclic groups of *G*. Conjecture 11.62 holds for instance for hyperbolic groups, see [532, Corollary 12], elementary amenable groups, see [582, Corollary 5.8], and linear groups, see [971].

# **11.9 On the Homotopy Type of the Quotient Space of the** Classifying Space for Proper Actions

One may think that there are more homotopy classes of CW-complexes than isomorphisms classes of groups. Namely, we can assign to any group G its classifying space BG and for two groups G and H the spaces BH and BG are homotopy equivalent if and only if G and H are isomorphic, and there are CW-complexes that are not homotopy equivalent to BG for any group G. However, here is a result due to Leary-Nucinkis [622, Theorem 1], which is in some sense the converse.

**Theorem 11.63 (Every** *CW*-complex occurs up to homotopy as a quotient of a classifying space for proper group actions). Let X be a CW-complex. Then there exists a group G such that  $G \setminus \underline{E}G$  is homotopy equivalent to X. Moreover one can arrange that G contains a torsionfree subgroup of index two.

**Exercise 11.64.** Let *X* be a *CW*-complex. Show that there exists a  $\mathbb{Z}/2$ -*CW*-complex *Y* such that *Y* is aspherical and *X* is homotopy equivalent to the  $\mathbb{Z}/2$ -quotient space of *Y*.

**Remark 11.65** (Metric Kan-Thurston Theorem). Leary proves a metric Kan-Thurston Theorem in [621, Theorem A]. It yields the following variant of Theorem 11.63, see [621, Theorem 8.3]. Given a group G and proper simplicial G-complex X with connected  $G \setminus X$ , there exists a group  $\widetilde{G}$ , a cubical CAT(0)complex E with simplicial G-action, an epimorphism of groups  $p: \widetilde{G} \to G$ , and a map  $f: E \to X$  such that E is a model for  $\underline{E}G$ , the map f is  $p: \widetilde{G} \to G$ -equivariant, and for any equivariant homology theory in the sense of Definition 12.9 the pair (p, f) induces for all  $n \in \mathbb{Z}$  isomorphisms  $\mathcal{H}_n^{\tilde{G}}(E) \to \mathcal{H}_n^G(X)$ . An application to Isomorphism Conjectures is discussed in [621, Section 10].

The understanding of  $G \setminus \underline{E}G$  and  $G \setminus \underline{E}G$  will be important for the computation of the left-hand side appearing in the Baum-Conjecture or the Farrell-Jones Conjecture, see Chapter 17.

In contrast to the trivial family  $\mathcal{TR}$  where EG and  $BG = G \setminus EG$  carry the same information, this is not true for  $\underline{E}G$  and  $G \setminus \underline{E}G$ . For instance,  $G \setminus \underline{E}G$  is contractible if G is the infinite dihedral group  $D_{\infty} \cong \mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$ , which can be seen by direct inspection, or if  $G = SL_3(\mathbb{Z})$ , see [921, Corollary on page 8].

## 11.10 Notes

The notion of a classifying space for a family was introduced by tom Dieck [951].

Classifying spaces for families play a role in computations of equivariant homology and cohomology for compact Lie groups such as equivariant bordism as explained in [952, Chapter 7] and [953, Chapter III].

Classifying spaces for topological groups and appropriate families of subgroups play a key role in the construction of classifying equivariant principal bundles in [687] or the construction of the topological *K*-cohomology for arbitrary proper equivariant *CW*-complexes in [670].

More information about classifying spaces for families can be found for instance in [1, 65, 109, 256, 297, 298, 299, 396, 592, 655, 668, 688, 819, 953, 971, 972].

## Chapter 12 Equivariant Homology Theory

## **12.1 Introduction**

This section is devoted to equivariant homology theories. They are a key input in the general formulations of the Baum-Connes Conjecture and the Farrell-Jones Conjecture. If one only wants to understand these conjectures, one only needs to browse through the Definition 12.1 of a *G*-homology theory, nothing more is needed from this chapter. Since *G*-homology theories are of general importance, we have added more material to this section. It will also be useful for concrete computations of *K*- and *L*-groups of group rings and group  $C^*$ -algebras based on the Farrell-Jones Conjecture and the Baum-Connes Conjecture.

For a fixed group G, the notion of a G-homology theory  $\mathcal{H}^G_*$  is the obvious generalization of the notion of a (generalized) homology theory in the non-equivariant sense. An important insight is to pass to an *equivariant homology theory*  $\mathcal{H}^P_*$ , see Definition 12.9. Roughly speaking, it assigns to every group G a G-homology theory  $\mathcal{H}^G_*$  and links for any group homomorphisms  $\alpha \colon H \to G$  the theories  $\mathcal{H}^H_*$  and  $\mathcal{H}^G_*$ by a so-called *induction structure*. This global point of view is the key for many applications and computations. Most of the interesting theories arise as equivariant homology theories.

Whenever one has a covariant functor from the category of small connected groupoids GROUPOIDS to the category of spectra SPECTRA, one obtains an associated equivariant homology theory, see Section 12.4. Thus one can construct our main examples for equivariant homology theories, which are based on K- and L-groups of group rings and group  $C^*$ -algebras, by extending these notions from groups to groupoids, see Section 12.5.

We will provide tools for computations, namely, the *equivariant Atiyah-Hirzebruch spectral sequence*, see Subsection 12.6.1, the *p-chain spectral sequence*, see Subsection 12.6.2, and the *equivariant Chern character*, see Section 12.7. We will present some concrete examples of such computations in Sections 12.8 and 12.9.

## 12.2 Basics about G-Homology Theories

In this section we describe the axioms of a (proper) *G*-homology theory and give some basic examples. The main examples for us will be the sources of the assembly maps appearing in the Baum-Connes Conjecture and the Farrell-Jones Conjecture.

Fix a discrete group G and an associative commutative ring  $\Lambda$  with unit.

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**Definition 12.1** (*G*-homology theory). A *G*-homology theory  $\mathcal{H}^G_*$  with values in  $\Lambda$ -modules is a collection of covariant functors  $\mathcal{H}^G_n$  from the category of *G*-*CW*-pairs to the category of  $\Lambda$ -modules indexed by  $n \in \mathbb{Z}$  together with natural transformations

$$\partial_n^G(X,A) \colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) \coloneqq \mathcal{H}_{n-1}^G(A,\emptyset)$$

for  $n \in \mathbb{Z}$  such that the following axioms are satisfied:

• *G*-homotopy invariance

If  $f_0$  and  $f_1$  are *G*-homotopic *G*-maps of *G*-*CW*-pairs  $(X, A) \to (Y, B)$ , then  $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$  for  $n \in \mathbb{Z}$ ;

Given a pair (X, A) of *G*-*CW*-complexes, there is a long exact sequence

$$\dots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \dots$$

where  $i: A \to X$  and  $j: X \to (X, A)$  are the inclusions; • *Excision* 

Let (X, A) be a *G*-*CW*-pair, and let  $f: A \to B$  be a cellular *G*-map of *G*-*CW*-complexes. Equip  $(X \cup_f B, B)$  with the induced structure of a *G*-*CW*-pair. Then the canonical map  $(F, f): (X, A) \to (X \cup_f B, B)$  induces an isomorphism

$$\mathcal{H}_n^G(F,f)\colon \mathcal{H}_n^G(X,A) \xrightarrow{\equiv} \mathcal{H}_n^G(X \cup_f B,B)$$

for all  $n \in \mathbb{Z}$ ;

• Disjoint union axiom

Let  $\{X_i \mid i \in I\}$  be a collection of *G*-*CW*-complexes. Denote by  $j_i \colon X_i \to \coprod_{i \in I} X_i$  the canonical inclusion. Then the map

$$\bigoplus_{i\in I} \mathcal{H}_n^G(j_i) \colon \bigoplus_{i\in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\Bigl(\bigsqcup_{i\in I} X_i\Bigr)$$

is bijective for all  $n \in \mathbb{Z}$ ;

If  $\mathcal{H}^G_*$  is defined or considered only for proper *G*-*CW*-pairs (*X*, *A*), we call it a proper *G*-homology theory  $\mathcal{H}^G_*$  with values in  $\Lambda$ -modules.

**Example 12.2 (Bredon Homology).** The most basic *G*-homology theory is *Bredon* homology, which was originally introduced in [162]. Recall that Or(G) denotes the orbit category of *G*. Let *X* be a *G*-*CW*-complex. It defines a contravariant functor from the orbit category Or(G) to the category of *CW*-complexes by sending *G*/*H* to map<sub>*G*</sub>(*G*/*H*, *X*) = *X*<sup>*H*</sup>. Composing it with the functor "cellular chain complex" yields a contravariant functor
12.2 Basics about G-Homology Theories

$$C^{\mathrm{Or}(G)}_*(X) \colon \mathrm{Or}(G) \to \mathbb{Z}\text{-CHCOM}$$

to the category of  $\mathbb{Z}$ -chain complexes. Let  $\Lambda$  be a commutative ring and let

$$M: \operatorname{Or}(G) \to \Lambda\operatorname{-MOD}$$

be a covariant functor to the abelian category of  $\Lambda$ -modules  $\Lambda$ -MOD. If  $N: Or(G) \rightarrow \mathbb{Z}$ -MOD is a contravariant functor, one can form the tensor product over the orbit category  $N \otimes_{\Lambda Or(G)} M$ , see for instance [644, 9.12 on page 166]. It is the quotient of the  $\Lambda$ -module

$$\bigoplus_{G/H \in ob(Or(G))} N(G/H) \otimes_{\mathbb{Z}} M(G/H)$$

by the  $\Lambda$ -submodule generated by

$$\{xf\otimes y - x\otimes fy \mid f \colon G/H \to G/K, x \in N(G/K), y \in M(G/H)\}$$

where xf stands for N(f)(x) and fy for M(f)(y). Since this is natural, we obtain a  $\Lambda$ -chain complex  $C^{\operatorname{Or}(G)}_*(X) \otimes_{\mathbb{ZOr}(G)} M$ . The homology of  $C^{\operatorname{Or}(G)}_*(X) \otimes_{\mathbb{ZOr}(G)} M$  is the Bredon homology of X with coefficients in M

(12.3) 
$$H_n^G(X;M) := H_n(C_*^{\operatorname{Or}(G)}(X) \otimes_{\mathbb{Z}\operatorname{Or}(G)} M).$$

This extends in the obvious way to *G*-*CW*-pairs. Thus we get a *G*-homology theory  $H^G_*$  with values in  $\Lambda$ -modules.

\* The description of  $C_*^{Or(G)}(X) \otimes_{\mathbb{Z}Or(G)} M$  in terms of the orbit category is conceptually the right one, since it is intrinsically defined and the basic properties are easily checked following closely the non-equivariant case. For computation, the following explicit description is useful.

Fix G-pushouts

$$\begin{array}{c|c} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} X_{n-1} \\ & \downarrow & \downarrow \\ & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} X_n \end{array}$$

as they appear in Definition 11.2. Then the *n*-th  $\Lambda$ -chain module of the  $\Lambda$ -chain complex  $C^{\operatorname{Or}(G)}_*(X) \otimes_{\mathbb{Z}\operatorname{Or}(G)} M$  can be identified with

$$C^{\operatorname{Or}(G)}_n(X)\otimes_{\mathbb{Z}\operatorname{Or}(G)} M=\bigoplus_{i\in I_n}M(G/H_i).$$

In order to define the *n*-th differential

$$c_n \colon \bigoplus_{i \in I_n} M(G/H_i) \to \bigoplus_{j \in I_{n-1}} M(G/H_j)$$

we specify for each pair  $(i, j) \in I_n \times I_{n-1}$  a  $\Lambda$ -homomorphism  $\alpha_{i,j} \colon M(G/H_i) \to M(G/H_j)$  such that for fixed  $i \in I_n$  there are only finitely many  $j \in I_{n-1}$  satisfying  $\alpha_{i,j} \neq 0$ .

We begin with the case n = 1. For  $i \in I_1$ , let j(i, +) and j(i, -) be the indices in  $I_0$  for which  $q_i^0(G/H_i \times \{\pm 1\}) \subseteq G/H_{j(i,\pm)}$  holds. Let  $f(i, \pm) \colon G/H_i \to G/H_{j(i,\pm)}$  be the map induced by  $q_i^0$ . Define for  $i \in I_1$  and  $j \in I_0$ 

$$\alpha_{i,j} = \begin{cases} M(f(i,+)) - M(f(i,-)) & \text{if } j = j(i,+) \text{ and } j = j(i,-); \\ M(f(i,+)) & \text{if } j = j(i,+) \text{ and } j \neq j(i,-); \\ -M(f(i,-)) & \text{if } j \neq j(i,+) \text{ and } j = j(i,-); \\ 0 & \text{if } j \neq j(i,+) \text{ and } j \neq j(i,-). \end{cases}$$

Next we deal with the case  $n \ge 2$ . Let  $X_{n-1,j}$  be the quotient of  $X_{n-1}$  where we collapse the (n-2)-skeleton and all the equivariant (n-1)-cells except the one for the index *j* to a point. The pushout above, but now for (n-1) instead of *n*, yields a *G*-homeomorphism

$$\overline{\mathcal{Q}_j^{n-1}}: \bigvee_{G/H_i} S^{n-1} = \left(G/H_j \times D^{n-1}\right) / \left(G/H_j \times S^{n-2}\right) \xrightarrow{\cong} X_{n-1,j}$$

where  $\bigvee_{G/H_i} S^{n-1}$  is the one-point union or wedge of as many copies of  $S^{n-1}$  as there are elements in  $G/H_i$ . If  $p_{gH_j} \colon \bigvee_{G/H_i} S^{n-1} \to S^{n-1}$  is the projection onto the summand belonging to  $gH_j \in G/H_j$ ,  $k \colon S^{n-1} \to G/H_i \times S^{n-1}$  is the obvious inclusion to the summand belonging to  $eH_i$ , and  $pr_j \colon X_{n-1} \to X_{n-1,j}$  the obvious projection, then we obtain a self-map of  $S^{n-1}$  by the following composite

$$S^{n-1} \xrightarrow{k} G/H_i \times S^{n-1} \xrightarrow{q_i^n} X_{n-1} \xrightarrow{\operatorname{pr}_j} X_{n-1,j} \xrightarrow{\overline{Q_j^{n-1}}^{-1}} \bigvee_{G/H_j} S^{n-1} \xrightarrow{P_{gH_j}} S^{n-1}.$$

Define  $d_{i,j,gH_j} \in \mathbb{Z}$  to be the mapping degree of the map above. For  $gH_j \in G/H_j^{H_i}$  we obtain a *G*-map

$$r_{gH_j}: G/H_i \to G/H_j, \quad g'H_i \mapsto g'gH_j.$$

Define

$$\alpha_{i,j} \colon M(G/H_i) \to M(G/H_j)$$

to be the sum of the maps  $\sum_{gH_j \in G/H_j^{H_i}} d_{i,j,gH_j} \cdot M(r_{gH_j})$ . Since because of the compactness of  $S^{n-1}$  there are for fixed  $i \in I_{n-1}$  only finitely many pairs  $(j, gH_j)$  for  $j \in I_{n-1}$  and  $gH_j \in G/H_j$  with  $d_{i,j,gH_j} \neq 0$ , the definition of  $\alpha_{i,j}$  makes sense and we can indeed define  $c_n$  by sending  $\{x_i \mid i \in I_n\}$  to  $\{\sum_{i \in I_n} \alpha_{i,j}(x_i) \mid j \in I_{n-1}\}$ .

12.2 Basics about G-Homology Theories

Obviously Bredon homology reduces for  $G = \{1\}$  to the cellular homology of a *CW*-complex with coefficients in the abelian group *M*. It is the obvious generalization of this concept to the equivariant setting if one keeps in mind that in the equivariant situation the building blocks are equivariant cells given by *G*-spaces  $G/H_i \times D^n$ .

**Exercise 12.4.** Let  $\mathbb{Z}/2$  act on  $S^2 := \{(x_0, x_1, x_2) \mid x_i \in \mathbb{R}, x_0^2 + x_1^2 + x_2^2 = 1\}$  by the involution that sends  $(x_0, x_1, x_2)$  to  $(x_0, x_1, -x_2)$ . Consider the covariant functor

$$R_{\mathbb{C}}: \operatorname{Or}(\mathbb{Z}/2) \to \mathbb{Z}$$
-MOD

that sends  $(\mathbb{Z}/2)/H$  to the complex representation ring  $R_{\mathbb{C}}(H)$ , any endomorphism in  $Or(\mathbb{Z}/2)$  to the identity and the morphism pr:  $(\mathbb{Z}/2)/\{1\} \rightarrow (\mathbb{Z}/2)/(\mathbb{Z}/2)$  to the homomorphism  $R_{\mathbb{C}}(\{1\}) \rightarrow R_{\mathbb{C}}(\mathbb{Z}/2)$  given by induction with the inclusion  $\{1\} \rightarrow \mathbb{Z}/2$ .

Show that  $S^2$  becomes a  $\mathbb{Z}/2$ -*CW*-complex if we take  $\{(1, 0, 0)\}$  as 0-skeleton,  $\{(x_0, x_1, 0) \mid x_0^2 + x_1^2 = 1\}$  as 1-skeleton, and  $S^2$  itself as 2-skeleton, and compute the abelian groups  $H_*^{\mathbb{Z}/2}(S^2; R_{\mathbb{C}})$ .

**Lemma 12.5.** Let  $\mathcal{H}^G_*$  be a *G*-homology theory. Let *X* be a *G*-CW-complex, and let  $\{X_i \mid i \in I\}$  be a directed system of *G*-CW-subcomplexes directed by inclusion such that  $X = \bigcup_{i \in I} X_i$ . Then for all  $n \in \mathbb{Z}$  the natural map

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

*Proof.* The non-equivariant case is treated [943, Proposition 7.53 on page 121] for  $I = \mathbb{N}$ . The proof extends directly to the equivariant case, provided that  $I = \mathbb{N}$ . The proof of the general case is left to the reader.

Let  $\mathcal{H}^G_*$  and  $\mathcal{K}^G_*$  be *G*-homology theories. A *natural transformation of G*-homology theories  $T_* : \mathcal{H}^G_* \to \mathcal{K}^G_*$  is a sequence of natural transformations  $T_n : \mathcal{H}^G_n \to \mathcal{K}^G_n$  of functors from the category of *G*-*CW*-pairs to the category of  $\Lambda$ -modules which are compatible with the boundary homomorphisms.

**Lemma 12.6.** Let  $T_*: \mathcal{H}^G_* \to \mathcal{K}^G_*$  be a natural transformation of *G*-homology theories. Suppose that  $T_n(G/H)$  is bijective for every homogeneous space G/H and  $n \in \mathbb{Z}$ .

Then  $T_n(X, A)$  is bijective for every *G*-*CW*-pair (X, A) and  $n \in \mathbb{Z}$ .

Note that one needs in Lemma 12.6 the existence of the natural transformation  $T_*$ . Namely, there exists two (non-equivariant) homology theories  $\mathcal{H}_*$  and  $\mathcal{K}_*$  such that  $\mathcal{H}(\{\bullet\}) \cong \mathcal{K}_n(\{\bullet\})$  holds for  $n \in \mathbb{Z}$  but there exists a *CW*-complex *X* and  $m \in \mathbb{Z}$  such that  $\mathcal{H}_m(X)$  and  $\mathcal{K}_m(X)$  are not isomorphic. An example is topological *K*-homology theory  $K_*$  and the homology theory  $\mathcal{H}_* = \bigoplus_{n \in \mathbb{Z}} H_{*+2n}$  for  $H_*$  singular homology.

Exercise 12.7. Give the proof of Lemma 12.6.

## **12.3 Basics about Equivariant Homology Theories**

In this section we describe the axioms of a (proper) equivariant homology theory and give some basic examples. The point is that an equivariant homology theory assigns to every group G a G-homology theory and links them by an induction structure. It will play a key role in computations, various proofs, and the construction of the equivariant Chern character.

Let  $\alpha: H \to G$  be a group homomorphism. Given an *H*-space *X*, define the *induction of X with*  $\alpha$  to be the *G*-space

(12.8) 
$$\operatorname{ind}_{\alpha} X = G \times_{\alpha} X$$

i.e., the quotient of  $G \times X$  by the right *H*-action  $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$  for  $h \in H$ and  $(g, x) \in G \times X$ . The *G*-actions comes from  $g' \cdot (g, x) = (g'g, x)$ . If  $\alpha : H \to G$ is an inclusion, we also write  $\operatorname{ind}_{H}^{G}$  instead of  $\operatorname{ind}_{\alpha}$ .

**Definition 12.9 (Equivariant homology theory).** A (proper) equivariant homology theory with values in  $\Lambda$ -modules  $\mathcal{H}^{?}_{*}$  assigns to each group G a (proper) G-homology theory  $\mathcal{H}^{G}_{*}$  with values in  $\Lambda$ -modules (in the sense of Definition 12.1) together with the following so-called *induction structure*:

Given a group homomorphism  $\alpha \colon H \to G$  and a (proper) *H*-*CW*-pair (*X*, *A*), there are for every  $n \in \mathbb{Z}$  natural homomorphisms

(12.10) 
$$\operatorname{ind}_{\alpha} \colon \mathcal{H}_{n}^{H}(X, A) \to \mathcal{H}_{n}^{G}(\operatorname{ind}_{\alpha}(X, A))$$

satisfying:

- Compatibility with the boundary homomorphisms
  - $\partial_n^G \circ \operatorname{ind}_{\alpha} = \operatorname{ind}_{\alpha} \circ \partial_n^H;$
- Functoriality

Let  $\beta: G \to K$  be another group homomorphism. Then we have for  $n \in \mathbb{Z}$ 

$$\operatorname{ind}_{\beta \circ \alpha} = \mathcal{H}_n^K(f_1) \circ \operatorname{ind}_{\beta} \circ \operatorname{ind}_{\alpha} \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^K(\operatorname{ind}_{\beta \circ \alpha}(X, A))$$

where  $f_1$ :  $\operatorname{ind}_{\beta} \operatorname{ind}_{\alpha}(X, A) \xrightarrow{\cong} \operatorname{ind}_{\beta \circ \alpha}(X, A), \ (k, g, x) \mapsto (k\beta(g), x)$  is the natural *K*-homeomorphism;

• Compatibility with conjugation

For  $n \in \mathbb{Z}$ ,  $g \in G$ , and a (proper) *G*-*CW*-pair (*X*, *A*) the homomorphism ind<sub>*c*(g): G \to G</sub>:  $\mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(g): G \to G}(X, A))$  agrees with  $\mathcal{H}_n^G(f_2)$ for the *G*-homeomorphism  $f_2: (X, A) \to \operatorname{ind}_{c(g): G \to G}(X, A)$  that sends *x* to  $(1, g^{-1}x)$  in  $G \times_{c(g)} (X, A)$ ;

• Bijectivity

If ker( $\alpha$ ) acts freely on  $X \setminus A$ , then  $\operatorname{ind}_{\alpha} : \mathcal{H}_{n}^{H}(X, A) \to \mathcal{H}_{n}^{G}(\operatorname{ind}_{\alpha}(X, A))$  is bijective for all  $n \in \mathbb{Z}$ .

**Exercise 12.11.** Let  $\mathcal{H}_*^?$  be an equivariant homology theory. Show for any group G, any  $g \in G$ , and any  $n \in \mathbb{Z}$  that induction with  $c(g): G \to G$  induces the identity on  $\mathcal{H}_n^G(\{\bullet\})$ .

**Lemma 12.12.** Let  $\mathcal{H}^{?}_{*}$  be a (proper) equivariant homology theory. Consider (finite) subgroups  $H, K \subset G$  and an element  $g \in G$  with  $gHg^{-1} \subset K$ . Let  $R_{g^{-1}} \colon G/H \to G/K$  be the G-map sending g'H to  $g'g^{-1}K$  and  $c(g) \colon H \to K$  be the homomorphism sending h to  $ghg^{-1}$ . Let pr:  $(\operatorname{ind}_{c(g)} \colon H \to K \{\bullet\}) \to \{\bullet\}$  be the projection. Then the following diagram commutes

$$\begin{array}{l} \mathcal{H}_{n}^{H}(\{\bullet\}) \xrightarrow{\mathcal{H}_{n}^{K}(\mathrm{pr}) \circ \mathrm{ind}_{c(g)}} \mathcal{H}_{n}^{K}(\{\bullet\}) \\ \cong \left| \mathrm{ind}_{H}^{G} & \cong \left| \mathrm{ind}_{K}^{G} \\ \mathcal{H}_{n}^{G}(G/H) \xrightarrow{\mathcal{H}_{n}^{G}(R_{g^{-1}})} \mathcal{H}_{n}^{G}(G/K). \end{array} \right.$$

*Proof.* Let  $f_1: \operatorname{ind}_{c(g): G \to G} \operatorname{ind}_H^G \{\bullet\} \to \operatorname{ind}_K^G \operatorname{ind}_{c(g): H \to K} \{\bullet\}$  be the bijective *G*-map sending  $(g_1, g_2, \{\bullet\})$  in  $G \times_{c(g)} G \times_H \{\bullet\}$  to  $(g_1gg_2g^{-1}, 1, \{\bullet\})$  in  $G \times_K K \times_{c(g)} \{\bullet\}$ . The condition that induction is compatible with composition of group homomorphisms means precisely that the composite

$$\mathcal{H}_{n}^{H}(\{\bullet\}) \xrightarrow{\operatorname{ind}_{H}^{G}} \mathcal{H}_{n}^{G}(\operatorname{ind}_{H}^{G}\{\bullet\}) \xrightarrow{\operatorname{ind}_{c(g): G \to G}} \mathcal{H}_{n}^{G}(\operatorname{ind}_{c(g): G \to G} \operatorname{ind}_{H}^{G}\{\bullet\}) \xrightarrow{\mathcal{H}_{n}^{G}(f_{1})} \mathcal{H}_{n}^{G}(\operatorname{ind}_{K}^{G} \operatorname{ind}_{c(g): H \to K}\{\bullet\})$$

agrees with the composite

$$\mathcal{H}_{n}^{H}(\{\bullet\}) \xrightarrow{\operatorname{ind}_{c(g): H \to K}} \mathcal{H}_{n}^{K}(\operatorname{ind}_{c(g): H \to K}\{\bullet\}) \xrightarrow{\operatorname{ind}_{K}^{G}} \mathcal{H}_{n}^{G}(\operatorname{ind}_{K}^{G} \operatorname{ind}_{c(g): H \to K}\{\bullet\}).$$

Naturality of induction implies  $\mathcal{H}_n^G(\operatorname{ind}_K^G \operatorname{pr}) \circ \operatorname{ind}_K^G = \operatorname{ind}_K^G \circ \mathcal{H}_n^K(\operatorname{pr})$ . Hence the following diagram commutes

$$\begin{aligned} \mathcal{H}_{n}^{H}(\{\bullet\}) & \xrightarrow{\mathcal{H}_{n}^{K}(\mathrm{pr})\circ\mathrm{ind}_{c(g)H\to K}} & \mathcal{H}_{n}^{K}(\{\bullet\}) \\ & \cong \bigvee_{i}\mathrm{ind}_{H}^{G} & \cong \bigvee_{i}\mathrm{ind}_{K}^{G} \\ \mathcal{H}_{n}^{G}(G/H) & \xrightarrow{\mathcal{H}_{n}^{G}(\mathrm{ind}_{K}^{G}\,\mathrm{pr})\circ\mathcal{H}_{n}^{G}(f_{1})\circ\mathrm{ind}_{c(g)\colon G\to G}} & \mathcal{H}_{n}^{G}(G/K). \end{aligned}$$

By the axioms  $\operatorname{ind}_{c(g): G \to G} : \mathcal{H}_n^G(G/H) \to \mathcal{H}_n^G(\operatorname{ind}_{c(g): G \to G} G/H)$  agrees with  $\mathcal{H}_n^G(f_2)$  for the map  $f_2 : G/H \to \operatorname{ind}_{c(g): G \to G} G/H$  that sends g'H to  $(g'g^{-1}, 1H)$  in  $G \times_{c(g)} G/H$ . Since the composite  $(\operatorname{ind}_K^G \operatorname{pr}) \circ f_1 \circ f_2$  is just  $R_{g^{-1}}$ , Lemma 12.12 follows.

**Example 12.13 (Borel homology).** Let  $\mathcal{K}_*$  be a homology theory for (non-equivariant) *CW*-pairs with values in  $\Lambda$ -modules. Examples are singular homology, oriented bordism theory, or topological *K*-homology. Then we obtain two equivariant homology theories with values in  $\Lambda$ -modules in the sense of Definition 12.9 by the following constructions

$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(G \setminus X, G \setminus A);$$
  
$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(EG \times_G (X, A)).$$

The second one is called the *equivariant Borel homology associated to*  $\mathcal{K}_*$ .

In both cases  $\mathcal{H}^G_*$  inherits the structure of a *G*-homology theory from the homology structure on  $\mathcal{K}_*$ . Induction for a group homomorphism  $\alpha: H \to G$  is induced by the following two maps *a* and *b*. Let  $a: H \setminus X \xrightarrow{\cong} G \setminus (G \times_\alpha X)$  be the homeomorphism sending Hx to G(1, x). Define  $b: EH \times_H X \to EG \times_G G \times_\alpha X$  by sending (e, x) to  $(E\alpha(e), 1, x)$  for  $e \in EH, x \in X$ , and  $E\alpha: EH \to EG$  the  $\alpha$ -equivariant map induced by  $\alpha$ . Induction for a group homomorphism  $\alpha: H \to G$  is induced by these maps *a* and *b*. If the kernel ker $(\alpha)$  acts freely on *X*, then the map *b* is a homotopy equivalence and hence in both cases ind<sub> $\alpha$ </sub> is bijective.

**Example 12.14 (Equivariant bordism).** For a proper *G*-*CW*-pair (*X*, *A*), one can define the *G*-bordism group  $\mathcal{N}_n^G(X, A)$  as the abelian group of *G*-bordism classes of *G*-maps  $f: (M, \partial M) \to (X, A)$  whose sources are smooth manifolds with cocompact proper smooth *G*-actions. *Cocompact* means that the quotient space  $G \setminus M$  is compact. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper *G*-homology theory. There is an obvious induction structure coming from induction of equivariant spaces which is, however, only defined if the kernel of  $\alpha$  acts freely on *X*. It is well-defined because of the following fact. If  $\alpha: H \to G$  is a group homomorphism, *M* is an smooth *H*-manifold with cocompact proper smooth *H*-action, and ker( $\alpha$ ) acts freely, then ind<sub> $\alpha$ </sub> *M* is a smooth *G*-manifold with cocompact proper smooth *G*-action. The boundary of ind<sub> $\alpha$ </sub> *M* is ind<sub> $\alpha$ </sub>  $\partial M$ .

**Example 12.15 (Equivariant topological** *K***-theory).** We have explained the notion of *equivariant topological K*-*theory*  $K_*^?$  in (10.67), where the induction structure, at least for injective group homomorphisms, comes from (10.63). If  $R_{\mathbb{C}}(H)$  denotes the complex representation ring of the finite subgroup  $H \subseteq G$ , then

$$K_n^G(G/H) \cong K_n^H(\{\bullet\}) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

There is a also a real version of it.

**Exercise 12.16.** Compute  $K_*^{D_{\infty}}(ED_{\infty})$ .

In the sequel we put

$$BG := G \setminus EG$$

12.4 Constructing Equivariant Homology Theories Using Spectra

**Lemma 12.18.** Let  $\mathcal{H}^{?}_{*}$  be an equivariant proper homology theory. Let G be any group. Let  $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$  be a ring such that the order of any finite subgroup of G is invertible in  $\Lambda$ .

(i) The map  $\mathcal{H}_n^{\{1\}}(BG) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^{\{1\}}(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$  is an isomorphism for all  $n \in \mathbb{Z}$ ; (ii) The map

$$\mathcal{H}_{n}^{G}(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_{n}^{\{1\}}(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$$

is split surjective, whereas the map

$$\mathcal{H}_n^G(EG) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda$$

is split injective.

*Proof.* (i) By the Atiyah-Hirzebruch spectral sequence it suffices to prove the bijectivity of the  $\Lambda$ -map  $H_p(BG; \mathcal{H}_n^{\{1\}}(\{\bullet\})) \otimes_{\mathbb{Z}} \Lambda \to H_p(\underline{B}G; \mathcal{H}_n^{\{1\}}(\{\bullet\})) \otimes_{\mathbb{Z}} \Lambda$  for  $p, q \in \mathbb{Z}$  with  $p \ge 0$ . The *G*-map  $EG \to \underline{E}G$  induces a homology equivalence of projective  $\Lambda G$ -chain complexes  $C_*(EG) \otimes_{\mathbb{Z}} \Lambda \to C_*(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda$ , which is therefore a  $\Lambda G$ -chain homotopy equivalence. Hence it induces a  $\Lambda$ -chain homotopy equivalence  $C_*(BG) \otimes_{\mathbb{Z}} \Lambda \to C_*(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$ .

(ii) Since the following diagram commutes

$$\begin{array}{ccc}
\mathcal{H}_{n}^{G}(EG) &\longrightarrow \mathcal{H}_{n}^{G}(\underline{E}G) \\
\operatorname{ind}_{G \to \{1\}} & & & & & & \\
\mathcal{H}_{n}^{\{1\}}(BG) &\longrightarrow \mathcal{H}_{n}^{\{1\}}(\underline{B}G)
\end{array}$$

and has a bijection as left vertical arrow, the claim follows from assertion (i).  $\Box$ 

**Example 12.19.** Note that Lemma 12.18 (ii) is not true if one just considers a *G*-homology theory  $\mathcal{H}^G_*$ . Here is a counterexample. Let *G* be a finite group. Let *M* be the covariant  $\mathbb{Z}\operatorname{Or}(G)$ -module which sends *G* to  $\mathbb{Z}$ , G/H for  $H \neq \{1\}$  to  $\{0\}$ , and every *G*-map  $f: G \to G$  to the identity on  $\mathbb{Z}$ . Then the Bredon homology  $H^G_n(EG; M)$  is  $H_n(BG)$  and the Bredon homology  $H^G_n(\underline{E}G; M) = H^G_n(G/G; M) = M(G/G)$  vanishes.

# 12.4 Constructing Equivariant Homology Theories Using Spectra

We briefly fix some conventions concerning spectra. Let SPACES<sup>+</sup> be the category of pointed compactly generated spaces. Here the objects are compactly generated spaces X, see Remark 11.1, with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by  $X_+$ 

the pointed space obtained from X by adding a disjoint base point. For two pointed spaces X = (X, x) and Y = (Y, y), define their *smash product* to be the pointed space

(12.20) 
$$X \wedge Y = X \times Y / (\{x\} \times Y \cup X \times \{y\}),$$

and the *reduced cone* to be the pointed space

(12.21) 
$$\operatorname{cone}(X) := X \times [0,1]/(X \times \{1\} \cup \{x\} \times [0,1]).$$

A spectrum  $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}\$  is a sequence of pointed spaces  $\{E(n) \mid n \in \mathbb{Z}\}\$  together with pointed maps called *structure maps*  $\sigma(n) : E(n) \land S^1 \longrightarrow E(n+1)$ . A map of spectra  $\mathbf{f} : \mathbf{E} \to \mathbf{E}'$  is a sequence of maps  $f(n) : E(n) \to E'(n)$  that are compatible with the structure maps  $\sigma(n)$ , i.e., we have  $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \land \operatorname{id}_{S^1})$  for all  $n \in \mathbb{Z}$ . Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category, see [13, III.2.]. The category of spectra and maps will be denoted SPECTRA. Recall that the homotopy groups of a spectrum are defined by

(12.22) 
$$\pi_i(\mathbf{E}) := \operatorname{colim}_{k \to \infty} \pi_{i+k}(E(k)),$$

where the *i*th structure map of the system  $\pi_{i+k}(E(k))$  is given by the composite

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism *S* and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map  $\mathbf{f} \colon \mathbf{E} \to \mathbf{F}$  of spectra inducing an isomorphism on all homotopy groups. A spectrum  $\mathbf{E}$  is called an  $\Omega$ -spectrum if the adjoint  $E_n \to \Omega E_{n+1}$  of each structure map is a weak homotopy equivalence.

Given a spectrum **E** and a pointed space *X*, we can define their smash product  $X \wedge \mathbf{E}$  by  $(X \wedge \mathbf{E})(n) := X \wedge E(n)$  with the obvious structure maps. It is a classical result that a spectrum **E** defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n \left( (X_+ \cup_{A_+} \operatorname{cone}(A_+)) \wedge \mathbf{E} \right).$$

We want to extend this to G-homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [280], where more details can be found.

In the sequel C is a small category. Our main example will be the orbit category Or(G).

**Definition 12.23.** A *covariant (contravariant) C-space X* is a covariant (contravariant) functor

$$X: \mathcal{C} \to SPACES.$$

A map between *C*-spaces is a natural transformation of such functors. Analogously a *pointed C-space* is a functor from *C* to SPACES<sup>+</sup> and a *C*-spectrum a functor to SPECTRA.

12.4 Constructing Equivariant Homology Theories Using Spectra

**Example 12.24.** Let *Y* be a left *G*-space. Define the associated *contravariant* Or(G)-space map<sub>*G*</sub>(-, Y) by

$$\operatorname{map}_{G}(-,Y) \colon \operatorname{Or}(G) \to \operatorname{SPACES}, \quad G/H \mapsto \operatorname{map}_{G}(G/H,Y) = Y^{H}$$

If *Y* has a *G*-invariant base point, then  $map_G(-, Y)$  takes values in pointed spaces.

Let *X* be a contravariant and *Y* be a covariant *C*-space. Define their *balanced product* to be the space

(12.25) 
$$X \times_C Y := \coprod_{c \in ob(C)} X(c) \times Y(c) / \sim$$

where ~ is the equivalence relation generated by  $(x\phi, y) \sim (x, \phi y)$  for all morphisms  $\phi: c \to d$  in *C* and points  $x \in X(d)$  and  $y \in Y(c)$ . Here  $x\phi$  stands for  $X(\phi)(x)$  and  $\phi y$  for  $Y(\phi)(y)$ . If *X* and *Y* are pointed, then one defines analogously their *balanced smash product* to be the pointed space

(12.26) 
$$X \wedge_C Y := \bigvee_{c \in ob(C)} X(c) \wedge Y(c) / \sim .$$

In [280] the notation  $X \otimes_C Y$  was used for this space. Performing the same construction levelwise, one defines the *balanced smash product*  $X \wedge_C \mathbf{E}$  of a contravariant pointed *C*-space and a covariant *C*-spectrum  $\mathbf{E}$ .

The proof of the next result is analogous to the non-equivariant case. Details can be found in [280, Lemma 4.4], where also cohomology theories are treated.

**Theorem 12.27 (Constructing** *G***-Homology Theories).** Let **E** be a covariant Or(G)-spectrum. It defines a *G*-homology theory  $H^G_*(-; \mathbf{E})$  by

$$H_n^G(X, A; \mathbf{E}) := \pi_n \left( \operatorname{map}_G \left( -, \left( X_+ \cup_{A_+} \operatorname{cone}(A_+) \right) \right) \wedge_{\operatorname{Or}(G)} \mathbf{E} \right).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) = \pi_n(\mathbf{E}(G/H)).$$

A version of the Brown representability theorem is proved for *G*-cohomology theories and Or(G)-spectra in [64], see also [295, Corollary 1.60 on page 36].

**Example 12.28 (Bredon homology in terms of spectra).** Consider a covariant functor M:  $Or(G) \rightarrow \mathbb{Z}$ -MOD. Composing it with the functor sending a  $\mathbb{Z}$ -module N to its Eilenberg-MacLane spectrum  $\mathbf{H}_N$ , which is a spectrum such that  $\pi_0(\mathbf{H}_N) \cong N$  and  $\pi_n(\mathbf{H}_N) = \{0\}$  for  $n \neq 0$ , yields a covariant functor

$$\mathbf{H}_M$$
:  $Or(G) \rightarrow SPECTRA$ .

Then the *G*-homology theory  $H^G_*(-; \mathbf{H}_M)$  associated to  $\mathbf{H}_M$  in Theorem 12.27 agrees with the Bredon homology  $H^G_*(-; M)$  defined in Example 12.2.

Recall that we seek an equivariant homology theory and not only a *G*-homology theory. If the Or(G)-spectrum in Theorem 12.27 is obtained from a GROUPOIDS-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let GROUPOIDS be the category of small connected groupoids with covariant functors as morphisms. Recall that a *groupoid* is a category in which all morphisms are isomorphisms and that it is called *connected* if for any two objects there exists an isomorphism between them. A covariant functor  $f: \mathcal{G}_0 \to \mathcal{G}_1$  of groupoids is called *injective* if for any two objects x, y in  $\mathcal{G}_0$  the induced map  $\operatorname{mor}_{\mathcal{G}_0}(x, y) \to$  $\operatorname{mor}_{\mathcal{G}_1}(f(x), f(y))$  is injective. (We are not requiring that the induced map on the set of objects is injective.) Let GROUPOIDS<sup>inj</sup> be the subcategory of GROUPOIDS with the same objects and injective functors as morphisms. For a *G*-set *S* we denote by  $\mathcal{G}^G(S)$  its associated *transport groupoid*. Its objects are the elements of *S*. The set of morphisms from  $s_0$  to  $s_1$  consists of those elements  $g \in G$  that satisfy  $gs_0 = s_1$ . Composition in  $\mathcal{G}^G(S)$  comes from the multiplication in *G*. It is connected if and only if *G* acts transitively on *S*. Thus we obtain for a group *G* a covariant functor

(12.29) 
$$\mathcal{G}^G \colon \operatorname{Or}(G) \to \operatorname{GROUPOIDS^{\operatorname{inj}}}, \quad G/H \mapsto \mathcal{G}^G(G/H).$$

A functor of small categories  $F: C \to \mathcal{D}$  is called an *equivalence* if there exists a functor  $G: \mathcal{D} \to C$  such that both  $F \circ G$  and  $G \circ F$  are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects  $x, y \in C$  the map  $\operatorname{mor}_{\mathcal{C}}(x, y) \to \operatorname{mor}_{\mathcal{D}}(F(x), F(y))$  induced by F is bijective.

**Theorem 12.30 (Constructing equivariant homology theories using spectra).** *Consider a covariant* GROUPOIDS-*spectrum* 

**E**: GROUPOIDS 
$$\rightarrow$$
 SPECTRA.

Suppose that **E** respects equivalences, i.e., it sends an equivalence of groupoids to a weak equivalence of spectra. Then **E** defines an equivariant homology theory

$$H_{*}^{?}(-;\mathbf{E})$$

whose underlying G-homology theory for a group G is the G-homology theory associated to the covariant Or(G)-spectrum  $\mathbf{E} \circ \mathcal{G}^G : Or(G) \to SPECTRA$  in the previous Theorem 12.27, i.e.,

$$H^G_*(X,A;\mathbf{E}) = H^G_*(X,A;\mathbf{E}\circ\mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\{\bullet\}; \mathbf{E}) \cong \pi_n(\mathbf{E}(I(H))),$$

where I(H) denotes H considered as a groupoid with one object. The whole construction is natural in **E**.

#### 12.4 Constructing Equivariant Homology Theories Using Spectra

*Proof.* We have to specify the induction structure for a homomorphism  $\alpha \colon H \to G$ . We only sketch the construction in the special case  $A = \emptyset$ .

The functor induced by  $\alpha$  on the orbit categories is denoted in the same way

$$\alpha \colon \operatorname{Or}(H) \to \operatorname{Or}(G), \quad H/L \mapsto \operatorname{ind}_{\alpha}(H/L) = G/\alpha(L).$$

There is an obvious natural transformation of covariant functors  $Or(H) \rightarrow GROUPOIDS$ 

$$T: \mathcal{G}^H \to \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the functor  $\mathcal{G}^H(H/L) \to \mathcal{G}^G(G/\alpha(L))$  that sends an object hL to the object  $\alpha(h)\alpha(L)$  and a morphism given by  $h \in H$  to the morphism given by  $\alpha(h) \in G$ . The desired homomorphism

$$\operatorname{ind}_{\alpha}: H_n^H(X; \mathbf{E} \circ \mathcal{G}^H) \to H_n^G(\operatorname{ind}_{\alpha} X; \mathbf{E} \circ \mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{split} & \operatorname{map}_{H}(-, X_{+}) \wedge_{\operatorname{Or}(H)} \mathbf{E} \circ \mathcal{G}^{H} \xrightarrow{\operatorname{id} \wedge \mathbf{E}(T)} \operatorname{map}_{H}(-, X_{+}) \wedge_{\operatorname{Or}(H)} \mathbf{E} \circ \mathcal{G}^{G} \circ \alpha \\ & \stackrel{\cong}{\leftarrow} (\alpha_{*} \operatorname{map}_{H}(-, X_{+})) \wedge_{\operatorname{Or}(G)} \mathbf{E} \circ \mathcal{G}^{G} \xrightarrow{\cong} \operatorname{map}_{G}(-, \operatorname{ind}_{\alpha} X_{+}) \wedge_{\operatorname{Or}(G)} \mathbf{E} \circ \mathcal{G}^{G} \end{split}$$

Here  $\alpha_* \operatorname{map}_H(-, X_+)$  is the pointed  $\operatorname{Or}(G)$ -space that is obtained from the pointed  $\operatorname{Or}(H)$ -space  $\operatorname{map}_H(-, X_+)$  by induction, i.e., by taking the balanced product over  $\operatorname{Or}(H)$  with the (discrete)  $\operatorname{Or}(H)$ - $\operatorname{Or}(G)$  biset  $\operatorname{mor}_{\operatorname{Or}(G)}(??, \alpha(?))$ , see [280, Definition 1.8]. Note that  $\mathbf{E} \circ \mathcal{G}^G \circ \alpha$  is the same as the restriction of the  $\operatorname{Or}(G)$ -spectrum  $\mathbf{E} \circ \mathcal{G}^G$  along  $\alpha$ , which is often denoted by  $\alpha^*(\mathbf{E} \circ \mathcal{G}^G)$  in the literature, see [280, Definition 1.8]. The second map is given by the adjunction homeomorphism of induction  $\alpha_*$  and restriction  $\alpha^*$ , see [280, Lemma 1.9]. The third map is the homeomorphism of  $\operatorname{Or}(G)$ -spaces that is the adjoint of the obvious map of  $\operatorname{Or}(H)$ -spaces  $\operatorname{map}_H(-, X_+) \to \alpha^* \operatorname{map}_G(-, \operatorname{ind}_{\alpha} X_+)$  whose evaluation at H/L is given by  $\operatorname{ind}_{\alpha}$ .

It remains to show  $\operatorname{ind}_{\alpha}$  is a weak equivalence, provided that  $\operatorname{ker}(\alpha)$  acts freely on X. Because the second and third maps appearing in the definition above are homeomorphisms, this boils down to proving that  $\operatorname{id} \wedge \mathbf{E}(T)$  is a weak equivalence, provided that  $\operatorname{ker}(\alpha)$  acts freely on X. This follows from the fact that T(H/L) is an equivalence of groupoids and hence  $\mathbf{E}(T)(G/L)$  is a weak equivalence of spectra for all subgroups  $L \subseteq G$  appearing as isotropy group in X, since for such L the restriction of  $\alpha$  to L induces a bijection  $L \to \alpha(L)$ .

**Remark 12.31.** In some cases the functor **E** to SPECTRA is only defined on GROUPOIDS<sup>inj</sup>. Then one still gets an equivariant homology theory with the exception that for the induction structure one has to require that the group homomorphisms  $\alpha: H \to G$  are injective. This does exclude the projection  $G \to \{1\}$ .

**Example 12.32 (Bredon homology).** Let M be a covariant functor from GROUPOIDS to  $\mathbb{Z}$ -MOD. Then Bredon homology yields an equivariant homology theory if we define its value at G as the Bredon homology with coefficients in

the covariant functor  $M^G$ :  $Or(G) \to \mathbb{Z}$ -MOD sending to G/H to  $M(\mathcal{G}^G(G/H))$ . This is the same as the equivariant homology theory we obtain from applying Theorem 12.30 to the functor GROUPOIDS  $\to$  SPECTRA that sends a groupoid  $\mathcal{G}$  to the Eilenberg-MacLane spectrum associated with  $M(\mathcal{G})$ .

**Example 12.33 (Borel homology in terms of spectra).** Let **E** be a spectrum. Let  $H(-; \mathbf{E})$  be the (non-equivariant) homology theory associated to **E**. Given a groupoid  $\mathcal{G}$ , denote by  $E\mathcal{G}$  its classifying space. If  $\mathcal{G}$  has only one object and the automorphism group of this object is G, then  $E\mathcal{G}$  is a model for EG. We obtain two covariant functors

$$c_{\mathbf{E}}$$
: GROUPOIDS  $\rightarrow$  SPECTRA,  $\mathcal{G} \mapsto \mathbf{E}$ ;  
 $b_{\mathbf{E}}$ : GROUPOIDS  $\rightarrow$  SPECTRA,  $\mathcal{G} \mapsto E\mathcal{G}_{+} \wedge \mathbf{E}$ .

Thus we obtain two equivariant homology theories  $H_*^*(-; c_E)$  and  $H_*^*(-; b_E)$  from Theorem 12.30. These coincide with the ones associated to  $\mathcal{K}_* = H(-; E)$  in Example 12.13. Namely, we get for any group *G* and any *G*-*CW*-complex *X* natural isomorphisms

(12.34) 
$$H_n^G(X;c_{\mathbf{E}}) \cong H_n(G\backslash X;\mathbf{E});$$

(12.35) 
$$H_n^G(X; b_{\mathbf{E}}) \cong H_n(EG \times_G X; \mathbf{E}).$$

**Exercise 12.36.** Let **E** and **F** be covariant functors from GROUPOIDS to SPECTRA. Let **t**:  $\mathbf{E} \to \mathbf{F}$  be a natural transformation such that for every  $\mathcal{G} \in ob(GROUPOIDS)$  the map  $\mathbf{t}(\mathcal{G}): \mathbf{E}(\mathcal{G}) \to \mathbf{F}(\mathcal{G})$  is a weak equivalence of spectra.

Show that the induced transformation of equivariant homology theories  $H_*^?(-; \mathbf{t}) : H_*^?(-; \mathbf{E}) \to H_*^?(-; \mathbf{F})$  is a natural equivalence.

# **12.5** Equivariant Homology Theories Associated to *K*- and *L*-Theory

In this section we explain our main examples for covariant functors from GROUPOIDS or GROUPOIDS<sup>inj</sup> to SPECTRA, at least for rings as coefficients. Later we will also consider additive categories and, more generally, right exact  $\infty$ -categories.

Let RINGS be the category of associative rings with unit. Let RINGS<sup>inv</sup> be the category of rings with involution. Let  $C^*$ -ALGEBRAS be the category of  $C^*$ -algebras. There are classical functors for  $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 1\}$ 

(12.38)	$\mathbf{L}^{\langle j \rangle}$ : RINGS <sup>inv</sup> $\rightarrow$	SPECTRA;
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(12.39)  $\mathbf{K}^{\text{TOP}}: C^*\text{-}\text{ALGEBRAS} \to \text{SPECTRA}.$ 

The construction of such a non-connective algebraic K-theory functor (12.37) goes back to Gersten [422] and Wagoner [973]. The spectrum for quadratic algebraic L-theory (12.38) is constructed by Ranicki in [839]. In a more geometric formulation it goes back to Quinn [822]. In the topological K-theory case a construction for (12.39) using Bott periodicity for  $C^*$ -algebras can easily be derived from the Kuiper-Mingo Theorem, see [904, Section 2.2]. The homotopy groups of these spectra give the algebraic K-groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L-theory groups, and the topological K-groups of  $C^*$ -algebras.

We emphasize that in all three cases we need the non-connective versions of the spectra, i.e., the homotopy groups in negative dimensions are non-trivial in general, in order to ensure later that the formulations of the various Isomorphism Conjectures do have a chance to be true.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors R(-): GROUPS  $\rightarrow$  RINGS, respectively R(-): GROUPS  $\rightarrow$  RINGS<sup>inv</sup>, where GROUPS denotes the category of groups. Let GROUPS<sup>inj</sup> be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group  $C^*$ -algebra defines a functor  $C_r^*$ : GROUPS<sup>inj</sup>  $\rightarrow C^*$ -ALGEBRAS. The composite of these functors with the functors (12.37), (12.38), and (12.39) above yields functors

(12.40)	$\mathbf{KR}(-)$ : GROUPS $\rightarrow$	SPECTRA;
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 $\mathbf{L}^{\langle j \rangle} R(-)$ : GROUPS  $\rightarrow$  SPECTRA; (12.41)

 $\mathbf{K}^{\text{TOP}}C_r^*(-,F)$ : GROUPS<sup>inj</sup>  $\rightarrow$  SPECTRA, (12.42)

where  $F = \mathbb{R}$  or  $\mathbb{C}$ . They satisfy

$$\pi_n(\mathbf{K}R(G)) = K_n(RG);$$
  

$$\pi_n(\mathbf{L}^{\langle j \rangle}R(G)) = L_n^{\langle j \rangle}(RG);$$
  

$$\pi_n(\mathbf{K}^{\text{TOP}}C_r^*(G,F)) = K_n(C_r^*(G,F)),$$

for every group G and every  $n \in \mathbb{Z}$ . The next result essentially says that these functors can be extended to groupoids.

Theorem 12.43 (K- and L-Theory Spectra over Groupoids). Let R be a ring (with involution). There exist covariant functors

(12.44)	$\mathbf{K}_{R}$ : GROUPOIDS -	$\rightarrow$ SPECTRA;
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(12.45) 
$$\mathbf{L}_{R}^{\langle j \rangle}$$
: GROUPOIDS  $\rightarrow$  SPECTRA;  
(12.46)  $\mathbf{K}_{F}^{\text{TOP}}$ : GROUPOIDS<sup>inj</sup>  $\rightarrow$  SPECTRA,

(12.46)

with the following properties:

(i) If  $F: \mathcal{G}_0 \to \mathcal{G}_1$  is an equivalence of (small) groupoids, then the induced maps  $\mathbf{K}_{R}(F)$ ,  $\mathbf{L}_{R}^{\langle j \rangle}(F)$ , and  $\mathbf{K}^{\mathsf{TOP}}(F)$  are weak equivalences of spectra;

(ii) Let I: GROUPS  $\rightarrow$  GROUPOIDS be the functor sending G to G considered as a groupoid, i.e., to  $\mathcal{G}^G(G/G)$ . This functor restricts to a functor GROUPS<sup>inj</sup>  $\rightarrow$  GROUPOIDS<sup>inj</sup>.

There are natural transformations from  $\mathbf{K}R(-)$  to  $\mathbf{K}_R \circ I$ , from  $\mathbf{L}^{\langle j \rangle}R(-)$  to  $\mathbf{L}_R^{\langle j \rangle} \circ I$ , and from  $\mathbf{K}C_r^*(-)$  to  $\mathbf{K}^{\mathsf{TOP}} \circ I$  such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra;

(iii) For every group G and all  $n \in \mathbb{Z}$  we have

$$\begin{aligned} \pi_n(\mathbf{K}_R \circ I(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle j \rangle} \circ I(G)) &\cong L_n^{\langle j \rangle}(RG); \\ \pi_n(\mathbf{K}_F^{\text{TOP}} \circ I(G)) &\cong K_n(C_r^*(G,F)). \end{aligned}$$

*Proof.* We only sketch the strategy of the proof. More details can be found in [280, Section 2].

Let G be a groupoid. Similar to the group ring RG one can define an R-linear category RG by taking the free R-modules over the morphism sets of G. Composition of morphisms is extended R-linearly. By formally adding finite direct sums one obtains an additive category  $R\mathcal{G}_{\oplus}$ . Pedersen-Weibel [800], see also [209] and [684], define a non-connective algebraic K-theory functor which digests additive categories and can hence be applied to  $R\mathcal{G}_{\oplus}$ . For the comparison result one uses that for every ring R (in particular for RG) the Pedersen-Weibel functor applied to  $R_{\oplus}$ (a small model for the category of finitely generated free *R*-modules) yields the nonconnective K-theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L-theory case  $RG_{\oplus}$  inherits an involution and one applies the construction of Ranicki [839, Example 13.6 on page 139] to obtain the  $L^{(1)} = L^h$ -version. The versions for  $j \leq 1$  can be obtained by a construction that is analogous to the Pedersen-Weibel construction for K-theory, compare Carlsson-Pedersen [214, Section 4], or by iterating the Shaneson splitting and then finally passing to a homotopy colimit, compare on the group level with [840, Section 17]. In the C<sup>\*</sup>-case one obtains from  $\mathcal{G}$  a C<sup>\*</sup>-category  $C_r^*(\mathcal{G})$  and assigns to it its topological K-theory spectrum. There is a construction of the topological K-theory spectrum of a C\*-category in Davis-Lück [280, Section 2]. However, the construction given there depends on two statements, which appeared in [387, Proposition 1 and Proposition 3], and those statements are incorrect, as already pointed out by Thomason in [948]. The construction in [280, Section 2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [524]. П

**Exercise 12.47.** Compute  $H_n^{D_{\infty}}(\underline{E}D_{\infty}; \mathbf{K}_R)$  for  $n \leq 0$  and  $R = \mathbb{Z}, \mathbb{C}$ .

## **12.6 Two Spectral Sequences**

In this section we state two spectral sequences, which are useful for computations of equivariant homology theories.

#### 12.6.1 The Equivariant Atiyah-Hirzebruch Spectral Sequence

**Theorem 12.48 (The equivariant Atiyah-Hirzebruch spectral sequence).** Let G be a group and  $\mathcal{H}^G_*$  be a G-homology theory with values in  $\Lambda$ -modules in the sense of Definition 12.1. Let X be a G-CW-complex.

Then there is a spectral (homology) sequence of  $\Lambda$ -modules

$$(E_{p,q}^r, d_{p,q}^r \colon E_{p,q}^r \to E_{p-r,q+r-1}^r)$$

whose  $E_2$ -term is given by the Bredon homology of Example 12.2

$$E_{p,q}^2 = H_p^G(X; \mathcal{H}_q^G(-))$$

for the coefficient system given by the covariant functor

$$Or(G) \to \Lambda$$
-MOD,  $G/H \mapsto \mathcal{H}_a^G(G/H)$ .

The  $E^{\infty}$ -term is given by

$$E_{p,q}^{\infty} = \operatorname{colim}_{r \to \infty} E_{p,q}^r.$$

This spectral sequence converges to  $\mathcal{H}_{p+q}^G(X)$ , i.e., there is an ascending filtration  $F_{p,m-p}\mathcal{H}_{p+q}^G(X)$  of  $\mathcal{H}_{p+q}^G(X)$  such that

$$F_{p,q}\mathcal{H}_{p+q}^G(X)/F_{p-1,q+1}\mathcal{H}_{p+q}^G(X) \cong E_{p,q}^{\infty}$$

The construction of the equivariant Atiyah-Hirzebruch spectral sequence is based on the filtration of X by its skeletons. More details, actually in the more general context of spaces over a category, and a version for cohomology can be found in [280, Theorem 4.7].

**Exercise 12.49.** Let *X* be a proper *G*-*CW*-complex such that *X*/*G* with the induced *CW*-structure has no odd-dimensional cells. Show that  $K_n^G(X) = 0$  for odd  $n \in \mathbb{Z}$ , where  $K_*^G$  denotes the equivariant topological complex *K*-homology. Show that  $K_n^G(X)$  for even  $n \in \mathbb{Z}$  is a finitely generated free abelian group if we additionally assume that *X*/*G* is finite.

#### 12.6.2 The *p*-Chain Spectral Sequence

Let *G* be a group. Recall that for a subgroup  $H \subseteq G$  we denote by  $N_G H$  its normalizer and define the Weyl group  $W_G H := N_G H/H$ . We obtain a bijection

$$W_GH \xrightarrow{\cong} \operatorname{aut}_G(G/H), \quad gH \mapsto \left(R_{g^{-1}} \colon G/H \to G/H\right)$$

where  $R_{g^{-1}}$  maps g'H to  $g'g^{-1}H$ . Hence for any two subgroups  $H, K \subseteq G$  the set  $\operatorname{map}_G(G/H, G/K)$  inherits the structure of a  $W_GK$ - $W_GH$ -biset.

A *p*-chain is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \cdots < (H_p)$$

where  $(H_{i-1}) < (H_i)$  means that  $H_{i-1}$  is subconjugate, but not conjugate to  $(H_i)$ . For  $p \ge 1$  define a  $W_G H_p$ - $W_G H_0$ -set associated to such a *p*-chain by

$$S((H_0) < \dots < (H_p))$$
  
:= map<sub>G</sub>(G/H<sub>p-1</sub>, G/H<sub>p</sub>) ×<sub>W<sub>G</sub>H<sub>p-1</sub></sub> ··· ×<sub>W<sub>G</sub>H<sub>1</sub></sub> map<sub>G</sub>(G/H<sub>0</sub>, G/H<sub>1</sub>).

For p = 0 put  $S(H_0) = W_G H_0$ .

Let X be a G-CW-complex. Then  $X^H = \max_G(G/H, X)$  inherits a right  $W_GH$ action. In particular, we get for a p-chain  $(H_0) < \cdots < (H_p)$  a right  $W_GH_0$ -space  $X(G/H_p) \times_{W_GH_p} S((H_0) < \cdots < (H_p))$ .

**Theorem 12.50** (The *p*-chain spectral sequence). Let *G* be a group and **E** be a covariant Or(G)-spectrum. Let *X* be a proper *G*-*CW*-complex.

Then there is a spectral sequence of  $\Lambda$ -modules, called the p-chain spectral sequence, which converges to  $H_{p+q}^G(X; \mathbf{E})$  and whose  $E^1$ -term is

$$E_{p,q}^{1} = \bigoplus_{(H_{0})<\cdots<(H_{p})} \pi_{q} \Big( \Big( EW_{G}H_{0} \times (X^{H_{p}} \times_{W_{G}H_{p}} S((H_{0}) < \cdots < (H_{p}))) \Big)_{+} \wedge_{W_{G}H_{0}} \mathbf{E}(G/H_{0}) \Big)$$

where  $(H_0) < \cdots < (H_p)$  runs through all p-chains consisting of finite subgroups  $H_i \subseteq G$  with  $X^{H_p} \neq \emptyset$ .

The *p*-chain spectral sequence is established in [281, Theorem 2.5 (a) and Example 2.14], actually more generally for spaces over a category. There is also a more complicated version where one drops the condition that X is proper. Since then the book-keeping gets more involved and in most applications X is proper, we only deal with the proper case here.

Note that the complexity of the equivariant Atiyah-Hirzebruch spectral sequence grows with the natural number *n* for which one wants to compute  $\mathcal{H}_n^G(X)$ . The complexity of the *p*-chain spectral sequence grows with the maximum over all natural

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numbers p for which there is a p-chain  $(H_0) < \cdots < (H_p)$  of finite subgroups such that  $X^{H_p}$  is non-empty.

**Example 12.51 (Free** *G-CW*-complex). Consider the situation of Theorem 12.50 and assume additionally that *X* is a free *G-CW*-complex. Then  $E_{p,q}^1 = 0$  for  $p \ge 1$  and hence the *p*-chain spectral sequence predicts

$$H_a^G(X; \mathbf{E}) = \pi_q \big( (EG \times X)_+ \wedge_G \mathbf{E}(G) \big).$$

But this is obviously true, since the right-hand side of the last equation is by definition  $H_q^G(EG \times X; \mathbf{E})$  and the projection  $EG \times X \to X$  is a *G*-homotopy equivalence and

induces an isomorphism  $H_q^G(EG \times X; \mathbf{E}) \xrightarrow{\cong} H_q^G(X; \mathbf{E}).$ 

**Example 12.52** (*G* is finite cyclic of prime order). Let *G* be a finite cyclic group of prime order. Then *G* has only two subgroups, namely, *G* and {1}. Let **E** be a covariant Or(G)-spectrum and *X* be a *G*-*CW*-complex. The *p*-chain spectral sequence of Theorem 12.50 satisfies  $E_{p,q}^1 = 0$  for  $p \ge 2$  and hence reduces to a long exact sequence

$$\cdots \to E_{1,n}^1 \xrightarrow{d_{1,n}^1} E_{0,n}^1 \to H_n^G(X; \mathbf{E}) \to E_{1,n-1}^1 \xrightarrow{d_{1,n-1}^1} E_{0,n-1}^1 \to \cdots$$

We get

$$E_{0,n}^{1} = \pi_n \big( (EG \times X)_+ \wedge_G \mathbf{E}(G) \big) \oplus \pi_n \big( X_+^G \wedge \mathbf{E}(G/G) \big);$$
  

$$E_{1,n}^{1} = \pi_n \big( (EG \times X^G)_+ \wedge_G \mathbf{E}(G) \big),$$

and the differential  $d_{1,n}^1$  is given by the homomorphism

$$\pi_n((EG \times X^G)_+ \wedge_G \mathbf{E}(G)) \to \pi_n((EG \times X)_+ \wedge_G \mathbf{E}(G))$$

that is induced by the inclusion  $X^G \to X$ , and the homomorphism (up to a sign)

$$\pi_n((EG \times X^G)_+ \wedge_G \mathbf{E}(G)) \to \pi_n(X^G_+ \wedge \mathbf{E}(G/G))$$

coming from the projection  $EG \times X^G \to X^G$ .

Now suppose additionally that **E** is the constant functor  $Or(G) \rightarrow SPECTRA$  with value the spectrum **F**. Let  $\mathcal{H}_*$  be the (non-equivariant) homology theory associated to **F**. Then  $H_n^G(X; \mathbf{E}) = \mathcal{H}_n(X/G)$  and the long exact sequence above reduces to the long exact sequence

(12.53) 
$$\dots \to \mathcal{H}_n(EG \times_G X^G) \xrightarrow{d_{1,n}^1} \mathcal{H}_n(EG \times_G X) \oplus \mathcal{H}_n(X^G) \xrightarrow{e_n} \mathcal{H}_n(X/G)$$
  
 $\to \mathcal{H}_{n-1}(EG \times_G X^G) \xrightarrow{d_{1,n-1}^1} \mathcal{H}_{n-1}(EG \times_G X) \oplus \mathcal{H}_{n-1}(X^G) \xrightarrow{e_{n-1}} \dots$ 

where the maps  $d_{n,1}^1$  and  $e_n$  are up to sign induced by the obvious map on space level.

**Exercise 12.54.** Give a direct construction of the long exact sequence (12.53).

# **12.7 Equivariant Chern Characters**

If we rationalize and have a Mackey structure on the coefficient system of an equivariant homology theory, then we can give a more direct and concrete computation via equivariant Chern characters which avoids all the difficulties concerning spectral sequences.

## 12.7.1 Mackey Functors

Let  $\Lambda$  be an associative commutative ring with unit. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let

$$M: FGINJ \rightarrow \Lambda-MOD$$

be a bifunctor, i.e., a pair  $(M_*, M^*)$  consisting of a covariant functor  $M_*$  and a contravariant functor  $M^*$  from FGINJ to  $\Lambda$ -MOD which agree on objects. We will often denote for an injective group homomorphism  $f: H \to G$  the map  $M_*(f): M(H) \to M(G)$  by  $\operatorname{ind}_f$  and the map  $M^*(f): M(G) \to M(H)$  by  $\operatorname{res}_f$  and write  $\operatorname{ind}_H^G = \operatorname{ind}_f$  and  $\operatorname{res}_G^H = \operatorname{res}_f$  if f is an inclusion of groups. We call such a bifunctor M a Mackey functor with values in  $\Lambda$ -modules if it satisfies:

- (i) For an inner automorphism  $c(g): G \to G$  we have  $M_*(c(g)) = \text{id}: M(G) \to M(G)$ ;
- (ii) For an isomorphism of groups  $f: G \xrightarrow{\cong} H$  the composites  $\operatorname{res}_f \circ \operatorname{ind}_f$  and  $\operatorname{ind}_f \circ \operatorname{res}_f$  are the identity;
- (iii) Double coset formula

We have for two subgroups  $H, K \subset G$ 

$$\operatorname{res}_G^K \circ \operatorname{ind}_H^G = \sum_{KgH \in K \setminus G/H} \operatorname{ind}_{c(g) \colon H \cap g^{-1}Kg \to K} \circ \operatorname{res}_H^{H \cap g^{-1}Kg}$$

where c(g) is conjugation with g, i.e.,  $c(g)(h) = ghg^{-1}$ .

Important examples of Mackey functors are  $\operatorname{Rep}_F(H)$ ,  $K_q(RH)$ ,  $L_q^{\langle j \rangle}(RH)$ , and  $K_q^{\operatorname{TOP}}(C_*^r(H,F))$ , where *R* is an associative ring with unit and  $F = \mathbb{R}, \mathbb{C}$ .

**Definition 12.55 (Extension to a Mackey functor).** Let  $\mathcal{H}^{?}_{*}$  be a proper equivariant homology theory with values in A-modules. It defines a covariant functor

$$\mathcal{H}_q^?(\{\bullet\})\colon \mathsf{FGINJ}\to \Lambda\text{-}\mathsf{MOD}, \quad H\mapsto \mathcal{H}_q^G(\{\bullet\}).$$

It sends an injective homomorphism  $i: H \to G$  to the composite  $\mathcal{H}_n^H(\{\bullet\}) \xrightarrow{\text{ind}_i} \mathcal{H}_n^G(G \times_H \{\bullet\}) \xrightarrow{\mathcal{H}_n^G(\text{pr})} \mathcal{H}_n^G(\{\bullet\})$  where pr:  $G \times_H \{\bullet\} \to \{\bullet\}$  is the projection. We say that *the coefficients of*  $\mathcal{H}_*^G$  *extend to a Mackey functor* if there exists a Mackey functor  $(M_*, M^*)$  such that  $M_*$  is the functor  $\mathcal{H}_q^G(\{\bullet\})$  above.

**Example 12.56.** The functors of (12.40), (12.41), and (12.42), which send a group to the algebraic *K*- or *L*-theory of *RG* or to the topological *K*-theory of  $C_r^*(G, F)$ , define Mackey functors with the obvious definition of induction and restriction.

#### 12.7.2 The Equivariant Chern Character

We can associate to a proper equivariant homology theory with values in  $\Lambda$ -modules  $\mathcal{H}^{?}_{*}$  another Bredon type equivariant homology theory with values in  $\Lambda$ -modules  $\mathcal{BH}^{?}_{*}$  as follows. For a group *G* we define

$$\mathcal{BH}_n^G(X) := \bigoplus_{p+q=n} H_p^G(X; \mathcal{H}_q^G(-))$$

where  $H_p^G(X; \mathcal{H}_q^G(-))$  is the Bredon homology of X with coefficients in the covariant functor  $Or(G) \rightarrow \Lambda$ -MOD sending G/H to  $\mathcal{H}_q^G(G/H)$ . Next we show that the collection of the G-homology theories  $\mathcal{BH}_*^G(X, A)$  inherits the structure of a proper equivariant homology theory. We have to specify the induction structure.

Let  $\alpha: H \to G$  be a group homomorphism and (X, A) be a proper *H*-*CW*-pair. Induction with  $\alpha$  yields a functor denoted in the same way

$$\alpha \colon \operatorname{Or}_{\mathcal{FIN}}(H) \to \operatorname{Or}_{\mathcal{FIN}}(G), \quad H/K \mapsto \operatorname{ind}_{\alpha}(H/K) = G/\alpha(K).$$

There is a natural isomorphism of  $Or_{\mathcal{FIN}}(G)$ -chain complexes

$$\operatorname{ind}_{\alpha} C^{\operatorname{Or}_{\mathcal{F}IN}(H)}_*(X,A) \xrightarrow{\cong} C^{\operatorname{Or}_{\mathcal{F}IN}(G)}_*(\operatorname{ind}_{\alpha}(X,A))$$

and a natural adjunction isomorphism, see [649, (2.5)]

$$\left( \operatorname{ind}_{\alpha} C^{\operatorname{Or}_{\mathcal{F}IN}(H)}_{*}(X, A) \right) \otimes_{\mathbb{Z}\operatorname{Or}_{\mathcal{F}IN}(G)} \mathcal{H}^{G}_{q}(-)$$

$$\xrightarrow{\cong} C^{\operatorname{Or}_{\mathcal{F}IN}(H)}_{*}(X, A) \otimes_{\mathbb{Z}\operatorname{Or}_{\mathcal{F}IN}(H)} \left( \operatorname{res}_{\alpha} \mathcal{H}^{G}_{q}(-) \right).$$

The induction structure on  $\mathcal{H}^{?}_{*}$  yields a morphism of  $ROr_{\mathcal{FIN}}(H)$ -modules

$$\mathcal{H}_q^H(H/?) \to \operatorname{res}_{\alpha} \mathcal{H}_q^G(-).$$

These maps or their inverses can be composed to a  $\Lambda$ -chain map

$$C^{\operatorname{Or}_{\mathcal{F}IN}(H)}_{*}(X,A) \otimes_{\mathbb{Z}\operatorname{Or}_{\mathcal{F}IN}(H)} \mathcal{H}^{H}_{q}(H/?) \xrightarrow{\cong} C_{*}(\operatorname{ind}_{\alpha}(X,A)) \otimes_{\mathbb{Z}\operatorname{Or}(G,\mathcal{F}IN)} \mathcal{H}^{G}_{q}(-).$$

Since *X* is proper and hence the Bredon homology can be defined over  $Or_{\mathcal{FIN}}(H)$  instead of Or(G), it induces a natural map

$$\operatorname{ind}_{\alpha} \colon H_p(X,A;\mathcal{H}^H_q(-)) \xrightarrow{\cong} H^G_p(\operatorname{ind}_{\alpha}(X,A);\mathcal{H}^G_q(-)).$$

Thus we obtain the required induction structure.

Define for a finite group *H* 

(12.57) 
$$S_H\left(\mathcal{H}_q^H(\{\bullet\})\right) := \operatorname{coker}\left(\bigoplus_{\substack{K \subset H \\ K \neq H}} \operatorname{ind}_K^H \colon \bigoplus_{\substack{K \subset H \\ K \neq H}} \mathcal{H}_q^K(\{\bullet\}) \to \mathcal{H}_q^H(\{\bullet\})\right)$$

Note that  $S_H(\mathcal{H}_q^H(\{\bullet\}))$  carries a natural left  $\Lambda[N_GH/H \cdot C_GH]$ -module structure where  $N_GH/H \cdot C_GH$  is the quotient of  $N_GH$  by the normal subgroup  $H \cdot C_GH :=$  $\{h \cdot g \mid h \in H, g \in C_GH\}$ . The obvious left-action of  $W_GH = N_GH/H$ -action on  $X^H$ yields a left  $N_GH/H \cdot C_GH$ -action on  $C_GH \setminus X^H$  and hence a right  $N_GH/H \cdot C_GH$ action by  $y \cdot k := k^{-1} \cdot y$  for  $y \in X^H$  and  $k \in N_GH/H \cdot C_GH$ .

The proof of the following result can be found in [649, Theorem 0.2 and 0.3].

**Theorem 12.58 (The equivariant Chern character).** Let  $\Lambda$  be a commutative ring with  $\mathbb{Q} \subset \Lambda$ . Let  $\mathcal{H}^{?}_{*}$  be a proper equivariant homology theory with values in  $\Lambda$ -modules in the sense of Definition 12.9. Suppose that its coefficients extend to a Mackey functor.

(i) There is an isomorphism of proper equivariant homology theories

$$\mathrm{ch}_*^?\colon \mathcal{BH}_*^?\xrightarrow{\cong}\mathcal{H}_*^?;$$

(ii) Let  $(\mathcal{FIN})$  be the set of conjugacy classes (H) of finite subgroups H of G. Then there is for any group G and any proper G-CW-pair (X, A) a natural isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(H)\in(\mathcal{FIN})} H_p(C_G H \setminus (X^H, A^H); \Lambda) \otimes_{\Lambda[N_G H/H \cdot C_G H]} S_H\left(\mathcal{H}_q^H(\{\bullet\})\right)$$
$$\xrightarrow{\cong} \mathcal{BH}_n^G(X, A).$$

Theorem 12.58 reduces the computation of  $\mathcal{H}_n^G(X, A)$  to the computation of the singular or cellular homology  $\Lambda$ -modules  $H_p(C_GH \setminus (X^H, A^H); \Lambda)$  of the *CW*pairs  $C_GH \setminus (X^H, A^H)$  including the obvious right  $N_GH/H \cdot C_GH$ -operation and of the left  $\Lambda[N_GH/H \cdot C_GH]$ -modules  $S_H\left(\mathcal{H}_q^H(\{\bullet\})\right)$  which only involve the values

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 $\mathcal{H}_q^G(G/H) = \mathcal{H}_q^H(\{\bullet\})$ . Note that  $N_G H/H \cdot C_G H$  is a finite group for any finite subgroup  $H \subseteq G$ .

**Exercise 12.59.** Let  $\Lambda$  be a commutative ring with  $\mathbb{Q} \subset \Lambda$ . Let  $\mathcal{H}^{?}_{*}$  be a proper equivariant homology theory with values in  $\Lambda$ -modules. Suppose that its coefficients extend to a Mackey functor. Consider a group *G* and a proper *G*-*CW*-complex *X*. Show that all differentials of the equivariant Atiyah-Hirzebruch spectral sequence converging to  $\mathcal{H}^{G}_{p+q}(X)$  vanish.

**Exercise 12.60.** Let  $\mathcal{H}_*^?$  be a proper equivariant homology theory with values in  $\mathbb{Q}$ -modules in the sense of Definition 12.9. Suppose that its coefficients extend to a Mackey functor. Let *G* be a group. Consider two families of subgroups  $\mathcal{F}$  and  $\mathcal{G}$  with  $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{FIN}$ . Let  $\iota_{\mathcal{F} \subseteq \mathcal{G}} : E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$  be the up to *G*-homotopy unique *G*-map. Show that for every *n* the induced map  $\mathcal{H}_n^G(\iota_{\mathcal{F} \subseteq \mathcal{G}}) : \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$  is injective.

**Remark 12.61 (Rationalizing an equivariant homology theory).** Let  $\mathcal{H}^{?}_{*}$  be an equivariant homology theory with values in  $\mathbb{Z}$ -modules. Suppose that its coefficients extend to a Mackey functor. Then we obtain an equivariant homology theory  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}^{?}_{*}$  with values in  $\mathbb{Q}$ -modules whose coefficients extend to a Mackey functor, since  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}^{?}_{*}$  is a flat functor and commutes with direct sums over arbitrary index sets. We can apply Theorem 12.58 to  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}^{?}_{*}$  and thus obtain a rational computation of  $\mathcal{H}^{?}_{*}$ .

## **12.8 Some Rational Computations**

### 12.8.1 Green Functors

Let  $\phi: \Lambda \to \Lambda'$  be a homomorphism of associative commutative rings with unit. Let *M* be a Mackey functor with values in  $\Lambda$ -modules, and let *N* and *P* be Mackey functors with values in  $\Lambda'$ -modules. A *pairing* with respect to  $\phi$  is a family of maps

$$m(G): M(G) \times N(G) \to P(G), \quad (x, y) \mapsto m(G)(x, y) =: x \cdot y$$

where *G* runs through the finite groups and we require the following properties for all injective group homomorphisms  $f: H \to G$  of finite groups:

$(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y$	for $x_1, x_2 \in M(H), y \in N(H);$
$x \cdot (y_1 + y_2) = x \cdot y_1 + x \cdot y_2$	for $x \in M(H), y_1, y_2 \in N(H);$
$(\lambda x) \cdot y = \phi(\lambda)(x \cdot y)$	for $\lambda \in \Lambda, x \in M(H), y \in N(H)$ ;
$x \cdot \lambda' y = \lambda' (x \cdot y)$	for $\lambda' \in \Lambda', x \in M(H), y \in N(H)$ ;
$\operatorname{res}_f(x \cdot y) = \operatorname{res}_f(x) \cdot \operatorname{res}_f(y)$	for $x \in M(G), y \in N(G)$ ;
$\operatorname{ind}_{f}(x) \cdot y = \operatorname{ind}_{f}(x \cdot \operatorname{res}_{f}(y))$	for $x \in M(H), y \in N(G)$ ;
$x \cdot \operatorname{ind}_{f}(y) = \operatorname{ind}_{f}(\operatorname{res}_{f}(x) \cdot y)$	for $x \in M(G), y \in N(H)$ .

A *Green functor* with values in  $\Lambda$ -modules is a Mackey functor U with values in  $\Lambda$ -modules together with a pairing with respect to  $\phi = \text{id} \colon \Lambda \to \Lambda$  and elements  $1_G \in U(G)$  for each finite group G such that for each finite group G the pairing  $U(G) \times U(G) \to U(G)$  induces the structure of an  $\Lambda$ -algebra on U(G) with unit  $1_G$  and for any morphism  $f \colon H \to G$  in FGINJ the map  $U^*(f) \colon U(G) \to U(H)$ is a homomorphism of  $\Lambda$ -algebras with unit. Let U be a Green functor with values in  $\Lambda$ -modules and M be a Mackey functor with values in  $\Lambda'$ -modules. A *(left)* U-module structure on M with respect to the ring homomorphism  $\phi \colon \Lambda \to \Lambda'$  is a pairing such that any of the maps  $U(G) \times M(G) \to M(G)$  induces the structure of a (left) module over the  $\Lambda$ -algebra U(G) on the  $\Lambda$ -module  $\phi^*M(G)$  that is obtained from the  $\Lambda'$ -module M(G) by  $\lambda x := \phi(\lambda)x$  for  $r \in \Lambda$  and  $x \in M(G)$ .

The importance of the notion of a Green functor is due to the following elementary lemma which allows to deduce induction theorems for all Mackey functors that are modules over a given Green functor from the corresponding statement for the given Green functor.

**Lemma 12.62.** Let  $\phi: \Lambda \to \Lambda'$  be a homomorphism of associative commutative rings with unit. Let U be a Green functor with values in  $\Lambda$ -modules and let M be a Mackey functor with values in  $\Lambda'$ -modules such that M comes with a U-module structure with respect to  $\phi$ . Let S be a set of subgroups of the finite group G. Suppose that the map

$$\bigoplus_{H\in \mathcal{S}} \mathrm{ind}_{H}^{G} \colon \bigoplus_{H\in \mathcal{S}} U(H) \to U(G)$$

is surjective. Then the map

$$\bigoplus_{H \in \mathcal{S}} \operatorname{ind}_{H}^{G} \colon \bigoplus_{H \in \mathcal{S}} M(H) \to M(G)$$

is surjective.

*Proof.* By hypothesis there are elements  $u_H \in U(H)$  for  $H \in S$  satisfying  $1_G = \sum_{H \in S} \operatorname{ind}_H^G u_H$  in U(G). This implies for  $x \in M(G)$ ,

$$x = 1_G \cdot x = \left(\sum_{H \in \mathcal{S}} \operatorname{ind}_H^G u_H\right) \cdot x = \sum_{H \in \mathcal{S}} \operatorname{ind}_H^G \left(u_H \cdot \operatorname{res}_G^H x\right).$$

**Example 12.63 (Burnside ring).** The *Burnside ring* A(G) of a (not necessarily finite) group *G* is the commutative associative ring with unit A(G) which is obtained by the additive Grothendieck construction applied to the commutative associative semiring with unit given by the *G*-isomorphism classes [*S*] of *G*-sets *S* of finite cardinality, i.e.,  $|S| < \infty$ , under disjoint union and cartesian product and the unit element given by [G/G]. For more information about the Burnside ring for not necessarily finite groups we refer to [652].

The Burnside ring defines a Mackey functor A(?) by induction and restriction. The ring structure and the Mackey structure fit together to the structure of a Green functor A(?) with values in  $\mathbb{Z}$ -modules.

**Exercise 12.64.** Let *M* be a Mackey functor with values in  $\Lambda$ -modules for an associative commutative ring  $\Lambda$  with unit. Let  $\phi : \mathbb{Z} \to \Lambda$  be the unique ring homomorphism. Show that *M* inherits the structure of a module over the Green functor given by the Burnside ring with respect to  $\phi$ .

**Definition 12.65 (Swan ring).** Let *G* be a (not necessarily finite) group. Let  $\Lambda$  be an associative commutative ring with unit. Denote by  $\operatorname{Sw}^p(G; \Lambda)$  the following abelian group. Generators are the isomorphism classes [M] of  $\Lambda G$ -modules *M* whose underlying  $\Lambda$ -module is finitely generated projective. For every short exact sequence  $0 \to M_0 \to M_1 \to M_2 \to 0$  of such  $\Lambda G$ -modules, we require the relation  $[M_0] - [M_1] + [M_2]$  in  $\operatorname{Sw}^p(G; \Lambda)$ . The tensor product over  $\Lambda$  with the diagonal *G*-action induces the structure of an associative commutative ring with unit  $[\Lambda]$ , where  $[\Lambda]$  is the class of  $\Lambda$  equipped with the trivial *G*-action. We call  $\operatorname{Sw}^p(G; \Lambda)$ the *Swan ring*. If  $\Lambda = \mathbb{Z}$ , we abbreviate  $\operatorname{Sw}^p(G) := \operatorname{Sw}^p(G; \mathbb{Z})$ .

If we replace finitely generated projective by finitely generated in the definition above, we denote the associated abelian group by  $Sw(G; \Lambda)$  and abbreviate  $Sw(G) := Sw(G; \mathbb{Z})$ .

**Lemma 12.66.** The canonical map  $e: Sw^p(G) \to Sw(G)$  is an isomorphism of abelian groups.

*Proof.* We only describe the definition of the inverse map  $e^{-1}$ : Sw(G)  $\rightarrow$  Sw<sup>*P*</sup>(G), more details can be found in [799, Lemma 2.2]. Consider a  $\mathbb{Z}G$ -module M such that the underlying abelian group is finitely generated. Since tors(M) is a finite G-set, we can find an exact sequence of  $\mathbb{Z}G$ -modules  $0 \rightarrow F_1 \rightarrow F_0 \rightarrow \text{tors}(M) \rightarrow 0$  such that the underlying abelian groups of  $F_0$  and  $F_1$  are finitely generated free. One may take for  $F_0$  the finitely generated free abelian group with the finite G-set tors(M) as  $\mathbb{Z}$ -basis. The  $\mathbb{Z}G$ -module M/tors(M) has as underlying abelian group a finitely generated free abelian group. We define  $e^{-1}([M]) = [F_0] - [F_1] + [M/\text{tors}(M)]$ .

**Example 12.67 (Swan ring).** Let *R* be an associative ring with unit. Let *M* be a  $\mathbb{Z}G$ -module whose underlying  $\mathbb{Z}$ -module is finitely generated free. It defines an exact functor RG-MOD  $\rightarrow RG$ -MOD by taking the tensor product  $M \otimes_{\mathbb{Z}}$ - with the diagonal *G*-action. It sends finitely generated free *RG*-modules to finitely generated free *RG*-modules by the following observations. We have the sheering *RG*-isomorphism

(12.68)  $\operatorname{sh}: M \otimes_2 \mathbb{Z}G \xrightarrow{\cong} M \otimes_d \mathbb{Z}G, \quad m \otimes g \mapsto gm \otimes g$ 

where  $M \otimes_2 RG$  and  $M \otimes_d RG$  are the *RG*-modules whose underlying *R*-module is  $M \otimes_R RG$  and on which  $g \in G$  acts by  $g \cdot (m \otimes x) = m \otimes gx$  and  $g \cdot (m \otimes x) = gm \otimes gx$ . Obviously  $M \otimes_2 RG$  is a finitely generated free *RG*-module since *M* is finitely generated free as an abelian group. If *P* is a finitely generated projective *RG*-module, then  $M \otimes_{\mathbb{Z}} P$  is a finitely generated projective *RG*-module provided that  $r \in R$  acts by  $r \cdot (m \otimes p) = m \otimes rp$  and  $g \in G$  acts by  $g \cdot (m \otimes p) := gm \otimes gp$ . We obtain a pairing

(12.69) 
$$\operatorname{Sw}^p(G) \otimes K_n(RG) \to K_n(RG).$$

Using induction and restriction  $Sw^p(?)$  defines a Green functor with values in  $\mathbb{Z}$ -modules. There is a natural homomorphism of Green functors with values in  $\mathbb{Z}$ -modules

$$A(G) \to \mathrm{Sw}^p(G)$$

sending the class of a finite *G*-set *S* to the  $\mathbb{Z}$ -module with *S* as basis equipped with the *G*-action coming from the *G*-action on *S*. Thanks to the pairing above, the Mackey functor given by  $K_n(R?)$  becomes a module over the Green functor given by  $Sw^p(?)$ .

**Example 12.70 (Rational representation ring).** An important example of a Green functor with values in  $\mathbb{Q}$ -modules is the rationalized representation ring of rational representations  $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(?)$ . It assigns to a finite group *G* the  $\mathbb{Q}$ -module  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(G)$ , where  $\operatorname{Rep}_{\mathbb{Q}}(G)$  denotes the rational representation ring of *G*. Note that  $\operatorname{Rep}_{\mathbb{Q}}(G)$  is the same as the projective class group  $K_0(\mathbb{Q}G)$  and also the same as  $\operatorname{Sw}^p(G; \mathbb{Q})$ . The Mackey structure comes from induction and restriction of representations. The pairing  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(G) \times \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(G)$  comes from the tensor product  $P \otimes_{\mathbb{Q}} Q$  of two  $\mathbb{Q}G$ -modules *P* and *Q* equipped with the diagonal *G*-action. The unit element is the class of  $\mathbb{Q}$  equipped with the trivial *G*-action.

Recall that  $\operatorname{class}_{\mathbb{Q}}(G)$  denotes the  $\mathbb{Q}$ -vector space of functions  $G \to \mathbb{Q}$  that are invariant under  $\mathbb{Q}$ -conjugation, i.e., we have  $f(g_1) = f(g_2)$  for two elements  $g_1, g_2 \in G$  if the cyclic subgroups  $\langle g_1 \rangle$  and  $\langle g_2 \rangle$  generated by  $g_1$  and  $g_2$  are conjugate in *G*. Elementwise multiplication defines the structure of a  $\mathbb{Q}$ -algebra on  $\operatorname{class}_{\mathbb{Q}}(G)$ with the function that is constant 1 as unit element. Taking the character of a rational representation yields an isomorphism of  $\mathbb{Q}$ -algebras [912, Theorem 29 on page 102]

(12.71) 
$$\chi^G : \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(G) \xrightarrow{=} \operatorname{class}_{\mathbb{Q}}(G).$$

We define a Mackey structure on  $class_{\mathbb{Q}}(?)$  as follows. Let  $f: H \to G$  be an injective group homomorphism. For a character  $\chi \in class_{\mathbb{Q}}(H)$  define its induction with f to be the character  $ind_f(\chi) \in class_{\mathbb{Q}}(G)$  given by

$$\operatorname{ind}_{f}(\chi)(g) = \frac{1}{|H|} \cdot \sum_{\substack{l \in G, h \in H \\ f(h) = l^{-1}gl}} \chi(h).$$

For a character  $\chi \in \text{class}_{\mathbb{Q}}(G)$  define its restriction with f to be the character  $\text{res}_{f}(\chi) \in \text{class}_{\mathbb{Q}}(H)$  given by

$$\operatorname{res}_{f}(\chi)(h) := \chi(f(h)).$$

One easily checks that this yields the structure of a Green functor on  $class_{\mathbb{Q}}(?)$  and that the family of isomorphisms  $\chi^G$  defined in (12.71) yields an isomorphism of Green functors from  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(?)$  to  $class_{\mathbb{Q}}(?)$ .

12.8 Some Rational Computations

#### **12.8.2 Induction Lemmas**

As already explained by Lemma 12.62, Green functors play a prominent role for induction theorems. In order to formulate two further versions, we have to introduce the following idempotents.

Let G be a finite group. There is a ring homomorphism

(12.72) 
$$\operatorname{card}: A(G) \to \prod_{H} \mathbb{Z}, \quad [S] \mapsto (|S^{H}|)_{(H)}$$

where the product is indexed over the conjugacy classes of subgroups of *G* and  $|S^H|$  is the cardinality of the *H*-fixed point set. The ring homomorphism card is injective and has a finite cokernel. In particular, it induces an isomorphism of  $\mathbb{Q}$ -algebras

$$\operatorname{card}_{\mathbb{Q}} \colon \mathbb{Q} \otimes_{\mathbb{Z}} A(G) \xrightarrow{\cong} \prod_{(H)} \mathbb{Q}.$$

Now let  $e_G \in \prod_{(H)} \mathbb{Q}$  be the idempotent whose value at (G) is 1 and whose value at (H) for  $H \neq G$  is 0. We then define the idempotent

(12.73) 
$$\Theta_G := \operatorname{card}_{\mathbb{O}}^{-1}(e_G) \in \mathbb{Q} \otimes_{\mathbb{Z}} A(G).$$

For a finite cyclic group *C*, define the idempotent

(12.74) 
$$\theta_C \in \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$$

to be the element whose image under the isomorphism of (12.71) is the class function that sends an element of *C* to 1 if it is a generator, and to 0 otherwise. The image of  $\Theta_C$  under the map  $\mathbb{Q} \otimes_{\mathbb{Z}} A(C) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$  that sends a finite *C*-set *S* to the associated permutation module  $\mathbb{Q}[S]$  is  $\theta_C$ .

**Lemma 12.75.** Let  $\phi$ :  $\mathbb{Q} \to \Lambda$  be a homomorphism of associative commutative rings with unit. Let M be a Mackey functor with values in  $\Lambda$ -modules which is a module over the Green functor  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(H)$  with respect to  $\phi$ . Then

(i) For a finite group H the map

$$\bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} \operatorname{ind}_{C}^{H} : \bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} M(C) \to M(H)$$

is surjective; (ii) Let C be a finite cyclic group. Let

$$\theta_C \colon M(C) \to M(C)$$

be the map induced by the  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(C)$ -module structure and multiplication by the idempotent  $\theta_C$  of (12.74). Then the inclusion of the image of  $\theta_C \colon M(C) \to M(C)$  into M(C) composed with the projection onto the cokernel of

$$\bigoplus_{\substack{D \subset C \\ D \neq C}} \operatorname{ind}_D^C : \bigoplus_{\substack{D \subset C \\ D \neq C}} M(D) \to M(C)$$

is an isomorphism.

*Proof.* Let  $C \subset H$  be a cyclic subgroup of the finite group H. Then we get for  $h \in H$ 

$$\frac{1}{[H:C]} \cdot \operatorname{ind}_{C}^{H} \theta_{C}(h) = \frac{1}{[H:C]} \cdot \frac{1}{|C|} \cdot \sum_{\substack{l \in H \\ l^{-1}hl \in C}} \theta_{C}(l^{-1}hl) = \frac{1}{|H|} \cdot \sum_{\substack{l \in H \\ \langle l^{-1}hl \rangle = C}} 1.$$

This implies in  $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Rep}_{\mathbb{Q}}(H) \cong \operatorname{class}_{\mathbb{Q}}(H)$ 

(12.76) 
$$1_{H} = \sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_{C}^{H} \theta_{C}$$

since for any  $l \in H$  and  $h \in H$  there is precisely one cyclic subgroup  $C \subset H$  with  $C = \langle l^{-1}hl \rangle$ . Now assertion (i) follows from the following calculation for  $x \in M(H)$ 

$$x = 1_H \cdot x = \left(\sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H \theta_C\right) \cdot x = \sum_{\substack{C \subset H \\ C \text{ cyclic}}} \frac{1}{[H:C]} \cdot \operatorname{ind}_C^H (\theta_C \cdot \operatorname{res}_H^C x).$$

It remains to prove assertion (ii). Obviously  $\theta_C$  is an idempotent for any cyclic group *C*. We get for  $x \in M(C)$  from (12.76)

$$(1_C - \theta_C) \cdot x = \left(\sum_{\substack{D \subset C \\ D \neq C}} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C \theta_D\right) \cdot x = \sum_{\substack{D \subset C \\ D \neq C}} \frac{1}{[C:D]} \cdot \operatorname{ind}_D^C (\theta_D \cdot \operatorname{res}_C^D x)$$

and for  $D \subset C, D \neq C$  and  $y \in M(D)$ 

$$\theta_C \cdot \operatorname{ind}_D^C y = \operatorname{ind}_D^C (\operatorname{res}_C^D \theta_C \cdot y) = \operatorname{ind}_D^C (0 \cdot y) = 0.$$

This finishes the proof of Lemma 12.75.

The proof of the next result is similar to that of Lemma 12.75. Details can be found in [674, Lemma 7.2 and Lemma 7.4]. Key ingredients are Lemma 12.62, Example 12.67, and the result of Swan [937, Corollary 4.2 on page 560] which implies together with [799, page 890] that for every finite group H the cokernel of the map

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$$\bigoplus_{\substack{C \subseteq H, \\ C \text{ cyclic}}} \operatorname{ind}_{C}^{H} \colon \bigoplus_{\substack{C \subseteq H \\ C \text{ cyclic}}} \operatorname{Sw}^{p}(C) \to \operatorname{Sw}^{p}(H)$$

is annihilated by  $|H|^2$ .

Lemma 12.77. Let R be an associative ring with unit. Then

(i) For a finite group H and  $n \in \mathbb{Z}$  the map

$$\bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} \operatorname{ind}_{C}^{H} : \bigoplus_{\substack{C \subset H \\ C \text{ cyclic}}} \mathbb{Q} \otimes_{\mathbb{Z}} K_{n}(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_{n}(RH)$$

is surjective;

(ii) Let C be a finite cyclic group. Let

$$\Theta_C \colon \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$$

be the map induced by the  $\mathbb{Q} \otimes_{\mathbb{Z}} A(C)$ -module structure and multiplication by the idempotent  $\Theta_C$  of (12.73).

Then the inclusion of the image of  $\Theta_C : \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$  into  $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$  with the projection onto the cokernel of

$$\bigoplus_{\substack{D \subset C \\ D \neq C}} \operatorname{ind}_D^C : \bigoplus_{\substack{D \subset C \\ D \neq C}} \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RD) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$$

is an isomorphism.

**Remark 12.78** (*L*-theory analog of Lemma 12.77). The *L*-theory analog of Lemma 12.77 is also true, one has to use instead of Swan [937, Corollary 4.2 on page 560] the corresponding *L*-theory analog of Dress [315, Theorem 2(a)].

For more information about Mackey and Green functors and induction theorems we refer for instance to [952, Section 6], [315], and [76].

### 12.8.3 Rational Computation of the Source of the Assembly Maps

**Theorem 12.79 (Rational computation of the source of the assembly maps appearing in the Farrell-Jones and Baum-Connes Conjecture).** Let R be an associative ring with unit and let F be  $\mathbb{R}$  or  $\mathbb{C}$ . Let G be a group. Denote by  $(\mathcal{FCY})$  the set of conjugacy classes (C) of finite cyclic subgroups C of G.

Then the rational Chern character of Theorem 12.58 induces isomorphisms

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC))$$

$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G; \mathbf{K}_R),$$

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FC}\mathcal{Y})} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot \left(\mathbb{Q} \otimes_{\mathbb{Z}} L_q^{\langle -\infty \rangle}(RC)\right)$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle})$$

and

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FC}\mathcal{Y})} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(C_r^*(C,F)))$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G;\mathbf{K}_F^{\mathrm{TOP}}).$$

*Proof.* This follows from Example 12.56, Theorem 12.58, Lemma 12.77, and Remark 12.78.

Remark 12.80. Explicit computations of the class

$$S_q(C; R) \in K_0(\mathbb{Q}[\operatorname{aut}(C)]) = R_{\mathbb{Q}}(\operatorname{aut}(C))$$

of the  $\mathbb{Q}[\operatorname{aut}(C)]$ -module  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC))$  for a finite cyclic group  $C, q \in \mathbb{Z}$ , and  $R = \mathbb{Z}$  or R a field of characteristic zero can be found in Patronas [793].

Let *C* be a non-trivial finite cyclic group. For a prime *p* denote by  $\operatorname{Gal}_p(C)$  the Galois group of the cyclotomic extension  $\mathbb{Q}_p^{\frown}(\mu_{|C|})$  of the *p*-adic rationals  $\mathbb{Q}_p^{\frown}$ . We can identify  $\operatorname{Gal}_p(C)$  in a canonical way with a subgroup of  $\operatorname{aut}(C)$ . Let *I* be the subgroup of  $\operatorname{aut}(C)$  generated by the automorphism of *C* sending *x* to  $x^{-1}$ . Let  $\mathbb{Q}$  be the  $\mathbb{Q}[\operatorname{aut}(C)]$ -module given by  $\mathbb{Q}$  equipped with the trivial  $\operatorname{aut}(C)$ -action.

Then we get for  $R = \mathbb{Z}$ 

$$S_{q}(C;\mathbb{Z}) = \begin{cases} -[\mathbb{Q}] + \sum_{p||C|} [\mathbb{Q}[\operatorname{aut}(C)/\operatorname{Gal}_{p}(C)]] & \text{if } q = -1; \\ [\mathbb{Q}[\operatorname{aut}(C)/I]] - [\mathbb{Q}] & \text{if } q = 1; \\ [\mathbb{Q}[\operatorname{aut}(C)/I]] & \text{if } q \ge 5, \ q \equiv 1 \mod 4; \\ [\mathbb{Q}[\operatorname{aut}(C)]] - [\mathbb{Q}[\operatorname{aut}(C)/I]] & \text{if } q \ge 3, \ q \equiv 3 \mod 4; \\ \{0\} & \text{otherwise,} \end{cases}$$

where p runs through all prime numbers dividing the order |C| in the case q = -1.

Let *F* be a field of characteristic zero. Denote by  $\operatorname{Gal}_F(C)$  the Galois group of the cyclotomic extension  $F[\mu_{|C|}]$  of *F*. Then there is an isomorphism of  $\mathbb{Q}[\operatorname{aut}(C)]$ -modules

$$\Theta_C \cdot \left( \mathbb{Q} \otimes_{\mathbb{Z}} K_q(FC) \right) \xrightarrow{\cong} \mathbb{Q}[\operatorname{aut}(C)/\operatorname{Gal}_F(C)]$$

where  $\operatorname{Gal}_F(C)$  is identified in a canonical way with a subgroup of  $\operatorname{aut}(C)$ .

**Exercise 12.81.** Let *C* be a non-trivial finite cyclic group. Let  $\varphi$  denote Euler's  $\varphi$ -function. Show:

(i) The dimension of the  $\mathbb{Q}$ -module  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(\mathbb{Z}C))$  is

$$\begin{cases} \varphi(|C|)/2 - 1 & |C| \ge 3 \text{ and } q = 1; \\ \varphi(|C|)/2 & |C| \ge 3, q \ge 3, \text{ and } q \equiv 1 \mod 2; \\ 1 & |C| = 2, q \ge 5, \text{ and } q \equiv 1 \mod 4; \\ s(|C|) & q = -1; \\ 0 & \text{otherwise,} \end{cases}$$

for  $s(n) := \sum_{i=1}^{s} \varphi(n/p_i^{e_i})/f_{p_i}$  where  $n = \prod_{i=1}^{s} p_i^{e_i}$  is the prime factorization of the positive integer *n* and  $f_{p_i}$  is the smallest positive integer such that  $p_i^{f_{p_i}} \equiv 1 \mod n/p^{e_i}$  holds;

(ii) The Q-module  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_1(\mathbb{Z}C))$  vanishes if and only if  $|C| \in \{1, 2, 3, 4, 6\}$ ; (iii) The Q-module  $\mathbb{Q} \otimes_{\mathbb{Q}[\operatorname{aut}(C)]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_m(\mathbb{Z}C))$  vanishes if  $m \neq -1$ .

The computations simplify even more if we consider the case  $R = \mathbb{C}$ , as the following example, which is taken from [649, Example 8.11], shows.

**Example 12.82 (Complex coefficients).** Let con(G) be the set of conjugacy classes (g) of elements  $g \in G$ . If we tensor with  $\mathbb{C}$  instead of  $\mathbb{Q}$  and take  $R = F = \mathbb{C}$ , then the isomorphisms appearing in Theorem 12.79 reduce to the isomorphisms

$$\begin{split} &\bigoplus_{p+q=n} \bigoplus_{(g)\in \operatorname{con}(G)} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G; \mathbf{K}_{\mathbb{C}}); \\ &\bigoplus_{p+q=n} \bigoplus_{(g)\in \operatorname{con}(G)} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} L_q^{\langle -\infty \rangle}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G; \mathbf{L}_{\mathbb{C}}^{\langle -\infty \rangle}); \\ &\bigoplus_{p+q=n} \bigoplus_{(g)\in \operatorname{con}(G)} H_p(C_G\langle g\rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\operatorname{TOP}}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G; \mathbf{K}_{\mathbb{C}}^{\operatorname{TOP}}), \end{split}$$

where we use in the definition of  $L_q(\mathbb{C})$  the involutions coming from complex conjugation. The targets of the maps above are isomorphic to  $\mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G)$ ,  $\mathbb{C} \otimes_{\mathbb{Z}} L_n^{\langle -\infty \rangle}(\mathbb{C}G)$ , and  $\mathbb{C} \otimes_{\mathbb{Z}} K_n(C_r^*(G, \mathbb{C}))$  if the Farrell-Jones Conjecture and the Baum-Connes Conjecture hold for *G*, where we use in the definition of  $L_q(\mathbb{C}) \cong L_q^{\langle -\infty \rangle}(\mathbb{C})$ the involution on  $\mathbb{C}G$  given by  $\sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \overline{\lambda_g} \cdot g^{-1}$ .

## **12.9 Some Integral Computations**

Integral computations are of course harder than rational computations. We have already provided basic tools such as the equivariant Atiyah-Hirzebruch spectral sequence and the p-chain spectral sequence in Section 12.6.

Often we are considering an equivariant homology theory and want to compute  $\mathcal{H}_n^G(\underline{E}G)$  or  $\mathcal{H}_n^G(\underline{E}G)$ . Sometimes one gets easy and useful computations if one has good models for  $\underline{E}G$  and  $\underline{E}G$ . We illustrate this in the following favorite case.

Let *G* be a discrete group. Let  $\mathcal{MFIN}$  be the subset of  $\mathcal{FIN}$  consisting of elements in  $\mathcal{FIN}$  that are maximal with respect to inclusion in  $\mathcal{FIN}$ . Throughout this subsection we suppose that *G* satisfies the conditions (<u>M</u>) and (<u>NM</u>) introduced in Subsection 11.6.12, where also examples of such groups *G* are given, see Example 11.33. Let  $\{M_i \mid i \in I\}$  be a complete set of representatives for the conjugacy classes of maximal finite subgroups of *G*. Consider an equivariant homology theory  $\mathcal{H}^2_*$ . Recall that we put <u>B</u>*G* =  $G \setminus \underline{E}G$ .

Then we obtain from Theorem 11.32 long exact sequences

$$(12.83) \quad \dots \to \bigoplus_{i \in I} \mathcal{H}_{n}^{\{1\}}(BM_{i}) \to \mathcal{H}_{n}^{\{1\}}(BG) \oplus \bigoplus_{i \in I} \mathcal{H}_{n}^{M_{i}}(\{\bullet\}) \to \mathcal{H}_{n}^{G}(\underline{E}G)$$
$$\bigoplus_{i \in I} \mathcal{H}_{n-1}^{\{1\}}(BM_{i}) \to \mathcal{H}_{n-1}^{\{1\}}(BG) \oplus \bigoplus_{i \in I} \mathcal{H}_{n-1}^{M_{i}}(\{\bullet\}) \to \dots;$$

$$(12.84) \quad \dots \to \bigoplus_{i \in I} \mathcal{H}_{n}^{\{1\}}(BM_{i}) \to \mathcal{H}_{n}^{\{1\}}(BG) \oplus \bigoplus_{i \in I} \mathcal{H}_{n}^{\{1\}}(\{\bullet\}) \to \mathcal{H}_{n}^{\{1\}}(\underline{B}G)$$
$$\bigoplus_{i \in I} \mathcal{H}_{n-1}^{\{1\}}(BM_{i}) \to \mathcal{H}_{n-1}^{\{1\}}(BG) \oplus \bigoplus_{i \in I} \mathcal{H}_{n-1}^{\{1\}}(\{\bullet\}) \to \dots .$$

We have the maps  $\mathcal{H}_n^{\{1\}}(\{\bullet\}) \to \mathcal{H}_n^{M_i}(\{\bullet\})$  induced by the inclusion  $\{1\} \to M_i$ and  $\mathcal{H}_n^{M_i}(\{\bullet\}) \to \mathcal{H}_n^{\{1\}}(\{\bullet\})$  induced by the projection  $M_i \to \{1\}$ . The composite is the identity. Define

(12.85) 
$$\widetilde{\mathcal{H}}_{n}^{M_{i}}(\{\bullet\}) := \ker\left(\mathcal{H}_{n}^{M_{i}}(\{\bullet\}) \to \mathcal{H}_{n}^{\{1\}}(\{\bullet\})\right).$$

Obviously we have an isomorphism

$$\mathcal{H}_n^{M_i}(\{\bullet\}) \cong \mathcal{H}_n^{\{1\}}(\{\bullet\}) \oplus \widetilde{\mathcal{H}}_n^{M_i}(\{\bullet\}).$$

One can splice the two long exact sequences (12.83) and (12.84) together to the long exact sequence

$$(12.86) \quad \dots \to \mathcal{H}_{n+1}^{\{1\}}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{\mathcal{H}}_{n}^{M_{i}}(\{\bullet\}) \to \mathcal{H}_{n}^{G}(\underline{E}G) \to \\ \to \mathcal{H}_{n}^{\{1\}}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{\mathcal{H}}_{n-1}^{M_{i}}(\{\bullet\}) \to \dots$$

The long exact sequence (12.86) splits after applying  $-\otimes_{\mathbb{Z}} \Lambda$ , more precisely,  $\mathcal{H}_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n^{\{1\}}(\underline{B}G) \otimes_{\mathbb{Z}} \Lambda$  is split surjective, see Lemma 12.18 (ii).

**Example 12.87 (Equivariant topological** *K*-theory of  $\underline{E}G$  for  $G = \mathbb{Z}^2 \rtimes \mathbb{Z}/4$ ). Consider the automorphism  $\phi \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ ,  $(x, y) \mapsto (-y, x)$ . It has order four. We want to show for the semidirect product  $G = \mathbb{Z}^2 \rtimes_{\alpha} \mathbb{Z}/4$ 

12.9 Some Integral Computations

$$K_n^G(\underline{E}G) \cong \begin{cases} \mathbb{Z}^9 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In this case we have the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_4 = \langle u, v, t \mid t^4 = 1, uv = vu, tut^{-1} = v, tvt^{-1} = u^{-1} \rangle.$$

The maximal finite subgroups are up to conjugacy given by

$$M_0 = \langle t \rangle;$$
  

$$M_1 = \langle ut \rangle;$$
  

$$M_2 = \langle ut^2 \rangle$$

We have  $M_0 \cong M_1 \cong \mathbb{Z}_4$  and  $M_2 \cong \mathbb{Z}_2$ . We obtain

$$\widetilde{K}_n^{\mathbb{Z}/m}(\{\bullet\}) \cong \begin{cases} \mathbb{Z}^{m-1} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Obviously <u>B</u>G is the same as  $\mathbb{Z}/4 \setminus T^2$  for the obvious  $\mathbb{Z}/4$ -action on the twodimensional torus  $T^2 = \mathbb{Z}^2 \setminus \underline{E}G = \mathbb{Z}^2 \setminus E\mathbb{Z}^2$ . This implies, because we are in dimension two, that <u>B</u>G has a model which is a compact 2-dimensional manifold. The rational cohomology  $H^*(\underline{B}G)$  agrees with  $H^*(T^2; \mathbb{Q})^{\mathbb{Z}/4}$ . Since  $\mathbb{Z}/4$  is a subgroup of  $SL_2(\mathbb{Z})$ , its action on  $T^2$  is orientation preserving. This implies that  $\mathbb{Z}/4$  acts trivially on  $H^p(T^2; \mathbb{Q})$  for p = 0, 2. Since  $\mathbb{Z}/4$  acts freely on  $\mathbb{Z}^2 = H_1(T^2; \mathbb{Z})$  outside  $\{0\}$ , we obtain  $H^1(T^2; \mathbb{Q})^{\mathbb{Z}/4} \cong \hom_{\mathbb{Z}}(H_1(T^2; \mathbb{Z})^{\mathbb{Z}/4}, \mathbb{Q}) \cong \{0\}$ . We conclude that  $\underline{B}G = \mathbb{Z}/4 \setminus T^2$  has the rational cohomology of  $S^2$  and hence is homeomorphic to  $S^2$ . This implies that  $K_0(\underline{B}G) \cong \mathbb{Z}^2$  and  $K_1(\underline{B}G) = 0$ .

The group *G* satisfies conditions (<u>M</u>) and (<u>NM</u>) by a direct check or because of Example 11.33, since the  $\mathbb{Z}/4$  action on  $\mathbb{Z}^2$  given by  $\alpha$  is free outside 0. Now the claim follows from the long exact sequence (12.86) applied in the case  $\mathcal{H}_*^2 = K_*^2$ .

Since G satisfies the Baum-Connes Conjecture, we have  $K_n(C_r^*(G)) \cong K_n^G(\underline{E}G)$ .

**Exercise 12.88.** Determine all finite subgroups  $F \subseteq SL_2(\mathbb{Z})$  and compute for any of these  $K_n^G(\underline{E}G)$  for  $n \in \mathbb{Z}$  and  $G = \mathbb{Z}^2 \rtimes F$ .

The long exact sequence (12.86) is a key ingredient in many computations of  $K_n(RG)$ ,  $L_n^{\langle -\infty \rangle}(RG)$ , and  $K_n(C_r^*(G))$ , provided that G satisfies the Farrell-Jones Conjecture and the Baum-Connes Conjecture, see Theorem 17.25.

Already for group homology the long exact sequence (12.86) contains valuable information, as we explain next.

**Example 12.89 (Group homology).** Suppose that *G* satisfies (<u>M</u>) and (<u>NM</u>). Let  $\mathcal{H}_*$  be Borel homology, i.e.,  $\mathcal{H}^G(X) := H_n(EG \times_G X)$  for  $H_n$  singular homology with coefficients in  $\mathbb{Z}$ , see Example 12.13. Then (12.86) reduces to the long exact sequence where  $H_n(G) := H_n(BG)$  is the group homology and  $\widetilde{H}_n(G) := \ker(H_n(G) \to H_n(\{1\}))$ 

$$\dots \to H_{n+1}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{H}_n(M_i) \to H_n(G)$$
$$\to H_n(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{H}_{n-1}(M_i) \to \dots$$

In particular, we get for  $n \ge \dim(BG) + 2$  an isomorphism

$$\bigoplus_{i \in I} H_n(M_i) \xrightarrow{\cong} H_n(G).$$

**Example 12.90 (The group homology of certain extensions**  $1 \to \mathbb{Z}^n \to G \to F \to 1$  **for finite** *F*). Consider an extension  $1 \to \mathbb{Z}^n \to G \to F \to 1$  for finite *F* such that the conjugation action of *F* on  $\mathbb{Z}^n$  is free outside  $0 \in \mathbb{Z}^n$ . Then the conditions (<u>M</u>) and (<u>NM</u>) are satisfied by Example 11.33 and there is an *n*-dimensional model for <u>E</u>G whose underlying space is  $\mathbb{R}^n$ .

Even in the case where F is a finite cyclic group, the computation of the homology of G is not at all easy. It is carried out in [282, Theorem 2.1], provided that |F| is a prime. More information in the case where there are no restrictions on |F| can be found in [618].

Based on the material of this section, we will compute the group homology of one-relators groups in Lemma 17.30 (iii) and Lemma 17.36.

# 12.10 Equivariant Homology Theory over a Group and Twisting with Coefficients

Next we present a slight variation of the notion of an equivariant homology theory introduced in Section 12.3. We have to treat this variation since we later want to study coefficients over a fixed group  $\Gamma$ , which we will then pullback via group homomorphisms with  $\Gamma$  as target. For instance, we may be interested in the algebraic *K*-theory of a twisted group ring  $R_{\alpha}G$  for some homomorphism  $\alpha: G \rightarrow \operatorname{aut}(R)$ . More generally, we will later consider additive *G*-categories as coefficients.

Fix a group  $\Gamma$ . A group  $(G,\xi)$  over  $\Gamma$  is a group G together with a group homomorphism  $\xi: G \to \Gamma$ . A map  $\alpha: (G_1,\xi_1) \to (G_2,\xi_2)$  of groups over  $\Gamma$  is a group homomorphisms  $\alpha: G_1 \to G_2$  satisfying  $\xi_2 \circ \alpha = \xi_1$ . Let  $\Lambda$  be an associative commutative ring with unit.

**Definition 12.91 (Equivariant homology theory over a group**  $\Gamma$ ). An *equivariant* homology theory  $\mathcal{H}^{?\downarrow\Gamma}_*$  with values in  $\Lambda$ -modules over a group  $\Gamma$  assigns to every group  $(G,\xi)$  over  $\Gamma$  a *G*-homology theory  $\mathcal{H}^{G,\xi}_*$  with values in  $\Lambda$ -modules and comes with the following so-called *induction structure*: given a homomorphism  $\alpha : (H, \mu) \to (G, \xi)$  of groups over  $\Gamma$  and an *H*-*CW*-pair (X, A), there are for each  $n \in \mathbb{Z}$  natural homomorphisms

(12.92) 
$$\operatorname{ind}_{\alpha} \colon \mathcal{H}_{n}^{H,\mu}(X,A) \to \mathcal{H}_{n}^{G,\xi}(\alpha_{*}(X,A))$$

satisfying

- Compatibility with the boundary homomorphisms  $\partial_n^{G,\xi} \circ \operatorname{ind}_{\alpha} = \operatorname{ind}_{\alpha} \circ \partial_n^{H,\mu};$
- Functoriality

Let  $\beta: (G, \xi) \to (K, \nu)$  be another morphism of groups over  $\Gamma$ . Then we have for  $n \in \mathbb{Z}$ 

$$\operatorname{ind}_{\beta \circ \alpha} = \mathcal{H}_n^{K,\nu}(f_1) \circ \operatorname{ind}_{\beta} \circ \operatorname{ind}_{\alpha} \colon \mathcal{H}^{H,\mu}H_n(X,A) \to \mathcal{H}_n^{K,\nu}((\beta \circ \alpha)_*(X,A))$$

where  $f_1: \beta_*\alpha_*(X, A) \xrightarrow{\cong} (\beta \circ \alpha)_*(X, A), \ (k, g, x) \mapsto (k\beta(g), x)$  is the natural *K*-homeomorphism;

• Compatibility with conjugation

Let  $(G, \xi)$  be a group over  $\Gamma$ . Fix  $g \in G$  such that  $\xi \circ c(g) = \xi$ . Then the conjugation homomorphisms  $c(g) \colon G \to G$  defines a morphism  $c(g) \colon (G, \xi) \to (G, \xi)$  of groups over  $\Gamma$ . Let  $f_2 \colon (X, A) \to c(g)_*(X, A)$  be the *G*-homeomorphism that sends *x* to  $(1, g^{-1}x)$  in  $G \times_{c(g)} (X, A)$ .

Then for every  $n \in \mathbb{Z}$  and every G-CW-pair (X, A) the homomorphism  $\operatorname{ind}_{c(g)} : \mathcal{H}_n^{G,\xi}(X, A) \to \mathcal{H}_n^{G,\xi}(c(g)_*(X, A))$  agrees with  $\mathcal{H}_n^G(f_2)$ ;

• Bijectivity

If ker( $\alpha$ ) acts freely on  $X \setminus A$ , then  $\operatorname{ind}_{\alpha} \colon \mathcal{H}_{n}^{H,\mu}(X,A) \to \mathcal{H}_{n}^{G,\xi}(\operatorname{ind}_{\alpha}(X,A))$  is bijective for all  $n \in \mathbb{Z}$ .

Definition 12.91 reduces to Definition 12.9 if one puts  $\Gamma = \{1\}$ .

The analog of Lemma 12.12 in this setting is obvious and easily checked.

The proof of Theorem 12.30 in this setting is explained in [71, Lemma 7.1].

**Theorem 12.93 (Constructing equivariant homology theories over a group using spectra).** Let  $\Gamma$  be a group. Denote by GROUPOIDS  $\downarrow I(\Gamma)$  the category of small connected groupoids over  $I(\Gamma)$  which is  $\Gamma$  considered as a groupoid with one object. Consider a covariant functor

**E**: GROUPOIDS 
$$\downarrow I(\Gamma) \rightarrow \text{SPECTRA}$$

that sends equivalences of groupoids to weak equivalences of spectra.

Then we can associate to it an equivariant homology theory  $\mathcal{H}^{?\downarrow\Gamma}_{*}(-; \mathbf{E})$  with values in  $\mathbb{Z}$ -modules over  $\Gamma$  such that for every group  $(G, \mu)$  over  $\Gamma$  and subgroup  $H \subseteq G$  we have a natural identification

$$\mathcal{H}_n^{H,\xi|_H}(\{\bullet\};\mathbf{E}) = \mathcal{H}_n^{G,\xi}(G/H,\mathbf{E}) = \pi_n(\mathbf{E}(H,\xi|_H)).$$

There are obvious twisted analogs of the functors mentioned in Section 12.5, see (13.10) together with Remark 13.12 and (13.18) together with Remark 13.20, and also [71, Theorem 6.1].

**Remark 12.94.** Equivariant Chern characters have only been constructed for equivariant homology theories but not for the more general notion of an equivariant homology theory over a group  $\Gamma$ . It is conceivable that they exist, provided that the coefficients of the homology theory  $\mathcal{H}^2_*$  over  $\Gamma$  extend to a Mackey functor over  $\Gamma$ , where we leave it to the reader to figure out what the latter condition means. For this claim there are many details to be checked and we have not done this. It also seems to be plausible that the equivariant homology theories over a group  $\Gamma$  given by the algebraic *K* and *L*-theory for a ring (with involution) coming with a homomorphism  $\Gamma \to \operatorname{aut}(R)$  have the property that the coefficients of the homology theories  $\mathcal{H}^2_*(-; \mathbf{K}_R)$  and  $\mathcal{H}^2_*(-; \mathbf{L}^{\langle -\infty \rangle}_R)$  over  $\Gamma$  extend to a Mackey functor over  $\Gamma$  and hence that there exists equivariant Chern characters for them.

**Remark 12.95.** Note that the proof of Lemma 12.18 (ii) does not extend to an equivariant homology theory over a non-trivial group  $\Gamma$  because we cannot pass to the quotient by *G* anymore. However, if the coefficients of the homology theory  $\mathcal{H}^{?}_{*}$  over  $\Gamma$  extend to a Mackey functor over  $\Gamma$  and we have an equivariant Chern character, then it is still true that the map  $\mathcal{H}^{G,\xi}_{n}(EG) \to \mathcal{H}^{G,\xi}_{n}(\underline{E}G)$  is rationally injective for every  $n \in \mathbb{Z}$  and every group  $\xi : G \to \Gamma$  over  $\Gamma$ . So this would yield the rational injectivity of the maps

$$H_n^{G,\xi}(EG;\mathbf{K}_R) \to H_n^{G,\xi}(\underline{E}G;\mathbf{K}_R);$$
  
$$H_n^{G,\xi}(EG;\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^{G,\xi}(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle}),$$

for every  $n \in \mathbb{Z}$  and every  $\xi: G \to \Gamma$ . The reader should note that we have proved this only in the case  $\Gamma = \{1\}$ , see Lemma 12.18 (ii).

**Exercise 12.96.** Let  $\Gamma$  be a group. Let *R* be a ring with a homomorphism  $\alpha \colon \Gamma \to \operatorname{aut}(R)$ . Let  $\xi \colon G \to \Gamma$  be a group over  $\Gamma$  such that *G* is finite.

Show that the map  $H_n^{G,\xi}(EG; \mathbf{K}_R) \to H_n^{G,\xi}(\underline{E}G; \mathbf{K}_R) = K_n(R_{\alpha \circ \xi}G)$  is rationally injective for every  $n \in \mathbb{Z}$ .

## 12.11 Notes

Equivariant stable cohomotopy has been introduced in [652] for arbitrary groups G and proper finite G-CW-complexes and extended to proper G-CW-complexes in [295, Example 3.43 on page 107]. A version of the Segal Conjecture in this setting is proved in [664]. A systematic study of the equivariant homotopy category for proper G-CW-complexes can be found in [295]. There it is explained in [295, Remark 3.44 on page 107] that the classical notion of an RO(G)-grading is taken over by a kind of  $K_G^0(\underline{E}G)$ -grading.

If one is dealing with equivariant topological K-theory, then there exists a Chern character where one does not have to fully rationalize, it suffices to invert the orders of all the isotropy groups of the proper G-CW-complex under consideration, see [651] or Theorem 10.69.

#### 12.11 Notes

There are also equivariant cohomology theories and a cohomological version of the equivariant Chern character, see [653]. It can be used to extend the Atiyah-Segal Completion Theorem for finite groups to infinite groups and proper *G*-*CW*-complexes, see [670, 671]. It also leads to rational computations of  $K^*(BG)$  for not necessarily finite groups, see [525, 656].

An equivariant Chern character for equivariant topological *K*-theory after complexification has been introduced in [107].
## Chapter 13 The Farrell-Jones Conjecture

### **13.1 Introduction**

In this chapter we discuss the Farrell-Jones Conjecture for *K*- and *L*-theory for arbitrary groups and rings. It predicts that certain assembly maps

$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_R) \to K_n(RG);$$
  
$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG).$$

are bijective for all  $n \in \mathbb{Z}$ . The targets are the algebraic *K*- or *L*-groups of the group ring *RG*, which one wants to understand. The source is an expression that depends only on the values of these *K*- and *L*-groups on virtually cyclic subgroups of *G* and is therefore much more accessible. The version above is often the one which is relevant in concrete applications, but nevertheless we will consider generalizations, for instance to twisted group rings and twisted involutions. The both most general and most important version will be the Full Farrell-Jones Conjecture 13.30. It implies all other variants of the Farrell-Jones Conjecture which appear in this book, see Section 13.11. It has very nice inheritance properties, see Section 13.7, which are in general not shared by the other variants.

A status report of the Full Farrell-Jones Conjecture 13.30 will be given in Theorem 16.1. It is known for a large class of groups.

The main point about the Full Farrell-Jones Conjecture 13.30 is that it implies a great variety of other prominent conjectures such as the ones due to Bass, Borel, Kaplansky, and Novikov, and leads to very deep and interesting results about manifolds and groups, as we will record and explain in Section 13.12. Often these applications are much more appealing and easier to comprehend than the rather technical Full Farrell-Jones Conjecture 13.30. The author's favorite is the Borel Conjecture, which predicts that two aspherical closed topological manifolds are homeomorphic if and only if their fundamental groups are isomorphic and any homotopy equivalence between them is homotopic to a homeomorphism.

Section 13.10 deals with the question whether one can reduce the family of virtually cyclic subgroups to a smaller family of subgroups, for instance to all finite subgroups or just to the family consisting of the trivial subgroup. Section 13.13 presents a short discussion of G-theory.

We have tried to keep this chapter independent of the other chapters as much as possible, so that one may start reading directly here.

#### **13.2** The Farrell-Jones Conjecture with Coefficients in Rings

Let *G* be a (discrete) group. Recall that a *G*-homology theory  $\mathcal{H}^G_*$  with values in  $\Lambda$ -modules for some commutative associative ring  $\Lambda$  assigns to every *G*-*CW*-pair (*X*, *A*) and integer  $n \in \mathbb{Z}$  a  $\Lambda$ -module  $\mathcal{H}^G_n(X, A)$  such that the obvious generalization to *G*-*CW*-pairs of the axioms of a (non-equivariant generalized) homology theory for *CW*-complexes holds, i.e., *G*-homotopy invariance, the long exact sequence of a *G*-*CW*-pair, excision, and the disjoint union axiom are satisfied. The precise definition of a *G*-homology theory can be found in Definition 12.1 and of a *G*-*CW*-complex in Definition 11.2, see also Remark 11.3.

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, to be a set of subgroups of G that is closed under conjugation with elements of G and passing to subgroups. Denote by  $E_{\mathcal{F}}(G)$  a model for the classifying space for the family  $\mathcal{F}$  of subgroups of G, i.e., a G-CW-complex  $E_{\mathcal{F}}(G)$  whose isotropy groups belong to  $\mathcal{F}$  and for which for each  $H \in \mathcal{F}$  the H-fixed point set  $E_{\mathcal{F}}(G)^H$  is weakly contractible. Such a model always exists and is unique up to G-homotopy, see Definition 11.18 and Theorem 11.19. Recall that  $\underline{E}G$  and  $\underline{E}G$  are abbreviations for  $E_{\mathcal{F}IN}(G)$  and  $E_{VCY}(G)$ , where  $\mathcal{FIN}$  is the family of finite subgroups and  $\mathcal{VCY}$  is the family of virtually cyclic subgroups, i.e., subgroups that are either finite or contain  $\mathbb{Z}$  as a subgroup of finite index.

#### 13.2.1 The K-Theoretic Farrell-Jones Conjecture with Coefficients in Rings

Given a ring *R*, there is a specific *G*-homology theory  $H_n^G(-; \mathbf{K}_R)$  with values in  $\mathbb{Z}$ -modules with the property that  $H_n^G(G/H; \mathbf{K}_R) \cong K_n(RH)$  holds for all  $n \in \mathbb{Z}$  and subgroups  $H \subseteq G$ , where  $K_n(RH)$  is the *n*th algebraic *K*-group of the group ring *RH*. Its construction can be used in the sequel as a black box. We have already specified some details, namely, it is given by the equivariant homology theory  $H_*^2(-; \mathbf{K}_R)$  evaluated at *G* that is associated to the covariant functor  $\mathbf{K}_R$ : GROUPOIDS  $\rightarrow$  SPECTRA of (12.44) in Theorem 12.30.

**Conjecture 13.1** (*K*-theoretic Farrell-Jones Conjecture with coefficients in the ring *R*). Given a group *G* and a ring *R*, we say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in the ring *R* if the assembly map induced by the projection pr:  $E_{VC,V}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

In many of the proofs the coefficients rings do not play a role, and therefore it is reasonable to consider the following stronger variant that is now a statement about the group G itself.

13.2 The Farrell-Jones Conjecture with Coefficients in Rings

**Conjecture 13.2** (*K*-theoretic Farrell-Jones Conjecture with coefficients in rings). We say that the group G satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in rings if the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in R holds for every ring R.

**Exercise 13.3.** Show that Conjecture 13.2 does not hold for  $G = \mathbb{Z}$  if one replaces  $\mathcal{VCY}$  by  $\mathcal{FIN}$  in Conjecture 13.1.

Conjecture 13.2 also makes sense for twisted group rings  $R_{\alpha}G$ , see Remark 13.12.

#### 13.2.2 The L-Theoretic Farrell-Jones Conjecture with Coefficients in Rings

The situation for *L*-theory is similar. Namely, given a ring with involution *R*, there is a specific *G*-homology theory  $H_n^G(-; \mathbf{L}_R^{\langle -\infty \rangle})$  with values in  $\mathbb{Z}$ -modules with the property that  $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RH)$  holds for all  $n \in \mathbb{Z}$  and subgroups  $H \subseteq G$ , where  $L_n^{\langle -\infty \rangle}(RH)$  is the *n*th quadratic *L*-group of the group ring with involution *RH* with decoration  $\langle -\infty \rangle$ . Its construction can be used in the sequel as a black box. We have already given some details, namely, it is given by the equivariant homology theory  $H_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle})$  evaluated at *G* that is associated to the covariant functor  $\mathbf{L}_R^{\langle -\infty \rangle}$ : GROUPOIDS  $\rightarrow$  SPECTRA of (12.45) in Theorem 12.30.

**Conjecture 13.4** (*L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R*). Given a group *G* and ring with involution *R*, we say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* if the assembly map induced by the projection  $E_{VCY}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

**Exercise 13.5.** Show that Conjecture 13.4 holds for  $G = \mathbb{Z}$  if one replaces  $\mathcal{VCY}$  by  $\mathcal{FIN}$ .

If we invert 2, it is expected that one can replace  $\mathcal{VCY}$  by  $\mathcal{FIN}$  in general.

**Conjecture 13.6** (*L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* after inverting 2). Given a group *G* and ring with involution *R*, we say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* after inverting 2 if the assembly map induced by the projection  $E_{\mathcal{FIN}}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle})\to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty\rangle})=L_n^{\langle -\infty\rangle}(RG)$$

is bijective for all  $n \in \mathbb{Z}$  after inverting 2.

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**Conjecture 13.7** (*L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution). A group *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution if the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution *R* holds for every ring with involution *R*.

**Conjecture 13.8** (*L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting 2). We say that a group *G* satisfies *the L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting* 2 if the *L*-theoretic Farrell-Jones Conjecture 13.6 with coefficients in the ring with involution *R* after inverting 2 holds for every ring with involution *R*.

**Remark 13.9 (The decoration**  $\langle -\infty \rangle$  **is necessary).** One can define for any decoration  $j \in \{n \in \mathbb{Z} \mid n \le 1\} \amalg \{-\infty\}$  an assembly map

$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle j \rangle}) \to H_n^G(G/G; \mathbf{L}_R^{\langle j \rangle}) = L_n^{\langle j \rangle}(RG).$$

But in general one can only hope that it is bijective if one chooses  $j = -\infty$ . Counterexamples for  $G = \mathbb{Z}^2 \times \mathbb{Z}/29$  for  $R = \mathbb{Z}$  and j = h, s are constructed in [371].

If we invert 2, the decorations do not play a role because of the Rothenberg sequences, see Subsection 9.10.4.

Conjectures 13.7 and 13.8 also make sense for twisted group rings  $R_{\alpha}G$ , see Remark 13.20.

## **13.3** The Farrell-Jones Conjecture with Coefficients in Additive Categories

There are situations where one wants to consider *twisted group rings*  $R_{\alpha}G$ , sometimes also denoted by  $R_{\alpha}[G]$ , for some group homomorphism  $\alpha : G \to \operatorname{aut}(R)$  to the group of ring automorphisms of R. Elements in  $R_{\alpha}G$  are given by formal finite sums  $\sum_{g \in G} r_g \cdot g$ , and addition and multiplication is given by

$$\begin{split} \left(\sum_{g \in G} r_g \cdot g\right) + \left(\sum_{g \in G} s_g \cdot g\right) &:= \sum_{g \in G} (r_g + s_g) \cdot g; \\ \left(\sum_{g \in G} r_g \cdot g\right) \cdot \left(\sum_{g \in G} s_g \cdot g\right) &:= \sum_{g \in G} \left(\sum_{\substack{h,k \in G, \\ g = hk}} r_h \cdot \alpha(h)(s_k)\right) \cdot g \end{split}$$

So the decisive relation for the multiplication is  $(r \cdot h) \cdot (s \cdot k) = (r \cdot \alpha(h)(s)) \cdot hk$ . Or even, more generally, one may want to consider crossed product rings, see for instance [77, Section 4].

In *L*-theory, we consider a ring with involution *R* and we wants to twist the involution on *RG* by an orientation homomorphism  $w: G \rightarrow \text{center}(R)$  satisfying  $\overline{w(g)} = w(g)$  resulting in the *w*-twisted involution on *RG* given by

$$\overline{\sum_{g \in G} r_g \cdot g} := \sum_{g \in G} w(g) \cdot \overline{r_g} \cdot g^{-1}.$$

The situation becomes even more involved if we want to consider crossed product rings with involution. The details are explained in [77, Section 4].

It turns out that one can nicely treat these generalizations of group rings and involutions by looking at additive *G*-categories (with involution).

There is another crucial reason why it is useful to look at coefficients in additive G-categories (with involution). These versions of the Farrell-Jones Conjecture with coefficients in additive G-categories (with involution) have much better inheritance properties than the one with coefficients in rings (with involution), as we will explain below in Section 13.7. For instance, they pass to subgroups.

The details are given for additive *G*-categories and *K*-theory in [92]. The case of additive *G*-categories with involution is treated for *K*-theory, taking the involution into account, and for *L*-theory in [77]. Since we can use this general approach essentially as a black box, we give only a brief summary here, following the notation of [77].

### 13.3.1 The K-Theoretic Farrell-Jones Conjecture with Coefficients in Additive G-Categories

Let  $\mathcal{A}$  be an additive *G*-category in the sense of [77, Definition 2.1], i.e., an additive category with *G*-action by functors of additive categories. Note that we use left actions here, whereas in [77] right actions are considered. Let GROUPOIDS  $\downarrow G$  be the category of connected groupoids over I(G). Recall that for a group *G* we denote by I(G) the groupoid with one object and *G* as its automorphism group. We obtain from [77, Section 5] a contravariant functor to the category ADDCAT of small additive categories

$$\mathsf{GROUPOIDS} \downarrow G \to \mathsf{ADDCAT}, \quad \mathrm{pr} \colon \mathcal{G} \to I(G) \mapsto \int_{\mathcal{G}} \mathcal{A} \circ \mathrm{pr}.$$

Composing it with the functor sending an additive category to its non-connective *K*-theory spectrum, see for instance [209, 684, 800], yields a functor

(13.10)  $\mathbf{K}_{\mathcal{A}}$ : GROUPOIDS  $\downarrow G \rightarrow \text{SPECTRA}$ .

By Theorem 12.93 we obtain an equivariant homology theory over *G* in the sense of Definition 12.91. In particular, its evaluation at *G* yields a *G*-homology theory  $H^G_*(-; \mathbf{K}_{\mathcal{R}})$ .

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Conjecture 13.11 (*K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories). We say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories if for every additive *G*-category  $\mathcal{A}$  and every  $n \in \mathbb{Z}$  the assembly map induced by the projection pr:  $E_{VCY}(G) \rightarrow G/G$ 

$$H_n^g(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n(\mathbf{K}_{\mathcal{A}}(I(G)))$$

is bijective.

Remark 13.12 (The setting with additive *G*-categories as coefficients encompasses the setting with rings as coefficients). Let  $\alpha: G \to \operatorname{aut}(R)$  be a group homomorphism. We have already introduced the twisted group ring  $R_{\alpha}[G]$  above. There is a covariant  $\operatorname{Or}(G)$ -spectrum  $\mathbf{K}_{R,\alpha}$  such that we have  $\pi_n(\mathbf{K}_{R,\alpha}(G/H)) =$  $K_n(R_{\alpha|_H}[H])$  for any subgroup  $H \subseteq G$  and integer  $n \in \mathbb{Z}$ . Thus we obtain a *G*-homology theory  $H_n^G(-; \mathbf{K}_{R,\alpha})$  for which  $H_n^G(G/H; \mathbf{K}_{R,\alpha}) \cong K_n(R_{\alpha|_H}[H])$ holds for all subgroups  $H \subseteq G$  and  $n \in \mathbb{Z}$ . For a suitable choice of an additive *G*-category  $\mathcal{A}$ , the *G*-homology theory  $H_n^G(-; \mathbf{K}_{\mathcal{A}})$  can be identified with the *G*-homology theory  $H_n^G(-; \mathbf{K}_{R,\alpha})$  In particular, the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

appearing in Conjecture 13.11 agrees with the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{R,\alpha}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{R,\alpha}) \to H_n^G(G/G; \mathbf{K}_{R,\alpha}) = K_n(R_\alpha G).$$

If  $\alpha$  is trivial, this is precisely the assembly map appearing in Conjecture 13.1. More details, even for crossed product rings, can be found in [77, Section 4 and 6].

In particular, we get that the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings holds for *G* if the *K*-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive *G*-categories holds for *G*.

In most cases we deal with untwisted group rings. The next example shall illustrate that for twisted group rings all the structural results remain true but computations become different, since one has to take the G-action on the K-theory of R into account.

**Example 13.13.** Let *G* be a torsionfree group satisfying Conjecture 13.11. Let *R* be a regular ring coming with a group homomorphism  $\alpha : G \to \operatorname{aut}(R)$ . Let  $R_{\alpha}[G]$  be the twisted group ring. Equip  $K_0(R)$  with the *G*-action by  $\mathbb{Z}$ -automorphisms coming from  $\alpha$ . Then the homomorphism  $K_0(R) \to K_0(R_{\alpha}[G])$  coming from the inclusion  $R \to R_{\alpha}[G]$  induces an isomorphism

(13.14) 
$$\mathbb{Z} \otimes_{\mathbb{Z}G} K_0(R) \xrightarrow{=} K_0(R_\alpha[G]).$$

This is proved as follows. We conclude from Remark 13.12 and Theorem 13.51 that the assembly map

$$H_n^G(EG; \mathbf{K}_{R,\alpha}) \to H_n(G/G; \mathbf{K}_{R,\alpha}) = K_n(R_\alpha[G])$$

is bijective for all  $n \in \mathbb{Z}$ . Since  $K_q(R)$  vanishes for  $q \leq -1$  by Theorem 4.7 the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, implies that we get an isomorphism

$$H_0^G(EG; \mathbf{K}_0(R)) \xrightarrow{\cong} H_0^G(EG; \mathbf{K}_{R,\alpha}).$$

The source of the latter isomorphism can be identified with  $\mathbb{Z} \otimes_{\mathbb{Z}G} K_0(R)$ .

**Exercise 13.15.** Give an example a torsionfree group *G* and of a regular ring *R* coming with a group homomorphism  $\alpha: G \to \operatorname{aut}(R)$  such that the canonical map  $K_0(R) \to \mathbb{Z} \otimes_{\mathbb{Z}G} K_0(R)$  is not injective.

**Exercise 13.16.** Let *R* be a ring. Define a category  $\underline{R}_{\oplus}$  as follows. For each integer  $m \in \mathbb{Z}$  with  $m \ge 0$  we have one object [m]. For  $m, n \ge 1$  the set of morphisms from [m] to [n] is the set  $M_{m,n}(R)$  of (m, n)-matrices with entries in *R*. The set of morphisms from [0] to [m] and from [m] to [0] consist of precisely one element. Composition is given by matrix multiplication.

Show that  $\underline{R}_{\oplus}$  can be equipped with the structure of a small additive category and that it is equivalent as an additive category to the category of finitely generated free *R*-modules.

**Remark 13.17 (Involutions and** *K***-theory).** Let  $\mathcal{A}$  be an additive *G*-category with involution in the sense of [77, Definition 4.22], i.e., an additive category with involution coming with *G*-action by functors of additive categories with involution.

Then the involution induces involutions on the source and target of the *K*-theoretic assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n(\mathbf{K}_{\mathcal{A}}(I(G)))$$

of Conjecture 13.11 and the assembly map is compatible with them.

### 13.3.2 The *L*-Theoretic Farrell-Jones Conjecture with Coefficients in Additive *G*-Categories with Involution

Let  $\mathcal{A}$  be an additive *G*-category with involution in the sense of [77, Definition 4.22]. We obtain from [77, Section 7] a contravariant functor to the category ADDCAT<sub>inv</sub> of small additive categories with involution

$$\mathsf{GROUPOIDS} \downarrow G \to \mathsf{ADDCAT}_{\mathrm{inv}}, \quad \mathrm{pr} \colon \mathcal{G} \to I(G) \mapsto \int_{\mathcal{G}} \mathcal{A} \circ \mathrm{pr} \, .$$

Composing it with the functor sending an additive category with involution  $\mathcal{A}$  to its *L*-theory spectrum  $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$  yields a functor

(13.18) 
$$\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}$$
: GROUPOIDS  $\downarrow G \rightarrow$  SPECTRA

where for the construction of the spectrum *L*-theory  $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$  associated to an additive category with involution  $\mathcal{A}$  we refer to Ranicki [839, Chapter 13]. By Theorem 12.93 we obtain an equivariant homology theory over *G* in the sense of Definition 12.91. In particular, its evaluation at *G* yields a *G*-homology theory  $H_n^G(-; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle})$ .

Conjecture 13.19 (*L*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with involution). We say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with involution if for every additive *G*-category with involution  $\mathcal{A}$  and every  $n \in \mathbb{Z}$  the assembly map given by the projection  $E_{VCY}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

is bijective.

Remark 13.20 (The setting of additive *G*-categories with involution as coefficients encompasses the setting with rings with involution as coefficients). Let *R* be a ring with involution. Consider a group homomorphism  $\alpha: G \rightarrow \text{aut}(R)$  satisfying  $\overline{\alpha(g)(r)} = \alpha(g)(\overline{r})$ , and a group homomorphism  $w: G \rightarrow \text{center}(R)$  satisfying  $\overline{w(g)} = w(g)$ . Then we have already introduced the twisted group ring  $R_{\alpha}(G)$  above. It inherits an involution by

$$\overline{\sum_{g \in G} r_g \cdot g} := \sum_{g \in G} w(g) \cdot \alpha(g^{-1})(\overline{r_g}) \cdot g^{-1},$$

and we denote this ring with involution by  $R_{\alpha,w}G$ . For a suitable choice of an additive *G*-category with involution  $\mathcal{A}$ , the assembly map

$$H_n^G(\mathrm{pr};\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})\to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle})$$

appearing in Conjecture 13.19 reduces to the assembly map

$$H_n^G(\mathrm{pr};\mathbf{L}_{R,\alpha,w}^{\langle-\infty\rangle})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_{R,\alpha,w}^{\langle-\infty\rangle})\to H_n^G(G/G;\mathbf{L}_{R,\alpha,w}^{\langle-\infty\rangle})$$

where for any subgroup  $H \subseteq G$  and integer  $n \in \mathbb{Z}$  we have  $\mathbf{L}_{R,\alpha,w}^{\langle -\infty \rangle}(I(H)) = L_n^{\langle -\infty \rangle}(R_{\alpha|_H}H, w|_H)$ . If  $\alpha$  and w are trivial, this is precisely the assembly map appearing in Conjecture 13.4. More details, even for crossed product rings, can be found in [77, Theorem 0.4, Section 4 and 8].

In particular, we get that the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution holds for *G* if the *L*-theoretic Farrell-Jones Conjecture 13.19 with coefficients in additive *G*-categories with involution holds for *G*. 13.4 The *K*-Theoretic Farrell-Jones Conjecture with Coefficients in Higher Categories 385

**Exercise 13.21.** Let  $F: \mathcal{A} \to \mathcal{B}$  be a functor of additive categories. Show that it is an equivalence of additive categories if and only if for every two objects A and B in  $\mathcal{A}$  the induced map  $\operatorname{mor}_{\mathcal{A}}(A_0, A_1) \to \operatorname{mor}_{\mathcal{B}}(F(A_0), F(A_1))$  sending f to F(f) is bijective and for each object B in  $\mathcal{B}$  there exists an object A in  $\mathcal{A}$  such that F(A) and B are isomorphic in  $\mathcal{B}$ .

**Remark 13.22 (Eilenberg swindle for** *L***-theory).** There is an obvious version of Theorem 6.37 (iii) for the algebraic *L*-theory  $\mathbf{L}^{\langle -\infty \rangle}(\mathcal{A})$  of an additive category  $\mathcal{A}$  with involution.

# **13.4** The *K*-Theoretic Farrell-Jones Conjecture with Coefficients in Higher Categories

Let C be a right exact G- $\infty$ -category in the sense of Definition 8.35. We obtain from (8.38) a covariant functor

 $\mathbf{K}_C$ : GROUPOIDS  $\downarrow I(G) \rightarrow Sp$ 

and from Theorem 8.36 (i) an equivariant homology theory  $H_*^{?\downarrow G}(-; \mathbf{K}_C)$  over G in the sense of Definition 12.91. In particular, its evaluation at G yields a G-homology theory  $H_*^G(-; \mathbf{K}_C)$ .

**Conjecture 13.23** (*K*-theoretic Farrell-Jones Conjecture with coefficients in higher *G*-categories). We say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in higher *G*-categories if for every right exact G-∞-category *C* and every  $n \in \mathbb{Z}$  the assembly map given by the projection

$$H_n^G(\operatorname{pr}; \mathbf{K}_{\mathcal{C}}) \colon H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{C}}) = \pi_n(\mathbf{K}_{\mathcal{C}}(I(G)))$$

is bijective.

**Remark 13.24 (For** *K***-theory the setting of higher** *G***-categories encompasses all other settings for** *K***-theory.).** The assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{C}}) \colon H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{C}}) = \pi_n(\mathbf{K}_{\mathcal{C}}(I(G)))$$

appearing in the *K*-theoretic Farrell-Jones Conjecture 13.23 with coefficients in higher *G*-categories reduces to the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

appearing in Conjecture 13.11 if we take for *C* the higher *G*-category  $\mathcal{K}^{b}(\mathcal{A})$ ), see Theorem 8.36

Moreover, the *K*-theoretic Farrell-Jones Conjecture 13.23 with coefficients in higher *G*-categories implies the Farrell-Jones Conjecture 15.61 for *A*-theory (with coefficients) by Subsection 8.5.4.

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Recall from Remark 13.12 that Conjecture 13.11 and hence also Conjecture 13.23 imply the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings.

A version of the Farrell-Jones Conjecture for ring spectra as coefficients appears in Subsection 8.5.5.

**Remark 13.25** (*L*-theory version of Conjecture 13.23). It has not been worked out in detail how to construct the assembly map for the *L*-theory version of the *K*-theoretic Farrell-Jones Conjecture 13.23 with coefficients in higher *G*-categories with Poincaré structure, or how to prove the conjecture for the same class of groups as has been done for the other versions. Such a version should of course imply the *L*-theoretic Farrell-Jones Conjecture 13.19 with coefficients in additive *G*-categories with involution. Christoph Winges is at the time of writing working on such a generalization to the setting of higher categories.

## **13.5 Finite Wreath Products**

The versions of the Farrell-Jones Conjecture discussed above do not carry over to overgroups of finite index. To handle this difficulty, we consider finite wreath products.

Let *G* and *F* be groups. Their *wreath product*  $G \wr F$  is defined as the semidirect product  $(\prod_F G) \rtimes F$  where *F* acts on  $\prod_F G$  by permuting the factors. For our purpose the following elementary lemma is crucial.

#### Lemma 13.26.

- (i) There is an embedding  $(H \wr F_1) \wr F_2 \rightarrow H \wr (F_1 \wr F_2)$ ;
- (ii) If  $F_1$  and  $F_2$  are finite, then  $F_1 \wr F_2$  is finite;
- (iii) Let G be an overgroup of H of finite index. Then there is subgroup  $N \subseteq H$  of H that satisfies  $[G:N] < \infty$  and is normal in G, and a finite group F such that G embeds into  $N \wr F$ .

*Proof.* (i) See [595, Lemma 1.21].

(ii) This is obvious.

(iii) Let *S* denote a system of representatives of the cosets *G*/*H*. Since *G*/*H* is by assumption finite,  $N := \bigcap_{s \in S} sHs^{-1}$  is a finite index normal subgroup of *G* and is contained in *H*. Now *G* can be embedded in  $N \wr G/N$ , see [311, Section 2.6].  $\Box$ 

**Conjecture 13.27** (*K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with finite wreath products). We say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with finite wreath products if for any finite group *F* the group  $G \wr F$  satisfies the *K*-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive  $G \wr F$ -categories.

Conjecture 13.28 (*L*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with involution with finite wreath products). We say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with involution with finite wreath products if for any finite group *F* the group  $G \wr F$  satisfies the *L*-theoretic Farrell-Jones Conjecture 13.19 with coefficients in additive in additive  $G \wr F$ -categories with involution.

**Conjecture 13.29** (*K*-theoretic Farrell-Jones Conjecture with coefficients in higher *G*-categories with finite wreath products). We say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in higher *G*-categories with finite wreath products if for any finite group *F* the group  $G \wr F$  satisfies the *K*-theoretic Farrell-Jones Conjecture 13.23 with coefficients in higher  $G \wr F$ -categories.

## **13.6 The Full Farrell-Jones Conjecture**

Next we can formulate the version of the Farrell-Jones Conjecture which is the most general one, implies all other ones, and has the best inheritance properties.

**Conjecture 13.30 (Full Farrell-Jones Conjecture).** We say that a group satisfies the *Full Farrell-Jones Conjecture* if *G* satisfies the following three conjectures:

- the *K*-theoretic Farrell-Jones Conjecture 13.27 with coefficients in additive *G*-categories with finite wreath products;
- the *L*-theoretic Farrell-Jones Conjecture 13.28 with coefficients in additive *G*-categories with involution with finite wreath products;
- the *K*-theoretic Farrell-Jones Conjecture 13.29 with coefficients in higher *G*-categories with finite wreath products.

Despite the fact that Conjecture 13.29 implies Conjecture 13.27, see Remark 13.24, we list Conjecture 13.27 above in Conjecture 13.30 for the reader's convenience. Recall that the version with rings as coefficients follow from the versions with additive categories as coefficients, see Remarks 13.12 and 13.20.

## 13.7 Inheritance Properties of the Farrell-Jones Conjecture

In this section we discuss the inheritance properties of the various versions of the Farrell-Jones Conjectures above. Both the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution do not have good inheritance properties. The reason why we have introduced the other variants is that they do have some remarkable inheritance properties.

**Definition 13.31 (Farrell-Jones groups).** Let  $\mathcal{FJ}$  be the class of groups that satisfy the Full Farrell-Jones Conjecture 13.30. We call a (discrete) group *G* a *Farrell-Jones group* if *G* belongs to  $\mathcal{FJ}$ .

#### Theorem 13.32 (Inheritance properties of the Full Farrell-Jones Conjecture).

- (i) Passing to subgroups Let H ⊆ G be an inclusion of groups and G ∈ FJ, then H ∈ FJ;
  (ii) Passing to overgroups of finite index Let G be an overgroup of H with finite index [G : H]. If H belongs to FJ, then G belongs to FJ;
- (iii) Passing to finite wreath products

If G belongs to  $\mathcal{FJ}$ , then  $G \wr F$  belongs to  $\mathcal{FJ}$  for any finite group F; (iv) Passing to finite direct products

- If the groups  $G_0$  and  $G_1$  belong to  $\mathcal{FJ}$ , then  $G_0 \times G_1$  belongs to  $\mathcal{FJ}$ ,
- (v) Group extensions

Let  $1 \to K \to G \to Q \to 1$  be an extension of groups. Suppose that the groups K and Q belong to  $\mathcal{FJ}$  and that for any infinite cyclic subgroup  $C \subseteq Q$  the group  $p^{-1}(C)$  belongs to  $\mathcal{FJ}$ . Then G belongs to  $\mathcal{FJ}$ ;

(vi) Colimits over directed systems

Let  $\{G_i \mid i \in I\}$  be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each  $i \in I$  the group  $G_i$  belongs to  $\mathcal{FJ}$ .

*Then the colimit*  $\operatorname{colim}_{i \in I} G_i$  *belongs to*  $\mathcal{FJ}$ *;* 

(vii) Passing to free products

Consider a collection of groups  $\{G_i \mid i \in I\}$  such that  $G_i$  belongs to  $\mathcal{FJ}$  for each  $i \in I$ . Then  $*_{i \in I}G_i$  belongs to  $\mathcal{FJ}$ .

*Proof.* (i) We begin with the case of additive *G*-categories as coefficients.

Assertion (i) is proved in [92, Theorem 4.5] for Conjecture 13.11, and in [77, Theorem 0.10] for Conjecture 13.19. Now assertion (i) follows for the version of Full Farrell-Jones Conjecture 13.30 for additive *G*-categories as coefficients since  $H \wr F$  is a subgroup of  $G \wr F$  for every subgroup  $H \subseteq G$ .

The proof of assertion (i) for the version with higher G-categories as coefficients is analogous and can be found in [185, Theorem 1.6 (1)].

- (ii) This follows from Lemma 13.26 and assertion (i).
- (iii) This follows from Lemma 13.26 and assertion (i).

(iv) We begin with the case of additive G-categories as coefficients.

The versions of the Farrell-Jones Conjecture 13.11 and 13.19 are true for virtually finitely generated abelian groups by [72, Theorem 3.1]. Hence they hold in particular for the product of two virtually cyclic subgroups. By inspecting the proof of [595, Lemma 3.15], we see that the assertion (iv) holds for the Farrell-Jones Conjectures 13.11 and 13.19.

#### 13.7 Inheritance Properties of the Farrell-Jones Conjecture

Next we prove assertion (iv) for the version of Full Farrell-Jones Conjecture 13.30 with additive *G*-categories as coefficients. Suppose it holds for  $G_1$  and  $G_2$ . Let *F* be any finite subgroup. We have to show that versions of the Farrell-Jones Conjecture 13.11 and 13.19 holds for  $(G_1 \times G_2) \wr F$ . By assumption they both hold for  $G_1 \wr F$  and  $G_2 \wr F$ . Since  $(G_1 \times G_2) \wr F$  is a subgroup of  $(G_1 \wr F) \times (G_2 \wr F)$  by [595, Lemma 1.197] and Conjecture 13.11 and 13.19 pass to subgroups by the argument given in assertion (i), assertion (iv) holds for the Full Farrell-Jones Conjecture 13.30 with additive *G*-categories as coefficients.

The proof of assertions (iv) for the version of the Full Farrell-Jones Conjecture 13.30 for higher G-categories is analogous and can be found in [185, Theorem 1.7 (11)].

(v) We begin with the case of additive G-categories as coefficients.

The following version of assertion (v) is proved in [77, Theorem 0.9] for Conjecture 13.19.

#### Property (E)

Let  $1 \to K \to G \to Q \to 1$  be an extension of groups. If for any virtually cyclic subgroup  $V \subseteq Q$  the group  $p^{-1}(V)$  and the group Q satisfy Conjecture 13.19, then G satisfies Conjecture 13.19.

The proof of property (E) for Conjecture 13.11 is analogous. Finally we conclude from [595, Lemma 3.16] and assertion (iv) that property (E) also holds for the Full Farrell-Jones Conjecture 13.30 for additive *G*-categories.

Because of assertion (ii), we can replace in property (E) the assumption that V is virtually cyclic by the assumption that V is trivial or infinite cyclic. This finishes the proof of assertion (v) for additive G-categories as coefficients.

The proof of assertion (v) for the version of the Full Farrell-Jones Conjecture 13.30 for higher *G*-categories is analogous and can be found in [185, Theorem 1.7 (13)].

(vi) We begin with the case of additive G-categories as coefficients.

Assertion (vi) is proved in [77, Theorem 0.8] for Conjecture 13.19, the proof for Conjecture 13.11 is completely analogous. Now assertion (vi) follows for the version of Full Farrell-Jones Conjecture 13.30 with additive *G*-categories, since there is an obvious isomorphism for a finite group *F*, see [595, Lemma 1.20],

$$\operatorname{colim}_{i\to\infty}(G_i \wr F) \xrightarrow{\cong} \left(\operatorname{colim}_{i\to\infty} G_i\right) \wr F.$$

The proof of assertion (vi) for the version of the Full Farrell-Jones Conjecture 13.30 for higher *G*-categories is analogous and can be found in [185, Theorem 1.6 (12)].

(vii) Because of assertion (vi) it suffices to consider the case where *I* is finite. An obvious induction argument over the cardinality of the finite set *I* reduces the claim to the case  $I = \{1, 2\}$ .

Let  $G_1$  and  $G_2$  be groups. Let pr:  $G_1 * G_2 \to G_1 \times G_2$  be the canonical projection. Let  $C \subseteq G_1 \times G_2$  be a cyclic subgroup. Then there exists a free group F and a finite group H such that  $pr^{-1}(C)$  is a subgroup of  $F \wr H$ , see [595, Lemma 3.21]. (In the statement of [595, Lemma 3.21] the assumption countable appears but the proof goes through in the general case without modification.) A finitely generated

free group satisfies the Full Farrell-Jones Conjecture 13.30 by [89, Remark 6.4] and [185, Theorem 1.6 (3)], since it is a hyperbolic group. Hence *F* satisfies the Full Farrell-Jones Conjecture 13.30 by assertion (vi). We conclude from assertion (iii) that  $F \wr H$  satisfies the Full Farrell-Jones Conjecture 13.30. Hence  $pr^{-1}(C)$  satisfies the Full Farrell-Jones Conjecture 13.30 for every cyclic subgroup  $C \subseteq G_1 \times G_2$  by assertion (i). The product  $G_1 \times G_2$  satisfies the Full Farrell-Jones Conjecture 13.30 by assertion (iv). Now assertion (v) implies that  $G_1 * G_2$  satisfies the Full Farrell-Jones Conjecture 13.30.

**Exercise 13.33.** Consider an epimorphism of groups  $G \rightarrow Q$  whose kernel is finite. Suppose that Q satisfies the Full Farrell-Jones Conjecture 13.30.

Show that G satisfies the Full Farrell-Jones Conjecture 13.30.

**Exercise 13.34.** Suppose that the Full Farrell-Jones Conjecture 13.30 holds for all groups that occur as fundamental groups of connected orientable closed 4-manifolds. Show that then the Full Farrell-Jones Conjecture 13.30 holds for all groups.

## 13.8 Splitting the Assembly Map from $\mathcal{FIN}$ to $\mathcal{VCY}$

In the sequel we denote for two families  $\mathcal{F} \subseteq \mathcal{G}$  by

(13.35) 
$$\iota_{\mathcal{F}\subseteq\mathcal{G}}\colon E_{\mathcal{F}}(G)\to E_{\mathcal{G}}(G)$$

the up to *G*-homotopy unique *G*-map. Note that  $\iota_{\mathcal{F}\subseteq \mathcal{ALL}}: E_{\mathcal{F}}(G) \to E_{\mathcal{ALL}}(G) = G/G$  is the projection.

**Theorem 13.36 (Splitting the** *K***-theoretic assembly map from**  $\mathcal{FIN}$  to  $\mathcal{VCY}$ ). *Let G be a group.* 

(i) Let A be an additive G-category. Let n be any integer. Then

$$H_n^G(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}})\to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{A}})$$

is split injective. In particular, we have an isomorphism

$$H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G);\mathbf{K}_{\mathcal{A}}) \xrightarrow{\cong} H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{A}}) \oplus H_n^G(\iota_{\mathcal{FIN}\subseteq\mathcal{VC}\mathcal{Y}};\mathbf{K}_{\mathcal{A}});$$

 (ii) Let C be a right exact G-∞-category. Let n be any integer. Then

$$H_n^G(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{K}_{\mathcal{C}})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathcal{C}})\to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathcal{C}})$$

is split injective.

*Proof.* (i) See [94, Theorem 1.3].

(ii) This follows from [183, Corollary 1.13] and [180, Theorem 1.1.5].

13.9 Rationally Splitting the Assembly Map from TR to FIN

For *L*-theory one has at least the following version that is mentioned after Theorem 1.3 in [94] for rings. The argument carries over to additive *G*-categories with involution.

**Theorem 13.37 (Splitting the** *L***-theoretic assembly map from**  $\mathcal{FIN}$  to  $\mathcal{VCY}$ ). Let  $\mathcal{A}$  be an additive *G*-category with involution such that there exists an integer Nwith the property that  $\pi_n(\mathbf{K}_{\mathcal{A}}(I(V))) = 0$  for all virtually cyclic subgroups *V* of *G* and all  $n \leq N$ .

Then

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq\mathcal{VCY}};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\colon H_n^G\big(\mathcal{E}_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\to H_n^G\big(\mathcal{E}_{\mathcal{VCY}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)$$

is split injective.

It is not clear whether the condition about  $\pi_n(\mathbf{K}_{\mathcal{A}}(I(V)))$  appearing in Theorem 13.37, which is needed for the proposed proof, is necessary. If we consider group rings *RG*, this condition is automatically satisfied if one of the following conditions holds:

- The ring *R* is regular and the order of every finite subgroup of *G* is invertible in *R*;
- The ring *R* is the ring of integers in an algebraic number field;
- The ring *R* is Artinian.

#### 13.9 Rationally Splitting the Assembly Map from TR to FIN

**Lemma 13.38.** *Let G be a group and let R be a ring (with involution). Then the relative assembly maps* 

$$H_{n}(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}};\mathbf{K}_{R}):H_{n}(E_{\mathcal{TR}}(G);\mathbf{K}_{R})\to H_{n}(E_{\mathcal{FIN}}(G);\mathbf{K}_{R});$$

$$H_{n}(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}};\mathbf{L}_{R}^{\langle-\infty}):H_{n}(E_{\mathcal{TR}}(G);\mathbf{L}_{R}^{\langle-\infty})\to H_{n}(E_{\mathcal{FIN}}(G);\mathbf{L}_{R}^{\langle-\infty});$$

$$K_{n}^{G}(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}}):K_{n}^{G}(E_{\mathcal{TR}}(G))\to K_{n}^{G}(E_{\mathcal{FIN}}(G));$$

$$KO_{n}^{G}(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}}):KO_{n}^{G}(E_{\mathcal{TR}}(G))\to KO_{n}^{G}(E_{\mathcal{FIN}}(G)),$$

are split injective after applying  $- \otimes_{\mathbb{Z}} \mathbb{Q}$  for  $n \in \mathbb{Z}$ .

Proof. This follows Lemma 12.18 (ii).

**Remark 13.39.** Note that Lemma 13.38 is only stated in the case when we consider the untwisted coefficients rings *R*. It is conceivable that it also holds in the case where we allow a twisting  $\alpha: G \rightarrow \operatorname{aut}(R)$ , but the details of a proof of this statement have not been worked out in detail and are definitely more complicated than the untwisted case, see Remark 12.95.

The proof of Lemma 13.38 carries over to additive categories and right-exact  $\infty$ -categories as coefficients provided that the *G*-actions on these are trivial.

Example 13.40 (The *L*-theory assembly map for the trivial family is not injective in general). Consider the group  $\mathbb{Z}/3$ . Then

$$H_1(B\mathbb{Z}/3; \mathbf{L}(\mathbb{Z})) \to L_1(\mathbb{Z}[\mathbb{Z}/3])$$

is not injective. Namely, the target is known to be trivial, but the source is non-trivial. This can be seen by inspecting the Atiyah-Hirzebruch spectral sequence converging to  $H_{p+q}(B\mathbb{Z}/3; \mathbf{L}(\mathbb{Z}))$  with  $E^2$ -term

$$H_p(B\mathbb{Z}/3, L_q(\mathbb{Z})) = \begin{cases} \mathbb{Z}/3 & p \ge 1, p \text{ odd}, q \equiv 0 \mod 4; \\ L_q(\mathbb{Z}) & p = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Note that Wh( $\mathbb{Z}/3$ ),  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$ , and  $K_n(\mathbb{Z}[\mathbb{Z}/3])$  for  $n \leq -1$  vanish by Theorem 3.115, Theorem 3.116 (iva), Theorem 4.22 (i) and (v), and Example 2.107 so that the decorations for the *L*-groups do not play a role by Theorem 9.106.

**Example 13.41 (The** *K***-theory assembly map for the trivial family is not injective in general).** An easy calculation using the Atyiah-Hirzebruch spectral sequence shows that the *K*-theoretic assembly map  $H_n(\iota_{\mathcal{TR}\subseteq\mathcal{FIN}}; \mathbf{K}_R) : H_n(E_{\mathcal{TR}}(G); \mathbf{K}_R) \to H_n(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$  is not injective if n = 2,  $G = \mathbb{Z}/2 \times \mathbb{Z}/2$  and  $R = \mathbb{F}_p$  for an odd prime p, see [961]. No such example is known to the author for  $R = \mathbb{Z}$ .

## 13.10 Reducing the Family of Subgroups for the Farrell-Jones Conjecture

Next we explain that one can sometimes reduce the family of virtually cyclic subgroups  $\mathcal{VCY}$  to a smaller family.

A virtually cyclic group V is called *of type I* if it admits an epimorphism to the infinite cyclic group, and *of type II* if it admits an epimorphism onto the infinite dihedral group. The elementary proof of the following result can be found in [685, Lemma 1.1].

Lemma 13.42. Let V be an infinite virtually cyclic group.

(i) V is either of type I or of type II;

- (ii) The following assertions are equivalent:
  - (a) V is of type I;
  - (b)  $H_1(V)$  is infinite;
  - (c)  $H_1(V)$ /tors(V) is infinite cyclic;
  - (d) *The center of V is infinite;*
- (iii) There exists a unique maximal normal finite subgroup  $K_V \subseteq V$ , i.e.,  $K_V$  is a finite normal subgroup and every normal finite subgroup of V is contained in  $K_V$ ;

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(iv) Let  $Q_V := V/K_V$ . Then we obtain a canonical exact sequence

$$1 \to K_V \xrightarrow{i_V} V \xrightarrow{p_V} Q_V \to 1.$$

Moreover,  $Q_V$  is infinite cyclic if and only if V is of type I and  $Q_V$  is isomorphic to the infinite dihedral group if and only if V is of type II;

(v) Let  $f: V \to Q$  be any epimorphism onto the infinite cyclic group or onto the infinite dihedral group. Then the kernel of f agrees with  $K_V$ ;

**Exercise 13.43.** Let  $\phi: V \to W$  be a homomorphism of infinite virtually cyclic groups with infinite image. Then  $\phi$  maps  $K_V$  to  $K_W$  and we obtain the following canonical commutative diagram with exact rows

with injective  $\phi_Q$ .

**Exercise 13.44.** Show that a group G is infinite virtually cyclic if and only if it admits a proper cocompact isometric action on  $\mathbb{R}$ .

In the sequel we denote by  $\mathcal{VCY}_I$  the family of subgroups that are either finite or infinite virtually cyclic of type I.

**Definition 13.45 (Hyperelementary group).** Let *p* be a prime. A (possibly infinite) group *G* is called *p*-hyperelementary if it can be written as an extension  $1 \rightarrow C \rightarrow G \rightarrow P \rightarrow 1$  for a cyclic group *C* and a finite group *P* whose order is a power of *p*. We call *G* hyperelementary if *G* is *p*-hyperelementary for some prime *p*.

If *G* is finite, this reduces to the usual definition. Note that for a finite *p*-hyperelementary group *G* one can arrange that the order of the finite cyclic group *C* appearing in the extension  $1 \rightarrow C \rightarrow G \rightarrow P \rightarrow 1$  for a finite *p*-group *P* is prime to *p*. Subgroups and quotient groups of *p*-hyperelementary groups are *p*-hyperelementary again. For a group *G* and a prime *p* let  $\mathcal{HE}_p$  and  $\mathcal{HE}$  respectively be the class of (possibly infinite) subgroups that are *p*-hyperelementary or hyperelementary respectively.

The following result is taken from [72, Theorem 8.2].

**Theorem 13.46 (Hyperelementary induction).** Let G be a group and let  $\mathcal{A}$  be an additive G-category (with involution). Then both relative assembly maps

$$H_n(\iota_{\mathcal{H}\mathcal{E},\mathcal{V}\mathcal{C}\mathcal{Y}};\mathbf{K}_{\mathcal{A}}):H_n^G(E_{\mathcal{H}\mathcal{E}}(G);\mathbf{K}_{\mathcal{A}})\to H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G);\mathbf{K}_{\mathcal{A}}) \text{ and}$$
$$H_n(\iota_{\mathcal{H}\mathcal{E},\mathcal{V}\mathcal{C}\mathcal{Y}};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}):H_n^G(E_{\mathcal{H}\mathcal{E}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})\to H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})$$

induced by the up to G-homotopy unique G-map  $\iota_{\mathcal{HE},\mathcal{VCY}}; E_{\mathcal{HE}}(G) \to E_{\mathcal{VCY}}(G)$ are bijective for all  $n \in \mathbb{Z}$ .

## 13.10.1 Reducing the Family of Subgroups for the Farrell-Jones Conjecture for *K*-Theory

**Theorem 13.47 (Passage from**  $\mathcal{VCY}_I$  to  $\mathcal{VCY}$  for K-theory). Let G be a group.

(i) Let A be an additive G-category. Then the relative assembly map

$$H_n^G(\iota_{\mathcal{VC}\mathcal{Y}_I\subseteq\mathcal{VC}\mathcal{Y}};\mathbf{K}_{\mathcal{A}}):H_n^G(E_{\mathcal{VC}\mathcal{Y}_I}(G);\mathbf{K}_{\mathcal{A}})\to H_n(E_{\mathcal{VC}\mathcal{Y}}(G);\mathbf{K}_{\mathcal{A}})$$

*is bijective for all*  $n \in \mathbb{Z}$ *;* 

(ii) Let C be a right exact G- $\infty$ -category. Then the relative assembly map

 $H_n^G(\iota_{\mathcal{VC}\mathcal{Y}_l\subseteq\mathcal{VC}\mathcal{Y}};\mathbf{K}_{\mathcal{C}}):H_n^G(E_{\mathcal{VC}\mathcal{Y}_l}(G);\mathbf{K}_{\mathcal{C}})\to H_n(E_{\mathcal{VC}\mathcal{Y}}(G);\mathbf{K}_{\mathcal{C}})$ 

is bijective for all  $n \in \mathbb{Z}$ ;

*Proof.* (i) See [285, Remark 1.6].

(ii) The argument for assertion (i) goes through, since the *K*-theoretic Farrell-Jones Conjecture 13.23 with coefficients in higher *G*-categories holds for finitely  $\mathcal{F}$ -amenable groups, actually for finitely homotopy  $\mathcal{F}$ -amenable groups, see [185, Theorem 5.1].

**Theorem 13.48 (Passage from**  $\mathcal{HE}_I$  **to**  $\mathcal{VCY}$  **for** *K*-theory and additive *G*-categories as coefficients). Let G be a group and  $\mathcal{A}$  be an additive G-category. Let  $\mathcal{HE}_I$  be the family of subgroups of G given by the intersection  $\mathcal{VCY}_I \cap \mathcal{HE}$ .

Then the relative assembly map

$$H_n^G(\iota_{\mathcal{H}\mathcal{E}_I \subseteq \mathcal{VC}\mathcal{Y}}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(\mathcal{E}_{\mathcal{H}\mathcal{E}_I}(G); \mathbf{K}_{\mathcal{A}}) \to H_n(\mathcal{E}_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_{\mathcal{A}})$$

*is bijective for all*  $n \in \mathbb{Z}$ *.* 

*Proof.* This follows from Theorem 13.46, Theorem 13.47 (ii), Theorem 15.9 (ii), and Lemma 15.14.  $\Box$ 

Theorem 13.48 implies that we get equivalent conjectures if we replace in Conjectures 13.1, 13.2, and 13.11 the family  $\mathcal{VCY}$  by the smaller family  $\mathcal{HE}_I$ .

**Exercise 13.49.** Fix a prime *p*. Show that an infinite subgroup  $H \subset G$  belongs to  $\mathcal{HE}_p \cap \mathcal{VCY}_I$  if and only if *H* is isomorphic to  $P \rtimes_{\phi} \mathbb{Z}$  for some finite *p*-group *P* and an automorphism  $\phi: P \to P$  whose order is a power of *p*.

**Exercise 13.50.** Let *p* be a prime. Let *G* be an infinite virtually cyclic group of type I that is *p*-hyperelementary. Let *R* be a regular ring.

Show that the map induced by the projection pr:  $E_{\mathcal{FIN}}(G) \to G/G$ 

$$H_n^{\mathcal{G}}(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^{\mathcal{G}}(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$  after applying  $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ .

13.10 Reducing the Family of Subgroups for the Farrell-Jones Conjecture

**Theorem 13.51 (Reduction to the family**  $\mathcal{FIN}$  **for algebraic** *K***-theory with regular rings as coefficients).** Let *G* be a group and let *R* be a regular ring coming with a homomorphism  $G \rightarrow \operatorname{aut}(R)$ . Let  $\mathcal{P}(G, R)$  be the set of primes which are not invertible in *R* and for which *G* contains an element of order *p*.

*Then for all*  $m \in \mathbb{Z}$  *the assembly map* 

$$H_m^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_m^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is an  $\mathcal{P}(G, R)$ -isomorphism, i.e., it becomes an isomorphism after inverting all primes in  $\mathcal{P}(G, R)$ .

*Proof.* See [666, Theorem 1.2]. Actually, additive categories with coefficients are treated in [666, Theorem 9.1].

**Exercise 13.52.** Let *G* be a group and let *R* be a regular ring. Suppose that  $\mathbb{Q} \subseteq R$  or that *G* is torsionfree.

Then for all  $m \in \mathbb{Z}$  the assembly map

$$H_m^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_m^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is an isomorphism.

One can reduce the families by extending the classical induction theorems for finite groups due to Dress to our setting. This is carried out in detail in [76]. There only rings as coefficients are treated but the proofs carry over to the setting of additive *G*-categories. For instance, for *K*-theory one has to extend the relevant pairing of the Swan group for group rings to additive categories. We leave the details to the reader and just record some results. Recall that  $\mathcal{FCY}$  is the family of finite cyclic subgroups.

**Theorem 13.53 (Reductions to families contained in**  $\mathcal{FIN}$  for algebraic *K*-theory with rings as coefficients). Let *G* be a group and *R* be a ring.

(i) Then the relative assembly map

$$H_n^G(\iota_{(\mathcal{H}\mathcal{E}\cap\mathcal{FIN})\subseteq\mathcal{FIN}};\mathbf{K}_R):H_n^G(E_{\mathcal{H}\mathcal{E}\cap\mathcal{FIN}}(G);\mathbf{K}_R)\xrightarrow{\cong} H_n(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$$

is bijective for all  $n \in \mathbb{Z}$ ;

(ii) Let p be a prime. Then the relative assembly map

$$H_n^G(\iota_{(\mathcal{H}\mathcal{E}_p\cap\mathcal{F}I\mathcal{N})\subseteq\mathcal{F}I\mathcal{N}};\mathbf{K}_R):H_n^G(\mathcal{E}_{\mathcal{H}\mathcal{E}_p\cap\mathcal{F}I\mathcal{N}}(G);\mathbf{K}_R)$$
$$\to H_n(\mathcal{E}_{\mathcal{F}I\mathcal{N}}(G);\mathbf{K}_R)$$

*is bijective for all*  $n \in \mathbb{Z}$  *after applying*  $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} -$ ; (iii) *The relative assembly map* 

$$H_n^G(\iota_{\mathcal{FC}\mathcal{Y}\subseteq\mathcal{FIN}};\mathbf{K}_R):H_n^G(E_{\mathcal{FC}\mathcal{Y}}(G);\mathbf{K}_R)\to H_n(E_{\mathcal{FIN}}(G);\mathbf{K}_R)$$

*is bijective for all*  $n \in \mathbb{Z}$  *after applying*  $\mathbb{Q} \otimes_{\mathbb{Z}} -$ *.* 

*Proof.* By the Transitivity Principle, see Theorem 15.13, it suffices to prove the assertions only in the special case where *G* is finite and in particular  $H_n(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$  reduces to  $K_n(RG)$ . Then the claim follows from [76, Theorem 2.9 and Lemma 4.1].

Note that in Theorem 13.53 we consider only rings with trivial G-action. It is conceivable that it carries over to twisted group rings and, more generally, to additive G-categories, but we have not checked the details of a proof of this claim.

Next we state and prove the following results, which will be needed for the proof of Theorem 13.65 (v).

**Lemma 13.54.** Consider a ring R, a group G, and  $m \in \mathbb{Z}$ . Suppose that, for every finite group H and every group automorphism  $\phi \colon H \xrightarrow{\cong} H$  with the property that the semidirect product  $H \rtimes_{\Phi} \mathbb{Z}$  is isomorphic to a subgroup of G, and every  $n \in \mathbb{Z}$ ,  $n \ge 0$ , the assembly map

$$H_m^{H \rtimes_{\phi} \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_m^{H \rtimes_{\phi} \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]}) = K_m((R[\mathbb{Z}^n])[H \rtimes_{\phi} \mathbb{Z}])$$

is an isomorphism where we consider the  $\mathbb{Z}$ -CW-complex  $E\mathbb{Z}$  as a  $H \rtimes_{\phi} \mathbb{Z}$ -CWcomplex by restriction with the projection  $H \rtimes_{\phi} \mathbb{Z} \to \mathbb{Z}$ .

Then the canonical map

$$H_i^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \xrightarrow{\cong} H_i^G(E_{\mathcal{VCY}}(G), \mathbf{K}_R)$$

is bijective for  $i \leq m$ .

*Proof.* Theorem 13.47 implies that for  $i \in \mathbb{Z}$  the map

$$H_i^G(E_{\mathcal{VCY}_I}(H \rtimes_{\phi} \mathbb{Z}), \mathbf{K}_R) \to H_i^G(E_{\mathcal{VCY}}(H \rtimes_{\phi} \mathbb{Z}), \mathbf{K}_R)$$

is bijective. Hence it suffices to show that, for  $i \in \mathbb{Z}$  with  $i \leq m$ , the canonical map

$$H_i^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \xrightarrow{\cong} H_i^G(E_{\mathcal{VCY}}(G), \mathbf{K}_R)$$

is bijective. Thanks to the Transitivity Principle appearing in Theorem 15.12, this has only to be done in the special case where G is a virtually cyclic group of type I.

Consider any finite group *H* and any group automorphism  $\phi: H \xrightarrow{\cong} H$ . Since  $E\mathbb{Z}$  with the  $H \rtimes_{\phi} \mathbb{Z}$  action coming from restriction with the projection  $H \rtimes_{\phi} \mathbb{Z} \to \mathbb{Z}$  is a model for  $E_{\mathcal{FIN}}(H \times_{\phi} \mathbb{Z})$  and  $\{\bullet\}$  is a model for  $E_{\mathcal{VCY}_I}(H \times_{\phi} \mathbb{Z})$ , it remains to show that the assembly map

$$H_i^{H \rtimes_{\phi} \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to H_i^{H \rtimes_{\phi} \mathbb{Z}}(\{\bullet\}; \mathbf{K}_R) = K_i(R[H \rtimes_{\phi} \mathbb{Z}])$$

is bijective for  $i \le m$ . This will be achieved by proving inductively for n = 0, 1, 2, ...that this map is bijective for  $m - n \le i \le m$  provided that  $H_m^{H \rtimes_{\phi} \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_m^{H \rtimes_{\phi} \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]})$  is bijective. The induction beginning n = 0 is trivial. The induction step from (n - 1) to n is done as follows. The Bass-Heller-Swan decomposition for the ring  $R[\mathbb{Z}^{n-1}]$  can be implemented on the spectrum level, see for instance [684, Theorem 4.2], and yields because of the identity  $(R[\mathbb{Z}^{n-1}])[\mathbb{Z}] = R[\mathbb{Z}^n]$  for every  $H \rtimes_{\phi} \mathbb{Z}$ -*CW*-complex *X* and every  $i \in \mathbb{Z}$  an isomorphism, natural in *X*,

$$\begin{split} H_m^{H \rtimes_{\phi} \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \oplus H_{m-1}^{H \rtimes_{\phi} \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \oplus H_m^{H \rtimes_{\phi} \mathbb{Z}}(X; \mathbf{N} \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \\ \oplus H_m^{H \rtimes_{\phi} \mathbb{Z}}(X; \mathbf{N} \mathbf{K}_{R[\mathbb{Z}^{n-1}]}) \xrightarrow{\cong} H_m^{H \rtimes_{\phi} \mathbb{Z}}(X; \mathbf{K}_{R[\mathbb{Z}^n]}). \end{split}$$

Since a direct sum of isomorphisms is again an isomorphism and we can apply the latter isomorphism to  $X = E\mathbb{Z}$  and  $X = \{\bullet\}$ , the map

$$H_k^{H\rtimes_{\phi}\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{R[\mathbb{Z}^{n-1}]})\to H_k^{H\rtimes_{\phi}\mathbb{Z}}(E\{\bullet\};\mathbf{K}_{R[\mathbb{Z}^{n-1}]})$$

is bijective for k = m - 1, m. Now the induction hypothesis implies that

$$H_i^{H \rtimes_{\phi} \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to H_i^{H \rtimes_{\phi} \mathbb{Z}}(\{\bullet\}; \mathbf{K}_R)$$

is bijective for  $m - n \le i \le m$ . This finishes the proof of Lemma 4.14.

Consider a ring *R* together with a ring automorphism  $\Psi: R \xrightarrow{\cong} R$ . We can think of  $\Psi$  as a group homomorphism  $\Psi: \mathbb{Z} \to \operatorname{aut}(R)$ . For a subgroup  $L \subseteq \mathbb{Z}$ , let  $\mathbf{K}(R_{\psi|_L}[L])$  be the non-connective algebraic *K*-theory spectrum of the  $\Psi|_L$ -twisted group ring of *L* with coefficient in *R* for the group homomorphism  $\Psi|_L: L \to \operatorname{aut}(R)$ . We obtain a covariant  $\operatorname{Or}(\mathbb{Z})$ -spectrum  $\mathbf{K}_{R,\Psi}$  by sending  $\mathbb{Z}/L$  to  $\mathbf{K}(R_{\Psi|_L}[L])$ . Note that for two subgroups  $L, L' \subseteq \mathbb{Z}$  the set  $\operatorname{mor}_{\operatorname{Or}(\mathbb{Z})}(\mathbb{Z}/L, \mathbb{Z}/L')$  is empty if  $L \nsubseteq L'$ , and consists of precisely one element, the canonical projection  $\mathbb{Z}/L \to \mathbb{Z}/L'$  if  $L \subseteq L'$ . In the case  $L \subseteq L'$  the functor  $\mathbf{K}_{R,\Psi}$  sends this morphism to the map of spectra induced by the inclusion of rings  $R_{\Psi|_L}[L] \to R_{\Psi|_{L'}}[L']$ .

**Lemma 13.55.** Let R be a regular ring and  $\Psi \colon R \to R$  be a ring automorphism. Then the map

$$H_m^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{R,\Psi})\to H_m^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_{R,\Psi})=K_m(R_{\Psi}[\mathbb{Z}])$$

is an isomorphisms for all  $m \in \mathbb{Z}$ .

*Proof.* This is a special case of Theorem 13.51 but we describe as an illustration a more elementary proof.

There is a twisted Bass-Heller-Swan decomposition for non-negative *K*-theory, see [686, Theorem 0.1], which reduces to the desired isomorphism if the twisted Nil terms  $NK_m(R, \Psi)$  vanish for  $m \in \mathbb{Z}$ . By inspecting the definition of the non-connective *K*-theory spectrum of [684] one sees that it suffices to show the bijectivity

$$H_m^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{R[\mathbb{Z}^n]},\Psi[\mathbb{Z}^n]) \to H_m^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_{R[\mathbb{Z}^n]},\Psi[\mathbb{Z}^n]) = K_m((R[\mathbb{Z}^n])_{\Psi[\mathbb{Z}^n]}[\mathbb{Z}])$$

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for all  $n, m \in \mathbb{Z}$  with  $m \ge 1$  and  $n \ge 0$ . Since *R* is regular, Theorem 3.80 (ii) shows that  $R[\mathbb{Z}^n]$  is regular for every  $n \ge 0$ . Hence it suffices to prove Lemma 13.55 only for  $m \ge 1$ . This has already be done by Waldhausen [974, Theorem 4 on page 138 and the Remark on page 216]. One may also refer to [436, Remark on page 362].

One may also refer for the proof of Lemma 13.55 to [85, Theorem 7.8 and Theorem 10.1], where more generally additive categories are treated.  $\Box$ 

Regard a group *H* together with an automorphism  $\phi: H \to H$ . Let  $p: H \rtimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$  be the projection. Then we get from the adjunction between  $p_*$  and  $p^*$ , see [280, Lemma 1.9], for every  $\mathbb{Z}$ -*CW*-complex *X* and all  $m, n \in \mathbb{Z}, n \ge 0$  an isomorphism, natural in *X* 

(13.56) 
$$H_m^{H \rtimes_{\phi} \mathbb{Z}}(p^*X; \mathbf{K}_{R[\mathbb{Z}^n]}) \xrightarrow{\cong} H_m^{\mathbb{Z}}(X; p_*\mathbf{K}_{R[\mathbb{Z}^n]}).$$

From the definitions we get

$$p_* \mathbf{K}_{R[\mathbb{Z}^n]}(\mathbb{Z}/L) = \mathbf{K}_{R[\mathbb{Z}^n]}((H \rtimes_{\phi} \mathbb{Z})/p^{-1}(L)) = \mathbf{K}(R[\mathbb{Z}^n][H \rtimes_{\phi|_L} L])$$

for any object  $\mathbb{Z}/L$  in  $Or(\mathbb{Z})$ . Let  $\Phi: RH \to RH$  be the ring automorphism induced by  $\phi$ . It yields a ring automorphism  $\Phi[\mathbb{Z}^n]: RH[\mathbb{Z}^n] \to RH[\mathbb{Z}^n]$ . We have defined a covariant  $Or(\mathbb{Z})$ -spectrum  $\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}$  before Lemma 13.55, just take  $\Psi = \Phi[\mathbb{Z}^n]$ . There is a weak equivalence of covariant  $Or(\mathbb{Z})$ -spectra

$$\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]} \xrightarrow{\cong} p_* \mathbf{K}_{R[\mathbb{Z}^n]}$$

coming from the identification  $R[H][\mathbb{Z}^n]_{\Phi|_L[\mathbb{Z}^n]}[L] = R[\mathbb{Z}^n][H \rtimes_{\phi|_L} L]$ . This implies using [280, Theorem 3.11] that the next lemma is true.

**Lemma 13.57.** We get for every  $\mathbb{Z}$ -*CW*-complex X and all  $m, n \in \mathbb{Z}, n \ge 0$  an isomorphism, natural in X

$$H_m^{\mathbb{Z}}(X; \mathbf{K}_{RH[\mathbb{Z}^n], \Phi[\mathbb{Z}^n]}) \xrightarrow{\cong} H_m^{H \times_{\phi} \mathbb{Z}}(p^*X; \mathbf{K}_{R[\mathbb{Z}^n]}).$$

**Lemma 13.58.** Let *H* be a finite group and let  $\phi$ :  $H \xrightarrow{\cong} H$  be an automorphism. Let *R* be an Artinian ring. Then the map

$$H_0^{H \rtimes_{\phi} \mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{R[\mathbb{Z}^n]}) \to H_0^{H \rtimes_{\phi} \mathbb{Z}}(\{\bullet\}; \mathbf{K}_{R[\mathbb{Z}^n]}) = K_0((R[\mathbb{Z}^n])[H \rtimes_{\phi} \mathbb{Z}])$$

is an isomorphisms for all  $n \in \mathbb{Z}, n \ge 0$ .

Proof. We conclude from Lemma 13.57 that it remains to show that the map

$$H_0^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) \to H_0^{\mathbb{Z}}(\{\bullet\};\mathbf{K}_{RH[\mathbb{Z}^n],\Phi[\mathbb{Z}^n]}) = K_0(RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}])$$

is bijective for all  $n \ge 1$ .

Denote by  $J \subseteq RH$  the Jacobson radical of *RH*. Since *RH* is Artinian, *J* is nilpotent, i.e., there exists a natural number *m* with  $J^m = \{0\}$ , see [610, Theorem 4.12 on page 56]. The ring *RH*/*J* is a semisimple Artinian ring, see [610, Definition 20.1 on page 311 and (20.3) on page 312], and in particular regular.

The ring automorphism  $\Phi: RH \to RH$  induced by  $\phi$  obviously satisfies  $\Phi(J) = J$ and hence induces a ring automorphism  $\overline{\Phi}: RH/J \to RH/J$ . Hence we get a commutative diagram induced by the projection  $RH \to RH/J$ .

(13.59) 
$$\begin{array}{ccc} H_{0}^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{RH[\mathbb{Z}^{n}], \Phi[\mathbb{Z}^{n}]}) &\longrightarrow K_{0}(RH[\mathbb{Z}^{n}]_{\Phi[\mathbb{Z}^{n}]}[\mathbb{Z}]) \\ & \downarrow \\ & \downarrow \\ & H_{0}^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{(RH/J)[\mathbb{Z}^{n}], \overline{\Phi}[\mathbb{Z}^{n}]}) &\longrightarrow K_{0}((RH/J)[\mathbb{Z}^{n}]_{\overline{\Phi}[\mathbb{Z}^{n}]}[\mathbb{Z}]). \end{array}$$

We have the short exact sequence of abelian groups  $0 \rightarrow J \rightarrow RH \rightarrow RH/J \rightarrow 0$ . It induces a short exact sequence of abelian groups

$$0 \to J[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]|_{J[\mathbb{Z}^n]}}[\mathbb{Z}] \to RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}] \to (RH/J)[\mathbb{Z}^n]_{\overline{\Phi}[\mathbb{Z}^n]}[\mathbb{Z}] \to 0.$$

Hence we can identify the ring  $(RH/J)[\mathbb{Z}^n]_{\overline{\Phi}[\mathbb{Z}^n]}[\mathbb{Z}]$  with the quotient of the ring  $RH[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}]$  by the ideal  $J[\mathbb{Z}^n]_{\Phi[\mathbb{Z}^n]}[\mathbb{Z}]$ . Recall that an ideal *I* in a ring is nilpotent if and only if there is a natural number *l* such that for any collection of *l* elements  $i_1, i_2, \ldots, i_l$  in *I* the product  $i_1 i_2 \cdots i_l$  vanishes. Since *J* is nilpotent, we conclude that the ideal  $J[\mathbb{Z}^n] \rtimes_{\Phi[\mathbb{Z}^n]|_{J[\mathbb{Z}^n]}}[\mathbb{Z}]$  is nilpotent. Hence the right vertical arrow in the diagram (13.59) is bijective by Lemma 2.125.

Next we show that the left vertical arrow in the diagram (13.59) is bijective. Since  $E\mathbb{Z}$  is a free  $\mathbb{Z}$ -*CW*-complex, we conclude from the equivariant Atiyah-Hirzebruch spectral sequence described in Theorem 12.48 that it suffices to show for every *i* that the map  $K_i(RH[\mathbb{Z}^n]) \rightarrow K_i((RH/J)[\mathbb{Z}^n])$  is bijective for all  $i \leq 0$ .

Since *J* is a nilpotent two-sided ideal of *RH*,  $J[\mathbb{Z}^n]$  is a nilpotent two-sided ideal of  $RH[\mathbb{Z}^n]$ . We can identify  $(RH/J)[\mathbb{Z}^n]$  with  $(RH[\mathbb{Z}^n])/(J[\mathbb{Z}^n])$ . Hence  $K_0(RH[\mathbb{Z}^n]) \rightarrow K_0((RH/J)[\mathbb{Z}^n])$  is bijective by Lemma 2.125. We conclude  $K_i(RH[\mathbb{Z}^n]) = 0$  for  $i \leq -1$  from Theorem 4.16 (ii). Since RH/J is regular and hence  $RH/J[\mathbb{Z}^n]$  is regular by Theorem 3.80 (ii), we conclude from Theorem 4.7 that  $K_i((RH/J)[\mathbb{Z}^n]) = 0$  for  $i \leq -1$ . Hence the left vertical arrow in the diagram (13.59) is bijective. The lower vertical arrow in the diagram (13.59) is bijective because of Lemma 13.55 applied to the automorphism  $\overline{\Phi}[\mathbb{Z}^n]$ . We conclude that the upper vertical arrow in the diagram (13.59) is bijective. This finishes the proof of Lemma 13.58

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#### 13.10.2 Reducing the Family of Subgroups for the Farrell-Jones Conjecture for *L*-Theory

**Theorem 13.60 (Passage from**  $\mathcal{FIN}$  to  $\mathcal{VCY}_I$  for *L*-theory). Let *G* be a group and let  $\mathcal{A}$  be an additive *G*-category with involution. Let *n* be any integer. Then

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq \mathcal{VCY}_I};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)\to H_n^G\big(E_{\mathcal{VCY}_I}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}\big)$$

#### is bijective.

*Proof.* The argument given in [654, Lemma 4.2] goes through since it is based on the Wang sequence for a semidirect product  $G \rtimes \mathbb{Z}$ , which can be generalized for additive *G*-categories with involutions as coefficients.

The last result is very useful when G does not contain virtually cyclic subgroups of type II since then one can replace in Conjectures 13.4, 13.7 and 13.19 the family  $\mathcal{VCY}$  by the family  $\mathcal{FIN}$ . (This is not true for Conjecture 13.28 since  $G \wr F$  for a finite group F may contain a virtually cyclic subgroup of type II even in the case when G does not contain a virtually cyclic subgroup of type II.)

**Exercise 13.61.** Consider the group extension  $1 \to F \to G \xrightarrow{f} \mathbb{Z}^d \to 1$  for a finite group *F*. Show that there exists a spectral sequence converging to  $L_{p+q}^{\langle -\infty \rangle}(\mathbb{Z}G)$  whose  $E^2$ -term is given by  $H_p(C_*(E\mathbb{Z}^d) \otimes_{\mathbb{Z}[\mathbb{Z}^d]} L_q^{\langle -\infty \rangle}(\mathbb{Z}F))$ , where the  $\mathbb{Z}^d$ -action on  $L_q^{\langle -\infty \rangle}(\mathbb{Z}F)$  is induced by the conjugation action of *G* on *F*.

Let p be a prime. A finite group G is called *p*-elementary if it is isomorphic to  $C \times P$  for a cyclic group C and a p-group P such that the order |C| is prime to p. Let  $\mathcal{E}_p$  be the class of finite subgroups that are p-elementary.

**Theorem 13.62 (Bijectivity of the** *L***-theoretic assembly map from**  $\mathcal{FIN}$  **to**  $\mathcal{VCY}$  **after inverting** 2). *Let G be a group and let R be a ring with involution.* 

(i) The relative assembly map

$$H_n^G\big(\iota_{\mathcal{FIN}\subseteq \mathcal{VCY}};\mathbf{L}_R^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)\to H_n^G\big(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)$$

*is bijective for all*  $n \in \mathbb{Z}$  *after applying*  $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} -$ ; (ii) *Put* 

$$\mathcal{F} = \bigcup_{p \text{ prime}, p \neq 2} \mathcal{E}_p$$

Then the relative assembly map

$$H_n^G\big(\iota_{\mathcal{F}\subseteq\mathcal{FIN}};\mathbf{L}_R^{\langle-\infty\rangle}\big)\colon H_n^G\big(E_{\mathcal{F}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)\to H_n\big(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle-\infty\rangle}\big)$$

*is bijective for all*  $n \in \mathbb{Z}$ 

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*Proof.* (i) See [673, Proposition 74 on page 747].

(ii) This is a variation of the proof of assertion (ii) of Theorem 13.53 taking [76, Section 15] into account.

Theorem 13.62 shows that Conjecture 13.4 implies Conjecture 13.6 and hence Conjecture 13.7 implies Conjecture 13.8.

Note that in Theorem 13.62 we consider only rings with involution with trivial G-action. It is conceivable that it carries over twisted group rings, but we have not checked the details of a proof of this claim. It is unclear whether it carries over to additive categories with involutions since UNil-terms have not been defined and investigated for additive categories.

## 13.11 The Full Farrell-Jones Conjecture Implies All Its Variants

Recall that the Full Farrell-Jones Conjecture 13.30 implies the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, see Remarks 13.12 and 13.20. In this section we give the proofs that Conjectures 13.2 and 13.7 imply all the variants we have stated before at various places for rings as coefficients. So the Full Farrell-Jones Conjecture is the "master" conjecture that implies all variants stated in this book for rings as coefficients.

For the reader's convenience we recall all these variants below before we show how they follow from Full Farrell-Jones Conjecture 13.30.

#### 13.11.1 List of Variants of the Farrell-Jones Conjecture

We begin with the *K*-theoretic variants.

**Conjecture 2.60** (Farrell-Jones Conjecture for  $K_0(R)$  for torsionfree *G* and regular *R*). Let *G* be a torsionfree group and let *R* be a regular ring. Then the map induced by the inclusion of the trivial group into *G* 

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective.

In particular, we get for any principal ideal domain R and torsionfree G

$$\overline{K}_0(RG) = 0.$$

Conjecture 2.67 (Farrell-Jones Conjecture for  $K_0(RG)$  for regular R with  $\mathbb{Q} \subseteq \mathbb{R}$ 

*R*). Let *R* be a regular ring with  $\mathbb{Q} \subseteq R$  and *G* be a group.

Then the homomorphism

 $I_{\mathcal{FIN}}(G,F)$ :  $\operatorname{colim}_{H\in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$ 

coming from the various inclusions of finite subgroups of G into G is a bijection.

Here is a stronger version of Conjecture 2.67.

**Conjecture 2.69.** (Farrell-Jones Conjecture for  $K_0(RG)$  for regular *R*). Let *R* be a regular ring and let *G* be a group. Let  $\mathcal{P}(G, R)$  be the set of primes which are not invertible in *R* and for which *G* contains an element of order *p*.

Then the homomorphism

$$I_{\mathcal{FIN}}(G,F)$$
: colim <sub>$H \in Sub_{\mathcal{FIN}}(G)$</sub>   $K_0(RH) \to K_0(RG)$ 

coming from the various inclusions of finite subgroups of G into G is a  $\mathcal{P}(G, R)$ -isomorphism, i.e., an isomorphism after inverting all primes in  $\mathcal{P}(G, R)$ .

**Conjecture 2.72 (Farrell-Jones Conjecture for**  $K_0(RG)$  **for an Artinian ring** *R***).** Let *G* be a group and *R* be an Artinian ring. Then the canonical map

$$I_{\mathcal{FIN}}(G,R)$$
: colim <sub>$H \in Sub_{\mathcal{FIN}}(G)$</sub>   $K_0(RH) \to K_0(RG)$ 

is an isomorphism.

**Conjecture 2.103 (The rational**  $\widetilde{K}_0(\mathbb{Z}G)$ **-to-** $\widetilde{K}_0(\mathbb{Q}G)$ **-Conjecture).** The change of ring maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Z}G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \widetilde{K}_0(\mathbb{Q}G)$$

is trivial.

**Conjecture 3.109 (Farrell-Jones Conjecture for**  $K_0(RG)$  **and**  $K_1(RG)$  **for regular** R **and torsionfree** G). Let G be a torsionfree group, and let R be a regular ring. Then the maps defined in (3.26) and (3.27)

$$A_0 \colon K_0(R) \xrightarrow{\cong} K_0(RG);$$
$$A_1 \colon G/[G,G] \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{\cong} K_1(RG),$$

are both isomorphisms. In particular, the groups  $Wh_0^R(G)$  and  $Wh_1^R(G)$ , see Definition 3.28, vanish.

Conjecture 3.110 (Farrell-Jones Conjecture for  $\widetilde{K}_0(\mathbb{Z}G)$  and Wh(G) for torsionfree *G*). Let *G* be a torsionfree group. Then  $\widetilde{K}_0(\mathbb{Z}G)$  and Wh(G) vanish.

**Conjecture 4.18 (The Farrell-Jones Conjecture for negative** *K***-theory and regular coefficient rings).** Let *R* be a regular ring and *G* be a group such that for every finite subgroup  $H \subseteq G$  the element  $|H| \cdot 1_R$  of *R* is invertible in *R*. Then

$$K_n(RG) = 0$$
 for  $n \le -1$ .

Conjecture 4.20 (The Farrell-Jones Conjecture for negative *K*-theory of the ring of integers in an algebraic number field). Let R be the ring of integers in an algebraic number field. Let R be the ring of integers in an algebraic number field. Then, for every group G, we have

$$K_n(RG) = 0$$
 for  $n \le -2$ ,

and the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_{-1}(RH) \xrightarrow{\cong} K_{-1}(RG)$$

is an isomorphism.

Conjecture 4.21 (The Farrell-Jones Conjecture for negative K-theory and Artinian rings as coefficient rings). Let G be a group and let R be an Artinian ring. Then

$$K_n(RG) = 0$$
 for  $n \le -1$ 

**Conjecture 5.22** (Farrell-Jones Conjecture for  $Wh_2(G)$  for torsionfree *G*). Let *G* be a torsionfree group. Then  $Wh_2(G)$  vanishes.

Conjecture 6.53 (Farrell-Jones Conjecture for *K*-theory for torsionfree groups and regular rings). Let G be a torsionfree group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for  $n \in \mathbb{Z}$ .

Conjecture 6.59 (Nil-groups for regular rings and torsionfree groups). Let G be a torsionfree group and let R be a regular ring. Then

$$NK_n(RG) = 0$$
 for all  $n \in \mathbb{Z}$ .

**Conjecture 6.74 (Farrell-Jones Conjecture for homotopy** *K***-theory for torsion-free groups).** Let *G* be a torsionfree group. Then the assembly map

$$H_n(BG; \mathbf{KH}(R)) \to KH_n(RG)$$

is an isomorphism for every  $n \in \mathbb{Z}$  and every ring *R*.

Conjecture 6.76 (Comparison of algebraic K-theory and homotopy K-theory for torsionfree groups). Let R be a regular ring and let G be a torsionfree group. Then the canonical map

$$K_n(RG) \to KH_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

**Conjecture 13.1** (*K*-theoretic Farrell-Jones Conjecture with coefficients in the ring *R*). Given a group *G* and a ring *R*, we say that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in the ring *R* if the assembly map induced by the projection pr:  $E_{VCY}(G) \rightarrow G/G$ 

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$$H_n^{\mathcal{G}}(\mathrm{pr}): H_n^{\mathcal{G}}(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^{\mathcal{G}}(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

0

**Conjecture 13.2** (*K*-theoretic Farrell-Jones Conjecture with coefficients in rings). We say that the group G satisfies the *K*-theoretic Farrell-Jones Conjecture with coefficients in rings if the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in R holds for every ring R.

Next we list the L-theoretic variants.

Conjecture 9.114 (Farrell-Jones Conjecture for *L*-theory for torsionfree groups).

Let G be a torsionfree group. Let R be any ring with involution.

Then the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for all  $n \in \mathbb{Z}$ .

**Conjecture 13.4** (*L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R*). Given a group *G* and ring with involution *R*, we say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* if the assembly map given by the projection  $E_{\mathcal{VC},\mathcal{V}}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

**Conjecture 13.6** (*L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* after inverting 2). Given a group *G* and ring with involution *R*, we say that *G* satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution *R* after inverting 2 if the assembly map given by the projection  $E_{\mathcal{FIN}}(G) \rightarrow G/G$ 

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle-\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle-\infty\rangle}) = L_n^{\langle-\infty\rangle}(RG)$$

is bijective for all  $n \in \mathbb{Z}$  after inverting 2.

**Conjecture 13.7** (*L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution). A group G satisfies the *L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution if the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R holds for every ring with involution R.

**Conjecture 13.8** (*L*-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting 2). We say that a group *G* satisfies *the L-theoretic Farrell-Jones Conjecture with coefficients in rings with involution after inverting* 2 if the *L*-theoretic Farrell-Jones Conjecture 13.6 with coefficients in the ring with involution *R* after inverting 2 holds for every ring with involution *R*.

13.11 The Full Farrell-Jones Conjecture Implies All Its Variants

Finally we mention the following Novikov type conjectures.

**Conjecture 13.63** (*K*-theoretic Novikov Conjecture). A group *G* satisfies the *K*-theoretic Novikov Conjecture if the assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) = H_n^G(EG; \mathbf{K}(\mathbb{Z})) \to H_n^G(G/G; \mathbf{K}(\mathbb{Z})) = K_n(\mathbb{Z}G)$$

is rationally injective for all  $n \in \mathbb{Z}$ .

**Conjecture 13.64** (*L*-theoretic Novikov Conjecture). A group *G* satisfies the *L*-theoretic Novikov Conjecture if the assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) = H_n^G(EG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))$$
$$\to H_n^G(G/G; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) = L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is rationally injective for all  $n \in \mathbb{Z}$ .

#### 13.11.2 Proof of the Variants of the Farrell-Jones Conjecture

#### Theorem 13.65 (The Full Farrell-Jones Conjecture implies all other variants).

- (i) The Full Farrell-Jones Conjecture 13.30 implies the K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution;
- (ii) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjecture 6.53, whereas the L-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involutions implies Conjecture 13.8 and Conjecture 9.114;
- (iii) Conjecture 6.53 implies Conjectures 2.60, 3.109, and 3.110;
- (iv) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 2.67, 2.69, 2.103, and 4.18;
- (v) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 2.72 and 4.21;
- (vi) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjecture 4.20;
- (vii) Conjecture 6.53 implies Conjecture 5.22;
- (viii) Conjecture 6.53 implies Conjecture 6.59;
- (ix) The K-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings implies Conjectures 6.74 and 6.76;
- (x) *The K-theoretic Farrell-Jones Conjecture* 13.1 *with coefficients in the ring* Z *implies the K-theoretic Novikov Conjecture* 13.63;
- (xi) The L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring  $\mathbb{Z}$  implies the L-theoretic Novikov Conjecture 13.64;

(xii) The Full Farrell-Jones Conjecture 13.30 implies all other variants of the Farrell-Jones Conjecture.

*Proof.* (i) See Remarks 13.12 and 13.20.

(ii) Conjecture 13.4 implies Conjecture 13.6 by Theorem 13.62 (i). Hence Conjecture 13.7 implies Conjecture 13.8.

Next we show why Conjecture 13.2 implies Conjecture 6.53 and why Conjecture 13.7 implies Conjecture 9.114. Every torsionfree virtually cyclic group is isomorphic to  $\mathbb{Z}$  by Lemma 13.42. By the Transitivity Principle 15.13 applied to  $\mathcal{TR} \subseteq \mathcal{VCY}$  it suffices to show that the assembly maps

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) \to K_n(R\mathbb{Z});$$
  
$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R\mathbb{Z}),$$

are bijective for  $n \in \mathbb{Z}$ . This follows for *K*-theory from the Bass-Heller-Swan decomposition, see Theorem 6.16, and for *L*-theory from the Shaneson splitting, see (9.109). (iii) Since *R* is regular, the negative *K*-groups of *R* vanish by Theorem 4.7. Hence the Atiyah-Hirzebruch spectral sequence, which has  $E^2$ -term  $E_{p,q}^2 = H_p(BG; K_q(R))$  and converges to  $H_{p+q}(BG; \mathbf{K}(R))$ , is a first quadrant spectral sequence. The edge homomorphism  $H_0(BG; K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}(R))$  at (0,0) is bijective. There is an obvious identification  $H_0(BG; K_0(R)) \cong K_0(R)$ . Under this identification the edge homomorphism composed with the assembly map appearing in Conjecture 6.53 turns out to be the change of rings map  $K_0(R) \to K_0(RG)$ . Hence we conclude from Conjecture 6.53 that  $K_0(R) \to K_0(RG)$  is bijective as predicted by Conjecture 2.60. Inspecting the Atiyah-Hirzebruch spectral yields an exact sequence  $0 \to H_0(BG; K_1(R)) \to H_1(BG; K_1(R)) \to H_1(BG; K_0(R)) \to 0$ . Under the obvious identification  $H_0(BG; K_1(R)) = K_1(R)$  the composite of  $H_0(BG; K_1(R)) \to H_1(BG; K(R)) = K_1(R)$  the composite of  $H_1(BG; K_0(R)) = G/[G, G] \otimes K_0(R)$ , we obtain an exact sequence

$$0 \to K_1(R) \to K_1(RG) \to G/[G,G] \otimes K_0(R) \to 0.$$

Next one checks that the composite of the map  $K_1(RG) \rightarrow G/[G,G] \otimes K_0(R)$ appearing in the sequence above with the map  $A_1$  appearing in Conjecture 3.109 is the obvious projection. This implies Conjecture 3.109, and hence also Conjecture 3.110. (iv) See [88, Theorem 1.5], [673, Proposition 87 on page 754] and [673, paragraph before Conjecture 79 on page 750] for the proof for Conjectures 2.67, 2.103, and 4.18. The proof of Conjecture 2.69 is analogous if one uses Theorem 13.51.

(v) We conclude from Lemma 13.54 and Lemma 13.58 that the assembly map  $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \to K_n(RG)$  is an isomorphism for  $n \le 0$ . We have  $K_i(RH) = 0$  for every finite group H and every  $i \le -1$  by Theorem 4.16 (ii). We conclude from the equivariant Atiyah-Hirzebruch spectral sequence described in Theorem 12.48 that  $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) = 0$  holds for  $n \le -1$  and that  $H_0^{H \rtimes_{\phi} \mathbb{Z}}(E_{\mathcal{FIN}}(H \times_{\phi} \mathbb{Z}), \mathbf{K}_R)$  is the 0-th Bredon homology of  $E_{\mathcal{FIN}}(H \times_{\phi} \mathbb{Z})$  with coefficients in the covariant

13.12 Summary of the Applications of the Farrell-Jones Conjecture

functor  $Or(G) \to \mathbb{Z}$ -MOD sending G/K to  $K_n(RK)$ . This 0-th Bredon homology can be identified with  $\operatorname{colim}_{G/H \in Or_{\mathcal{FIN}}(G)} K_0(RH)$ . Under this identification the bijective assembly map  $H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}_R) \to K_n(RG)$  becomes the canonical map  $\operatorname{colim}_{G/H \in Or_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$ .

(vi) See [673, page 749]. The proof goes through if we replace  $\mathbb{Z}$  by the ring *R* of integers in an algebraic number field since the results appearing in [368] for  $\mathbb{Z}$  have been extended to *R* by Juan-Pineda [530].

(vii) We conclude from [634] that the second Whitehead group can be identified with the cokernel of the assembly map

$$H_2(\mathrm{pr}; \mathbf{K}_R) \colon H_2^G(EG; \mathbf{K}_R) = H_2(BG; \mathbf{K}(\mathbb{Z})) \to H_2^G(EG; \mathbf{K}_R) = K_2(\mathbb{Z}G).$$

(viii) We conclude from Theorem 3.80 that R[t] is regular. We have the obvious commutative diagram

whose horizontal arrows are bijective by the assumption that Conjecture 13.2 holds and whose left vertical arrow is bijective since  $K_n(R[t]) \rightarrow K_n(R)$  is bijective for all  $n \in \mathbb{Z}$  by Theorem 6.16 (ii). Hence the right vertical arrow is bijective, which implies by definition  $NK_n(RG) = 0$ .

(ix) This follows from [75, Theorem 8.4 and Remark 8.6].

(x) This follows from Theorem 13.36 and Lemma 13.38.

(xi) The *L*-theoretic Novikov Conjecture 13.64 follows from the *L*-theoretic Farrell-Jones Conjecture 13.4 because of Theorem 13.37 and Lemma 13.38.

(xii) The Full Farrell-Jones Conjecture 13.30 implies Conjectures 13.2 and 13.7, see Remarks 13.12 and 13.20. Now the claim follows from all the other assertions which we have already proved. □

## 13.12 Summary of the Applications of the Farrell-Jones Conjecture

We have discussed at various places applications and consequences of the various versions of the Farrell-Jones Conjecture. In Theorem 13.65 we have explained that the Full Farrell-Jones Conjecture 13.30 implies all of these variants of the Farrell-Jones Conjecture and hence all these applications and consequences. For the reader's convenience we list now all these applications and where they are treated in this book or in the literature.

• Wall's Finiteness Obstruction

Wall's finiteness obstruction of a connected finitely dominated *CW*-complex *X* takes values in  $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$  and vanishes if and only if *X* is homotopy equivalent to a finite *CW*-complex, see Section 2.5. For torsionfree  $\pi_1(X)$  Conjecture 2.60 predicts that  $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$  vanishes and hence *X* is always homotopy equivalent to a finite *CW*-complex, see Remark 2.61.

• Kaplansky's Idempotent Conjecture

Kaplansky's Idempotent Conjecture 2.73 predicts for an integral domain R and a torsionfree group G that all idempotents of RG are trivial. See Section 2.9.

• The Bass Conjectures

The Bass Conjecture 2.92 for fields of characteristic zero as coefficients says for a field F of characteristic zero and a group G that the Hattori-Stallings homomorphism of (2.88) induces an isomorphism

$$\operatorname{HS}_{FG} \colon K_0(FG) \otimes_{\mathbb{Z}} F \to \operatorname{class}_F(G)_f$$

This essentially generalizes character theory for finite-dimensional representations over finite groups to finitely generated projective modules over infinite groups.

The Bass Conjecture 2.99 for integral domains as coefficients predicts for a commutative integral domain R, a group G, and  $g \in G$  that for every finitely generated projective RG-module the value of its Hattori-Stallings rank  $HS_{RG}(P)$  at (g) is trivial provided that either the order |g| is infinite or that the order |g| is finite and not invertible in R.

For more information about the Bass Conjectures, we refer to Section 2.10.

Whitehead torsion

One can assign to a homotopy equivalence  $f: X \to Y$  of connected finite *CW*-complexes its Whitehead torsion  $\tau(f)$ , which takes values in the Whitehead group Wh( $\pi_1(Y)$ ), see Sections 3.3. It vanishes if and only f is a simple homotopy equivalence, see Section 3.4

An *h*-cobordism of dimension  $\geq 6$  is trivial if and only if its Whitehead torsion vanishes, see Theorem 3.47.

If the group *G* is torsionfree, then Conjecture 3.110 predicts that Wh(*G*) vanishes. Hence Conjecture 3.110 implies that a homotopy equivalence of connected finite *CW*-complexes is simple if  $\pi_1(Y)$  is torsionfree, and that every connected *h*-cobordism *W* of dimension  $\geq 6$  with torsionfree  $\pi_1(W)$  is trivial, see Remark 3.112.

• Bounded h-cobordisms

There are so-called bounded *h*-cobordisms, controlled over  $\mathbb{R}^k$ , for  $k \ge 1$ . They are trivial (for dimension  $\ge 6$ ) if and only if certain elements in negative *K*-groups  $\widetilde{K}_{1-k}(\mathbb{Z}G)$  vanish, see Section 4.3. Conjecture 4.18 predicts for a torsionfree group *G* the vanishing of  $\widetilde{K}_n(\mathbb{Z}G)$  for  $n \le 0$ .

• Pseudoisotopy and the second Whitehead group

There is a certain obstruction for pseudoisotopies to be trivial, which takes values in  $Wh_2(G)$ , see Section 5.6. Conjecture 5.22 predicts for a torsionfree group *G* the vanishing of  $Wh_2(G)$ .

- 13.12 Summary of the Applications of the Farrell-Jones Conjecture
- Whitehead spaces and pseudoisotopy spaces

One can assign to a compact manifold M its pseudoisotopy spaces  $\mathcal{P}(M)$  and  $\mathcal{P}^{\text{DIFF}}(M)$ , Whitehead spaces Wh<sup>PL</sup>(X) and Wh<sup>DIFF</sup>(X), and its A-theory A(X) in the sense of Waldhausen, see Section 7.2 and 7.3. There also exist non-connective versions. There are various relations between these spaces. The homotopy groups of A(M) are related to the K-groups  $K_n(\mathbb{Z}[\pi_1(M)])$ .

Conjecture 6.53 predicts for a torsionfree group G and a regular ring R that the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for  $n \in \mathbb{Z}$ . It implies for an aspherical closed manifold *M* for all  $n \ge 0$ , see Theorem 7.27 and Theorem 7.32,

$$\pi_{n}(\operatorname{Wh}^{\operatorname{PL}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0;$$
  

$$\pi_{n}(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0;$$
  

$$\pi_{n}(\operatorname{Wh}^{\operatorname{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(M; \mathbb{Q});$$
  

$$\pi_{n}(\mathcal{P}^{\operatorname{DIFF}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M; \mathbb{Q}).$$

#### • Automorphisms of manifolds

If Conjecture 6.53 and Conjecture 9.114 hold for the torsionfree group *G* and the ring  $R = \mathbb{Z}$ , then some rational computations of the homotopy groups of the automorphism group of an aspherical closed manifold *M* with  $G = \pi_1(M)$  can be found in Theorems 9.195 and 9.196.

Novikov Conjecture

The Novikov Conjecture 9.137 for a group G predicts the homotopy invariants of the higher signatures

(13.66) 
$$\operatorname{sign}_{\mathbf{x}}(M, u) := \langle \mathcal{L}(M) \cup u^* x, [M]_{\mathbb{O}} \rangle \in \mathbb{Q}$$

of a closed oriented manifold *M* coming with a reference map  $f: M \to BG$  for an element  $x \in \prod_{k\geq 0} H^k(BG; \mathbb{Q})$ , see Subsection 9.14.1.

For the proof that the *L*-theoretic Novikov Conjecture 13.64 for *G* implies the Novikov Conjecture 9.137 for *G*, we refer to [587, Lemma 23.2 on page 192] and [841, Proposition 6 on page 300]. Or just take a look at Remark 9.143 and use the fact that under the Chern character the assembly map

$$\operatorname{asmb}_n^G \colon \bigoplus_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Q}) \to L_n^h(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q},$$

appearing in Remark 9.143 can be identified with the assembly map appearing in the *L*-theoretic Novikov Conjecture 13.64 for G.

• Borel Conjecture

The Borel Conjecture 9.163 predicts that any aspherical closed topological manifold M is topologically rigid, i.e, if N is another aspherical closed topological manifold with  $\pi_1(M) \cong \pi_1(N)$ , then M and N are homeomorphic and any homotopy equivalence  $M \to N$  is homotopic to a homeomorphism.

Let *G* be a finitely presented torsionfree group. Suppose that it satisfies the versions of the *K*-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the *L*-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ . Then Theorem 9.171 shows that every aspherical closed topological manifold of dimension  $\geq 5$  with *G* as fundamental group is topologically rigid.

• Poincaré duality groups

Conjecture 9.183 predicts that a finitely presented group is an *n*-dimensional Poincaré duality group if and only if it is the fundamental group of an aspherical closed *n*-dimensional topological manifold.

Suppose that the torsionfree group *G* is a finitely presented Poincaré duality group of dimension  $n \ge 6$  and satisfies the versions of the *K*-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the *L*-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ . Let *X* be a Poincaré complex of dimension  $\ge 6$  with  $\pi_1(X) \cong G$ . Suppose that its Spivak normal fibration has a TOP-reduction.

Then X is homotopy equivalent to a compact homology ANR-manifold satisfying the disjoint disk property, see Theorem 9.184.

• Boundaries of hyperbolic groups

As a consequence of the Farrell-Jones Conjecture, we get Theorem 9.188, which says for a torsionfree hyperbolic group G and  $n \ge 6$  that the following statements are equivalent:

- The boundary  $\partial G$  is homeomorphic to  $S^{n-1}$ ;
- There is an aspherical closed topological manifold M such that  $G \cong \pi_1(M)$ , its universal covering  $\widetilde{M}$  is homeomorphic to  $\mathbb{R}^n$ , and the compactification of  $\widetilde{M}$  by  $\partial G$  is homeomorphic to  $D^n$ .

Moreover the aspherical closed topological manifold M appearing above is unique up to homeomorphism.

• Stable Cannon Conjecture

Let *G* be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to  $S^2$ . The Cannon Conjecture 9.191 predicts that *G* is the fundamental group of a closed hyperbolic 3-manifold.

The Cannon Conjecture 9.191 is open at the time of writing, but a stable version of the Cannon Conjecture is known to be true, see Theorem 9.192. It says that for any closed manifold N of dimension  $\geq 2$  the product  $BG \times N$  is simple homotopy equivalent to a closed manifold M which is uniquely determined by this property up to homeomorphism.

If one could take  $N = \{\bullet\}$  above, the Cannon Conjecture 9.191 would follow.

- 13.12 Summary of the Applications of the Farrell-Jones Conjecture
- Product decompositions of aspherical closed manifolds

Theorem 9.194 deals with the question when for an aspherical closed topological manifold M a given algebraic decomposition  $\pi_1(M) = G_1 \times G_2$  comes from the topological decomposition  $M = M_1 \times M_2$ . Theorem 9.194 is a consequence of the K-theoretic Farrell-Jones Conjecture stated in 3.110 and 4.20 and the version of the L-theoretic Farrell-Jones Conjecture stated in 9.114 for the ring  $R = \mathbb{Z}$ .

• Classification of manifolds homotopy equivalent to certain torus bundles over lens spaces.

The *K*-theoretic Farrell-Jones Conjecture 13.1 and the *L*-theoretic Farrell-Jones Conjecture 13.4 play a key role in the paper [283], where a classification of manifolds homotopy equivalent to certain torus bundles over lens spaces is presented. See also [988].

• Fibering manifolds

The *K*-theoretic Farrell-Jones Conjecture 13.1 and the *L*-theoretic Farrell-Jones Conjecture 13.4 play a key role in the paper [375], which considers the question of when, for an aspherical closed manifold *B* and a map  $p: M \to B$  from some closed connected manifold *M*, the map *p* is homotopic to Manifold Approximate Fibration.

• The Atiyah Conjecture

Conjecture 2.67 is related to the Atiyah Conjecture, which makes predictions about the possibly values of the  $L^2$ -Betti numbers of coverings of closed Riemannian manifolds, see Remark 2.71.

• Homotopy invariance of  $\tau^{(2)}(M)$ 

Suppose that the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring *R* with involution is rationally true for  $R = \mathbb{Z}$ , i.e., the rationalized assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for  $n \in \mathbb{Z}$ .

Then the Hirzebruch-type invariant  $\tau^{(2)}(M)$  is a homotopy invariant, see Remark 14.60.

• Homotopy invariance of the (twisted)  $L^2$ -torsion

The *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $R = \mathbb{Z}$  implies the homotopy invariance of  $L^2$ -torsion and of the  $L^2$ -torsion function, see [662, Theorem 7.5 (4)]. The twisted  $L^2$ -torsion function is related to the Thurston norm for appropriate 3-manifolds in [405].

- Vanishing of  $\kappa$ -classes for aspherical closed manifolds The vanishing of  $\kappa$ -classes for aspherical closed manifolds is analyzed in [475] using as one input the Full Farrell-Jones Conjecture 13.30.
- *Classification of 4-manifolds* Sometimes the Farrell-Jones Conjecture is needed as input in the (stable) classification of certain 4-manifolds, see for instance [455, 456, 569].

• Group actions on manifolds

Applications of the Farrell-Jones Conjecture to manifolds with group actions are given for instance in [198, 251, 252, 257, 277, 665].

## 13.13 G-Theory

Instead of considering finitely generated projective modules, one may apply the standard *K*-theory machinery to the category of finitely generated modules. This leads to the definition of the groups  $G_n(R)$  for  $n \ge 0$ . One can also define them for negative *n* using [900]. We have described  $G_0(R)$  and  $G_1(R)$  already in Definitions 2.1 and 3.1. One may ask whether versions of the Farrell-Jones Conjectures for *G*-theory instead of *K*-theory might be true. The answer is negative, as the following discussion explains.

For a finite group *H* the ring  $\mathbb{C}H$  is semisimple. Hence any finitely generated  $\mathbb{C}H$ -module is automatically projective and  $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$ . Recall that a group *G* is called *virtually poly-cyclic* if there exists a subgroup of finite index  $H \subseteq G$  together with a filtration  $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r = H$  such that  $H_{i-1}$  is normal in  $H_i$  and the quotient  $H_i/H_{i-1}$  is cyclic. More generally for all  $n \in \mathbb{Z}$  the forgetful map

$$f: K_n(\mathbb{C}G) \to G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic, since then  $\mathbb{C}G$  is regular by [880, Theorem 8.2.2 and Theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [860, Corollary 53.26 on page 293]. In particular, this applies to virtually cyclic groups and so the left-hand side of the Farrell-Jones assembly map does not see the difference between K- and G-theory if we work with complex coefficients. We obtain a commutative diagram

where, as indicated, the left-hand vertical map is an isomorphism. Conjecture 2.67, which follows from the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{C}$ , predicts that the upper horizontal arrow is an isomorphism. A *G*-theoretic analog of Conjecture 2.67 would say that the lower horizontal map is an isomorphism. There are however cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map *f* on the right is not injective or not surjective, and hence the lower vertical arrow cannot be injective or surjective.
#### 13.14 Notes

If *G* contains a non-abelian free subgroup, then the class  $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [650, Theorem 9.66 on page 364] and hence the map  $f: K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$  has an infinite kernel since  $[\mathbb{C}G]$  generates an infinite cyclic subgroup in  $K_0(\mathbb{C}G)$ . Note that Conjecture 13.1 is known for non-abelian free groups.

Conjecture 13.1 is also known for  $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$  and hence  $K_0(\mathbb{C}A)$  is countable, whereas  $G_0(\mathbb{C}A)$  is not countable [650, Example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know the answer to the following questions:

Question 13.67 (Amenability and the passage from  $K_0(\mathbb{C}G)$  to  $G_0(\mathbb{C}G)$ ). If *G* is an amenable group for which there is an upper bound on the orders of its finite subgroups, then is the forgetful map  $f: K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$  an isomorphism?

Question 13.68 (Amenability and the vanishing of  $G_0(\mathbb{C}G)$ ). If the group G is not amenable, then is  $G_0(\mathbb{C}G)$  equal to  $\{0\}$ ?

To our knowledge the answer to Question 13.68 is not even known in the special case  $G = \mathbb{Z} * \mathbb{Z}$ .

For more information about  $G_0(\mathbb{C}G)$ , we refer for instance to [650, Subsection 9.5.3].

**Exercise 13.69.** Let  $H \subseteq G$  be a subgroup of *G* possessing an epimorphism  $f: H \to \mathbb{Z}$ . Show that the class of  $\mathbb{C}[G/H]$  in  $G_0(\mathbb{C}G)$  is trivial.

#### 13.14 Notes

The original formulation of the Farrell-Jones Conjecture with rings as coefficients appears in [366, 1.6 on page 257]. Our formulation differs from the original one, but is equivalent, see Remark 15.44.

Proofs of some of the inheritance properties above are also given in [464, 878].

The inheritance properties of the Farrell-Jones Conjecture under actions of trees is discussed in [75], see also Section 6.9 and Section 15.7. The situation is much more complicated than for the Baum-Connes Conjecture 14.11 with coefficients, where the optimal result holds, see Theorem 14.31 (v) and Remark 14.35.

In the sequel we consider classes *C* of groups that are closed under taking subgroups and passing to isomorphic groups. Examples are the classes of virtually cyclic or of finite groups. Given a group *G*, let C(G) be the family of subgroups of *G* that belong to *C*. The relevant family of subgroups appearing in Conjectures 13.1, 13.2, 13.4, 13.7, 13.11, 13.19, 13.27, 13.28, and 13.30 is always given by C(G), where *C* is the class of virtually cyclic subgroups. We have proved various theorems where *C* could be chosen to be smaller, for instance to be the class of virtually cyclic groups of type I or of hyperelementary groups, see Theorems 13.46, 13.47, and 13.48. One may ask whether there is always a class  $C_{\min}$  for which such a conjecture holds for all groups *G* and which is minimal. Of course for the class of all groups

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such a conjecture will hold for trivial reasons. In the worst case  $C_{\min}$  may just be the class of all groups. A candidate for  $C_{\min}$  may be the intersection of all the classes C of groups for which the conjecture is true for all groups, but we do not know whether this intersection satisfies the conjecture for all groups, see also Section 15.15 and in particular Lemma 15.103. At least we know that the intersection of two classes of groups  $C_0$  and  $C_1$  for which one of the Conjectures 13.11, 13.19, 13.27, 13.28, and 13.30 holds for all groups, satisfies this conjecture for all groups as well. We also know for two classes of subgroups  $C \subseteq D$  that D satisfies one of the Conjectures 13.11, 13.19, 13.27, 13.28, and 13.30 for all groups if C does. These claims follow from Theorem 15.9 (ii) and (iv), Theorem 15.13 (ii) and Lemma 15.14.

Further variants of the Farrell-Jones Conjecture for other theories such as *A*-theory, topological cyclic homology and Hochschild homology, homotopy *K*-theory, and the *K*-theory of Hecke algebras of totally disconnected groups will be discussed in Sections 15.10, 15.11, 15.12, and 15.13.

A coarse version of the Farrell-Jones Conjecture is treated in [1032].

A version of the Farrell-Jones Conjecture for polyhedra is proved in [149].

## Chapter 14 The Baum-Connes Conjecture

#### **14.1 Introduction**

In this chapter we discuss the Baum-Connes Conjecture 14.9 for the topological *K*-theory of the reduced group  $C^*$ -algebra  $C^*_r(G, F)$  for  $F = \mathbb{R}, \mathbb{C}$ . It predicts that certain assembly maps

$$K_n^G(E_{\mathcal{FIN}}(G)) \to K_n(C_r^*(G,\mathbb{C}));$$
  
$$KO_n^G(E_{\mathcal{FIN}}(G)) \to KO_n(C_r^*(G,\mathbb{R})),$$

are bijective for all  $n \in \mathbb{Z}$ . The target is the topological *K*-theory of  $C_r^*(G, F)$ , which one wants to understand. The source is an expression that depends only on the values of these topological *K*-groups on finite subgroups of *G* and is therefore much more accessible. The version above is often the one which is relevant in concrete applications, but there is also a more general version, the Baum-Connes Conjecture 14.11 with coefficients, where one allows coefficients in a *G*-*C*<sup>\*</sup>-algebra. Note that in contrast to the Full Farrell-Jones Conjecture 13.30 it suffices to consider finite subgroups instead of virtually cyclic subgroups.

A status report of the Baum-Connes Conjecture 14.9 and its version 14.11 with coefficients will be given in Section 16.4.

The main point about the Baum-Connes Conjecture 14.9 is that it implies a great variety of other prominent conjectures such as the ones due to Kadison and Novikov, and leads to very deep and interesting results about manifolds and  $C^*$ -algebras, as we will record and explain in Section 14.8.

Variants of the Baum-Connes Conjecture 14.9 and its versions 14.11 with coefficient are presented in Section 14.5.

We will discuss the inheritance properties of the Baum-Connes Conjecture 14.11 with coefficients in Section 14.6.

We have tried to keep this chapter independent of the other chapters as much as possible, so that one may start reading directly here.

#### 14.2 The Analytic Version of the Baum-Connes Assembly Map

Let *A* be a *G*-*C*<sup>\*</sup>-algebra over  $F = \mathbb{R}, \mathbb{C}$ . Denote by  $A \rtimes_r G$  the *C*<sup>\*</sup>-algebra over *F* given by the reduced crossed product, see [802, 7.7.4 on page 262]. If *A* is  $\mathbb{R}$  or  $\mathbb{C}$  with the trivial *G*-action, this is the reduced real or complex reduced group

 $C^*$ -algebra  $C_r^*(G, \mathbb{R})$  or  $C_r^*(G, \mathbb{C})$ , see Subsection 10.3.1. Denote by  $K_n(A \rtimes_r G)$  and  $KO(A \rtimes_r G)$  their topological *K*-theory, as introduced in Subsection 10.3.2.

Let *X* be a proper *G*-*CW*-complex. Denote by  $K^G_*(X; A)$  and  $KO^G_*(X; A)$  the complex and real equivariant topological *K*-theory of *X* with coefficients in *A*, see Section 10.6. Note that  $K^G_*(-; A)$  and  $KO^G_*(-; A)$  are *G*-homology theories in the sense of Definition 12.1 such that  $K^G_n(G/H; A) = K_n(A \rtimes_r H)$  and  $KO^G_n(G/H; A) = KO_n(A \rtimes_r H)$  hold for any finite subgroup  $H \subseteq G$  and  $n \in \mathbb{Z}$ , provided that we consider proper *G*-*CW*-complexes only.

We want to explain the analytic Baum-Connes assembly map

(14.1) 
$$\operatorname{asmb}_{A}^{G,\mathbb{C}}(X)_{n} \colon K_{n}^{G}(X;A) \to K_{n}(A \rtimes_{r} G);$$

(14.2) 
$$\operatorname{asmb}_{A}^{G,\mathbb{R}}(X)_{n} \colon KO_{n}^{G}(X;A) \to KO_{n}(A \rtimes_{r} G)$$

We will only treat the case  $F = \mathbb{C}$ , the case  $F = \mathbb{R}$  is analogous.

We first consider the special case where X is proper and cocompact and then explain how the map extends by a colimit argument to arbitrary proper *G*-*CW*-complexes. Note that for a proper and cocompact *G*-*CW*-complex X we can identify  $K_n^G(X; A)$  with the equivariant *KK*-groups  $KK_n^G(C_0(X), A)$ , see Section 10.6.

One description is in terms of indices with values in  $C^*$ -algebras. Namely, one assigns to a Kasparov cycle representing an element in  $KK_n^G(C_0(X), A)$  its  $C^*$ -valued index in  $K_n(A \rtimes G)$  in the sense of Mishchenko-Fomenko [738], thus defining a map  $KK_n^G(C_0(X), A) \rightarrow K_n(A \rtimes G)$ , provided that X is proper and cocompact. This is the approach appearing in [109].

The other equivalent approach is based on the Kasparov product. Given a proper cocompact *G-CW*-complex *X*, one can assign to it an element  $[p_X] \in KK_0^G(\mathbb{C}, C_0(X) \rtimes_r G)$ . Now define the map (14.1) by the composite of a descent map and a map coming from the Kasparov product

$$\begin{split} \textit{KK}_n^G(C_0(X), A) & \xrightarrow{j_r^G} \textit{KK}_n(C_0(X) \rtimes_r G, A \rtimes_r G) \\ & \xrightarrow{[p_X]\widehat{\otimes}_{C_0(X) \rtimes_r G} -} \textit{KK}_n(\mathbb{C}, A \rtimes_r G) = \textit{K}_n(A \rtimes G). \end{split}$$

For some information about these two approaches and their identification, we refer to [612] in the torsionfree case and to [184, 585] in the general case.

This extends to arbitrary proper *G*-*CW*-complexes *X* by the following argument. If  $f: X \to Y$  is a *G*-map of proper cocompact *G*-*CW*-complexes, then *f* is a proper map (after forgetting the group action). Hence composition with *f* defines a homomorphism of *G*-*C*<sup>\*</sup>-algebras  $C_0(f): C_0(Y) \to C_0(X)$ . We denote by  $KK_n^G(C_0(f), \mathrm{id}_A): KK_n^G(C_0(X), A) \to KK_n^G(C_0(Y), A)$  the induced map on the equivariant *KK*-groups. It is not hard to check that  $\operatorname{asmb}^{G,\mathbb{C}}(Y)_n \circ KK_n^G(C_0(f), \mathrm{id}_A) = \operatorname{asmb}^{G,\mathbb{C}}(X)_n$  holds. We conclude by inspecting Definition (10.67) that for any proper *G*-*CW*-complex *X* the canonical map

$$\operatorname{colim}_{C \subseteq X} K_n^G(C) \xrightarrow{\cong} K_n^G(X)$$

is an isomorphism where *C* runs through the finite *G*-*CW*-subcomplexes of *X* directed by inclusion. Hence by a colimit argument over the directed systems of proper cocompact *G*-*CW*-subcomplexes the definition above for proper compact *G*-*CW*-complexes extends to the desired assembly maps (14.1) for any proper *G*-*CW*-complex *X*. Moreover, for any *G*-map of proper *G*-*CW*-complexes  $f: X \to Y$ , we obtain again by passing to the colimit a homomorphism  $K_n^G(f): K_n^G(X; A) \to K_n^G(Y; A)$  satisfying

(14.3) 
$$\operatorname{asmb}^{G,\mathbb{C}}(Y)_n \circ K_n^G(f;A) = \operatorname{asmb}^{G,\mathbb{C}}(X)_n;$$

(14.4)  $\operatorname{asmb}^{G,\mathbb{R}}(Y)_n \circ KO_n^G(f;A) = \operatorname{asmb}^{G,\mathbb{R}}(X)_n.$ 

# 14.3 The Version of the Baum-Connes Assembly Map in Terms of Spectra

There is also a version of the Baum-Connes assembly map, which is very close to the construction of the one for the Farrell-Jones Conjecture. Namely, if we apply Theorem 12.30, taking Remark 12.31 into account, to the functor

$$\mathbf{K}_{F}^{\text{TOP}}$$
: GROUPOIDS<sup>inj</sup>  $\rightarrow$  SPECTRA,

of (12.46) for  $F = \mathbb{R}$ ,  $\mathbb{C}$ , then we obtain an equivariant homology theory  $H_*^?(-; \mathbf{K}_F^{\text{TOP}})$ in the sense of Definition 12.9 such that we get for every inclusion  $H \subseteq G$  of groups natural identifications

$$H_n^G(G/H; \mathbf{K}_F^{\mathsf{TOP}}) \cong H_n^H(H/H; \mathbf{K}_F^{\mathsf{TOP}}) \cong \pi_n(\mathbf{K}_F^{\mathsf{TOP}} \circ I(H)) = K_n(C_r^*(H, F)).$$

Note that  $H_n^?(X; \mathbf{K}_F^{\text{TOP}})$  is defined for any *G*-*CW*-complex *X*, whereas the definition of  $K_n^G(X)$  and  $KO_n(X)$  in terms of *KK*-theory only makes sense for proper *G*-*CW*-complexes.

We get assembly maps induced by the projection pr:  $X \rightarrow G/G$ 

(14.5) 
$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}}) \colon H_n^G(X; \mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}}) \to H_n^G(G/G; \mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}}) = K_n(C_r^*(G, \mathbb{C}));$$
  
(14.6) 
$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}) \colon H_n^G(X; \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}) \to H_n^G(G/G; \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}) = K_n(C_r^*(G, \mathbb{R})).$$

The assembly maps (14.1) and (14.5) are identified in [280, Section 6]. Unfortunately, the proof is based on an unpublished preprint by Carlsson-Pedersen-Roe. Another proof of the identification is given in [461, Corollary 8.4] and [745].

The identification above, in the general case where one allows coefficients in a G- $C^*$ -algebra A, is carried out in [184, 585].

Consider a proper G-CW-complex X. One sometimes finds in the literature the notation

(14.7) 
$$RK_n^G(X) := \operatorname{colim}_{C \subseteq X} KK_n^G(C_0(X), \mathbb{C}),$$

where *C* runs through the finite *G*-*CW*-subcomplexes of *X* directed by inclusion. By definition and by the discussion above we get for every proper *G*-*CW*-complex *X* identifications, natural in *X*,

(14.8) 
$$RK_n^G(X) = K_n^G(X) = H_n^G(X; \mathbf{K}_{\mathbb{C}}^{\text{TOP}}),$$

and analogously in the real case.

#### 14.4 The Baum-Connes Conjecture

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Recall that a model for the *classifying space for proper G-actions* is a *G-CW*complex  $\underline{E}G = E_{\mathcal{FIN}}(G)$  such that  $\underline{E}G^H$  is non-empty and contractible for each finite subgroup  $H \subseteq G$  and empty for each infinite subgroup  $H \subseteq G$ . Two such models are *G*-homotopy equivalent. See Definition 11.18 and Theorem 11.19.

**Conjecture 14.9 (Baum-Connes Conjecture).** A group *G* satisfies the *Baum-Connes Conjecture* if the assembly maps

$$\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_r^*(G,\mathbb{C}));$$
  
$$\operatorname{asmb}^{G,\mathbb{R}}(\underline{E}G)_n \colon KO_n^G(\underline{E}G) \to KO_n(C_r^*(G,\mathbb{R})),$$

defined in (14.1) and (14.2) are bijective for all  $n \in \mathbb{Z}$  in the special case when A is  $\mathbb{C}$  or  $\mathbb{R}$ , respectively, with the trivial G-action.

**Exercise 14.10.** Show  $K_n^G(\underline{E}G) \cong \mathbb{Z}^k$  for  $k, n \in \mathbb{Z}, k \ge 1$  and  $G = \mathbb{Z} \times \mathbb{Z}/k$ .

**Conjecture 14.11 (Baum-Connes Conjecture with coefficients).** A group *G* satisfies the *Baum-Connes Conjecture with coefficients* if the assembly maps

$$\operatorname{asmb}_{A}^{G,\mathbb{C}}(\underline{E}G)_{n} \colon K_{n}^{G}(\underline{E}G;A) \to K_{n}(A \rtimes_{r} G);$$
$$\operatorname{asmb}_{A}^{G,\mathbb{R}}(EG)_{n} \colon KO_{n}^{G}(EG;A) \to KO_{n}(A \rtimes_{r} G),$$

defined in (14.1) and (14.2) are bijective for all  $n \in \mathbb{Z}$  and all G- $C^*$ -algebras A over  $F = \mathbb{R}, \mathbb{C}$ .

**Remark 14.12 (Counterexample to the Baum-Connes Conjecture 14.11 with coefficients and a modified version).** We will discuss the status and further applications of the Baum-Connes Conjecture 14.11 with coefficients in Section 14.8 and 16.4, but immediately want to point out that there exist counterexamples to the version 14.11 with coefficients, see [487], but no counterexample to the Baum-Connes Conjecture 14.9 is known.

In [111] a new formulation of the Baum-Connes Conjecture with coefficients is given by considering a different crossed product for which the counterexamples mentioned above are not counterexamples anymore, see also [192], and no counterexample is known to the author's knowledge. The new version takes care

of the problem that there exist groups G together with short exact sequences of G- $C^*$ -algebras  $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$  for which the induced sequence  $0 \rightarrow I \rtimes G \rightarrow A \rtimes G \rightarrow B \rtimes G \rightarrow 0$  is not exact anymore and it is hence not clear that there exists a long exact sequence

$$\cdots \to K_n(I \rtimes G) \to K_n(A \rtimes G) \to K_n(B \rtimes G)$$
$$\to K_{n-1}(I \rtimes G) \to K_{n-1}(A \rtimes G) \to K_{n-1}(B \rtimes G) \to \cdots$$

whose existence is a consequence of the Baum-Connes Conjecture 14.11 with coefficients. The new version still has the flaw that the left-hand side of the assembly map is functorial under arbitrary group homomorphism, whereas this is unknown for the right-hand side, compare Remark 14.20.

The original source of the Baum-Connes Conjecture (with coefficients) is [109, Conjecture 3.15 on page 254].

**Remark 14.13 (The complex case implies the real case).** The complex version of the Baum-Connes Conjecture 14.9 and 14.11 automatically implies the real version, see [116, 897].

**Remark 14.14 (The torsionfree case).** There are canonical isomorphisms  $K^G_*(EG) \xrightarrow{\cong} K_*(BG)$  and  $KO^G_*(EG) \xrightarrow{\cong} KO_*(BG)$ . Suppose that *G* is torsion-free. Then *EG* is a model for <u>*EG*</u> and under the identification above the assembly map appearing in the Baum-Connes Conjecture 14.9 agrees with the one appearing in the Baum-Connes Conjecture for torsionfree groups 10.44. Hence the Baum-Connes Conjecture 14.9.

**Exercise 14.15.** Let  $f: H \to G$  be a group homomorphism of torsionfree groups. Suppose that H and G satisfy the Baum-Connes Conjecture 14.9 and the induced map on group homology  $H_n(f): H_n(H) \to H_n(G)$  is bijective for  $n \in \mathbb{Z}$ . Show that then  $K_n(C_r^*(G, \mathbb{C})) \cong K_n(C_r^*(H, \mathbb{C}))$  and  $KO_n(C_r^*(G, \mathbb{R})) \cong KO_n(C_r^*(H, \mathbb{R}))$  holds for all  $n \in \mathbb{Z}$ .

#### 14.5 Variants of the Baum-Connes Conjecture

In this section we discuss some variants of the Baum-Connes Conjecture.

#### 14.5.1 The Baum-Connes Conjecture for Maximal Group C\*-Algebras

There are also versions of the Baum-Connes assembly map using the maximal crossed product  $A \rtimes_m G$ , see [802, 7.6.5 on page 257] for a *G*-*C*<sup>\*</sup>-algebra *A* over *F* and the maximal group *C*<sup>\*</sup>-algebra  $C_m^*(G, F)$  for  $F = \mathbb{R}, \mathbb{C}$ . Namely, there are

assembly maps

(14.16) 
$$\operatorname{asmb}_{A}^{G,\mathbb{C},m}(X)_{*} \colon K_{*}^{G}(X;A) \to K_{*}(A \rtimes_{m} G);$$

(14.17) 
$$\operatorname{asmb}_{A}^{G,\mathbb{K},m}(X)_{*} \colon KO_{*}^{G}(X;A) \to KO_{*}(A \rtimes_{m} G)_{*}$$

which reduce for  $A = \mathbb{R}, \mathbb{C}$  equipped with the trivial *G*-action to assembly maps

(14.18) 
$$\operatorname{asmb}^{G,\mathbb{C},m}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_m^*(G,\mathbb{C}));$$

(14.19) 
$$\operatorname{asmb}^{G,\mathbb{R},m}(\underline{E}G)_n \colon KO_n^G(\underline{E}G) \to KO_n(C_m^*(G,\mathbb{R}))$$

In the sequel we only consider the complex case, the corresponding statements are true over  $\mathbb R$  as well.

There is always a *C*<sup>\*</sup>-homomorphism  $p: A \rtimes_m G \to A \rtimes_r G$ , and we obtain the following factorization of the Baum-Connes assembly map of (14.1)

$$\operatorname{asmb}_{A}^{G,\mathbb{C}}(X)_{*} \colon K_{n}^{G}(X;A) \xrightarrow{\operatorname{asmb}_{A}^{G,\mathbb{C},m}(X)_{*}} K_{*}(A \rtimes_{m} G) \xrightarrow{K_{*}(p)} K_{*}(A \rtimes_{r} G).$$

The Baum-Connes Conjecture 14.11 implies that the map  $\operatorname{asmb}_{A}^{G,\mathbb{C},m}(\underline{E}G)_*$  is always injective, and that it is surjective if and only if the map  $K_*(p)$  is bijective.

**Remark 14.20 (Functoriality of the Baum-Connes assembly map).** Note that the source of the assembly maps  $\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_r^*(G))$  and  $\operatorname{asmb}^{G,\mathbb{C},m}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_m^*(G))$  are functorial in G. The target  $K_n(C_m^*(G))$  is also functorial in G since  $C_m^*(G)$  is functorial in G, and the assembly map  $\operatorname{asmb}^{G,\mathbb{C},m}(\underline{E}G)_n \colon K_n^G(\underline{E}G) \to K_n(C_m^*(G))$  is natural in G.

However, it is not known whether the target  $K_n(C_r^*(G))$  is functorial in G and we have already explained in Subsection 10.3.1 that not every group homomorphism  $\alpha: G \to H$  induces a homomorphism of  $C^*$ -algebras  $C_r^*(G) \to C_r^*(H)$ . This is irritating since the Baum-Connes Conjecture 14.9 implies that  $K_n(C_r^*(G))$  is also functorial in G.

The same problem is still present in the new formulation of the Baum-Connes Conjecture with coefficients in [111].

Remark 14.21 (The Baum-Connes Conjecture does not hold in general for the maximal group  $C^*$ -algebra). It is known that the assembly map  $\operatorname{asmb}_A^{G,\mathbb{C},m}(\underline{E}G)_*$  of (14.16) is in general *not* surjective. Namely,  $K_0(p)$  is not injective if G is any infinite group with property (T), compare for instance the discussion in [536]. There are infinite groups with property (T) for which the Baum-Connes Conjecture is known, see [599] and also [918]. Hence there are counterexamples to the conjecture that  $\operatorname{asmb}_{G,\mathbb{C},m}(\underline{E}G)_n$  is surjective.

**Remark 14.22 (The Baum-Connes Conjecture for the maximal group**  $C^*$ -algebra holds for A-T-menable groups). A countable group *G* is called *K-amenable* if the map  $p: C^*_{max}(G) \to C^*_r(G)$  induces a *KK*-equivalence, see [267]. This implies in particular that the map  $K_n(p)$  above is an isomorphism for all  $n \in \mathbb{Z}$ . A-T-menable groups are *K*-amenable, see [486] and they satisfy the Baum-Connes

Conjecture 14.9, see Theorem 16.7 (ia). Hence for A-T-menable groups the assembly map  $\operatorname{asmb}_{A}^{G,\mathbb{C},m}(\underline{E}G)_*$  of (14.16) is bijective for all  $n \in \mathbb{Z}$ . This is also true for the real version of the assembly map (14.19).

#### 14.5.2 The Bost Conjecture

Some of the strongest results about the Baum-Connes Conjecture are proven using the so-called Bost Conjecture, see [601, page 798]. The Bost Conjecture is the version of the Baum-Connes Conjecture where one replaces the reduced group  $C^*$ -algebra  $C^*_r(G, F)$  by the Banach algebra  $L^1(G, F)$ . One still can define the topological *K*-theory of  $L^1(G, F)$  and the assembly map in this context.

Conjecture 14.23 (Bost Conjecture). The assembly maps

$$\operatorname{asmb}^{G,\mathbb{C},L^{1}}(\underline{E}G)_{n} \colon K_{n}^{G}(\underline{E}G) \to K_{n}(L^{1}(G,\mathbb{C}));$$
  
$$\operatorname{asmb}^{G,\mathbb{R},L^{1}}(\underline{E}G)_{n} \colon KO_{n}^{G}(\underline{E}G) \to KO_{n}(L^{1}(G,\mathbb{R})),$$

are isomorphism for all  $n \in \mathbb{Z}$ .

In the sequel we only consider the complex case, the corresponding statements are true over  $\mathbb{R}$  as well.

Again the left-hand side coincides with the left-hand side of the Baum-Connes assembly map. There is a canonical map of Banach \*-algebras  $q: L^1(G) \to C_r^*(G)$ . We obtain a factorization of the Baum-Connes assembly map appearing in the Baum-Connes Conjecture 14.9

(14.24) 
$$\operatorname{asmb}_{A}^{G,\mathbb{C}}(\underline{E}G)_{*} \colon K_{n}^{G}(\underline{E}G) \xrightarrow{\operatorname{asmb}^{G,\mathbb{C},L^{1}}(\underline{E}G)_{*}} K_{*}(L^{1}(G)) \xrightarrow{K_{*}(q)} K_{*}(C_{r}^{*}(G))$$

Every group homomorphism  $G \to H$  induces a homomorphism of Banach algebras  $L^1(G) \to L^1(H)$  and the assembly map appearing in Conjecture 14.23 is natural in G.

However, the disadvantage of  $L^1(G)$  is that indices of operators tend to take values in the topological *K*-theory of the group  $C^*$ -algebras, not in  $K_n(L^1(G))$ . Moreover the representation theory of *G* is closely related to the group  $C^*$ -algebra, whereas the relation to  $L^1(G)$  is not well understood.

There is also a version of the Bost Conjecture with coefficients in a  $C^*$ -algebra:

(14.25) 
$$\operatorname{asmb}_{A}^{G,\mathbb{C},L^{1}}(\underline{E}G)_{*} \colon K^{G}_{*}(\underline{E}G;A) \to K_{*}(A \rtimes_{L^{1}} G).$$

For more information about the Bost Conjecture 14.23, we refer for instance to [71, 601, 603, 789, 790, 918].

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#### 14.5.3 The Strong and the Integral Novikov Conjecture

We mention the following conjectures, which actually follow from the Baum-Connes Conjecture 14.9.

**Conjecture 14.26 (Strong Novikov Conjecture).** A group *G* satisfies the *Strong Novikov Conjecture* if the assembly maps appearing in (10.42) or (10.43)

$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_* \colon K_n(BG) \to K_n(C_r^*(G,\mathbb{C}));$$
$$\operatorname{asmb}^{G,\mathbb{R}}(BG)_* \colon KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})),$$

are rationally injective for all  $n \in \mathbb{Z}$ .

**Conjecture 14.27 (Integral Novikov Conjecture).** A torsionfree group *G* satisfies the *Integral Novikov Conjecture* if the assembly maps appearing in (10.42) or (10.43) are injective for all  $n \in \mathbb{Z}$ .

The assembly maps appearing in the Integral Novikov Conjecture 14.26 agree with the assembly maps appearing in the Baum-Connes Conjecture for torsionfree groups.

The Integral Novikov Conjecture only makes sense for torsionfree groups.

**Exercise 14.28.** Find a finite group *G* for which there cannot be an injective map from  $K_1(BG)$  to  $K_1(C_r^*(G))$ .

**Theorem 14.29 (The Baum-Connes Conjecture implies the Novikov Conjecture).** Given a group G, the Baum-Connes Conjecture 14.9 for G implies the Strong *Novikov Conjecture 14.26 for G and the Strong Novikov Conjecture 14.26 for G implies the Novikov Conjecture 9.137 for G.* 

*Proof.* The implication that the Baum-Connes Conjecture 14.9 implies the Strong Novikov Conjecture 14.26 follows from Lemma 13.38. For proofs that the Strong Novikov Conjecture 14.26 implies the Novikov Conjecture 9.137 we refer to Kasparov [564, § 9], [556] or Kaminker-Miller [542].

#### 14.5.4 The Coarse Baum-Connes Conjecture

We briefly explain the Coarse Baum-Connes Conjecture, a variant of the Baum-Connes Conjecture that applies to metric spaces and not only to groups. Its importance lies in the fact that isomorphism results about the Coarse Baum-Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological *K*-theory, see Theorem 16.15.

A metric X is *proper* if for each r > 0 and  $x \in X$  the closed ball of radius r centered at x is compact. Let X be a proper metric space. Let  $H_X$  a separable Hilbert space with a faithful nondegenerate \*-representation of  $C_0(X)$ . Let  $T: H_X \to H_X$  be

a bounded linear operator. Its *support* supp  $T \subset X \times X$  is defined as the complement of the set of all pairs (x, x') for which there exist functions  $\phi$  and  $\phi' \in C_0(X)$ satisfying  $\phi(x) \neq 0$ ,  $\phi'(x') \neq 0$ , and  $\phi'T\phi = 0$ . The operator *T* is said to be a *finite propagation operator* if there exists a constant  $\alpha$  such that  $d(x, x') \leq \alpha$  holds for all pairs in the support of *T*. The operator is said to be *locally compact* if  $\phi T$  and  $T\phi$  are compact for every  $\phi \in C_0(X)$ . An operator is called *pseudolocal* if  $\phi T\psi$  is a compact operator for all pairs of continuous functions  $\phi$  and  $\psi$  with compact and disjoint supports.

The *Roe algebra*  $C^*(X)$  is the operator-norm closure of the \*-algebra of all locally compact finite propagation operators on  $H_X$ . The algebra  $D^*(X)$  is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological *K*-theory of the quotient  $K_*(D^*(X)/C^*(X))$  agrees with *K*-homology  $K_{*-1}(X)$ . A metric space is called *uniformly contractible* if for every R > 0 there exists an S > R such that for every  $x \in X$  the inclusion of open balls  $B_R(x) \to B_S(x)$  is nullhomotopic. For a uniformly contractible proper metric space the coarse assembly map  $K_n(X) \to K_n(C^*(X))$  is the boundary map in the long exact sequence associated to the short exact sequence of  $C^*$ -algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0.$$

For general metric spaces one first approximates the metric space by spaces with nice local behavior, compare [854].

For simplicity we only explain the case where X is a discrete metric space. Let  $P_d(X)$  be the Rips complex for a fixed distance d, i.e., the geometric realization of the abstract simplicial complex with vertex set X where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip  $P_d(X)$  with the spherical metric, compare [1025].

A discrete metric space has *bounded geometry* if for each r > 0 there exists a natural number N(r) such that for all x the open ball of radius r centered at  $x \in X$  contains at most N(r) elements.

**Conjecture 14.30 (Coarse Baum-Connes Conjecture).** Let *X* be a discrete metric space of bounded geometry. Then for  $n \in \mathbb{Z}$  the coarse assembly map

$$\operatorname{colim}_{d\to\infty} K_n(P_d(X)) \to \operatorname{colim}_{d\to\infty} K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

A counterexample to the surjectivity part is constructed in [487, Section 6]. The injectivity part of this conjecture is false if one drops the bounded geometry hypothesis, see [313, 1026].

The Coarse Baum-Connes Conjecture for a finitely generated discrete group G (considered as a metric space) can be interpreted as a case of the Baum-Connes Conjecture 14.11 with coefficients for the group G with a certain specific choice of coefficients, see [1030].

Further information about the coarse Baum-Connes Conjecture can be found for instance in [239, 389, 410, 411, 488, 489, 491, 769, 854, 1013, 1014, 1025, 1026, 1027, 1024].

#### 14.6 Inheritance Properties of the Baum-Connes Conjecture

Similar to the Farrell-Jones Conjecture, the Baum-Connes Conjecture 14.11 with coefficients has much better inheritance properties than the Baum-Connes Conjecture 14.9. This is illustrated by the next theorem.

## Theorem 14.31 (Inheritance properties of the Baum-Connes Conjecture with coefficients).

(i) Passing to subgroups

Let  $H \subseteq G$  be an inclusion of groups. If G satisfies the Baum-Connes Conjecture 14.11 with coefficients, then H satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(ii) Group extensions

Let  $1 \to K \to G \xrightarrow{p} Q \to 1$  be an extension of groups. Suppose that for any finite subgroup  $H \subseteq Q$  the group  $p^{-1}(H)$  satisfies the Baum-Connes Conjecture 14.11 with coefficients and that the group Q satisfies the Baum-Connes Conjecture 14.11 with coefficients.

*Then G satisfies the Baum-Connes Conjecture* **14.11** *with coefficients;* (iii) Passing to finite direct products

If the groups  $G_0$  and  $G_1$  satisfy the Baum-Connes Conjecture 14.11 with coefficients, then  $G_0 \times G_1$  satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(iv) Directed unions

Let G be a union of the directed system of subgroups  $\{G_i | i \in I\}$ . If each group  $G_i$  satisfies the Baum-Connes Conjecture 14.11 with coefficients, then G satisfies the Baum-Connes Conjecture 14.11 with coefficients;

(v) Actions on trees

Suppose that G acts on a tree without inversion. Assume that the Baum-Connes Conjecture 14.11 with coefficients holds for the stabilizers of any of the vertices. Then the Baum-Connes Conjecture 14.11 with coefficients holds for G;

(vi) Amalgamated free products

Let  $G_0$  be a subgroup of  $G_1$  and  $G_2$  and G be the amalgamated free product  $G = G_1 *_{G_0} G_2$ . Suppose  $G_i$  satisfies the Baum-Connes Conjecture 14.11 with coefficients for i = 0, 1, 2.

Then G satisfies Baum-Connes Conjecture 14.11 with coefficients;

(vii) HNN extension

Let G be an HNN extension of the group H. Suppose that G satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Then G satisfies the Baum-Connes Conjecture 14.11 with coefficients.

14.6 Inheritance Properties of the Baum-Connes Conjecture

*Proof.* (i) This has been stated in [109], a proof can be found for instance in [221, Theorem 2.5].

(ii) See [786, Theorem 3.1].

(iii) This follows from assertion (ii).

(iv) See [71, Theorem 1.8 (ii)].

(v) This is proved by Oyono-Oyono [787, Theorem 1.1].

(vi) and (vii) These are special case of assertion (v).

**Exercise 14.32.** Show that the Baum-Connes Conjecture 14.11 with coefficients holds for any abelian group and any free group.

**Exercise 14.33.** Let G be the fundamental group of the orientable closed surface of genus  $g \ge 1$ . Show

$$K_n(C_r^*(G,\mathbb{C})) = \begin{cases} \mathbb{Z}^2 & n \text{ is even;} \\ \mathbb{Z}^g & n \text{ is odd.} \end{cases}$$

Remark 14.34 (The Baum-Connes Conjecture with coefficients is not compatible with colimits in general). The Baum-Connes Conjecture with coefficients is not compatible with colimits in general. This is in contrast to the Full Farrell-Jones Conjecture 13.30, see Theorem 13.32 (vi) and to the Bost Conjecture (14.25) with coefficients, see [71, Theorem 1.8 (i)]. The Baum-Connes Conjecture 14.11 with coefficients is known for hyperbolic groups, see [599, 918]. Now let *G* be a colimit of a directed system of hyperbolic groups { $G_i | i \in I$ } (whose structure maps  $G_i \rightarrow G_j$  are not injective). Suppose that the Baum-Connes Conjecture 14.11 with coefficients passes to colimits of directed systems of groups. Then the Baum-Connes Conjecture 14.11 with coefficients holds for *G* as well. However, there exists a group *G* which is a colimit of hyperbolic groups and contains appropriate expanders so that [487] applies and hence the Baum-Connes Conjecture 14.11 with coefficients does not hold for *G*. The construction of such a group is described in [39, 783].

**Remark 14.35 (The Farrell-Jones Conjecture and actions on trees).** The inheritance properties of the Baum-Connes Conjecture 14.11 with coefficients for actions on trees, see Theorem 14.31 (v), is very useful. It is not known to hold for the Full Farrell-Jones Conjecture 13.30. The main reason is that in the Baum-Connes setting the family  $\mathcal{FIN}$  suffices, whereas in the Farrell-Jones setting we have to use the family  $\mathcal{VCY}$ , since in the Farrell-Jones setting Nil-phenomena occur which are not present in the Baum-Connes setting can be found in [75]. Alternatively, one uses actions on trees to compute  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$ , see Section 15.7, and treats the relative group  $H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R)$  separately, for which the results of Section 13.10 are very useful. Thanks to the splitting results of Section 13.8 one can put these two computations together to get a full description of  $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$ . The analogous remark applies to *L*-theory.

**Remark 14.36 (Passing to overgroups of finite index).** It is not known whether the Baum-Connes Conjecture 14.11 with coefficients passes to overgroups of finite index. The same is true for the K- and L-theoretic Farrell-Jones Conjecture with

coefficients in additive *G*-categories (with involution), see Conjecture 13.11 and Conjecture 13.19. This was the reason why we introduced in Section 13.5 the versions "with finite wreath products". One can do the same in the Baum-Connes setting.

#### 14.7 Reducing the Family of Subgroups for the Baum-Connes Conjecture

The following result is proved in [76, Theorem 0.5] based on a Completion Theorem, see [670, Theorem 6.5] and a Universal Coefficient Theorem, see [146, 525]. An argument for the complex case using equivariant Euler classes is given by Mislin and Matthey [709] for the complex case. It is not clear to us whether it is possible to extend the methods of [709] to the real case.

**Theorem 14.37 (Reducing the family of subgroups for the Baum-Connes Conjecture).** *For any group G the relative assembly maps* 

$$K_n^G(E_{\mathcal{FC}\mathcal{Y}}(G)) \to K_n^G(E_{\mathcal{FI}\mathcal{N}}(G));$$
  
$$KO_n^G(E_{\mathcal{FC}\mathcal{Y}}(G)) \to KO_n^G(E_{\mathcal{FI}\mathcal{N}}(G)),$$

are bijective for all  $n \in \mathbb{Z}$  where  $\mathcal{FCY}$  is the family of finite cyclic subgroups.

**Remark 14.38** ( $\mathcal{FCY}$  is the smallest family for the Baum-Connes Conjecture). Let *C* be a finite cyclic group and  $\mathcal{F}$  be a family of subgroups of *C*. Then the assembly map

$$K_0^C(E_{\mathcal{F}}(C)) \to K_0^C(C/C) = \operatorname{Rep}_{\mathbb{C}}(C)$$

is surjective if and only if  $\mathcal{F}$  consists of all subgroups. This follows from [651, Theorem 0.7 and Lemma 3.4] since they predict that the homomorphism induced by the various inclusions

$$\bigoplus_{D \in \mathcal{F}} \operatorname{Rep}_{\mathbb{C}}(D) \to \operatorname{Rep}_{\mathbb{C}}(C)$$

is rationally surjective and hence C must be contained in  $\mathcal{F}$ .

Let *C* be a class of groups that is closed under taking subgroups and passing to isomorphic groups. Examples are the classes of finite cyclic groups or of finite groups. Given a group *G*, let C(G) be the family of subgroups of *G* that belong to *G*. Suppose that for any group *G* the assembly map

$$K_n^G(E_{\mathcal{C}(G)}(G)) \to K_n^G(G/G)$$

is bijective. The considerations above imply that *C* has to contain all finite cyclic subgroups. So, roughly speaking,  $\mathcal{FCY}$  is the smallest family for which one can hope that the Baum-Connes Conjecture 14.9 is true for all groups.

#### 14.8 Applications of the Baum-Connes Conjecture

#### 14.8.1 The Kadison Conjecture and the Trace Conjecture for Torsionfree Groups

The Baum-Connes Conjecture for torsionfree groups 10.44 follows from the Baum-Connes Conjecture 14.9, see Remark 14.14. If the Baum-Connes Conjecture for torsionfree groups 10.44 holds for the torsionfree group G, then the following conjectures hold for G, see Subsections 10.4.1 and 10.4.2.

• *Trace Conjecture* 10.50 *for torsionfree groups* For a torsionfree group *G* the image of

$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

• *Kadison Conjecture 10.52* If G is a torsionfree group, then the only idempotent elements in  $C_r^*(G)$  are 0 and 1.

#### 14.8.2 The Novikov Conjecture and the Zero-in-the-Spectrum Conjecture

If the Baum-Connes Conjecture 14.9 holds for the group G, then the following conjecture holds for G by Theorem 14.29.

• *Strong Novikov Conjecture* 14.26 The assembly maps

$$asmb^{G,\mathbb{C}}(BG)_* \colon K_n(BG) \to K_n(C_r^*(G,\mathbb{C}));$$
$$asmb^{G,\mathbb{R}}(BG)_* \colon KO_n(BG) \to KO_n(C_r^*(G,\mathbb{R})).$$

of (10.42) and (10.43) are rationally injective for all  $n \in \mathbb{Z}$ .

The strong Novikov Conjecture 14.26 for the group G (and hence also the Baum-Connes Conjecture 14.9 for the group G) implies the next conjecture, see Subsection 10.4.3.

• Zero-in-the-Spectrum Conjecture 10.55

If  $\widetilde{M}$  is the universal covering of an aspherical closed Riemannian manifold M with fundamental group isomorphic G, then zero is in the spectrum of the minimal closure of the *p*th Laplacian on  $\widetilde{M}$  for some  $p \in \{0, 1, ..., \dim M\}$ .

Moreover, we have already shown in Theorem 14.29 that the Baum-Connes Conjecture 14.9 for the group G implies the following conjecture.

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• Novikov Conjecture 9.137
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Higher signatures over G are homotopy invariant.

#### 14.8.3 The Modified Trace Conjecture

Denote by  $\Lambda^G$  the subring of  $\mathbb{Q}$  that is obtained from  $\mathbb{Z}$  by inverting all orders |H| of finite subgroups H of G, i.e.,

(14.39) 
$$\Lambda^G = \mathbb{Z}\left[|H|^{-1} \mid H \subset G, \ |H| < \infty\right].$$

The following conjecture generalizes Conjecture 10.50 to the case where the group need no longer be torsionfree. For the standard trace see (10.48).

**Conjecture 14.40 (Trace Conjecture, modified).** Let *G* be a group. Then the image of the homomorphism induced by the standard trace

(14.41) 
$$\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

is contained in  $\Lambda^G$ .

The following result is proved in [651, Theorem 0.3].

**Theorem 14.42.** Let G be a group. Then the image of the composite

$$K_0^G(E_{\mathcal{FIN}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n \otimes_{\mathbb{Z}} \operatorname{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\operatorname{tr}_{C_r^*(G)}} \mathbb{R}$$

is  $\Lambda^G$ . Here  $\operatorname{asmb}^{G,\mathbb{C}}(\underline{E}G)_n$  is the map appearing in the Baum-Connes Conjecture 14.9. In particular, the Baum-Connes Conjecture 14.9 implies the Modified Trace Conjecture 14.40.

The original version of the Trace Conjecture due to Baum and Connes [108, page 21] makes the stronger claim that the image of  $\operatorname{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \to \mathbb{R}$  is the additive subgroup of  $\mathbb{Q}$  generated by all numbers  $\frac{1}{|H|}$  where  $H \subset G$  runs though all finite subgroups of *G*. Roy has constructed a counterexample to this version in [881] based on her article [882]. The examples of Roy do *not* contradict the Modified Trace Conjecture 14.40 or the Baum-Connes Conjecture 14.9.

**Exercise 14.43.** The *G* be a finite group. Show that the image of the trace map  $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$  is  $\{n \cdot |G|^{-1} \mid n \in \mathbb{Z}\}$ .

#### 14.8.4 The Stable Gromov-Lawson-Rosenberg Conjecture

The Stable Gromov-Lawson-Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. It is related to the real version of the Baum-Connes Conjecture 14.9.

Let  $\Omega_n^{\text{Spin}}(BG)$  be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG. Given an element  $[u: M \to BG] \in \Omega_n^{\text{Spin}}(BG)$ , we can take the  $C_r^*(G, \mathbb{R})$ -valued index of the equivariant Dirac operator associated to the G-covering  $\overline{M} \to M$  determined by u. Thus we get a homomorphism

(14.44) 
$$\operatorname{ind}_{C_r^*(G,\mathbb{R})} \colon \Omega_n^{\operatorname{Spin}}(BG) \to KO_n(C_r^*(G,\mathbb{R})).$$

A *Bott manifold* is any simply connected closed Spin-manifold *B* of dimension 8 whose  $\widehat{A}$ -genus  $\widehat{A}(B)$  is 1. We fix such a choice, the particular choice does not matter for the sequel. Note that  $\operatorname{ind}_{C_r^*(\{1\},\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$  is a generator and the product with this element induces the Bott periodicity isomorphisms  $KO_n(C_r^*(G,\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G,\mathbb{R}))$ . In particular,

(14.45) 
$$\operatorname{ind}_{C_r^*(G,\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(G,\mathbb{R})}(M \times B),$$

if we identify  $KO_n(C_r^*(G,\mathbb{R})) = KO_{n+8}(C_r^*(G,\mathbb{R}))$  via Bott periodicity.

**Conjecture 14.46 (Stable Gromov-Lawson-Rosenberg Conjecture).** Let *M* be a connected closed Spin-manifold of dimension  $n \ge 5$ . Let  $u_M : M \to B\pi_1(M)$  be the classifying map of its universal covering. Then  $M \times B^k$  carries for some integer  $k \ge 0$  a Riemannian metric with positive scalar curvature if and only if

$$\operatorname{ind}_{C_r^*(\pi_1(M),\mathbb{R})}([M, u_M]) = 0 \in KO_n(C_r^*(\pi_1(M),\mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner-Lichnerowicz formula [857]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [931, Section 3].

**Theorem 14.47 (The Baum-Connes Conjecture implies the Stable Gromov-**Lawson-Rosenberg Conjecture). If the assembly map for the real version of the Baum-Connes Conjecture 14.9 is injective for the group G, then the Stable Gromov-Lawson-Rosenberg Conjecture 14.46 is true for all closed Spin-manifolds of dimension  $\geq 5$  with  $\pi_1(M) \cong G$ .

The requirement  $\dim(M) \ge 5$  is essential in the Stable Gromov-Lawson-Rosenberg Conjecture since in dimension four new obstructions, the Seiberg-Witten invariants, occur. The unstable version of this conjecture says that M carries a

Riemannian metric with positive scalar curvature if and only if  $\operatorname{ind}_{C_r^*(\pi_1(M),\mathbb{R})}([M, u_M]) = 0$ . Schick [895] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau, see also [321]. There are counterexamples with  $\pi \cong \mathbb{Z}^4 \times \mathbb{Z}/3$ . However for appropriate  $\rho: \mathbb{Z}/3 \to \operatorname{aut}(\mathbb{Z}^4)$  the unstable version does hold for  $\pi \cong \mathbb{Z}^4 \rtimes_{\rho} \mathbb{Z}/3$  and  $\dim(M) \ge 5$ , see [282, Theorem 0.7 and Remark 0.9]. More infinite groups for which the unstable version holds are presented in [507, Theorem 6.3].

Since the Baum-Connes Conjecture 14.9 is true for finite groups (for the trivial reason that  $E_{\mathcal{FIN}}(G) = \{\bullet\}$  for finite groups *G*), Theorem 14.47 implies that the Stable Gromov-Lawson Conjecture 14.46 holds for finite fundamental groups, see also [869]. It is not known at the time of writing whether the unstable version is true for finite fundamental groups.

The index map appearing in (14.44) can be factorized as a composite

(14.48) 
$$\operatorname{ind}_{C_r^*(G,\mathbb{R})} \colon \Omega_n^{\operatorname{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{\operatorname{asmb}^{G,\mathbb{C}}(BG)_n} KO_n(C_r^*(G,\mathbb{R}))$$

where *D* sends [M, u] to the class of the *G*-equivariant Dirac operator of the *G*-manifold  $\overline{M}$  given by *u* and  $\operatorname{asmb}^{G,\mathbb{C}}(BG)_n$  is the real version of the classical assembly map. The homological Chern character defines an isomorphism

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG; \mathbb{Q}).$$

Recall that associated to *M* there is the  $\widehat{A}$ -class

(14.49) 
$$\widehat{\mathcal{A}}(M) \in \prod_{p \ge 0} H^p(M; \mathbb{Q})$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (14.48) sends the class of  $u: M \to BG$  to  $u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}})$ , i.e., the image of the Poincaré dual of  $\widehat{\mathcal{A}}(M)$  under the map induced by u in rational homology. Hence D([M, u]) = 0 if and only if  $u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}})$  vanishes. For  $x \in \prod_{k \ge 0} H^k(BG; \mathbb{Q})$ define the *higher*  $\widehat{A}$ -genus of (M, u) associated to x to be

$$(14.50) \quad \widehat{A}_x(M,u) = \langle \widehat{\mathcal{A}}(M) \cup u^* x, [M]_{\mathbb{Q}} \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]_{\mathbb{Q}}) \rangle \in \mathbb{Q}.$$

The vanishing of  $\widehat{\mathcal{A}}(M)$  is equivalent to the vanishing of all higher  $\widehat{A}$ -genera  $\widehat{A}_x(M, u)$ . The following conjecture is a weak version of the Stable Gromov-Lawson-Rosenberg Conjecture.

**Conjecture 14.51 (Homological Gromov-Lawson-Rosenberg Conjecture).** Let *G* be a group. Then for any closed Spin-manifold *M* which admits a Riemannian metric with positive scalar curvature, the  $\widehat{A}$ -genus  $\widehat{A}_x(M, u)$  vanishes for all maps  $u: M \to BG$  and elements  $x \in \prod_{k \ge 0} H^k(BG; \mathbb{Q})$ .

From the discussion above we obtain the following result.

14.8 Applications of the Baum-Connes Conjecture

Lemma 14.52. If the assembly map

 $KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to KO_n(C_r^*(G,\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$ 

is injective for all  $n \in \mathbb{Z}$ , then the Homological Gromov-Lawson-Rosenberg Conjecture 14.51 holds for G.

The following conjecture is due to Gromov-Lawson [443, page 313].

**Conjecture 14.53 (Aspherical closed manifolds carry no Riemannian metric with positive scalar curvature).** An aspherical closed manifold carries no Riemannian metric with positive scalar curvature.

Conjecture 14.53 is known to be true in dimensions 4 and 5 by Chodosh-Li-Liokumovich [241] and Gromov [442].

**Lemma 14.54.** *Let M* be an aspherical closed Spin-manifold whose fundamental group satisfies the Homological Gromov-Lawson-Rosenberg Conjecture 14.51.

Then M satisfies Conjecture 14.53, i.e., M carries no Riemannian metric with positive scalar curvature.

*Proof.* Suppose *M* carries a Riemannian metric of positive scalar curvature. Since *M* is aspherical, we can take M = BG for  $G = \pi_1(M)$  and  $f = id_G$  in Conjecture 14.51. Since  $\widehat{\mathcal{A}}(M)_0 = 1$ , we get for all  $x \in H^{\dim(M)}(M; \mathbb{Q})$  that  $\langle x, [M] \rangle = 0$  holds, a contradiction.

**Exercise 14.55.** Let  $F \to M \to S^1$  be a fiber bundle such that F is an orientable closed surface and M is a closed spin-manifold. Show that M carries a Riemannian metric with positive scalar curvature if and only if F is  $S^2$ .

The (moduli) space of metrics of positive scalar curvature of closed spin manifolds is studied in [156, 157, 264, 323, 466, 899].

#### 14.8.5 $L^2$ -Rho-Invariants and $L^2$ -Signatures

Let *M* be an orientable connected closed Riemannian manifold. Denote by  $\eta(M) \in \mathbb{R}$  the *eta-invariant* of *M* and by  $\eta^{(2)}(\widetilde{M}) \in \mathbb{R}$  the *L*<sup>2</sup>*-eta-invariant* of the  $\pi_1(M)$ -covering given by the universal covering  $\widetilde{M} \to M$ . Let  $\rho^{(2)}(M) \in \mathbb{R}$  be the  $L^2$ -*rho-invariant* that is defined to be the difference  $\eta^{(2)}(M) - \eta(M)$ . These invariants were studied by Cheeger and Gromov [237, 238]. They show that  $\rho^{(2)}(M)$  depends only on the diffeomorphism type of *M* and is in contrast to  $\eta(M)$  and  $\eta^{(2)}(\widetilde{M})$  independent of the choice of Riemannian metric on *M*. The following conjecture is taken from Mathai [708].

Conjecture 14.56 (Homotopy invariance of the  $L^2$ -Rho-invariant for torsionfree groups). If  $\pi_1(M)$  is torsionfree, then  $\rho^{(2)}(M)$  is a homotopy invariant.

**Theorem 14.57 (Homotopy Invariance of**  $\rho^{(2)}(M)$ ). Let M be an oriented connected closed manifold Mof odd dimension such that  $G = \pi_1(M)$  is torsionfree. Suppose that the assembly map  $K_0(BG) \to K_0(C^*_{\max}(G))$  for the maximal group  $C^*$ -algebra, see Subsection 14.5.1, is surjective.

Then  $\rho^{(2)}(M)$  is a homotopy invariant.

*Proof.* This is proved by Keswani [577, 578].

**Remark 14.58** ( $L^2$ -signature Theorem). Let *X* be a 4*n*-dimensional Poincaré space over  $\mathbb{Q}$ . Let  $\overline{X} \to X$  be a normal covering with torsionfree covering group *G*. Suppose that the assembly map  $K_0(BG) \to K_0(C^*_{\max}(G))$  for the maximal group  $C^*$ -algebra is surjective, see Subsection 14.5.1, or suppose that the rationalized assembly map for *L*-theory

$$H_{4n}(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_{4n}^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective. Then the following  $L^2$ -signature theorem is proved in Lück-Schick [681, Theorem 0.3]

(14.59) 
$$\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X).$$

If one drops the condition that *G* is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space *X* with a finite *G* covering  $\overline{X} \to X$  for which  $\operatorname{sign}(\overline{X}) \neq |G| \cdot \operatorname{sign}(X)$  holds, see [839, Example 22.28], [985, Corollary 5.4.1]. If *X* is a closed topological manifold, then (14.59) is true for all groups *G*, see [681, Theorem 0.2].

**Remark 14.60.** Chang-Weinberger [225] assign to an oriented connected closed (4k - 1)-dimensional manifold M a Hirzebruch-type invariant  $\tau^{(2)}(M) \in \mathbb{R}$  as follows. By a result of Hausmann [473] there is an oriented connected closed 4k-dimensional smooth manifold W with  $M = \partial W$  such that the inclusion  $\partial W \to W$  induces an injection on the fundamental groups. Define  $\tau^{(2)}(M)$  as the difference  $\operatorname{sign}^{(2)}(\widetilde{W}) - \operatorname{sign}(W)$  of the  $L^2$ -signature of the  $\pi_1(W)$ -covering given by the universal covering  $\widetilde{W} \to W$  and the signature of W. This is indeed independent of the choice of W. We conjecture that  $\rho^{(2)}(M) = \tau^{(2)}(M)$  is always true. Chang-Weinberger [225] use  $\tau^{(2)}$  to prove that, if  $\pi_1(M)$  is not torsionfree, there are infinitely many diffeomorphically distinct smooth manifolds which are tangentially simple homotopy equivalent to M, if M is an oriented connected closed (4k - 1)-dimensional manifold for  $k \geq 2$ .

If  $\pi_1(M)$  is not torsionfree, then  $\tau^{(2)}(M)$  is not a homotopy invariant. Suppose that *G* is torsionfree and the rationalized assembly map for *L*-theory

$$H_{4n}(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_{4n}^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for  $n \in \mathbb{Z}$ . Then it is conceivable that  $\tau^{(2)}(M)$  is a homotopy invariant if  $G = \pi_1(M)$ .

**Remark 14.61 (Obstructions for knots to be slice).** Cochran-Orr-Teichner give in [246] new obstructions for a knot to be slice, which are sharper than the Casson-Gordon invariants. They use  $L^2$ -signatures and the Baum-Connes Conjecture 14.9. We also refer to the survey article [245] about non-commutative geometry and knot theory.

#### **14.9 Notes**

The Baum-Connes Conjecture has also been formulated and proved for (not necessarily discrete) topological groups, see for instance [109, 112, 222, 603]. It is interesting for representation theory, see for instance [113].

The Baum-Connes assembly maps in terms of localizations of triangulated categories are considered in [514, 515, 516, 716, 717, 718].

Certain so-called Cuntz-Lie  $C^*$ -algebras, see [268, 269], were classified in [629, Corollary 1.3]. The main difficulty is to compute the topological *K*-theory of these  $C^*$ -algebras, which boils down to the computation of the topological  $C^*$ -algebra of certain crystallographic groups. This in turn leads via the Baum-Connes Conjecture to an open conjecture about group homology which was solved in the case needed for this application, see [618, 619].

Other classification results whose proof uses the Baum-Connes Conjecture 14.9, can be found in [324, Theorem 0.1].

We propose that one should also construct a Baum-Connes assembly map for the Fréchet algebra  $\mathcal{R}(G)$  associated to a group G. This will lead to the intriguing factorization of the Baum-Connes assembly map

$$K_n^G(\underline{E}G) \to K_n(\mathcal{R}(G)) \to K_n(L^1(G)) \to K_n(C_r^*(G)).$$

There is some hope that the methods of proof for the *K*-theoretic Farrell-Jones Conjecture will carry over to group Fréchet algebras. This would lead for instance to the proof of the bijectivity of  $K_n^G(\underline{E}G) \to K_n(\mathcal{R}(G))$  for (not necessarily cocompact) lattices in second countable locally compact Hausdorff groups with finitely many path components. Note that the Baum-Connes Conjecture 14.9 is open for  $SL_n(\mathbb{Z})$  for  $n \ge 3$ .

For more information about the Baum-Connes Conjecture and its applications we refer for instance to [109, 426, 484, 492, 493, 494, 601, 602, 673, 742, 807, 866, 896, 963].

## Chapter 15 The (Fibered) Meta- and Other Isomorphism Conjectures

#### **15.1 Introduction**

In this section we deal with Isomorphism Conjectures in their most general form. Namely, given a *G*-homology theory  $\mathcal{H}^G_*$ , the Meta-Isomorphism Conjecture 15.2 predicts that, for a group *G* and a family  $\mathcal{F}$  of subgroups of *G*, the map induced by the projection  $E_{\mathcal{F}}(G) \to G/G$ 

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

is bijective for all  $n \in \mathbb{Z}$ .

If we take special examples for  $\mathcal{H}^G_*$  and  $\mathcal{F}$ , then we obtain the Farrell-Jones Conjecture for a ring *R* (with involution), see Conjectures 13.1 and 13.4, and the Baum-Connes Conjecture 14.9. We will also introduce a Fibered Meta-Isomorphism Conjecture 15.8, which is more general and has much better inheritance properties, see Section 15.6. The versions of the Farrell-Jones Conjecture with coefficients in additive categories, see Conjectures 13.11 and 13.19, and the Baum-Connes Conjecture 14.11 with coefficients are automatically fibered, see Theorem 15.9, and hence have good inheritance properties.

The main tool to reduce the family of subgroups is the Transitivity Principle, which we discuss in Section 15.5.

Section 15.7 is devoted to actions on trees and their implications, such as the existence of Mayer-Vietoris sequences associated to amalgamated free products and Wang sequences associated to semidirect products with  $\mathbb{Z}$ , or more generally to HNN-extensions.

In Section 15.8 we pass to the special case where the homology theory comes from a functor from spaces to spectra which respects weak homotopy equivalences and disjoint unions, and discuss inheritance properties in this framework.

By specifying the functor from spaces to spectra, we obtain the Farrell-Jones Conjecture for Waldhausen's A-theory for pseudoisotopy and Whitehead spaces in Section 15.10. We also deal with topological Hochschild homology and cyclic homology in Section 15.11. We explain the Farrell-Jones Conjecture for homotopy K-theory in Section 15.12. The only instance where we will consider not necessarily discrete groups is the Farrell-Jones Conjecture 15.80 for the algebraic K-theory of the Hecke algebra of a totally disconnected locally compact second countable Hausdorff group.

In Section 15.14 interesting relations between these conjectures are discussed, namely, between the Farrell-Jones Conjecture for the *K*-theory of group rings, for the *A*-theory of classifying spaces of groups, and for pseudoisotopy of classifying

spaces of groups, see Subsections 15.14.1 and 15.14.2, between the Farrell-Jones Conjecture for the *K*-theory and for the topological cyclic homology of integral group rings, see Subsection 15.14.3, between the Farrell-Jones Conjecture of the *L*-theory of group rings and the Baum-Connes Conjecture for the topological *K*-theory of reduced group  $C^*$ -algebras, see Subsection 15.14.4, between the Bost Conjecture for the topological *K*-theory of group Banach algebras and the Baum-Connes Conjecture for the topological *K*-theory of reduced group  $C^*$ -algebras, see Subsection 15.14.6, and between the Farrell-Jones Conjecture for *K*-theory and the homotopy *K*-theory of group rings, see Subsection 15.14.7. We will briefly also relate the geometric Surgery Exact Sequence in the topological category to an analytic Surgery Exact Sequence in Subsection 15.14.5.

#### 15.2 The Meta-Isomorphism Conjecture

Let *G* be a (discrete) group. Let  $\mathcal{H}^G_*$  be a *G*-homology theory with values in  $\Lambda$ -modules for some commutative associative ring with unit  $\Lambda$ . Recall that it assigns to every *G*-*CW*-pair (*X*, *A*) and integer  $n \in \mathbb{Z}$  a  $\Lambda$ -module  $\mathcal{H}^G_n(X, A)$  such that the obvious generalization to *G*-*CW*-pairs of the axioms of a (non-equivariant generalized) homology theory for *CW*-complexes holds, i.e., *G*-homotopy invariance, the long exact sequence of a *G*-*CW*-pair, excision, and the disjoint union axiom are satisfied. The precise definition of a *G*-homology theory can be found in Definition 12.1 and of a *G*-*CW*-complex in Definition 11.2, see also Remark 11.3.

Recall that we have defined the notion of a family of subgroups of a group G in Definition 2.62, namely, it is a set of subgroups of G that is closed under conjugation with elements of G and passing to subgroups. Let  $\mathcal{F}$  be a family of subgroups of G. Denote by  $E_{\mathcal{F}}(G)$  a model for the classifying G-CW-complex for the family  $\mathcal{F}$  of subgroups of G, i.e., a G-CW-complex  $E_{\mathcal{F}}(G)$  whose isotropy groups belong to  $\mathcal{F}$  and for which for each  $H \in \mathcal{F}$  the H-fixed point set  $E_{\mathcal{F}}(G)^H$  is weakly contractible. Such a model always exists and is unique up to G-homotopy, see Definition 11.18 and Theorem 11.19.

The projection pr:  $E_{\mathcal{F}}(G) \to G/G$  induces for all integers  $n \in \mathbb{Z}$  a homomorphism of  $\Lambda$ -modules

(15.1) 
$$\mathcal{H}_n^G(\mathrm{pr}) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G),$$

which is sometimes called an assembly map.

**Conjecture 15.2 (Meta-Isomorphism Conjecture).** The group *G* satisfies the *Meta-Isomorphism Conjecture* with respect to the *G*-homology theory  $\mathcal{H}^G_*$  and the family  $\mathcal{F}$  of subgroups of *G* if the assembly map

$$\mathcal{H}_n(\mathrm{pr}): \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

of (15.1) is bijective for all  $n \in \mathbb{Z}$ .

#### 15.3 The Fibered Meta-Isomorphism Conjecture

If we choose  $\mathcal{F}$  to be the family  $\mathcal{ALL}$  of all subgroups, then G/G is a model for  $E_{\mathcal{ALL}}(G)$  and the Meta-Isomorphism Conjecture 15.2 is obviously true. The point is to find a family  $\mathcal{F}$  that is as small as possible. The idea of the Meta-Isomorphism Conjecture 15.2 is that one wants to compute  $\mathcal{H}_n^G(G/G)$ , which is the unknown and the interesting object, by assembling it from the values  $\mathcal{H}_n^G(G/H)$  for  $H \in \mathcal{F}$ , which are usually much more accessible since the structure of the groups H is easy. For instance,  $\mathcal{F}$  could be the family  $\mathcal{FIN}$  of finite subgroups or the family  $\mathcal{VCY}$  of virtually cyclic subgroups.

The various Isomorphism Conjectures are now obtained by specifying the *G*-homology theory  $\mathcal{H}_*^G$  and the family  $\mathcal{F}$ . For instance, the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* and the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution *R* are equivalent to the Meta-Isomorphism Conjecture 15.2 if we choose  $\mathcal{F}$  to be  $\mathcal{VCY}$  and  $\mathcal{H}_n^G$  to be  $\mathcal{H}_n^G(-; \mathbf{K}_R)$  and  $\mathcal{H}_n^G(-; \mathbf{L}_R^{\langle -\infty \rangle})$ . The Baum-Connes Conjecture 14.9 is equivalent the Meta-Isomorphism Conjecture 15.2 if we choose  $\mathcal{F}$  to be  $\mathcal{FIN}$  and  $\mathcal{H}_n^G$  to be  $\mathcal{K}_n^G(-) = \mathcal{H}_n^G(-; \mathbf{K}_{\mathbb{C}}^{\mathrm{TOP}})$  or  $\mathcal{KO}_n^G(-) = \mathcal{H}_n^G(-; \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}})$ . The analogous statement holds for the versions with coefficients in additive *G*-categories (with involutions), Conjectures 13.11, 13.19, and 14.11, and for the version with coefficients in higher *G*-categories, see Conjecture 13.23.

**Exercise 15.3.** Let  $\mathcal{H}^{?}_{*}$  be an equivariant homology theory with values in  $\Lambda$ -modules in the sense of Definition 12.9. Fix a class of groups *C* that is closed under isomorphisms, taking subgroups, and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group *G* let C(G) be the family of subgroups of *G* that belong to *C*. Then we obtain for each group *G* an assembly map induced by the projection  $E_{C(G)}(G) \rightarrow G/G$ 

$$\mathcal{H}_n^G(E_{\mathcal{C}(G)}(G)) \to \mathcal{H}_n^G(G/G).$$

By using the induction structure of  $\mathcal{H}^{?}_{*}$  explain how we can turn the source and target to be functors from the category of groups to the category of  $\Lambda$ -modules such that the assembly maps yield a natural transformation of such functors.

#### 15.3 The Fibered Meta-Isomorphism Conjecture

Given a group homomorphism  $\phi \colon K \to G$  and a family  $\mathcal{F}$  of subgroups of *G*, define the family of subgroups of *K* 

(15.4) 
$$\phi^* \mathcal{F} := \{ H \subseteq K \mid \phi(H) \in \mathcal{F} \}.$$

If  $\phi$  is an inclusion of subgroups, we also write

(15.5) 
$$\mathcal{F}|_K := \phi^* \mathcal{F} = \{ H \subseteq K \mid H \in \mathcal{F} \}.$$

If  $\psi: H \to K$  is another group homomorphism, then

(15.6) 
$$\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}$$

**Exercise 15.7.** Let  $\phi: K \to G$  be a group homomorphism. Consider a family  $\mathcal{F}$  of subgroups of G and a G-CW-model  $E_{\mathcal{F}}(G)$ . Show that its restriction to K by  $\phi: K \to G$  is a K-CW-complex which is a model for  $E_{\phi^* \mathcal{F}}(K)$ .

Consider an equivariant homology theory  $\mathcal{H}^{?\downarrow\Gamma}_*$  over the group  $\Gamma$  with values in  $\Lambda$ -modules in the sense of Definition 12.91.

**Conjecture 15.8 (Fibered Meta-Isomorphism Conjecture).** A group  $(G, \xi)$  over  $\Gamma$  satisfies the *Fibered Meta-Isomorphism Conjecture with respect to*  $\mathcal{H}_*^{?\downarrow\Gamma}$  *and the family*  $\mathcal{F}$  *of subgroups of* G if for each group homomorphism  $\phi: K \to G$  the group K satisfies the Meta-Isomorphism Conjecture 15.2 with respect to the K-homology theory  $\mathcal{H}_*^{K,\xi\circ\phi}$  and the family  $\phi^*\mathcal{F}$  of subgroups of K.

# 15.4 The Farrell-Jones Conjecture with Coefficients in Additive or Higher Categories is Fibered

We will see that it is important for inheritance properties to pass to the fibered version. It turns out that the fibered version is automatically built into the versions of the Farrell-Jones Conjecture with coefficients in additive *G*-categories (with involution) and in higher categories.

#### **Theorem 15.9 (The Farrell-Jones Conjecture with coefficients in additive** *G*-categories (with involutions) is automatically fibered).

(i) Let  $\phi: K \to G$  be a group homomorphism. Let  $\mathcal{F}$  be a family of subgroups of *G*. Suppose that the assembly map

 $H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n(\mathbf{K}_{\mathcal{A}}(I(G)))$ 

is bijective for every  $n \in \mathbb{Z}$  and every additive *G*-category  $\mathcal{A}$ . Then the assembly map

$$H_n^K(\mathrm{pr}): H_n^K(E_{\phi^*\mathcal{F}}(K);\mathbf{K}_{\mathcal{B}}) \to H_n^K(K/K;\mathbf{K}_{\mathcal{B}}) = \pi_n(\mathbf{K}_{\mathcal{B}}(I(K)))$$

is bijective for every  $n \in \mathbb{Z}$  and every additive K-category B. The analogous statement holds for higher categories as coefficients;

(ii) Suppose that G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories.

Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for the group  $(G, \mathrm{id}_G)$  over G, the family  $\mathcal{VCY}$ , and the equivariant homology theory  $H^2_*(-; \mathbf{K}_{\mathcal{A}})$  over G for every additive G-category  $\mathcal{A}$ .

The analogous statement holds for higher categories as coefficients;

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- (iii) Let  $\phi: K \to G$  be a group homomorphism. Let  $\mathcal{F}$  be a family of subgroups of *G*. Suppose that the assembly map

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})\to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})=\pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}(I(G))\big)$$

is bijective for every  $n \in \mathbb{Z}$  and every additive *G*-category with involution  $\mathcal{A}$ . Then the assembly map

$$H_n^K(\mathrm{pr}): H_n^K(E_{\phi^*\mathcal{F}}(G); \mathbf{L}_{\mathcal{B}}^{\langle -\infty \rangle}) \to H_n^K(K/K; \mathbf{L}_{\mathcal{B}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{B}}^{\langle -\infty \rangle}(I(K)) \big)$$

is bijective for every  $n \in \mathbb{Z}$  and every additive K-category with involution  $\mathcal{B}$ ;

(iv) Suppose that G satisfies the L-theoretic Farrell-Jones Conjecture 13.19 with coefficients in additive G-categories with involution.

Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for the group  $(G, \mathrm{id}_G)$  over G, the family  $\mathcal{VCY}$ , and the equivariant homology theory  $H^2_*(-; \mathbf{L}_{\mathcal{A}}^{(-\infty)})$  over G for every additive G-category with involution  $\mathcal{A}$ .

#### *Proof.* (i) See [92, Corollary 4.3] and [185, Corollary 8.2].

(ii) This follows from assertion (i) by taking  $\mathcal{B} = \phi^* \mathcal{A}$  since a direct inspection of the definitions in [77, Section 9] shows that the *K*-homology theory obtained by taking in  $H^2_*(-; \mathbf{K}_{\mathcal{A}})$  the variable ? to be  $\phi$  is the same as the *K*-homology theory  $\mathbf{H}^K_*(-; \mathbf{K}_{\phi^* \mathcal{A}})$  associated to the additive *K*-category  $\phi^* \mathcal{A}$ .

(iii) See [77, Theorem 11.3].

(iv) This follows from (iii) by the same proof as for assertion (ii).

It is useful to have the Fibered Meta Conjecture 15.8 available since there are other situations where it is not known how to formulate it with adequate coefficients, as is possible in the Farrell-Jones setting for *K*- and *L*-theory.

#### **15.5 Transitivity Principles**

In this subsection we treat only equivariant homology theories  $\mathcal{H}^{?}_{*}$  to keep the notation and exposition simple. The generalizations to an equivariant homology theory over a group  $\Gamma$  are obvious, just equip each group occurring below with the appropriate reference map to  $\Gamma$ .

**Lemma 15.10.** Let G be a group, and let  $\mathcal{F}$  be a family of subgroups of G. Let m be an integer. Let Z be a G-CW-complex. For  $H \subseteq G$  let  $\mathcal{F}|_H$  be the family of subgroups of H given by  $\{L \subseteq H \mid L \in \mathcal{F}\}$ . Suppose for each  $H \subseteq G$  occurring as an isotropy group in Z that the maps induced by the projection  $pr_H: E_{\mathcal{F}|_H}(H) \to H/H$ 

$$\mathcal{H}_n^H(\mathrm{pr}_H): \mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

satisfy one of the following conditions

- (i) They are bijective for  $n \in \mathbb{Z}$  with  $n \leq m$ ;
- (ii) They are bijective for  $n \in \mathbb{Z}$  with  $n \le m 1$  and surjective for n = m.

Then the maps induced by the projection  $pr_2: E_{\mathcal{F}}(G) \times Z \to Z$ 

$$\mathcal{H}_n^G(\mathrm{pr}_2) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \to \mathcal{H}_n^G(Z)$$

satisfies the same condition.

*Proof.* We first prove the claim for finite-dimensional *G*-*CW*-complexes by induction over  $d = \dim(Z)$ . The induction beginning  $\dim(Z) = -1$ , i.e.  $Z = \emptyset$ , is trivial. In the induction step from (d - 1) to *d* we choose a *G*-pushout



If we cross it with  $E_{\mathcal{F}}(G)$ , we obtain another *G*-pushout of *G*-*CW*-complexes. The various projections induce a map from the Mayer-Vietoris sequence of the latter *G*-pushout to the Mayer-Vietoris sequence of the first *G*-pushout. By the Five Lemma (or its obvious variant if we consider assumption (ii)) it suffices to prove that the following maps

$$\mathcal{H}_{n}^{G}(\mathrm{pr}_{2}) \colon \mathcal{H}_{n}^{G}\left(E_{\mathcal{F}}(G) \times \bigsqcup_{i \in I_{d}} G/H_{i} \times S^{d-1}\right) \to \mathcal{H}_{n}^{G}\left(\bigsqcup_{i \in I_{d}} G/H_{i} \times S^{d-1}\right);$$
$$\mathcal{H}_{n}^{G}(\mathrm{pr}_{2}) \colon \mathcal{H}_{n}^{G}(E_{\mathcal{F}}(G) \times Z_{d-1}) \to \mathcal{H}_{n}^{G}(Z_{d-1});$$
$$\mathcal{H}_{n}^{G}(\mathrm{pr}_{2}) \colon \mathcal{H}_{n}^{G}\left(E_{\mathcal{F}}(G) \times \bigsqcup_{i \in I_{d}} G/H_{i} \times D^{d}\right) \to \mathcal{H}_{n}^{G}\left(\bigsqcup_{i \in I_{d}} G/H_{i} \times D^{d}\right),$$

satisfy condition (i) or (ii) respectively. This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and *G*-homotopy invariance of  $\mathcal{H}^{?}_{*}$  the claim follows for the third map if we can show for any  $H \subseteq G$  which occurs as an isotropy group in *Z* that the maps

(15.11) 
$$\mathcal{H}_n^G(\mathrm{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \to \mathcal{H}^G(G/H)$$

satisfy condition (i) or (ii) respectively. The G-map

$$G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G) \to G/H \times E_{\mathcal{F}}(G) \quad (g, x) \mapsto (gH, gx)$$

is a *G*-homeomorphism where  $\operatorname{res}_G^H$  denotes the restriction of the *G*-action to an *H*-action. Since  $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$ , the *H*-space  $\operatorname{res}_G^H E_{\mathcal{F}}(G)$  is a model for  $E_{\mathcal{F}|_H}(H)$ . We conclude from the induction structure that the map (15.11) can be identified with the map

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$$\mathcal{H}_{n}^{H}(\mathrm{pr}_{H}) \colon \mathcal{H}_{n}^{H}(E_{\mathcal{F}|_{H}}(H)) \to \mathcal{H}^{H}(H/H),$$

which satisfies condition (i) or (ii) respectively by assumption. This finishes the proof in the case that Z is finite-dimensional. The general case follows by a colimit argument using Lemma 12.5. 

**Theorem 15.12 (Transitivity Principle for equivariant homology).** Suppose  $\mathcal{F} \subseteq$ G are two families of subgroups of the group G. Suppose for every  $H \in G$  that the maps induced by the projection

$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

satisfy one of the following conditions:

(i) They are bijective for  $n \in \mathbb{Z}$  with  $n \leq m$ ;

...

(ii) They are bijective for  $n \in \mathbb{Z}$  with  $n \leq m - 1$  and surjective for n = m.

Then the maps induced by the up to G-homotopy unique G-map  $\iota_{\mathcal{F}\subseteq \mathcal{G}}: E_{\mathcal{F}}(G) \rightarrow$  $E_G(\mathcal{G})$ 

$$\mathcal{H}_n^G(\iota_{\mathcal{F}\subseteq\mathcal{G}})\colon\mathcal{H}_n^G(E_{\mathcal{F}}(G))\to\mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

satisfy the same condition.

*Proof.* If we equip  $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G)$  with the diagonal G-action, it is a model for  $E_{\mathcal{F}}(G)$ . Now apply Lemma 15.10 in the special case  $Z = E_{\mathcal{G}}(G)$ . 

This implies the following transitivity principle for the Fibered Isomorphism Conjecture. At the level of spectra this transitivity principle is due to Farrell and Jones [366, Theorem A.10].

**Theorem 15.13 (Transitivity Principle).** Suppose  $\mathcal{F} \subseteq \mathcal{G}$  are two families of subgroups of G.

(i) Assume that for every element  $L \in \mathcal{G}$  the group L satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively for  $\mathcal{F}|_L$ .

Then the group G satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to G if and only if G satisfies the Meta-Isomorphism Conjecture 15.2 or the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to  $\mathcal{F}$ ;

(ii) The group G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to G if G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 respectively with respect to  $\mathcal{F}$ .

*Proof.* (i) We first treat the (slightly harder) case of the Fibered Meta-Isomorphism Conjecture 15.8.

Consider a group homomorphism  $\phi: K \to G$ . Then for every subgroup H of K we conclude

$$(\phi|_H)^*(\mathcal{F}|_{\phi(H)}) = (\phi^*\mathcal{F})|_H$$

from (15.6), where  $\phi|_H \colon H \to \phi(H)$  is the group homomorphism induced by  $\phi$ . For every element  $H \in \phi^* \mathcal{G}$  the map

$$\mathcal{H}_n^H(E_{(\phi|_H)^*(\mathcal{F}|_{\phi(H)})}(H)) = \mathcal{H}_n^H(E_{\phi^*\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(H/H)$$

is bijective for all  $n \in \mathbb{Z}$  by the assumption that the element  $\phi(H) \in \mathcal{G}$  satisfies the Fibered Isomorphism Conjecture for  $\mathcal{F}|_{\phi(H)}$ . Hence by Theorem 15.12 applied to the inclusion  $\phi^* \mathcal{F} \subseteq \phi^* \mathcal{G}$  of families of subgroups of *K* we get an isomorphism

$$\mathcal{H}_{n}^{K}(\iota_{\phi^{*}\mathcal{F}\subseteq\phi^{*}\mathcal{G}})\colon\mathcal{H}_{n}^{K}(E_{\phi^{*}\mathcal{F}}(K))\xrightarrow{=}\mathcal{H}_{n}^{K}(E_{\phi^{*}\mathcal{G}}(K)).$$

Therefore the map  $\mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \to \mathcal{H}_n^K(K/K)$  is bijective for all  $n \in \mathbb{Z}$  if and only if the map  $\mathcal{H}_n^K(E_{\phi^*\mathcal{G}}(K)) \to \mathcal{H}_n^K(K/K)$  is bijective for all  $n \in \mathbb{Z}$ .

The argument for the Meta-Isomorphism Conjecture 15.8 is analogous, just specialize the argument above to the case  $\phi = id_G$ .

(ii) We want to apply assertion (i). We have to show that for every element  $H \in \mathcal{G}$  the group H satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for  $\mathcal{F}|_H$ , provided that G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to  $\mathcal{F}$ . This follows from the elementary Lemma 15.16 below since  $\mathcal{F}|_H = i^*\mathcal{F}$  for the inclusion  $i: H \to G$ .

Note that assertion (ii) of Theorem 15.13 is only formulated for the fibered version. The F is a set of the fibered version.

The Fibered Isomorphism Conjecture is also well behaved with respect to finite intersections of families of subgroups.

**Lemma 15.14.** Let G be a group, and let  $\mathcal{F}$  and G be families of subgroups. Suppose that G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for both  $\mathcal{F}$  and G.

Then G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the family  $\mathcal{F} \cap \mathcal{G} := \{H \subseteq G \mid H \in \mathcal{F} \text{ and } H \in \mathcal{G}\}.$ 

*Proof.* Obviously  $\mathcal{F} \cup \mathcal{G} := \{H \subseteq G \mid H \in \mathcal{F} \text{ or } H \in \mathcal{G}\}$  is a family of subgroups of *G*.

Consider a group homomorphism  $\phi: K \to G$ . We have to show that the Meta-Isomorphism Conjecture 15.2 holds for G with respect to  $\phi^*(\mathcal{F} \cap \mathcal{G})$ .

Choose *G*-*CW*-models  $E_{\mathcal{F}\cap\mathcal{G}}(G)$ ,  $E_{\mathcal{F}}(G)$ , and  $E_{\mathcal{G}}(G)$  such that  $E_{\mathcal{F}\cap\mathcal{G}}(G)$  is a *G*-*CW*-subcomplex of both  $E_{\mathcal{F}}(G)$  and  $E_{\mathcal{G}}(G)$ . This can be arranged by a mapping cylinder construction. Define a *G*-*CW*-complex

$$X = E_{\mathcal{F}}(G) \cup_{E_{\mathcal{F} \cap \mathcal{G}}(G)} E_{\mathcal{G}}(G).$$

For any subgroup  $H \subseteq G$  we get

$$X^{H} = E_{\mathcal{F}}(G)^{H} \cup_{E_{\mathcal{F} \cap \mathcal{C}}(G)^{H}} E_{\mathcal{G}}(G)^{H}.$$

If  $E_{\mathcal{F}}(G)^H$  and  $E_{\mathcal{G}}(G)^H$  are empty, the same is true for  $X^H$ . If  $E_{\mathcal{F}}(G)^H$  is empty, then  $E_{\mathcal{G}}(G)^H = X^H$ . If  $E_{\mathcal{G}}(G)^H$  is empty, then  $E_{\mathcal{F}}(G)^H = X^H$ . If  $E_{\mathcal{F}\cap\mathcal{G}}(G)^H$ is empty,  $E_{\mathcal{F}}(G)^H$  or  $E_{\mathcal{G}}(G)^H$  is empty. If  $E_{\mathcal{F}}(G)^H$ ,  $E_{\mathcal{G}}(G)^H$ , and  $E_{\mathcal{F}\cap\mathcal{G}}(G)^H$  are all non-empty and hence weakly contractible, the same is true for  $X^H$ . Hence X is a model for  $E_{\mathcal{F}\cup\mathcal{G}}(G)$ . If we apply restriction with  $\phi$ , we get a decomposition of  $E_{\phi^*\mathcal{F}}(\mathcal{F}) = \phi^* E_{\mathcal{F}\cup\mathcal{G}}(G)$  as the union of  $E_{\phi^*\mathcal{F}}(K) = \phi^* E_{\mathcal{F}}(G)$  and  $E_{\phi^*\mathcal{G}}(K) = \phi^* E_{\mathcal{G}}(G)$  such that the intersection of  $E_{\phi^*\mathcal{F}}(K)$  and  $E_{\phi^*\mathcal{G}}(K)$  is  $E_{\phi^*(\mathcal{F}\cap\mathcal{G})}(K) = \phi^* E_{\mathcal{F}\cap\mathcal{G}}(G)$ . By assumption and by Theorem 15.13 (ii) the Fibered Meta-Isomorphism Conjecture 15.8 holds for G with respect to  $\mathcal{F}, \mathcal{G}$ , and  $\mathcal{F} \cup \mathcal{G}$ . Hence the Meta-Isomorphism Conjecture 15.2 holds for G with respect to  $\phi^*(\mathcal{F}\cup\mathcal{G}), \phi^*\mathcal{F}$ , and  $\phi^*\mathcal{G}$ . Using the Mayer-Vietoris sequence for the decomposition of  $E_{\phi^*\mathcal{F}\cup\phi^*\mathcal{G}}(K)$  above and the Five Lemma, we conclude that Meta-Isomorphism Conjecture 15.2 holds for G is an arbitrary group homomorphism with target G, the group G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the family  $\mathcal{F} \cap \mathcal{G}$ .

**Exercise 15.15.** Assume that the Fibered Meta-Isomorphism Conjecture 15.8 holds for  $G = \mathbb{Z}$ , the family  $\mathcal{F} = \mathcal{FIN}$ , and the equivariant homology theory  $H_*^?(-; \mathbf{K}_R)$  for a given ring R.

Show that then we have  $NK_n(RG) = 0$  for every group G and  $n \in \mathbb{Z}$ .

#### 15.6 Inheritance Properties of the Fibered Meta-Isomorphism Conjecture

The Fibered Meta-Isomorphism Conjecture 15.8 has better inheritance properties than the Meta-Isomorphism Conjecture 15.2.

In this subsection we treat only equivariant homology theories  $\mathcal{H}^{?}_{*}$  for simplicity. The generalizations to an equivariant homology theory over a group  $\Gamma$  are obvious.

**Lemma 15.16.** Let  $\phi: K \to G$  be a group homomorphism and  $\mathcal{F}$  be a family of subgroups. If  $(G, \mathcal{F})$  satisfies the Fibered Meta-Isomorphism Conjecture 15.8 then  $(K, \phi^* \mathcal{F})$  satisfies the Fibered Meta-Isomorphism Conjecture 15.8.

*Proof.* If  $\psi: L \to K$  is a group homomorphism, then  $\psi^*(\phi^* \mathcal{F}) = (\phi \circ \psi)^* \mathcal{F}$  by (15.6).

**Exercise 15.17.** Fix a class of groups *C* that is closed under isomorphisms and taking subgroups, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group *G* let C(G) be the family of subgroups of *G* that belong to *C*. Suppose that the Fibered Meta-Isomorphism Conjecture 15.8 holds for (G, C(G)). Let  $H \subseteq G$  be a subgroup.

Show that (H, C(H)) satisfies the Fibered Meta-Isomorphism Conjecture 15.8.

**Lemma 15.18.** Fix a class of groups C that is closed under isomorphisms, taking subgroups, and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let C(G) be the family of subgroups of G that belong to C. Let  $1 \to K \to G \xrightarrow{P} Q \to 1$  be an extension of groups. Suppose

that (Q, C(Q)) and  $(p^{-1}(H), C(p^{-1}(H)))$  for every  $H \in C(Q)$  satisfy the Fibered Meta-Isomorphism Conjecture 15.8.

Then (G, C(G)) satisfies the Fibered Meta-Isomorphism Conjecture 15.8.

*Proof.* By Lemma 15.16 the pair  $(G, p^*C(Q))$  satisfies the Fibered Meta-Isomorphism Conjecture 15.8. Obviously  $C(G) \subseteq p^*C(Q)$ . Because of the Transitivity Principle 15.13 (i) it remains to show for each  $L \in p^*C(Q)$  that the pair (L, C(L))satisfies the Fibered Meta-Isomorphism Conjecture 15.8. Since  $L \subseteq p^{-1}(p(L))$ and  $p(L) \in C(Q)$  holds, we conclude from Exercise 15.17 that this follows from the assumption that  $(p^{-1}(H), C(p^{-1}(H)))$  satisfy the Fibered Meta-Isomorphism Conjecture 15.8 for every  $H \in C(Q)$ .

Fix an equivariant homology theory  $\mathcal{H}^{?}_{*}$  with values in  $\Lambda$ -modules. Let X be a G-CW-complex. Let  $\alpha \colon H \to G$  be a group homomorphism. Denote by  $\alpha^{*}X$  the H-CW-complex obtained from X by *restriction with*  $\alpha$ . Recall that  $\alpha_{*}Y$  denotes the induction of an H-CW-complex Y and is a G-CW-complex. The functors  $\alpha_{*}$  and  $\alpha^{*}$  are adjoint to one another. In particular, the adjoint of the identity on  $\alpha^{*}X$  is a natural G-map

(15.19) 
$$f(X,\alpha): \alpha_*\alpha^*X \to X.$$

It sends an element in  $G \times_{\alpha} \alpha^* X$  given by (g, x) to gx. Define the A-map

$$a_n = a_n(X, \alpha) \colon \mathcal{H}_n^H(\alpha^* X) \xrightarrow{\text{ind}_\alpha} \mathcal{H}_n^G(\alpha_* \alpha^* X) \xrightarrow{\mathcal{H}_n^G(f(X, \alpha))} \mathcal{H}_n^G(X)$$

If  $\beta: G \to K$  is another group homomorphism, then by the axioms of an induction structure the composite  $\mathcal{H}_n^H(\alpha^*\beta^*X) \xrightarrow{a_n(\beta^*X,\alpha)} \mathcal{H}_n^G(\beta^*X) \xrightarrow{a_n(X,\beta)} \mathcal{H}_n^K(X)$ agrees with  $a_n(X,\beta\circ\alpha): \mathcal{H}_n^H(\alpha^*\beta^*X) = \mathcal{H}_n^H((\beta\circ\alpha)^*X) \to \mathcal{H}_n^G(X)$  for a *K*-*CW*complex *X*.

Consider a directed system of groups  $\{G_i \mid i \in I\}$  with  $G = \operatorname{colim}_{i \in I} G_i$  and structure maps  $\psi_i \colon G_i \to G$  for  $i \in I$  and  $\phi_{i,j} \colon G_i \to G_j$  for  $i, j \in I, i \leq j$ . (The group homomorphism  $\phi_{i,j}$  are *not* required to be injective or to be surjective.) We obtain for every *G*-*CW*-complex *X* a system of  $\Lambda$ -modules  $\{\mathcal{H}_n^{G_i}(\psi_i^*X) \mid i \in I\}$ with structure maps  $a_n(\psi_j^*X, \phi_{i,j}) \colon \mathcal{H}_n^{G_i}(\psi_i^*X) \to \mathcal{H}_n^{G_j}(\psi_j^*X)$ . We get a map of  $\Lambda$ -modules

(15.20)  
$$t_n^G(X,A) := \operatorname{colim}_{i \in I} a_n(X,\psi_i) : \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*(X,A)) \to \mathcal{H}_n^G(X,A).$$

**Definition 15.21 ((Strongly) continuous equivariant homology theory).** An equivariant homology theory  $\mathcal{H}_*^?$  is called *continuous* if for every group *G* and every directed system of subgroups  $\{G_i \mid i \in I\}$  of *G* with  $G = \bigcup_{i \in I} G_i$  the  $\Lambda$ -map, see (15.20),

$$t_n^G(\{\bullet\})$$
: colim <sub>$i \in I$</sub>   $\mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$ 

is an isomorphism for every  $n \in \mathbb{Z}$ .

15.6 Inheritance Properties of the Fibered Meta-Isomorphism Conjecture

An equivariant homology theory  $\mathcal{H}_*^?$  is called *strongly continuous* if for every directed system of groups  $\{G_i \mid i \in I\}$  with  $G = \operatorname{colim}_{i \in I} G_i$  and (not necessarily injective or surjective) structure maps  $\psi_i : G_i \to G$  for  $i \in I$  the  $\Lambda$ -map

$$t_n^G(\{\bullet\})$$
: colim <sub>$i \in I$</sub>   $\mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$ 

is an isomorphism for every  $n \in \mathbb{Z}$ .

The next result is taken from [71, Lemma 3.4].

**Lemma 15.22.** Consider a directed system of groups  $\{G_i \mid i \in I\}$  with  $G = \text{colim}_{i \in I} G_i$  and structure maps  $\psi_i : G_i \to G$  for  $i \in I$ . Let (X, A) be a G-CW-pair. Suppose that  $\mathcal{H}^{?}_*$  is strongly continuous.

*Then the*  $\Lambda$ *-homomorphism, see* (15.20)

$$t_n^G(X, A)$$
: colim <sub>$i \in I$</sub>   $\mathcal{H}_n^{G_i}(\psi_i^*(X, A)) \xrightarrow{=} \mathcal{H}_n^G(X, A)$ 

is bijective for every  $n \in \mathbb{Z}$ .

The proof of the next result is based on Lemma 15.22.

**Lemma 15.23.** Fix a class of groups C that is closed under isomorphisms, taking subgroups, and taking quotients, e.g., the class of finite groups or the class of virtually cyclic subgroups. For a group G let C(G) be the family of subgroups of G that belong to C. Let G be a group.

(i) Let G be the directed union of subgroups  $\{G_i \mid i \in I\}$ . Suppose that  $\mathcal{H}^?_*$  is continuous and for every  $i \in I$  the Meta-Isomorphism Conjecture 15.2 holds for  $G_i$  and  $C(G_i)$ .

Then the Meta-Isomorphism Conjecture 15.2 holds for G and C(G);

(ii) Let G be the directed union of subgroups  $\{G_i \mid i \in I\}$ . Suppose that  $\mathcal{H}^{*}_{*}$  is continuous and for every  $i \in I$  the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for  $G_i$  and  $C(G_i)$  is injective for all  $n \in \mathbb{Z}$ . Then the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for G and C(G) is injective for all  $n \in \mathbb{Z}$ .

The same statement is true if we replace "injective" by "surjective".

(iii) Let  $\{G_i \mid i \in I\}$  be a directed system of groups with  $G = \operatorname{colim}_{i \in I} G_i$  and structure maps  $\psi_i : G_i \to G$ . Suppose that  $\mathcal{H}^{?}_*$  is strongly continuous and for every  $i \in I$  the Fibered Meta-Isomorphism Conjecture 15.8 holds for  $G_i$  and  $C(G_i)$ .

Then the Fibered Meta-Isomorphism Conjecture 15.8 holds for G and C(G).

*Proof.* (i) is proved in [75, Proposition 3.4].

(ii) The proof of [75, Proposition 3.4] for isomorphisms also yields a proof for the injectivity or surjectivity version since the colimit over a directed system is an exact functor and hence preserves injectivity and surjectivity.

(iii) See [71, Theorem 5.6].

**Remark 15.24 (Injectivity, surjectivity, and the Transitivity Principle).** For colimits over a directed system of subgroups, we got a statement about injectivity or surjectivity in Lemma 15.23 (ii), essentially since the colimit over a directed system is an exact functor. We cannot prove such injectivity or surjectivity statement for assertion (iii) since its proof uses the Transitivity Principle 15.13, for which the injectivity or surjectivity version is not true in general, essentially, because the Five Lemma does not have a version for injectivity or surjectivity.

#### 15.7 Actions on Trees

In this subsection we treat only equivariant homology theories  $\mathcal{H}^{?}_{*}$  for simplicity. The generalizations to an equivariant homology theory over a group  $\Gamma$  are obvious.

Given a subgroup  $H \subseteq G$ , we obtain a *G*-homeomorphism  $G \times_H \underline{E}G|_H \xrightarrow{\cong} G/H \times \underline{E}G$  sending (g, z) to (gH, gz), where *G* acts diagonally on the target. The inverse sends (gH, z) to  $(g, g^{-1}z)$ . Since  $\underline{E}G|_H$  is a model for  $\underline{E}H$ , we obtain a *G*-homotopy equivalence

(15.25) 
$$\mu(H): G \times_H EH \xrightarrow{-} G/H \times EG$$

Recall that we obtain for any subgroup  $H \subseteq G$  and  $n \in \mathbb{Z}$  from the induction structure an isomorphism

(15.26) 
$$\operatorname{ind}_{H}^{G} \colon \mathcal{H}_{n}^{H}(\underline{E}H) \xrightarrow{\cong} \mathcal{H}_{n}^{G}(G \times_{H} \underline{E}H).$$

In the sequel we denote by pr the obvious projection and by  $\iota$  the obvious inclusion.

**Lemma 15.27.** Suppose that G acts on the tree T by automorphisms of trees without inversion. Let  $\mathcal{H}^{?}_{*}$  be an equivariant homology theory.

(i) We can write T as a G-pushout

$$\begin{array}{c|c} \coprod_{j \in J} G/K_j \times S^0 & \xrightarrow{q} & \coprod_{i \in I} G/H_i \\ & & & & \downarrow_{i \in I} G/K_j \\ & & & & \downarrow_{\overline{k}} \\ & & & & \downarrow_{\overline{k}} \\ & & & & & \downarrow_{\overline{q}} \end{array}$$

where there are for every  $j \in J$  two elements i(j, +) and i(j, -) in I such that the restriction of q to  $G/K_j$  considered as G-subspace of  $\coprod_{j \in J} G/K_j \times S^0$  by

$$G/K_j = G/K_j \times \{\pm 1\} \subseteq G/K_j \times S^0 \subseteq \prod_{j \in J} G/K_j \times S^0$$

is given by the composite of a G-map  $\widehat{q}_{j,\pm 1}$ :  $G/K_j \rightarrow G/H_{i(j,\pm)}$  with the canonical inclusion  $G/H_{i(j,\pm)} \rightarrow \coprod_{i \in I} G/H_i$ ;

#### 15.7 Actions on Trees

(ii) We obtain a long exact sequence

$$\cdots \to \bigoplus_{j \in J} \mathcal{H}_{n}^{K_{j}}(\underline{E}K_{j}) \xrightarrow{t_{n}(j,+)-t_{n}(j,-)} \bigoplus_{i \in I} \mathcal{H}_{n}^{H_{i}}(\underline{E}H_{i}) \xrightarrow{s_{n}} \mathcal{H}_{n}^{G}(\underline{E}G)$$
$$\to \bigoplus_{j \in J} \mathcal{H}_{n-1}^{K_{j}}(\underline{E}K_{j}) \xrightarrow{t_{n-1}(j,+)-t_{n-1}(j,-)} \bigoplus_{i \in I} \mathcal{H}_{n-1}^{H_{i}}(\underline{E}H_{i}) \xrightarrow{s_{n-1}} \cdots$$

where  $t_n(j, \pm)$  is given by the composite

$$\begin{array}{c} \mathcal{H}_{n}^{K_{j}}(\underline{E}K_{j}) \xrightarrow{\operatorname{ind}_{K_{j}}^{G}} \mathcal{H}_{n}^{G}(G \times_{K_{j}} \underline{E}K_{j}) \xrightarrow{\mathcal{H}_{n}^{G}(\mu(K_{j}))} \mathcal{H}_{n}^{G}(G/K_{j} \times \underline{E}G) \\ \xrightarrow{\mathcal{H}_{n}^{G}(\widehat{q}_{j,\pm1} \times \operatorname{id}_{\underline{E}G})} \mathcal{H}_{n}^{G}(G/H_{i(j,\pm)} \times \underline{E}G) \xrightarrow{\mathcal{H}_{n}^{G}(\mu(H_{i(j,\pm)}))^{-1}} \mathcal{H}_{n}^{G}(G \times_{H_{i(j,\pm)}} \underline{E}H_{i(j,\pm)}) \\ \xrightarrow{\operatorname{(ind}_{H_{i(j,\pm)}}^{G})^{-1}} \mathcal{H}_{n}^{H_{i(j,\pm)}}(\underline{E}H_{i(j,\pm)}) \xrightarrow{\iota} \bigoplus_{i \in I} \mathcal{H}_{n}^{H_{i}}(\underline{E}H_{i})
\end{array}$$

and  $s_n$  is the direct sum of the maps for  $i \in I$ 

$$\mathcal{H}_{n}^{H_{i}}(\underline{E}(H_{i})) \xrightarrow{\mathrm{ind}_{H_{i}}^{G}} \mathcal{H}_{n}^{G}(G \times_{H_{i}} \underline{E}(H_{i})) \\ \xrightarrow{\mathcal{H}_{n}^{G}(\mu(H_{i}))} \mathcal{H}_{n}^{G}(G/H \times \underline{E}(G)) \xrightarrow{\mathcal{H}_{n}^{G}(\mathrm{pr})} \mathcal{H}_{n}^{G}(\underline{E}G)$$

*Proof.* (i) Since *G* acts on *T* by automorphisms of trees without inversion, *T* is a 1-dimensional *G*-*CW*-complex and the *G*-pushout just describes how the 1-skeleton is obtained from the 0-skeleton  $\prod_{i \in I} G/H_i$ .

(ii) If we cross the G-pushout of assertion (i) with  $\underline{E}G$  using the diagonal G-action, we obtain the G-pushout

(15.28) 
$$\begin{aligned} & \coprod_{j \in J} G/K_j \times \underline{E}G \times S^0 \xrightarrow{q \times \mathrm{id}_{\underline{E}G}} & \coprod_{i \in I} G/H_i \times \underline{E}G \\ & & \downarrow_{k \times \mathrm{id}_{\underline{E}G}} \\ & & \downarrow_{\overline{k} \times \mathrm{id}_{\underline{E}G}} \\ & & \coprod_{j \in J} G/K_j \times \underline{E}G \times D^1 \xrightarrow{\overline{q} \times \mathrm{id}_{\underline{E}G}} T \times \underline{E}G. \end{aligned}$$

The *H*-fixed point set  $T^H$  is a non-empty subtree and therefore contractible for every finite subgroup  $H \subseteq G$ , see [911, Theorem 15 in 6.1 on page 58 and 6.3.1 on page 60]. We conclude that the projection  $\underline{E}G \times T \rightarrow \underline{E}G$  is a *G*-homotopy equivalence from the Equivariant Whitehead Theorem, see for instance [644, Theorem 2.4 on page 36]. The desired long exact sequence can be derived from the Mayer-Vietoris sequence associated to the *G*-pushout (15.28) using the identifications (15.25) and (15.26).

**Lemma 15.29.** Suppose that G acts on the tree T by automorphisms of trees without inversion. Let  $\mathcal{H}^{?}_{*}$  be an equivariant homology theory. Suppose that the Meta-Isomorphism Conjecture 15.2 holds for G with respect to FIN. Assume that for any isotropy group H of the G-action on T the Meta-Isomorphism Conjecture 15.2 holds for H with respect to  $\mathcal{FIN}$ .

(i) The projection  $T \to \{\bullet\}$  induces for all  $n \in \mathbb{Z}$  an isomorphism

$$\mathcal{H}_n^G(T) \xrightarrow{\cong} \mathcal{H}_n^G(\{\bullet\});$$

(ii) Write T as a G-pushout as described in Lemma 15.27 (i). Let  $g(j, \pm)$  be an element in G such that  $g(j, \pm)K_jg(j, \pm))^{-1} \subseteq H_{i(j,\pm)}$  and the G-map  $\widehat{q}_{j,\pm 1}: G/K_j \to G/H_{i_{\pm}(j)}$  is given by  $gK_j \mapsto gg(j, \pm)^{-1}H_{i(j,\pm)}$ . Let  $c(g(j,\pm)):$  $K_j \rightarrow H_{i(j,\pm)}$  be the group homomorphism sending k to  $g(j,\pm) kg(j,\pm)^{-1}$ . We get a long exact sequence

$$\cdots \to \bigoplus_{j \in J} \mathcal{H}_{n}^{K_{j}}(\{\bullet\}) \xrightarrow{t_{n}^{\prime}(j,+)-t_{n}^{\prime}(j,-)} \bigoplus_{i \in I} \mathcal{H}_{n}^{H_{i}}(\{\bullet\}) \xrightarrow{s^{\prime}} \mathcal{H}_{n}^{G}(\{\bullet\})$$
$$\to \bigoplus_{j \in J} \mathcal{H}_{n-1}^{K_{j}}(\{\bullet\}) \xrightarrow{t_{n-1}^{\prime}(j,+)-t_{n-1}^{\prime}(j,-)} \bigoplus_{i \in I} \mathcal{H}_{n-1}^{H_{i}}(\{\bullet\}) \xrightarrow{s_{n-1}^{\prime}} \cdots$$

where  $t'_n(j, \pm)$  is given by the composite

$$\mathcal{H}_{n}^{K_{j}}(\{\bullet\}) \xrightarrow{\operatorname{ind}_{c(g(j,\pm))}} \mathcal{H}_{n}^{H_{i}(j,\pm)}(\operatorname{ind}_{c(g(j,\pm))}\{\bullet\})$$
$$\xrightarrow{\operatorname{pr}} \mathcal{H}_{n}^{H_{i}(j,\pm)}(\{\bullet\}) \xrightarrow{\iota} \bigoplus_{i \in I} \mathcal{H}_{n}^{H_{i}}(\{\bullet\})$$

and  $s'_n$  is the direct sum of the maps for  $i \in I$ 

$$\mathcal{H}_{n}^{H_{i}}(\{\bullet\}) \xrightarrow{\operatorname{ind}_{H_{i}}^{G}} \mathcal{H}_{n}^{G}(G \times_{H_{i}} \{\bullet\}) \xrightarrow{\mathcal{H}_{n}^{G}(\operatorname{pr})} \mathcal{H}_{n}^{G}(\{\bullet\}).$$

Proof. (i) We have already explained in the proof of Lemma 15.27 (ii) that the projection  $\underline{E}G \times T \rightarrow \underline{E}G$  is a G-homotopy equivalence. By assumption the projection  $\underline{E}G \to \{\bullet\}$  induces for all  $n \in \mathbb{Z}$  isomorphisms  $\mathcal{H}_n^G(\underline{E}G) \to \mathcal{H}_n^G(\{\bullet\})$ . Hence the projection  $\underline{E}G \times T \to \{\bullet\}$  induces for all  $n \in \mathbb{Z}$  isomorphisms  $\mathcal{H}_n^G(\underline{E}G \times T) \to$  $\mathcal{H}_n^G(\{\bullet\})$ . By Lemma 15.10 and the assumptions on T the projection  $\underline{E}G \times T \to T$ induces for all  $n \in \mathbb{Z}$  isomorphisms  $\mathcal{H}_n^G(\underline{EG} \times T) \to \mathcal{H}_n^G(T)$ . Hence the projection  $T \to \{\bullet\}$  induces for all  $n \in \mathbb{Z}$  isomorphisms  $\mathcal{H}_n^G(T) \to \mathcal{H}_n^G(\{\bullet\})$ . 

(ii) This follows from Lemma 12.12 and Lemma 15.27 (ii).

**Example 15.30** (Amalgamated free products). Let  $\mathcal{H}^{?}_{*}$  be an equivariant homology theory with values in  $\Lambda$ -modules. Let G be the amalgamated free product  $G_1 *_{G_0} G_2$ for a common subgroup  $G_0$  of the groups  $G_1$  and  $G_2$ . Suppose that  $G_i$  for i = 0, 1, 2and G satisfy the Meta-Isomorphism Conjecture 15.2 with respect to the family
#### 15.7 Actions on Trees

 $\mathcal{FIN}$ . Then there is a long exact sequence

$$(15.31) \quad \dots \to \mathcal{H}_{n}^{G_{0}}(\{\bullet\}) \to \mathcal{H}_{n}^{G_{1}}(\{\bullet\}) \oplus \mathcal{H}_{n}^{G_{1}}(\{\bullet\}) \to \mathcal{H}_{n}^{G}(\{\bullet\}) \\ \to \mathcal{H}_{n-1}^{G_{0}}(\{\bullet\}) \to \mathcal{H}_{n-1}^{G_{1}}(\{\bullet\}) \oplus \mathcal{H}_{n-1}^{G_{1}}(\{\bullet\}) \to \cdots .$$

Namely, there is a 1-dimensional G-CW-complex T whose underlying space is a tree such that the 1-skeleton is obtained from the 0-skeleton by the G-pushout

$$\begin{array}{c} G/G_0 \times S^0 \xrightarrow{q} G/G_1 \coprod G/G_2 \\ \downarrow & \downarrow \\ G/G_0 \times D^1 \xrightarrow{q} T \end{array}$$

where q is the disjoint union of the canonical projection  $G/G_0 \rightarrow G/G_1$  and  $G/G_0 \rightarrow G/G_2$ , see [911, Theorem 7 in §4.1 on page 32]. Now the desired long exact sequence is the one appearing in Lemma 15.29 (ii).

Suppose that  $G_0, G_1, G_2$ , and G satisfy the Baum-Connes Conjecture 14.9, which is equivalent to the Meta-Isomorphism Conjecture 15.2 if we choose  $\mathcal{F}$  to be  $\mathcal{FIN}$  and  $\mathcal{H}_n^G$  to be  $H_n^G(-; \mathbf{K}_{\mathbb{C}}^{\text{TOP}})$ . Then we obtain a long exact sequence

$$(15.32) \qquad \cdots \xrightarrow{\partial_{n+1}} K_n(C_r^*(G_0)) \xrightarrow{K_n(C_r^*(i_1)) \oplus K_n(C_r^*(i_2))} K_n(C_r^*(G_1)) \oplus K_n(C_r^*(G_2)) \xrightarrow{K_n(C_r^*(j_1)) - K_n(C_r^*(j_2))} K_n(C_r^*(G)) \xrightarrow{\partial_n} K_{n-1}(C_r^*(G_0)) \xrightarrow{K_{n-1}(C_r^*(i_1)) \oplus K_{n-1}(C_r^*(i_2))} K_{n-1}(C_r^*(G_2)) \oplus K_{n-1}(C_r^*(G_1)) \xrightarrow{K_{n-1}(C_r^*(j_1)) - K_{n-1}(C_r^*(j_2))} K_{n-1}(C_r^*(G)) \xrightarrow{\partial_{n-1}} \cdots$$

where  $i_1, i_2, j_1$ , and  $j_2$  are the obvious inclusions. Actually, such long exact Mayer-Vietoris sequence exists always for an amalgamated free product  $G = G_1 *_{G_0} G_2$ , see Pimsner [812, Theorem 18 on page 632].

Suppose that  $G_0$ ,  $G_1$ ,  $G_2$ , and G satisfy the *K*-theoretic Farrell Conjecture 13.1 with coefficients in the regular ring R with  $\mathbb{Q} \subseteq R$ . Then we obtain using Theorem 13.51 a long exact sequence

$$(15.33) \qquad \cdots \xrightarrow{\partial_{n+1}} K_n(RG_0) \xrightarrow{K_n(Ri_1) \oplus K_n(Ri_2)} K_n(RG_1) \oplus K_n(RG_2)$$

$$\xrightarrow{K_n(Rj_1) - K_n(Rj_2)} K_n(RG) \xrightarrow{\partial_n} K_{n-1}(RG_0)$$

$$\xrightarrow{K_{n-1}(Ri_1) \oplus K_{n-1}(Ri_2)} K_{n-1}(RG_2) \oplus K_{n-1}(RG_1)$$

$$\xrightarrow{K_{n-1}(Rj_1) - K_{n-1}(Rj_2)} K_{n-1}(RG) \xrightarrow{\partial_{n-1}} \cdots$$

Without extra assumptions on R the long exact sequence above does not exist, certain Nil-terms enter, see Theorem 6.62.

Suppose that  $G_0$ ,  $G_1$ ,  $G_2$ , and G satisfy the *L*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* with involution. Then we obtain using Theorem 13.62 (i) a long exact sequence

$$(15.34) \quad \cdots \xrightarrow{\partial_{n+1}} L_n(RG_0)[1/2] \\ \xrightarrow{L_n(Ri_1)[1/2] \oplus L_n(Ri_2)[1/2]} L_n(RG_1)[1/2] \oplus L_n(RG_2)[1/2] \\ \xrightarrow{L_n(Rj_1)[1/2] - L_n(Rj_2)[1/2]} L_n(RG)[1/2] \xrightarrow{\partial_n} L_{n-1}(RG_0)[1/2] \\ \xrightarrow{L_{n-1}(Ri_1)[1/2] \oplus L_{n-1}(Ri_2)[1/2]} L_{n-1}(RG_2)[1/2] \oplus L_{n-1}(RG_1)[1/2] \\ \xrightarrow{L_{n-1}(Rj_1)[1/2] - L_{n-1}(Rj_2)[1/2]} L_{n-1}(RG)[1/2] \xrightarrow{\partial_{n-1}} \cdots .$$

Note that the decoration of the *L*-groups does not play a role since we invert 2. Actually, such long exact Mayer-Vietoris sequences always exist for an amalgamated free product  $G = G_1 *_{G_0} G_2$ , see Cappell [204]. Without inverting 2 the long exact sequence above does not exist, certain UNil-terms enter.

**Exercise 15.35.** Let  $\mathcal{H}^?_*$  be an equivariant homology theory. Let  $\phi: G \to G$  be a group automorphism. Let  $G \times_{\phi} \mathbb{Z}$  be the associated semidirect product. Denote by  $i: G \to G \rtimes_{\phi} \mathbb{Z}$  the obvious inclusion. Suppose that *G* and  $G \times_{\phi} \mathbb{Z}$  satisfy the Meta-Isomorphism Conjecture 15.2 with respect to the family  $\mathcal{FIN}$ .

Prove the existence of a long exact sequence

$$\cdots \to \mathcal{H}_{n}^{G}(\{\bullet\}) \xrightarrow{\phi_{*} - \mathrm{id}} \mathcal{H}_{n}^{G}(\{\bullet\}) \xrightarrow{k_{*}} \mathcal{H}_{n}^{G \rtimes_{\phi} \mathbb{Z}}(\{\bullet\})$$
$$\to \mathcal{H}_{n-1}^{G}(\{\bullet\}) \xrightarrow{\phi_{*} - \mathrm{id}} \mathcal{H}_{n-1}^{G}(\{\bullet\}) \xrightarrow{k_{*}} \cdots$$

where  $\phi_* : \mathcal{H}_n^G(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$  and  $k_*$  come from the induction structure and the identification  $\operatorname{ind}_{\phi}\{\bullet\} = \{\bullet\}$  and the projection  $\operatorname{ind}_i\{\bullet\} \to \{\bullet\}$ .

Explain that this reduces in the case of the Baum-Connes Conjecture to the long exact sequence

(15.36)  

$$\cdots \to K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(\phi)) - \mathrm{id}} K_n(C_r^*(G)) \xrightarrow{K_n(C_r^*(k))} K_n(C_r^*(G \rtimes_{\phi} \mathbb{Z}))$$

$$\to K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(\phi)) - \mathrm{id}} K_{n-1}(C_r^*(G)) \xrightarrow{K_{n-1}(C_r^*(k))} \cdots,$$

and similarly for the *K*-theoretic Farrell-Jones Conjecture for a regular ring *R* with  $\mathbb{Q} \subseteq R$  and the *L*-theoretic Farrell-Jones Conjecture after inverting 2.

# **15.8** The Meta-Isomorphism Conjecture for Functors from Spaces to Spectra

Let  $S: SPACES \rightarrow SPECTRA$  be a covariant functor. Throughout this section we will assume that it respects weak equivalences and disjoint unions, i.e., a weak homotopy equivalence of spaces  $f: X \rightarrow Y$  is sent to a weak homotopy equivalence of spectra  $S(f): S(X) \rightarrow S(Y)$  and for a collection of spaces  $\{X_i \mid i \in I\}$  for an arbitrary index set *I* the canonical map

$$\bigvee_{i\in I} \mathbf{S}(X_i) \to \mathbf{S}\left(\bigsqcup_{i\in I} X_i\right)$$

is weak homotopy equivalence of spectra.

We obtain a covariant functor

(15.37)  $S^B: GROUPOIDS \rightarrow SPECTRA, \mathcal{G} \mapsto S(B\mathcal{G})$ 

where BG is the classifying space of the category G, which is the geometric realization of the simplicial set given by its nerve and denoted by  $B^{\text{bar}}G$  in [280, page 227]. Let  $H_n^?(-; \mathbf{S}^B)$  be the equivariant homology theory in the sense of Definition 12.9, which is associated to  $\mathbf{S}_B$  by the construction of Theorem 12.30. It has the property that for any group G and subgroup  $H \subseteq G$  we have canonical identifications

$$H_n^G(G/H; \mathbf{S}^B) \cong H_n^H(H/H; \mathbf{S}^B) \cong \pi_n(\mathbf{S}(BH)).$$

Conjecture 15.38 (Meta-Isomorphism Conjecture for functors from spaces to spectra). Let S: SPACES  $\rightarrow$  SPECTRA be a covariant functor that respects weak equivalences and disjoint unions. The group *G* satisfies the *Meta-Isomorphism Conjecture for* S with respect to the family  $\mathcal{F}$  of subgroups of *G* if it satisfies the Meta-Isomorphism Conjecture 15.2 for the *G*-homology theory  $H^G_*(-; S^B)$ , i.e., the assembly map

$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{F}}(G); \mathbf{S}^B) \to H_n^G(G/G; \mathbf{S}^B)$$

is bijective for all  $n \in \mathbb{Z}$ .

Example 15.39 (The Farrell-Jones Conjecture in the setting of functors from spaces to spectra). In the sequel  $\Pi(X)$  denotes the fundamental groupoid of a space X. If we take the covariant functor S to be the one that sends a space X to  $\mathbf{K}_R(\Pi(X))$ ,  $\mathbf{L}_R^{\langle -\infty \rangle}(\Pi(X))$ , or  $\mathbf{K}_F^{\mathsf{TOP}}(\Pi(X))$  respectively, see Theorem 12.43, then the Meta-Isomorphism Conjecture 15.38 for **S** for a group G and the family  $\mathcal{VCY}, \mathcal{VCY}$ , or  $\mathcal{FIN}$  respectively is equivalent to the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R, the L-theoretic Farrell-Jones Conjecture 14.9 respectively. This follows from the obvious natural weak equivalence of groupoids  $\mathcal{G} \xrightarrow{\simeq} \Pi(\mathcal{BG})$ .

Let G be a group and Z be a G-CW-complex. Define a covariant Or(G)-spectrum

(15.40) 
$$\mathbf{S}_Z^G \colon \operatorname{Or}(G) \to \operatorname{SPECTRA}, \quad G/H \mapsto \mathbf{S}(G/H \times_G Z)$$

where  $G/H \times_G Z$  is the orbit space of the diagonal left *G*-action on  $G/H \times S$ . Note that there is an obvious homeomorphism  $G/H \times_G Z \xrightarrow{\cong} H \setminus Z$ .

Conjecture 15.41 (Meta-Isomorphism Conjecture for functors from spaces to spectra with coefficients). Let  $S: SPACES \rightarrow SPECTRA$  be a covariant functor that respects weak equivalences and disjoint unions. The group *G* satisfies the *Meta-Isomorphism Conjecture for* **S** *with coefficients* with respect to the family  $\mathcal{F}$  of subgroups of *G* if for any free *G-CW*-complex *Z* the pair ( $G, \mathcal{F}$ ) satisfies the Meta-Isomorphism Conjecture 15.2 for  $H_*^G(-; \mathbf{S}_Z^G)$ , i.e., the assembly map

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{S}_{\mathbf{Z}}^G) \to H_n^G(G/G; \mathbf{S}_{\mathbf{Z}}^G)$$

is bijective for all  $n \in \mathbb{Z}$ .

**Exercise 15.42.** Let **S**: SPACES  $\rightarrow$  SPECTRA be a covariant functor that respects weak equivalences. Suppose that it satisfies the Meta-Isomorphism Conjecture 15.41 for every group *G* and the trivial family  $\mathcal{TR}$  consisting of one element, the trivial subgroup. Let *X* be a connected *CW*-complex. Prove:

(i) We obtain a weak homotopy equivalence

$$E\pi_1(X)_+ \wedge_{\pi_1(X)} \mathbf{S}(X) \to \mathbf{S}(X);$$

- (ii)  $\pi_n(E\pi_1(X)_+ \wedge_{\pi_1(X)} \mathbf{S}(\widetilde{X}))$  and  $\pi_n(B\pi_1(X)_+ \wedge \mathbf{S}(\{\bullet\}))$  are not isomorphic in general;
- (iii)  $\pi_n(E\pi_1(X)_+ \wedge_{\pi_1(X)} \mathbf{S}(\widetilde{X}))$  and  $\pi_n(B\pi_1(X)_+ \wedge \mathbf{S}(\{\bullet\}))$  are isomorphic, provided that  $\widetilde{X}$  is contractible or **S** is of the shape  $Y \mapsto \mathbf{T}(\Pi(Y))$  for some covariant functor **T**: GROUPOIDS  $\rightarrow$  SPECTRA.

**Example 15.43** (Z = EG). If we take Z = EG in Conjecture 15.41, then Conjecture 15.41 reduces to Conjecture 15.38, since there is a natural homotopy equivalence  $G/H \times_G EG \xrightarrow{\simeq} B\mathcal{G}^G(G/H)$  and hence we get a weak homotopy equivalence of Or(G)-spectra  $\mathbf{S}_{EG}^G \xrightarrow{\simeq} \mathbf{S}^B(\mathcal{G}^G(G/?))$ .

**Remark 15.44 (Relation to the original formulation).** In [366, Section 1.7 on page 262] Farrell and Jones formulate a fibered version of their conjectures for a covariant functor **S**: SPACES  $\rightarrow$  SPECTRA for every (Serre) fibration  $\xi: Y \rightarrow X$  over a connected CW-complex *X*. In our setup this corresponds to choosing *Z* to be the total space of the fibration obtained from  $Y \rightarrow X$  by pulling back along the universal covering  $\widetilde{X} \rightarrow X$ . This space *Z* is a free *G*-*CW*-complex for  $G = \pi_1(X)$ . Note that an arbitrary free *G*-*CW*-complex *Z* can always be obtained in this fashion from the fiber bundle  $EG \times_G Z \rightarrow BG$  up to *G*-homotopy, compare [366, Corollary 2.2.1 on page 263].

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We sketch the proof of this identification. Let *A* be a *G*-*CW*-complex. Let  $\mathcal{E}(X)$  be the *G*-quotient of the diagonal  $G = \pi_1(X)$ -action on  $A \times \widetilde{X}$  and let  $f : \mathcal{E}(X) \to X$  be the obvious projection. Denote by  $\widehat{p} : \mathcal{E}(\xi) \to \mathcal{E}(X)$  the pullback of  $\xi$  with *f*. Let  $q : \mathcal{E}(\xi) \to A/G$  be the composite of  $\widehat{p}$  with the map  $\mathcal{E}(X) \to A/G$  induced by the projection  $A \times X \to A$ . This is a stratified fibration, and one can consider the spectrum  $\mathbb{H}(A/G; \mathcal{S}(q))$  in the sense of Quinn [824, Section 8]. Put

$$\mathcal{H}_n^G(A;\xi) := \pi_n(\mathbb{H}(A/G;\mathcal{S}(q))).$$

The projection pr:  $A \rightarrow G/G$  induces a map

(15.45) 
$$a_n(A): \mathcal{H}_n^G(A;\xi) \to \mathcal{H}_n^G(G/G;\xi) = \pi_n(\mathbf{S}(Y))$$

which is the assembly map in [366, Section 1.7 on page 262] if we take  $A = E_{VCY}(G)$ . The construction of  $\mathcal{H}_n^G(A;\xi) := \mathbb{H}(A/G;S(q))$  is very complicated, but, fortunately, for us only two facts are relevant. We obtain by  $\mathcal{H}_*^G(-;\xi)$  a *G*-homology theory in the sense of Definition 12.1, and for every  $H \subseteq G$  we get a natural identification  $\mathcal{H}_n^G(G/H;\xi) = \mathbf{S}_Z^G(G/H)$ . Hence the functor *G*-CW-COM  $\rightarrow$  SPECTRA sending  $A \rightarrow \mathbb{H}(A/G;S(q))$  is weakly excisive and its restriction to Or(G) is the functor  $\mathbf{S}_Z^G$ . Corollary 18.16 implies that the map (15.45) can be identified with the map induced by the projection  $A \rightarrow G/G$ 

$$H_n^G(A; \mathbf{S}_Z^G) \to H_n^G(G/G; \mathbf{S}_Z^G) = \pi_n(\mathbf{S}(Z/G)) = \pi_n(\mathbf{S}(Y))$$

which appears in Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients.

**Remark 15.46 (The condition free is necessary in Conjecture 15.41).** In general Conjecture 15.41 is not true if we drop the condition that *Z* is free. Take for instance Z = G/G. Then Conjecture 15.41 predicts that the projection  $E_{\mathcal{F}}(G)/G \to G/G$  induces for all  $n \in \mathbb{Z}$  an isomorphism

$$H_n(\operatorname{pr}; \mathbf{S}(\{\bullet\})) \colon H_n(E_{\mathcal{F}}(G)/G; \mathbf{S}(\{\bullet\})) \to H_n(\{\bullet\}, \mathbf{S}(\{\bullet\}))$$

where  $H_*(-; \mathbf{S}(\{\bullet\}))$  is the (non-equivariant) homology theory associated to the spectrum  $\mathbf{S}(\{\bullet\})$ . This statement is in general wrong, except in extreme cases such as  $\mathcal{F} = \mathcal{ALL}$ .

The proof of the next theorem will be given at the end of Section 15.9.

**Theorem 15.47 (Inheritance properties of the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients).** Let  $S: SPACES \rightarrow$ SPECTRA be a covariant functor that respects weak equivalences and disjoint unions. Fix a class of groups C that is closed under isomorphisms, taking subgroups, and taking quotients.

- (i) Suppose that the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients holds for the group G and the family of subgroups C(G) := {K ⊆ G, K ∈ C} of G. Let H ⊆ G be a subgroup. Then Conjecture 15.41 holds for (H, C(H));
- (ii) Let  $1 \to K \to G \xrightarrow{p} Q \to 1$  be an extension of groups. Suppose that (Q, C(Q))and  $(p^{-1}(H), C(p^{-1}(H))$  for every  $H \in C(Q)$  satisfy Conjecture 15.41. Then (G, C(G)) satisfies Conjecture 15.41;
- (iii) Suppose that Conjecture 15.41 is true for  $(H_1 \times H_2, C(H_1 \times H_2))$  for every  $H_1, H_2 \in C$ .

Then for two groups  $G_1$  and  $G_2$  Conjecture 15.41 is true for the direct product  $G_1 \times G_2$  and the family  $C(G_1 \times G_2)$  if and only it is true for  $(G_k, C(G_k))$  for k = 1, 2;

(iv) Suppose that, for any directed system of spaces  $\{X_i \mid i \in I\}$  indexed over an arbitrary directed set I, the canonical map

$$\operatorname{hocolim}_{i \in I} \mathbf{S}(X_i) \to \mathbf{S}(\operatorname{hocolim}_{i \in I} X_i)$$

is a weak homotopy equivalence. Let  $\{G_i \mid i \in I\}$  be a directed system of groups over a directed set I (with arbitrary structure maps). Put  $G = \operatorname{colim}_{i \in I} G_i$ . Suppose that Conjecture 15.41 holds for  $(G_i, C(G_i))$  for every  $i \in I$ ; Then Conjecture 15.41 holds for (G, C(G)).

**Exercise 15.48.** Let S: SPACES  $\rightarrow$  SPECTRA be a covariant functor which respects weak equivalences and disjoint unions. Suppose that, for any directed system of spaces  $\{X_i \mid i \in I\}$  indexed over an arbitrary directed set *I*, the canonical map

$$\operatorname{hocolim}_{i \in I} \mathbf{S}(X_i) \to \mathbf{S}(\operatorname{hocolim}_{i \in I} X_i)$$

is a weak homotopy equivalence. Let *C* be the class of finite groups or let *C* be the class of virtually cyclic subgroups. Suppose that Conjecture 15.41 holds for (H, C(H)) if *H* contains a subgroup *K* of finite index such that *K* is a finite product of finitely generated free groups.

Show that for a collection of groups  $\{G_i \mid i \in I\}$  Conjecture 15.41 is true for the free product  $*_{i \in I}G_i$  and the family  $C(*_{i \in I}G_i)$  if and only it is true for  $(G_i, C(G_i))$  for every  $i \in I$ .

**Lemma 15.49.** Suppose that there is a spectrum  $\mathbf{E}$  such that  $\mathbf{S}$ : SPACES  $\rightarrow$  SPECTRA is given by  $Y \mapsto Y_+ \wedge \mathbf{E}$ .

(i) Then, for any group G, any G-CW-complex X that is contractible (after forgetting the G-action), and any free G-CW-complex Z, the projection  $X \to G/G$  induces for all  $n \in \mathbb{Z}$  an isomorphism

$$H_n^G(X; \mathbf{S}_Z^G) \xrightarrow{\cong} H_n^G(G/G; \mathbf{S}_Z^G);$$

(ii) Both Conjecture 15.38 and Conjecture 15.41 for **S** hold for every group G and every family  $\mathcal{F}$  of subgroups of G.

*Proof.* (i) There are natural isomorphisms of spectra

$$\begin{split} & \operatorname{map}_{G}(G/?, X)_{+} \wedge_{\operatorname{Or}(G)} \left( (G/? \times_{G} Z)_{+} \wedge \mathbf{E} \right) \\ & \xrightarrow{\cong} \left( (\operatorname{map}_{G}(G/?), X) \times_{\operatorname{Or}(G)} G/?) \times_{G} Z \right)_{+} \wedge \mathbf{E} \\ & \xrightarrow{\cong} (X \times_{G} Z)_{+} \wedge \mathbf{E} \end{split}$$

where the second isomorphism comes from the G-homeomorphism

$$\operatorname{map}_{G}(G/?), X) \times_{\operatorname{Or}(G)} G/? \xrightarrow{=} X$$

of [280, Theorem 7.4 (1)]. Since *Z* is a free *G*-*CW*-complex and *X* is contractible (after forgetting the group action), the projection  $X \times_G Z \to G/G \times_G Z$  is a homotopy equivalence and hence induces a weak homotopy equivalence

$$(X \times_G Z)_+ \wedge \mathbf{E} \xrightarrow{-} (G/G \times_G Z)_+ \wedge \mathbf{E}.$$

Thus we get a weak homotopy equivalence

$$\operatorname{map}_{G}(G/?), X)_{+} \wedge_{\operatorname{Or}(G)} \left( (G/? \times_{G} Z)_{+} \wedge \mathbf{E} \right) \to (G/G \times_{G} Z)_{+} \wedge \mathbf{E}.$$

Under the identifications coming from the definitions

$$H_n^G(X; \mathbf{S}_Z^G) := \pi_n \big( \operatorname{map}_G(G/?, X)_+ \wedge_{\operatorname{Or}(G)} ((G/? \times_G Z)_+ \wedge \mathbf{E}) \big);$$
  
$$H_n^G(G/G; \mathbf{S}_Z^G) = \pi_n \left( (G/G \times_G Z)_+ \wedge \mathbf{E} \right),$$

this weak homotopy equivalence induces on homotopy groups the isomorphism  $H_n^G(X; \mathbf{S}_Z^G) \to H_n^G(G/G; \mathbf{S}_Z^G)$ .

(ii) This follows from assertion (i).

**Lemma 15.50.** Let **S**, **T**, **U**: SPACES  $\rightarrow$  SPECTRA be covariant functors that respect weak equivalences and disjoint unions. Let **i**: **S**  $\rightarrow$  **T** and **p**: **T**  $\rightarrow$  **U** be natural transformations such that for any space *Y* the map of spectra **S**(*Y*)  $\xrightarrow{\mathbf{i}(Y)}$  **T**(*Y*)  $\xrightarrow{\mathbf{p}(Y)}$  **U**(*Y*) is up to weak homotopy equivalence a cofibration of spectra.

(i) Then we obtain for every group G and all G-CW-complexes X and Z a natural long exact sequence

$$\cdots \to H_n^G(X; \mathbf{S}_Z^G) \to H_n^G(X; \mathbf{T}_Z^G) \to H_n^G(X; \mathbf{U}_Z^G) \to H_{n-1}^G(X; \mathbf{S}_Z^G) \to H_{n-1}^G(X; \mathbf{T}_Z^G) \to H_{n-1}^G(X; \mathbf{U}_Z^G) \to \cdots ;$$

(ii) Let G be a group and F be a family of subgroups of G. Then Conjecture 15.38 or Conjecture 15.41 respectively holds for all three functors S, T, and U for (G, F) if Conjecture 15.38 or Conjecture 15.41 respectively holds for two of the functors S, T, and U for (G, F).

*Proof.* (i) This is a consequence of the fact following from the version for spectra of [280, Theorem 3.11] that we obtain an up to weak homotopy equivalence cofiber sequence of spectra

$$\operatorname{map}_{G}(G/?, X)_{+} \wedge_{\operatorname{Or}(G)} \mathbf{S}(G/? \times_{G} Z) \to \operatorname{map}_{G}(G/?, X)_{+} \wedge_{\operatorname{Or}(G)} \mathbf{T}(G/? \times_{G} Z)$$
$$\to \operatorname{map}_{G}(G/?, X)_{+} \wedge_{\operatorname{Or}(G)} \mathbf{U}(G/? \times_{G} Z).$$

(ii) This follows from assertion (i) and the Five Lemma.

#### **15.9 Proof of the Inheritance Properties**

This section is entirely devoted to the proof of Theorem 15.47.

Let S: SPACES  $\rightarrow$  SPECTRA be a covariant functor. Throughout this section we will assume that it respects weak equivalences and disjoint unions.

**Lemma 15.51.** Let  $\psi$ :  $K_1 \rightarrow K_2$  be a group homomorphism.

(i) If Z is a  $K_1$ -CW-complex and X is a  $K_2$ -CW-complex, then there is a natural isomorphism

$$H_n^{K_1}(\psi^*X;\mathbf{S}_Z^{K_1}) \xrightarrow{\cong} H_n^{K_2}(X;\mathbf{S}_{\psi_*Z}^{K_2});$$

(ii) If Z is a  $K_2$ -CW-complex and X is a  $K_1$ -CW-complex, then there is a natural isomorphism

$$H_n^{K_1}(X; \mathbf{S}_{\psi^* Z}^{K_1}) \xrightarrow{\cong} H_n^{K_2}(\psi_* X; \mathbf{S}_Z^{K_2})$$

*Proof.* (i) The fourth isomorphism appearing in [280, Lemma 1.9] implies that it suffices to construct a natural weak homotopy equivalence of  $Or(K_2)$ -spectra

$$u(\psi, Z) \colon \psi_* \mathbf{S}_Z^{K_1} \xrightarrow{\cong} \mathbf{S}_{\psi_* Z}^{K_2}$$

where  $\psi_* \mathbf{S}_Z^{K_1}$  is the  $\operatorname{Or}(K_2)$ -spectrum obtained by induction in the sense of [280, Definition 1.8] with the functor  $\operatorname{Or}(\psi)$ :  $\operatorname{Or}(K_1) \to \operatorname{Or}(K_2)$ ,  $K_1/H \mapsto \psi_*(K_1/H)$  applied to the  $\operatorname{Or}(K_1)$ -spectrum  $\mathbf{S}_Z^{K_1}$ . For a homogeneous space  $K_2/H$ , we define  $u(\psi, Z)(K_2/H)$  to be the composite

$$\begin{split} \psi_* \mathbf{S}_Z^{K_1}(K_2/H) &:= \operatorname{map}_{K_2}(\psi_*(K_1/?), K_2/H) \times_{\operatorname{Or}(K_1)} \mathbf{S}\left(K_1/? \times_{K_1} Z\right) \\ & \xrightarrow{\cong} \operatorname{map}_{K_1}(K_1/?, \psi^*(K_2/H)) \times_{\operatorname{Or}(K_1)} \mathbf{S}(K_1/? \times_{K_1} Z) \\ & \xrightarrow{\cong} \mathbf{S}(\psi^*(K_2/H) \times_{K_1} Z) \\ & \xrightarrow{\cong} \mathbf{S}(K_2/H \times_{K_2} \psi_* Z) \\ & =: \mathbf{S}_{\psi_* Z}^{K_2}(K_2/H). \end{split}$$

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Here the first map comes from the adjunction isomorphism

$$\operatorname{map}_{K_2}(\psi_*(K_1/?), K_2/H) \xrightarrow{=} \operatorname{map}_{K_1}(K_1/?, \psi^*(K_2/H))$$

and the third map comes from the canonical homeomorphism

$$\psi^*(K_2/H) \times_{K_1} Z \xrightarrow{=} K_2/H \times_{K_2} \psi_* Z.$$

The second map is the special case  $T = \psi^* K_2/?$  of the natural weak homotopy equivalence defined for any  $K_1$ -set T

$$\kappa(T): \operatorname{map}_{K_1}(K_1/?, T) \times_{\operatorname{Or}(K_1)} \mathbf{S}(K_1/? \times_{K_1} Z) \xrightarrow{=} \mathbf{S}(T \times_{K_1} Z)$$

that is given by  $(u: K_1/? \to T) \times s \mapsto \mathbf{S}(u \times_{K_1} \operatorname{id}_Z)(s)$ . If *T* is a homogeneous  $K_1$ -set, then  $\kappa(T)$  is an isomorphism by the Yoneda Lemma. Since  $\psi$  is compatible with disjoint unions, **S** is compatible with disjoint unions up to weak homotopy equivalence by assumption, and every  $K_1$ -set is the disjoint union of homogeneous  $K_1$ -sets,  $\kappa(T)$  is a weak homotopy equivalence for all  $K_1$ -sets *T*.

(ii) The third isomorphism appearing in [280, Lemma 1.9] implies that it suffices to construct a natural weak homotopy equivalence of  $Or(K_1)$ -spectra

$$v(\psi, Z) \colon \psi^* \mathbf{S}_Z^{K_2} \xrightarrow{\simeq} \mathbf{S}_{\psi^* Z}^{K_1}$$

where  $\psi^* \mathbf{S}_Z^{K_2}$  is the  $Or(K_1)$ -spectrum obtained by restriction in the sense of [280, Definition 1.8] with the functor  $Or(\psi)$ :  $Or(K_1) \to Or(K_2)$ ,  $K_1/H \mapsto \psi_*(K_1/H)$  applied to the  $Or(K_2)$ -spectrum  $\mathbf{S}_Z^{K_2}$ . Actually, we obtain even an isomorphism  $v(\psi, Z)$  using the adjunction

$$\psi_*(K_1/H) \times_{K_2} Z \cong K_1/H \times_{K_1} \psi^* Z$$

for any subgroup  $H \subseteq K_1$ .

Note that for a homomorphism  $\phi: H \to G$  the restriction  $\phi^*Z$  of a free *G-CW*-complex *Z* is free again if and only if  $\phi$  is injective. We have already explained in Remark 15.46 that the assumption that *Z* is free is needed in Conjecture 15.41. In the Fibered Meta-Isomorphism Conjecture 15.8 it is crucial not to require that  $\phi: H \to G$  is injective, since we want to have good inheritance properties such as the one appearing in assertion (iii) of Lemma 15.23, which will be crucial for the proof of assertion (iv) of Theorem 15.47. Therefore we are forced to introduce the following construction.

Consider a group *G* and a *G*-*CW*-complex *Z*. We want to define an equivariant homology theory  $H^{?}_{*}(-; \mathbf{S}_{Z}^{\downarrow G})$  over *G* in the sense of Definition 12.91. Given a group homomorphism  $\phi: K \to G$ , define the associated *K*-homology theory

$$H^{K,\phi}_*(-;\mathbf{S}^{\downarrow G}_Z) := H^K_*(-;\mathbf{S}^K_{EK\times\phi^*Z}).$$

Given group homomorphisms  $\psi : K_1 \to K_2$ ,  $\phi_1 : K_1 \to G$ , and  $\phi_2 : K_2 \to G$  with  $\phi_2 \circ \psi = \phi_1$ , a  $K_1$ -*CW*-complex *X*, and  $n \in \mathbb{Z}$ , we have to define a natural map

$$H_n^{K_1}(X; \mathbf{S}_{EK_1 \times \phi_1^* Z}^{K_1}) \to H_n^{K_2}(\psi_* X; \mathbf{S}_{EK_2 \times \phi_2^* Z}^{K_2})$$

We get the isomorphism  $H_n^{K_2}(\psi_*X; \mathbf{S}_{EK_2 \times \phi_2^*Z}^{K_2}) = H_n^{K_1}(X; \mathbf{S}_{\psi^*(EK_2 \times \phi_2^*Z)}^{K_2})$  from Lemma 15.51 (ii). Hence it suffices to specify a  $K_1$ -map

$$EK_1 \times \phi_1^* Z \longrightarrow \psi^* (EK_2 \times \phi_2^* Z) = \psi^* (EK_2) \times \phi_1^* Z.$$

The homomorphism  $\psi: K_1 \to K_2$  induces a  $K_1$ -map  $EK_1 \to \psi^*(EK_2)$  and we can take its product with  $\mathrm{id}_{\phi^*Z}$ .

The proof of the next lemma is left to the reader.

**Lemma 15.52.** Given a group G and a G-CW-complex Z, all the axioms of an equivariant homology theory over G, see Definition 12.91, are satisfied by  $H_*^?(-; \mathbf{S}_Z^{\downarrow G})$ .

Exercise 15.53. Let G be a group and Z be a G-CW-complex. Consider the functor

**E**: GROUPOIDS 
$$\downarrow G \rightarrow$$
 SPECTRA,  $p: \mathcal{G} \rightarrow I(G) \mapsto \mathbf{S}(E\mathcal{G} \times_{\mathcal{G}} p^*Z).$ 

Here  $E\mathcal{G}$  is the classifying  $\mathcal{G}$ -CW-complex associated to  $\mathcal{G}$ , see [280, Definition 3.8], for which we use the functorial model  $E^{\text{bar}}\mathcal{G}$  of [280, page 230], we consider Z as an I(G)-CW-complex and hence get a  $\mathcal{G}$ -CW-complex  $p^*Z$  by restriction with  $p: \mathcal{G} \to I(G)$ , and the space  $E\mathcal{G} \times_{\mathcal{G}} p^*Z$  is defined in [280, Definition 1.4].

Show that the equivariant homology theory  $H^{?}_{*}(-; \mathbf{E})$  over *G* associated to **E** in Theorem 12.93 is isomorphic to  $H^{?}_{*}(-; \mathbf{S}_{Z}^{\downarrow G})$ .

**Lemma 15.54.** Let  $\phi: H \to K$  and  $\psi: K \to G$  be group homomorphisms.

(i) Let X be a G-CW-complex and let Z be a K-CW-complex. Then we obtain a natural isomorphism

$$H_n^{H,\phi}(\phi^*\psi^*X;\mathbf{S}_Z^{\downarrow K})\xrightarrow{\cong} H_n^G(X;\mathbf{S}_{(\psi\circ\phi)_*(EH\times\phi^*Z)}^G);$$

(ii) Let X be a H-CW-complex and let Z be a G-CW-complex. Then we obtain a natural isomorphism

$$H_n^{H,\phi}(X;\mathbf{S}_{\psi^*Z}^{\downarrow K}) \xrightarrow{\cong} H_n^{H,\psi\circ\phi}(X;\mathbf{S}_Z^{\downarrow G}).$$

*Proof.* (i) We have by definition

$$H_n^{H,\phi}(\phi^*\psi^*X;\mathbf{S}_Z^{\downarrow K}) := H_n^H(\phi^*\psi^*X;\mathbf{S}_{EH\times\phi^*Z}^H).$$

Now apply Lemma 15.51 (i).

15.9 Proof of the Inheritance Properties

(ii) We get by definition

$$\begin{split} H_n^{H,\phi}(X;\mathbf{S}_{\psi^*Z}^{\downarrow K}) &:= H_n^H(X;\mathbf{S}_{EH\times\phi^*\psi^*Z}^H) \\ &= H_n^H(X;\mathbf{S}_{EH\times(\psi\circ\phi)^*Z}^H) =: H_n^{H,\psi\circ\phi}\big(X;\mathbf{S}_Z^{\downarrow G}\big). \end{split}$$

Conjecture 15.55 (Fibered Meta-Isomorphism Conjecture for functors from spaces to spectra with coefficients). We say that S satisfies the Fibered Meta-Isomorphism Conjecture for a functor from spaces to spectra with coefficients for the group G and the family of subgroups  $\mathcal{F}$  of G if for any G-CW-complex Z the equivariant homology theory  $H^{?}_{*}(-; \mathbf{S}_{Z}^{\downarrow G})$  over G satisfies the Fibered Meta-Isomorphism Conjecture 15.8 for the group  $(G, \operatorname{id}_{G})$  over G and the family  $\mathcal{F}$ .

Note that Conjecture 15.41 is a statement about  $H^G_{Z}(-\mathbf{S}^G_{Z})$  and Z is required to be a free *G*-*CW*-complex, whereas Conjecture 15.55 is a statement about  $H^G_{*}(-; \mathbf{S}^{\downarrow G}_{Z})$ and Z can be any *G*-*CW*-complex. Moreover, we introduce Conjecture 15.55 only for technical reasons.

**Lemma 15.56.** Let  $\psi : K \to G$  be a group homomorphism.

- (i) Suppose that the Meta Conjecture 15.41 with coefficients holds for the group G and the family F. Then the Fibered Meta Conjecture 15.55 with coefficients holds for the group K and the family ψ\*F;
- (ii) If the Fibered Meta Conjecture 15.55 with coefficients holds for the group G and the family  $\mathcal{F}$ , then the Meta Conjecture 15.41 with coefficients holds for the group G and the family  $\mathcal{F}$ ;
- (iii) Suppose that the Fibered Meta Conjecture 15.55 with coefficients holds for K and the family  $\mathcal{F}$ . Then for every G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_n(-; \mathbf{S}_Z^{\downarrow G})$  over G for the group  $(K, \psi)$  over G and the family  $\mathcal{F}$  of subgroups of K.

*Proof.* (i) This follows from Lemma 15.54 (i) since in the notation used there we have  $\phi^*\psi^*E_{\mathcal{F}}(G) = \phi^*E_{\psi^*\mathcal{F}}(K)$  and  $\phi^*\psi^*G/G = H/H$ , and  $(\psi \circ \phi)_*(EH \times \phi^*Z)$  is a free *G*-*CW*-complex.

(ii) This follows from the fact that for a free *G*-*CW*-complex *Z* the projection  $EG \times Z \rightarrow Z$  is a *G*-homotopy equivalence and hence we get a natural isomorphism

$$H_n^{G,\mathrm{id}_G}(X;\mathbf{S}_Z^{\downarrow G}):=H_n^G(X;\mathbf{S}_{EG\times Z}^G)\xrightarrow{\cong} H_n^G(X;\mathbf{S}_Z^G)$$

for every *G*-*CW*-complex *X* and  $n \in \mathbb{Z}$ .

(iii) This follows from Lemma 15.54 (ii).

**Lemma 15.57.** Suppose that, for any directed system of spaces  $\{X_i \mid i \in I\}$  indexed over an arbitrary directed set *I*, the canonical map

$$\operatorname{hocolim}_{i \in I} \mathbf{S}(X_i) \to \mathbf{S}(\operatorname{hocolim}_{i \in I} X_i)$$

is a weak homotopy equivalence.

Then for every group G and G-CW-complex Z the equivariant homology theory over G given by  $H^{?}_{*}(-\mathbf{S}_{Z}^{\downarrow G})$  is strongly continuous.

*Proof.* We only treat the case  $id_G: G \to G$ , the case of a group  $\psi: K \to G$  over *G* is completely analogous. Consider a directed system of groups  $\{G_i \mid i \in I\}$  with  $G = \operatorname{colim}_{i \in I} G_i$ . Let  $\psi_i: G_i \to G$  be the structure map for  $i \in I$ .

The canonical map

(15.58) hocolim<sub>*i* \in I</sub>  $\mathbf{S}(EG_i \times_{G_i} \psi_i^* Z) \to \mathbf{S}(\text{hocolim}_{i \in I}(EG_i \times_{G_i} \psi_i^* Z))$ 

is by assumption a weak homotopy equivalence. We have the homeomorphisms

$$EG_i \times_{G_i} \psi_i^* Z \xrightarrow{\cong} (\psi_i)_* EG_i \times_G Z;$$
  
(hocolim<sub>i \in I</sub>(\psi\_i)\_\* EG\_i) \times\_G Z \xrightarrow{\cong} hocolim\_{i \in I} ((\psi\_i)\_\* EG\_i \times\_G Z).

They induce a homeomorphism

(15.59)  $\mathbf{S}(\operatorname{hocolim}_{i \in I}(EG_i \times_{G_i} \psi_i^* Z)) \xrightarrow{\cong} \mathbf{S}(\operatorname{hocolim}_{i \in I}(\psi_i)_* EG_i) \times_G Z)$ 

The canonical map

$$\operatorname{hocolim}_{i \in I}(\psi_i)_* EG_i \to EG$$

is a G-homotopy equivalence. The proof of this fact is a special case of the argument appearing in the proof of [688, Theorem 4.3 on page 516]. It induces a weak homotopy equivalence

(15.60) 
$$\mathbf{S}((\operatorname{hocolim}_{i \in I}(\psi_i)_* EG_i) \times_G Z) \to \mathbf{S}(EG \times_G Z).$$

Hence we get by taking the composite of the maps (15.58), (15.59), and (15.60) a weak homotopy equivalence

hocolim<sub>$$i \in I$$</sub> **S**( $EG_i \times_{G_i} \psi_i^* Z$ )  $\rightarrow$  **S**( $EG \times_G Z$ ).

It induces after taking homotopy groups for every  $n \in \mathbb{Z}$  an isomorphism

$$\operatorname{colim}_{i\in I} \pi_n \big( \mathbf{S}(EG_i \times_{G_i} \psi_i^* Z) \big) \to \pi_n \big( \mathbf{S}(EG \times_G Z) \big)$$

which is by definition the same as the canonical map

$$\operatorname{colim}_{i \in I} H_n^{G_i, \psi_i}(G_i/G_i; \mathbf{S}_Z^{\downarrow G}) \to H_n^{G, \operatorname{id}_G}(G/G; \mathbf{S}_Z^{\downarrow G}).$$

This finishes the proof of Lemma 15.57.

*Proof of Theorem* 15.47. (i) Consider a free *H*-*CW*-complex *Z*. Let  $i: H \to G$  be the inclusion. Then  $i_*Z$  is a free *G*-*CW*-complex,  $i^*E_{C(G)}(G)$  is a model for  $E_{C(H)}(H)$ , and  $i^*G/G = K/K$ . From Lemma 15.51 (i) we obtain a commutative diagram with isomorphisms as vertical maps

$$\begin{array}{c} H_n^H(E_{\mathcal{C}(H)}(H); \mathbf{S}_Z^H) \longrightarrow H_n^H(H/H; \mathbf{S}_Z^G) \\ \cong & & \downarrow & \downarrow \\ H_n^G(E_{\mathcal{C}(G)}(G); \mathbf{S}_{i,Z}^G) \longrightarrow H_n^G(G/G; \mathbf{S}_{i,Z}^G) \end{array}$$

where the horizontal maps are induced by the projections. The lower map is bijective by assumption. Hence the upper map is bijective as well.

(ii) Since (Q, C(Q)) and  $(p^{-1}(H), C(p^{-1}(H)))$  for every  $H \in C(Q)$  satisfy the Meta-Isomorphism Conjecture Conjecture 15.41 with coefficients by assumption, we conclude from Lemma 15.56 (i) that the Fibered Meta-Isomorphism Conjecture 15.55 with coefficients holds for the group G and the family  $p^*C(Q)$  and that for every  $H \in C(Q)$  the Fibered Meta-Isomorphism Conjecture 15.55 with coefficients holds for  $p^{-1}(H)$  and the family  $C(p^{-1}(H)) = C(G)|_{p^{-1}(H)}$ . Lemma 15.56 (iii) implies that for every  $H \in C(Q)$  and G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_n^?(-; \mathbf{S}_Z^{\downarrow G})$  over G for the group  $(p^{-1}(H) \subseteq G)$  over G and the family  $C(G)|_{p^{-1}(H)}$ . Since for every  $L \in p^*C(Q)$  we have  $p(L) \in C(Q)$  and hence  $L \subseteq p^{-1}(p(L))$ , we conclude from Lemma 15.16 that for every  $L \in p^*C(Q)$  and *G*-*CW*-complex *Z* the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_n^?(-; \mathbf{S}_{\mathbf{Z}}^{\downarrow G})$ over G for the group  $(L \subseteq G)$  over G and the family  $C(G)|_L$ . The Transitivity Principle 15.13 (i) implies that for every G-CW-complex Z the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_n^?(-; \mathbf{S}_z^{\downarrow G})$  over G for the group  $(G, id_G)$  over G and the family C(G), in other words, the Fibered Meta-Isomorphism Conjecture 15.55 with coefficients holds for G and the family C(G). We conclude from Lemma 15.56 (ii) that the Meta-Isomorphism Conjecture 15.41 holds for the group G and the family C(G).

(iii) If the Meta-Isomorphism Conjecture 15.41 with coefficients holds for  $(G_1 \times G_1, C(G_1 \times G_2))$ , it holds for  $G_k$  and the family  $C(G_k) = C(G_1 \times G_2)|_{G_k}$  for k = 1, 2 by assertion (i).

Suppose that the Meta-Isomorphism Conjecture 15.41 with coefficients holds for  $(G_k, C(G_k))$  for k = 1, 2. By assertion (ii) applied to the obvious exact sequence  $1 \rightarrow H_2 \rightarrow G_1 \times H_2 \rightarrow G_1 \rightarrow 1$ , Conjecture 15.41 holds for  $(G_1 \times H_2, C(G_1 \times H_2))$  for every  $H_2 \in C(G_2)$ . By assertion (ii) applied to the obvious exact sequence  $1 \rightarrow G_1 \rightarrow G_1 \times G_2 \rightarrow G_2 \rightarrow 1$  Conjecture 15.41 with coefficients holds for  $(G_1 \times G_2, C(G_1 \times G_2))$ .

(iv) Since the Meta-Isomorphism Conjecture 15.41 with coefficients holds for  $G_i$  and  $C(G_i)$  for every  $i \in I$  by assumption, we conclude from Lemma 15.56 (i) that the Fibered Meta-Isomorphism Conjecture 15.55 with coefficients holds for the group

 $G_i$  and the family  $C(G_i)$  for every  $i \in I$ . Lemma 15.56 (iii) implies that for every  $i \in I$ and *G*-*CW*-complex *Z* the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_n(-; \mathbf{S}_Z^{\downarrow G})$  over *G* for the group  $\psi_i : G_i \to G$  over *G* and the family  $C(G_i)$ . We conclude from Lemma 15.23 (iii) and Lemma 15.57 that for every *G*-*CW*-complex *Z* the Fibered Meta-Isomorphism Conjecture 15.8 holds for the equivariant homology theory  $H_*^2(-; \mathbf{S}_Z^{\downarrow G})$  over *G* for the group  $(G, \text{id}_G)$  over *G* and the family C(G), in other words, the Fibered Meta-Isomorphism Conjecture 15.55 with coefficients holds for the group *G* and the family C(G). We conclude from Lemma 15.56 (ii) that the Meta-Isomorphism Conjecture Conjecture 15.41 with coefficients holds for the group *G* and the family C(G). This finishes the proof of Theorem 15.47.

### **15.10** The Farrell-Jones Conjecture for *A*-Theory, Pseudoisotopy, and Whitehead Spaces

**Conjecture 15.61 (Farrell-Jones Conjecture for** *A***-theory (with coefficients)).** A group *G* satisfies the *Farrell-Jones Conjecture for A-theory* if the Meta-Isomorphism Conjecture 15.38 for functors from spaces to spectra applied to the case  $\mathbf{S} = \mathbf{A}$  for the functor non-connective *A*-theory  $\mathbf{A}$  introduced in (7.12) holds for (*G*,  $\mathcal{VCY}$ ).

A group *G* satisfies the *Farrell-Jones Conjecture for A-theory with coefficients* if the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients applied to the case S = A for the functor non-connective *A*-theory *A* introduced in (7.12) holds for  $(G, \mathcal{VCY})$ .

Note that A respects weak equivalences and disjoint unions, see Theorem 7.16.

**Exercise 15.62.** Suppose that *G* is torsionfree and satisfies the Farrell-Jones Conjecture 15.61 for *A*-theory. Show that  $\pi_n(\mathbf{A}(BG)) = 0$  for  $n \le -1$  and  $\pi_0(\mathbf{A}(BG)) \cong \mathbb{Z}$ .

Conjecture 15.63 (Farrell-Jones Conjecture for (smooth) pseudoisotopy (with coefficients)). A group *G* satisfies the *Farrell-Jones Conjecture for (smooth) pseudoisotopy* if the Meta-Isomorphism Conjecture 15.38 for functors from spaces to spectra applied to the case  $\mathbf{S} = \mathbf{P}$  or  $\mathbf{S} = \mathbf{P}^{\text{DIFF}}$  for the functor non-connective (smooth) pseudoisotopy  $\mathbf{P}$  and  $\mathbf{P}^{\text{DIFF}}$  of Definition 7.1 holds for (*G*,  $\mathcal{VCY}$ ).

A group *G* satisfies the *Farrell-Jones Conjecture for (smooth) pseudoisotopy with coefficients* if the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients applied to the case  $\mathbf{S} = \mathbf{P}$  or  $\mathbf{S} = \mathbf{P}^{\text{DIFF}}(X)$  for the functor non-connective (smooth) pseudoisotopy  $\mathbf{P}$  and  $\mathbf{P}^{\text{DIFF}}$  of Definition 7.1 holds for  $(G, \mathcal{VCY})$ .

**Conjecture 15.64 (Farrell-Jones Conjecture for (smooth) Whitehead spectra** (with coefficients)). A group *G* satisfies the *Farrell-Jones Conjecture for (smooth) Whitehead spectra* if the Meta-Isomorphism Conjecture 15.38 for functors from

spaces to spectra applied to the case S = Wh or  $S = Wh^{DIFF}$  for the functor nonconnective (smooth) Whitehead spectra Wh and Wh<sup>DIFF</sup> of Remark 7.34 holds for  $(G, \mathcal{VCY})$ .

A group *G* satisfies the *Farrell-Jones Conjecture for (smooth) Whitehead spectra with coefficients* if the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients applied to the case  $\mathbf{S} = \mathbf{W}\mathbf{h}$  or  $\mathbf{S} = \mathbf{W}\mathbf{h}^{\text{DIFF}}$  for the functor non-connective (smooth) Whitehead spectra  $\mathbf{W}\mathbf{h}$  and  $\mathbf{W}\mathbf{h}^{\text{DIFF}}$  of Remark 7.34 holds for  $(G, \mathcal{VCY})$ .

Note that **P** and **P**<sup>DIFF</sup> respect weak equivalences and disjoint unions, see Theorem 7.3. Moreover, **Wh** and **Wh**<sup>DIFF</sup> also have these properties because of Theorem 7.16 and the homotopy (co)fibration sequences (7.35) and (7.36).

#### Theorem 15.65.

- (i) *The following assertions are equivalent for a group G:* 
  - (a) The Farrell-Jones Conjecture 15.61 for A-theory holds for G;
  - (b) The Farrell-Jones Conjecture 15.63 for pseudoisotopy holds for G;
  - (c) The Farrell-Jones Conjecture 15.63 for smooth pseudoisotopy holds for G;
  - (d) The Farrell-Jones Conjecture 15.64 for Whitehead spectra holds for G;
  - (e) *The Farrell-Jones Conjecture* 15.64 *for smooth Whitehead spectra holds for G*;
- (ii) Assertion (i) holds also for the versions of the conjectures with coefficients;
- (iii) Suppose that the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories, see Conjecture 13.23 holds for G. Then the versions with coefficients of the Conjectures 15.61, 15.63, and 15.64 holds for G.

*Proof.* Assertions (i) and (ii) are proved in [344, Lemma 3.3]. They are direct consequences of (7.35), (7.36), the non-connective versions of (7.26) and (7.30), Lemma 15.49 (ii), and Lemma 15.50 (ii).

(iii) is proved in [180, Example 1.1.11 and Corollary 7.5.6]

# 15.11 The Farrell-Jones Conjecture for Topological Hochschild and Cyclic Homology

There are the notions of Hochschild homology and cyclic homology of algebras, which are defined in the algebraic setting, see for instance Connes [250] or Loday [636]. One of the important insights of Waldhausen was that one can define an analog of algebraic *K*-theory for rings where one "spacifies" the constructions. This led to *A*-theory, which we have described in Chapter 7. This circle of ideas also motivated the definition of topological Hochschild homology by Bökstedt and then of topological cyclic homology by Bökstedt-Hsiang-Madsen [150], which are better approximations of the algebraic *K*-theory than their original algebraic counterparts.

A systematic study of how much algebraic cyclic homology detects from the algebraic *K*-theory of group rings is presented in [674], showing that the topological versions are much more effective. Roughly speaking, in the topological versions one replaces rings by ring spectra and tensor products by (highly structured and strictly commutative) smash products. The role of the ring  $\mathbb{Z}$  of integers, which is initial in the category of rings, is now played by the sphere spectrum  $\mathbb{S}$ , which is initial in the category of ring spectra. For further information, we refer to the book by Dundas-Goodwillie-McCarthy [316] and the survey article by Madsen [696].

Given a symmetric ring spectrum A and a prime p, one can define functors see [675, (14.1) and Example 14.3]

(15.66) **THH**<sub>A</sub>: GROUPOIDS 
$$\rightarrow$$
 SPECTRA;

(15.67) 
$$TC_{\mathbb{A};p}$$
: GROUPOIDS  $\rightarrow$  SPECTRA,

such that for a group *G* considered as the groupoid I(G) the value of these functors is the topological Hochschild homology and the topological cyclic homology with respect to the prime *p* of the group ring spectrum  $\mathbb{A}[G] := \mathbb{A} \wedge G_+$ . From Theorem 12.30 we obtain equivariant homology theories  $\mathcal{H}^{?}_{*}(-; \mathbf{THH}_{\mathbb{A}})$  and  $\mathcal{H}^{?}_{*}(-; \mathbf{TC}_{\mathbb{A};p})$  satisfying for any group *G* and subgroup  $H \subseteq G$ 

$$\mathcal{H}_{n}^{G}(G/H; \mathbf{THH}_{\mathbb{A}}) = \mathcal{H}_{n}^{H}(H/H; \mathbf{THH}_{\mathbb{A}}) = \pi_{n}(\mathbf{THH}(\mathbb{A}[H]));$$
  
$$\mathcal{H}_{n}^{G}(G/H; \mathbf{TC}_{\mathbb{A};p}) = \mathcal{H}_{n}^{H}(H/H; \mathbf{TC}_{\mathbb{A};p}) = \pi_{n}(\mathbf{TC}(\mathbb{A}[H]; p)).$$

#### 15.11.1 Topological Hochschild Homology

The following theorem is taken from [675, Theorem 1.19]. The notion of a very well pointed spectrum and of a connective<sup>+</sup>-spectrum are introduced in [675, Subsection 4J]. These are mild condition that are satisfied by the sphere spectrum S and the Eilenberg-MacLane spectrum of a discrete ring.

**Theorem 15.68 (The Farrell-Jones Conjecture holds for topological Hochschild homology).** Let G be a group and  $\mathcal{F}$  be a family of subgroups. Let  $\mathbb{A}$  be a very well pointed symmetric ring spectrum. Then the map induced by the projection pr:  $E_{\mathcal{F}}(G) \to G/G$ 

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{THH}_{\mathbb{A}}) \to H_n^G(G/G; \mathbf{THH}_{\mathbb{A}}) = \pi_n(\mathbf{THH}(\mathbb{A}[G]))$$

is split injective for all  $n \in \mathbb{Z}$ . If  $\mathcal{F}$  contains all cyclic subgroups, then it is bijective for all  $n \in \mathbb{Z}$ .

Topological Hochschild homology is one of the rare instances where an Isomorphism Conjecture is known for all groups and an interesting family of subgroups, namely the family of all cyclic subgroups, and the reasons are not completely elementary.

#### 15.11.2 Topological Cyclic Homology

For the rest of this subsection we assume that  $\mathbb{A}$  is connective<sup>+</sup>.

The assembly map for topological cyclic homology

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G; \mathbf{TC}_{\mathbb{A}.p}) = \pi_n(\mathbf{TC}_p(\mathbb{A}[G]))$$

for the family  $\mathcal{VCY}$  of virtually cyclic subgroups is not bijective in general. For instance, it is not surjective for n = -1 if  $\mathbb{A}$  is the Eilenberg-MacLane spectrum associated to the (discrete) ring  $\mathbb{Z}_{(p)}$  and *G* is either finitely generated free abelian or torsionfree hyperbolic, but not cyclic, see [676, Theorem 1.5]. More counterexamples against surjectivity are presented in [676, Remark 6.7]. Counterexamples against rational injectivity are described in [676, Remark 1.9] based on [675, Remark 3.7].

There are also some positive results.

**Theorem 15.69 (Bijectivity of the assembly map for topological cyclic homology for finite groups and the family of cyclic subgroups).** If G is finite, then the assembly map for the family of cyclic subgroups

$$H_n^G(E_{\mathcal{CYC}}(G); \mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G; \mathbf{TC}_{\mathbb{A}.p}) = \pi_n l(\mathbf{TC}_p(\mathbb{A}[G]))$$

*is bijective for all*  $n \in \mathbb{Z}$ *.* 

Proof. See [676, Theorem 1.1].

**Exercise 15.70.** Let  $S_3$  be the symmetric group on the set  $\{1, 2, 3\}$ . Let  $C_2$  and  $C_3$  be any cyclic subgroups of  $S_3$  of order 2 and 3.

Show that for any prime *p* there is a weak equivalence

$$\mathbf{TC}(\mathbb{A}[C_2]; p) \lor ((EC_2)_+ \land_{C_2} \mathbf{TC}(\mathbb{A}[C_3]; p)) \xrightarrow{\simeq} \mathbf{TC}(\mathbb{A}[S_3]; p)$$

where  $C_2$  acts on  $C_3$  by sending the generator to its inverse and  $\mathbf{TC}(\mathbb{A}[G]; p)$  is the homotopy cofiber of the map  $\mathbf{TC}(\mathbb{A}; p) \to \mathbf{TC}(\mathbb{A}[G]; p)$  induced by the inclusion.

**Theorem 15.71.** Let G be a group and p be a prime.

(i) Assume that there is a G-CW-model for  $E_{\mathcal{FIN}}(G)$  of finite type. Then the map induced by the projection pr:  $E_{\mathcal{FIN}}(G) \to G/G$ 

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{TC}_{\mathbb{A},p}) \to H_n^G(G/G; \mathbf{TC}_{\mathbb{A};p}) = \pi_n(\mathbf{TC}(\mathbb{A}[G]; p))$$

is split injective for all  $n \in \mathbb{Z}$ ;

(ii) Assume that G is hyperbolic or virtually finitely generated abelian. Then the map induced by the projection pr:  $E_{VCY}(G) \rightarrow G/G$ 

$$H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{TC}_{\mathbb{A}.p}) \to H_n^G(G/G; \mathbf{TC}_{\mathbb{A};p}) = \pi_n(\mathbf{TC}(\mathbb{A}[G]; p))$$

*is injective for all*  $n \in \mathbb{Z}$ *;* 

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*Proof.* See [676, Theorem 1.4],

A more general result about rational injectivity of the assembly map for topological cyclic homology can be found in [676, Theorem 1.8].

One of the reasons why topological cyclic homology is much harder than topological Hochschild homology is that in the construction of topological cyclic homology a homotopy inverse limits occurs and taking the smash product does not commute with homotopy inverse limits in general, see [677]. This is the main reason for the existence of the counterexamples above.

**Remark 15.72 (Pro-systems).** If one does not pass to the assembly maps but argues on the level of pro-systems, then there is a kind of assembly map for pro-systems for any group *G* and the family  $C\mathcal{Y}C$  of cyclic subgroups which is indeed a proisomorphism, see [676, Theorem 1.3]. In other words, a pro-system version of the Farrell-Jones Conjecture for topologically cyclic homology holds for any group *G* and any connective<sup>+</sup> spectrum  $\mathbb{A}$  for the family  $C\mathcal{Y}C$  of cyclic subgroups.

More information about topological cyclic homology and its applications to algebraic *K*-theory via the cyclotomic trace can be found for instance in [316, 479, 759].

### 15.12 The Farrell-Jones Conjecture for Homotopy K-Theory

Let E: ADDCAT  $\rightarrow$  SPECTRA be a (covariant) functor from the category ADDCAT of small additive categories. In [684, Definition 8.1] its *homotopy stabilization* is constructed, which consists of a covariant functor

**EH**: ADDCAT  $\rightarrow$  SPECTRA

together with a natural transformation

 $h \colon E \to EH$ 

We call **E** homotopy stable if  $\mathbf{h}(\mathcal{A})$  is an equivalence for any object  $\mathcal{A}$  in ADDCAT.

This construction has the following basic properties. Given an automorphism  $\Phi: \mathcal{A} \to \mathcal{A}$ , let  $\mathcal{A}_{\Phi}[t]$  be the additive category of twisted polynomials with coefficients in  $\mathcal{A}$ , see [686, Definition 1.2]. Let  $ev_0^+: \mathcal{A}_{\Phi}[t] \to \mathcal{A}$  be the functor of additive categories given by taking t = 0 and let  $\mathbf{i}^+: \mathcal{A} \to \mathcal{A}_{\Phi}[t]$  be the obvious inclusion see [686, (1.10) and (1.12)].

**Lemma 15.73.** Let  $E: ADDCAT \rightarrow SPECTRA$  be a covariant functor.

(i) **EH** *is homotopy stable;* 

(ii) Suppose that **E** is homotopy stable. Let  $\mathcal{A}$  be any additive category with an automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ . Then the maps

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$$\begin{split} \mathbf{E}(\mathrm{ev}_0^+) \colon \mathbf{E}(\mathcal{A}_{\Phi}[t]) &\xrightarrow{\simeq} \mathbf{E}(\mathcal{A}); \\ \mathbf{E}(\mathbf{i}^+) \colon \mathbf{E}(\mathcal{A}) \xrightarrow{\simeq} \mathbf{E}(\mathcal{A}_{\Phi}[t]), \end{split}$$

are weak homotopy equivalences;

(iii) The functor **E** is homotopy stable if and only if for every additive category  $\mathcal{A}$  the inclusion  $\mathcal{A} \to \mathcal{A}[t]$  induces a weak homotopy equivalence  $\mathbf{E}(\mathcal{A}) \to \mathbf{E}(\mathcal{A}[t])$ .

*Proof.* (i) and (ii) See [684, Lemma 8.2].

(iii) The only if statement follows from assertion (ii). The if-statement is a direct consequence of the definition of **EH**, see [684, Definition 8.1].  $\Box$ 

Lemma 15.73 (ii) essentially says that homotopy stable automatically implies homotopy stable in the twisted sense.

**Remark 15.74 (Universal property of EH).** Note that Lemma 15.73 (i) says that up to weak homotopy equivalence the transformation  $h: E \rightarrow EH$  is universal (from the left) among transformations  $f: E \rightarrow F$  to homotopy stable functors  $F: ADDCAT \rightarrow SPECTRA$ , since we obtain a commutative square whose lower vertical arrow is a weak homotopy equivalence



**Definition 15.75 (Homotopy** *K***-theory).** Let **K**: ADDCAT  $\rightarrow$  SPECTRA be the covariant functor that sends an additive category to its non-connective *K*-theory spectrum, see for instance [209, 684, 800]. Define the *homotopy K*-theory functor

**KH**: ADDCAT 
$$\rightarrow$$
 SPECTRA

to be the homotopy stabilization of K.

The next result is taken from [684, Lemma 8.6].

**Theorem 15.76 (Bass-Heller-Swan decomposition for homotopy** *K***-theory).** *Let*  $\mathcal{A}$  *be an additive category with an automorphism*  $\Phi \colon \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ *. Then we get for all*  $n \in \mathbb{Z}$  *a weak homotopy equivalence* 

**a**: 
$$\mathbf{T}_{\mathbf{KH}(\Phi^{-1})} \xrightarrow{\simeq} \mathbf{KH}(\mathcal{A}_{\Phi}[t, t^{-1}])$$

where  $T_{KH(\Phi^{-1})}$  is the mapping torus of the self-map  $KH(\Phi^{-1})$ :  $KH(\mathcal{A}) \rightarrow KH(\mathcal{A})$ .

**Remark 15.77 (Identification with Weibel's definition).** Weibel has defined a version of homotopy K-theory for a ring R by a simplicial construction in [996]. It is not hard to check using Remark 15.74, which applies also to the constructions

of [996] instead of **H**, that  $\pi_i(\mathbf{KH}(\mathcal{R}))$  can be identified with the one in [996] if  $\mathcal{R}$  is a skeleton of the category of finitely generated free *R*-modules.

**Conjecture 15.78 (Farrell-Jones Conjecture for homotopy** *K***-theory with coefficients in additive** *G***-categories).** We say that *G* satisfies the *Farrell-Jones Conjecture with coefficients for homotopy K-theory in additive G-categories* if for every additive *G*-category  $\mathcal{A}$  and every  $n \in \mathbb{Z}$  the assembly map given by the projection pr:  $E_{\mathcal{FIN}}(G) \rightarrow G/G$ 

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{KH}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{KH}_{\mathcal{A}}) = \pi_n(\mathbf{KH}_{\mathcal{A}}(I(G)))$$

is bijective, where  $\mathbf{KH}_{\mathcal{R}}$ : GROUPOIDS  $\downarrow G \rightarrow$  SPECTRA is analogously defined as the functor appearing in (13.10) but with **K** replaced by **KH**.

A version of Conjecture 15.78 has been treated for rings in [75].

Conjecture 15.79 (Farrell-Jones Conjecture for homotopy *K*-theory with coefficients in additive *G*-categories with finite wreath products). We say that *G* satisfies the *Farrell-Jones Conjecture with coefficients for homotopy K-theory in additive G-categories with finite wreath products* if for any finite group *F* the group  $G \wr F$  satisfies the Farrell-Jones Conjecture with coefficients for homotopy *K*-theory in additive  $G \wr F$ -categories 15.78.

#### 15.13 The Farrell-Jones Conjecture for Hecke Algebras

There is one instance where one can formulate the Farrell-Jones Conjecture for nondiscrete groups, namely, for the algebraic *K*-theory of a Hecke algebra  $\mathcal{H}(G)$  of a totally disconnected locally compact second countable Hausdorff group *G*.

Denote by  $\mathcal{H}(G)$  the *Hecke algebra* of *G* that consists of locally constant functions  $G \to \mathbb{C}$  with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra  $\mathcal{H}(G)$  plays the same role for *G* as the complex group ring  $\mathbb{C}G$  for a discrete group *G* and reduces to this notion if *G* happens to be discrete. There is a *G*-homology theory  $\mathcal{H}^G_*$  with the property that for any open and closed subgroup  $H \subseteq G$  and all  $n \in \mathbb{Z}$  we have  $\mathcal{H}^G_n(G/H) = K_n(\mathcal{H}(H))$ , where  $K_n(\mathcal{H}(H))$  is the algebraic *K*-group of the Hecke algebra  $\mathcal{H}(H)$ . There is also the notion of a classifying space  $E_{\mathcal{K}O}(G)$  for the family of compact-open subgroups of *G*. Note that  $\mathcal{K}O$  is not closed under passing to subgroups but at least under finite intersections, which suffices to our purposes. The space  $E_{\mathcal{K}O}(G)$  is characterized by the property that for any *G*-*CW*-complex *X* whose isotropy groups are compact-open, there is up to *G*-homotopy precisely one *G*-map from *X* to  $E_{\mathcal{K}O}(G)$ . More information about this space and the comparison with the classifying space for numerable *G*-spaces  $J_{\mathcal{K}O}(G)$  can be found in [655]. The following conjecture has appeared already in [673, Conjecture 119 on page 773].

15.14 Relations among the Isomorphism Conjectures

**Conjecture 15.80 (The Farrell-Jones Conjecture for the algebraic** *K***-theory of Hecke-Algebras).** For a totally disconnected locally compact second countable Hausdorff group *G* the assembly map

(15.81) 
$$\mathcal{H}_n^G(E_{\mathcal{K}O}(G)) \to \mathcal{H}^G(\{\bullet\}) = K_n(\mathcal{H}(G))$$

induced by the projection pr:  $E_{\mathcal{KO}}(T) \to \{\bullet\}$  is an isomorphism for all  $n \in \mathbb{Z}$ .

In the case n = 0 this reduces to the statement that

(15.82) 
$$\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{KO}}(G)} K_0(\mathcal{H}(H)) \to K_0(\mathcal{H}(G))$$

is an isomorphism. Some evidence for this comes for instance from [274], where the bijectivity of (15.82) has been proved rationally for a reductive *p*-adic group *G*. For  $n \leq -1$  the conjecture predicts  $K_n(\mathcal{H}(G)) = 0$ . The Hecke algebra  $\mathcal{H}(G)$ and its projective class group  $K_0(\mathcal{H}(G))$  are closely related to the theory of smooth representations of *G*, see for instance [129, 901, 902]. The *G*-homology theory can be constructed using an appropriate functor  $\mathbf{K}_{\mathcal{H}}$ :  $Or_{\mathcal{K}O}(G) \rightarrow SPECTRA$  and the recipe explained in Theorem 12.27. The desired functor  $\mathbf{K}_{\mathcal{H}}$  is constructed in [890].

All this is explained and carried out in the papers by Bartels and Lück [81, 83, 84], actually also for twisted Hecke algebras with respect to a central character and more general coefficient than  $\mathbb{C}$ . Moreover, the following result is proved in [81, Corollaries 1.8 and 1.18] and [84, Theorem 1.1].

**Theorem 15.83 (The Farrell-Jones Conjecture for the algebraic** *K***-theory of Hecke algebras).** Suppose that *G* is modulo a compact subgroup isomorphic to a closed subgroup of a reductive p-adic group. Then Conjecture 15.80 is true, the map (15.82) is bijective, and  $K_n(\mathcal{H}(G))$  vanishes for  $n \leq -1$ .

The Farrell-Jones Conjecture for the algebraic *K*-theory of Hecke algebras for completed Kac-Moody groups will be treated in Bartels-Lück-Witzel [91].

#### 15.14 Relations among the Isomorphism Conjectures

#### 15.14.1 The Farrell-Jones Conjecture for K-Theory and for A-Theory

Let *G* be a group and let *X* be a *G*-*CW*-complex. We get from the linearization map of (7.17) a natural map

(15.84) 
$$L_n^G(X): H_n^G(X; \mathbf{A}^B) \to H_n^G(X; \mathbf{K}_{\mathbb{Z}})$$

if we take Example 15.39 into account and  $\mathbf{A}^B$  is defined by (15.37) for  $\mathbf{S} = \mathbf{A}$  for the functor  $\mathbf{A}$  of (7.12). We conclude from Theorem 7.18 and the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, that  $L_n^G(X)$  is bijective for  $n \le 1$ , surjective for n = 2 and rationally bijective for all  $n \in \mathbb{Z}$ . If we take  $X = E_{VCY}(G)$ 

and X = G/G, we obtain a commutative diagram where the horizontal maps are assembly maps and the vertical maps are given by the maps (15.84)

We conclude that for  $n \in \mathbb{Z}$  with  $n \le 1$  the upper arrow is bijective if and only if the lower arrow is bijective. We also conclude for every  $n \in \mathbb{Z}$  that the lower arrow is rationally bijective if and only if the lower arrow is rationally bijective. This gives some interesting relations between the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$  and the Farrell-Jones Conjecture 15.61 for *A*-theory (without coefficients). For instance, they are equivalent in degrees  $n \le 1$ , and they are rationally equivalent.

The case where we allow coefficients in the Farrell-Jones Conjecture 15.61 for *A*-theory is more complicated since in Theorem 7.18 (ii) the assumption occurs that the space under consideration has to be aspherical. Consider a free *G*-*CW*-complex *Z* that is simply connected (but not necessarily contractible). Then  $\pi_1(G/H \times_G Z) \cong H$ . We still get a commutative diagram

and we know that the vertical arrows are bijective for  $n \le 1$  and surjective for n = 2, but *not* anymore that they are rationally bijective for all  $n \in \mathbb{Z}$ .

# **15.14.2** The Farrell-Jones Conjecture for *A*-Theory, Pseudoisotopy, and Whitehead Spaces

The Farrell Jones Conjecture 15.61 for *A*-theory (with coefficients), the Farrell-Jones Conjecture 15.63 for (smooth) pseudoisotopy (with coefficients) and the Farrell-Jones Conjecture 15.64 for (smooth) Whitehead spaces are equivalent by Theorem 15.65.

#### 15.14.3 The Farrell-Jones Conjecture for *K*-Theory and for Topological Cyclic Homology

The basic reason why topological cyclic homology is a powerful approximation of algebraic *K*-theory is the cyclotomic trace due to Bökstedt-Hsiang-Madsen [150]. It can be extended to the equivariant setting and thus be used together with the linearization map (7.17) to construct the following commutative diagram, which is closely related to the main diagram in [675, (3.1)] for  $n \ge 0$ ,

$$(15.85) \qquad H_{n}^{G}(E_{\mathcal{V}C\mathcal{Y}}(G); \mathbf{K}_{\mathbb{Z}}^{\geq 0}) \longrightarrow H_{n}^{G}(G/G; \mathbf{K}_{\mathbb{Z}}^{\geq 0}) = K_{n}(\mathbb{Z}G)$$

$$H_{n}(\iota_{\mathcal{F}C\mathcal{Y}\subseteq\mathcal{V}C\mathcal{Y}}; \mathbf{K}_{\mathbb{Z}}^{\geq 0}) \stackrel{\cong_{\mathbb{Q}}}{\cong_{\mathbb{Q}}} \stackrel{\cong_{\mathbb{Q}}}{\stackrel{$$

Here  $\mathcal{FCY}$  is the family of finite cyclic subgroups of *G*, the superscript  $\geq 0$  indicates that we consider the 0-connective covers, the vertical arrows from the third row to the second row come from the linearization map, and the vertical arrows from the third row to the fourth row come from the cyclotomic trace. All arrows marked with  $\cong_{\mathbb{Q}}$  are known to be rationally bijective. This follows from the maps induced by the linearization from Theorem 7.18. For the map  $H_n(\iota_{\mathcal{FCY}\subseteq \mathcal{VCY}}; \mathbf{K}_{\mathbb{Z}}^{\geq 0})$ , this follows from Theorem 13.51 and further computations based on equivariant Chern characters using Theorem 12.79 and [675, Example 12.12].

Rationally the natural map  $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}}^{\geq 0}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}})$  is split injective and has a cokernel that is given by an expression involving the groups  $K_{-1}(\mathbb{Z}C)$  for finite cyclic subgroups  $C \subseteq G$ . Hence the rational injectivity of the uppermost horizontal arrow in the diagram (15.85) implies that the *K*-theoretic Farrell-Jones assembly map is rationally injective, ignoring certain contributions from the collection of the groups  $K_{-1}(\mathbb{Z}C)$  for finite cyclic subgroups  $C \subseteq G$ ,

The uppermost horizontal arrow in the diagram (15.85) is rationally injective, provided that the composite of the lowermost horizontal arrow and  $H_n(E_{\mathcal{FCY}}(G); \mathbf{ct}^{\geq 0})$  is rationally injective. This is the basic idea in the proof of rational injectivity results for the *K*-theoretic Farrell-Jones assembly map presented in [675, Theorem 1.13], where the actual argument is more involved and uses the *C*-functors as well.

A rational computation of  $K_n(\mathbb{Z}G)$  is given in Theorem 17.1, provided that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$ . With the methods mentioned above, one can detect under certain conditions, without assuming the Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$ ,

that the map appearing in Theorem 17.1 is injective if one ignores in the source the summand for q = -1. An interesting special case is Theorem 16.33, which deals with Whitehead groups.

# **15.14.4** The *L*-Theoretic Farrell-Jones Conjecture and the Baum-Connes Conjecture

In the sequel [1/2] stands for inverting 2 at the level of spectra or abelian groups. Note that for a spectrum **E** we have a natural isomorphism  $\pi_n(\mathbf{E})[1/2] \xrightarrow{\cong} \pi_n(\mathbf{E}[1/2])$ .

One can construct the following commutative diagram

$$(15.86) \qquad H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{Z}}^{(-\infty)})[1/2] \longrightarrow L_{n}^{(-\infty)}(\mathbb{Z}G)[1/2]$$

$$i \downarrow^{\cong} \qquad \text{id} \downarrow^{\cong}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{Z}}^{(-\infty)}[1/2]) \longrightarrow L_{n}^{(-\infty)}(\mathbb{Z}G)[1/2]$$

$$i_{0} \uparrow^{\cong} \qquad \cong \uparrow_{j_{0}}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{Z}}^{p}[1/2]) \longrightarrow L_{n}^{p}(\mathbb{Z}G)[1/2]$$

$$i_{1} \downarrow^{\cong} \qquad \cong \downarrow_{j_{1}}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{Q}}^{p}[1/2]) \longrightarrow L_{n}^{p}(\mathbb{Q}G)[1/2]$$

$$i_{2} \downarrow^{\cong} \qquad \qquad \downarrow_{j_{2}}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{R}}^{p}[1/2]) \longrightarrow L_{n}^{p}(\mathbb{R}G)[1/2]$$

$$i_{3} \downarrow^{\cong} \qquad \qquad \downarrow_{j_{3}}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{L}_{\mathbb{C}^{r}(?,\mathbb{R})}^{p}[1/2]) \longrightarrow L_{n}^{p}(\mathbb{C}^{*}(G,\mathbb{R}))[1/2]$$

$$i_{4} \uparrow^{\cong} \qquad \cong \uparrow_{j_{4}}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{K}_{\mathbb{R}}^{\text{TOP}}[1/2]) \longrightarrow K_{n}^{\text{TOP}}(\mathbb{C}^{*}(G,\mathbb{R}))[1/2]$$

$$\cong \uparrow_{l} \qquad \text{id} \uparrow^{\cong}$$

$$H_{n}^{G}(\underline{E}G; \mathbf{K}_{\mathbb{C}}^{\text{TOP}})[1/2] \longrightarrow K_{n}^{\text{TOP}}(\mathbb{C}^{*}(G,\mathbb{R}))[1/2]$$

$$i_{5} \downarrow$$

$$H_{n}^{G}(\underline{E}G; \mathbf{K}_{\mathbb{C}}^{\text{TOP}})[1/2] \longrightarrow K_{n}^{(\text{TOP}}(\mathbb{C}^{*}(G,\mathbb{R}))[1/2]$$

where all horizontal maps are assembly maps and the vertical arrows are induced by transformations of functors GROUPOIDS  $\rightarrow$  SPECTRA. These transformations are induced by change of rings maps except the one from  $\mathbf{K}_{\mathbb{R}}^{\text{TOP}}[1/2]$  to  $\mathbf{L}_{C_r^*(?,\mathbb{R})}^p[1/2]$ , which is much more complicated and carried out in [614, 615]. Actually, it does not exist without inverting two on the spectrum level. Since it is a weak equivalence, the maps  $i_4$  and  $j_4$  are bijections.

On the level of homotopy groups the comparison between the algebraic *L*-theory and the topological *K*-theory of a real and of a complex  $C^*$ -algebra have already been explained in Theorem 10.78, namely we obtain isomorphisms

- (15.87)  $KO_n(A)[1/2] \xrightarrow{\cong} L_n^p(A)[1/2]$ , if A is a real C\*-algebra;
- (15.88)  $K_n(A) \xrightarrow{\cong} L_n^p(A)$ , if A is a complex  $C^*$ -algebra.

Since for any finite group *H* each of the following maps is known to be a bijection because of [839, Proposition 22.34 on page 252] and  $\mathbb{R}H = C_r^*(H, \mathbb{R})$ 

$$L_n^p(\mathbb{Z}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{Q}H)[1/2] \xrightarrow{\cong} L_n^p(\mathbb{R}H)[1/2] \xrightarrow{\cong} L_n^p(C_r^*(H,\mathbb{R})),$$

we conclude from the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, that the vertical arrows  $i_1$ ,  $i_2$ , and  $i_3$  are isomorphisms. The arrow  $j_1$  is bijective by [837, page 376]. The maps *l* are isomorphisms for general results about localizations.

The lowermost vertical arrows  $i_5$  and  $j_5$  are known to be split injective, a splitting comes by restriction with the inclusions  $C_r^*(G, \mathbb{R}) \to C_r^*(G, \mathbb{C})$ .

The following conjecture is already raised as a question in [587, Remark 23.14 on page 197], see also [614, Conjecture 1 in Subsection 5.2].

**Conjecture 15.89 (Passage for** *L***-theory from**  $\mathbb{Q}G$  **to**  $\mathbb{R}G$  **to**  $C_r^*(G, \mathbb{R})$ ). The maps  $j_2$  and  $j_3$  appearing in diagram (15.86) are bijective.

#### Lemma 15.90. Let G be a group.

- (i) Suppose that G satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring R for  $R = \mathbb{Q}$  and  $R = \mathbb{R}$  and the complex version of the Baum-Connes Conjecture 14.9. Then G satisfies Conjecture 15.89;
- (ii) Suppose that G satisfies Conjecture 15.89. Then G satisfies the L-theoretic Farrell-Jones Conjecture 13.4 for the ring Z after inverting 2 if and only if G satisfies the real version of the Baum-Connes Conjecture 14.9 after inverting 2;
- (iii) Suppose that the assembly map appearing in the complex version of the Baum-Connes Conjecture 14.9 is (split) injective after inverting 2. Then the assembly map appearing in L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring for  $R = \mathbb{Z}$  is (split) injective after inverting 2.

*Proof.* This follows from Theorem 13.62 (i), Remark 14.13, and the diagram (15.86).  $\Box$ 

#### 15.14.5 Mapping Surgery to Analysis

Let X be a connected CW-complex with fundamental group  $\pi$ . Let  $\widetilde{X} \to X$  be its universal covering. Denote by  $\epsilon$  one of the decorations s, h or p. We have constructed functors  $\mathbf{L}_{\mathbb{Z}}^{\epsilon}$  and  $\mathbf{K}_{\mathbb{R}}^{\text{TOP}}$ : GROUPOIDS  $\to$  SPECTRA in Theorem 12.43. We obtain maps of spectra

$$X_{+} \wedge \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\{*\}) \xleftarrow{\simeq} \widetilde{X}_{+} \wedge_{\pi} \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\mathcal{G}^{\pi}(\pi)) \longrightarrow \mathbf{L}_{\mathbb{Z}}^{\epsilon}(\mathcal{G}^{\pi}(\pi/\pi));$$
$$X_{+} \wedge \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}(\{*\}) \xleftarrow{\simeq} \widetilde{X} \wedge_{\pi} \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}(\mathcal{G}^{\pi}(\pi)) \longrightarrow \mathbf{K}_{\mathbb{R}}^{\mathrm{TOP}}(\mathcal{G}^{\pi}(\pi/\pi))$$

Here {\*} denotes the trivial groupoid with one object, the horizontal arrows pointing to the left are defined in the obvious way and are weak homotopy equivalences since  $\widetilde{X}$  is a free  $\pi$ -*CW*-complex with  $\pi \setminus \widetilde{X} = X$  and  $\mathcal{G}^{\pi}(\pi) \to \{*\}$  is an equivalence of groupoids, and the horizontal arrows to the right are assembly maps composed with maps induced by a fixed  $\pi$ -map  $\widetilde{X} \to E\pi$ . (If one wants to get rid of the dependency of a choice of  $\pi$ -map  $\widetilde{X} \to E\pi$ , one can consider  $\Pi(\pi/H \times_{\pi} \widetilde{X})$  instead of  $\mathcal{G}^{\pi}(\pi/H)$ for objects  $\pi/H$  in  $Or(\pi)$ .)

Denote by  $S^{\epsilon}(X)$  and D(X) respectively the homotopy fiber of the arrow pointing to the right in the first and second row above.

After taking homotopy groups we obtain long exact sequences

(15.91) 
$$\cdots \to H_{n+1}(X; \mathbf{L}^{\epsilon}_{\mathbb{Z}}(\{*\})) \to L^{\epsilon}_{n+1}(\mathbb{Z}\pi) \to \pi_n(\mathbf{S}^{\epsilon}(X))$$
  
 $\to H_n(X; \mathbf{L}^{\epsilon}_{\mathbb{Z}}(\{*\})) \to L^{\epsilon}_n(\mathbb{Z}\pi) \to \cdots,$ 

and

(15.92) 
$$\cdots \to KO_{n+1}(X) \to KO_{n+1}(C_r^*(\pi,\mathbb{R})) \to \pi_n(\mathbf{D}(X))$$
  
 $\to KO_n(X) \to KO_n(C_r^*(\pi,\mathbb{R})) \to \cdots$ .

After inverting 2 there is a zigzag of natural transformation from  $\mathbf{K}_{\mathbb{R}}^{\text{TOP}}[1/2]$  $\mathbf{L}_{\mathbb{Z}}^{\epsilon}[1/2]$  as explained in Subsection 15.14.4. It yields a map between long exact sequences



**Lemma 15.94.** Suppose that  $\pi$  satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution  $\mathbb{Z}$  and the Baum-Connes Conjecture 14.9 for the real group C<sup>\*</sup>-algebra.

Then the map

$$\pi_n(\mathbf{D}(M))[1/2] \xrightarrow{=} \pi_n(\mathbf{S}^{\epsilon}(M))[1/2]$$

is bijective for  $n \in \mathbb{Z}$ .

*Proof.* The first and fourth horizontal arrow in the diagram 15.93 are bijective since they are given by transformation of homology theories and their evaluation at  $\{\bullet\}$  is known to be bijective. The Rothenberg sequences of Subsection 9.10.4, Theorem 13.62 (i), and the diagram (15.86) together with the assumption that  $\pi$  satisfies the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution  $\mathbb{Z}$  and the Baum-Connes Conjecture 14.9 for the real group  $C^*$ -algebra imply that the second and fifth horizontal arrow in the diagram 15.93 are bijective. Now apply the Five Lemma to the diagram (15.93).

Now consider the case X = M for a closed orientable topological manifold M of dimension d. Then the part of the sequence (15.91) for  $n \ge d$  can identified with the long Surgery Exact Sequence in the topological category appearing in Theorem 9.130, see for instance [839, Theorem 18.5 on page 198] or [596]. Some extra care is necessary at the end in degree d since one has to pass to the 1-connective cover of the *L*-theory spectrum. In particular, we get an identification of  $\pi_d(\mathbf{S}^s(M))$ 

with the topological structure set  $S_d^{\text{TOP},s}(M)$ , see Subsection 9.12.1, which is the central object of study in the classification of topological manifolds. Note that in view of Lemma 15.94 one can hope for an identification of  $S_d^{\text{TOP},s}(M)$  after inverting 2 with  $\pi_d(\mathcal{D}(M))$ , which is an object related to the topological *K*-theory of spaces and *C*\*-algebras. An analytic Surgery Exact Sequence in terms of the topological *K*-theory of *C*\*-algebras associated to *M* is constructed in [494, Section 1].

**Problem 15.95 (Identification of analytic Surgery Exact Sequences).** Identify the real version of the analytic Surgery Exact Sequence appearing in [494, Section 1] with the exact sequence (15.92) for a closed orientable manifold of dimension *d*.

Note that Higson-Roe have to work with smooth manifolds, since they want to apply index theory. So they have to consider the Surgery Exact Sequence in the smooth category. They construct a diagram relating the Surgery Exact Sequence in the smooth category to their analytic Surgery Exact Sequence.

A more direct approach to the map comparing the Surgery Exact Sequence in the smooth category to the analytic Surgery Exact Sequence is given in Piazza-Schick [808].

A comparison map starting with the Surgery Exact Sequence in the topological category is constructed in Zenobi [1033] using the approach of [808] and Lipschitz structures.

Recall that the Surgery Exact Sequence in the topological category is an exact sequence of abelian groups, which is not true for the smooth category. It is not clear whether the construction in Zenobi [1033] is compatible with the abelian group structures on the topological and analytic structure sets.

Note that the comparison maps appearing in [494, 808, 1033] go in the opposite direction, namely, from *L*-theory to *KO*-theory, in comparison with the transformations appearing in [614, 615].

So one can state the following problem after Problem 15.95 has been solved:

**Problem 15.96 (Identification of transformations from the Surgery Exact Sequence to its analytic counterpart).** Identify the comparison map (15.93) from the Surgery Exact Sequence in the topological category to the analytic Surgery Exact Sequence appearing in [494, Section 5] with the comparison map appearing in Zenobi [1033].

#### 15.14.6 The Baum-Connes Conjecture and the Bost Conjecture

We have the a factorization of the Baum-Connes assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients

$$\operatorname{asmb}_{A}^{G,\mathbb{C}}(\underline{E}G)_{*} \colon K_{n}^{G}(\underline{E}G;A) \xrightarrow{\operatorname{asmb}_{A}^{G,\mathbb{C},L^{1}}(\underline{E}G)_{*}} K_{*}(A \rtimes_{L^{1}} G) \xrightarrow{K_{*}(q)} K_{*}(A \rtimes_{r} G).$$

Recall that the Bost Conjecture with coefficients predicts the bijectivity of the first map. We have also mentioned that there are counterexamples to the Baum-Connes Conjecture Conjecture 14.11 with coefficients. The group *G* involved in these counterexamples can be constructed as colimits of hyperbolic groups. For such colimits the Bost Conjecture with coefficients is known to be true. Hence for such a group *G* the map  $K_*(q): K_*(A \rtimes_{L^1} G) \to K_*(A \rtimes_r G)$  fails to be bijective. More details about this discussion can be found in [71, Section 1.5].

# **15.14.7** The Farrell-Jones Conjecture for *K*-Theory and for Homotopy *K*-theory

**Theorem 15.97 (The K-theoretic Farrell-Jones Conjecture implies the Farrell-Jones Conjecture for homotopy K-theory).** If G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories, then G also satisfies the Farrell-Jones Conjecture 15.78 for homotopy K-theory with coefficients in additive G-categories.

*Proof.* See [684, Theorem 9.1 (iii)].

**Remark 15.98 (Implications of the homotopy** *K***-theory version to the** *K***-theory version).** Next we discuss some cases where the Farrell-Jones Conjecture 15.78 for homotopy *K*-theory with coefficients in additive *G*-categories gives implications for the injectivity part of the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R*. These all follow by inspecting for a ring *R* the following commutative diagram

where the two vertical arrows pointing downwards are induced by the transformation **h**: **K**  $\rightarrow$  **KH**, the map  $\iota_{\mathcal{FIN} \subseteq \mathcal{VCY}}$  is induced by the inclusion of families  $\mathcal{FIN} \subseteq \mathcal{VCY}$ , and the two horizontal arrows are the assembly maps for *K*-theory and homotopy *K*-theory.

Suppose that *R* is regular and the order of any finite subgroup of *G* is invertible in *R*. Then the two left vertical arrows are known to be bijections. This follows for  $\iota_{\mathcal{FIN}\subseteq \mathcal{VCY}}$  from [673, Proposition 70 on page 744] and for *h* from [280, Lemma 4.6] and the fact that *RH* is regular for all finite subgroups *H* of *G* and hence  $K_n(RH) \rightarrow KH_n(RH)$  is bijective for all  $n \in \mathbb{Z}$  by Theorem 6.16. Hence the (split) injectivity of the lower horizontal arrow implies the (split) injectivity of the upper horizontal arrow.

Suppose that *R* is regular. Then the two left vertical arrows are rational bijections. This follows for  $\iota_{\mathcal{FIN}\subseteq \mathcal{VCY}}$  from Theorem 13.51. To prove it for *h*, it suffices because of [280, Lemma 4.6] to show that  $K_n(RH) \to KH_n(RH)$  is rationally bijective for each finite group *H* and  $n \in \mathbb{Z}$ . By the version of the spectral sequence appearing in [996, 1.3] for non-connective *K*-theory, it remains to show that  $N^p K_n(RH)$ vanishes rationally for all  $n \in \mathbb{Z}$ . Since R[t] is regular if *R* is, this boils down to showing that  $NK_p(RH)$  is rationally trivial for any regular ring *R* and any finite group *H*. The proof that  $NK_p(RH)$  is rationally trivial for any regular ring *R* and any finite group *H* can be found for instance in [685, Theorem 9.4]. Hence the upper horizontal arrow is rationally injective if the lower horizontal arrow is rationally injective.

The next conjecture generalizes Conjecture 6.76 from torsionfree groups to arbitrary groups.

**Conjecture 15.99** (*K*-theory versus homotopy *K*-theory for regular rings). Let G be a group. Suppose that R is regular and the order of any finite subgroup of G is invertible in R.

Then the natural map

$$K_n(RG) \rightarrow KH_n(RG)$$

is an isomorphism for all  $n \in \mathbb{Z}$ .

**Exercise 15.100.** Suppose that G satisfies the K-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive G-categories. Then G satisfies Conjecture 15.99.

**Exercise 15.101.** Let *G* be a group. Suppose that *R* is regular and the order of any finite subgroup of *G* is invertible in *R*. Suppose that Conjecture 15.99 is true for *G*. Show that then  $NK_n(RG) = 0$  holds for all  $n \in \mathbb{Z}$ .

#### 15.15 Notes

One can also define a version of the Meta-Isomorphism Conjecture 15.2 or of the Fibered Meta-Isomorphism Conjecture 15.8 with finite wreath products, compare Section 13.5. Let *C* be a class of groups closed under isomorphisms and taking subgroups and quotients. Let  $\mathcal{H}_*^2$  be an equivariant homology theory.

**Definition 15.102 (Fibered Meta-Isomorphism Conjecture with finite wreath products).** A group *G* satisfies the *Fibered Isomorphism Conjecture with finite wreath products* with respect to  $\mathcal{H}^2_*$  and *C* if for any finite group *F* the wreath product  $G \wr F$  satisfies the Fibered Meta-Isomorphism Conjecture 15.8 with respect to  $\mathcal{H}^2_*$  and the family  $C(G \wr F)$  consisting of subgroups of  $G \wr F$  that belong to *C*.

#### 15.15 Notes

The inheritance properties for the Fibered Meta-Isomorphism Conjecture 15.8 plus the passage to overgroups of finite index also hold for the Fibered Meta-Isomorphism Conjecture 15.102 with finite wreath products, see [595, Section 3].

Proofs of some of the inheritance properties above are also given in [464, 878].

One may ask whether one can find abstractly for the Fibered Meta-Isomorphism Conjecture 15.8 a smallest family for which it is true. For instance what happens if one takes the intersection of all families for which the Fibered Meta-Isomorphism Conjecture 15.8 is true. This questions turns out to be equivalent to the difficult and unsolved question whether the Fibered Meta-Isomorphism Conjecture 15.8 holds for an infinite product of groups, provided that for each of these groups the Fibered Meta-Isomorphism Conjecture 15.8 is true,

The following observation is taken from [819, Section 7]. Fix an equivariant homology theory  $\mathcal{H}^2$ . Take for simplicity  $\Gamma$  to be the trivial group when considering the Fibered Meta Isomorphism Conjecture 15.8.

We consider the following properties:

(P) For any set {(G<sub>i</sub>, F<sub>i</sub>) | i ∈ I} for G<sub>i</sub> a group and F<sub>i</sub> a family of subgroups of G<sub>i</sub> such that (G<sub>i</sub>, F<sub>i</sub>) satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every i ∈ I, the group ∏<sub>i∈I</sub> G<sub>i</sub> with respect to the family

$$\prod_{i \in I} \mathcal{F}_i := \left\{ H \subseteq \prod_{i \in I} G_i \; \middle| \; \exists H_i \in \mathcal{F}_i \text{ for every } i \in I \text{ with } H \subseteq \prod_{i \in I} H_i \right\}$$

satisfies the Fibered Meta Isomorphism Conjecture 15.8.

(I) For any group G and families of subgroups  $\{\mathcal{F}_i \mid i \in I\}$  of G such that  $(G, \mathcal{F}_i)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every  $i \in I$ , the pair  $(G, \bigcap_{i \in I} \mathcal{F}_i)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8.

#### Lemma 15.103. The properties (I) and (P) are equivalent.

#### Exercise 15.104. Prove Lemma 15.103 using Lemma 15.16.

Recall that for the Baum-Connes Conjecture 14.9 for a group *G* the smallest family for which it can be true is the family of finite cyclic subgroups  $\mathcal{FCY}$ , see Theorem 14.37 and Remark 14.38. For the *K*-theoretic Farrell-Jones Conjecture for a group *G* with arbitrary rings as coefficients 13.2, one expects that  $\mathcal{HE}_I$  is the smallest family for which it can be true, see Theorem 13.48. For topological Hochschild homology one expects the family  $\mathcal{FCY}$  of infinite cyclic group to be smallest family for which the Farrell-Jones Conjecture can be true, see Theorem 15.68.

# Chapter 16 Status

### **16.1 Introduction**

In this chapter we give a status report about the class of groups for which the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1, the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7, the Baum-Connes Conjecture 14.9, see Theorem 16.12, and the Novikov Conjecture 9.137, see Section 16.7, have been proved. We discuss injectivity results in Sections 16.5 and 16.6. In order to restrict the length of the exposition, we do not present the long history of these results and concentrate only on the current state of the art, although this unfortunately means that certain papers, which were spectacular breakthroughs at the time of their writing and had a big impact on the following papers, may not appear here.

A review of and a status report for some classes of groups is given in Section 16.8. This may be helpful for a reader who is interested in a certain class of groups, although this means that there are some repetitions of statements of results.

At the time of writing no counterexamples to the Full Farrell-Jones Conjecture 13.30, the Baum-Connes Conjecture 14.9 without coefficients, and the Novikov Conjecture 9.137 are known to the author. These conjectures are open in general. In Section 16.10 we explain that the search for counterexamples is not easy at all. In Subsection 16.10.5 we mention a few results which are consequences of the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* and for which there exist proofs for all groups.

## 16.2 Status of the Full Farrell-Jones Conjecture

The most general form of the Farrell-Jones Conjecture is the Full Farrell-Jones Conjecture 13.30. It has the best inheritance properties and all variants of the Farrell-Jones Conjecture presented in this book are special cases of it, see Section 13.11.

**Theorem 16.1 (Status of the Full Farrell-Jones Conjecture 13.30).** Let  $\mathcal{FJ}$  be the class of groups for which the Full Farrell-Jones Conjecture 13.30 is true.

- (i) The following classes of discrete groups belong to  $\mathcal{FJ}$ :
  - (a) Hyperbolic groups;
  - (b) *Finite-dimensional* CAT(0)-*groups;*
  - (c) Virtually solvable groups;

(d) (Not necessarily cocompact) lattices in path connected second countable locally compact Hausdorff groups.

More generally, if L is a (not necessarily cocompact) lattice in a second countable locally compact Hausdorff group G such that  $\pi_0(G)$  is discrete and belongs to  $\mathcal{FJ}$ , then L belongs to  $\mathcal{FJ}$ ;

- (e) Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;
- (f) The groups  $\operatorname{GL}_n(\mathbb{Q})$  and  $\operatorname{GL}_n(F(t))$  for F(t) the function field over a finite field F;
- (g) *S*-arithmetic groups;
- (h) The mapping class group  $\Gamma_{g,r}^s$  of a closed orientable surface of genus g with r boundary components and s punctures for  $g, r, s \ge 0$ ;
- (i) Fundamental groups of graphs of abelian groups;
- (j) Fundamental groups of graphs of virtually cyclic groups;
- (k) Artin's full braid groups  $B_n$ ;
- (1) *Coxeter groups;*
- (m) Groups in the class AC(VSOLV) defined in (20.46) for the class VSOLV of virtually solvable groups;
- (n) Groups which acts properly and cocompactly on a finite product of hyperbolic graphs.
- (ii) The class  $\mathcal{FJ}$  has the following inheritance properties:
  - (a) Passing to subgroups

Let  $H \subseteq G$  be an inclusion of groups. If G belongs to  $\mathcal{FJ}$ , then H belongs to  $\mathcal{FJ}$ ;

(b) Passing to finite direct products *If the groups G*<sub>0</sub> and G<sub>1</sub> belong to FJ, then G<sub>0</sub> × G<sub>1</sub> also belongs to FJ;
(c) Group extensions

Let  $1 \to K \to G \to Q \to 1$  be an extension of groups. Suppose that for any infinite cyclic subgroup  $C \subseteq Q$  the group  $p^{-1}(C)$  belongs to  $\mathcal{FJ}$  and that the groups K and Q belong to  $\mathcal{FJ}$ . Then G belongs to  $\mathcal{FJ}$ ;

- (d) Group extensions with virtually torsionfree hyperbolic groups as kernel Let  $1 \to K \to G \to Q \to 1$  be an extension of groups such that K is virtually torsionfree hyperbolic and Q belongs to  $\mathcal{FJ}$ . Then G belongs to  $\mathcal{FJ}$ ;
- (e) Group extensions with countable free groups as kernel Let 1 → K → G → Q → 1 be an extension of groups such that K is a countable free group (of possibly infinite rank) and Q belongs to FJ. Then G belongs to FJ;
- (f) Colimits over directed systems  $Lat \{G_i \mid i \in I\}$  be a direct system of

Let  $\{G_i \mid i \in I\}$  be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each  $i \in I$  the group  $G_i$  belongs to  $\mathcal{FJ}$ .

*Then the colimit*  $\operatorname{colim}_{i \in I} G_i$  *belongs to*  $\mathcal{FJ}$ *;* 

- 16.2 Status of the Full Farrell-Jones Conjecture
  - (g) Passing to free products
     Consider a collection of groups {G<sub>i</sub> | i ∈ I} such that G<sub>i</sub> belongs FJ for each i ∈ I. Then \*<sub>i∈I</sub>G<sub>i</sub> belongs to FJ;
  - (h) Passing to overgroups of finite index
    Let G be an overgroup of H with finite index [G : H]. If H belongs to FJ,
    then G belongs to FJ;
  - (i) Graph products
     A graph product of groups, each of which belongs to FJ, belongs to FJ again.

*Proof.* We begin with assertion (i) about classes of groups belonging to  $\mathcal{FJ}$ .

(ia) This is proved for *K*-theory with coefficients in additive *G*-categories in [87, Main Theorem] and for *L*-theory with coefficients in additive *G*-categories in [78, Theorem B], but not including the "with finite wreath product" property. How this can be included is explained in [89, Remark 6.4]. The proof for *K*-theory with coefficients in higher *G*-categories can be found in [185, Theorem 1.7 (3)].

(ib) This is proved for *K*-theory with coefficients in additive *G*-categories in degree  $\leq 1$  and for *L*-theory with coefficients in additive *G*-categories in all degrees in [78, Theorem B]. The argument why the *K*-theory case with coefficients in additive *G*-categories holds in all degrees can be found in [992, Theorem 1.1 and Theorem 3.4]. Note that for a finite-dimensional CAT(0)-group *G* and a finite group *F* the wreath product  $G \wr F$  is a finite-dimensional CAT(0)-group again so that the passage to the version with finite wreath products is automatically true. The proof for *K*-theory with coefficients in higher *G*-categories can be found in [185, Theorem 1.71.7 (2)].

(ic) For coefficients in additive categories see [993, Theorem 1.1]. (The special case of certain nearly crystallographic groups is treated in [374, Main Theorem].) For coefficients in higher categories we refer to [185, Theorem 1.7 (4)].

(id) See [543, Theorem 8] whose proof is based on the case of cocompact lattices in an almost connected Lie groups handled in [72, Theorem 1.2 and Remark 1.4] and [185, Theorem 1.7 (6)].

(ie) In dimension 3 this is proved in [72, Corollary 1.3 and Remark 1.4], where [878, 879] are used, and in [185, Theorem 1.7 (7)]. The dimensions 1 and 2 can be handled directly or reduced to dimension 3 by crossing with  $D^1$ .

(if) See [884, Theorem 8.13] and [185, Theorem 1.7 (5)].

(ig) This follows from assertion (if) and the inheritance property passing to subgroups, see assertion (iia), since any S-arithmetic group is a subgroup of  $GL_n(\mathbb{Q})$  or of  $GL_n(F(t))$  for F(t) the function field over a finite field F.

(ih) See [70, Theorem A and Remark 9.4] and [185, Theorem 1.7 (9)].

(ii) See [415, Main Theorem] and [185, Theorem 1.7 (8)].

(ij) See [1019, Theorem A] and [185, Theorem 1.7 (8)].

(ik) The pure Artin braid group  $P_n$  is a strongly poly-surface group in the sense of Definition 16.24 by [37, Theorem 2.1]. Hence it satisfies the Full Farrell-Jones

Conjecture 13.30 by Theorem 16.25. Since the full braid group  $B_n$  contains  $P_n$  as a subgroup of finite index,  $B_n$  satisfies the Full Farrell-Jones Conjecture 13.30 by assertion (iih).

(il) The argument in [78, page 636] for the version without "finite wreath products" extends directly to the case with "finite wreath products".

(im) See Theorem 20.47.

(in) Such a group G is strongly transfer reducible in the sense of Definition 20.38 by inspecting the proof of [571, Theorem 6.1 and Example 2.9] and Theorem 22.45. Now apply Theorem 20.39.

Finally we deal with the assertion (ii) about inheritance properties. Here we can refer to Theorem 13.32 except for assertions (iid), (iie), and (iii).

In the sequel we give references only where additive categories are considered. The arguments carry over to the setting of higher *G*-categories as coefficients, since they are based only on inheritance properties which also hold for higher *G*-categories as coefficients.

Assertion (iii) is proved in [416].

Assertion (iid) follows from assertion (im) and from [136, Theorem 2.3].

Assertion (iie), follows from [173, Theorem 2.5].

This finishes the proof of Theorem 16.1.

**Exercise 16.2.** Let *G* be a cocompact torsionfree lattice in an almost connected Lie group *L* with dim(*L*)  $\geq$  5. Let *M* be an aspherical closed manifold with fundamental group *G*. Let  $K \subseteq L$  be a maximal compact subgroup. Show that then *M* is homeomorphic to  $G \setminus L/K$ .

**Exercise 16.3.** Let *U* be a group that is universal finitely presented, i.e., any finitely presented group is isomorphic to a subgroup of *G*. (Such a group exists by Higman [480, page 456], and there is even a universal finitely presented group which is the fundamental group of a complement of an embedded  $S^3$  in  $S^5$ , see [429, Corollary 3.4].) Show that the Full Farrell-Jones Conjecture 13.30 holds for all groups if and only if it holds for *U*.

**Exercise 16.4.** Let  $S \subseteq R$  be a subring of R such that R as a right S-module is finitely generated free. Suppose that for every natural number m the group  $GL_m(S)$  belongs to  $\mathcal{FJ}$ . Show that  $GL_n(R)$  belongs to  $\mathcal{FJ}$  for every natural number n.

## 16.3 Status of the Farrell-Jones Conjecture for Homotopy *K*-Theory

**Theorem 16.5 (Status of the Farrell-Jones Conjecture for homotopy** *K***-theory).** Let *FJKH* be the class of groups for which the Farrell-Jones Conjecture 15.79 for homotopy *K*-theory with coefficients in additive *G*-categories with finite wreath products is true.
- 16.3 Status of the Farrell-Jones Conjecture for Homotopy K-Theory
- (i) The class FJKH contains the class FJ of groups for which the Full Farrell-Jones Conjecture 13.30 holds. (The class FJ is analyzed in Theorem 16.1.) Moreover, FJKH contains all elementary amenable groups and all one-relator groups;
- (ii) The class FJKH has the following inheritance properties:
  - (a) Passing to subgroups
     Let H ⊆ G be an inclusion of groups. If G belongs to FJKH, then H also belongs to FJKH;
  - (b) Passing to finite direct products If the groups G<sub>0</sub> and G<sub>1</sub> belong to FJKH, then G<sub>0</sub>×G<sub>1</sub> belongs to FJKH;
  - (c) Group extensions Let  $1 \to K \to G \to Q \to 1$  be an extension of groups. If K and Q belong to FJKH, then G belongs to FJKH;
  - (d) Directed colimits
    Let {G<sub>i</sub> | i ∈ I} be a direct system of subgroups indexed by the directed set I (with arbitrary structure maps). Suppose that for each i ∈ I the group G<sub>i</sub> belongs to FJKH, then colim<sub>i∈I</sub> G<sub>i</sub> belongs to FJKH;
  - (e) Passing to free products
     Consider a collection of groups {G<sub>i</sub> | i ∈ I} such that G<sub>i</sub> belongs FJKH for each i ∈ I. Then \*<sub>i∈I</sub>G<sub>i</sub> belongs to FJKH;
  - (f) Passing to overgroups of finite index
     Let G be an overgroup of H with finite index [G : H]. If H belongs to
     FJKH, then G belongs to FJKH;
  - (g) Graph products A graph product of groups each of which belongs to FJKH belongs to FJKH again;
  - (h) Actions on trees

If G acts on a tree T without inversion such that every stabilizer group  $G_x$  of any vertex x in T belongs to FJKH. Then G belongs to FJKH.

*Proof.* This follows from Theorem 15.97 and [684, Remark 9.3] except for assertion (iig). Here the arguments of [416] apply also directly to homotopy *K*-theory, the situation is actually easier because of assertion (iic).

The class of groups  $\mathcal{FJKH}$  is larger and has better inheritance properties than the class  $\mathcal{FJ}$ . The decisive difference is that we can use for the homotopy *K*-theory the family  $\mathcal{FIN}$  instead of the family  $\mathcal{VCY}$ . This is essentially a consequence of and reflected by Theorem 15.76.

**Exercise 16.6.** Let G be a torsionfree elementary amenable group and let R be regular.

Show that then the assembly map  $H_n(BG; \mathbf{K}(R)) \to K_n(RG)$  is split injective.

One can also construct a version of homotopy *K*-theory for higher categories. For instance one could generalize the construction in [684, Section 8].

# **16.4 Status of the Baum-Conjecture (with Coefficients)**

We have introduced the Baum-Connes Conjecture 14.11 with coefficients in Section 14.4.

**Theorem 16.7 (Status of the Baum-Connes 14.11 with coefficients).** Let BC be the class of groups for which the Baum-Connes Conjecture 14.11 with coefficients holds.

- (i) The following classes of groups belong to BC.
  - (a) A-T-menable groups;
  - (b) CAT(0)-cubical groups in the sense of [168], i.e., groups which act properly and cocompactly on a finite-dimensional CAT(0)-cubical complex with bounded geometry;
  - (c) countable subgroups of  $GL_2(F)$  for a field F;
  - (d) Hyperbolic groups;
  - (e) One-relator groups;
  - (f) Fundamental groups of compact 3-manifolds (possibly with boundary);
  - (g) Artin's full braid groups  $B_n$ ;
  - (h) *Thompson's groups F*, *T*, and *V*;
  - (i) Coxeter groups;
- (ii) The class BC has the following inheritance properties:
  - (a) Passing to subgroups Let  $H \subseteq G$  be an inclusion of groups. If G belongs to  $\mathcal{B}C$ , then H belongs to  $\mathcal{B}C$ :
  - (b) Passing to finite direct products

If the groups  $G_0$  and  $G_1$  belong to  $\mathcal{BC}$ , then  $G_0 \times G_1$  also belongs to  $\mathcal{BC}$ ; (c) Group extensions

Let  $1 \to K \to G \to Q \to 1$  be an extension of groups. Suppose that for any finite subgroup  $F \subseteq Q$  the group  $p^{-1}(F)$  belongs to  $\mathcal{BC}$  and that the group Q belongs to  $\mathcal{BC}$ .

Then G belongs to  $\mathcal{BC}$ ;

(d) Directed unions

Let  $\{G_i \mid i \in I\}$  be a direct system of subgroups of G indexed by the directed set I such that  $G = \bigcup_{i \in I} G_i$ . Suppose that  $G_i$  belongs to  $\mathcal{B}C$  for every  $i \in I$ . Then G belongs to  $\mathcal{B}C$ ;

(e) Actions on trees

Let G be a countable group acting without inversion on a tree T. Then G belongs to  $\mathcal{BC}$  if and only if the stabilizers of each of the vertices of T belong to  $\mathcal{BC}$ .

In particular, *BC* is closed under amalgamated free products and HNNextensions. *Proof.* We begin with assertion (i) about classes of groups belong to  $\mathcal{BC}$ .

(ia). This is proved in [486, Theorem 1.1].

(ib) See [168]. This also follows from assertion (ia).

(ic) Such groups are a-T-menable by [446, Theorem 4]. Now apply assertion (ia).

(id) This is proved in [604, Théorème 0.4], see also [818]. (The proof without coefficients can be found in [732].)

(ie) See [787, Corollary 1.3].

(if) Let *M* be a closed Seifert manifold. Then there is an extension  $1 \to \mathbb{Z} \to \pi_1(M) \to Q \to 1$  such that *Q* contains a subgroup *H* of finite index that is isomorphic to the fundamental group of a closed surface *S*, see [477, Theorem 12.2 on page 118]. If *S* carries the structure of a hyperbolic manifold,  $\pi_1(S)$  and hence *Q* are hyperbolic and belongs to  $\mathcal{BC}$  by assertion (id). If *S* does not carry the structure of a hyperbolic manifold, its fundamental group and hence *Q* are virtually finitely generated abelian and hence belong to  $\mathcal{BC}$  by assertion (ia). Now assertions (ia) and (iic) imply that  $\pi_1(M)$  belongs to  $\mathcal{BC}$ .

Let *M* be a closed hyperbolic 3-manifold. Then its fundamental group is hyperbolic and hence belongs to  $\mathcal{BC}$  by assertion (id).

Let *M* be a compact irreducible manifold with infinite fundamental group such that its boundary is non-trivial or is Haken. Then  $\pi_1(M)$  can be obtained from the trivial group by a finite number of HNN extensions and amalgamated free products. See [976, proof of Proposition 19.5 (6) on page 253] where the condition orientable is only assumed for simplicity, or see [477, Theorem 13.3 on page 141]. Hence  $\pi_1(M)$  belongs to  $\mathcal{BC}$  by assertion (iie). Let *M* be an irreducible closed 3-manifold. If it does not contain an incompressible torus, it is either Seifert or hyperbolic by the proof of Thurston's Geometrization Conjecture due to Perelman, see for instance [751], and hence belongs to  $\mathcal{BC}$  by the argument above. We conclude that  $\pi_1(M)$  belongs to  $\mathcal{BC}$  for any compact irreducible 3-manifold. Since any prime 3-manifold that is not irreducible is an  $S^1$ -bundle over  $S^2$ , see [477, Lemma 3.13 on page 28], and hence belongs to  $\mathcal{BC}$  by assertion (ia), any compact prime 3-manifold *M* belongs to  $\mathcal{BC}$ . Since any compact 3-manifold is a connected sum of prime compact 3-manifolds, see [477, Theorem 3.15 on page 31], assertion (if) follows from assertion (iie).

(ig) See [898, Theorem 20].

(ih) These groups are a-T-menable by Farley [348], and hence we can apply assertion (ia).

(ii) Since a finitely generated Coxeter group is a-T-menable, it satisfies the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ia). By a colimit argument based on Theorem 16.7 (iid) every Coxeter group satisfies the Baum-Connes Conjecture 14.11 with coefficients.

Finally we deal with the assertion (ii) about inheritance properties.

(iia) See [221, Theorem 2.5].

(iib) See [221, Theorem 3.17], or [786, Corollary 7.12].

(iic) See [786, Theorem 3.1].

(iid) This follows from [71, Theorem 5.6 (i) and Lemma 6.2].

(iie) This is proved in [787, Theorem 1.1].

**Exercise 16.8.** Let  $1 \to K \to G \to Q \to 1$  be an extension of groups such that *K* and *Q* satisfy the Baum-Connes Conjecture 14.11 with coefficients and *Q* is torsionfree. Show that then *G* satisfies the Baum-Connes Conjecture 14.11 with coefficients.

**Exercise 16.9.** Let *G* be a torsionfree group. Suppose that  $\mathbb{C}G$  has an idempotent different from 0 and from 1. Show that then *G* cannot be a subgroup of a hyperbolic group, a finite-dimensional CAT(0)-group, a lattice in an almost connected Lie group, the fundamental group of a manifold of dimension  $\leq 3$ , an amenable group, a mapping class group, or a one-relator group.

**Remark 16.10 (Passing to overgroups of finite index).** It is not known in general whether a group G belongs to  $\mathcal{BC}$ , i.e., G satisfies the Baum-Connes Conjecture 14.11 with coefficients if a subgroup of finite index does. Partial answers to this question are given by Schick [898, Theorem 20].

This suggests to systematically implement the with "finite wreath product version" in the Baum-Connes setting, as we did in the Farrell-Jones setting, see Section 13.5.

**Remark 16.11 (The Status of the Baum-Connes Conjecture for topological groups).** We have only dealt with the Baum-Connes Conjecture for discrete groups. The Baum Connes Conjecture (with coefficients) also makes sense for second countable locally compact Hausdorff groups. Here are some results in this setting.

Higson-Kasparov [486] treat the Baum-Connes Conjecture with coefficients for second countable locally compact Hausdorff groups which are a-T-menable

Julg-Kasparov [538, Theorem 5.4 (i)] prove the Baum-Connes Conjecture with coefficients for connected Lie groups *L* whose Levi-Malcev decomposition L = RS into the radical *R* and semisimple part *S* is such that *S* is locally of the form

 $S = K \times SO(n_1, 1) \times \cdots \times SO(n_k, 1) \times SU(m_1, 1) \times \cdots \times SU(m_l, 1)$ 

for a compact group *K*. The Baum-Connes Conjecture with coefficients for Sp(n, 1) is proved by Julg [537].

The Baum-Connes Conjecture without coefficients has been proven by Chabert-Echterhoff-Nest [222] for second countable almost connected Hausdorff groups, based on the work of Higson-Kasparov [486] and Lafforgue [603].

Next we deal with the Baum-Connes Conjecture 14.9 without coefficients for (discrete) groups. Recall that all groups which satisfy the Baum-Connes Conjecture 14.11 with coefficients in particular satisfy the Baum-Connes Conjecture 14.9. We mention a few cases below, some of which are not covered by this implication.

A *length function* on *G* is a function  $L: G \to \mathbb{R}_{\geq 0}$  such that L(1) = 0,  $L(g) = L(g^{-1})$  for  $g \in G$  and  $L(g_1g_2) \leq L(g_1) + L(g_2)$  for  $g_1, g_2 \in G$  holds. The word length metric  $L_S$  associated to a finite set *S* of generators is an example. A length

function *L* on *G* has property (*RD*) ("rapid decay") if there exist *C*, s > 0 such that for any  $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$  we have

$$||\rho_G(u)||_{\infty} \le C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s}\right)^{1/2}$$

where  $||\rho_G(u)||_{\infty}$  is the operator norm of the bounded *G*-equivariant operator  $l^2(G) \rightarrow l^2(G)$  coming from right multiplication by *u*. A group *G* has *property* (*RD*) if there is a length function which has property (RD). This notion is due to Jolissaint [529]. More information about property (RD) can be found for instance in [233, 235], [600], and [963, Chapter 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. A simply connected complete Riemannian manifold with non-positive sectional curvature is bolic. We refer to [557, Section 2] for a precise definition.

**Theorem 16.12 (Status of the Baum-Connes Conjecture (without coefficients)).** A group G satisfies the Baum-Connes Conjecture 14.9 (without coefficients) if it satisfies one of the following conditions:

(i) The group G is a discrete subgroup of a connected Lie groups L whose Levi-Malcev decomposition L = RS into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \cdots \times SO(n_k, 1) \times SU(m_1, 1) \times \cdots \times SU(m_l, 1)$$

for a compact group K;

- (ii) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space;
- (iii) The group G is a discrete finite covolume subgroup of the isometry groups of a simply connected complete Riemannian manifold with pinched negative sectional curvature;
- (iv) The group G is a discrete subgroup of Sp(n, 1).

*Proof.* (i) See [538].

- (ii) See [599] or [918].
- (iii) See [235, Corollary 0.3].

(iv) See [537].

## 

## 16.5 Injectivity Results in the Baum-Connes Setting

There are cases where one can show that the assembly maps appearing in the Farrell-Jones setting or Baum-Connes setting are injective without knowing that they are bijective. There is no case where one can prove surjectivity but not prove bijectivity as well. This is a common phenomenon in algebraic topology where surjectivity

arguments often contain an injectivity argument. Essentially one applies the surjectivity argument to a cycle whose boundary is the image of a cycle representing an element in the kernel of the assembly map. Moreover, this shows that in general surjectivity results are harder than injectivity results.

The main value of surjectivity statements is that they allow us to interpret elements in the K- or L-groups homologically and thus to obtain valuable information. The injectivity statements are interesting since they imply the Novikov Conjecture or give some idea how large the K- and L-groups are.

**Theorem 16.13 (Split injectivity of the assembly map appearing in the Baum-Connes Conjecture 14.9 (without coefficients) for fundamental groups of complete Riemannian manifolds with non-positive sectional curvature).** *The assembly map appearing in the Baum-Connes Conjecture 14.9 is split injective if G is the fundamental group of a complete Riemannian manifold with non-positive sectional curvature.* 

*Proof.* See Kasparov [563, Theorem 6.7].

More general results for bolic spaces are proved in Kasparov-Skandalis [558].

A metric space (X, d) admits a *uniform embedding in a Hilbert space*, sometimes also called a *coarse embedding in a Hilbert space*, if there exist a separable Hilbert space H, a map  $f: X \to H$ , and non-decreasing functions  $\rho_1$  and  $\rho_2$  from  $[0, \infty) \to \mathbb{R}$  such that  $\rho_1(d(x, y)) \le ||f(x) - f(y)|| \le \rho_2(d(x, y))$  for  $x, y \in X$  and  $\lim_{r\to\infty} \rho_i(r) = \infty$  for i = 1, 2. Recall that a metric space X is proper if for each r > 0 and  $x \in X$  the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G-invariant length metric dadmits a uniform embedding in a Hilbert space is independent of the choice of d, since the induced coarse structure does not depend on d, see [919, page 808]. We mention that for a finitely generated group any left invariant word length metric is an example of a proper left G-invariant length metric.

For more information about groups admitting a uniform embedding in a Hilbert space we refer to [314, 446].

The next result is due to Yu [1027, Theorem 2.2 and Proposition 2.6].

**Theorem 16.14 (Status of the Coarse Baum-Connes Conjecture).** *The Coarse Baum-Connes Conjecture 14.30 is true for a discrete metric space X of bounded geometry if X admits a uniform embedding in a Hilbert space. In particular, a countable group G satisfies the Coarse Baum-Connes Conjecture 14.30 if G equipped with a proper left G-invariant length metric admits a uniform embedding in a Hilbert space.* 

**Theorem 16.15 (Split injectivity of the assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients).** Let G be a countable group. Then for any  $C^*$ -algebra A the assembly map appearing in the Baum-Connes Conjecture 14.11

$$K_n^G(\underline{E}G;A) \to K_n(A \rtimes_r G)$$

is split injective if the group G has one of the following properties:

- 16.5 Injectivity Results in the Baum-Connes Setting
- (i) The group G admits a proper left G-invariant length metric for which G admits a uniform embedding in a Hilbert space;
- (ii) The group G admits a proper left G-invariant length metric for which G admits a uniform embedding in a Banach space with property (H);
- (iii) The group G is a subgroup of  $GL_n(F)$  for some field F and natural number n;
- (iv) *The group G is a subgroup of an almost connected Lie group.*

*Proof.* (i) This is proved by Skandalis-Tu-Yu [919, Theorem 6.1] using ideas of Higson [485] and Theorem 16.14.

(ii) See Kasparov-Yu [559, Theorem 1.3].

(iii) Assertion (i) applies to G by Guentner-Higson-Weinberger [446, Theorem 2 and 3].

(iv) Assertion (i) applies to G by Guentner-Higson-Weinberger [446, Theorem 7].  $\Box$ 

**Exercise 16.16.** Let *G* be a group such that for any finitely generated subgroup  $H \subseteq G$  and every H- $C^*$ -algebra *A* the assembly map  $K_n^H(\underline{E}H; A) \to K_n(A \rtimes_r H)$  is injective.

Show that then the assembly map  $K_n^G(\underline{E}G; A) \to K_n(A \rtimes_r G)$  is injective for every G- $C^*$ -algebra A. Prove the analogous statement for the K-theoretic and L-theoretic assembly maps with coefficients in additive categories (with involution) and the family of virtually cyclic subgroups.

Split injectivity of the Baum-Connes assembly map (for trivial coefficients) is proved under certain conditions about the compactifications of the model for the space for proper G-actions by Rosenthal [875] based on techniques developed by Carlsson-Pedersen [214].

**Remark 16.17 (Groups Acting Amenably on a Compact Space).** A continuous action of a discrete group *G* on a compact space *X* is called *topologically amenable* if there exists a sequence

$$p_n: X \to M^1(G) = \{f: G \to [0,1] \mid \sum_{g \in G} f(g) = 1\}$$

of weak-\*-continuous maps such that for each  $g \in G$  one has

$$\lim_{n\to\infty}\sup_{x\in X}||g*(p_n(x)-p_n(g\cdot x))||_1=0.$$

More information about this notion can be found for instance in [25, 26]. It should not be confused with the notion of an *amenable action* of a group G on a set X, where amenable in this context means that there exists a G-invariant mean on X. Note that the following statements are equivalent:

- The group G is amenable;
- The action of G on G by multiplication is amenable;
- The obvious action on G on the one-point-space is topologically amenable.

A group G is called *boundary amenable* if it admits a topologically amenable action on a compact metric space in the sense above.

Higson-Roe [490, Theorem 1.1 and Proposition 2.3] show that a finitely generated group is boundary amenable if and only if it belongs to the class A defined in [1027, Definition 2.1], and hence admits a uniform embedding in a Hilbert space. Hence Theorem 16.15 (i) implies the result of Higson [485, Theorem 1.1] that the assembly map  $K_n^G(\underline{E}G; A) \rightarrow K_n(A \rtimes_r G)$  appearing in the Baum-Connes Conjecture 14.11 with coefficients is split injective if G is boundary amenable.

Finally we mention that a finitely generated group *G* is boundary amenable if and only if the reduced group  $C^*$ -algebra  $C_r^*(G)$  is exact, i.e., the minimal tensor product with it preserves short exact sequences of  $C^*$ -algebras, see for instance [426, Proposition 9.9].

# 16.6 Injectivity Results in the Farrell-Jones Setting

Theorem 16.18 (Split injectivity of the assembly map appearing in the *L*-theoretic Farrell Jones Conjecture with coefficients in the ring  $\mathbb{Z}$  for fundamental groups of complete Riemannian manifolds with non-positive sectional curvature). The assembly map appearing in the *L*-theoretic Farrell Jones Conjecture 13.4 with coefficients in the ring  $\mathbb{Z}$  is split injective if *G* is the fundamental group of a complete Riemannian manifold with non-positive sectional curvature.

*Proof.* See [386, Theorem 2.3].

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any R > 0 there exists a cover  $\mathcal{U}$  of X with the property that the diameter of the members of  $\mathcal{U}$  is uniformly bounded and every open ball of radius R intersects at most (n+1) elements of  $\mathcal{U}$ , see [441, page 29]. The asymptotic dimension of a finitely generated group is the asymptotic dimension of its Cayley graph (and is independent of the choice of set of finite generators.)

For a torsionfree group *G* with finite asymptotic dimension and a finite model for *BG* and any ring *R* the split injectivity of  $H_n(BG; \mathbf{K}(R)) \to K_n(RG)$  is proved by Bartels [95, Theorem 1.1] and by Carlsson-Goldfarb [213, Main Theorem on page 406]. The *L*-theory version is proved in Bartels [95, Section 7] as well, provided that there exists a natural number *N* with  $K_{-i}(R) = 0$  for  $i \ge N$ .

The notion of finite decomposition complexity was introduced and studied by Guentner-Tessera-Yu [447, 448]. It is a weaker notion than finite asymptotic dimension. The split injectivity of the assembly maps  $H_n(BG; \mathbf{K}(R)) \to K_n(RG)$  and of  $H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$  for a torsionfree group *G* with finite model for *BG* and finite decomposition complexity is proved by Ramras-Tessera-Yu [831, Theorem 1.1] and Guentner-Tessera-Yu [447, page 334] for any ring *R* (with involution), provided that in the *L*-theory case there exists a natural number *N* with  $K_{-i}(R) = 0$  for  $i \ge N$ .

Kasprowski [566, Theorem 8.1] proved, for a group *G* with finite-dimensional model for  $E_{\mathcal{FIN}}(G)$  and finite quotient finite decomposition complexity, a strengthening of the notion of finite decomposition complexity, and a global upper bound on the orders of the finite subgroups that the assembly map  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$  is split injective for all  $n \in \mathbb{Z}$ . An *L*-theory version is proved in [566, Theorem 9.1].

The paper [566] uses ideas of [93]. Kasprowski [566, page 566] points out a gap in the proof of [93] which has the consequence that the results in [93] are only proved under the additional assumption that there is a finite model for  $E_{\mathcal{FIN}}(G)$ .

The papers by Kasprowski [567, 568] are based on [566] and lead to the following two results.

**Theorem 16.19 (Injectivity of the Farrell-Jones assembly map for**  $\mathcal{FIN}$  for **subgroups of almost connected Lie groups).** Let G be a subgroup of an almost connected Lie group L. (We do not assume that G with the subspace topology is discrete.) Equip G with the discrete topology.

(i) Let  $\mathcal{A}$  be an additive G-category. Suppose that G admits a finite-dimensional model for the classifying space  $E_{\mathcal{FIN}}(G)$ . Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n \big( \mathbf{K}_{\mathcal{A}}(I(G)) \big)$$

is split injective for all  $n \in \mathbb{Z}$ ;

(ii) Let A be an additive G-category with involution. Suppose that G admits a finitedimensional model for the classifying space E<sub>FIN</sub>(G). Suppose that there exists an N ≥ 0 such that π<sub>-i</sub>(**K**<sub>A</sub>(I(A))) = 0 holds for all i ≥ N and all virtually abelian subgroups A ⊆ G. Then the assambly map

Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

*is split injective for all*  $n \in \mathbb{Z}$ *;* 

(iii) Let C be a right exact  $G \cdot \infty$ -category. Suppose that G admits a finite-dimensional model for the classifying space  $E_{\mathcal{FIN}}(G)$ . Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathcal{C}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{C}}) = \pi_n(\mathbf{K}_{\mathcal{C}}(I(G)))$$

*is split injective for all*  $n \in \mathbb{Z}$ *;* 

- (iv) The group G admits a finite dimensional model for  $E_{\mathcal{FIN}}(G)$  if and only if there exists an  $N \in \mathbb{N}$  such that every finitely generated abelian subgroup of G has rank at most N;
- (v) If G is a discrete subgroup of L, then G possesses a finite-dimensional model for  $E_{\mathcal{FIN}}(G)$ .

*Proof.* (i) and (ii) If G is finitely generated, this is proved in [567, Theorem 1.1 and Theorem 6.1]. Since every group is the union of its finitely generated subgroups, the general case for injectivity follows from Lemma 15.23 (ii). One even obtains split injectivity since the retraction is natural, see [567, Section 7].

(iii) This is proved in [180, Theorem 1.1.6 (2)] provided that G is finitely generated. The general case follows from Lemma 15.23 (ii).

- (iv) See [567, Proposition 1.3].
- (**v**) See Theorem 11.24.

**Theorem 16.20 (Injectivity of the Farrell-Jones assembly map for**  $\mathcal{FIN}$  **for linear groups).** Let R be a commutative ring with unit and let  $G \subseteq GL_n(R)$  be a subgroup. Equip G with the discrete topology. Suppose that G admits a finite dimensional model for the classifying space  $E_{\mathcal{FIN}}(G)$ .

(i) Let  $\mathcal{A}$  be any additive G-category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n(\mathbf{K}_{\mathcal{A}}(I(G)))$$

*is split injective for all*  $n \in \mathbb{Z}$ *;* 

(ii) Let  $\mathcal{A}$  be any additive G-category with involution. Suppose that there exists an  $N \ge 0$  such that  $\pi_{-i}(\mathbf{K}_{\mathcal{A}}(I(H)) = 0$  holds for all  $i \ge N$  and all virtually nilpotent subgroups  $H \subseteq G$ .

Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

is split injective for all  $n \in \mathbb{Z}$ ;

(iii) Let C be a right exact G- $\infty$ -category. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_C) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_C(I(G))\big)$$

is split injective for all  $n \in \mathbb{Z}$ ;

*Proof.* If *G* is finitely generated, this is proved in [568, Theorem 1.1]. Since every group is the union of its finitely generated subgroups, the general case for injectivity follows from Lemma 15.23 (ii). One even obtains split injectivity since the retraction is natural, as explained in [567, Section 7]. The case of higher *G*-categories is proved in [180, Theorem 1.1.6 (2)].

Split injectivity of the *K*- and *L*-theoretic Farrell-Jones assembly map (for trivial coefficients) is proved under certain conditions about the compactifications of the model for the space for proper *G*-actions by Rosenthal [871, 872, 873], based on techniques developed by Carlsson-Pedersen [214].

We will present further injectivity results based on cyclotomic traces in Subsection 16.8.30.

# **16.7 Status of the Novikov Conjecture**

Recall that the Novikov Conjecture 9.137 holds for a group G if one of the following conditions is satisfied:

• The assembly map

$$\begin{split} H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) &= H_n^G(EG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \\ &\to H_n^G(G/G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G) \end{split}$$

is rationally injective for all  $n \in \mathbb{Z}$ , see Theorem 13.65 (xi);

• The assembly map

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z}}) \to H_n^G(G/G;\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z}}) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G)$$

is rationally injective, see Lemma 13.38 and Theorem 13.65 (xi);

- The *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring ℤ holds, see Theorem 13.65 (xi);
- The assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is rationally injective for all  $n \in \mathbb{Z}$ , see Theorem 14.29;

• The assembly map

$$K_n^G(E_{\mathcal{FIN}}(G))) \to K_n(C_r^*(G))$$

is rationally injective for all  $n \in \mathbb{Z}$ , see Lemma 13.38 and Theorem 14.29;

• The Baum-Connes Conjecture 14.9 holds for *G*, see Theorem 14.29.

Hence all groups appearing in Theorems 16.1, 16.7, 16.12, 16.13, 16.15, 16.18, and 16.19 satisfy the Novikov Conjecture 9.137. In particular, a group G satisfies the Novikov Conjecture 9.137 if G is a countable discrete subgroup of one of the following type of groups:

- Hyperbolic groups (or more generally directed colimits of hyperbolic groups);
- Finite-dimensional CAT(0)-groups;
- Almost connected Lie groups;
- (Not necessarily cocompact) lattices in second countable locally compact Hausdorff groups G for which π<sub>0</sub>(G) is discrete and belongs to FJ;
- $GL_n(F)$  for a field F and some natural number n;
- *S*-arithmetic groups;
- Mapping class groups;
- Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;
- A-T-menable groups and hence also amenable and elementary amenable groups;
- One-relator groups;
- Coxeter groups;
- Thompson's groups *F*, *T* and *V*;

- Artin's full braid groups  $B_n$ ;
- $Out(F_n)$  or more generally,  $Out(\Gamma)$  for a torsionfree hyperbolic group or a rightangled Artin group  $\Gamma$ , see [137].

Furthermore, the Novikov Conjecture 9.137 is satisfied for a countable group *G* if one of the following conditions is satisfied:

- *G* is the fundamental group of a complete Riemannian manifold with non-positive sectional curvature;
- The group G admits a proper left G-invariant length metric for which G admits a uniform embedding in a Hilbert space;
- The group G admits a proper left G-invariant length metric for which G admits a uniform embedding in a Banach space with property (H);
- *G* has a finite model for *BG* and finite asymptotic dimension, see [1026], or, more generally, has a finite model for *BG* and finite decomposition complexity, Guentner-Tessera-Yu [447, page 334];
- G is a geometrically discrete subgroup of a volume preserving diffeomorphism of any smooth compact manifold, see [428]. See also [427], where the volume preserving condition is no longer assumed.

A Banach version of the strong Novikov conjecture is proved in [341] for groups having polynomially bounded higher-order combinatorial functions. This includes all automatic groups. If the group G is of type  $F_{\infty}$ , is polynomially contractible, and has property (RD), it satisfies the strong Novikov Conjecture 14.26.

More information about the Novikov Conjecture and its status can be found for instance in [1029].

# 16.8 Review of and Status Report for Some Classes of Groups

## 16.8.1 Hyperbolic Groups

Almost all conjectures in this book about groups are satisfied for hyperbolic groups, since they satisfy both the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia), and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (id).

#### 16.8.2 Lacunary Hyperbolic Groups

A finitely generated group is a *lacunary hyperbolic group* if one of its asymptotic cones is an  $\mathbb{R}$ -tree, see Olshanskii-Osin-Sapir [780]. Since they are directed colimits of hyperbolic groups, see [780, Theorem 1.1], they satisfy the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia) and (iif). It is not known whether lacunary hyperbolic groups satisfy the Baum-Connes Conjecture 14.9.

A lacunary hyperbolic group is finitely presented if and only if it is hyperbolic. This is due to Kapovich-Kleiner, see [780, Theorem 8.1].

There are rather exotic examples of lacunary hyperbolic groups. For instance an infinite finitely generated torsionfree non-cyclic group all of whose proper subgroups are all infinite cyclic is constructed by Ol'shanskii [778]. It is a lacunary hyperbolic group. This follows from [780, Theorem 1.1].

Other examples of lacunary hyperbolic groups are constructed in [39]. These finitely generated groups contain (in a weak sense) an infinite expander. Hence they admit no uniform embedding in a Hilbert space (or in any  $l^p$  with  $1 \le p < \infty$ ) and any infinite-dimensional linear representation of these groups has infinite image. Note that for these groups a counterexample to the Baum-Connes Conjecture 14.11 with coefficients is constructed by Higson-Lafforgue-Skandalis [487]. This led Baum-Guentner-Willet [111] to reformulate the Baum-Connes Conjecture 14.11 with coefficients by introducing a new crossed product, see also [192], for which no counterexamples are known so far.

The class of lacunary groups contains some non–virtually cyclic elementary amenable groups and some infinite torsion groups. More examples of exotic lacunary hyperbolic groups are discussed in [780] and [889, Section 4].

## 16.8.3 Hierarchically hyperbolic groups

Durham-Minsky-Sisto [320] deal with so-called hierarchically hyperbolic groups. For instance the Full Farrell-Jones Conjecture 13.30 holds for decomposable hierarchically hyperbolic groups, see [320, Corollay D].

## 16.8.4 Relatively Hyperbolic Groups

For the definition and basic information about relatively hyperbolic groups we refer for instance to [160, 178, 346, 440, 785, 944, 945]. We use the notion of relatively hyperbolic groups of Bowditch [160].

**Theorem 16.21 (The Full Farrell-Jones Conjecture and relatively hyperbolic groups).** Let G be a countable group which is relatively hyperbolic to the subgroups  $P_1, P_2, \ldots, P_n$ . If  $P_1, P_2, \ldots, P_n$  satisfy the Full Farrell-Jones Conjecture 13.30, then G satisfies the Full Farrell-Jones Conjecture 13.30.

*Proof.* The case of coefficients in additive categories follows from Bartels [68, Remark 4.7]. The case of higher categories as coefficients follows by the same argument using [68, Theorem 3.1 and Remark 4.7] and Theorem 20.45.  $\Box$ 

The analog of assertion (iid) of Theorem 16.1, which is due to Bestvina-Fujiwara-Wigglesworth [136, Theorem 2.3], has been studied for certain relatively hyperbolic groups by Andrew-Guerch-Hughes [31]. The strategy of the proof of Bartels [68] is used by Knopf [581, Corollary 4.2] to study the Farrell-Jones Conjecture for groups acting acylindrically on a simplicial tree.

## 16.8.5 Systolic Groups

Let *G* be a group which acts cocompactly and properly on a systolic complex with the Isolated Flats Property by simplicial automorphisms. Then *G* is relatively hyperbolic to the family of virtually abelian groups by Elsner [335, Theorem B]. Hence Theorem 16.1 (ic) and Theorem 16.21 imply that *G* satisfies the Full Farrell-Jones Conjecture 13.30.

## 16.8.6 Finite-Dimensional CAT(0)-Groups

A CAT(0)-group is a group admitting a cocompact proper isometric action on a CAT(0)-space X. We call it a *finite-dimensional* CAT(0)-group if we can additionally arrange that X has finite topological dimension. Basic properties of this notion can be found for instance in [165, 657]. Examples of finite-dimensional CAT(0)-groups are fundamental groups of closed Riemannian manifolds with non-positive sectional curvature.

A finite-dimensional CAT(0)-group satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ib).

It is not known whether every finite-dimensional CAT(0)-group satisfies the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9. If *G* admits a cocompact proper isometric action on a CAT(0)-space with the Isolated Flats Property in the sense of [235, Definition 3.1], then the Baum-Connes Conjecture 14.9 holds for *G*, see [235, Corollary 0.3 b]. If *G* is a CAT(0)-cubical groups in the sense of [168], then the Baum-Connes Conjecture 14.9 holds for *G*, see [168].

## 16.8.7 Limit Groups

Limit groups as they appear, for instance, in [907] have been a focus of geometric group theory for the last few years. Expositions about limit groups include, for instance, [224, 794]. Alibegović-Bestvina [22] have shown that limit groups are CAT(0)-groups. It is not hard to check that their proof shows that a limit group is

even a finite-dimensional CAT(0)-group. Hence every limit group satisfies the Full Farrell-Jones Conjecture 13.30.

# 16.8.8 Fundamental Groups of Complete Riemannian Manifolds with Non-Positive Sectional Curvature

Let  $\pi$  be the fundamental group of a complete Riemannian manifold M. Let sec denote its sectional curvature.

• M closed and sec(M) < 0

If *M* is closed and has negative sectional curvature, then  $\pi$  is hyperbolic and hence satisfies both the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia), and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (id).

- *M* closed and sec(*M*)  $\leq 0$ If *M* is closed and has non-positive sectional curvature, then  $\pi$  is a finitedimensional CAT(0)-group and satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ib). It is not known whether all such  $\pi$  satisfy the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9.
- $C_1 \leq \sec(M) \leq C_2 < 0$  and finite volume

Let *M* be a complete Riemannian manifold which is pinched negatively curved and has finite volume. Then  $\pi$  satisfies the Full Farrell-Jones Conjecture 13.30 since  $\pi$  is relatively hyperbolic with respect to the family of virtually finitely generated nilpotent groups, see [160], or [346, Theorem 4.11], and we can apply Theorem 16.1 (ic) and Theorem 16.21.

If we additionally assume that the curvature tensor has bounded derivatives, then also the Baum-Connes Conjecture 14.9 holds for *G* by Chatterji-Ruan [235, Corollary 0.3 a]. Lattices in rank one Lie groups are examples for  $\pi$ .

• *M A*-regular and  $sec(M) \le 0$ 

A complete Riemannian manifold M is called A-regular if there exists a sequence of positive real numbers  $A_0, A_1, A_2, \ldots$  such that  $||\nabla^n K|| \le A_n$  holds for  $n \ge 0$ , where  $||\nabla^n K||$  is the supremum-norm of the *n*-th covariant derivative of the curvature tensor K. Every locally symmetric space is A-regular since  $\nabla K$  is identically zero.

Let *M* be a complete Riemannian manifold with non-positive sectional curvature that is *A*-regular. Then  $\pi = \pi_1(M)$  satisfies the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$  in degree  $n \leq 1$  and the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution  $\mathbb{Z}$ , see Farrell-Jones [370, Proposition 0.10 and Lemma 0.12]. Since  $\pi$  is torsionfree, this implies that Wh( $\pi$ ),  $\tilde{K}_0(\mathbb{Z}\pi)$ , and  $K_n(\mathbb{Z}\pi)$  for  $n \leq -1$  all vanish and Conjecture 9.114 holds for  $R = \mathbb{Z}$ .

•  $C_1 \leq \sec(M) \leq C_2 < 0$ 

Let M be a complete Riemannian manifold with pinched negative curvature. Then there is another Riemannian metric for which M is complete, negatively curved, and A-regular. This fact is mentioned in Farrell-Jones [370, page 216] and attributed there to Abresch [2] and Shi [914]. Hence the conclusions above for complete Riemannian manifolds with non-positive sectional curvature that are *A*-regular also hold for pinched negatively curved complete Riemannian manifolds.

•  $\sec(M) \le 0$ 

If *M* is a complete Riemannian manifold with non-positive sectional curvature, we have already stated some injectivity results for  $\pi$  in Theorem 16.13 and Theorem 16.18.

In particular,  $\pi$  satisfies the Novikov Conjecture 9.137 by Theorem 13.65 (xi) or Theorem 14.29.

## 16.8.9 Lattices

A discrete subgroup G of a locally compact second countable Hausdorff group  $\Gamma$  is called a *lattice* if the quotient space  $\Gamma/G$  has finite covolume with respect to the Haar measure of  $\Gamma$ .

Every lattice *G* in  $\Gamma$  satisfies the Full Farrell-Jones Conjecture 13.30 if  $\pi_0(\Gamma)$  is discrete and belongs to the class  $\mathcal{FJ}$  introduced and analyzed in Theorem 16.1, for instance, if  $\Gamma$  is path connected or an almost connected Lie group. This follows from Theorem 16.1 (id).

It is a prominent open problem to decided whether lattices satisfy the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9. This is not even known for lattices in almost connected Lie groups. The case  $SL_n(\mathbb{Z})$  is still open for  $n \ge 3$ . By [235, Corollary 0.3 a] lattices *G* in rank one Lie groups satisfy the Baum-Connes Conjecture 14.9. Some other lattices satisfying the Baum-Connes Conjecture 14.9.

#### 16.8.10 S-Arithmetic Groups

Every S-arithmetic group satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ig). This is not known for the Baum-Connes Conjecture 14.11 with coefficients or the Baum-Connes Conjecture 14.9, the group  $SL_n(\mathbb{Z})$  for  $n \ge 3$  is still an open problem.

## 16.8.11 Linear Groups

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are in general open for

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linear groups, i.e.,  $GL_n(F)$  for some field F. The same statement holds for the Baum-Connes Conjecture.

The Novikov-Conjecture holds by Theorem 14.29, Theorem 16.15 (iii), and Exercise 16.16 for any countable subgroup of  $GL_n(F)$  for a field *F*.

## 16.8.12 Subgroups of Almost Connected Lie Groups

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for discrete subgroups of almost connected Lie groups in general. The same statement holds for the Baum-Connes Conjecture 14.9.

The Novikov-Conjecture holds by Theorem 14.29 and Theorem 16.15 (iv) and Exercise 16.16 for any countable subgroup of an almost connected Lie group.

#### 16.8.13 Virtually Solvable Groups

Virtually solvable groups satisfy both the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ic) and the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (ia).

#### 16.8.14 A-T-menable, Amenable and Elementary Amenable Groups

A group *G* is called *amenable* if there is a (left) *G*-invariant linear operator  $\mu: L^{\infty}(G, \mathbb{R}) \to \mathbb{R}$  with  $\mu(1) = 1$  that satisfies for all  $f \in l^{\infty}(G, \mathbb{R})$ 

$$\inf\{f(g) \mid g \in G\} \le \mu(f) \le \sup\{f(g) \mid g \in G\}.$$

The latter condition is equivalent to the condition that  $\mu$  is bounded and  $\mu(f) \ge 0$  if  $f(g) \ge 0$  for all  $g \in G$ .

The *class of elementary amenable* groups is defined as the smallest class of groups that has the following properties:

- (i) It contains all finite and all abelian groups;
- (ii) It is closed under taking subgroups;
- (iii) It is closed under taking quotient groups;
- (iv) It is closed under extensions, i.e., if  $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$  is an exact sequence of groups and *H* and *K* belong to the class, then also *G*;

(v) It is closed under *directed unions* i.e., if  $\{G_i \mid i \in I\}$  is a directed system of subgroups such that  $G = \bigcup_{i \in I} G_i$  and each  $G_i$  belongs to the class, then G belongs to the class.

Since the class of amenable groups has all the properties mentioned above, every elementary amenable group is amenable. The converse is not true. For more information about amenable and elementary amenable groups, we refer for instance to [650, Section 6.4.1] or [792].

A group *G* is *a*-*T*-menable, or, equivalently, has the Haagerup property, if *G* admits a metrically proper isometric action on some affine Hilbert space. Metrically proper means that for any bounded subset *B* the set  $\{g \in G \mid gB \cap B \neq \emptyset\}$  is finite.

An extensive treatment of such groups is presented in [240, 964]. Any a-Tmenable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients, and under finite products. It is not closed under semidirect products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of SO(n, 1) and SU(n, 1), Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes. A group *G* has Kazhdan's *property* (*T*) if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. For more information about this property we refer for instance to [119]. An infinite a-T-menable group does not have property (T). Since  $SL_n(\mathbb{Z})$  for  $n \ge 3$ has property (T), it cannot be a-T-menable.

Every a-T-menable, every amenable, and every elementary-amenable group satisfies the Baum-Connes Conjecture 14.11 with coefficients. This follows from Theorem 16.7 (ia) in the a-T-menable case. Since every group is the directed union of its finitely generated subgroups, every finitely generated group is countable, and every countable amenable group is a-T-menable, the claim follows for amenable groups and hence also for elementary amenable groups from Theorem 16.7 (iid).

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for elementary amenable groups. The main problem in the Farrell-Jones setting is that one has to deal with virtually cyclic subgroups in its formulation and for the inheritance property under extensions, see Theorem 16.7 (iic), whereas in the Baum-Connes setting finite subgroups suffice. This also explains why elementary amenable groups satisfy the Farrell-Jones Conjecture 15.79 for homotopy *K*-theory with coefficients in additive *G*-categories with finite wreath products, see Theorem 16.5 (i).

The *L*-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2 holds for elementary amenable groups by [475, Theorem 5.2.1].

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#### 16.8.15 Three-Manifold Groups

Let *M* be a (not necessarily compact) manifold (possibly with boundary) of dimension  $\leq 3$ . Then  $\pi_1(M)$  satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ie).

If we additionally assume that *M* is compact, then  $\pi_1(M)$  satisfies the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (if).

In the Farrell-Jones setting the reason why we do not need compactness is the inheritance property under directed colimits of directed systems of subgroups, see Theorem 16.1 (iif), which is not available in the Baum-Connes setting, where we need that all structure maps are injective, see Theorem 16.7 (iid).

Exercise 16.22. Let G be the fundamental group of a knot complement.

Show for any regular ring *R* that the projection pr:  $G \to G/[G, G] \cong \mathbb{Z}$  induces for every ring *R* an isomorphism  $K_n(RG) \to K_n(R[G/[G, G]])$  and we get an isomorphism  $K_n(RG) \cong K_n(R) \oplus K_{n-1}(R)$ .

Show for any ring *R* with involution  $L_n^{\langle -\infty \rangle}(RG) \cong L_n^{\langle -\infty \rangle}(R) \oplus L_n^{\langle -\infty \rangle}(R)$ .

#### 16.8.16 One-Relator Groups

The Baum-Connes Conjecture 14.11 with coefficients holds for one-relator groups by Theorem 16.7 (ie).

A consequence of Newman's spelling theorem, see [757], is that a one-relator group which is not torsionfree is hyperbolic and hence satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia).

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for torsionfree one-relator groups. Note that not all one-relator groups are solvable, hyperbolic, or finite-dimensional CAT(0)-groups, so that we cannot apply Theorem 16.1 in general.

Nevertheless the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* is known if *R* is regular and *G* is a subgroup of a torsionfree one-relator group by Waldhausen [976, Theorem 19.4 on page 249] in the connective case and by Bartels-Lück [75, Theorem 0.11] for the non-connective version. Recall that in this special case Conjecture 13.1 boils down to Conjecture 6.53.

The *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in any ring with involution *R* holds after inverting two for torsionfree one-relator groups by Cappell [204, Corollary 8].

All Baumslag-Solitar groups satisfy the Full Farrell-Jones Conjecture 13.30, see Farrell-Wu [376] for the version without "finite wreath products" and Gandini-Meinert-Rüping [415, Corollary 1.1].

#### 16.8.17 Self-Similar Groups

We use the notion of self-similar group as presented in [96, Section 3], which is slightly more general than the classical notion defined for instance in [97, 756]. Self-similar groups are groups acting in a recursive manner on a regular rooted tree  $RT_d$ . If the recursion of every element involves only a linearly growing subtree of  $T_d$ , the group is said to be bounded.

The Full Farrell-Jones Conjecture 13.30 is proved by Bartholdi [96, Theorem A] for bounded self-similar groups since these are subgroups of finite-dimensional CAT(0)-groups and hence Theorem 16.1 (ib) and (iia) applies. Using Theorem 16.1 (ib) and (iib) Bartholdi [96, Theorem C] proves the Full Farrell-Jones Conjecture 13.30 for Aleshin-Grigorchuk groups, Gupta-Sidki groups, and generalized Grigorchuk groups, whose definition and intriguing properties are reviewed in [96, Section 4].

## 16.8.18 Virtually Torsionfree Hyperbolic by Infinite Cyclic Groups

If *H* is a virtually torsionfree hyperbolic group and  $\phi: H \to H$  is an automorphism, then  $G = H \rtimes_{\phi} \mathbb{Z}$  satisfies the Full Farrell-Jones Conjecture 13.30.

This follows from [136, Proposition 2.2 and Theorem 2.3] using [70, Remark 9.4]. Note that this implies the more general assertion (iid) appearing in Theorem 16.1.

There is no counterexample to the conjecture that every hyperbolic group is virtually torsionfree.

**Exercise 16.23.** Let *G* be a group with a filtration  $\{1\} = G_0 \subseteq G_1 \subseteq ... \subseteq G_d = G$  such that  $G_{i-1}$  is normal in *G* and  $G_i/G_{i-1}$  is a virtually torsionfree hyperbolic group for i = 1, 2, ..., d.

Show that then the Full Farrell-Jones Conjecture 13.30 holds for G.

## 16.8.19 Countable Free Groups by Infinite Cyclic Groups

If *F* is a countable free group (of possibly infinite rank) and  $\phi: F \to F$  is an automorphism, then  $G = F \rtimes_{\phi} \mathbb{Z}$  satisfies the Full Farrell-Jones Conjecture 13.30.

This follows from [173, Theorem 2.5] using [136]. Note that this implies the more general assertion (iie) appearing in Theorem 16.1.

The condition that F is countable ensures that F has a countable basis. It is conceivable that it is not necessary.

#### 16.8.20 Strongly Poly-Surface Groups

**Definition 16.24 (Strongly poly-surface group).** Let *G* be a group with a finite filtration  $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_d = G$ .

We call *G* strongly poly-surface if the filtration satisfies the following conditions:

- (i)  $G_i$  is normal in *G* for i = 0, 1, 2, ..., d;
- (ii) For every  $i \in \{1, 2, ..., d\}$  and  $g \in G$ , there is a (not necessarily compact) surface *S* (possibly with boundary) with torsionfree  $\pi_1(S)$ , a diffeomorphism  $f: S \to S$ , and an isomorphism  $\alpha: G_i/G_{i-1} \xrightarrow{\cong} \pi_1(S)$  such that the following diagram commutes

$$\begin{array}{c|c} G_i/G_{i-1} & \xrightarrow{c_g} & G_i/G_{i-1} \\ & \alpha \\ & & & & \downarrow \alpha \\ & & & & \\ \pi_1(S) & \xrightarrow{\pi_1(f)} & \pi_1(S) \end{array}$$

where  $c_g$  is induced by conjugation with  $g \in G$ .

Note that condition (ii) is automatically satisfied if S is a closed surface.

**Theorem 16.25 (The Full Farrell-Jones Conjecture for strongly poly-surface groups).** A strongly poly-surface group G satisfies the Full Farrell-Jones Conjecture 13.30.

*Proof.* Fix a filtration  $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_d = G$  as it occurs in Definition 16.24. We show by induction over  $i = 0, 1, 2, \ldots, d$  that  $G/G_{d-i}$  satisfies the Full Farrell-Jones Conjecture 13.30. The induction beginning i = 0 is trivial, the induction step from (i - 1) to i done as follows.

We have the exact sequence  $1 \rightarrow G_{d-i+1}/G_{d-i} \rightarrow G/G_{d-i} \xrightarrow{p} G/G_{d-i+1} \rightarrow 1$ . By induction hypothesis  $G/G_{d-i+1}$  satisfies the Full Farrell-Jones Conjecture 13.30. Since  $G_{d-i+1}/G_{d-i} \cong \pi_1(S)$ , the group  $G_{d-i+1}/G_{d-i}$  satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ie). Consider any infinite cyclic subgroup  $C \subseteq G/G_{d-i+1}$ . Choose  $g \in G$  such that the image of g under  $p: G/G_{d-i} \rightarrow G/G_{d-i+1}$ sends g to a generator of C. Hence  $p^{-1}(C)$  is isomorphic to  $G_{d-i+1}/G_{d-i} \rtimes_{c_g} \mathbb{Z}$ . From the assumptions about G, we get a diffeomorphism  $f: S \rightarrow S$  of a surface Ssuch that  $p^{-1}(C)$  is isomorphic to  $\pi_1(T_f)$ . Since  $T_f$  is a 3-manifold,  $\pi_1(T_f)$  satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ie). We conclude from Theorem 16.1 (iic) that  $G/G_{d-i}$  satisfies the Full Farrell-Jones Conjecture 13.30.  $\Box$ 

**Exercise 16.26.** Let *G* be a group with a filtration  $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_d = G$  such that  $G_{i-1}$  is normal in  $G_i$  and  $G_i/G_{i-1}$  is torsionfree and isomorphic to the fundamental group of a compact manifold of dimension  $\leq 3$  (possibly with boundary) for  $i = 1, 2, \ldots, d$ .

Show that then the Baum-Connes Conjecture 14.11 with coefficients holds for G.

## 16.8.21 Normally Poly-Free Groups

A group *G* is called *poly-free* if there is a finite filtration  $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_d = G$  such that  $G_{i-1} \subseteq G_i$  is normal and  $G_i/G_{i-1}$  is countable and free (of possibly infinite rank) for  $i = 1, 2, \dots, d$ . The Baum-Connes Conjecture 14.11 with coefficients holds for poly-free groups *G* by Theorem 16.7 (iic), (iid), and (iie).

The Full Farrell-Jones Conjecture 13.30 is not known for all poly-free groups.

We call a group a *normally poly-free group* if there is a finite filtration  $\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_d = G$  such that  $G_{i-1} \subseteq G$  is normal and  $G_i/G_{i-1}$  is countable and free (of possibly infinite rank) for i = 1, 2, ..., d.

**Theorem 16.27 (The Full Farrell-Jones Conjecture for normally poly-free groups).** A normally poly-free group satisfies the Full Farrell-Jones Conjecture 13.30.

*Proof.* This is proved by Brück-Kielak-Wu [173] using the proof for the case of a finitely generated free group extended by  $\mathbb{Z}$  due to Bestvina-Fujiwara-Wigglesworth [136].

**Exercise 16.28.** Let  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  be an extension of groups such that *K* is the fundamental group of a compact connected manifold (possibly with boundary) of dimension  $\leq 2$ .

Show that G satisfies the Full Farrell-Jones Conjecture 13.30 if Q does.

## 16.8.22 Coxeter Groups

For the definition of and information about Coxeter groups we refer to [289]. Every Coxeter group satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (il) and the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ii).

## 16.8.23 Right-Angled Artin groups

Every right-angled Artin group can be embedded in a right-angled Coxeter groups as a subgroup of finite index, see [292]. Hence every right-angled Artin group satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (il) and (iia) and the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ii) and (iia).

For more information about Right-Angled Artin groups we refer for instance to [231].

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#### 16.8.24 Artin groups

The Full Farrell-Jones Conjecture 13.30 and the Baum-Connes Conjecture 14.11 are open for Artin groups, only some partial results are known.

It is an open problem whether every Artin group admits a cocompact proper isometric action on a complete CAT(0)-space. This is known in some cases, see for instance Haettel [452, 453]. It seems to be also an open question whether Artin groups are A-T-menable.

Even Artin groups of type FC satisfy the Full Farrell-Jones Conjecture 13.30 by Huang-Osajda [506, Corollary], see also [173, Corollary B] and [1020].

Extra large type Artin groups satisfy the Full Farrell-Jones Conjecture 13.30, see [320, Corollary E].

The Baum-Connes Conjecture 14.11 with coefficients is proved for some Artin groups by Haettel [453, Corollary C].

## 16.8.25 Braid Groups

Artin's full braid groups  $P_n$  satisfy both the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ik) and the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (ig).

## 16.8.26 Mapping Class Groups

Let  $F_{g,r}^s$  be the orientable compact surface of genus g with r boundary components and s punctures where s punctures means the choice of s pairwise distinct points. Let Diff $(F_{g,r}^s$ , rel) be the group of orientation preserving diffeomorphisms  $F_{g,r}^s \rightarrow$  $F_{g,r}^s$  that leave the boundary and the punctures pointwise fixed. Then the *mapping* class group  $\Gamma_{g,r}^s$  is defined to be  $\pi_0(\text{Diff}(S_{g,r}^s, \text{rel}))$ , the group of isotopy classes of such diffeomorphisms. All mapping class groups satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 ih.

The Baum-Connes Conjecture 14.9 does not seem to be known to be true for all mapping class groups.

## 16.8.27 $Out(F_n)$

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for  $Out(F_n)$  for  $n \ge 3$ . The same statement holds for the Baum-Connes Conjecture.

The group  $Out(F_n)$  is boundary amenable by a result of Bestvina-Guiardel-Horbez [137]. Hence the assembly map appearing in the Baum-Connes Conjecture 14.11 with coefficients is rationally injective, see Remark 16.17, and therefore also the Novikov Conjecture holds for any subgroup of  $Out(F_n)$ , see Section 14.8. Actually, in [137] groups other than  $F_n$ , for instance torsionfree hyperbolic groups, and right-angled Artin groups, are also treated.

At least the rational injectivity of the *K*-theoretic Farrell-Jones assembly map with coefficients in  $\mathbb{Z}$  (disregarding some  $K_{-1}$ -term contribution) follows from [675] for  $Out(F_n)$ .

## 16.8.28 Thompson's Groups

Thompson defined the groups F, T, and V in some handwritten notes from 1965. Thompson's group V is the group of right-continuous automorphisms f of [0, 1] that map dyadic rational numbers to dyadic rational numbers, that are differentiable except at finitely many dyadic rational numbers, and such that, on each interval on which f is differentiable, f is affine with derivative a power of 2. The group F is the subgroup of V consisting of homeomorphisms. The group T is the subgroup of V consisting of those elements that induce homeomorphisms of the circle where the circle is regarded as [0, 1] with 0 and 1 identified. These groups have some unusual properties. It is an open question whether F is amenable. It is known that F is not elementary amenable.

Farley [348] has shown that F, T, and V are a-T-menable and hence satisfy the Baum-Connes Conjecture 14.11 with coefficients, see Theorem 16.7 (ia).

The Full Farrell-Jones Conjecture 13.30, and actually even the *K*-theoretic Farrell-Jones Conjecture 13.2 with coefficients in rings and the *L*-theoretic Farrell-Jones Conjecture 13.7 with coefficients in rings with involution, are open for F, T, and V.

At least the rational injectivity of the *K*-theoretic Farrell-Jones assembly map with coefficients in  $\mathbb{Z}$  (disregarding some  $K_{-1}$ -term contribution) follows from [675] for *T* using [420].

## 16.8.29 Helly Groups

The Full Farrell-Jones Conjecture 13.30 is proved for Helly groups by Chalopin-Chepoi-Genevois-Osajda [223, Section 7.5] using [571]. This implies that the Full Farrell-Jones Conjecture 13.30 holds also for weak Garside groups of finite type, see Huang-Osajda [506, Theorem and Corollary].

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#### 16.8.30 Groups Satisfying Homological Finiteness Conditions

So far the groups for which we were able to prove the Farrell-Jones Conjecture or the Baum-Connes Conjecture have satisfied some geometric conditions, often reminiscent of non-positive sectional curvature. At least for the *K*-theoretic Farrell-Jones Conjecture there are results where no geometric conditions but some finiteness conditions are required. The celebrated prototype of such a result is the following theorem due to Boekstedt-Hsiang-Madsen [150].

**Theorem 16.29 (Bökstedt-Hsiang-Madsen Theorem).** Let G be a group such that  $H_i(G;\mathbb{Z})$  is finitely generated for all  $i \ge 0$ . Then G satisfies the K-theoretic Novikov Conjecture 13.63, i.e., the assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is rationally injective for all  $n \in \mathbb{Z}$ .

This raises the question under which finiteness conditions one can show that the assembly map appearing in the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$  is rationally injective. Recall from Theorem 13.51 that for a group *G* and a regular ring *R* the map

(16.30) 
$$H_n^G(\iota_{\mathcal{FIN}\subseteq \mathcal{VCY}};\mathbf{K}_R): H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \xrightarrow{\cong} H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$$

is bijective for all  $n \in \mathbb{Z}$  after applying  $\mathbb{Q} \otimes_{\mathbb{Z}} -$ .

The source of the map (16.30) has already been computed rationally using equivariant Chern characters in Theorem 12.79

(16.31) 
$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_R K_q(RC))$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_R H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R).$$

By the isomorphisms (16.30) and (16.31), the assembly map appearing in the Farrell-Jones Conjecture 13.1 with coefficients in the regular ring R becomes rationally a map

(16.32) 
$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_GC; \mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_R K_q(RC)) \rightarrow \mathbb{Q} \otimes_R K_n(RG).$$

So the question above is equivalent to the question whether the map (16.32) is rationally injective.

From now on we consider the special case  $R = \mathbb{Z}$ . The restriction of the map (16.32) to the summand corresponding to  $C = \{1\}$  is rationally the same as the map appearing in Theorem 16.29. Hence a positive answer to the question above implies Theorem 16.29.

The main result of [675] says that under certain finiteness assumptions, which are for instance satisfied if there is a model for  $E_{\mathcal{FIN}}(G)$  of finite type, and certain number theoretic conditions, which are implied by the Leopoldt-Schneider Conjecture, the assembly map (16.32) is rational injective if we ignore the summands for q = -1. This summand cannot be detected since topological cyclic homology does not see  $K_{-1}$ . Note that Theorem 16.31 just detects the summand for  $C = \{1\}$  and does not see the ones for non-trivial C. Nevertheless, the methods and proofs of [675] are based on the ideas of [150].

As an illustration we mention two easy to formulate consequences of the results of [675, Main Theorem 1.13] where the necessary input from number theory is known to be true and therefore does not appear in the assumptions, similar to the situation in Theorem 16.29.

**Theorem 16.33 (Rationally injectivity of the colimit map for finite subgroups for the Whitehead group).** Let G be a group. Assume that for every finite cyclic subgroup C of G the abelian groups  $H_1(BC_GC;\mathbb{Z})$  and  $H_2(BC_GC;\mathbb{Z})$  associated to their centralizers  $C_GC$  are finitely generated.

Then the canonical map

$$\operatorname{colim}_{H \in \operatorname{Sub}G(\mathcal{FIN})} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Wh}(H) \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Wh}(G)$$

is injective.

Proof. See [675, Theorem 1.1]

Note that the Q-module  $\operatorname{colim}_{H \in \operatorname{Sub}G(\mathcal{FIN})} \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Wh}(H)$  above can be identified with  $\bigoplus_{(C) \in J} \mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_1(\mathbb{Z}C))$ , where *J* is the set of conjugacy classes of finite cyclic subgroups of *G*. This is the portion for p = 0 appearing in the source of the isomorphism with  $\mathbb{Q} \otimes_{\mathbb{Z}} H_n^G(\underline{E}G; \mathbf{K}_R)$  as target of Theorem 12.79. Moreover, the  $\mathbb{Q}[\operatorname{aut}(C)]$ -module  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_1(\mathbb{Z}C))$  is described explicitly in Remark 12.80.

**Theorem 16.34 (Eventual injectivity of the rational** *K***-theoretic assembly map** for  $R = \mathbb{Z}$ ). Let G be a group. Assume that there is a finite G-CW-model for  $E_{\mathcal{FIN}}(G)$ .

Then there exists an integer L > 0 such that the rationalized Farrell-Jones assembly map (16.32) is injective for all  $n \ge L$ . The bound L only depends on the dimension of  $E_{\mathcal{FIN}}(G)$  and on the orders of the finite cyclic subgroups of G.

Proof. See [675, Theorem 1.15]

## 16.9 Open Cases

Here is a list of interesting groups for which the Full Farrell-Jones Conjecture 13.30 is open in general:

16.10 How Can We Find Counterexamples?

- elementary amenable, amenable, or a-T-menable groups;
- $\operatorname{Out}(F_n)$  for  $n \ge 3$ ;
- Artin groups;
- Thompson's groups *F*, *V*, and *T*;
- Torsionfree one-relator groups;
- Linear groups;
- Subgroups of almost connected Lie groups;
- Residual finite groups;
- (Bi-)Automatic groups;
- Locally indicable groups.

Here is a list of interesting groups for which the Baum-Connes Conjecture 14.11 with coefficients is open in general:

- Finite-dimensional CAT(0)-groups.
- Fundamental groups of closed Riemannian manifolds with non-positive sectional curvature;
- Lattices in almost connected Lie groups, for instance  $SL_n(\mathbb{Z})$  for  $n \ge 3$ ;
- S-arithmetic groups;
- $\operatorname{Out}(F_n)$  for  $n \ge 3$ ;
- Artin groups;
- Mapping class groups (of higher genus);
- Linear groups;
- Subgroups of almost connected Lie groups;
- Residual finite groups;
- (Bi-)Automatic groups;
- Locally indicable groups.

# 16.10 How Can We Find Counterexamples?

We are not aware of any group for which the Full Farrell-Jones Conjecture 13.30 is known to be false. The same statement holds for the Baum-Connes Conjecture 14.9 without coefficients and the Novikov Conjecture 9.137.

## 16.10.1 Is the Full Farrell-Jones Conjecture True for All Groups?

It is hard to believe that the Full Farrell-Jones Conjecture 13.30 is true for all groups since there have been so many prominent conjectures about groups which were open for some time and for which counterexamples were finally found. On the other hand, the conjecture is known for so many groups that we currently have no strategy to find counterexamples, as we will illustrate below.

We have already mentioned that the groups that come from the construction of Arzhantseva-Delzant [39], see also Osajda [783], yield counterexamples to the Baum-Connes Conjecture 14.11 with coefficients by Higson-Lafforgue-Skandalis [487]. These groups are colimits of hyperbolic groups and hence satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia) and (iif).

Baum-Guentner-Willet [111] give a reformulation of the Baum-Connes Conjecture 14.11 with coefficients by introducing a new crossed product, see also [192], for which no counterexamples are known so far.

We have already discussed the problem concerning the Baum-Connes Conjecture 14.9, which does not occur for the Full Farrell-Jones Conjecture 13.30, that the left-hand side of the Baum-Connes Conjecture is functorial under group homomorphism and there is no reason why the right-hand side should have this property, see Remark 14.12. The new version of Baum-Guentner-Willet [111] still faces this problem. This sheds additional doubts on the Baum-Connes Conjecture.

#### 16.10.2 Exotic Groups

One does not know of a property of a group for which one may expect that groups with this property are automatically counterexamples to the Full Farrell-Jones Conjecture 13.30 or to the Baum-Connes Conjecture 14.9. Next we list some groups with an exotic property for which the Full Farrell-Jones Conjecture 13.30 is known to be true at least for some groups satisfying this property.

• Finitely generated infinite torsion *p*-groups

Given a large enough prime p, there exists an infinite finitely generated group all of whose proper subgroups are finite cyclic groups of order p, see [779]. These groups are lacunary hyperbolic groups and hence satisfy the Full Farrell-Jones Conjecture 13.30, see Subsection 16.8.2.

Other examples of finitely generated infinite torsion p-groups are mentioned in Subsection 16.8.17;

• Groups with expanders

There exists a group G that is a colimit of hyperbolic groups and contains appropriate expanders, see [39]. It satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia) and (iif);

- Self-similar groups
  - See Subsection 16.8.17.
- Infinite torsionfree simple groups

There exists finitely presented torsionfree simple CAT(0)-groups, see [188, Corollary 5.4 and Theorem 5.5]. They satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ib);

• Groups which do not possess a finite-dimensional model or a model of finite type for *BG* or *BG* 

Examples of such groups satisfying the Full Farrell-Jones Conjecture 13.30 can easily be constructed using Theorem 16.1 (iig);

• Groups with property (T)

There are hyperbolic groups that have property (T). They satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia);

• Groups for which certain decision problems are unsolvable.

A lot of groups for which the Full Farrell-Jones Conjecture 13.30 is known and some decision problems such as the isomorphism problem, conjugacy problem and membership problem are unsolvable can be found in Bridson [164].

The results about groups with some homological finiteness conditions of Subsection 16.8.30 also indicate that the search for counterexamples for the Farrell-Jones Conjecture is not easy.

In order to find counterexamples one seems to need completely new ideas, maybe from random groups or logic. It is unlikely that the counterexample is a concrete group, but rather a group with certain strange properties, for which existence can be shown by abstract methods but not by a concrete construction.

It is probably easier to find counterexamples to surjectivity than to injectivity.

## 16.10.3 Infinite Direct Products

Nothing is known about infinite products. It would be very interesting if one can show that for a family of groups  $\{G_i \mid i \in I\}$  (with infinite *I*) the Full Farrell-Jones Conjecture 13.30 is true for the direct product  $\prod_{i \in I} G_i$  if it holds for each  $G_i$ . (Note that the corresponding statement is true for the direct sum  $\bigoplus_{i \in I} G_i$  by Theorem 16.1 (iib) and (iif).) In view of Theorem 16.1 (iia) this would imply that the Full Farrell-Jones Conjecture 13.30 is stable under inverse limits over directed systems of groups. This would have the immediate consequence that the Full Farrell-Jones Conjecture 13.30 is true for all residually finite groups. On the other hand it may be worthwhile to look at infinite direct products in order to find a counterexample.

For this discussion see also Lemma 15.103.

## 16.10.4 Exotic Aspherical Closed Manifolds

One may also look for counterexamples to one of the conjecture which follow from the Full Farrell-Jones Conjecture 13.30, for instance to the Borel Conjecture 9.163. There are indeed aspherical closed manifolds with unusual properties, but the fundamental groups of some of them do satisfy the Full Farrell-Jones Conjecture 13.30 and hence the Borel Conjecture. Note that we have already discussed aspherical closed manifolds with exotic properties in Subsection 9.15.1.

Davis constructed for every  $n \ge 4$  aspherical closed manifolds of dimension n whose universal covering is not homeomorphic to Euclidean space [288, Corollary 15.8]. In particular, these manifolds do not support metrics of non-positive sectional curvature. The fundamental groups of these examples are finite index

subgroups of Coxeter groups W. Thus they satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (il) and (iia). In particular, these manifolds are indeed topologically rigid, provided that  $n \ge 5$ .

Davis and Januszkiewicz [291, Theorem 5b.1] used Gromov's hyperbolization technique to construct for every  $n \ge 5$  an aspherical closed *n*-dimensional manifold M such that the universal covering  $\tilde{M}$  is a finite-dimensional CAT(0)-space whose fundamental group at infinity is non-trivial. In particular, these universal covers are not homeomorphic to Euclidean space. Because these examples are in addition non-positively curved polyhedra, their fundamental groups are finite-dimensional CAT(0)-groups. There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [232] and produces an aspherical closed manifold Mwhose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic. The fundamental groups of these manifolds M satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia) and (ib). In particular, these manifolds M are topologically rigid.

Davis-Januszkiewicz [291, Theorem 5a.1 and Corollary 5a.4] constructed a 4-manifold N such that  $\pi_1(N)$  is a finite-dimensional CAT(0)-group and  $N \times T^k$  for  $k \ge 1$  is not homotopy equivalent to a PL-manifold. Since  $\pi_1(N \times T^k)$  is a finite-dimensional CAT(0)-group and dim $(N \times T^k) \ge 5$  for  $k \ge 1$ , the manifolds  $N \times T^k$  for  $k \ge 1$  are topologically rigid by Theorem 16.1 (ib).

Davis-Fowler-Lafont [290] constructed using the work of Manolescu [706, 705] non-triangulable aspherical closed manifolds with hyperbolic fundamental group in all dimensions  $\geq 6$ . In particular, these manifolds *M* are topologically rigid since hyperbolic groups satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia).

Some proofs of the results above are based on the *reflection group trick* as it appears for instance in [286, Sections 8,10 and 13]. It can be summarized as follows.

**Theorem 16.35 (Reflection group trick).** Let G be a group that possesses a finite model for BG. Then there is an aspherical closed manifold M and a map  $i: BG \to M$  and  $r: M \to BG$  such that  $r \circ i = id_{BG}$ .

An interesting immediate consequence of the reflection group trick is that many well-known conjectures about groups hold for every group that possesses a finite model for BG if and only if it holds for the fundamental group of every aspherical closed manifold, see also [286, Sections 11].

**Exercise 16.36.** Suppose that the Farrell-Jones Conjecture 6.53 for torsionfree groups and regular rings holds for the fundamental group of any aspherical closed manifold. Show that it then holds for all groups G with a finite model for BG.

Prove the analogous statement for the *L*-theoretic Farrell-Jones Conjecture 9.114 for torsionfree groups.

The upshot of the discussion is that one does not know of a property of aspherical closed manifolds, such as being not triangulable, for which one may expect that the Borel Conjecture 9.163 automatically fails if this property is satisfied.

16.11 Notes

## 16.10.5 Some Results Which Hold for All Groups

Here is a result which holds for all (discrete) groups, is non-trivial, and is related to the Farrell-Jones Conjecture. Let  $i: H \to G$  be the inclusion of a normal subgroup  $H \subset G$ . It induces a homomorphism  $i_0: Wh(H) \to Wh(G)$ . The conjugation actions of G on H and on G induce G-actions on Wh(H) and on Wh(G) which turns out to be trivial on Wh(G). Hence  $i_0$  induces homomorphisms

(16.37) 
$$i_1: \mathbb{Z} \otimes_{\mathbb{Z}G} Wh(H) \to Wh(G);$$

(16.38) 
$$i_2 \colon \mathrm{Wh}(H)^G \to \mathrm{Wh}(G)$$

**Theorem 16.39 (Rational injectivity of**  $\mathbb{Z} \otimes_{\mathbb{Z}G} Wh(H) \to Wh(G)$  **for normal finite**  $H \subseteq G$ ). Let  $i: H \to G$  be the inclusion of a normal finite subgroup H into an arbitrary group G. Then the maps  $i_1$  and  $i_2$  defined in (16.37) and (16.38) have finite kernel.

*Proof.* See [650, Theorem 9.38 on page 354].

We omit the details of the proof that the result of Theorem 16.39 can be deduced from the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{Z}$ .

In Lück-Oliver [672] there is a systematic study of those finite groups H for which Theorem 16.39 implies that for every group G with  $\mathbb{Q} \otimes_{\mathbb{Z}} Wh(G) = \{0\}$  the group H cannot occur as a normal subgroup in G.

**Exercise 16.40.** Let *G* be a group with vanishing Whitehead group. Show that then each element in the center has order 1, 2, 3, 4, or 6.

We have already stated a more advanced detection result for  $\mathbb{Q}$  and  $\mathbb{C}$  as coefficients, see Theorem 6.78, which also holds for all groups. Recall also Theorem 16.33, which requires only very mild conditions on the group *G*.

Another non-trivial consequence of the Farrell-Jones Conjecture which concerns the Hattori-Stallings rank of idempotents in FG for fields F and groups G and holds for all groups G has been discussed in Remark 2.98.

Furthermore, Yu [1028, Theorem 1.1], see also Cortinas-Tartaglia [261], proved that the *K*-theoretic assembly map  $H_n^G(E_{VCY}(G); \mathbf{K}_S) \to K_n(SG)$  appearing in the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* is rationally injective for every group *G*, provided that *R* is the ring *S* of Schatten class operators of an infinite-dimensional separable Hilbert space.

# 16.11 Notes

There are groups for which the Full Farrell-Jones Conjecture 13.30 is not known to be true but weaker versions of it have been proved. For example, the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R is known if R is regular

and G belongs to the class  $C\mathcal{L}'$  described in [75, Definition 0.10]. The class  $C\mathcal{L}'$  contains for instance all torsionfree 1-relator groups.

The class of groups for which the *L*-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2 is analyzed in [475, Proposition 5.2.2 and Lemma 5.2.3]; actually the more general fibered version is treated. It contains for instance all elementary amenable groups. The result and its proof is analogous to Theorem 16.5.

A proof of the Full Farrell-Jones Conjecture 13.30 for finite-dimensional CAT(0)groups has been extended to a larger class of groups which also contains all hyperbolic groups by Kasprowski-Rüping [571, Theorem 6.1]. In particular, they prove it for all groups acting properly and cocompactly on a finite product of hyperbolic graphs, see [571, Theorem 1.1], as already mentioned in Theorem 16.1 (in).

The bijectivity of the algebraic *K*-theoretic assembly map for certain coefficients coming from  $C^*$ -algebras is proved by Cortinas-Tartaglia [259, Corollary 1.5] for a-T-menable groups *G* by reducing it to the Baum-Connes Conjecture.

Gonzalez-Acuna-Gordon-Simon [429, Theorem 5.6, Corollary 5.7, Theorem 5.8] show that the problem whether the projective class group, the Whitehead group, or the *L*-group of a group is trivial, cannot be decided. So it is possible that the problem whether a group *G* satisfies the Farrell-Jones Conjecture holds cannot be decided.

# Chapter 17 Guide for Computations

# **17.1 Introduction**

One major goal is to compute *K*- and *L*-groups such as  $K_n(RG)$ ,  $L_n^{\langle -\infty \rangle}(RG)$ , and  $K_n(C_r^*(G))$ . Assuming that the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R*, the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution *R*, or the Baum-Connes Conjecture 14.9 hold for *G*, this reduces to the computation of the left-hand side of the corresponding assembly maps, namely, of  $H_n^G(E_{VCY}(G); \mathbf{K}_R)$ ,  $H_n^G(E_{VCY}(G); \mathbf{L}_R^{\langle -\infty \rangle})$ , or  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\text{TOP}}) = K_n^G(E_{\mathcal{FIN}}(G))$ . This is much easier, since here we can use standard methods from algebraic topology. The main general tools are the equivariant Atiyah-Hirzebruch spectral sequence, see Theorem 12.48, the *p*-chain spectral sequence, see Theorem 12.50, and equivariant Chern characters, see Theorem 12.58. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a reasonable answer after rationalization, but integral computations have only been done case by case, and there seems to be no pattern for a general answer. Often the key is a good understanding of how one can built  $E_{\mathcal{FIN}}(G)$  from *EG* and how one can built  $E_{\mathcal{VCY}}(G)$  from  $E_{\mathcal{FIN}}(G)$ . These passages have already been studied in Theorems 11.32 and 11.37.

# 17.2 K- and L-Groups for Finite Groups

For the computations of  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$ ,  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle})$ , and  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathsf{TOP}}) = K_n^G(E_{\mathcal{FIN}}(G))$ , one has to understand  $K_n(RH)$ ,  $L_n^{\langle -\infty \rangle}(RH)$ , and  $K_n(C_r^*(H))$  for finite groups H, since these are the values of  $H_n^G(G/H; \mathbf{K}_R)$ ,  $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle})$ , and  $H_n^G(G/H; \mathbf{K}^{\mathsf{TOP}}) = K_n^G(G/H)$  for homogeneous spaces G/H for finite subgroups  $H \subseteq G$ .

For a finite group *G* we have given information about  $K_0(\mathbb{Z}G)$  in Section 2.12, about  $K_1(\mathbb{Z}G)$  and Wh(G) in Section 3.12, about  $K_n(\mathbb{Z}G)$  for  $n \leq -1$  in Example 4.12, Section 4.5, and Example 5.15, about  $K_2(\mathbb{Z}G)$  and  $Wh_2(G)$  in Section 5.8, about  $L_n^{\langle j \rangle}(\mathbb{Z}G)$  in Section 9.22, and about  $K_n(C_r^*G)$  and  $KO_n(C_r^*(G; \mathbb{R}))$  in Section 10.9.

Let us summarize what we know for a finite group G. There is a complete calculation of the finitely generated abelian group  $K_{-1}(\mathbb{Z}G)$ , and one knows  $K_n(\mathbb{Z}G) = 0$ for  $n \leq -2$ . One has a very good understanding of Wh(G). The group  $\widetilde{K}_0(\mathbb{Z}G)$ is finite, but a complete computation of  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$  for arbitrary primes p is out of reach. A complete computation of  $K_n(\mathbb{Z})$  is not known for all  $n \in \mathbb{Z}$ . We have already mentioned Borel's formula for  $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  for all  $n \in \mathbb{Z}$  in Theorem 6.24. The *L*-groups of  $\mathbb{Z}G$  are pretty well understood. The finitely generated abelian groups  $K_n(C_r^*(G))$  and  $K_n(C_r^*(G;\mathbb{R}))$  are explicitly known. They are torsionfree in the complex case. In the real case every non-trivial element of finite order has order 2.

# 17.3 The Passage from $\mathcal{FIN}$ to $\mathcal{VCY}$

In the Baum-Connes setting it is enough to consider the family  $\mathcal{FIN}$ . In the Farrell-Jones Conjecture we have to pass from  $\mathcal{FIN}$  to  $\mathcal{VCY}$ . This passage has been discussed in detail already in Section 13.8. We get a splitting

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \\ \cong H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \oplus H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathcal{A}})$$

and under mild K-theoretic assumptions a splitting

$$\begin{split} H_n^G \big( E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle} \big) \\ &\cong H_n^G \big( E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle} \big) \oplus H_n^G \big( E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle} \big). \end{split}$$

We have also explained in Theorem 13.47 that in *K*-theory it suffices to replace  $\mathcal{VCY}$  by  $\mathcal{VCY}_I$  and in Theorem 13.60 that in *L*-theory there is no difference between  $\mathcal{FIN}$  and  $\mathcal{VCY}_I$ .

If we are only interested in rational information, then there is no difference between  $\mathcal{FIN}$  and  $\mathcal{VCY}$  when we are dealing with the algebraic *K*-theory of group rings *RG* for regular rings *R*, see Theorem 13.51, and when we are dealing with *L*-theory, see Theorem 13.62 (i).

For *L*-theory the Tate cohomology of the *K*-theory is important when one is comparing different decorations, see Subsection 9.10.4.

In general the *L*-theoretic Farrell-Jones assembly map is not an isomorphism if one replaces the decoration  $\langle -\infty \rangle$  by the decoration *p*, *h*, or *s*, see Remark 13.9. This can be very unpleasant since for applications one needs the decorations *s* or *h*. The situation is better when *G* is torsionfree, as explained in Theorem 9.106.

# **17.4 Rational Computations for Infinite Groups**

Next we state what is known rationally about the *K*- and *L*-groups of an infinite (discrete) group, provided the Farrell-Jones Conjectures 13.1 or 13.4 or the Baum-Connes Conjecture 14.9 hold.

17.4 Rational Computations for Infinite Groups

#### 17.4.1 Rationalized Algebraic K-Theory

The next result follows from Theorem 12.79 and Theorem 13.51. For  $R = \mathbb{Z}$  see also Grunewald [444, Corollary on page 165].

**Theorem 17.1 (Rational computations of**  $K_n(RG)$  **for regular** R). Let R be a regular ring, e.g., R is  $\mathbb{Z}$ . Suppose that the group G satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R.

Then we have for all  $n \in \mathbb{Z}$  a natural isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC))$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG)$$

where we use the notation from Theorem 12.79.

Computations of  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC))$  as  $\mathbb{Q}[\operatorname{aut}(C)]$ -module for finite cyclic groups *C* and  $R = \mathbb{Z}$  or *R* a field of characteristic zero can be found in [793], see Remark 12.80.

**Exercise 17.2.** If in Theorem 17.1 we drop the condition that R is regular, show that then we still know that the map appearing there is split injective.

**Example 17.3 (A Formula for**  $K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$  **for** *R* **the ring of integers in an algebraic number field).** Let *R* be the ring of integers in an algebraic number field, e.g.,  $R = \mathbb{Z}$ . Note that then *R* is regular by Theorems 2.21 and 2.23. Suppose that the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R* is true for *G*. Then Conjecture 4.20 is true by Theorem 13.65. Hence we obtain from Theorem 2.105, Theorem 4.22 (i), and Theorem 17.1 an isomorphism

$$\widetilde{K}_{0}(RG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{\substack{(C) \in (\mathcal{FC}\mathcal{Y}) \\ C \neq \{1\}}} H_{1}(BC_{G}C; \mathbb{Q}) \otimes_{\mathbb{Q}[N_{G}C/C_{G}C]} \theta_{C} \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that  $\widetilde{K}_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$  contains only contributions from  $K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$  for finite cyclic subgroups  $C \subseteq G$ .

Next we give criteria for the rational vanishing of middle and lower *K*-theory, where the homology of the centralizers of the cyclic subgroups does not occur.

**Theorem 17.4 (Criteria for the rational vanishing of middle and lower** *K***-theory of integral group rings).** Let *G* be a Farrell-Jones group. Let *I* be the subgroup of  $\operatorname{aut}(C)$  generated by the automorphism of *C* sending *x* to  $x^{-1}$ . Denote by  $D_GC$  the subgroup of  $\operatorname{aut}(C)$  which is the image of the injective group homomorphism  $N_GC/C_GC \rightarrow \operatorname{aut}(C)$  given by conjugation with elements in  $N_GC$ . Consider the following conditions:

- (P) The order of every finite cyclic subgroup C is a prime power;
- (A) For every non-trivial finite cyclic subgroup C of G, we have

$$\operatorname{aut}(C) = I \cdot D_G C.$$

Then:

- (i)  $K_n(\mathbb{Z}G) = 0$  for  $n \leq -2$ ;
- (ii) We have  $\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(\mathbb{Z}G) = 0$  if and only if condition (P) holds;
- (iii) We have  $\mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{Z}G) = 0$  if condition (P) is satisfied;
- (iv) We have  $\mathbb{Q} \otimes_{\mathbb{Z}} Wh(G) = 0$ , if conditions (P) and (A) are satisfied;
- (v) If  $\mathbb{Q} \otimes_{\mathbb{Z}} Wh(G) = 0$  holds, then condition (A) is satisfied.

*Proof.* (i) This is a consequence of Conjecture 4.20, which follows from the Full Farrell-Jones Conjecture by Theorem 13.65 (i) and (vi).

(ii) For the sequel note that Theorem 4.22 (i) and (v) imply that  $K_{-n}(\mathbb{Z}C) = 0$  holds for every cyclic group *C* and every  $n \leq -2$  and that  $K_{-1}(\mathbb{Z}C) = 0$  holds for every cyclic subgroup *C*, whose order is a prime power order. We conclude from Theorem 17.1 that  $\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(\mathbb{Z}G) = 0$  is trivial, if condition (P) is satisfied.

Now suppose that  $\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(\mathbb{Z}G) = 0$  holds. Theorem 17.1 implies that  $\mathbb{Q} \otimes_{\mathbb{Q}[N_G C/C_G C]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(RC))$  vanishes for every finite cyclic subgroup *C* of *G*. Fix a non-trivial finite cyclic subgroup *C*. Because of Remark 12.80 we get an isomorphism of  $\mathbb{Q}$ -modules

$$\mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_{-1}(\mathbb{Z}C)) \oplus \mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \mathbb{Q}$$
$$\cong \bigoplus_{p||C|} \mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \mathbb{Q}[\operatorname{aut}(C)/\operatorname{Gal}_p(C)].$$

Hence we get

$$\mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \mathbb{Q} \cong_{\mathbb{Q}} \bigoplus_{p \mid |C|} \mathbb{Q} \otimes_{\mathbb{Q}[N_GC/C_GC]} \mathbb{Q}[\operatorname{aut}(C)/\operatorname{Gal}_p(C)].$$

Since each  $\mathbb{Q}[N_G C/C_G C]$ -module  $\mathbb{Q}[\operatorname{aut}(C)/\operatorname{Gal}_p(C)]$  is non-trivial, there is at most one prime *p* dividing |C|. Hence condition (P) is satisfied.

(iii) This follows from Example 17.3.

(iv) and (v) Remark 12.80 implies for a non-trivial finite cyclic subgroup *C* of *G* that  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_1(\mathbb{Z}C)) = 0$  holds if and only if  $\operatorname{aut}(C) = I \cdot D_G C$  is true and that we have  $\Theta_C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathbb{Z}C)) = 0$ . Now the claim follows from Theorem 17.1.  $\Box$
17.4 Rational Computations for Infinite Groups

# 17.4.2 Rationalized Algebraic L-Theory

The next result follows from Subsection 9.10.4, Theorem 12.79, and Theorem 13.62 (i).

**Theorem 17.5 (Rational computation of algebraic** *L***-theory).** Suppose that the group G satisfies the L-theoretic Farrell-Jones Conjecture 13.6 with coefficients in the ring with involution R after inverting 2.

*Then we get for every decoration*  $\langle j \rangle$  *and every*  $n \in \mathbb{Z}$  *an isomorphism* 

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(C_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot \left(\mathbb{Q} \otimes_{\mathbb{Z}} L_q^{\langle j \rangle}(RC)\right)$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(RG).$$

**Exercise 17.6.** Let *F* be a finite group of odd order. Put  $G = F \wr \mathbb{Z}$ . Show for all decorations  $j \in \mathbb{Z}, j \leq 2$  and odd  $n \in \mathbb{Z}$ 

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Q} & n \equiv 1 \mod 4; \\ \{0\} & n \equiv 3 \mod 4. \end{cases}$$

## 17.4.3 Rationalized Topological K-Theory

**Theorem 17.7 (Rational computation of topological** *K***-theory).** Suppose that the group *G* satisfies the Baum-Connes Conjecture 14.9. Let  $\Lambda$  be the ring  $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$  that is obtained from  $\mathbb{Z}$  by inverting the orders of the finite subgroups of *G*. Then there is an isomorphism

$$\bigoplus_{(C)\in(\mathcal{FC}\mathcal{Y})}\Lambda\otimes_{\mathbb{Z}}K_n(BC_GC)\otimes_{\Lambda[N_GC/C_GC]}\theta_C\cdot\Lambda\otimes_{\mathbb{Z}}\operatorname{Rep}_{\mathbb{C}}(C)$$

$$\xrightarrow{\cong} \Lambda \otimes_{\mathbb{Z}} K_n(C_r^*(G))$$

where  $(\mathcal{FCY})$  is the set of conjugacy classes of finite cyclic subgroups of *G*. If we tensor with  $\mathbb{Q}$ , we get an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BC_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GC/C_GC]} \Theta_C \cdot \mathbb{Q} \otimes_{\mathbb{Z}} K_q(C_r^*(C))$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} K_n(C_r^*(G)).$$

*Proof.* This follows from Theorem 10.69 and Theorem 12.79.

# 17.4.4 The Complexified Comparison Map from Algebraic to Topological *K*-Theory

If we consider  $R = \mathbb{C}$  as coefficient ring and apply  $- \otimes_{\mathbb{Z}} \mathbb{C}$  instead of  $- \otimes_{\mathbb{Z}} \mathbb{Q}$ , the formulas simplify. Suppose that *G* satisfies the Baum-Connes Conjecture 14.9 and the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{C}$ . Recall that  $\operatorname{con}(G)_f$  is the set of conjugacy classes (g) of elements  $g \in G$  of finite order. We denote for  $g \in G$  by  $\langle g \rangle$  the cyclic subgroup generated by g.

Then we get the following commutative square whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homomorphisms, see [649, Theorem 0.5],

Suslin [935, Theorem 4.9] has proved that the algebraic *K*-theory of  $\mathbb{C}$  in dimensions 2n for  $n \ge 1$  has a unique divisible group and hence admits no non-trivial map to  $\mathbb{Z}$ . This implies that the canonical map  $K_q^{ALG}(\mathbb{C}) \to K_q^{TOP}(\mathbb{C})$  from the algebraic *K*-theory of  $\mathbb{C}$  to the topological *K*-theory of  $\mathbb{C}$  is trivial in dimensions  $q \ne 0$  and a bijection in dimension q = 0. Thus rationally we understand by the diagram above the comparison map from algebraic *K*-theory of the complex group ring to the topological *K*-theory of the group  $C^*$ -algebra provided that *G* satisfies the Baum-Connes Conjecture 14.9 and the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring  $\mathbb{C}$ .

**Remark 17.8 (Separation of Variables).** In Theorems 17.1, 17.5, and 17.7 and in Subsection 17.4.4 we see a principle which we call *separation of variables*: There is a group homology part that is independent of the coefficient ring *R* and the *K*- or *L*-theory under consideration and a part depending only on the values of the theory under consideration on *RC* or  $C_r^*(C)$  for all finite cyclic subgroups  $C \subseteq G$ .

# **17.4.5** Rationalized Topological *K*-Theory of Virtually $\mathbb{Z}^n$ Groups

Consider a group extension  $1 \to A \to G \xrightarrow{p} Q \to 1$  of a finitely generated free abelian group *A* by a finite group *Q*. Our goal is to compute the rank  $\operatorname{rk}_{\mathbb{Z}}(K_n(C_r^*(G)))$  of the finitely generated abelian group  $K_n(C_r^*(G))$ . This turns out to be rather difficult. The most applicable result will be Theorem 17.21.

As A is abelian, we get a well-defined group homomorphism

(17.9) 
$$\rho: Q \to \operatorname{aut}_{\mathbb{Z}}(A)$$

to the group of automorphisms of the abelian group A by sending q to the  $\mathbb{Z}$ -automorphism of A given by conjugation with any lift of q to G under p.

Consider a cyclic subgroup  $C \subseteq Q$ . Choose a generator  $q \in C$ . Recall that  $H^1(C; A)$  is the quotient of the kernel of the  $\mathbb{Z}$ -homomorphism  $N_q: A \to A$  sending a to  $\sum_{i=0}^{|q|-1} \rho(q)^i(a)$  by the image of the  $\mathbb{Z}$ -homomorphism  $\rho(q) - \mathrm{id}_A: A \to A$  sending a to  $\rho(q)(a) - a$ . Note that  $N_q$  is independent of the choice of q. Moreover, the image of  $\rho(q) - \mathrm{id}_A: A \to A$  is independent of the choice of the generator q.

For a cyclic subgroup  $C \subseteq Q$  let  $C_Q C$  be the centralizer of C, i.e.,  $C_Q C = \{q_1 \in Q \mid q_1 c = cq_1 \text{ for all } c \in C\}$ . Let  $\overline{C}_Q C$  be the quotient of  $C_Q C$  by C, i.e.,  $\overline{C}_Q C = C_Q C/C$ . For  $q_1 \in C_Q C$  let  $\overline{q_1} \in \overline{C}_G C$  be the element in represented by  $q_1$ .

For any element  $a \in \ker(N_q)$  we denote by  $\overline{a}$  its class in  $H^1(C; A)$ . The group homomorphism  $\rho: Q \to \operatorname{aut}_{\mathbb{Z}}(A)$  induces by restriction to  $C_QC$  a group homomorphism  $\rho|_{C_QC}: C_QC \to \operatorname{aut}_{\mathbb{Z}C}(A)$ , where we consider A as a  $\mathbb{Z}C$ -module by restricting  $\rho: Q \to \operatorname{aut}_{\mathbb{Z}}(A)$  to C. One easily checks that  $\rho|_{C_QC}$  induces a group homomorphism

(17.10) 
$$\overline{\rho}_C \colon \overline{C}_Q C \to \operatorname{aut}_{\mathbb{Z}}(H^1(C; A))$$

which sends  $\overline{q_1}$  for  $q_1 \in C_Q C$  to the automorphism  $H^1(C; A) \xrightarrow{\cong} H^1(C; A)$  which maps  $\overline{a}$  to  $\overline{\rho(q_1)(a)} = \overline{q_1 a q_1^{-1}}$  for any  $a \in \ker(N_q)$  and any  $\overline{q_1} \in p^{-1}(q_1)$ . Note that the composite of  $\overline{\rho}_C : \overline{C}_Q C \to \operatorname{aut}_{\mathbb{Z}}(H^1(C; A))$  with the projection  $C_Q C \to \overline{C}_Q C$  is given by the group homomorphism  $\rho|_{C_Q C} : C_Q C \to \operatorname{aut}_{\mathbb{Z}C}(A)$  and the functoriality of  $H^1(C; A)$  in A. In order to show that  $\overline{\rho}_C$  is well-defined, one needs to show for  $c \in C$  that the  $\mathbb{Z}C$ -automorphism  $l_c : A \to A$  sending a to  $\rho(c)(a)$  induces the identity on  $H^1(C; A)$ . This follows from the fact that for any projective  $\mathbb{Z}C$ resolution  $P_*$  of the trivial  $\mathbb{Z}C$ -module  $\mathbb{Z}$  the  $\mathbb{Z}C$ -chain map  $(l_c)_* : P_* \to P_*$  given by multiplication by c is  $\mathbb{Z}C$ -chain homotopic to the identity  $id_{P_*}$ .

Fix an element  $q \in Q$  together with a lift  $\tilde{q} \in G$ , i.e.,  $p(\tilde{q}) = q$ . We assume that  $\tilde{q}$  has finite order. Then we have  $|\tilde{q}| = |q|$ . We want to define a map

(17.11) 
$$\mu_{\widetilde{q}} \colon \overline{C}_Q\langle q \rangle \to H^1(\langle q \rangle; A)$$

sending  $\overline{q_1} \in \overline{C}_Q\langle q \rangle$  represented by  $q_1 \in C_Q\langle q \rangle$  to  $\overline{\tilde{q}^{-1}\tilde{q_1}\tilde{q}\tilde{q_1}^{-1}}$  for any lift  $\tilde{q_1}$  of  $q_1$ . We need to check that  $\mu$  is well-defined.

Obviously  $p(\tilde{q}^{-1}\tilde{q}_1\tilde{q}_1\tilde{q}_1^{-1}) = q^{-1}q_1qq_1^{-1} = q_1q^{-1}qq_1^{-1} = e_Q$  holds, which implies  $\tilde{q}^{-1}\tilde{q}_1\tilde{q}_1\tilde{q}_1^{-1} \in A$ . One easily proves by induction over n = 1, 2, ...

$$\sum_{i=0}^{n-1} \widetilde{q}^i (\widetilde{q}^{-1} \widetilde{q}_1 \widetilde{q} \widetilde{q}_1^{-1}) \widetilde{q}^{-i} = \widetilde{q}^{-1} \widetilde{q}_1 \widetilde{q}^n \widetilde{q}_1^{-1} \widetilde{q}^{-n+1}.$$

This implies that  $\tilde{q}^{-1}\tilde{q}_1\tilde{q}\tilde{q}_1^{-1}$  lies in the kernel of ker $(N_q)$  and hence defines an element  $\overline{\tilde{q}^{-1}\tilde{q}_1\tilde{q}\tilde{q}_1^{-1}}$  in  $H^1(\langle q \rangle; A)$ . Next one has to show that this is independent of the choice of the lift  $\tilde{q}_1$  of  $q_1$ . This is a consequence of the following calculation

for  $a \in A$ 

$$\begin{split} \widetilde{q}^{-1}(\widetilde{q_1}a)\widetilde{q}(\widetilde{q_1}a)^{-1} &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q_1}^{-1}\widetilde{q_1}\widetilde{q}^{-1}a\widetilde{q}a^{-1}\widetilde{q_1}^{-1}\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q_1}^{-1} + \widetilde{q_1}(\widetilde{q}^{-1}a\widetilde{q}a^{-1})\widetilde{q_1}^{-1}\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1} + \rho(q_1)(\rho(q^{-1})(a) - a)\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1} + \rho(q_1)\circ\rho(q^{-1})(a) - \rho(q_1)(a)\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1} + \rho(q_1)\circ\rho(q^{-1})(a) - \rho(qq_1q^{-1})(a)\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1} + \rho(q_1)\circ\rho(q^{-1})(a) - \rho(q)\circ\rho(q_1)\circ\rho(q^{-1})(a)\\ &= \widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1} + (\rho(q) - \mathrm{id})(\rho(q_1)\circ\rho(q^{-1})(-a)), \end{split}$$

which implies  $\overline{\tilde{q}^{-1}(\tilde{q_1}a)\tilde{q}(\tilde{q_1}a)^{-1}} = \overline{\tilde{q}^{-1}\tilde{q_1}\tilde{q}\tilde{q}_1^{-1}}$ . The choice of the representative  $q_1$  of  $\overline{q_1}$  does not matter, since we have for  $i \ge 0$ 

$$\widetilde{q}^{-1}(\widetilde{q}_1\widetilde{q}^i)\widetilde{q}(\widetilde{q}_1\widetilde{q}^i)^{-1} = \widetilde{q}^{-1}\widetilde{q}_1\widetilde{q}^i\widetilde{q}\widetilde{q}^{-i}\widetilde{q}_1^{-1} = \widetilde{q}^{-1}\widetilde{q}_1\widetilde{q}\widetilde{q}\widetilde{q}_1^{-1}.$$

Note that  $\mu_{\tilde{q}}$  depends on the choice of the lift  $\tilde{q}$  of q. Moreover, it is not necessarily a group homomorphism. Namely, for  $q_1, q_2 \in C_Q\langle q \rangle$  we get

(17.12) 
$$\mu_{\widetilde{q}}(\overline{q_1q_2}) = \mu_{\widetilde{q}}(\overline{q_1}) + \overline{\rho}_{\langle q \rangle}(\overline{q_1})(\mu(\overline{q_2}))$$

from the following calculation, since  $\tilde{q_1}\tilde{q_2}$  is a lift of  $q_1q_2$ 

$$\begin{split} \mu_{\widetilde{q}}(q_1q_2) &= \overline{\widetilde{q}^{-1}(\widetilde{q_1}\widetilde{q_2})\widetilde{q}(\widetilde{q_1}\widetilde{q_2})^{-1}} \\ &= \overline{\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}\widetilde{q}^{-1}\widetilde{q}_1^{-1}} \\ &= \overline{\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}^{-1}} + \overline{\widetilde{q}_1}\widetilde{q}^{-1}\widetilde{q}\widetilde{q}\widetilde{q}\widetilde{q}^{-1}\widetilde{q}_1^{-1}} \\ &= \mu_{\widetilde{q}}(q_1) + \overline{\rho}_{\langle q \rangle}(q_1)(\mu_{\widetilde{q}}(q_2)). \end{split}$$

The composite of  $\mu_{\tilde{q}}$  with the map induced by p from  $C_G \langle \tilde{q} \rangle$  to  $C_Q \langle q \rangle$  is obviously trivial. If p has a splitting  $s: Q \to G$  and we consider  $\tilde{q} = s(q)$ , then the map  $\mu_{\tilde{q}}$  is trivial because we get for  $q_1 \in C_Q\langle q \rangle$ 

$$\mu_{\widetilde{q}}(q_1) = \overline{s(q)^{-1}s(q_1)s(q)s(q_1)^{-1}} = \overline{s(q^{-1}q_1qq_1^{-1})} = \overline{s(e_Q)} = 0.$$

**Lemma 17.13.** We get for  $a \in A$ :

$$p(C_G\langle \widetilde{q}a \rangle) = \{q_1 \in C_Q\langle q \rangle \mid \mu_{\widetilde{q}}(\overline{q_1}) = \overline{a} - \overline{\rho}_{\langle q \rangle}(\overline{q_1})(\overline{a})\}.$$

*Proof.* Consider  $q_1 \in p(C_G \langle \tilde{q}a \rangle)$ . Let  $g_1 \in C_G \langle \tilde{q}a \rangle$  be an element with  $p(g_1) = q_1$ . Then  $g_1 \tilde{q}a g_1^{-1} = \tilde{q}a$ . Hence  $q_1 \in C_Q \langle q \rangle$  and we get:

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$$\begin{split} \overline{a} &= \overline{\widetilde{q}^{-1}g_1\widetilde{q}ag_1^{-1}} \\ &= \overline{(\widetilde{q}^{-1}g_1\widetilde{q}g_1^{-1})(g_1ag_1^{-1})} \\ &= \overline{(\widetilde{q}^{-1}g_1\widetilde{q}g_1^{-1})} + \overline{g_1ag_1^{-1}} \\ &= \mu_{\widetilde{q}}(\overline{q_1}) + \overline{\rho}_{\langle q \rangle}(\overline{q_1})(\overline{a}). \end{split}$$

Consider an element  $q_1 \in C_Q\langle q \rangle$  satisfying  $\mu_{\tilde{q}}(\overline{q_1}) = \overline{a} - \overline{\rho}_{\langle q \rangle}(\overline{q_1})(\overline{a})$ . Choose  $g_1 \in G$  with  $p(g_1) = q_1$ . Then we get from the calculation above  $\overline{a} = \overline{\tilde{q}^{-1}g_1\tilde{q}ag_1^{-1}}$ . Hence there is an element  $b \in A$  satisfying

$$a = \tilde{q}^{-1}g_1\tilde{q}ag_1^{-1} + (\rho(q) - \mathrm{id})(b)$$

Consider any  $c \in A$ . Put  $g_2 = cg_1$ . Then

$$p(g_2) = p(cg_1) = p(c)p(g_1) = p(g_1) = q_1$$

and

$$\begin{split} \widetilde{q}^{-1}g_{2}\widetilde{q}ag_{2}^{-1} &= \widetilde{q}^{-1}cg_{1}\widetilde{q}a(cg_{1})^{-1} \\ &= \widetilde{q}^{-1}c\widetilde{q}\widetilde{q}^{-1}g_{1}\widetilde{q}ag_{1}^{-1}c^{-1} \\ &= \rho(q^{-1})(c)\widetilde{q}^{-1}g_{1}\widetilde{q}ag_{1}^{-1}c^{-1} \\ &= \widetilde{q}^{-1}g_{1}\widetilde{q}ag_{1}^{-1} + \rho(q^{-1})(c) - c \\ &= a - \rho(q)(b) + b + \rho(q^{-1})(c) - c. \end{split}$$

If we take  $c = -\rho(q)(b)$ , then  $-\rho(q)(b) + b + \rho(q^{-1})(c) - c = 0$  and hence  $\tilde{q}^{-1}g_2\tilde{q}ag_2^{-1} = a$ . This implies  $g_2\tilde{q}ag_2^{-1} = \tilde{q}a$  and therefore  $g_2 \in C_G\langle \tilde{q}a \rangle$ . Hence  $q_1 \in \rho(C_G\langle \tilde{q}a \rangle)$  holds.

As a special case we get  $p(C_G\langle \tilde{q} \rangle) = \ker(\mu_{\tilde{q}})$  since  $\overline{\rho}_{\langle q \rangle}(\overline{q_1})(0) = 0$  holds for all  $\overline{q_1} \in \overline{C}_Q\langle q \rangle$ . Moreover, if p has a section s and we put  $\tilde{q} = s(q)$ , then  $\mu_{\tilde{q}}$  is trivial and hence  $p(C_G\langle s(q) \rangle) = C_Q\langle q \rangle$ .

- **Lemma 17.14.** (i) Consider  $a \in A$ . Then  $\tilde{q}a$  has finite order if and only if  $a \in \ker(N_q)$ ;
- (ii) Consider  $a_0, a_1 \in \text{ker}(N_q)$ . Then we have  $(\tilde{q}a_0) = (\tilde{q}a_1)$  if and only if there exists a  $q_1 \in C_Q\langle q \rangle$  with  $\overline{a_1} \overline{\rho}_{\langle q \rangle}(q_1)(\overline{a_0}) = \mu_{\widetilde{q}}(q_1)$ .

*Proof.* (i) If  $|\tilde{q}a| < \infty$  holds, then we have  $|\tilde{q}a| = |\tilde{q}| = |q|$ . Now the claim follows from the equality  $(\tilde{q}a)^{|q|} = N_q(a)$ .

(ii) Suppose  $(\tilde{q}a_0) = (\tilde{q}a_1)$ . Then there is an element  $g_1 \in G$  satisfying  $g_1 \tilde{q} a_0 g_1^{-1} = \tilde{q}a_1$ . This implies

$$p(g_1)qp(g_1)^{-1} = p(g_1)qp(a_0)p(g_1)^{-1} = p(g_1\tilde{q}a_0g_1^{-1})$$
  
=  $p(\tilde{q}a_1) = p(\tilde{q})p(a_1) = q.$ 

If we put  $q_1 = p(g_1)$ , then  $q_1 \in C_Q(q)$  and we take as a lift  $\tilde{q_1}$  of  $q_1$  the element  $g_1$ . We get

$$\overline{a_1} = \overline{\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}a_0\widetilde{q_1}^{-1}} = \overline{\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q_1}^{-1}\widetilde{q_1}a_0\widetilde{q_1}^{-1}} = \overline{\widetilde{q}^{-1}\widetilde{q_1}\widetilde{q}\widetilde{q}\widetilde{q}\widetilde{q}}^{-1} + \overline{\widetilde{q}_1a_0\widetilde{q}}^{-1} = \mu_{\widetilde{q}}(q_1) + \overline{\rho}_{\langle q \rangle}(q_1)(\overline{a_0}).$$

Now suppose that there exists a  $q_1 \in Q$  satisfying  $\overline{a_1} - \overline{\rho}_{\langle q \rangle}(q_1)(\overline{a_0}) = \mu_{\widetilde{q}}(q_1)$ . Let  $g_1$  be any lift of  $q_1$  to G. Then the same computation above shows  $\overline{\widetilde{q}^{-1}g_1\widetilde{q}a_0g_1^{-1}} = \overline{a_1}$ . We can arrange by replacing  $g_1$  by  $g_2 = cg_1$  for  $c \in A$  that  $\widetilde{q}^{-1}g_2\widetilde{q}a_0g_2^{-1} = a_1$  holds by the same argument as in the proof of Lemma 17.13. Now we get

$$(\tilde{q}a_1) = (\tilde{q}\tilde{q}^{-1}g_2\tilde{q}a_0g_2^{-1}) = (g_2\tilde{q}a_0g_2^{-1}) = (\tilde{q}a_0).$$

**Lemma 17.15.** For every finite cyclic subgroup  $D \subseteq G$  and every  $n \in \mathbb{Z}$  we get a  $\mathbb{Q}$ -isomorphism

$$H_n(C_G D; \mathbb{Q}) \cong H_n(A^{p(D)}; \mathbb{Q})^{p(C_G D)}$$

where  $A^{p(D)}$  is to be understood with respect to the Q-action  $\rho$  on A and the  $p(C_G D)$ -fixed point set  $H_n(A^{p(D)}; \mathbb{Q})^{p(C_G D)}$  is understood with respect to the  $C_Q p(D)$ -action on  $H_n(A^{p(D)}; \mathbb{Q})$  induced by  $\rho|_{C_Q p(D)}$  and the inclusion  $p(C_G D) \subseteq C_Q p(D)$ .

*Proof.* This follows from the Lyndon-Hochschild-Serre spectral sequence applied to the group extension

$$1 \to A^{p(D)} = A \cap C_G D \to C_G D \to p(C_G D) \to 1.$$

Denote by  $\operatorname{con}_f(G)$  the conjugacy classes of elements of finite order in G. Let  $\operatorname{con}_f(p)$ :  $\operatorname{con}_f(G) \to \operatorname{con}_f(Q)$  be the map sending (g) to (p(g)). Then we have

$$\operatorname{con}_f(G) = \bigsqcup_{\{(q) \in \operatorname{con}_f(Q) | \exists \widetilde{q} \in p^{-1}(q), |\widetilde{q}| < \infty\}} \operatorname{con}_f(p)^{-1}((q)).$$

We conclude from [649, Example 8.1]

(17.16) 
$$\operatorname{rk}_{\mathbb{Z}}(K_n(C_r^*(G))) = \sum_{m \in \mathbb{Z}} \sum_{(g) \in \operatorname{con}_f(g)} \dim_{\mathbb{Q}}(H_{n+2m}(C_G\langle g \rangle; \mathbb{Q})).$$

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Hence Lemma 17.15 implies for every  $n \in \mathbb{Z}$ ,

(17.17) 
$$\operatorname{rk}_{\mathbb{Z}}(K_{n}(C_{r}^{*}(G))) = \sum_{\substack{m \in \mathbb{Z} \\ \exists \widetilde{q} \in p^{-1}(q), |\widetilde{q}| < \infty}} \sum_{\substack{(g) \in \operatorname{con}_{f}(G) \\ (p(g)) = (q)}} \dim_{\mathbb{Q}}(H_{n+2m}(A^{\langle q \rangle}; \mathbb{Q})^{p(C_{G}\langle g \rangle)})$$

where  $A^{\langle q \rangle}$  is to be understood with respect to the *Q*-action  $\rho$  and the  $p(C_G \langle g \rangle)$ -fixed point set  $H_m(A^{\langle q \rangle}; \mathbb{Q})^{p(C_G \langle g \rangle)}$  is understood with respect to the  $p(C_G \langle g \rangle)$ -action on  $A^{\langle q \rangle}$  induced by  $\rho|_{p(C_G \langle g \rangle)}$ .

Next we define a group homomorphism

(17.18) 
$$\sigma_{\widetilde{q}} \colon \overline{C}_G \langle q \rangle \to \operatorname{aut}(H^1(\langle q \rangle; A)),$$

where  $\operatorname{aut}(H^1(\langle q \rangle; A))$  is the set of automorphisms of the set  $H^1(\langle q \rangle; A)$ . It sends  $\widetilde{q_1} \in \overline{C}_Q\langle q \rangle$  to the automorphism of the set  $H^1(\langle q \rangle; A)$  mapping  $\overline{a}$  to  $\overline{\rho}_{\langle q \rangle}(\overline{q_1})(\overline{a}) + \mu_{\overline{q}}(\overline{q_1})$ . This is indeed a group homomorphism by the following calculation for  $q_1, q_2 \in C_G\langle q \rangle$  and  $\overline{a} \in H^1(\langle q \rangle; A)$ 

$$\begin{split} \sigma_{\widetilde{q}}(\overline{q_1}) \circ \sigma_{\widetilde{q}}(\overline{q_2})(\overline{a}) &= \sigma_{\widetilde{q}}(q_1) \left( \overline{\rho}_{\langle q \rangle}(\overline{q_2})(\overline{a}) + \mu_{\widetilde{q}}(\overline{q_2}) \right) \\ &= \overline{\rho}_{\langle q \rangle}(\overline{q_1}) \left( \overline{\rho}_{\langle q \rangle}(\overline{q_2})(\overline{a}) + \mu_{\widetilde{q}}(\overline{q_2}) \right) + \mu_{\widetilde{q}}(\overline{q_1}) \\ &= \overline{\rho}_{\langle q \rangle}(\overline{q_1}) \circ \overline{\rho}_{\langle q \rangle}(\overline{q_2})(\overline{a}) + \overline{\rho}_{\langle q \rangle}(\overline{q_1}) (\mu_{\widetilde{q}}(\overline{q_2})) + \mu_{\widetilde{q}}(\overline{q_1}) \\ \begin{pmatrix} 17.12 \\ = \end{array} \right) \overline{\rho}_{\langle q \rangle}(\overline{q_1}q_2)(\overline{a}) + \mu_{\widetilde{q}}(\overline{q_1}q_2) \\ &= \sigma_{\widetilde{q}}(\overline{q_1q_2})(\overline{a}) \end{split}$$

and by  $\sigma_{\widetilde{q}}(\overline{e})(\overline{a}) = \overline{\rho}_{\langle q \rangle}(\overline{e})(\overline{a}) + \mu_{\widetilde{q}}(\overline{e}) = \overline{a} + 0 = \overline{a}$  for  $e \in C_Q\langle q \rangle$  the unit element. Note for  $\overline{q_1} \in \overline{C}_G\langle q \rangle$  that  $\sigma_{\widetilde{q}}(\overline{q_1}) \colon H^1(\langle q \rangle; A) \to H^1(\langle q \rangle; A)$  is not necessarily a  $\mathbb{Z}$ -homomorphism, whereas  $\overline{\rho}_{\widetilde{q}}(\overline{q_1}) \colon H^1(\langle q \rangle; A) \to H^1(\langle q \rangle; A)$  is always a  $\mathbb{Z}$ -homomorphism. If p has a section and we put  $\widetilde{q} = s(q)$ , then  $\mu$  is trivial and hence the two  $C_G\langle q \rangle$ -operations  $\overline{\rho}_{\langle q \rangle}$  and  $\sigma_{\widetilde{q}}$  on aut $(H^1(\langle q \rangle; A))$  agree.

**Lemma 17.19.** Consider  $q \in Q$  and  $\tilde{q} \in G$  with  $p(\tilde{q}) = q$  and  $|\tilde{q}| < \infty$ .

(i) There is a bijection

$$t_{\widetilde{q}} \colon \overline{C}_G \langle q \rangle \backslash H^1(\langle q \rangle; A) \xrightarrow{\cong} \operatorname{con}_f(p)^{-1}((q)), \quad \overline{C}_G \langle q \rangle \overline{a} \mapsto (\widetilde{q}a)$$

for any choice  $a \in \ker(N_q - id)$  representing  $\overline{a}$ , where the quotient  $\overline{C}_G\langle q \rangle \backslash H^1(\langle q \rangle; A)$  is to be understood with respect to the operation  $\sigma_{\widetilde{q}}$ ;

(ii) Consider  $a \in \ker(N_q - id)$ . Then we get

$$p(C_G\langle \widetilde{q}a \rangle) = \{q_1 \in C_G \langle q \rangle \mid \overline{a} = \sigma_{\widetilde{q}}(\overline{q_1})(\overline{a})\}.$$

*Proof.* (i) This follows from Lemma 17.14.(ii) This follows from Lemma 17.13.

**Theorem 17.20 (Rational computation of**  $K_*(C_r^*(G))$  **for a virtually**  $\mathbb{Z}^n$ -group *G*). Consider a group extension  $1 \to A \to G \xrightarrow{p} Q \to 1$  of a finitely generated free abelian group A by a finite group Q. Then we get in the setup and the notation above for  $n \in \mathbb{Z}$ 

$$\operatorname{rk}_{\mathbb{Z}}(K_{n}(C_{r}^{*}(G))) = \sum_{m \in \mathbb{Z}} \sum_{\substack{(q) \in \operatorname{con}(Q) \\ \exists \widetilde{q} \in p^{-1}(q), |\widetilde{q}| < \infty}} \sum_{\substack{\overline{C}_{Q} \langle q \rangle \overline{a} \in \\ \overline{C}_{Q} \langle q \rangle \setminus H^{1}(\langle q \rangle; A)}} \dim_{\mathbb{Q}}(H_{n+2m}(A^{\langle q \rangle}; \mathbb{Q})^{(C_{G} \langle q \rangle)_{\overline{a}}}),$$

where  $A^{\langle q \rangle}$  is to be understood with respect to the Q-action  $\rho$  and the  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$ -fixed point set  $H_m(A^{\langle q \rangle}; \mathbb{Q})^{(\overline{C}_Q\langle q \rangle)_{\overline{a}}}$  is understood with respect to the  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$ -action on  $A^{\langle q \rangle}$  induced by  $\rho|_{(\overline{C}_Q\langle q \rangle)_{\overline{a}}}$  and  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$  is the isotropy of  $\overline{a} \in H^1(\langle q \rangle, A)$  with respect to the  $C_G\langle q \rangle$ -operation  $\sigma_{\widetilde{q}}$  of (17.18).

*Proof.* This follows (17.17) and Lemma 17.19.

Note that  $(C_G \langle q \rangle)_{\overline{a}}$  depends on the choice of  $\tilde{q}$  as  $\sigma_{\tilde{q}}$  depends on this choice.

The situation improves drastically if we assume that p has a section  $s: Q \to G$ , since then  $\mu_{\tilde{q}}$  is trivial and  $\overline{\rho}_{\langle q \rangle} = \sigma_{\tilde{q}}$ . Namely, Theorem 17.20 reduces to the following result.

**Theorem 17.21 (Rational computation of**  $K_*(C_r^*(G))$  **for a virtually**  $\mathbb{Z}^n$ **-group** G **in the split case).** Consider a group extension  $1 \to A \to G \xrightarrow{p} Q \to 1$  of a finitely generated free abelian group A by a finite group Q. Suppose that there is a group homomorphism  $s: Q \to G$  with  $p \circ s = \operatorname{id}_Q$ . Then we get

$$\operatorname{rk}_{\mathbb{Z}}(K_{n}(C_{r}^{*}(G))) = \sum_{m \in \mathbb{Z}} \sum_{(q) \in \operatorname{con}(Q)} \sum_{\substack{\overline{C}_{Q} \langle q \rangle \overline{a} \in \\ \overline{C}_{Q} \langle q \rangle \setminus H^{1}(\langle q \rangle; A)}} \operatorname{dim}_{\mathbb{Q}}(H_{n+2m}(A^{\langle q \rangle}; \mathbb{Q})^{(\overline{C}_{Q} \langle q \rangle)_{\overline{a}}}),$$

where  $A^{\langle q \rangle}$  is to be understood with respect to the Q-action  $\rho$ , the  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$ -fixed point set  $H_m(A^{\langle q \rangle}; \mathbb{Q})^{(\overline{C}_Q\langle q \rangle)_{\overline{a}}}$  is understood with respect to the  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$ -action on  $A^{\langle q \rangle}$  induced by  $\rho|_{(\overline{C}_Q\langle q \rangle)_{\overline{a}}}$ , and  $(\overline{C}_Q\langle q \rangle)_{\overline{a}}$  is the isotropy of  $\overline{a} \in H^1(\langle q \rangle, A)$ with respect to the  $\overline{C}_Q\langle q \rangle$ -operation  $\overline{\rho}_{\langle q \rangle}$  of (17.10).

**Remark 17.22.** Note that in the situation of Theorem 17.21  $(C_G \langle q \rangle)_{\overline{a}}$  depends only on  $\langle q \rangle$  as  $\overline{\rho}_{\langle q \rangle}$  depends only on  $\langle q \rangle$ . Hence the formula appearing in Theorem 17.21 can be rewritten in terms of conjugacy classes of finite cyclic subgroups as

(17.23) 
$$\operatorname{rk}_{\mathbb{Z}}(K_n(C_r^*(G)))$$
  
=  $\sum_{m \in \mathbb{Z}} \sum_{\substack{C \subseteq Q \text{ cyclic}}} |\operatorname{gen}(C)/N_Q C| \cdot \sum_{\substack{\overline{C}_Q C \overline{a} \in \\ \overline{C}_Q C \setminus H^1(C;A)}} \dim_{\mathbb{Q}}(H_{n+2m}(A^C; \mathbb{Q})^{(\overline{C}_Q C)_{\overline{a}}}),$ 

where gen(*C*) is the set of generators of *C* and  $N_QC$  is the normalizer of *C* in *Q*, which acts on gen(*C*) by conjugation. This follows from the observation that for a cyclic group *C* of *Q* the number of elements  $(q) \in \operatorname{con}(Q)$  with  $(\langle q \rangle) = (C)$  is  $|\operatorname{gen}(C)/N_QC|$ .

**Exercise 17.24.** Apply Theorem 17.21 to the infinite dihedral group  $D_{\infty} = \mathbb{Z} \rtimes \mathbb{Z}/2$ .

# **17.5 Integral Computations for Infinite Groups**

As mentioned above, no general pattern for integral calculations is known or expected. We give some examples where computations are possible and which will illustrate the techniques.

## 17.5.1 Groups Satisfying Conditions (M) and (NM)

We mention at least one situation where a certain class of groups can be treated systematically. Let MFIN be the subset of FIN consisting of elements in FIN that are maximal in FIN.

Consider the following assertions concerning G which we have already introduced in Subsection 11.6.12.

- ( $\underline{M}$ ) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;
- (NM)  $M \in \mathcal{MFIN}, M \neq \{1\} \implies N_G M = M.$

Classes of groups satisfying conditions ( $\underline{M}$ ) and ( $\underline{NM}$ ) have been given in Example 11.33.

Denote by  $\widetilde{K}_n(C_r^*(G))$  the cokernel of  $K_n(C_r^*(\{1\})) \to K_n(C_r^*(G))$ , by  $\widetilde{KO}_n(C_r^*(G;\mathbb{R}))$  the cokernel of  $KO_n(C_r^*(\{1\};\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R}))$ , and by  $\overline{L}_n^{\langle j \rangle}(RG)$  the cokernel of  $L_n^{\langle j \rangle}(R) \to L_n^{\langle j \rangle}(RG)$ . This coincides with  $\widetilde{L}_n^{\langle j \rangle}(R)$ , which is defined for any ring R with involution to be the cokernel of  $L_n^{\langle j \rangle}(\mathbb{R}) \to L_n^{\langle j \rangle}(R)$  if  $R = \mathbb{Z}G$ , but not in general if we replace  $\mathbb{Z}$  by other coefficients. Recall that  $\operatorname{Wh}_n^R(G)$  is the (n-1)-th homotopy group of the homotopy fiber of the assembly map  $BG_+ \wedge \mathbf{K}(R) \to \mathbf{K}(RG)$ . Recall that we abbreviate  $\underline{E}G = E_{\mathcal{FIN}}(G)$  and  $\underline{B}G = G \setminus E_{\mathcal{FIN}}(G)$ .

**Theorem 17.25 (Fundamental exact sequences for groups satisfying conditions** (<u>M</u>) and (<u>NM</u>)). Let  $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$  be a ring such that the order of any finite subgroup of *G* is invertible in  $\Lambda$ . Suppose that the group *G* satisfies conditions (<u>M</u>) and (<u>NM</u>). Let { $M_i \mid i \in I$ } be a complete set of representatives for the conjugacy classes of maximal finite subgroups of *G*. Then:

(i) If G satisfies the Baum-Connes Conjecture 14.9, then there is a short exact sequence of topological K-groups

$$0 \to \bigoplus_{i \in I} \widetilde{K}_n(C_r^*(M_i)) \to K_n(C_r^*(G)) \to K_n(\underline{B}G) \to 0$$

where the maps  $\widetilde{K}_n(C_r^*(M_i)) \to K_n(C_r^*(G))$  are induced by the inclusions  $H \to G$ .

*It splits after applying*  $- \otimes_{\mathbb{Z}} \Lambda$ *;* 

(ii) If G satisfies the Baum-Connes Conjecture 14.9, then there is a long exact sequence of topological K-groups

$$\dots \to KO_{n+1}(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{KO}_n(C_r^*(M_i; \mathbb{R})) \to KO_n(C_r^*(G; \mathbb{R}))$$
$$\to KO_n(\underline{B}G) \to \bigoplus_{i \in I} \widetilde{KO}_{n-1}(C_r^*(M_i; \mathbb{R})) \to \dots$$

where the maps  $\widetilde{KO}_n(C_r^*(H;\mathbb{R})) \to KO_n(C_r^*(G;\mathbb{R}))$  are induced by the inclusions  $H \to G$ .

It splits after applying  $- \otimes_{\mathbb{Z}} \Lambda$ . More precisely, the  $\Lambda$ -homomorphism

$$KO_n(C_r^*(G;\mathbb{R}))\otimes_{\mathbb{Z}}\Lambda \to KO_n(\underline{B}G)\otimes_{\mathbb{Z}}\Lambda$$

is split surjective;

(iii) Suppose that every infinite virtually cyclic subgroup of G is of type I, and G satisfies the L-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution R.

*Then for all*  $n \in \mathbb{Z}$  *there is an exact sequence* 

$$\dots \to H_{n+1}(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(R)) \to \bigoplus_{i \in I} \overline{L}_n^{\langle -\infty \rangle}(RM_i)$$
$$\to L_n^{\langle -\infty \rangle}(RG) \to H_n(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(R)) \to \dots$$

where the maps  $\overline{L}_n^{\langle -\infty \rangle}(RH) \to L_n^{\langle -\infty \rangle}(RG)$  are induced by the inclusions  $H \to G$ .

*It splits after applying*  $- \otimes_{\mathbb{Z}} \Lambda$ *, more precisely* 

$$L_n^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \Lambda \to H_n(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of  $\Lambda$ -modules;

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- (iv) If G satisfies the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R, then there is for  $n \in \mathbb{Z}$  an isomorphism

$$H_n(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R) \oplus \bigoplus_{i \in I} \operatorname{Wh}_n^R(M_i) \xrightarrow{\cong} \operatorname{Wh}_n^R(G)$$

where the homomorphisms  $\operatorname{Wh}_n^R(H) \to \operatorname{Wh}_n^R(G)$  are induced by the inclusions  $H \to G$ .

*Proof.* This follows from the existence of a nice model for  $E_{\mathcal{FIN}}(G)$ , see Theorem 11.32, the long exact sequence (12.86), and Lemma 12.18 (ii).

**Remark 17.26 (Role of** <u>*BG*</u>). Theorem 17.25 illustrates that for such computations a good understanding of the geometry of the orbit space <u>*BG*</u> is necessary. This can be hard to figure out, even for what seem at first glance to be nice groups with pleasant geometric properties such as crystallographic groups. In general <u>*BG*</u> can be very complicated, see Theorem 11.63.

Many of the following results are based on Theorem 17.25.

## 17.5.2 Torsionfree One-Relator Groups

Let  $G = \langle s_i, i \in I | r \rangle$  be the presentation of a one-relator group *G*. Denote by *F* the free group on the set of generators  $\{s_i | i \in I\}$ . Note that *r* is an element in *F*. The group *G* is torsionfree if and only if for any element  $s \in F$  and natural number *m* satisfying  $r = s^m$  we get m = 1, see [553] or [693, Proposition 5.17 on page 107].

We begin with the following lemma.

Lemma 17.27. Let X be the 2-dimensional CW-complex given by the pushout

Let  $d_i \in \mathbb{Z}$  be the degree of the composition  $S^1 \xrightarrow{q} \bigvee_{i \in I} S^1 \xrightarrow{\text{pr}_i} S^1$  where  $\text{pr}_i$  is the projection onto the *i*-th summand. Let  $\mathcal{H}_*$  be any (non-equivariant) generalized homology theory satisfying the disjoint union axiom.

(i) Suppose that  $d_i = 0$  holds for all  $i \in I$ . Then we get for  $n \in \mathbb{Z}$  an isomorphism

$$\mathcal{H}_n(X) \cong \mathcal{H}_n(\{\bullet\}) \oplus \bigoplus_{i \in I} \mathcal{H}_{n-1}(\{\bullet\}) \oplus \mathcal{H}_{n-2}(\{\bullet\});$$

(ii) Suppose that there is one  $i \in I$  with  $d_i \neq 0$ . Then we have an isomorphism

$$\mathcal{H}_n(X) \xrightarrow{\cong} \mathcal{H}_n(X, \{\bullet\}) \oplus \mathcal{H}_n(\{\bullet\}),$$

and a short exact sequence

as |I| if I is infinite.

$$0 \to H_1(X) \otimes_{\mathbb{Z}} \mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_n(X, \{\bullet\}) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(X), \mathcal{H}_{n-2}(\{\bullet\})) \to 0$$

(iii) Let d be the common greatest divisor of the finite set  $\{|d_i| \mid i \in I, d_i \neq 0\}$ , provided that  $\{|d_i| \mid i \in I, d_i \neq 0\}$  is non-empty. Then  $H_1(X) \cong \bigoplus_{i \in I} \mathbb{Z}$  and  $H_2(X)$  is infinite cyclic if  $d_i = 0$  holds for all  $i \in I$ . If there is one  $i \in I$  with  $d_i \neq 0$ , then  $H_2(X)$  is trivial and  $H_1(X) \cong \mathbb{Z}/d \bigoplus_{i \in J} \mathbb{Z}$ where the set |J| has cardinality |I| - 1 if |I| is finite, and the same cardinality

*Proof.* We can assume without loss of generality that the pushout (17.28) above consists of base point preserving maps, otherwise change q up to homotopy to be base point preserving. From the Mayer-Vietoris sequence of the pair  $(X, \{\bullet\})$  and the projection  $X \to \{\bullet\}$ , we obtain an isomorphism

$$\mathcal{H}_n(X) \xrightarrow{=} \mathcal{H}_n(X, \{\bullet\}) \oplus \mathcal{H}_n(\{\bullet\}).$$

(i) If we apply  $S^1 \wedge -$  to (17.28), we obtain a pushout of pointed spaces



Since  $S^2$  is simply connected, one gets using the Hurewicz Theorem an isomorphism  $\bigoplus_{i \in I} H_2(S^2) \xrightarrow{\cong} \pi_2(\bigvee_{i \in I} S^2)$ . We conclude that  $\operatorname{id}_{S^1} \land q$  is pointed nullhomotopic. Hence we obtain a pointed homotopy equivalence  $S^1 \land X \xrightarrow{\cong} S^3 \lor \bigvee_{i \in I} S^2$ . Now assertion (i) follows from the suspension isomorphism.

(ii) Since  $S^1$  is compact, only finitely many of the numbers  $d_i$  are different from zero. We get for any abelian group A a group homomorphism

$$D(A): A \to \bigoplus_{i \in I} A, \quad a \mapsto (d_i \cdot a)_{i \in I}.$$

The long exact sequence

(17.29) 
$$\cdots \to \mathcal{H}_{n-1}(\{\bullet\}) \xrightarrow{D(\mathcal{H}_{n-1}(\{\bullet\}))} \bigoplus_{i \in I} \mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_n(X, \{\bullet\})$$
  
 $\to \mathcal{H}_{n-2}(\{\bullet\}) \xrightarrow{D(\mathcal{H}_{n-2}(\{\bullet\}))} \bigoplus_{i \in I} \mathcal{H}_{n-2}(\{\bullet\}) \to \cdots$ 

comes from the long Mayer-Vietoris sequence of the pushout of pointed spaces (17.28) above and the identification derived from the disjoint union axiom and the suspension isomorphism

$$\bigoplus_{i\in I} \mathcal{H}_{n-1}(\{\bullet\}) \xrightarrow{\cong} \mathcal{H}_n\Big(\bigvee_{i\in I} S^1, \{\bullet\}\Big).$$

If we take  $\mathcal{H}_*$  to be singular homology with integer coefficients, we see that D(A) is obtained from  $D(\mathbb{Z})$  by  $D(A) = D(\mathbb{Z}) \otimes_A \operatorname{id}_A$  and there is a short exact sequence  $0 \to \mathbb{Z} \xrightarrow{D(\mathbb{Z})} \bigoplus_{i \in I} \mathbb{Z} \to H_1(X) \to 0$ . This implies

$$\operatorname{coker}(D(A)) = H_1(X) \otimes_{\mathbb{Z}} A;$$
$$\operatorname{ker}(D(A)) = \operatorname{Tor}_1^{\mathbb{Z}}(H_1(X), A).$$

(iii) If  $d_i = 0$  holds for all  $i \in I$ , this follows from assertion (i). If there is one  $i \in I$  with  $d_i \neq 0$ , we conclude  $H_2(X) = \{0\}$  from assertion (ii) and the claim about  $H_1(X)$  follows from the exact sequence  $0 \to \mathbb{Z} \xrightarrow{D(\mathbb{Z})} \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \to H_1(X) \to 0$  established in the proof of assertion (ii).

We denote by  $H_n(Y; A)$  the singular homology of a space Y with coefficients in the abelian group A and abbreviate  $H_n(Y) := H_n(Y; \mathbb{Z})$ . Note that the group homology  $H_n(G)$  is  $H_n(BG)$  and  $H_1(G) = G/[G, G]$ .

**Lemma 17.30.** Suppose that the one-relator-group G is torsionfree. Let  $\mathcal{H}_*$  be any (non-equivariant) generalized homology theory.

(i) If r lies in [F, F], we get isomorphisms

$$\mathcal{H}_n(BG) \cong \mathcal{H}_n(\{\bullet\}) \oplus \bigoplus_{i \in I} \mathcal{H}_{n-1}(\{\bullet\}) \oplus \mathcal{H}_{n-2}(\{\bullet\});$$

(ii) If r does not lie in [F, F], then we get isomorphisms

$$\mathcal{H}_n(BG) \cong \mathcal{H}_n(\{\bullet\}) \oplus \mathcal{H}_n(BG, \{\bullet\}),$$

and a short exact sequence

$$0 \to H_1(BG) \otimes_{\mathbb{Z}} \mathcal{H}_{n-1}(\{\bullet\}) \to \mathcal{H}_n(BG, \{\bullet\})$$
$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(BG); \mathcal{H}_{n-2}(\{\bullet\})) \to 0;$$

(iii)

$$H_n(BG; A) \cong \begin{cases} A & n = 0; \\ \bigoplus_{i \in I} A & n = 1 \text{ and } r \in [F, F]; \\ H_1(BG) \otimes_{\mathbb{Z}} A & n = 1 \text{ and } r \notin [F, F]; \\ A & n = 2 \text{ and } r \in [F, F]; \\ \text{Tor}_1^{\mathbb{Z}}(H_1(BG); A) & n = 2 \text{ and } r \notin [F, F]; \\ 0 & n \ge 3. \end{cases}$$

Proof. Consider the pushout

$$\begin{array}{c|c} S^1 & \xrightarrow{q} & \bigvee_{i \in I} S^1 \\ \downarrow & & & \downarrow_{\bar{i}} \\ D^2 & \xrightarrow{Q} & Z \end{array}$$

where the upper vertical arrow is given by the word  $r \in *_{i \in I} \mathbb{Z} = \pi_1 (\bigvee_{i \in I} S^1)$ . Then *Z* is a model for *BG*, see [693, Chapter III §§9 -11].

- (i) This follows from Lemma 17.27 (i).
- (ii) This follows from Lemma 17.27 (ii).

(iii) This follows from assertions (i) and (ii) applied to the special case when  $\mathcal{H}_*$  is singular homology with coefficients in the abelian group A.

Recall that the Baum-Connes-Conjecture 10.44 for torsionfree groups holds for every torsionfree one-relator group *G* predicting isomorphisms

$$\operatorname{asmb}^{G,\mathbb{C}}(BG)_n \colon K_n(BG) \to K_n(C_r^*(G;\mathbb{C}));$$
  
$$\operatorname{asmb}^{G,\mathbb{R}}(BG)_n \colon KO_n(BG) \to KO_n(C_r^*(G;\mathbb{R})).$$

Hence we get from Lemma 17.30 (i) in the case when r belongs to [F, F]

$$K_n(C_r^*(G;\mathbb{C})) \cong K_n(\{\bullet\}) \oplus \bigoplus_{i \in I} K_{n-1}(\{\bullet\}) \oplus K_{n-2}(\{\bullet\}) \cong \begin{cases} \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} & n \text{ even}; \\ \mathbb{Z}^2 & n \text{ odd}, \end{cases}$$

and

$$KO_n(C_r^*(G;\mathbb{R})) \cong KO_n(\{\bullet\}) \oplus \bigoplus_{i \in I} KO_{n-1}(\{\bullet\}) \oplus KO_{n-2}(\{\bullet\}).$$

If r does not belong to [F, F], then get from Lemma 17.30 (ii)

$$K_n(C_r^*(G; \mathbb{C})) \cong \begin{cases} \mathbb{Z} & n \text{ even;} \\ H_1(G) & n \text{ odd,} \end{cases}$$

$$KO_n(C_r^*(G;\mathbb{R})) \cong KO_n(\{\bullet\}) \oplus KO_n(BG,\{\bullet\}),$$

and a short exact sequence

$$0 \to H_1(G) \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}) \to KO_n(BG, \{\bullet\})$$
$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(G), KO_{n-2}(\{\bullet\})) \to 0.$$

The computation for  $K_*(C_r^*(G))$  agrees with the one in [117].

Recall that the Farrell-Jones Conjecture 6.53 for torsionfree groups and regular rings for *K*-theory holds for torsionfree one-relator groups predicting for a regular ring *R* an isomorphism  $H_n(BG; \mathbf{K}(R)) \to K_n(RG)$  for  $n \in \mathbb{Z}$ , and one can apply Lemma 17.30 to  $H_n(BG; \mathbf{K}(R))$ . Moreover, the Farrell-Jones Conjecture 9.114 for torsionfree groups for *L*-theory predicts that the assembly map  $H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$  is bijective for  $n \in \mathbb{Z}$ , and is known for torsionfree one-relator groups to be true after inverting 2. So Lemma 17.30 can also be used to compute  $K_n(RG)$  and  $L_n^{\langle -\infty \rangle}(RG)[1/2]$  if one understands  $K_n(R)$  and  $L_n^{\langle -\infty \rangle}(R)[1/2]$ .

**Exercise 17.31.** Let  $G = \langle s_1, s_2, ..., s_n | r \rangle$  be a finitely generated (not necessarily torsionfree) one-relator group where *r* is given by the word  $s_{i_1}^{m_1} s_{i_2}^{m_2} \cdots s_{i_l}^{m_l}$  for  $i_j \in \{1, 2, ..., n\}$  and  $m_j \in \mathbb{Z}$ . Define for j = 1, 2, ..., n

$$d_j = \sum_{\substack{k \in \{1, 2, \dots, n\}\\i_k = j}} m_k$$

Show that  $H_1(G) \cong \mathbb{Z}^n$  if all the numbers  $d_j$  are trivial, and  $H_1(G) \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}/d$  if not all the numbers  $d_j$  are zero and d is the greatest common divisor of  $\{|d_j| \mid j = 1, 2, ..., l, d_j \neq 0\}$ .

**Exercise 17.32.** Consider the 1-relator group  $G = \langle s_1, s_2 | s_1 s_2 s_1 s_2^{-1} s_1^{-2} \rangle$ . Compute the algebraic *K*-groups  $K_n(\mathbb{C}[\mathbb{Z}/m \times G])$  for  $n \le 1$  and any natural number *m*.

**Exercise 17.33.** Let *G* be the non-trivial semidirect product  $\mathbb{Z} \rtimes \mathbb{Z}$ . Compute  $L_n^s(\mathbb{Z}[G])$  for  $n \in \mathbb{Z}$ .

#### 17.5.3 One-Relator Groups with Torsion

Let  $G = \langle s_i, i \in I | r \rangle$  be the presentation of a one-relator group G. For the remainder of this subsection we assume that G is not torsionfree.

Then there exists a maximal non-trivial finite subgroup  $C \subseteq G$ , unique up to conjugation. It is cyclic. Let  $m \ge 2$  be its order. Denote by F the free group on the set of generators  $\{s_i \mid i \in I\}$ . Note that then r is an element in F. The natural number m can be characterized as the largest natural number for which there exists a word  $s \in F$  with  $r = s^m$ . Note that for such s the cyclic group C of order m is generated

by the class  $\overline{s}$  in *G* represented by *s* and every torsion element in *G* is conjugated to some power of  $\overline{s}$ . This was proved by Karras-Magnus-Solitar, see [553] or [693, Proposition 5.17 on page 107].

Let  $p: BG \to \underline{B}G$  be the up to homotopy unique canonical map and let  $i: C \to A$  be the inclusion. The Mayer-Vietoris sequence of the *G*-quotient of the *G*-pushout appearing in Theorem 11.32 yields the long exact sequence

$$(17.34) \cdots \to \mathcal{H}_{n}(BC, \{\bullet\}) \xrightarrow{\mathcal{H}_{n}(Bi)} \mathcal{H}_{n}(BG, \{\bullet\}) \xrightarrow{\mathcal{H}_{n}(p)} \mathcal{H}_{n}(\underline{B}G, \{\bullet\})$$
$$\to \mathcal{H}_{n-1}(BC, \{\bullet\}) \xrightarrow{\mathcal{H}_{n-1}(Bi)} \mathcal{H}_{n-1}(BG, \{\bullet\}) \xrightarrow{\mathcal{H}_{n-1}(p)} \mathcal{H}_{n-1}(\underline{B}G, \{\bullet\}) \to \cdots$$

for any (non-equivariant) generalized homology theory  $\mathcal{H}_*$ . Let  $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$  be a ring such that the order of any finite subgroup of G is invertible in  $\Lambda$ . Then sequence (17.34) splits into short split exact sequences after applying  $-\otimes_{\mathbb{Z}} \Lambda$ , more precisely, the  $\Lambda$ -map  $\mathcal{H}_n(BG, \{\bullet\}) \otimes_{\mathbb{Z}} \Lambda \to \mathcal{H}_n(\underline{B}G, \{\bullet\}) \otimes_{\mathbb{Z}} \Lambda$  is split surjective for every  $n \in \mathbb{Z}$ . The proof is the same as the proof of Lemma 12.18 (i).

By inspecting the model for  $\underline{E}G$  of Subsection 11.6.13 and dividing out the *G*-action, we obtain a pushout

Note that we can apply Lemma 17.30 and get information about  $\mathcal{H}_n(\underline{B}G)$  for any (non-equivariant) generalized homology theory  $\mathcal{H}_*$ . As an illustration we investigate the singular homology groups  $H_n(BG, A)$ .

**Lemma 17.36.** *Let G be a one-relator group with presentation*  $\langle s_i, i \in I | r \rangle$  *and let A be an abelian group.* 

(i) The inclusion  $C \rightarrow G$  induces isomorphisms

$$H_n(Bi; A): H_n(BC; A) \xrightarrow{\cong} H_n(BG; A)$$

for  $n \ge 3$ ;

(ii) We obtain an exact sequence

$$0 \to H_2(BC; A) \xrightarrow{H_1(Bi;A)} H_2(BG; A) \xrightarrow{H_2(p;A)} H_2(\underline{B}G; A) \to H_1(BC; A)$$
$$\xrightarrow{H_1(Bi;A)} H_1(BG; A) \xrightarrow{H_1(p;A)} H_1(\underline{B}G; A) \to 0;$$

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- (iii) Take  $A = \mathbb{Z}$ . Then precisely one of the following two cases occurs:
  - Both  $H_2(\underline{B}G)$  and  $H_2(BG)$  are infinite cyclic and we have  $r \in [F, F]$ . Moreover, we get a short exact sequence

$$0 \to H_2(BG) \xrightarrow{H_2(p;A)} H_2(\underline{B}G) \to C \to 0$$

and the homomorphism  $H_1(p): H_1(BG) \xrightarrow{\cong} H_1(BG)$  is bijective;

• Both  $H_2(\underline{B}G)$  and  $H_2(BG)$  are trivial and  $r \notin [F, F]$ . Moreover, we get a short exact sequence

$$0 \to C = H_1(BC) \xrightarrow{H_1(Bi)} H_1(BG) \xrightarrow{H_2(p)} H_1(\underline{B}G) \to 0.$$

*Proof.* (i) and (ii) These follow from the long exact sequence (17.34) and the conclusion of (17.35) that there is a 2-dimensional *CW*-model for <u>*BG*</u>.

(iii) Since we conclude from (17.35) that there is a 2-dimensional *CW*-model for <u>*BG*</u> with precisely one 2-cell,  $H_2(\underline{B}G)$  is either trivial or infinite cyclic.

Since  $H_2(BC)$  vanishes and  $H_1(BC)$  is finite,  $H_2(BG)$  is a subgroup of finite index of  $H_2(\underline{B}G)$ . This implies that  $H_2(BG)$  is infinite cyclic if and only if  $H_2(\underline{B}G)$  is infinite cyclic and that  $H_2(BG)$  is trivial if and only if  $H_2(\underline{B}G)$  is trivial.

Suppose that  $H_2(\underline{B}G)$  is infinite cyclic. Let X be the presentation complex of G for some presentation  $\langle s_i, i \in I | r \rangle$ . Then the classifying map  $f: X \to BG$  is 2-connected and induces an epimorphism  $H_2(X) \to H_2(BG)$  and an isomorphism  $H_1(X) \xrightarrow{\cong} H_1(BG)$ . Since X is a 2-dimensional CW-complex with precisely one 2-cell.  $H_2(X)$  is either trivial or infinite cyclic. Hence  $H_2(X)$  is infinite cyclic. Lemma 17.27 (iii) implies that  $r \in [F, F]$  and  $H_1(X) \cong H_1(BG)$  is torsionfree. Hence any map  $H_1(C) \to H_1(BG)$  is trivial. Now the claim follows from assertion (ii).

Suppose that  $H_2(\underline{B}G)$  is trivial. Then Lemma 17.27 (iii) implies that  $r \notin [F, F]$  and the claim follows from assertion (ii).

Recall that the Baum-Connes Conjecture 14.11 with coefficients holds for onerelator groups. Hence the assembly maps

$$K_n^G(\underline{E}G) \to K_n(C_r^*(G;\mathbb{C}));$$
  
$$KO_n^G(\underline{E}G) \to KO_n(C_r^*(G;\mathbb{R})),$$

are bijective for all  $n \in \mathbb{Z}$ .

Recall that the Full Farrell-Jones Conjecture 13.30 holds for one-relator groups with torsion. If *R* is a regular ring with  $\mathbb{Q} \subseteq R$  then we obtain an isomorphism for every  $n \in \mathbb{Z}$ , see Theorem 13.51

$$H_n^G(\underline{E}G;\mathbf{K}_R) \xrightarrow{=} K_n(RG).$$

If *m* is odd, any virtually cyclic subgroup of *G* is of type I, and we obtain for any ring with involution and  $n \in \mathbb{Z}$  an isomorphism, see Theorem 13.60,

$$H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})\xrightarrow{\cong} L_R^{\langle -\infty\rangle}(RG).$$

If *m* is even, we know at least that this map is bijective after inverting two.

In any case we want to compute the source of the assembly maps. A far reaching strategy is to use Theorem 17.25 after one has computed  $K^G(\underline{B}G)$ ,  $KO^G(\underline{B}G)$ ,  $H_n(BG; \mathbf{K}(R))$ , or  $H_n(BG; \mathbf{L}^{\langle -\infty \rangle})$  by applying Lemma 17.27 to (17.35)

**Example 17.37 (Topological** *K*-theory in the complex case). Given a one-relator group *G* with presentation  $\langle s_i, i \in I | r \rangle$ , we carry this out for  $K_n(C_r^*(G))$ . Let  $C \subseteq G$  be the maximal cyclic subgroup of *G* and put m = |C|. Since  $K_n(\{\bullet\})$  is  $\mathbb{Z}$  for even *n* and trivial for odd *n*, we get from Lemma 17.27 applied to (17.35) and Lemma 17.36 (iii)

$$K_n(\underline{B}G) \cong \begin{cases} \mathbb{Z}^2 & r \in [F,F] \text{ and } n \text{ even;} \\ \bigoplus_{i \in I} \mathbb{Z} & r \in [F,F] \text{ and } n \text{ odd;} \\ \mathbb{Z} & r \notin [F,F] \text{ and } n \text{ even;} \\ H_1(\underline{B}G) \cong \operatorname{coker}(H_1(C) \to H_1(G)) & r \notin [F,F] \text{ and } n \text{ odd.} \end{cases}$$

We get from Theorem 17.25 (i) the short exact sequence

$$0 \to \widetilde{K}_n(C_r^*(C)) \to K_n(C_r^*(G)) \to K_n(\underline{B}G) \to 0$$

which splits after inverting *m*. Since  $\widetilde{K}_n(C_r^*(\mathbb{Z}/m)) \cong \mathbb{Z}^{m-1}$  for even *n* and is {0} for odd *n*, we get

$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^{m+1} & r \in [F,F] \text{ and } n \text{ even}; \\ \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} & r \in [F,F] \text{ and } n \text{ odd}; \\ \mathbb{Z}^m & r \notin [F,F] \text{ and } n \text{ even}; \\ \operatorname{coker}(H_1(i) \colon H_1(C) \to H_1(G)) & r \notin [F,F] \text{ and } n \text{ odd}. \end{cases}$$

This computation for  $K_*(C_r^*(G))$  agrees with the one in [117].

The following example is illuminating, since it combines a lot of the material and methods we have presented so far in this book.

Example 17.38. Consider the finitely generated one-relator group

$$G = \langle s_1, s_2, s_3 | r \rangle \text{ for } r = s_1^6 s_2^9 s_1^{21} s_2^9 s_1^{21} s_2^9 s_1^{15}$$

Put  $s = s_1^6 s_2^9 s_1^{15}$ . Then  $r = s^3$ . If *m* is a natural number for which  $r = s'^m$  for some word *s'*, then m = 1, 3. Hence *G* has a maximal finite subgroup *C* generated by the element  $\overline{s} \in G$  represented by *s* and *C* is cyclic of order 3. We can compute  $H_1(G)$  using the recipe stated in Exercise 17.31 and obtain  $H_1(G) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/9$ . Since *r* does not belong to [F, F], we get from Lemma 17.36

$$H_n(G) \cong \begin{cases} \mathbb{Z}/3 & n \ge 3 \text{ and } n \text{ odd}; \\ 0 & n \ge 2 \text{ and } n \text{ even}; \\ \mathbb{Z}^2 \oplus \mathbb{Z}/9 & n = 1; \\ \mathbb{Z} & n = 0. \end{cases}$$

We get from Example 17.37

$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^3 & n \text{ even}; \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 & n \text{ odd.} \end{cases}$$

We conclude from Theorem 10.79 (ii) that  $\widetilde{KO}_n(C_r^*(C;\mathbb{R}))$  is  $\mathbb{Z}$  for *n* even and  $\{0\}$  for *n* odd.

We conclude from Lemma 17.27 (ii) in the case  $\mathcal{H}_* = KO_*$  applied to the pushout (17.35) an isomorphism

$$KO_n(\underline{B}G) \xrightarrow{=} KO_n(\underline{B}G, \{\bullet\}) \oplus KO_n(\{\bullet\}),$$

and a short exact sequence

$$0 \to H_1(\underline{B}G) \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}) \to KO_n(\underline{B}G, \{\bullet\})$$
$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(\underline{B}G), KO_{n-2}(\{\bullet\})) \to 0.$$

Since  $H_1(\underline{B}G) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3$  by Lemma 17.36 (iii), this implies

$$KO_n(\underline{B}G) \cong KO_n(\{\bullet\}) \oplus KO_{n-1}(\{\bullet\}) \oplus KO_{n-1}(\{\bullet\}) \oplus \mathbb{Z}/3 \otimes_{\mathbb{Z}} KO_{n-1}(\{\bullet\}).$$

Since  $\widetilde{KO}_n(C_r^*(C; \mathbb{R}))$  is  $\mathbb{Z}$  or trivial, we obtain from Theorem 17.25 (ii) for every  $n \in \mathbb{Z}$  a short exact sequence

$$0 \to \widetilde{KO}_n(C^*_r(C;\mathbb{R})) \to KO_n(C^*_r(G;\mathbb{R})) \to KO_n(\underline{B}G) \to 0$$

which splits after inverting 3.

When *n* is odd,  $\widetilde{KO}_n(C_r^*(C; \mathbb{R}))$  vanishes, and we obtain an isomorphism  $KO_n(C_r^*(G; \mathbb{R})) \cong KO_n(\underline{B}G)$ . Thus we get

$$KO_n(C_r^*(G;\mathbb{R})) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 & n \equiv 1 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3 & n \equiv 5 \mod 8; \\ \{0\} & n \equiv 7 \mod 8. \end{cases}$$

When n is even,  $KO_n(BG)$  contains no 3-torsion and we get

$$\begin{split} KO_n(C_r^*(G;\mathbb{R})) &\cong \widetilde{KO}_n(C_r^*(C;\mathbb{R})) \oplus KO_n(\underline{B}G) \\ &\cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \equiv 0 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 2 \mod 8; \\ \mathbb{Z} \oplus \mathbb{Z} & n \equiv 4 \mod 8; \\ \mathbb{Z} & n \equiv 6 \mod 8. \end{cases} \end{split}$$

Let  $V \subseteq G$  be an infinite virtually cyclic subgroup of type I. Then we can find an exact sequence  $1 \to H \to V \to \mathbb{Z} \to 0$ . Any finite subgroup of G is subconjugate to C and hence we can find  $g \in G$  with  $gHg^{-1} \subseteq C$ . Since  $gVg^{-1} \subseteq N_G(gHg^{-1})$ and  $N_G C = C$  by Example 11.33, we get  $H = \{1\}$  and hence  $V \cong \mathbb{Z}$ .

Suppose that there exists an infinite virtually cyclic subgroup  $V \subseteq G$  of type II. It contains an infinite cyclic subgroup V' of type I satisfying [V:V'] = 2. Since we have already proved that V' is infinite cyclic. V' must be  $\mathbb{Z}/2 * \mathbb{Z}/2$ . This contradicts the fact that any finite subgroup of G is subconjugate to  $C \cong \mathbb{Z}/3$ . Thus we have shown that any infinite virtually cyclic subgroup of G is infinite cyclic.

We conclude from Theorem 6.16 and the Transitivity Principle 15.12 that the assembly map  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}})$  is bijective for all  $n \in \mathbb{Z}$ . We conclude from Theorem 13.60 that the assembly map  $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^{(-\infty)}) \to$  $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$  is bijective for all  $n \in \mathbb{Z}$ . We conclude from Theorem 17.25 (iv) that the inclusion  $C \to M$  induces for all

 $n \in \mathbb{Z}$  an isomorphism

$$\operatorname{Wh}_n^{\mathbb{Z}}(C) \xrightarrow{=} \operatorname{Wh}_n^{\mathbb{Z}}(G).$$

Since Wh( $\mathbb{Z}/3$ ) by Theorem 3.115 and Theorem 3.116 (iii),  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3])$  by Theorem 2.113 (i), and  $K_n(\mathbb{Z}[\mathbb{Z}/3])$  for  $n \leq -1$  by Theorem 4.10 all vanish, the groups Wh(*G*),  $\widetilde{K}_0(\mathbb{Z}G)$ , and  $K_n(\mathbb{Z}G)$  for  $n \leq -1$  also vanish.

We conclude from Theorem 9.106 that the L-groups of  $\mathbb{Z}G$  are independent of the decoration, namely, for every  $j \in \mathbb{Z}$ ,  $j \leq -1$  and every  $n \in \mathbb{Z}$  the forgetful maps induce isomorphisms

$$L_n^s(\mathbb{Z}G) \xrightarrow{\cong} L_n^h(\mathbb{Z}G) \xrightarrow{\cong} L_n^p(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

The same statement is true for the L-groups of  $\mathbb{Z}C$ . We conclude from Theorem 9.204

$$\overline{L}_{n}^{\langle -\infty \rangle}(\mathbb{Z}C) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (4); \\ 0 & n \equiv 1 \mod (4); \\ \mathbb{Z} & n \equiv 2 \mod (4); \\ 0 & n \equiv 3 \mod (4). \end{cases}$$

Hence we get from Theorem 17.25 (iii) for  $n \in \mathbb{Z}$  a short exact sequence

$$0 \to \overline{L}_n^{\langle -\infty \rangle}(\mathbb{Z}C) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \to H_n(\underline{B}G; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0$$

which splits after inverting 3.

We obtain from Theorem 17.27 (ii) an isomorphism

$$H_n(\underline{B}G;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \cong H_n(\underline{B}G,\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z})) \oplus H_n(\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))$$

and the short exact sequence

$$0 \to H_1(\underline{B}G) \otimes_{\mathbb{Z}} L_{n-1}^{\langle -\infty \rangle}(\mathbb{Z}) \to H_n(\underline{B}G, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(\underline{B}G), L_{n-2}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0.$$

We get from Lemma 17.36 (ii) and (iii)

$$H_n(\underline{B}G) \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 & n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Hence we get for every decoration j

$$L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (4); \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3 & n \equiv 1 \mod (4); \\ \mathbb{Z}/2 & n \equiv 2 \mod (4); \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod (4). \end{cases}$$

# 17.5.4 Fuchsian Groups

Let F be a cocompact Fuchsian group with presentation

$$F = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_t |$$
  
$$c_1^{\gamma_1} = \dots = c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1} [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

for integers  $g, t \ge 0$  and  $\gamma_i > 1$ . Then <u>B</u>F is an orientable closed surface of genus g. The following is essentially a consequence of Theorem 17.25 and Example 11.33, see [683], in particular [683, Remark 4.10] for assertion (iv), for the details. Lower algebraic *K*-theory has also been computed in [127].

## Theorem 17.39 (K-and L-groups of Fuchsian groups).

(i) There are isomorphisms

$$K_n(C_r^*(F)) \cong \begin{cases} \left(2 + \sum_{i=1}^t (\gamma_i - 1)\right) \cdot \mathbb{Z} & n = 0; \\ (2g) \cdot \mathbb{Z} & n = 1; \end{cases}$$

(ii) The inclusions of the maximal subgroups  $\mathbb{Z}/\gamma_i = \langle c_i \rangle$  induce an isomorphism

$$\bigoplus_{i=1}^{l} \operatorname{Wh}_{n}(\mathbb{Z}/\gamma_{i}) \xrightarrow{\cong} \operatorname{Wh}_{n}(F)$$

for  $n \leq 1$ ;

(iii) There are isomorphisms

$$L_{n}(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1 + \sum_{i=1}^{t} \left\lfloor \frac{\gamma_{i}}{2} \right\rfloor\right) \cdot \mathbb{Z}[1/2] & n \equiv 0 \quad (4); \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \quad (4); \\ \left(1 + \sum_{i=1}^{t} \left\lfloor \frac{\gamma_{i}-1}{2} \right\rfloor\right) \cdot \mathbb{Z}[1/2] & n \equiv 2 \quad (4); \\ 0 & n \equiv 3 \quad (4), \end{cases}$$

where  $\lfloor r \rfloor$  for  $r \in \mathbb{R}$  denotes the largest integer less than or equal to r. (iv) From now on suppose that each  $\gamma_i$  is odd. Then we get for  $\epsilon = p$  and s

$$L_n^{\epsilon}(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & n \equiv 0 \quad (4); \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \quad (4); \\ \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 2 \quad (4); \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \quad (4). \end{cases}$$

For  $\epsilon = h$  we do not know an explicit formula for  $L_n^{\epsilon}(\mathbb{Z}F)$ . The problem is that no general formula is known for the 2-torsion contained in  $\widetilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$ , for *m* odd, since it is given by the term  $\widehat{H}^2(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$ , see [60, Theorem 2].

**Exercise 17.40.** Let *F* be a Fuchsian group as above. Show that its Whitehead group Wh(*F*) is a free abelian group of rank  $\bigoplus_{i=1}^{t} \lfloor \gamma_i/2 \rfloor + 1 - \delta(\gamma_i)$  where  $\delta(\gamma_i)$  is the number of divisors of  $\gamma_i$ .

# 17.5.5 Torsionfree Hyperbolic Groups

**Theorem 17.41 (Farrell-Jones Conjecture for torsionfree hyperbolic groups for** *K***-theory).** *Let G be a non-trivial torsionfree hyperbolic group.* 

(i) We obtain for all  $n \in \mathbb{Z}$  an isomorphism

$$H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_C \left( NK_n(R) \oplus NK_n(R) \right) \xrightarrow{\cong} K_n(RG)$$

where *C* runs through a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups;

- (ii) The abelian groups  $K_n(\mathbb{Z}G)$  for  $n \leq -1$ ,  $\widetilde{K}_0(\mathbb{Z}G)$ , and Wh(G) vanish;
- (iii) We get for every ring R with involution and  $n \in \mathbb{Z}$  an isomorphism

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(R)) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

For every  $j \in \mathbb{Z}$ ,  $j \leq 2$ , and  $n \in \mathbb{Z}$ , the natural map

$$L_n^{\langle j \rangle}(\mathbb{Z}G) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is bijective;

(iv) We get for any  $n \in \mathbb{Z}$  isomorphisms

$$K_n(BG) \xrightarrow{\cong} K_n(C_r^*(G));$$
  
$$KO_n(BG) \xrightarrow{\cong} KO_n(C_r^*(G;\mathbb{R})).$$

*Proof.* (i) By [688, Corollary 2.11, Theorem 3.1 and Example 3.6], see also Theorem 11.37, there is a *G*-pushout

where *i* is an inclusion of *G*-*CW*-complexes and *p* is the obvious projection. Hence we obtain using Theorem 6.16 an isomorphism

$$H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VCY}}(G); \mathbf{K}_R) \cong \bigoplus_C H_n^G(G \times_C EC \to G/C; \mathbf{K}_R)$$
$$\cong \bigoplus_C H_n^C(EC \to \{\bullet\}; \mathbf{K}_R)$$
$$\cong \bigoplus_C (NK_n(R) \oplus NK_n(R)).$$

We obtain from Theorem 13.36 an isomorphism

$$H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_R) \cong H_n^G(EG; \mathbf{K}_{\mathcal{R}}) \oplus H_n^G(E_{\mathcal{FIN}}(G) \to E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_R)$$
$$\cong H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_C (NK_n(R) \oplus NK_n(R)).$$

Since *G* satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia), Theorem 13.65 implies that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring *R*.

(ii) Since G satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia) and hence Conjectures 3.110 and 4.18 by Theorem 13.65, assertion (ii) follows.

(iii) Since G satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (ia), Theorem 13.65 implies that G satisfies Conjecture 9.114. Now assertion (iii) follows from assertion (ii) and Theorem 9.106.

(iv) This follows from the fact that G satisfies the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (id) and from Remark 14.14.

## 17.5.6 Hyperbolic Groups

Not necessarily torsionfree hyperbolic groups are treated in [679, Theorem 1.1], which says the following.

**Theorem 17.42 (Hyperbolic groups).** Let G be a hyperbolic group, and let  $\mathcal{M}$  be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of G.

(i) For each  $n \in \mathbb{Z}$  there is an isomorphism

$$H_n^G(\underline{E}G;\mathbf{K}_R) \oplus \bigoplus_{V \in \mathcal{M}} H_n^V(\underline{E}V \to \{\bullet\};\mathbf{K}_R) \xrightarrow{\cong} K_n(RG);$$

(ii) For each  $n \in \mathbb{Z}$  there is an isomorphism

$$H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) \oplus \bigoplus_{V\in\mathcal{M}} H_n^V(\underline{E}V \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG),$$

provided that there exists an  $n_0 \leq -2$  such that  $K_n(RV) = 0$  holds for all  $n \leq n_0$ and all virtually cyclic subgroups  $V \subseteq G$ . (The latter condition is satisfied if  $R = \mathbb{Z}$  or if R is regular with  $\mathbb{Q} \subseteq R$ .)

## 17.5.7 L-Theory of Torsionfree Groups

Throughout this subsection, let G be a torsionfree group satisfying Conjecture 9.114, i.e., we have the isomorphism

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G).$$

Thus we obtain from Subsection 15.14.4 an isomorphism

(17.43) 
$$KO(BG)[1/2] \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}G)[1/2].$$

**Example 17.44** (*p*-torsion in *L*-groups). Let  $n \ge 3$  be an odd natural number. Consider the group automorphism

$$\alpha \colon \mathbb{Z}^2 \to \mathbb{Z}^2, (a, b) \mapsto (a + nb, b).$$

Let *G* be the semidirect product  $\mathbb{Z}^2 \rtimes \mathbb{Z}$ . Obviously there is an orientable aspherical closed smooth 3-manifold *M* that is the total space in a locally trivial fiber bundle  $T^2 \to M \to S^1$  whose fundamental group is *G*, namely, the mapping torus of the self-diffeomorphism  $S^1 \times S^1 \to S^1 \times S^1$  sending  $(z_1, z_2)$  to  $(z_1 z_2^n, z_2)$ . The group *G* satisfies the Full Farrell-Jones Conjecture and hence Conjecture 9.114. One easily computes

$$H_k(M;\mathbb{Z}) \cong H_k(G) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 3; \\ \mathbb{Z} \oplus \mathbb{Z}/n & k = 1; \\ 0 & \text{otherwise.} \end{cases}$$

An elementary spectral sequence argument shows

$$L_n^{\langle -\infty \rangle}(\mathbb{Z}G)[1/2] \cong KO_k(M;\mathbb{Z})[1/2] \cong \begin{cases} \mathbb{Z}[1/2] & k = 0, 2, 3 \mod 4; \\ \mathbb{Z}[1/2] \oplus \mathbb{Z}/n & k = 1 \mod 4; \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$  can contain *p*-torsion for any odd prime *p*. Recall that for finite groups *G* only 2-torsion occurs in  $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$  by Theorem 9.204 (ii).

**Exercise 17.45.** Let *p* be a prime. Show for every  $n \ge 6$  and every decoration  $j \in \{2, 1, 0, -1, ...\}$  II  $\{-\infty\}$  that there is an orientable aspherical closed smooth manifold *M* of dimension *n* such that  $L_k^{\langle j \rangle}(\mathbb{Z}\pi_1(M))$  contains non-trivial *p*-torsion for every  $k \in \mathbb{Z}$ .

Since we have the decomposition of spectra after localization at 2

$$\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})_{(2)} = \prod_{k \in \mathbb{Z}} \mathbf{K}(\mathbb{Z}_{(2)}, 4k) \times \prod_{k \in \mathbb{Z}} \mathbf{K}(\mathbb{Z}/2, 4k-2),$$

see Remark 9.133 in the connective case and [946, Theorem  $A_{(2)}$  on page 178] in the periodic case, we obtain for any torsionfree group *G* satisfying Conjecture 9.114

(17.46) 
$$L_n^{\langle -\infty \rangle}(\mathbb{Z}G)_{(2)} \cong \prod_{k \in \mathbb{Z}} H_{n+4k}(BG; \mathbb{Z}_{(2)}) \times \prod_{k \in \mathbb{Z}} H_{n+4k-2}(BG; \mathbb{Z}/2)$$

#### 17.5.8 Cocompact NEC-Groups

A calculation of  $Wh_n(G)$ ,  $L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$ , and  $K_n(C_*^r(G))$  for 2-dimensional crystallographic groups *G* and more general cocompact NEC-groups *G* is presented in [683], see also [795]. For these groups the orbit spaces <u>*B*</u>*G* are compact surfaces possibly with boundary.

# 17.5.9 Crystallographic Groups

A crystallographic group of dimension n is a discrete group that acts cocompactly, properly, and isometrically on the Euclidean space  $\mathbb{R}^n$  for some  $n \ge 0$ . A group G is a crystallographic group of dimension n if and only if there exists an extension  $1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$  such that A is isomorphic to  $\mathbb{Z}^n$ , Q is a finite group, and the centralizer of A in G is A itself. One does not have a complete calculation of K- and L-groups of integral group rings or reduced group C\*-algebras of crystallographic groups except in dimension two, as mentioned above in Subsection 17.5.8. Computations of the lower and middle algebraic K-theory of the integral group ring of split three-dimensional crystallographic groups are carried out by Farley-Ortiz [349], see also [24].

As an illustration we mention the following result taken from [619, Theorem 0.1].

**Theorem 17.47 (Computation of the topological** *K***-theory of**  $\mathbb{Z}^n \rtimes \mathbb{Z}/m$  **for a free conjugation action).** Consider the extension of groups  $1 \to \mathbb{Z}^n \to G \to \mathbb{Z}/m \to 1$  such that the conjugation action of  $\mathbb{Z}/m$  on  $\mathbb{Z}^n$  is free outside the origin  $0 \in \mathbb{Z}^n$ . Let  $\mathcal{M}$  be the set of conjugacy classes of maximal finite subgroups of G.

(i) We obtain an isomorphism

$$\omega_1 \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(\underline{B}G).$$

*Restriction with the inclusion*  $k : \mathbb{Z}^n \to G$  *induces an isomorphism* 

$$k^* \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}$$

Induction with the inclusion k yields a homomorphism

$$\overline{k_*}: \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(G)).$$

It fits into an exact sequence

$$0 \to \widehat{H}^{-1}(\mathbb{Z}/m, K_1(C_r^*(\mathbb{Z}^n))) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\overline{k_*}} K_1(C_r^*(G)) \to 0$$

where  $\widehat{H}^*(\mathbb{Z}/m; M)$  denotes the Tate cohomology of  $\mathbb{Z}/m$  with coefficients in a  $\mathbb{Z}[\mathbb{Z}/m]$ -module M. In particular,  $\overline{k_*}$  is surjective and its kernel is annihilated by multiplication with m;

(ii) There is an exact sequence

$$0 \to \bigoplus_{(M) \in \mathcal{M}} \widetilde{\operatorname{Rep}}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_M} K_0(C_r^*(G)) \xrightarrow{\omega_0} K_0(\underline{B}G) \to 0$$

where  $\operatorname{Rep}_{\mathbb{C}}(M)$  is the kernel of the map  $\operatorname{Rep}_{\mathbb{C}}(M) \to \mathbb{Z}$  sending the class [V]of a complex *M*-representation *V* to  $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$  and the map  $i_M$  comes from the inclusion  $M \to G$  and the identification  $\operatorname{Rep}_{\mathbb{C}}(M) = K_0(C_r^*(M))$ . We obtain a homomorphism

$$\overline{k_*} \oplus \bigoplus_{(M) \in \mathcal{M}} i_M \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{(M) \in \mathcal{M}} \widetilde{\operatorname{Rep}}_{\mathbb{C}}(M) \to K_0(C_r^*(G)).$$

*It is injective. It is bijective after inverting m;* (iii) *We have* 

$$K_i(C_r^*(G)) \cong \mathbb{Z}^{s_i}$$

where

$$s_{i} = \begin{cases} \left( \sum_{(M) \in \mathcal{M}} (|M| - 1) \right) + \sum_{l \in \mathbb{Z}} \operatorname{rk}_{\mathbb{Z}} \left( (\Lambda^{2l} \mathbb{Z}^{n})^{\mathbb{Z}/m} \right) & \text{if } i \text{ even;} \\ \sum_{l \in \mathbb{Z}} \operatorname{rk}_{\mathbb{Z}} \left( (\Lambda^{2l+1} \mathbb{Z}^{n})^{\mathbb{Z}/m} \right) & \text{if } i \text{ odd;} \end{cases}$$

(iv) If m is even, then  $s_1 = 0$  and

$$K_1(C_r^*(G)) \cong \{0\}.$$

The numbers  $s_i$  can be made more explicit, see [619]. For instance, if m = p for a prime number p, then there exists a natural number k that is determined by the property  $n = (p - 1) \cdot k$ , and we get:

(17.48) 
$$s_{i} = \begin{cases} p^{k} \cdot (p-1) + \frac{2^{n}+p-1}{2p} + \frac{p^{k-1} \cdot (p-1)}{2} & p \neq 2 \text{ and } i \text{ even;} \\ \frac{2^{n}+p-1}{2p} - \frac{p^{k-1} \cdot (p-1)}{2} & p \neq 2 \text{ and } i \text{ odd;} \\ 3 \cdot 2^{k-1} & p = 2 \text{ and } i \text{ even;} \\ 0 & p = 2 \text{ and } i \text{ odd.} \end{cases}$$

**Exercise 17.49.** The automorphism  $\phi \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ ,  $(a, b) \mapsto (b, -a - b)$  satisfies  $\phi^3 = \text{id. Show}$ 

$$K_i(C_r^*(\mathbb{Z}^2 \rtimes_{\phi} \mathbb{Z}/3)) \cong \begin{cases} \mathbb{Z}^8 & i \text{ even;} \\ \{0\} & i \text{ odd.} \end{cases}$$

Theorem 17.47 in the special case where m is a prime number is treated in [282]. The groups appearing in Theorem 17.47 are crystallographic groups, see [619, Lemma 3.1].

The proof of Theorem 17.47 is surprisingly complicated. It is based on computations of the group homology of  $\mathbb{Z}^n \rtimes \mathbb{Z}/m$  by Langer-Lück [618, Theorem 0.5]. They prove a conjecture of Adem-Ge-Pan-Petrosyan [17, Conjecture 5.2], which says that the associated Lyndon-Hochschild-Serre spectral sequence collapses in the strongest sense, in the special case when the conjugation action of  $\mathbb{Z}/m$  of  $\mathbb{Z}^n$  is free outside the origin  $0 \in \mathbb{Z}^n$ . Moreover, it uses generalizations of the Atiyah-Segal Completion Theorem for finite groups to infinite groups, see Lück-Oliver [670, 671]. Interestingly the conjecture of Adem-Ge-Pan-Petrosyan is disproved in general by Langer-Lück [618, Theorem 0.6].

The computation of  $K_*(C_r^*(\mathbb{Z}^n \rtimes \mathbb{Z}/m))$  for square free *m* is carried out by Sánchez-Velásquez [887].

## **17.5.10** Virtually $\mathbb{Z}^n$ Groups

One does not have a complete calculation of the *K*-groups and *L*-groups of integral group rings or group  $C^*$ -algebras of crystallographic groups and hence not of virtually finitely generated abelian groups. This has already been illustrated in Subsection 17.4.5, where a rather complicated recipe for the computation of the rank is given. The favorite situation is the one occurring in Example 11.33, when one considers groups *G* occurring in an extensions  $1 \to \mathbb{Z}^n \to G \to F \to 1$  for finite *F* such that the conjugation action of *F* on  $\mathbb{Z}^n$  is free outside  $0 \in \mathbb{Z}^n$ . The computation of Wh<sub>n</sub>(*G*; *R*) can be found in [679, Theorem 1.7], provided that the *Q*-action on  $\mathbb{Z}^n$  is free outside the origin.

**Question 17.50** (Is  $K_i(C_r^*(\mathbb{Z}^n \rtimes \mathbb{Z}/m))$  torsionfree?). For which groups *G* of the shape  $\mathbb{Z}^n \rtimes \mathbb{Z}/m$  is the topological complex *K*-theory  $K_i(C_r^*(G))$  torsionfree for every  $i \in \mathbb{Z}$ ?

The answer to Question 17.50, which is also stated in [887, Question 1.3], is positive if the  $\mathbb{Z}/m$ -action of  $\mathbb{Z}^n$  is free outside the origin, see Theorem 17.47, or if *m* is square-free, see [887]. Note that in these cases the conjecture of Adem-Ge-Pan-Petrosyan [17, Conjecture 5.2] is known to be true. Counterexamples to it can be found in Langer-Lück [618, Theorem 0.6] and these are potential candidates for a negative answer to Question 17.50.

The next example shows that one cannot expect in general a positive answer to Question 17.50 if one considers an extension  $1 \to \mathbb{Z}^{n+1} \to G \to \mathbb{Z}/m \to 1$  which does not split.

**Example 17.51.** In this example we give an elementary computation of the topological *K*-theory of  $K_n(C^*(G))$  of the semi-direct product  $G = \mathbb{Z}^n \rtimes_{\rho} \mathbb{Z}$  for a natural number *n* and the group homomorphism  $\rho : \mathbb{Z} \to \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^n)$  sending a generator of  $\mathbb{Z}$  to  $\varphi_n := -\operatorname{id}_{\mathbb{Z}^n} : \mathbb{Z}^n \to \mathbb{Z}^n$ .

The associated Wang sequence, see [812, Theorem 18 on page 632] looks like

$$\cdots \to K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\operatorname{id} - K_1(C_r^*(\varphi_n))} K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(G))$$

$$\to K_0(C_r^*(\mathbb{Z}^n)) \xrightarrow{\operatorname{id} - K_0(C_r^*(\varphi_n))} K_0(C_r^*(\mathbb{Z}^n)) \to K_0(C_r^*(G))$$

$$\to K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\operatorname{id} - K_1(C_r^*(\varphi_n))} K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(G)) \to \cdots .$$

It yields the short exact sequence

(17.52)  

$$0 \to \operatorname{coker}(\operatorname{id} - K_i(C_r^*(\varphi_n)) \colon K_i(C_r^*(\mathbb{Z}^n)) \to K_i(C_r^*(\mathbb{Z}^n))) \to K_i(C_r^*(G))$$

$$\to \operatorname{ker}(\operatorname{id} - K_{i-1}(C_r^*(\varphi_n)) \colon K_{i-1}(C_r^*(\mathbb{Z}^n)) \to K_{i-1}(C_r^*(\mathbb{Z}^n))) \to 0$$

for  $i \in \mathbb{Z}$ . The external product on topological *K*-theory induces an isomorphism natural in  $\mathbb{Z}^n$  and  $\mathbb{Z}$ 

$$\left(K_i(C_r^*(\mathbb{Z}^n)) \otimes K_0(C_r^*(\mathbb{Z}))\right) \oplus \left(K_{i-1}(C_r^*(\mathbb{Z}^n)) \otimes K_1(C_r^*(\mathbb{Z}))\right) \xrightarrow{\cong} K_i(C_r^*(\mathbb{Z}^{n+1})).$$

Since  $K_i(\varphi_1) \colon K_i(C^*(\mathbb{Z})) \to K_i(C^*(\mathbb{Z}))$  is  $(-1)^i \cdot id$ , we get a commutative diagram with isomorphisms as horizontal arrows

Now one shows by induction over n = 0, 1, 2, ... that we obtain for  $m \in \mathbb{Z}$  a commutative diagram with isomorphisms as horizontal arrows

$$\begin{array}{c} \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}} \xrightarrow{\cong} K_{i}(C_{r}^{*}(\mathbb{Z}^{n})) \\ \oplus_{j=0}^{n}(^{-1)^{j}} \cdot \mathrm{id} \\ \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}} \xrightarrow{\cong} K_{i}(C_{r}^{*}(\mathbb{Z}^{n})). \end{array}$$

This implies

$$\operatorname{coker}(\operatorname{id} - K_{i}(C_{r}^{*}(\varphi_{n})) \colon K_{i}(C_{r}^{*}(\mathbb{Z}^{n})) \to K_{i}(C_{r}^{*}(\mathbb{Z}^{n})))$$

$$\cong \bigoplus_{j=0}^{n} \operatorname{coker}\left(\operatorname{id} - (-1)^{j} \cdot \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}} \to \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}}\right)$$

$$\cong \bigoplus_{\substack{j=0,1,\ldots,n\\i-j \text{ even}}} \operatorname{coker}\left(\operatorname{id} - (-1)^{j} \cdot \mathbb{Z}^{\binom{n}{j}} \to \mathbb{Z}^{\binom{n}{j}}\right)$$

$$\cong \left(\bigoplus_{\substack{j=0,1,\ldots,n\\i,j \text{ even}}} \mathbb{Z}^{\binom{n}{j}}\right) \oplus \left(\bigoplus_{\substack{j=0,1,\ldots,n\\i,j \text{ odd}}} \mathbb{Z}/2^{\binom{n}{j}}\right)$$

and

$$\ker\left(\operatorname{id} - K_{i}(C_{r}^{*}(\varphi_{n})) \colon K_{i}(C_{r}^{*}(\mathbb{Z}^{n})) \to K_{i}(C_{r}^{*}(\mathbb{Z}^{n}))\right)$$

$$\cong \bigoplus_{j=0}^{n} \ker\left(\operatorname{id} - (-1)^{j} \cdot \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}} \to \bigoplus_{j=0}^{n} K_{i-j}(\mathbb{C})^{\binom{n}{j}}\right)$$

$$\cong \bigoplus_{\substack{j=0,1,\dots,n\\i-j \text{ even}\\i,j \text{ even}}} \mathbb{Z}^{\binom{n}{j}}.$$

Since ker(id  $-K_i(C_r^*(\varphi_n)): K_i(C_r^*(\mathbb{Z}^n)) \to K_i(C_r^*(\mathbb{Z}^n)))$  is torsionfree for all  $i \in \mathbb{Z}$ , we conclude from the short exact sequence (17.52)

$$K_{i}(C_{r}^{*}(G)) \cong \left(\bigoplus_{\substack{j=0,1,\dots,n\\i,j \text{ even}}} \mathbb{Z}^{\binom{n}{j}}\right) \oplus \left(\bigoplus_{\substack{j=0,1,\dots,n\\i,j \text{ odd}}} \mathbb{Z}/2^{\binom{n}{j}}\right) \oplus \left(\bigoplus_{\substack{j=0,1,\dots,n\\i \text{ odd},j \text{ even}}} \mathbb{Z}^{\binom{n}{j}}\right)$$
$$\cong \left(\bigoplus_{\substack{j=0,1,\dots,n\\j \text{ even}}} \mathbb{Z}^{\binom{n}{j}}\right) \oplus \left(\bigoplus_{\substack{j=0,1,\dots,n\\i,j \text{ odd}}} \mathbb{Z}/2^{\binom{n}{j}}\right).$$

Since we have

$$\sum_{\substack{j=0,1,...,n\\ j \text{ even}}} \binom{n}{j} = \sum_{\substack{j=0,1,...,n\\ j \text{ odd}}} \binom{n}{j} = 2^{n-1}$$

we conclude for  $n \ge 1$  and  $G = \mathbb{Z}^n \rtimes_{\rho} \mathbb{Z}$ 

(17.53) 
$$K_i(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^{2^{n-1}} \oplus (\mathbb{Z}/2)^{2^{n-1}} & \text{if } i \text{ is odd;} \\ \mathbb{Z}^{2^{n-1}} & \text{if } i \text{ is even.} \end{cases}$$

Note that  $K_i(C_r^*(G))$  does contain torsion. Obviously *G* is a crystallographic group. It can be written as an extension

$$1 \to \mathbb{Z}^{n+1} \to G \to \mathbb{Z}/2 \to 1$$

if we identify  $\mathbb{Z}^{n+1}$  with the preimage of  $2\mathbb{Z}$  under the canonical projection  $G = \mathbb{Z}^n \rtimes_{\rho} \mathbb{Z} \to \mathbb{Z}$ . The  $\mathbb{Z}/2$ -action on  $\mathbb{Z}^{n+1}$  given by the extension above is not free outside the origin. So we get no contradiction with Theorem 17.47 but we see that Theorem 17.47 does not hold if we drop the assumption that the conjugation action of  $\mathbb{Z}^m$  on  $\mathbb{Z}^n$  is free outside the origin. Note that Theorem 17.47 applies to the semidirect product  $\mathbb{Z}^n \rtimes_{\rho} \mathbb{Z}/2$  where  $\rho \colon \mathbb{Z}/2 \to \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^n)$  sends the generator to  $\varphi_n = -\operatorname{id}_{\mathbb{Z}^n}$ .

# 17.5.11 Mayer-Vietoris Sequences and Wang Sequences

We have explained in Section 15.7 how an action of *G* on a tree *T* yields a long exact sequence involving the isotropy groups. In particular, we get for an amalgamated free product a Mayer-Vietoris sequence and for a semidirect product with  $\mathbb{Z}$ , or, more generally, for an HNN-extension, a long exact Wang sequence, computing the value at <u>*E*</u>*G*.

# 17.5.12 SL<sub>2</sub>(Z)

We want to illustrate this in the case  $G = SL_2(\mathbb{Z})$ . We have explained in Subsection 11.6.11 that  $SL_2(\mathbb{Z})$  is the amalgamated free product  $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ . Since the inclusion  $\mathbb{Z}/2 \to \mathbb{Z}/6$  is split injective, we obtain from the long exact sequence appearing in Theorem 15.27 (ii) for every equivariant homology theory  $\mathcal{H}^2_*$  an isomorphism

$$\mathcal{H}_{n}^{\mathbb{Z}/4}(\{\bullet\}) \oplus \operatorname{coker}(\mathcal{H}_{n}^{\mathbb{Z}/2}(\{\bullet\}) \to \mathcal{H}_{n}^{\mathbb{Z}/6}(\{\bullet\})) \xrightarrow{\cong} \mathcal{H}_{n}^{\operatorname{SL}_{2}(\mathbb{Z})}(\underline{E}\operatorname{SL}_{2}(\mathbb{Z})).$$

Since  $SL_2(\mathbb{Z})$  is hyperbolic, it satisfies the Baum-Connes Conjecture 14.11 with coefficients by Theorem 16.7 (id) and the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ia). In particular, we get isomorphisms

$$K_n(C_r^*(\mathbb{Z}/4;\mathbb{C})) \oplus \operatorname{coker} \left( K_n(C_r^*(\mathbb{Z}/2;\mathbb{C})) \to K_n(C_r^*(\mathbb{Z}/6;\mathbb{C})) \right)$$
$$\cong K_n^{\operatorname{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z})) \cong K_n(C_r^*(\operatorname{SL}_2(\mathbb{Z});\mathbb{C}));$$

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$$KO_n(C_r^*(\mathbb{Z}/4;\mathbb{R})) \oplus \operatorname{coker}(KO_n(C_r^*(\mathbb{Z}/2;\mathbb{R})) \to KO_n(C_r^*(\mathbb{Z}/6;\mathbb{R})))$$
$$\cong KO_n^{\operatorname{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z})) \cong KO_n(C_r^*(\operatorname{SL}_2(\mathbb{Z});\mathbb{R}));$$

$$L_n^{\langle j \rangle}(R[\mathbb{Z}/4])[1/2] \oplus \operatorname{coker}(L_n^{\langle j \rangle}(R[\mathbb{Z}/2])[1/2] \to L_n^{\langle j \rangle}(R[\mathbb{Z}/6])[1/2])$$
$$\cong H_n^{\operatorname{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z}); \mathbf{L}_R^{\langle j \rangle})[1/2] \cong L_n^{\langle j \rangle}(R[\operatorname{SL}_2(\mathbb{Z})])[1/2];$$

$$L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/4]) \oplus \operatorname{coker}(L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \to L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/6]))$$
$$\cong H_n^{\operatorname{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z}); \mathbf{L}_R^{\langle -\infty \rangle});$$

$$K_n(R[\mathbb{Z}/4]) \oplus \operatorname{coker}(K_n(R[\mathbb{Z}/2]) \to K_n(R[\mathbb{Z}/6])) \cong H_n^{\operatorname{SL}_2}(\mathbb{Z})(\underline{E}\operatorname{SL}_2(\mathbb{Z}); \mathbf{K}_R).$$

Since every infinite cyclic subgroup of type I of  $SL_2(\mathbb{Z})$  is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}\times\mathbb{Z}/2$ , we conclude from Theorem 4.3 and Theorem 6.21 that  $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{K}_{\mathbb{Z}})$  vanishes for  $n \leq 1$  for any infinite virtually cyclic subgroup of type I of  $SL_2(\mathbb{Z})$ . The Transitivity Principal of Theorem 15.12 and Theorem 13.47 imply that the relative assembly map  $H_n^V(\underline{E} SL_2(\mathbb{Z}); \mathbf{K}_{\mathbb{Z}}) \to H_n^V(\underline{E} SL_2(\mathbb{Z}); \mathbf{K}_{\mathbb{Z}})$  is bijective for  $n \leq 1$ . Hence the map

$$H_n^V(\underline{E}\operatorname{SL}_2(\mathbb{Z}); \mathbf{K}_{\mathbb{Z}}) \to H_n^V(\{\bullet\}; \mathbf{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}[\operatorname{SL}_2(\mathbb{Z})])$$

is bijective for  $n \le 1$ . So we get for  $n \le 1$  an isomorphism

$$K_n(\mathbb{Z}[\mathbb{Z}/4]) \oplus \operatorname{coker}(K_n(\mathbb{Z}[\mathbb{Z}/2]) \to K_n(\mathbb{Z}[\mathbb{Z}/6])) \cong K_n(\mathbb{Z}[\operatorname{SL}_2(\mathbb{Z})]).$$

Recall that Wh( $\mathbb{Z}/m$ ) vanishes for m = 2, 4, 6 and that  $\widetilde{K}_n(\mathbb{Z}[\mathbb{Z}/m])$  vanishes for  $n \le 0$  and m = 2, 4, 6 except for n = -1 and m = 6, where  $K_{-1}(\mathbb{Z}[\mathbb{Z}/6])$  turns out to be infinite cyclic, see Theorem 2.113 (i), Theorem 3.115, Theorem 3.116 (iv), Example 4.12, and Theorem 4.22 (i) and (v). We conclude that Wh(SL<sub>2</sub>( $\mathbb{Z})$ ),  $\widetilde{K}_0(\mathbb{Z}[SL_2(\mathbb{Z})])$ , and  $\widetilde{K}_n(\mathbb{Z}[SL_2(\mathbb{Z})])$  for  $n \le -2$  vanish and that the inclusion  $\mathbb{Z}/6 \to SL_2(\mathbb{Z})$  induces an isomorphism  $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \xrightarrow{\cong} K_{-1}(\mathbb{Z}[SL_2(\mathbb{Z})])$ .

For L-theory we get using [204] an isomorphism

$$H_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z}); \mathbf{L}_R^{\langle -\infty \rangle}) \oplus \operatorname{UNil}_n(\mathbb{Z}/2; \mathbb{Z}/4, \mathbb{Z}/6; R) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}[\operatorname{SL}_2(\mathbb{Z})])$$

where  $\text{UNil}_n(\mathbb{Z}/2; \mathbb{Z}/4, \mathbb{Z}/6; R)$  is a certain UNil-term which is known to be a (not necessarily finitely generated) 2-primary abelian group and vanishes if 2 is invertible in *R*. Hence we get for  $n \in \mathbb{Z}$  an isomorphism

$$L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/4]) \oplus \operatorname{coker}(L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/2]) \to L_n^{\langle -\infty \rangle}(R[\mathbb{Z}/6]))$$
$$\oplus \operatorname{UNil}_n(\mathbb{Z}/2; \mathbb{Z}/4, \mathbb{Z}/6; R) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}[\operatorname{SL}_2(\mathbb{Z})]).$$

We get from Theorem 11.37 and Theorem 13.60 isomorphisms

$$\begin{split} H_n^{\mathrm{SL}_2(\mathbb{Z})}(\underline{E}\operatorname{SL}_2(\mathbb{Z});\mathbf{L}_R^{\langle -\infty\rangle}) \oplus \bigoplus_V H_n^V(\underline{E}V \to \{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}) \\ &\cong L_n^{\langle -\infty\rangle}(R[\operatorname{SL}_2(\mathbb{Z})]) \end{split}$$

for  $n \in \mathbb{Z}$ , where *V* runs through a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of type II. Note that every infinite virtually cyclic subgroup of type II is isomorphic to the semidirect product  $\mathbb{Z} \rtimes \mathbb{Z}/4$ , where the generator of  $\mathbb{Z}/4$  acts on  $\mathbb{Z}$  by – id. In particular, we get an isomorphism

$$\mathrm{UNil}_n(\mathbb{Z}/2;\mathbb{Z}/4,\mathbb{Z}/6;R) \cong \bigoplus_V H_n^{\mathbb{Z} \rtimes \mathbb{Z}/4}(\underline{E}(\mathbb{Z} \rtimes \mathbb{Z}/4) \to \{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}).$$

Exercise 17.54. Prove

$$K_n(C_r^*(\mathrm{SL}_2(\mathbb{Z});\mathbb{C})) \cong \begin{cases} \mathbb{Z}^8 & n \text{ even}; \\ 0 & n \text{ odd}, \end{cases}$$

and

$$KO_n(C_r^*(\mathrm{SL}_2(\mathbb{Z});\mathbb{R})) \cong \begin{cases} \mathbb{Z}^5 & n \equiv 0 \mod(8); \\ (\mathbb{Z}/2)^2 & n \equiv 1 \mod(8); \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}^3 & n \equiv 2 \mod(8); \\ \{0\} & n \equiv 3 \mod(8); \\ \mathbb{Z}^5 & n \equiv 4 \mod(8); \\ \{0\} & n \equiv 5 \mod(8); \\ \mathbb{Z}^3 & n \equiv 6 \mod(8); \\ \{0\} & n \equiv 7 \mod(8). \end{cases}$$

**Exercise 17.55.** Let  $D_8$  be the dihedral group of order eight and C be its center, which is a group of order two. Let G be the group  $D_8 *_C D_8$ . Prove

$$K_0(\mathbb{C}G) \cong \mathbb{Z}^8 \oplus \mathbb{Z}/2;$$
  

$$K_n(C_r^*(G)) \cong \begin{cases} \mathbb{Z}^8 \oplus \mathbb{Z}/2 & \text{if } n \text{ is even}; \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

17.5.13 SL<sub>3</sub>(Z)

Since  $SL_3(\mathbb{Z})$  satisfies the Full Farrell-Jones Conjecture 13.30, see Theorem 16.1 (id), Theorem 13.65 implies that *G* satisfies the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in  $\mathbb{Z}$ . Using this fact the following result is proved in [926] and [962]. **Theorem 17.56 (Lower and middle** *K*-theory of the integral group ring of  $SL_3(\mathbb{Z})$ ). The groups  $K_n(\mathbb{Z}[SL_3(\mathbb{Z})])$  for  $n \leq -2$ ,  $\widetilde{K}_0(\mathbb{Z}[SL_3(\mathbb{Z})])$ , and  $Wh(SL_3(\mathbb{Z}))$  are trivial. For an appropriate subgroup  $C_6 \subseteq SL_3(\mathbb{Z})$ , which is cyclic of order six, the inclusion  $C_6 \rightarrow SL_3(\mathbb{Z})$  induces an isomorphism

$$\mathbb{Z} \cong K_{-1}(\mathbb{Z}[C_6]) \xrightarrow{=} K_{-1}(\mathbb{Z}[\operatorname{SL}_3(\mathbb{Z})]).$$

The following result is taken from [885, Corollary 2] in the complex case and from [507, Theorem 4.2] in the real case.

# **Theorem 17.57 (Topological equivariant** *K***-theory of** $E_{\mathcal{FIN}}(SL_3(\mathbb{Z}))$ ).

- (i) The abelian group  $K_n^{\mathrm{SL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{SL}_3(\mathbb{Z})))$  is  $\mathbb{Z}^8$  for even *n* and vanishes for odd *n*;
- (ii) We have for n = 0, 1, 2, ..., 7

$$KO_n^{\mathrm{SL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\mathrm{SL}_3(\mathbb{Z}))) = \mathbb{Z}^8, \mathbb{Z}/2^8, \mathbb{Z}/2^8, \{0\}, \mathbb{Z}^8, \{0\}, \{0\}, \{0\}, \{0\}\}$$

and the remaining groups are given by 8-fold Bott periodicity.

The groups  $K_n^{\operatorname{GL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\operatorname{GL}_3(\mathbb{Z})))$  are determined in [885, Corollary 4], and the groups  $KO_n^{\operatorname{GL}_3(\mathbb{Z})}(E_{\mathcal{FIN}}(\operatorname{GL}_3(\mathbb{Z})))$  are determined in [507, Corollary 3.3].

Recall that the Baum-Connes Conjecture is not known to be true for  $SL_3(\mathbb{Z})$ . So it would be interesting to compute  $K_n(C_r^*(SL_3(\mathbb{Z});\mathbb{C}))$  and  $KO_n(C_r^*(SL_3(\mathbb{Z});\mathbb{R}))$  directly and to compare the result with the computations of Theorem 17.57.

### 17.5.14 Right Angled Artin Groups

The group homology, the algebraic *K*- and *L*-groups, and the topological *K*-groups of right-angled Artin groups, and, more generally, of graph products are computed in [570, Section 6].

Let *X* be a finite simplicial graph on the vertex set *V* and suppose that we are given a collection of groups  $\mathcal{W} = \{W_v \mid v \in V\}$ . Then the graph product  $W(X, \mathcal{W})$  is defined as the quotient of the free product  $*_{v \in V} W_v$  of the collection of groups  $\mathcal{W}$  by introducing the relations

 $\{[g,g'] = 1 \mid v, v' \in V, \text{ there is an edge joining } v \text{ and } v', g \in W_v, g' \in W_{v'}\}.$ 

In other words, elements of subgroups  $W_v$  and  $W_{v'}$  commute if there is an edge joining v and v'. This notion is due to Green [437].

A *right-angled Artin group* is a graph product W = W(X, W) for which each of the groups  $W_v$  is infinite cyclic. For general information about right-angled Artin groups, we refer for instance to Charney [231]. Denote by  $\Sigma$  the flag complex associated to the finite simplicial graph X. Let  $\mathcal{P}$  be the poset of simplices of  $\Sigma$ , both ordered by inclusion where the empty subcomplex and the empty simplex are

allowed and the dimension of the empty simplex is defined to be -1. Note that W is torsionfree. In the sequel we denote by  $r_k$  the number of k-simplices in  $\mathcal{P}$ .

Let  $\mathcal{K}_*$  be any generalized non-equivariant homology theory with values in  $\Lambda$ -modules. Then

$$\bigoplus_{\sigma \in \mathcal{P}} \mathcal{K}_{n-\dim(\sigma)-1}(\{\bullet\}) \xrightarrow{\cong} \mathcal{K}_n(BW).$$

If we take for  $\mathcal{K}_*$  singular homology  $H_*(-; A)$  with coefficients in A, this boils down to the well-known isomorphism

(17.58) 
$$A^{r_{n-1}} \xrightarrow{=} H_n(BW; A).$$

In particular, we get the following relation for the Euler characteristics

$$\chi(BW) = 1 - \chi(\Sigma).$$

Theorem 17.59 (The algebraic *K*-theory and *L*-theory of right-angled Artin groups).

(i) Let R be a regular ring. Then there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(R) \xrightarrow{\cong} K_n(RW).$$

In particular, we get  $K_n(RW) = 0$  for  $n \le -1$ .

If we take  $R = \mathbb{Z}$ , we conclude that  $K_n(\mathbb{Z}W)$  for  $n \leq -1$ ,  $\widetilde{K}_0(\mathbb{Z}W)$ , and Wh(W) vanish;

(ii) Let *R* be a ring with involution. Then there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} L_{n-\dim(\sigma)-1}^{\langle -\infty \rangle}(R) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RW).$$

**Theorem 17.60 (The topological** *K***-theory of right-angled Artin groups).** *There are explicit isomorphisms of abelian groups* 

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(\mathbb{C}) \xrightarrow{\cong} K_n(C_m^*(W)) \cong K_n(C_r^*(W));$$
$$\bigoplus_{\sigma \in \mathcal{P}} KO_{n-\dim(\sigma)-1}(\mathbb{R}) \xrightarrow{\cong} KO_n(C_m^*(W;\mathbb{R})) \cong KO_n(C_r^*(W;\mathbb{R})).$$

In particular, we get an isomorphism of abelian groups

$$K_n(C_m^*(W)) \cong K_n(C_r^*(W)) \cong \mathbb{Z}^{t_n},$$

*if we put*  $t_n = \sum_{k \in \{-1,0,1,2,...,\dim(\Sigma)\}} r_k$ .

**Exercise 17.61.** Let *G* be  $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$  where we consider  $\mathbb{Z}$  as a subgroup of  $\mathbb{Z}^2$  by sending *n* to (n, 0). Compute  $H_*(G)$ ,  $K_*(C_r^*(G; \mathbb{C}))$ , and  $KO_*(C_r^*(G; \mathbb{R}))$ .

#### 17.5.15 Right Angled Coxeter Groups

Recall that a *right-angled Coxeter group* is a graph product W = W(X, W) for which each of the groups  $W_v$  is cyclic of order two. The group homology, the algebraic *K*- and *L*-groups, and the topological *K*-groups of right-angled Coxeter groups, and, more generally, of graph products are computed in [570, Section 7]. The results are nearly as explicit as in the case of right-angled Artin groups which we have presented in Subsection 17.5.14.

For instance, the integral group homology  $H_n(W; \mathbb{Z})$  is in degree  $n \ge 1$  an explicit  $\mathbb{F}_2$ -vector space,  $K_n(\mathbb{Z}W) = 0$  for  $n \le -1$ ,  $\widetilde{K}_0(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = 0$ , and  $K_1(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = 0$ . Next we state the result for the topological *K*-theory.

**Theorem 17.62 (The topological** *K***-theory of right-angled Coxeter groups).** *There are for every*  $n \in \mathbb{Z}$  *isomorphisms* 

$$\bigoplus_{\sigma \in \mathcal{P}} K_n(\mathbb{C}) \xrightarrow{\cong} K_n(C_m^*(W)) \cong K_n(C_r^*(W));$$
$$\bigoplus_{\sigma \in \mathcal{P}} KO_n(\mathbb{R}) \xrightarrow{\cong} KO_n(C_m^*(W;\mathbb{R})) \cong KO_n(C_r^*(W;\mathbb{R})).$$

In particular, there are isomorphisms of abelian groups

$$K_n(C_m^*(W)) \cong K_n(C_r^*(W)) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \text{ is even;} \\ \{0\} & \text{otherwise;} \end{cases}$$
$$KO_n(C_m^*(W;\mathbb{R})) \cong KO_n(C_r^*(W;\mathbb{R})) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \equiv 0 \mod 4; \\ (\mathbb{Z}/2)^r & \text{if } n \equiv 1,2 \mod 8; \\ \{0\} & \text{otherwise,} \end{cases}$$

where r is the number of simplices (including the empty simplex) in  $\mathcal{P}$ .

The computation of the topological *K*-theory of the complex reduced group  $C^*$ -algebra of a right-angled Coxeter group is also done by Sanchez-Garcia [886] using the Davis complex as a model for <u>*E*</u>*W*. The real case is treated by Fuentes [409].

**Exercise 17.63.** Let *G* be a group that is isomorphic to some amalgamated free product of the form  $(\mathbb{Z}/2)^3 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$ . Compute  $K_n(C_r^*(G; \mathbb{C}))$  and  $KO_n(C_r^*(G; \mathbb{R}))$  for  $n \in \mathbb{Z}$ .

# 17.5.16 Fundamental Groups of 3-Manifolds

The algebraic *K*-theory  $K_n(R[\pi_1(M)])$  has been computed for a compact connected 3-manifold *M* in [534] based on Theorem 16.1 (ie) and [527] modulo Nil-terms of the ring *R*. We at least present the computation for an already interesting special case, also including the algebraic *L*-theory.
**Theorem 17.64** (*K*-and *L*-groups of 3-manifolds). Let *M* be a compact connected orientable 3-manifold with fundamental group  $\pi$  and prime decomposition  $M \cong M_1 \# M_2 \# \cdots \# M_r$ .

(i) Suppose that *R* is a regular ring. Then we get for  $n \in \mathbb{Z}$ 

$$\overline{K}_n(R\pi) \cong \bigoplus_{i=1}^n \overline{K}_n(R[\pi_1(M_i)]);$$
$$K_n(R\pi) \cong 0 \quad if \ n \le -1,$$

where  $\overline{K}_n(RG)$  is the cokernel of the split injective map  $K_n(R) \to K_n(RG)$ . If  $\pi$  is torsionfree, then there is an isomorphism

$$H_n(B\pi; \mathbf{K}_R) \xrightarrow{\cong} K_n(R\pi);$$

(ii) Let *R* be a ring with involution. Suppose that  $\pi$  contains no 2-torsion. We get for  $n \in \mathbb{Z}$ 

$$\overline{L}_{n}^{\langle -\infty \rangle}(R\pi) \cong \bigoplus_{i=1}^{n} \overline{L}_{n}^{\langle -\infty \rangle}(R[\pi_{1}(M_{i})])$$

where  $\overline{L}_n^{\langle -\infty \rangle}(RG)$  is the cokernel of the split injective map  $L_n^{\langle -\infty \rangle}(R) \to L_n^{\langle -\infty \rangle}(RG)$ .

If  $\pi$  is torsionfree, then there is an isomorphism

$$H_n(B\pi; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(R\pi).$$

*Proof.* We conclude from Theorem 16.1 (ie) that  $\pi$  satisfies the Full Farrell-Jones Conjecture 13.30.

Note that  $\pi \cong *_{i=1}^r \pi_1(M_i)$ . The Kurosh Subgroup Theorem, see [693, Theorem 1.10 on page 178], says for a subgroup  $H \subseteq \pi$  that  $H \cong (*_{j \in J}H_j) * F$  where each  $H_j$  is the intersection of H with some conjugate of  $\pi_1(M_i)$  and F is a free group. Note that  $\pi_1(M_i)$  is either finite or torsionfree since every irreducible 3-manifold with infinite fundamental group is aspherical by the Sphere Theorem, see [477, 4.3 on page 40], and a prime 3-manifold that is not irreducible is a  $S^2$  bundle over  $S^1$ , see [477, Lemma 3.13 on page 28]. Every torsionfree virtually cyclic group is isomorphic to  $\mathbb{Z}$ . A virtually cyclic group V is isomorphic to a non-trivial free product  $L_1 * L_2$  if and only if V is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}/2 * \mathbb{Z}/2$ .

(i) Since *R* is regular, we conclude from Lemma 13.54 and Lemma 13.55 that the assembly map

$$H_n^{\pi}(\underline{E}\pi;\mathbf{K}_R)\to K_n(R\pi)$$

is an isomorphism for  $n \in \mathbb{Z}$ . We conclude from Example 15.30 that the obvious map  $\bigoplus_{i=1}^{n} \overline{K}_n(R[\pi_1(M_i)]) \to \overline{K}_n(R\pi)$  is bijective. The claim in the special case that  $\pi$  is torsionfree follows from Conjecture 6.53, which holds for  $\pi$  by Theorem 13.65 (xii).

(ii) The assembly map

$$H_n^{\pi}(\underline{E}\pi; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R\pi)$$

is an isomorphism by Theorem 13.60 since every virtually cyclic subgroup of  $\pi$  is isomorphic to  $\mathbb{Z}$ . We conclude from Example 15.30 that the obvious map  $\bigoplus_{i=1}^{n} \overline{L}_{n}^{\langle -\infty \rangle}(R[\pi_{1}(M_{i})]) \rightarrow \overline{L}_{n}^{\langle -\infty \rangle}(R\pi)$  is bijective. The claim in the special case that  $\pi$  is torsionfree follows from Conjecture 9.114, which holds for  $\pi$  by Theorem 13.65 (xii).

**Exercise 17.65.** Let *M* be a connected orientable irreducible closed 3-manifold with infinite fundamental group  $\pi$ . Show that  $L_n^{\langle i \rangle}(\mathbb{Z}\pi)$  is independent of the decoration and that we have isomorphisms

$$L_0(\mathbb{Z}\pi) \cong \mathbb{Z} \oplus \hom_{\mathbb{Z}}(\pi, \mathbb{Z}/2);$$
  

$$L_1(\mathbb{Z}\pi) \cong \pi/[\pi, \pi] \oplus \mathbb{Z}/2;$$
  

$$L_2(\mathbb{Z}\pi) \cong \mathbb{Z}/2 \oplus \hom_{\mathbb{Z}}(\pi, \mathbb{Z});$$
  

$$L_3(\mathbb{Z}\pi) \cong \mathbb{Z} \oplus (\pi/[\pi, \pi] \otimes_{\mathbb{Z}} \mathbb{Z}/2)$$

#### **17.6 Applications of Some Computations**

#### 17.6.1 Classification of Some C\*-algebras

Theorem 17.47 is an important input in the classification of certain  $C^*$ -algebras associated to number fields by Li-Lück [629]. Here the key point is the rather surprising result that the topological *K*-groups are all torsionfree, which is not the case for the group homology. Actually, it is intriguing that the topological complex *K*-groups are finitely generated free abelian groups in many of the examples presented in Subsection 17.5, see also Question 17.50.

Another application of the computation of the topological *K*-theory of group  $C^*$ -algebras can be found in [324], namely, to the structure of crossed products of irrational rotation algebras by finite subgroups of  $SL_2(\mathbb{Z})$ .

#### 17.6.2 Unstable Gromov-Lawson Rosenberg Conjecture

We have already discussed in Subsection 14.8.4 that Schick [895] constructed counterexamples to the unstable version of the Gromov-Lawson-Rosenberg Conjecture with fundamental group  $\pi \cong \mathbb{Z}^4 \times \mathbb{Z}/3$ . However for appropriate  $\rho: \mathbb{Z}/3 \to \operatorname{aut}(\mathbb{Z}^4)$ 

the unstable version does hold for  $\pi \cong \mathbb{Z}^4 \rtimes_{\rho} \mathbb{Z}/3$  and dim $(M) \ge 5$ . This is proved by Davis-Lück [282, Theorem 0.7 and Remark 0.9] based on explicit calculations of the topological *K*-theory of the reduced real group  $C^*$ -algebra of  $\mathbb{Z}^4 \rtimes_{\rho} \mathbb{Z}/3$ . More infinite groups for which the unstable version holds are presented in [507, Theorem 6.3].

# 17.6.3 Classification of Certain Manifolds with Infinite Not Torsionfree Fundamental Groups

Manifolds homotopy equivalent to the total space of certain fiber bundles over lens spaces with tori as fiber are classified by Davis-Lück [283]; see also [988]. Here the key input is the calculation of algebraic *K*-and *L*-groups of integral group rings of groups of the shape  $\pi = \mathbb{Z} \rtimes_p \mathbb{Z}/p$  for odd primes *p* where the conjugation action of  $\mathbb{Z}/p$  on  $\mathbb{Z}^n$  is free outside the origin. Note that  $\pi$  is infinite and not torsionfree. This is one of the few classification result about a class of closed manifolds whose fundamental group is not obtained from torsionfree and finite groups using amalgamated free products and HNN-extensions.

# **17.7 Notes**

The lower and middle algebraic *K*-theory of integral group rings of certain reflection groups has been computed by Lafont-Ortiz [607] and by Lafont-Margurn-Ortiz [605], of  $\Gamma_3 := O^+(3, 1) \cap \text{GL}_4(\mathbb{Z})$  by Ortiz [781, 782], of Bianchi groups by Berkove-Farrell-Pineda-Pearson [125], and of pure braid groups by Aravinda-Farrell-Roushon [37]. The lower and middle algebraic *K*-theory of integral group rings of mapping class group of genus 1 is computed in [126]. The topological *K*-theory of the complex group *C*\*-algebra of a cocompact 3-dimensional hyperbolic reflection group is computed by Lafont-Ortiz-Rahm-Sanchez-Gracia [608]. Computations of the algebraic *K*-groups  $K_n(RG)$  for Artin groups *G* of dihedral type can be found in [33].

Some necessary conditions on a group G for which  $\mathbb{Q} \otimes_{\mathbb{Z}} Wh(G)$  vanishes can be found in Lück-Oliver [672].

# Chapter 18 Assembly Maps

# **18.1 Introduction**

In this chapter we discuss assembly maps and the assembly principle in general.

We recall the homological approach in Section 18.2, which we have used in this book.

We give the version in terms of spectra in Section 18.3. Actually, in all concrete situations, such as in the Farrell-Jones Conjecture for K- and L-theory and pseudoisotopy or the Baum-Connes Conjecture, the assembly map can be implemented in terms of spectra. This can easily be identified with the elementary approach in terms of homotopy colimits, which nicely illustrates the name assembly, but works only if we confine ourselves to classifying spaces of families of subgroups, see Section 18.4. The approach in terms of homotopy colimits is the quickest and most natural approach for a homotopy theorist.

The universal property of assembly is explained in Section 18.5. Roughly speaking, it says that the assembly map is the best approximation of a weakly homotopy invariant functor  $E: G-CW-COM \rightarrow SPECTRA$  from the left by a weakly excisive functor  $G-CW-COM \rightarrow SPECTRA$ , where weakly excisive essentially means that after taking homotopy groups the functor yields a *G*-homology theory. This is very helpful to identify the various versions of the assembly maps appearing in the literature with our homological approach, since the constructions of the assembly maps can be very complicated and it is much easier to use the universal property to establish the desired identifications than to go through the actual definitions. The universal property will be exploited to identify the various assembly maps in Section 18.6.

This universal approach explains the philosophical background of assembly and presents a uniform approach to the assembly map in all cases, such as the Farrell-Jones Conjecture or the Baum-Conjecture. It is important to have the other more geometric or operator-theoretic definitions of assembly maps in terms of surgery theory or index theory at hand, in order to apply the Farrell-Jones Conjecture and the Baum-Connes Conjecture to geometric problems, such as the topological rigidity of closed aspherical manifolds or the existence of a Riemannian metric with positive scalar curvature.

The homological or homotopy theoretic approach to assembly maps is best suited for computations based on the Isomorphism Conjectures and for proofs of inheritance properties, but not necessarily for their proofs for specific classes of groups such as hyperbolic groups or CAT(0)-groups, where the approach using index theory or controlled topology come into play.

#### **18.2 Homological Approach**

The homological version of assembly is manifested in the Meta-Isomorphism Conjecture 15.2. Recall that it predicts for a group G, a family  $\mathcal{F}$  of subgroups of G, and a G-homology theory  $\mathcal{H}^G_*$  in the sense of Definition 12.1 that the map induced by the projection pr:  $E_{\mathcal{F}}(G) \to G/G$ , for  $E_{\mathcal{F}}(G)$  the classifying space of the family  $\mathcal{F}$  in the sense of Definition 11.18,

(18.1) 
$$\mathcal{H}_n(\mathrm{pr}): \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

is bijective for all  $n \in \mathbb{Z}$ . The various conjectures due to Baum-Connes and Farrell-Jones are special cases where one explicitly specifies  $\mathcal{F}$  and  $\mathcal{H}^G_*$ .

#### **18.3** Extension from Homogenous Spaces to *G*-*CW*-Complexes

Let **E** be a covariant Or(G)-spectrum, i.e., a covariant functor **E**:  $Or(G) \rightarrow$ SPECTRA. We get an extension of **E** to the category *G*-CW-COM of *G*-CWcomplexes by

(18.2)  $\mathbf{E}_{\mathbb{Y}_{0}}: G\text{-}CW\text{-}COM \to SPECTRA, \quad X \mapsto \operatorname{map}_{G}(-, X)_{+} \wedge_{\operatorname{Or}(G)} \mathbf{E},$ 

where  $\operatorname{map}_{G}(-, X)$  and  $\wedge_{\operatorname{Or}(G)}$  have been defined in Example 12.24 and in (12.25). The projection pr:  $E_{\mathcal{F}}(G) \to G/G$  for  $E_{\mathcal{F}}(G)$  induces a map of spectra

(18.3)  $\mathbf{E}_{\mathscr{V}}(\mathbf{pr}) \colon \mathbf{E}_{\mathscr{V}}(E_{\mathscr{F}}(G)) \to \mathbf{E}_{\mathscr{V}}(G/G).$ 

After taking homotopy groups we get for all  $n \in \mathbb{Z}$  a homomorphism

(18.4) 
$$\pi_n(\mathbf{E}_{\%}(\mathbf{pr})): \pi_n(\mathbf{E}_{\%}(E_{\mathcal{F}}(G))) \to \pi_n(\mathbf{E}_{\%}(G/G)).$$

We have constructed a *G*-homology theory  $H^G_*(-; \mathbf{E})$  with the property that  $H^G_n(G/H; \mathbf{E}) \cong \pi_n(\mathbf{E}(G/H))$  holds for all  $n \in \mathbb{Z}$  and all subgroups  $H \subseteq G$  in Theorem 12.27. The *G*-homology theories relevant for the Baum-Connes and the Farrell-Jones Conjecture are given by specifying such covariant functors  $\mathbf{E}$ . It follows essentially from the definitions that the map (18.1) for  $\mathcal{H}^G_* = H^G_*(-; \mathbf{E})$  agrees with the map (18.4).

# **18.4 Homotopy Colimit Approach**

Consider a covariant functor  $\mathbf{E} \colon \operatorname{Or}(G) \to \operatorname{SPECTRA}$ . Recall that  $\operatorname{Or}_{\mathcal{F}}(G)$  denotes the  $\mathcal{F}$ -restricted orbit category, see Definition 2.64. If the *G*-homology theory  $\mathcal{H}^G_*$ is given by  $H^G_*(-; \mathbf{E})$ , one can identify the assembly map (18.4) with the map 18.5 Universal Property

(18.5) 
$$\pi_n(\mathbf{p}) \colon \pi_n (\operatorname{hocolim}_{\operatorname{Or}_{\mathcal{F}}(G)} \mathbf{E}) \to \pi_n(\mathbf{E}(G/G))$$

where the map of spectra

**p**:  $\operatorname{hocolim}_{\operatorname{Or}_{\mathcal{F}}(G)} \mathbf{E} \to \operatorname{hocolim}_{\operatorname{Or}(G)} \mathbf{E} = \mathbf{E}(G/G)$ 

comes from the inclusion of categories  $\operatorname{Or}_{\mathcal{F}}(G) \to \operatorname{Or}(G)$  and the fact that G/G is a terminal object in  $\operatorname{Or}(G)$ . For more information about homotopy colimits and the identification of the maps (18.1), (18.4), and (18.5), we refer to [280, Sections 3 and 5].

This interpretation is one explanation for the name *assembly*. If the assembly map (18.5) is bijective for all  $n \in \mathbb{Z}$ , or, equivalently, the map **p** above is a weak homotopy equivalence, we have a recipe to assemble  $\mathbf{E}(G/G)$  from its values  $\mathbf{E}(G/H)$ , where H runs through  $\mathcal{F}$ . The idea is that  $\mathcal{F}$  consists of well-understood subgroups, for which one knows the values  $\mathbf{E}(G/H)$  for  $H \subseteq G$  and hence hocolim $O_{\Gamma_{\mathcal{F}}(G)} \mathbf{E}$ , whereas  $\mathbf{E}(G/G)$  is the object which one wants to understand and is very hard to access.

# **18.5 Universal Property**

In this section we characterize assembly maps by a universal property. This is useful for identifying different constructions of assembly maps.

**Lemma 18.6.** Let **E** be a covariant Or(G)-spectrum. Then:

(i) The canonical map

$$\mathbf{E}_{\%}(X) \cup_{\mathbf{E}_{\%}(f)} \mathbf{E}_{\%}(Y) \to \mathbf{E}_{\%}(X \cup_{f} Y)$$

is an isomorphism of spectra where (X, A) is a G-CW-pair and  $f : A \to Y$  is a cellular G-map;

(ii) The canonical map

$$\operatorname{colim}_{i \in I} \mathbf{E}_{\%}(X_i) \to \mathbf{E}_{\%}(X)$$

is an isomorphism of spectra where  $\{X_i \mid i \in I\}$  is a directed system of *G*-*C*W-subcomplexes of the *G*-*C*W-complex *X* directed by inclusion and satisfying  $X = \bigcup_{i \in I} X_i$ ;

(iii) The canonical map

$$Z_+ \wedge \mathbf{E}_{\%}(X) \to \mathbf{E}_{\%}(Z \times X)$$

*is an isomorphism of spectra where Z is a CW-complex (with trivial G-action) and X is a G-CW-complex;* 

(iv) The canonical map

 $\mathbf{E}_{\mathbb{G}}(G/H) \to \mathbf{E}(G/H)$ 

is an isomorphism of spectra for all  $H \in \mathcal{F}$ .

*Proof.* One easily checks that the *H*-fixed point set functor  $\operatorname{map}_G(G/H, -)$  commutes with the passage from a *G*-*CW*-pair (*X*, *A*) and a cellular *G*-map  $f: A \to Y$  to  $X \cup_f Y$  and with directed unions of *G*-*CW*-subcomplexes. Now assertions (i) and (ii) follow from the fact that  $- \wedge_{\operatorname{Or}(G)} \mathbf{E}$  commutes with colimits, since it has a right adjoint, see [280, Lemma 1.5]. Assertions (iii) and (iv) follow by inspecting the definition of  $\mathbf{E}_{\mathcal{G}}$ .

**Lemma 18.7.** Let  $\mathbf{E}$  be a covariant Or(G)-spectrum. Then the extension  $\mathbf{E} \mapsto \mathbf{E}_{\mathcal{H}}$  is uniquely determined up to isomorphism of G-CW-COM-spectra by the properties of Lemma 18.6.

*Proof.* Let  $\mathbf{E} \mapsto \mathbf{E}_{\$}$  be another such extension. There is an (a priori not necessarily continuous) set-theoretic natural transformation

$$\mathbf{T}(X) : \mathbf{E}_{\%}(X) = X_{+} \wedge_{\operatorname{Or}(G)} \mathbf{E} \longrightarrow \mathbf{E}_{\$}(X)$$

which sends an element represented by  $(x: G/H \longrightarrow X, e)$  in  $\operatorname{map}_G(G/H, X) \times \mathbf{E}(G/H)$  to  $\mathbf{E}_{\$}(x)(e)$ . Since any *G*-*CW*-complex is constructed from orbits G/H with  $H \in \mathcal{F}$  via products with disks and disjoint unions, attaching a *G*-space to a *G*-space along a *G*-map, and is the directed union over its skeletons, and  $\mathbf{T}(G/H)$  is an isomorphism of spectra for  $H \subseteq G$ ,  $\mathbf{T}(X)$  is an isomorphism of spectra for all *G*-*CW*-complexes *X*.

Lemma 18.7 is a characterization of  $\mathbf{E} \mapsto \mathbf{E}_{\mathscr{G}}$  up to isomorphism. Next we give a homotopy theoretic characterization.

Definition 18.8 ((Weakly) excisive). We call a covariant functor

$$E: G-CW-COM \rightarrow SPECTRA$$

(*weakly*) homotopy invariant if it sends G-homotopy equivalences to (weak) homotopy equivalences of spectra.

The functor **E** is (*weakly*) *excisive* if it has the following four properties:

- It is (weakly) homotopy invariant;
- The spectrum  $\mathbf{E}(\emptyset)$  is (weakly) contractible;
- It respects homotopy pushouts up to (weak) homotopy equivalence, i.e., if the *G*-*CW*-complex X is the union of *G*-*CW*-subcomplexes  $X_1$  and  $X_2$  with intersection  $X_0$ , then the canonical map from the homotopy pushout of  $\mathbf{E}(X_2) \leftarrow \mathbf{E}(X_0) \longrightarrow \mathbf{E}(X_2)$  to  $\mathbf{E}(X)$  is a (weak) homotopy equivalence of spectra;
- It respects disjoint unions up to (weak) homotopy, i.e., the natural map  $\bigvee_{i \in I} \mathbf{E}(X_i) \to \mathbf{E}(\coprod_{i \in I} X_i)$  is a (weak) homotopy equivalence for all index sets *I*.

**Exercise 18.9.** Let E: CW-COM  $\rightarrow$  SPECTRA be an excisive functor for the trivial group. Show that the functor *G*-CW-COM  $\rightarrow$  SPECTRA sending *X* to  $\mathbf{E}(X/G)$  is excisive.

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Notation 18.10. If E: *G*-CW-COM  $\rightarrow$  SPECTRA is a covariant functor, we denote  $(\mathbf{E}|_{Or(G)})_{\%}$  by  $\mathbf{E}_{\%}$  again where  $\mathbf{E}|_{Or(G)}$  is the composite of E with the obvious inclusion  $Or(G) \rightarrow G$ -CW-COM.

The following result has been proved for  $G = \{1\}$  in Weiss-Williams [1002].

#### Theorem 18.11 (Universal Property of assembly).

- (i) Suppose that  $\mathbf{E}: \operatorname{Or}(G) \to \operatorname{SPECTRA}$  is a covariant functor. Then  $\mathbf{E}_{\mathbb{Y}_0}$  is excisive;
- (ii) Suppose that  $\mathbf{E} \colon \operatorname{Or}(G) \to \operatorname{SPECTRA}$  is a covariant functor. Then we obtain a G-homology theory  $H_n^G(-; \mathbf{E})$  in the sense of Definition 12.1 from Theorem 12.27, and we get for every pair (X, A) of G-CW-complexes (X, A) a natural isomorphism

$$H_n^G(X, A; \mathbf{E}) \cong \operatorname{coker}(\pi_n(\mathbf{E}_{\mathbb{V}}(\emptyset_+))) \to \pi_n(\mathbf{E}_{\mathbb{V}}(X/A))).$$

If  $A = \emptyset$ , this becomes an isomorphism

$$H_n^G(X;\mathbf{E}) \cong \pi_n(\mathbf{E}_{\%}(X));$$

(iii) Let  $\mathbf{T} \colon \mathbf{E} \to \mathbf{F}$  be a transformation of (weakly) excisive functors  $\mathbf{E}$  and  $\mathbf{F}$  from *G*-CW-COM to SPECTRA so that  $\mathbf{T}(G/H)$  is a (weak) homotopy equivalence of spectra for all  $H \subseteq G$ .

Then  $\mathbf{T}(X)$  is a (weak) homotopy equivalence of spectra for all G-CWcomplexes X;

(iv) For every (weakly) homotopy invariant functor **E** from G-CW-COM to SPECTRA, there is a (weakly) excisive functor

$$\mathbf{E}^{\vee_0}$$
: G-CW-COM  $\rightarrow$  SPECTRA

and natural transformations

$$\begin{aligned} \mathbf{A}_{\mathbf{E}} \colon \mathbf{E}^{\%} &\to \mathbf{E}; \\ \mathbf{B}_{\mathbf{E}} \colon \mathbf{E}^{\%} &\to \mathbf{E}_{\%} \end{aligned}$$

which induce (weak) homotopy equivalences of spectra  $\mathbf{A}_{\mathbf{E}}(G/H)$  for all  $H \subseteq G$  and (weak) homotopy equivalences of spectra  $\mathbf{B}_{\mathbf{E}}(X)$  for all G-CW-complexes X.

The constructions  $\mathbf{E}_{\%}$ ,  $\mathbf{E}^{\%}$ ,  $\mathbf{A}_{\mathbf{E}}$ , and  $\mathbf{B}_{\mathbf{E}}$  are natural in  $\mathbf{E}$ .

Moreover, **E** is (weakly) excisive if and only if  $A_E(X)$  is a (weak) homotopy equivalence of spectra for all *G*-CW-complexes *X*.

*Proof.* (i) follows from Lemma 18.6 after one has shown that in the situation of Lemma 18.6 (i) the canonical map from the homotopy pushout of spectra to the pushout of spectra is a weak homotopy equivalence. This follows from the fact that the inclusion of  $\mathbf{E}_{\mathbb{Y}_{0}}(A) \to \mathbf{E}_{\mathbb{Y}_{0}}(X)$  is on each level a cofibration of spaces.

(ii) There is an obvious *G*-homotopy equivalence of pointed *G*-*CW*-complexes  $X_+ \cup_{A_+} \operatorname{cone}(A_+) \to X/A$ . Hence we get from the definitions

$$H_n^G(X, A; \mathbf{E}) = \pi_n (\operatorname{map}_G(-, X/A) \wedge_{\operatorname{Or}(G)} \mathbf{E}).$$

Now the assertion follows from the cofibration sequence of spectra

$$\mathbf{E}_{\%}(\emptyset_{+}) = \operatorname{map}_{G}(-, \emptyset_{+})_{+} \wedge_{\operatorname{Or}(G)} \mathbf{E} \to \mathbf{E}_{\%}(X/A) = \operatorname{map}_{G}(-, X/A)_{+} \wedge_{\operatorname{Or}_{\mathcal{F}}(G)} \mathbf{E} \to \operatorname{map}_{G}(-, X/A) \wedge_{\operatorname{Or}_{\mathcal{F}}(G)} \mathbf{E}.$$

(iii) Use the fact that a (weak) homotopy colimit of homotopy equivalences of spectra is again a (weak) homotopy equivalence of spectra.

(iv) See [280, Theorem 6.3].

**Exercise 18.12.** Show that a covariant functor  $\mathbf{E}: G$ -CW-COM  $\rightarrow$  SPECTRA is weakly excisive if and only if the assignment sending a pair (X, A) of *G*-CW-complexes to coker $(\pi_n(\mathbf{E}(\emptyset_+)) \rightarrow \pi_n(\mathbf{E}(X/A)))$  defines a *G*-homology in the sense of Definition 12.1.

**Exercise 18.13.** Let E: *G*-CW-COM  $\rightarrow$  SPECTRA be a weakly excisive functor such that  $\pi_n(\mathbf{E}(G/H))$  is finitely generated for every  $H \subseteq G$  and  $n \in \mathbb{Z}$ . Then  $\pi_n(\mathbf{E}(X))$  is finitely generated for every finite *G*-CW-complex *X* and  $n \in \mathbb{Z}$ .

**Definition 18.14 (Homotopy theoretic assembly transformation).** Given a covariant functor E: G-CW-COM  $\rightarrow$  SPECTRA, we call the transformation appearing in Theorem 18.11 (iv)

$$A_E: E^{\circ} \to E$$

the homotopy theoretic assembly transformation.

**Remark 18.15 (No continuity condition E).** One may be tempted to define a natural transformation  $\mathbf{S} \colon \mathbf{E}_{\%} \to \mathbf{E}$  as indicated in the proof of Lemma 18.7. Then  $\mathbf{S}(X)$  is a well-defined bijection of sets but is not necessarily continuous because we do not want to assume that  $\mathbf{E}$  is continuous, i.e., that the induced map from  $\hom_C(X,Y)$  to  $\hom_C(\mathbf{E}(X), \mathbf{E}(Y))$  is continuous for all *G-CW*-complexes *X* and *Y*. This is the reason why we have to pass to the more complicated construction of  $\mathbf{E}^{\%}$  and only obtain a zigzag

$$\mathbf{E}_{\%} \xleftarrow{\mathbf{B}_{\mathbf{E}}} \mathbf{E}^{\%} \xrightarrow{\mathbf{A}_{\mathbf{E}}} \mathbf{E},$$

which suffices for all our purposes. The construction of this zigzag uses the (weak) homotopy invariance of **E** and does not require any continuity condition for **E**.

Theorem 18.11 implies the following corollary.

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18.5 Universal Property

**Corollary 18.16.** Let  $\mathbf{E}$ : *G*-CW-COM  $\rightarrow$  SPECTRA be a weakly excisive functor. Denote by  $\mathbf{E}|_{O_{\mathsf{T}}(G)}$  its restriction to a covariant functor  $Or(G) \rightarrow$  SPECTRA.

Then we obtain for every  $n \in \mathbb{Z}$  and every G-CW-complex X an isomorphism, natural in X,

$$\pi_n(\mathbf{E}(X)) \xrightarrow{=} H_n^G(X; \mathbf{E}|_{\mathrm{Or}(G)}).$$

In particular, we get for every family of subgroups  $\mathcal{F}$  and  $n \in \mathbb{Z}$  a commutative diagram with isomorphisms as vertical arrows

Exercise 18.17. Consider the covariant functor

**E**: *G*-CW-COM 
$$\rightarrow$$
 SPECTRA,  $X \mapsto \mathbf{K}_R(\Pi(EG \times_G X))$ 

where  $\Pi(EG \times_G X)$  is the fundamental groupoid of the space  $EG \times_G X$  and  $\mathbf{K}_R$ : GROUPOIDS  $\rightarrow$  SPECTRA has been defined in (12.44). Suppose that **E** is weakly excisive.

Show that then for every family  $\mathcal{F}$  of subgroups the assembly map induced by the projection  $E_{\mathcal{F}}(G) \to G/G$ 

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective for all  $n \in \mathbb{Z}$ .

**Remark 18.18 (Universal property of the homotopy theoretic assembly transformation).** Next we explain why Theorem 18.11 characterizes the homotopy theoretic assembly map

$$A_E : E^{\%} \longrightarrow E$$

in the sense that it is the universal approximation from the left by a (weakly) excisive functor of a (weakly) homotopy invariant functor  $\mathbf{E}$  from *G*-CW-COM to SPECTRA up to (weak) homotopy equivalence. Namely, let  $\mathbf{T} \colon \mathbf{F} \to \mathbf{E}$  be a transformation of covariant functors from *G*-CW-COM to SPECTRA such that  $\mathbf{F}$  is (weakly) excisive. Then for any *G*-CW-complex *X* the following diagram commutes



and  $A_F(X)$  is a (weak) homotopy equivalence by Theorem 18.11 (iv). Hence one may say that T(X) factorizes over  $A_E(X)$  up to (weak) homotopy equivalence.

Suppose additionally that T(G/H) is a (weak) homotopy equivalence for all  $H \subseteq G$ . Then both  $A_F(X)$  and  $T^{\%}(X)$  are (weak) homotopy equivalences by Theorem 18.11 (iii) and (iv). In particular, we obtain for every *G*-*CW*-complex *X* a commutative diagram with an isomorphism as vertical arrow



# **18.6 Identifying Assembly Maps**

In this section we explain and summarize that we can identify all the various assembly maps we have studied so far.

We recall that we have the following versions of assembly maps.

• The Meta-Isomorphism Conjecture 15.2 with respect to the *G*-homology theory  $\mathcal{H}^G_*$  and the family  $\mathcal{F}$  of subgroups of *G*, where the assembly map

$$\mathcal{H}_n(\mathrm{pr}) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(G/G)$$

comes from the projection pr:  $E_{\mathcal{F}}(G) \to G/G$ ;

- The Meta-Isomorphism Conjecture 15.2, where  $\mathcal{H}^G_*$  is the value at G of the equivariant homology theory  $\mathcal{H}^2_*$  coming from a functor GROUPOIDS  $\rightarrow$  SPECTRA respecting equivalences, see Theorem 12.30 and Section 12.5;
- The Meta-Isomorphism Conjecture 15.38 for functors from spaces to spectra;
- The homotopy theoretic assembly transformation in the sense of Definition 18.14;
- For the *L*-theoretic Farrell-Jones Conjecture and *G* the fundamental group of an aspherical closed manifold, the assembly map given by taking surgery obstructions, see the sketch of the proof of Theorem 9.171 in Subsection 9.15.3;
- For the Baum-Connes Conjecture in terms of index theory, see Section 14.2.

Remark 18.19 (The homotopy theoretic assembly transformation and the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients). Consider a functor  $S: SPACES \rightarrow SPECTRA$  which respects weak equivalences and disjoint unions. Given a group *G* and a free *G*-*CW*-complex *Z*, we get a functor

$$\mathbf{S}_Z^G: G\text{-}CW\text{-}COM \to SPECTRA, \quad X \mapsto \mathbf{S}(X \times_G Z)$$

#### 18.6 Identifying Assembly Maps

whose restriction to Or(G) is denoted in the same way and has already been introduced in (15.40). The Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients predicts for a family  $\mathcal{F}$  of subgroups of G that the map

(18.20) 
$$H_n^G(\mathrm{pr}; \mathbf{S}_Z^G) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{S}_Z^G) \to \mathcal{H}_n^G(G/G; \mathbf{S}_Z^G)$$

induced by the projection pr:  $E_{\mathcal{F}}(G) \to G/G$  is bijective for all  $n \in \mathbb{Z}$ . This map can be identified with the map

(18.21) 
$$\pi_n\left((\mathbf{S}_Z^G)^{\mathscr{G}_0}(\mathrm{pr})\right) \colon \pi_n\left((\mathbf{S}_Z^G)^{\mathscr{G}_0}(E_{\mathcal{F}}(G))\right) \to \pi_n\left((\mathbf{S}_Z^G)^{\mathscr{G}_0}(G/G)\right)$$

and with the map induced on homotopy groups by the homotopy theoretic assembly map of Definition 18.14

(18.22) 
$$\pi_n \left( \mathbf{A}_{\mathbf{S}_Z^G}(E_{\mathcal{F}}(G)) \right) \colon \pi_n \left( (\mathbf{S}_Z^G)^{\mathbb{V}}(E_{\mathcal{F}}(G)) \right) \to \pi_n \left( \mathbf{S}_Z^G(E_{\mathcal{F}}(G)) \right)$$

by the following argument.

Because of Theorem 18.11 (ii) the map (18.20) can be identified with the map induced by the projection pr:  $E_{\mathcal{F}}(G) \to G/G$ 

$$\pi_n\big((\mathbf{S}_Z^G)_{\%}(\mathrm{pr})\big)\colon \pi_n\big((\mathbf{S}_Z^G)_{\%}(E_{\mathcal{F}}(G))\big) \to \pi_n\big((\mathbf{S}_Z^G)_{\%}(G/G)\big),$$

and hence by Theorem 18.11 (iv) with the map (18.21).

We have the following commutative diagram

The right vertical arrow is a weak homotopy equivalence by Theorem 18.11 (iv). Since *Z* is a free *G*-*CW*-complex and  $E_{\mathcal{F}}(G)$  is contractible (after forgetting the group action), the map id  $\times_G$  pr:  $Z \times_G E_{\mathcal{F}}(G) \to Z \times_G G/G$  is a homotopy equivalence and hence the lower horizontal arrow is a weak homotopy equivalence. Hence we get an identification of the maps (18.22) and (18.21). Thus we have identified the maps (18.20), (18.21), and (18.22).

Example 18.23 (The Farrell-Jones Conjecture and the Baum-Connes Conjecture in the setting of the homotopy theoretic assembly transformation). In the sequel  $\Pi(X)$  denotes the fundamental groupoid of a space X. If we take in Remark 18.19 the covariant functor S: SPACES  $\rightarrow$  SPECTRA to be the one which sends a space X to  $\mathbf{K}_R(\Pi(X))$  or  $\mathbf{L}_R^{\langle -\infty \rangle}(\Pi(X))$  respectively, see Theorem 12.43, then we conclude from Example 15.39 and Remark 18.19 that the assembly map appearing in the K-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R

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$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

or the assembly map appearing in the *L*-theoretic Farrell-Jones Conjecture 13.4 with coefficients in the ring with involution *R* 

$$H_n^G(\mathrm{pr})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(G/G;\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

respectively can be identified with the map

$$\pi_n \big( \mathbf{S}^G_{EG}(E_{\mathcal{VC}\mathcal{Y}}(G))^{\mathbb{Q}}(\mathrm{pr}) \big) \colon \pi_n \big( (\mathbf{S}^G_{EG})^{\mathbb{Q}}(E_{\mathcal{VC}\mathcal{Y}}(G)) \big) \to \pi_n \big( (\mathbf{S}^G_{EG})^{\mathbb{Q}}(G/G) \big)$$

and with the map induced on homotopy groups by the homotopy theoretic assembly map of Definition 18.14

$$\pi_n\big(\mathbf{A}_{\mathbf{S}_{EG}^G}(E_{\mathcal{VC}\mathcal{Y}}(G))\big):\pi_n\big((\mathbf{S}_{EG}^G)^{\mathscr{G}}(E_{\mathcal{VC}\mathcal{Y}}(G))\big)\to\pi_n\big(\mathbf{S}_{EG}^G(E_{\mathcal{VC}\mathcal{Y}}(G))\big).$$

The claims above directly extend to additive or higher categories as coefficients. In the Baum-Connes setting we get an identification of the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}^{\mathrm{TOP}}) \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{TOP}}) \to H_n^G(G/G; \mathbf{K}^{\mathrm{TOP}}) = K_n(C_r^*(G))$$

with the map

$$\pi_n \big( \mathbf{S}^G_{EG}(E_{\mathcal{F}IN}(G))^{\mathscr{V}_0}(\mathrm{pr}) \big) \colon \pi_n \big( (\mathbf{S}^G_{EG})^{\mathscr{V}_0}(E_{\mathcal{F}IN}(G)) \big) \\ \to \pi_n \big( (\mathbf{S}^G_{EG})^{\mathscr{V}_0}(G/G) \big)$$

and with the map induced on homotopy groups by the homotopy theoretic assembly map of Definition 18.14

$$\pi_n \big( \mathbf{S}^G_{EG}(E_{\mathcal{F}IN}(G))^{\varsigma_0}(\mathrm{pr}) \big) \colon \pi_n \big( (\mathbf{S}^G_{EG})^{\varsigma_0}(E_{\mathcal{F}IN}(G)) \big) \\ \to \pi_n \big( \mathbf{S}^G_{EG}(E_{\mathcal{F}IN}(G)) \big),$$

if we take  $\mathbf{S} = \mathbf{K}^{\text{TOP}}(\Pi(X))$ , see Theorem 12.43, and analogously in the real case.

We have explained in Remark 15.44 the identification of the original formulation of the fibered Farrell-Jones Conjecture for covariant functors from SPACES to SPECTRA, e.g., for pseudoisotopy, *K*-theory and *L*-theory, due to Farrell-Jones [366, Section 1.7 on page 262] with the setting we are using in the Meta-Isomorphism Conjecture 15.41 for functors from spaces to spectra with coefficients.

We have discussed the various Baum-Connes assembly maps and their relations already in Sections 14.2 and 14.3.

We have explained the relationship between the L-theoretic assembly map in terms of spectra, which we are using here, and the surgery obstruction map appearing in the geometric Surgery Exact Sequence in the sketch of the proof of Theorem 9.171 in Subsection 9.15.3.

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18.7 Notes

# **18.7** Notes

The Baum-Connes assembly maps in terms of localizations of triangulated categories are considered in [514, 515, 516, 716, 717, 718]. A categorial approach in terms of codescent is presented in [62].

Chain complex versions of the *L*-theoretic assembly map for additive categories are intensively studied by Ranicki [839] and Kühl-Macko-Mole [596, Section 11] emphasizing the aspect of comparing local Poincaré duality and global Poincaré duality.

The idea of the geometric assembly map is due to Quinn [823, 828] and its algebraic counterpart was introduced by Ranicki [839]. See also Loday [635]. The basic and uniform approach to assembly as presented in this chapter is sometimes called the Davis-Lück approach and was developed in [280].

For more information about assembly maps we refer for instance to the survey article [663].

# **Chapter 19 Motivation, Summary, and History of the Proofs of the Farrell-Jones Conjecture**

# **19.1 Introduction**

The purpose of this chapter is to present basic ideas and motivations for the proofs of the Farrell-Jones Conjecture and some information about their long history without getting lost in technical details. So it will be a soft introduction to the methods of proofs conveying only ideas. Moreover, we also want to provide some insight into why some input such as controlled topology, transfers, and flows occurs, which one might not expect at first glance since so far the assembly maps have been purely homotopy theoretic notions. We refer the interested reader, who wants to see more details, to Chapters 21, 22, 23, and 24.

We also want to explain why it is rather difficult to say something about all the proofs in full detail since the proofs and their methods have been moving targets. Many new ideas and technical modifications have been introduced over the last few decades, up to the present, so that sometimes the original ideas can hardly be recognized, and the overwhelming variety of different proofs cannot be presented in detail in a single book. The most advanced presentation of a framework of a proof will be given in Chapter 24, where we will work in the setting of higher categories as coefficients, which is more general than considering additive categories or rings as coefficients. We will not deal with the Farrell-Jones Conjecture for reductive p-adic groups, see Bartels-Lück [81, 83], which is the next level of complexity, since we confine ourselves in this book to discrete groups and do not consider topological groups.

The original formulation of the Farrell-Jones Conjecture appears in [366, 1.6 on page 257]. Of course it had many previous versions, some of them can be found in Subsection 13.11.1.

# **19.2 Homological Aspects**

We have already explained in the introduction of this book, see Chapter 1, how homological aspects concerning the topological *K*-theory  $K_*(C_r^*(G))$  of the reduced group  $C^*$ -algebra  $C_r^*(G)$  of *G* and the algebraic *K*-theory  $K_*(RG)$  and algebraic *L*-theory  $L_*^{\langle -\infty \rangle}(RG)$  of the group ring *RG* lead to the assembly maps 19 Motivation, Summary, and History of the Proofs of the Farrell-Jones Conjecture

$$\begin{split} K_n^G(E_{\mathcal{VC}\mathcal{Y}}(G)) &\xrightarrow{\cong} K_n(C_r^*(G)); \\ H_n^G(E_{\mathcal{VC}\mathcal{Y}}(G); \mathbf{K}_R) &\xrightarrow{\cong} K_n(RG); \\ H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) &\xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG). \end{split}$$

They appear in the Baum-Connes Conjecture 1.1 and the Farrell-Jones Conjectures 1.2 and 1.3, which predict that these assembly maps are bijections for all  $n \in \mathbb{Z}$ . Moreover we have explained in Chapter 18 that, after passing to the spectrum version of  $K_*(C_r^*(G))$ ,  $K_*(RG)$ , and  $L_*^{(-\infty)}(RG)$  as functors from the category of *G*-*CW*complexes to the category of spectra, these assembly maps are characterized by the universal property that they are the best approximation from the left by an excisive functor and do have interpretations in terms of homotopy colimits over the orbit category. So the first attempt to prove these conjectures is to show that these functors are excisive. However, this direct strategy has never really worked out, at least not without further sophisticated input. The problem is to isolate the reason why these functors are excisive in general. It is unclear which basic properties of the *K*- and *L*-theory of group rings or reduced group  $C^*$ -algebras guarantee excisiveness.

## **19.3** Constructing Detection maps

The next idea is just to construct an inverse to these assembly maps. In the Baum-Connes setting this is a successful strategy relying on the equivariant Kasparov product and the Dirac-Dual Dirac Method, see Section 25.2. In the Farrell-Jones setting this has nearly never worked out. The main reason is that it is hard to construct detecting maps with the algebraic K- or L-theory of group rings as source. There are interesting attempts to do this, most prominently the cyclotomic trace for the algebraic K-theory of group rings, or Chern characters for the topological K-theory of  $C^*$ -algebras with values in cyclic homology, but these give inverses to the assembly maps only in a very few instances. However, they can be used to show injectivity results, as explained in Sections 16.5 and 16.6. Note that surjectivity results are more valuable than injectivity results, since they give some insight about elements in the K- or L-groups under consideration and imply many other conjectures, whereas injectivity results only describe some portion of the K- or L-groups under consideration and do not have so many consequences, with the exception of the Novikov Conjecture, which is essentially an injectivity claim about assembly maps. Moreover, surjectivity results can often be turned into bijectivity results by considering relative versions.

In the Farrell-Jones setting the most successful method for proving bijectivity results is controlled topology, as motivated and explained next.

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# **19.4 Controlled Topology**

#### **19.4.1** Two Classical Results

Let  $\alpha$  be an open cover of a space Y. Two maps  $f, g: X \to Y$  are called  $\alpha$ -close if for every  $x \in X$  there is a  $U_x \in \alpha$  satisfying  $f(x), g(x) \in U_x$ . They are called  $\alpha$ -homotopic if there exists a homotopy  $h: X \times [0, 1] \to Y$  such that  $h_0 = f$  and  $h_1 = g$  hold and for every  $x \in X$  there is a  $U_x \in \alpha$  satisfying  $h(\{x\} \times [0, 1]) \subseteq U_x$ . A map  $f: X \to Y$  is an  $\alpha$ -domination if there is a map  $g: Y \to X$  such that  $f \circ g$  is  $\alpha$ -homotopic to the identity id<sub>Y</sub>. In such a situation, g is called a *right*  $\alpha$ -homotopy *inverse* for f. We call  $f: X \to Y$  an  $\alpha$ -homotopy equivalence if f is an  $\alpha$ -domination and, for some right  $\alpha$ -homotopy inverse g, the composite  $g \circ f$  is  $f^{-1}(\alpha)$ -homotopic to the identity id<sub>X</sub>, where  $f^{-1}(\alpha)$  denotes the cover  $\{f^{-1}(U) \mid U \in \alpha\}$  of X. We call g an  $\alpha$ -homotopy inverse of f.

Recall that a map  $f: X \to Y$  is *proper* if  $f^{-1}(C)$  is compact for every compact subset  $C \subseteq Y$ .

Obviously a homeomorphism  $f: X \to Y$  is an  $\alpha$ -homotopy equivalence for every  $\alpha$  and a proper map.

The next result is due to Chapman and Ferry, see [230].

**Theorem 19.1** ( $\alpha$ -Approximation Theorem). Let N be a topological manifold of dimension n and let  $\alpha$  be an open cover of N. Then there is an open cover  $\beta$  of N with the following property: If M is a topological manifold and  $f: (M, \partial M) \rightarrow (N, \partial N)$  is a proper  $\beta$ -homotopy equivalence of pairs such that either  $n \ge 6$  or  $(n \ge 5$  and  $\partial f$  is a homeomorphism) hold, then f is  $\alpha$ -close to a homeomorphism.

The following result is a special case of a theorem due to Ferry [379, Theorem 1]. Its proof relies on the  $\alpha$ -Approximation Theorem 19.1.

**Theorem 19.2.** Let M be a closed topological manifold of dimension  $n \ge 5$ . Equip M with a metric generating the given topology. Then there is an  $\epsilon > 0$  with the following property: Every surjective map  $f: M \to N$  to some closed manifold N of dimension n for which the diameter of  $f^{-1}(y)$  for every  $y \in N$  is less than  $\epsilon$  is homotopic to a homeomorphism.

The next result follows from Quinn [824, Theorem 2.7], which is closely related to the work of Chapman and Ferry [229, 230, 378].

Let  $M_0$  be a closed topological manifold of dimension  $n \ge 5$ . Equip M with a metric generating the given topology. An h-cobordism  $(W; M_0, M_1, f_0, f_1)$  is called  $\epsilon$ -controlled if for i = 0, 1 the composite  $M_i \xrightarrow{f_i} \partial_i W \xrightarrow{j_i} W$  for  $j_i$  the inclusion possesses a retraction  $r_i: W \to M_i$  coming with a homotopy  $H_i: j_i \circ r_i \simeq id_W$  such that for every  $w \in W$  the subset of  $M_0$  given by  $r_0 \circ H_i(\{w\} \times [0, 1])$  has a diameter less than  $\epsilon$ , in other words, the images of all the tracks of the two homotopies  $H_0$  and  $H_1$  under  $r_0$  have diameter less than  $\epsilon$ .

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An  $\epsilon$ -controlled *h*-cobordism  $(W; M_0, M_1, f_0, f_1)$  has an  $\epsilon$ -product structure if there is additionally a homeomorphism  $F: W \xrightarrow{\cong} M_0 \times [0, 1]$  such that  $F \circ j_0 \circ f_0$  sends  $x \in M_0$  to (x, 0) and  $r_0$  and  $\operatorname{pr}_{M_0} \circ F$  for the projection  $\operatorname{pr}_{M_0}: M_0 \times [0, 1] \to M_0$  are  $\epsilon$ -homotopic in the sense that there exists a homotopy  $L: W \times [0, 1] \to M_0$  between them such that the diameter of the subset  $L(\{w\} \times [0, 1])$  of  $M_0$  is less than  $\epsilon$  for every  $w \in W$ . In particular every  $\epsilon$ -controlled *h*-cobordism  $(W; M_0, M_1, f_0, f_1)$  with an  $\epsilon$ -product structure is trivial and hence has vanishing Whitehead torsion.

**Theorem 19.3 (Thin** *h***-Cobordism Theorem).** Let  $M_0$  be a closed topological manifold of dimension  $n \ge 5$ . Equip  $M_0$  with a metric generating the given topology.

Then for every  $\epsilon > 0$  there exists a  $\delta$  with  $0 < \delta < \epsilon$  such that every topological *h*-cobordism (W;  $M_0, M_1, f_0, f_1$ ) over  $M_0$  which is  $\delta$ -controlled has an  $\epsilon$ -product structure. In particular, there exists a  $\delta > 0$  such that every topological *h*-cobordism (W;  $M_0, M_1, f_0, f_1$ ) over  $M_0$  which is  $\delta$ -controlled is trivial.

#### 19.4.2 The Strategy of Gaining Control

Let *N* and  $M_0$  be closed topological manifolds of dimension  $n \ge 5$  equipped with a metric generating the given topology. Then there exists an  $\epsilon > 0$  with the following properties:

- Let *M* be a closed manifold and  $f: M \to N$  be a homotopy equivalence which is  $\epsilon$ -controlled in the sense that it is an  $\alpha$ -homotopy equivalence for the open covering  $\alpha$  of *N* consisting of all open balls of radius  $\epsilon/2$ . Then by the  $\alpha$ -Approximation Theorem 19.1 *f* is homotopic to a homeomorphism and in particular has trivial Whitehead torsion. So in order to prove that *N* is topological rigid in the sense of Definition 9.162, it suffices to show that a given homotopy equivalence  $g: M \to N$  is homotopic to an  $\epsilon$ -controlled homotopy equivalence. Roughly speaking, to achieve up to homotopy a homeomorphism, it suffices to gain  $\epsilon$ -control.
- An *h*-cobordism (W; M<sub>0</sub>, M<sub>1</sub>, f<sub>0</sub>, f<sub>1</sub>) over M<sub>0</sub> is trivial and hence has vanishing Whitehead torsion if we can show that it is ε-controlled. This follows from the Thin *h*-Cobordism Theorem 19.3. In particular, in order to show that Wh(π<sub>1</sub>(N)) vanishes, it suffices to show because of the *s*-Cobordism Theorem 3.47 that, for any *h*-cobordism (W; M<sub>0</sub>, M<sub>1</sub>, f<sub>0</sub>, f<sub>1</sub>) over M<sub>0</sub>, we can find another an *h*-cobordism (W'; M<sub>0</sub>, M'<sub>1</sub>, f'<sub>0</sub>, f'<sub>1</sub>) over M<sub>0</sub> such that (W; M<sub>0</sub>, M<sub>1</sub>, f<sub>0</sub>, f<sub>1</sub>) and (W'; M<sub>0</sub>, M'<sub>1</sub>, f'<sub>0</sub>, f'<sub>1</sub>) have the same Whitehead torsion and the new *h*-cobordism (W'; M<sub>0</sub>, M'<sub>1</sub>, f'<sub>0</sub>, f'<sub>1</sub>) is ε-controlled.

Hence to prove the Farrell-Jones Conjecture 3.110 for the Whitehead group Wh(G) for torsionfree G, which predicts the vanishing of Wh(G), or the Borel Conjecture 9.163 for G, which predicts the topological rigidity of an aspherical closed manifold with fundamental group G, a promising strategy is to gain control, i.e., turning an *h*-cobordism or a homotopy equivalence to an  $\epsilon$ -controlled one without changing the class associated to the *h*-cobordism in the Whitehead group.

19.4 Controlled Topology

With this strategy one can also achieve the  $\widetilde{K}_n(\mathbb{Z}G)$  part for  $n \le 0$  of Conjecture 3.110 and Conjecture 4.18 using the Bass-Heller-Swan decomposition 3.73 and replacing N by  $N \times T^n$ .

This turns out to be a major breakthrough since it allows us to bring in completely new methods, namely, geometric methods, into the play. This was pioneered by Farrell and Jones, in particular in their seminal papers [359, 360]. They used the Thin *h*-Cobordism Theorem 19.3, which does not play a role anymore in more recent proofs but inspired them.

#### 19.4.3 Controlled Algebra

Fix an infinite cardinal  $\kappa$ . Let  $\mathcal{F}^{\kappa}(R)$  be a small model for the category of all free R-modules which admit a basis B with card $(B) \leq \kappa$  such that  $\mathcal{F}^{\kappa}(R)$  possesses direct sums over index sets of cardinality  $\leq \kappa$ .

We have to consider this cardinal  $\kappa$  and  $\mathcal{F}^{\kappa}(R)$  and consider only countable groups and spaces whose cardinality is less than or equal to  $\kappa$  for set theoretic reasons which the reader may ignore in the sequel. Denote by  $\mathcal{F}^{f}(R) \subseteq \mathcal{F}^{\kappa}(R)$  the full subcategory consisting of all free *R*-modules which admit a finite basis *B*. For more information about these issues and  $\mathcal{F}^{\kappa}(R)$  see for instance [92, Lemma 9.2].

**Definition 19.4 (Geometric modules).** Let G be a group, R be a ring, and X be a free G-space with  $card(X) \le \kappa$ . We define the *additive category*  $GM^G(X)$  of geometric modules over X as follows.

An object *M* is a collection  $\{M_x \mid x \in X\}$  of objects in  $\mathcal{F}^{\kappa}(R)$  such that  $M_{gx} = M_x$  holds for every  $x \in X$  and  $g \in G$ . Define the *support* of an object

$$\operatorname{supp}(M) = \{x \in X \mid M_x \neq \{0\}\} \subseteq X.$$

For two objects  $M = \{M_x \mid x \in X\}$  and  $N = \{N_y \mid y \in X\}$ , a morphism  $f: M \to N$ consists of a collection of *R*-homomorphisms  $f = \{f_{x,y}: M_x \to N_y \mid x, y \in X\}$ such that  $f_{gx,gy} = f_{x,y}$  holds for  $x, y \in X$  and  $g \in G$  and for every  $x \in x$  the set  $\{y \in Y \mid f_{x,y} \neq 0\}$  is finite and for every  $y \in X$  the set  $\{x \in x \mid f_{x,y} \neq 0\}$  is finite. Define the *support* of a morphism

$$\operatorname{supp}(f) = \{(x, y) \in X \times X \mid f_{x, y} \neq \{0\}\} \subseteq X \times X.$$

If  $P = \{P_z \mid z \in X\}$  is an object and  $g: N \to P$  is a morphism, define the composite

$$g \circ f = \{(g \circ f)_{x,z} \colon M_x \to P_z \mid x, z \in X\} \colon M \to P$$

by  $(g \circ f)_{x,z} = \sum_{y \in Y} g_{y,z} \circ f_{x,y}$ . Define the identity

$$\mathrm{id}_M = (\mathrm{id}_M)_{x,y} \mid x, y \in X\} \colon M_x \to M_y$$

of the object M by  $(id_M)_{x,y} = id_{M_x}$  for x = y and by  $(id_M)_{x,y} = 0$  for  $x \neq y$ .

Given two morphisms  $f, g: M \to N$  and  $m, n \in \mathbb{Z}$ , define the morphism  $m \cdot f + n \cdot g: M \to N$  by  $(m \cdot f + n \cdot g)_{x,y} = m \cdot f_{x,y} + n \cdot g_{x,y}$  for  $x, y \in X$ . The direct sum of two objects M and N is defined by  $(M \oplus N)_x = M_x \oplus N_x$  for  $x \in X$ .

Denote by  $GM^G(X)^f$  the full additive subcategory of  $GM^G(X)$  consisting of those objects  $M = \{M_x \mid x \in X\}$  such that  $M_x$  belongs to  $\mathcal{F}^f(R)$  for all  $x \in X$  and the support supp $(M) = \{x \in X \mid M_x \neq \{0\}\}$  is *G*-cofinite, i.e., there is a finite subset *S* of *X* with supp $(M) = G \cdot S$ , or, equivalently,  $G \setminus \text{supp}(M)$  is finite.

The additive category  $GM^G(X)$  is equivalent to the additive category  $\mathcal{F}^{\kappa}(RG)$ . Namely, there is an equivalence of additive categories

(19.5) 
$$F: \mathsf{GM}^{\mathsf{G}}(X) \to \mathcal{F}^{\kappa}(RG)$$

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defined as follows. Given an object  $M = \{M_x \mid x \in X\}$  in  $GM^G(X)$ , we obtain an *RG*-module whose underlying *R*-module is  $\bigoplus_{x \in X} M_x$  and  $g \in G$  acts by sending  $\{m_x \mid x \in X\}$  to  $\{m_{g^{-1}x} \mid x \in X\}$ . The *G*-action is well-defined since  $M_x = M_{gx}$  holds for  $x \in X$  and  $g \in G$  by assumption. Since *G* acts freely on *X*, each  $M_x$  is free as *R*-module, and F(M) is isomorphic to the free *RG*-module  $RG \otimes_R (\bigoplus_{y \in S} M_y)$  for a set *S* with supp $(M) = G \cdot S$ . Hence we can choose an object  $V_M$  in  $\mathcal{F}^{\kappa}(RG)$  and an *RG*-isomorphism  $\xi_M : \bigoplus_{x \in X} M_x \xrightarrow{\cong} V_M$  and define  $F(M) = V_M$ . Given a morphism  $f = \{f_{x,y} : M_x \to N_y \mid x, y \in X\}$ :  $M \to N$ , we get an

Given a morphism  $f = \{f_{x,y} \colon M_x \to N_y \mid x, y \in X\} \colon M \to N$ , we get an *RG*-homomorphism  $\eta_f \colon \bigoplus_{x \in X} M_x \to \bigoplus_{y \in X} N_y$  by sending  $(u_x \mid x \in X)$  to  $\{v_y \mid y \in X\}$  with  $v_y = \sum_{x \in X} f_{x,y}(u_x)$ . Now define F(f) to be the composite  $\xi_N \circ \eta_f \circ \xi_M^{-1}$ .

The functor F induces an equivalence of additive categories

(19.6) 
$$F^f \colon \mathsf{GM}^G(X)^f \to \mathcal{F}^f(RG).$$

**Exercise 19.7.** Show that the functors F and  $F^f$  are equivalences of additive categories.

The additive categories  $GM^G(X)$  and  $GM^G(X)^f$  become much more interesting than  $\mathcal{F}^{\kappa}(RG)$  and  $\mathcal{F}^f(RG)$  if we bring the notion of control into play. Namely, suppose that we have a metric space Z = (Z, d) with free isometric *G*-action together with a *G*-map  $p: X \to Z$ . Given  $\epsilon \ge 0$ , we call a morphism  $f = \{f_{x,y}: M_x \to N_y \mid x, y \in X\}: M \to N \epsilon$ -controlled if the implication  $x, y \in X, f_{x,y} \neq 0 \Longrightarrow$  $d(p(x), p(y)) \le \epsilon$  holds. An automorphism  $f: M \xrightarrow{\cong} M$  is called an  $\epsilon$ -controlled automorphism if both f and  $f^{-1}$  are  $\epsilon$ -controlled.

Geometric modules were introduced by Connell-Hollowingsworth [248]. Their theory was developed further by, among others, Pedersen and Quinn and is sometimes referred to as *controlled algebra*. More information can be found in the survey article [798]. One can find an algebraic proof of the topological invariance of Whitehead torsion in [798, Section 5].

Next we give a kind of algebraic version of the Thin *h*-Cobordism Theorem 19.3 taken from [67, Theorem 1.2.8].

#### 19.4 Controlled Topology

An *abstract simplicial complex*  $\Sigma = (\Sigma, V)$  consists of a set V and a family  $\Sigma$ of non-empty finite subsets of V such that for every element  $\sigma$  in  $\Sigma$ , and every non-empty subset  $\tau \subseteq \sigma$ , the subset  $\tau$  also belongs to  $\Sigma$  and for each  $v \in V$  the subset  $\{v\}$  belongs to  $\Sigma$ . In the sequel we will often identify  $v \in V$  with  $\{v\} \in \Sigma$ . The *dimension* dim( $\sigma$ ) of a simplex is defined to be  $|\sigma| - 1$ . The dimension dim( $\Sigma$ ) is the supremum of the dimension of all simplices of  $\Sigma$ . A *map of simplicial complexes*  $f: (\Sigma, V) \to (\Sigma', V')$  is a map  $f: V \to V'$  such that for any element  $\sigma \in \Sigma$  the subset  $f(\sigma) \subseteq V'$  belongs to  $\Sigma'$ . The *barycentric subdivision*  $\Sigma'$  of an abstract simplicial complex  $\Sigma$  is the abstract simplicial complex whose set of vertices is  $\Sigma$ and whose set of simplices consists of non-empty finite subsets of  $\Sigma$  which are totally ordered with respect to inclusion. Note that a *d*-simplex in  $\Sigma'$  is the same as a flag  $\sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_d$  of elements  $\sigma_i \in \Sigma$ .

We equip the geometric realization  $|\Sigma|$  of  $\Sigma$ , which consists of functions  $b: V \to [0, 1]$  whose support supp $(b) = \{v \in V \mid f(v) \neq 0\}$  is finite and belongs to  $\Sigma$  and satisfies  $\sum_{v \in V} b(v) = 1$ , with the  $L^1$ -metric given by  $d_{L^1}(b, b') = \sum_{v \in V} |b(v) - b'(v)|$ .

An *abstract simplicial G-complex* is an abstract simplicial complex  $\Sigma$  together with *G*-action by simplicial automorphisms. The *G*-action on  $\Sigma$  induces an isometric *G*-action on  $|\Sigma|$  equipped with its  $L^1$ -metric. Let  $\mathcal{F}$  be a family of subgroups. We call  $\Sigma$  an *abstract simplicial*  $(G, \mathcal{F})$ -*complex* if the isotropy group  $G_b = \{g \in G \mid gb = b\}$  for every  $b \in |\Sigma|$  belongs to  $\mathcal{F}$ . Note that  $|\Sigma|$  is not necessarily a *G*-*CW*complex, but  $|\Sigma'|$  for the barycentric subdivision  $\Sigma'$  of  $\Sigma$  is. If the isotropy group of each vertex  $v \in V$  belongs to  $\mathcal{F}$  and  $\mathcal{F}'$  is the family of subgroups of *G* consisting of those subgroups which contain a subgroup of finite index belonging to  $\mathcal{F}$ , then  $\Sigma$  and  $\Sigma'$  are abstract simplicial  $(G, \mathcal{F}')$ -complexes and  $|\Sigma'|$  is a *G*-*CW*-complex whose isotropy groups belong to  $\mathcal{F}'$ .

**Theorem 19.8 (Algebraic Thin** *h***-Cobordism Theorem).** *Given a natural number N*, *there exists an*  $\epsilon_N > 0$  *with the following property. Consider:* 

- (i) A family  $\mathcal{F}$  of subgroups of G;
- (ii) An abstract simplicial  $(G, \mathcal{F})$ -complex Z of dimension  $\leq N$ ;
- (iii) A free G-space X together with a G-map  $p: X \to |Z|$ ;
- (iv) An automorphism  $a: M \to M$  in  $GM^G(X)^f$  which is  $\epsilon_N$ -controlled with respect to p and the  $L^1$ -metric on |Z|.

Then the class  $[F^f(a)] \in K_1(\mathbb{Z}G)$  of the RG-automorphism  $F^f(a) : F^f(M) \xrightarrow{\cong} F^f(M)$  of the finitely generated free RG-module  $F^f(M)$  for the functor  $F^f$  of (19.6) is contained in the image of the assembly map  $H_1(E_{\mathcal{T}}(G); \mathbb{K}_{\mathbb{Z}}) \to K_1(\mathbb{Z}G)$ .

The Algebraic Thin *h*-Cobordism Theorem 19.8 follows from [78, Theorem 5.3] and implies the Thin *h*-Cobordism Theorem 19.3, as explained in [67, Remark 1.2.11 and Remark 1.2.9]. There is also a converse to the Algebraic Thin *h*-Cobordism Theorem 19.8, as discussed in [67, Remark 1.2.11 and Remark 1.2.15]. It says, roughly speaking, that any element appearing in the image of the assembly map can be realized as  $[F^f(a)]$  for appropriate *Z*, *X*, *p*, and *a*.

**Remark 19.9 (Control-Strategy).** The considerations above lead to the following *Control-Strategy* for proving the Farrell-Jones Conjecture.

- (i) Interpret each element in the target group  $K_n(\mathbb{Z}G)$  of the assembly map as a kind of cycle and the elements of the source of the assembly map  $H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)$ as *controlled cycles*, i.e., cycles satisfying certain control conditions related to the family  $\mathcal{F}$ ;
- (ii) Identify the assembly map as a kind of *forget control map*;
- (iii) For a specific group *G* and a specific family  $\mathcal{F}$ , develop a strategy to change a cocycle without changing its class in  $K_n(\mathbb{Z}G)$  such that the new representative satisfies the necessary control conditions to ensure that it defines an element in  $H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)$ . This proves surjectivity of the assembly map. One may call this process *gaining control*;
- (iv) Use a relative version of part (iii) to prove injectivity of the assembly map. One may call this process *gaining relative control*;

The strategy for *L*-theory is completely analogous.

**Example 19.10 (Singular homology).** Next we illustrate this strategy in a much easier and classical instance, namely, singular homology, by repeating how one proves excision for it.

Let X be a topological space, and let  $C_*^{\text{sing}}(X; R)$  be the singular chain complex of X with coefficients in the ring R. Let  $\mathcal{U} = \{U_i \mid i \in I\}$  be a cover of X, i.e., a collection of subsets  $U_i$  such that the union of their interiors  $U_i^{\circ}$  is X. Denote by  $S_n^{\mathcal{U}}(X)$  the subset of the set  $S_n(X)$  of those singular *n*-simplices  $\sigma : \Delta_n \to X$ for which there exists an  $i \in I$  satisfying  $\operatorname{im}(\sigma) \subseteq U_i$ . Let  $C_*^{\operatorname{sing},\mathcal{U}}(X; R)$  be the R subchain complex of  $C_*^{\operatorname{sing}}(X; R)$  whose *n*th chain module consists of elements of the shape  $\sum_{\sigma \in S_n^{\mathcal{U}}(X)} r_{\sigma} \cdot \sigma$ . Let  $i_*^{\mathcal{U}} : C_*^{\operatorname{sing},\mathcal{U}}(X; R) \to C_*^{\operatorname{sing}}(X; R)$  be the inclusion. The main ingredient in the proof of excision is to show that  $i_*$  is a homology equivalence. Then excision follows by applying the result above to  $\mathcal{U} = \{X \setminus A, B\}$  for  $A \subseteq B \subseteq X$ with  $\overline{A} \subseteq B^{\circ}$ .

The proof that  $i_*^{\mathcal{U}}: C_*^{\operatorname{sing},\mathcal{U}}(X; R) \to C_*^{\operatorname{sing}}(X; R)$  is a homology equivalence is based on the construction of the subdivision operator which subdivides  $\Delta_n$  into a bunch of smaller copies of  $\Delta_n$  and replaces the singular simplex  $\sigma: \Delta_n \to X$ by the sum of the singular simplices obtained by restricting to these smaller pieces. This process does not change the homology class but can be used to arrange that the representing cycle lies in  $C_*^{\operatorname{sing},\mathcal{U}}(X; R)$ . This implies surjectivity of  $H_n(i_*^{\mathcal{U}}): H_n(C_*^{\operatorname{sing},\mathcal{U}}(X; R)) \to H_n(C_*^{\operatorname{sing}}(X; R))$ . One obtains injectivity by applying this construction to an (n+1)-simplex  $\tau: \Delta_{n+1} \to X$ , provided that the restriction of  $\tau$  to faces of  $\Delta_{n+1}$  already lies in  $S_n^{\mathcal{U}}(X)$ .

Roughly speaking, the process of gaining control is realized by subdivision.

#### 19.4.4 Controlled Algebra Defined Using the Open Cone

In order to carry out the Control Strategy discussed in Remark 19.9, one needs to find the equivalent setup of the homotopy theoretic construction of  $H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R)$ , but now in the controlled setting. The basic idea is to construct additive categories (with involution) which encode  $\mathcal{F}$  and the relevant control conditions and to consider their *K*- or *L*-groups.

An obvious drawback of the notion of  $\epsilon$ -controlled morphisms between geometric modules, see Subsection 19.4.3, is that they do not form a subcategory of the additive category of geometric modules. The composite of two  $\epsilon$ -controlled morphisms is  $2\epsilon$ -controlled but not necessary  $\epsilon$ -controlled. The same applies to  $\epsilon$ -controlled automorphisms. In order to fix this problem, Pedersen-Weibel [801] considered for a finite PL-subcomplex X of  $S^n$  (for large n) the open cone  $O(X) = \{sx \mid s \in \mathbb{R}, s > 0, x \in X\} \subseteq \mathbb{R}^{n+1}$  with the metric induced by the maximum metric on  $\mathbb{R}^{n+1}$  and introduced a quotient category in which every morphism has for any  $\epsilon > 0$  a representative that is  $\epsilon$ -controlled. They used this construction to produce a geometric homology theory digesting these finite PL-complexes  $X \subseteq S^n$  with coefficients in the K-theory spectrum  $\mathbf{K}_R$  of a ring R, which is a delooping of the homology theory associated to the algebraic K-theory spectrum  $\mathbf{K}_R$  sending X to the homotopy groups of the spectrum  $X_+ \wedge \mathbf{K}_R$ . This construction can easily be extended to additive categories  $\mathcal{A}$  as coefficients instead of a ring R as coefficient.

The idea of the open cone O(X) is that, given a constant C > 0, for two points x and y in X the implication  $d(sx, sy) \le C \implies d(x, y) \le \frac{C}{s}$  holds for s > 0. Hence d(x, y) becomes arbitrarily small if  $d(sx, sy) \le C$  holds for large enough s. More generally, given constants C > 0 and R > 0, we can find for every  $\epsilon > 0$  a real number T > 0 such that for  $x, y \in X$  and s, t > 0 the implication

$$d(sx, ty) \le C, |t - s| \le R, t \ge T \implies d(x, y) \le \epsilon$$

holds. These points *sx* and *ty* will be points contained in the support supp $(f) = \{sx, ty\} \in O(X) \times O(X) \mid f_{sx,ty} \neq 0\}$  of a morphism  $f = \{f_{sx,ty}\}$  in GM<sup>{1}</sup>(O(X)). Our setup ensures that we get an additive subcategory of GM<sup>{1}</sup>(O(X)) if we consider only those morphisms  $f = \{f_{sx,ty}\}$  for which there exist constants C > 0 and  $R \ge 0$  satisfying  $d(tx, sy) \le C$  and  $|t-s| \le R$  for every  $(sx, ty) \in \text{supp}(f)$ . One may think of the inclusion of this subcategory to GM<sup>{1}</sup>(O(X)) as a forget control functor.

#### 19.4.5 Continuous Control

Roughly speaking, the idea is to introduce a new non-compact coordinate, for instance the distance from the origin in  $\mathbb{R}^{n+1}$  in the open cone O(X) appearing in Subsection 19.4.4, so that bounded control for objects or morphisms over the given space *X* correspond to  $\epsilon$ -controlled morphisms in the new extended space for which the  $\epsilon$  can be chosen to be smaller and smaller the farer out the objects and morphisms are with respect to this new coordinate. In principle one uses the observation that bounded plus bounded is bounded (in contrast to the wrong statement  $\epsilon$  plus  $\epsilon$  is  $\epsilon$ ) so that bounded controlled morphisms form a subcategory. One has to consider germs of morphisms where it is allowed to ignore everything which is bounded in this new coordinate, or, equivalently, where only the asymptotic behavior at  $\infty$  matters. Therefore one takes the quotient by the category of those objects and morphisms that live in a bounded region with respect to the new coordinate, in other word, do not get arbitrarily close to  $\infty$ . This quotient has the desired property that for every morphism and  $\epsilon > 0$ , we can find a representative that is  $\epsilon$ -controlled. Taking this quotient has the side effect that one deals with a delooping of the desired homology theory.

The constructions of Pedersen-Weibel [801] have undergone a long lasting mutation through various steps, in order to get a better and better setting. For instance, one needs to design equivariant versions, and the theory should digest arbitrary *G-CW*-complexes and no conditions such as an embedding into  $S^n$  as above should occur.

For these development we refer to the papers by Bartels-Farrell-Jones-Reich [73, 74], Bartels-Lück-Reich [87], and Bartels-Lück [78]. The most advanced setup is presented in Bartels-Lück [81] where for the first time the Farrell-Jones Conjecture is considered for topological groups, namely, for totally disconnected groups such as reductive *p*-adic groups. We will not discuss this long process but we will give details about the constructions in [81] in the discrete case in Chapter 21, where we also give the full proof that we indeed get a *G*-homology theory digesting arbitrary *G-CW*-complexes. The construction of the TOD-sequence in Section 21.5 is the detailed and mathematically complete manifest of the discussion above.

As an illustration we want to describe the notion of continuous control (in the non-equivariant setting), which will replace the open cone construction, can digest any CW-complex X without any embedding into  $S^n$ , and does not need a choice of a metric.

We define an additive subcategory O(X) of  $GM^{\{1\}}(X \times \mathbb{N})$  as follows, where  $\mathbb{N}$  denotes the natural numbers. The support of an object  $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$  is defined to be  $\operatorname{supp}(M) = \{(x,s) \in X \times \mathbb{N} \mid M_{x,s} \neq \{0\}\}$ . We require for an object M in O(X):

• Compact support over X

The set  $\{x \in X \mid \exists s \in \mathbb{N} \text{ with } (x, s) \in \text{supp}(M)\}$  is contained in a compact subset of *X*;

• Locally finiteness over  $\mathbb{N}$ 

For every  $n \in \mathbb{N}$  the set  $\{x \in X \mid (x, n) \in \text{supp}(M)\}$  is finite.

We require for the support

 $\operatorname{supp}(f) = \{((x, s), (y, t)) \in (X \times \mathbb{N}) \times (X \times \mathbb{N}) \mid \{f_{(x, s), (y, t)} \neq 0\}$ 

of a morphism  $f = \{f_{(x,s),(y,t)}\}$  in O(X):

#### 19.4 Controlled Topology

• Bounded control in the  $\mathbb{N}$  direction

There is an  $N \in \mathbb{N}$  such that  $|t - s| \le N$  holds for  $((x, s), (y, t)) \in \text{supp}(f)$ ;

Continuous control

For every  $z \in X$  and every open neighborhood V of z, there exists an open neighborhood U of z with  $U \subseteq V$  and  $r \in \mathbb{N}$  such that the implications

$$\begin{split} &((x,s),(y,t))\in \mathrm{supp}(f), x\in U, s\geq r \implies y\in V;\\ &((x,s),(y,t))\in \mathrm{supp}(f), y\in U, t\geq r \implies x\in V, \end{split}$$

hold.

The condition above ensures that the morphisms become more and more controlled in the X direction the further we go out in the N-direction. The other conditions will be needed to construct the transfer or certain quotient categories. One may also consider the full additive subcategory  $\mathcal{T}(X)$  of O(X) where we additionally require for an object M that there exists a natural number  $m \in \mathbb{N}$  for which the implication  $(x, s) \in \text{supp}(M) \implies s \leq m$  holds. Then the quotient category  $\mathcal{D}(X) = O(X)/\mathcal{T}(X)$  can be thought of as equivalence classes of objects and morphism in O(X) where we identify two of them if they agree outside of a bounded region in the N-direction. In this category  $\mathcal{D}(X)$  we can always find representatives in O(X) which are with respect to the X direction arbitrarily well controlled since we can set all modules and morphism to be zero in any region bounded in the N-direction. The definition of the quotient category  $\mathcal{D}(X)$  has been given in Definition 8.42 and we get the weak homotopy fibration sequence of non-connective spectra

(19.11) 
$$\mathbf{K}(\mathcal{T}(X)) \to \mathbf{K}(\mathcal{O}(X)) \to \mathbf{K}(\mathcal{D}(X))$$

from Theorem 8.46. It will be the key ingredient to show that the functor sending X to  $\mathbf{K}(\mathcal{D}(X))$  is weakly excisive in the sense of Definition 18.8 for  $G = \{1\}$ , or, equivalently, that we get a homology theory with values in abelian groups by sending a *CW*-complex X to  $K_{n+1}(\mathcal{D}(X))$  for  $n \in \mathbb{Z}$ . An Eilenberg swindle towards infinity in the  $\mathbb{N}$ -direction will show that  $K_n(O(\{\bullet\}))$  vanishes for all  $n \in \mathbb{Z}$ . It is not hard to see that  $\mathcal{T}(X)$  is equivalent to  $\mathbb{GM}^{\{1\}}(X)^f$  and hence we get from the equivalence (19.6) an identification  $K_n(\mathcal{T}(X)) = K_n(R)$ . Thus we obtain an identification  $K_n(R) = K_{n+1}(\mathcal{D}(\{\bullet\}))$ . We conclude from the universal property of assembly maps, see Theorem 18.11 and Remark 18.18, that we get natural identifications  $H_n(X; \mathbf{K}(R)) \cong \pi_{n+1}(\mathcal{D}(X))$ . Furthermore, if we take X = BG, the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

appearing in Conjecture 6.53 can be identified with a map

$$\pi_{n+1}(\mathcal{D}(BG)) \to K_n(RG)$$

which can be thought of as a forget control map.

All this will be fully explained in Chapter 21, also in the equivariant setting. In particular, there is for any *G*-*CW*-complex X an equivariant version of (19.11)

$$\mathbf{K}(\mathcal{T}^G(X)) \to \mathbf{K}(\mathcal{O}^G(X)) \to \mathbf{K}(\mathcal{D}^G(X))$$

such that the assembly map

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$$H_n^G(\mathrm{pr}): H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

appearing in the *K*-theoretic Farrell-Jones Conjecture 13.1 with coefficients in the ring R can be identified with the homomorphism

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{VC}\mathcal{M}}(G))) \to K_{n+1}(\mathcal{D}^G(G/G))$$

induced by the projection pr:  $E_{\mathcal{VCY}}(G) \to G/G$  which can be thought of as a forget control map.

**Exercise 19.12.** Show that the inclusion  $I: GM^{\{1\}}(X) \to \mathcal{T}(X)$  coming from the inclusion  $X \to X \times \mathbb{N}$  sending *x* to (x, 0) is an equivalence of additive categories.

# 19.5 Gaining Control by Using Flows and Transfers

In this section we briefly sketch the basic ideas appearing in the seminal papers by Farrell-Jones [359, 360]. These papers do of course rely on earlier work by Farrell and Jones and other mathematicians, which we will not explain here. For us it is important to explain briefly the main idea in these two papers to prove the vanishing of the Whitehead group Wh(*G*) for torsionfree groups *G* which occur as fundamental groups of certain closed manifolds. For simplicity we only consider the case  $G = \pi_1(M)$  for an orientable hyperbolic closed smooth Riemannian manifold of dimension  $d \ge 5$ .

The key ingredient is to lift an element x in the Whitehead group  $Wh(\pi_1(M))$ to the Whitehead group  $Wh(\pi_1(STM))$  of the fundamental group of the total space STM of the sphere tangent bundle  $p: STM \to M$  by a transfer map  $p^*: Wh(\pi_1(M)) \to Wh(\pi_1(STM))$  and to use the geometric flow on STM and the hyperbolic structure on M to show that this element  $p^*(x)$  has a representative with good enough control ensuring that  $p^*(x)$  vanishes. The composite of the transfer  $p^*$  with the obvious map  $p_*: Wh(\pi_1(STM)) \to Wh(\pi_1(M))$  induced by the isomorphism  $\pi_1(p): \pi_1(STM) \to \pi_1(M)$  satisfies  $p_* \circ p^* = 2 \cdot id_{Wh(\pi_1(M))}$ if d is odd, since the fiber of p is an even-dimensional sphere  $S^{d-1}$  and hence has Euler characteristic 2. This implies 2x = 0, if d is odd. To get rid of the factor 2, Farrell and Jones replaced the sphere bundle  $p: STM \to M$  by a kind of upper hemisphere bundle  $p_+: S_+TM \to M$  whose fiber is the upper hemisphere  $S_{+}^{d-1}$ and hence contractible and therefore has Euler characteristic 1. Then the composite  $Wh(\pi_1(M)) \xrightarrow{(p_+)^*} Wh(\pi_1(S_+TM)) \xrightarrow{(p_+)_*} Wh(\pi_1(M))$  is the identity for all  $d \ge 5$ , and one can still show using the geometric flow on  $S_+TM$  and the hyperbolic structure on M that  $(p_+)^*(x)$  vanishes if  $d \ge 5$ . (All these claims about the transfers will be explained in Example 23.14, which is a consequence of Theorem 23.13.)

We will give more information about the transfer in Chapter 23 and will confine ourself for the remainder of this section to explaining why every element in Wh(*STM*) vanishes if *M* is a hyperbolic closed smooth Riemannian manifold. Farrell and Jones used the Algebraic Thin *h*-Cobordism Theorem 19.8 and the fact that every element in the Whitehead group Wh( $\pi_1(STM)$ ) can be realized by the Whitehead torsion of an *h*-cobordism over *STM*, see Theorem 3.47 (i). The main ingredients in proof of Farrell and Jones was to use the geodesic flow and its specific properties due to the hyperbolic structure to convert an arbitrary *h*-cobordism into a thin one without changing its Whitehead torsion in Wh( $\pi_1(M)$ ). Having the Algebraic Thin *h*-Cobordism Theorem 19.8 in mind, we just will explain how the geodesic flow can be used to turn automorphisms of geometric modules into  $\epsilon$ -controlled ones without changing their Whitehead torsion in the Whitehead group. For this we look at the very specific case, namely, the geodesic flow on the half plane model  $\mathbb{H}^2$  for the two-dimensional hyperbolic space.

Consider two points with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $\mathbb{H}^2$ . We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the two geodesics given by the vertical lines through these points, i.e., towards infinity in the y-direction. However, there is a fundamental problem: if  $y_1 \neq y_2$ , then the distance between these points will be bounded from below by a constant C > 0, regardless how long we let them flow to infinity. Therefore we make the following prearrangement. Suppose that  $y_1 < y_2$ . Then we first let the point  $(x_1, y_1)$  flow so that it reaches a position where  $y_1 = y_2$  and do nothing to the point  $(x_2, y_2)$ , and then we let both points flow simultaneously. Inspecting the hyperbolic metric, one sees that the distance between the two points  $(x_1, \tau)$  and  $(x_2, \tau)$  goes to zero if  $\tau$  goes to infinity. This is the basic idea to gain control in the negatively curved case. In some sense we will see this wait and then flow together principle in the more general theorems about flows, which we will present in Chapter 22. Note that moving along a flow is a continuous process and therefore should not change the associated homology class or element in the Whitehead group. It should also be clear what it means for instance to move an object or a morphism in  $GM^{\{1\}}(X \times \mathbb{N})$  along a flow, just move the positions of the modules  $M_{(x,s)}$  and morphisms  $f_{((x,s),(y,t))}$ accordingly. All of this also works in the case where M is a closed Riemannian manifold with strictly negative sectional curvature.

**Exercise 19.13.** Consider two points  $(x_1, y_1)$  and  $(x_2, y_2)$  in the half plane model  $\mathbb{H}^2$ . Denote by  $\gamma_{(x_k, y_k)}(t)$  the point obtained by flowing upwards starting with  $(x_k, x_k)$  along the two geodesics given by the vertical lines though  $(x_k, y_k)$  for k = 1, 2. Show for the hyperbolic metric  $d_{hyp}$ 

$$\lim_{t \to \infty} d_{\text{hyp}}(\gamma_{(x_1, y_1)}(t), \gamma_{x_2, y_2}(t)) = |\ln(y_2) - \ln(y_1)|.$$

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Later Farrell-Jones could also deal with the case where *M* is a closed Riemannian manifold with non-positive sectional curvature, see for instance [365]. This case is significantly harder, as illustrated next. Again, consider the half plane model, but this time equip it with the flat Riemannian metric coming from the Euclidean inner product on  $\mathbb{R}^2$ . Then the same construction makes sense, but the distance between two points  $(x_1, \tau)$  and  $(x_2, \tau)$  is unchanged if we change  $\tau$ . The basic first idea is to choose a so-called focus point far away, say  $f := ((x_1 + x_2)/2, \tau + 169356991)$ , and then let  $(x_1, \tau)$  and  $(x_2, \tau)$  flow along the rays emanating from them and passing through the focus point f. In the beginning the effect is indeed that the distance becomes smaller, but as soon as we have passed the focus point towards infinity while the points  $x_1$  and  $x_2$  flow towards it, as Farrell and Jones did, or one stops the flow when has reached the focus point. We will use the second solution. In particular, we want to fix a base point  $x_0$  and carry out all the constructions inside the closed ball  $\overline{B}_R(x_0)$  for large R > 0.

The problem with this idea is obvious, we must describe this process in a functorial way and carefully check all the estimates to guarantee the desired effects.

Another problem is that we later need to make everything equivariant. So if the group *G* acts isometrically (and does not necessarily leave the origin  $x_0$  fixed), there are points  $x \in \overline{B}_R(x_0)$  and  $g \in G$  such that gx lands outside  $\overline{B}_R(x)$ . Then we have to use the radial projection to pull back gx to  $\overline{B}_R(x_0)$ . With this modification we of course do not get a strict *G*-action on  $\overline{B}_R(x_0)$  but an up to homotopy (and actually up to higher homotopies) well-defined *G*-action. This is the reason why in the CAT(0)-setting one has to deal with these kind of non-strict *G*-actions. Moreover, we also have to deal with the problem that the focus point *f* may also not be fixed under the *G*-action.

Next we give a quantitative version of the sketch of ideas above for  $\mathbb{R}^n$  with the Euclidian metric *d*. For two distinct points  $a, b \in \mathbb{R}$ , define

$$c_{a,b} \colon \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto \begin{cases} a & t \le 0; \\ a + \frac{t}{d(a,b)} \cdot (b-a) & 0 \le t \le d(a,b); \\ b & t \ge d(a,b). \end{cases}$$

Note that the restriction of *c* to [0, d(a, b)] is the geodesic line starting at *a* and ending at *b* and is constant for  $t \le 0$  and  $t \ge d(a, b)$ .

**Lemma 19.14.** Fix  $x_0 \in \mathbb{R}^n$  and real numbers  $r', r'', \beta$ , and L satisfying  $r', L, \beta > 0$  and  $r'' > 2\beta$ . Put T := r'' + r'. Fix  $x_1, x_2 \in \overline{B}_{\beta}(x_0)$ . Let x be any point in  $B_{r'+r''+L}(x_0)$ . Put  $\tau := d(x_2, x) - d(x_1, x)$ .

Then we get for all  $t \in [T - r', T + r']$ 

$$d(c_{x_{1},x}(t), c_{x_{2},x}(t+\tau)) \leq \frac{4 \cdot \beta \cdot (r'+\beta+L)}{r''};$$
  
$$c_{x_{1},x}(t) \in \overline{B}_{2r'+r''+2\beta}(x_{1});$$
  
$$c_{x_{2},x}(t+\tau) \in \overline{B}_{2r'+r''+2\beta}(x_{2}).$$

#### 19.6 Notes

Note that the larger we take r'' (without changing r',  $\beta$ , and L), the smaller  $d(c_{x_1,x}(t), c_{x_2,x}(t+\tau))$  becomes for  $t \in [\underline{T} - r', T + r']$  and that the geodesic triangle with  $x, x_1$ , and  $x_2$  as vertices lies in  $\overline{B}_{r'+r''+\beta+L}(x_0)$ . Actually, the obvious analogue of Lemma 19.14 holds in any CAT(0)-space. The contents of Lemma 19.14 will be stated in more generality in Proposition 22.30 and Theorem 22.34.

In the situation of Lemma 19.14 the points  $x_1$  and  $x_2$  flow towards the focal point x and everything takes place in a fixed ball around a fixed base point  $x_0$ . The wait and then flow together principle is reflected in Lemma 19.14 by the appearance of  $\tau$ .

Exercise 19.15. Give a proof of Lemma 19.14.

More details about the discussion of this subsection will be given in Chapter 22.

#### **19.6 Notes**

Farrell-Hsiang used in [355] a beautiful combination of controlled topology and induction theory à la Dress to prove that the Whitehead group of fundamental groups of compact flat Riemannian manifolds is trivial. This general method, often called the Farrell-Hsiang method, has been refined and used further, see for example [79, 357, 358, 361, 829, 960, 1015]. This will be explained in some more detail in Chapter 20, notably in Sections 20.2 and 20.9.

There are survey articles about continuously controlled algebra by Rosenthal [874] and about controlled K-theory by Quinn [830].

# Chapter 20 Conditions on a Group Implying the Farrell-Jones Conjecture

# **20.1 Introduction**

In this chapter we want to isolate geometric properties of a group G which guarantee that the strategy of proofs discussed in Chapter 19 works out. So we want to describe a bunch of geometric conditions on G which imply the Farrell-Jones Conjecture but do not contain any K-theoretic or homotopy theoretic data. This may be useful for someone who wants to prove the Farrell-Jones Conjecture for a new class of groups, since she or he needs only to check that this class satisfies one of the properties (or some appropriate variation or generalization) appearing below without having to deal with the proofs relying on homotopy theory and K-theory that these properties imply the Farrell-Jones Conjecture.

We do this in chronological order taking into account that these conditions have been reformulated, been generalized and evolved over the last decades. Here is a list of the different notions which we will treat:

- Farrell-Hsiang groups in Section 20.2;
- Strictly transfer reducible groups almost equivariant version in Section 20.3;
- Strictly transfer reducible groups cover version in Section 20.4;
- *Transfer reducible groups* in Section 20.5;
- Strongly transfer reducible groups in Section 20.6;
- *Finitely F-amenable groups* in Section 20.7;
- *Finitely homotopy F-amenable groups* in Section 20.8;
- Dress-Farrell-Hsiang groups in Section 20.9;
- Dress-Farrell-Hsiang-Jones groups in Section 20.10.

**Remark 20.1.** These various notions come in two flavors, in terms of covers or in terms of almost equivariant maps, where in general the first version implies the second. This is essentially a consequence of results such as Proposition 20.22 or Lemma 20.42.

Some of the notions above imply one another, as the next result shows.

#### Lemma 20.2.

- (i) Strictly transfer reducible groups cover version ⇒ strictly transfer reducible groups – almost equivariant version;
- (ii) Strictly transfer  $\mathcal{F}$ -reducible almost equivariant version *and* finitely presented  $\implies$  transfer  $\mathcal{F}$ -reducible;
- (iii) Strictly transfer reducible groups cover version  $\implies$  finitely  $\mathcal{F}$ -amenable;

- (iv) Strongly transfer  $\mathcal{F}$ -reducible  $\implies$  finitely homotopy  $\mathcal{F}$ -amenable;
- (v) Strongly transfer  $\mathcal{F}$ -reducible  $\implies$  transfer  $\mathcal{F}$ -reducible;
- (vi) Finitely  $\mathcal{F}$ -amenable  $\implies$  finitely homotopy  $\mathcal{F}$ -amenable;
- (vii) Farrell-Hsiang over  $\mathcal{F} \implies$  Dress-Farrell-Hsiang over  $\mathcal{F}$ ;
- (viii) Dress-Farrell-Hsiang over  $\mathcal{F} \implies$  Dress-Farrell-Hsiang-Jones over  $\mathcal{F}$ ;
- (ix) Finitely homotopy  $\mathcal{F}$ -amenable  $\implies$  Dress-Farrell-Hsiang-Jones over  $\mathcal{F}$ .

*Proof.* (i) see Lemma 20.25.

(ii) This follows directly from the definitions.

(iii) An *N*-transfer space is a compact metrizable finite-dimensional contractible ANR by Lemma 20.15. A compact metrizable topological space *X* is an ER if and only if it is a finite-dimensional contractible ANR, see [154, Theorem V.10.1 on page 122]. Hence any *N*-transfer space is a compact ER. Now the assertion follows from Lemma 20.42.

(iv) This follows from the argument appearing in the proof of assertion (iii) about *N*-transfer spaces using a variation of Lemma 20.42 and the fact that an ANR is an AR if and only if it is contractible, see [505, Theorem 7.1 and Proposition 7.2 in Chapter III on page 96].

(v) This follows directly from the definitions.

(vi) This follows from Lemma 20.42.

(vii) This follows directly from the definitions.

(viii) see [185, Remark 7.2 (2)].

(ix) see [185, Remark 7.2 (1)].

**Remark 20.3.** Note that by Lemma 20.2 the notion of a *Dress-Farrell-Hsiang-Jones* group is the most general one if we ignore *transfer reducible groups*. Namely every *Farrell-Hsiang group, strictly transfer reducible group – almost equivariant version,* strictly transfer reducible group – cover version, strongly transfer reducible group, finitely  $\mathcal{F}$ -amenable group, finitely homotopy  $\mathcal{F}$ -amenable group, or Dress-Farrell-Hsiang group is a Dress-Farrell-Hsiang-Jones group.

The notion of *transfer reducible groups* deals only with homotopy *G*-actions and not with strong homotopy *G*-actions or strict *G*-actions and therefore does not imply *Dress-Farrell-Hsiang-Jones group*. Note that the conclusions for *transfer reducible groups* predicts only that the *K*-theoretic assembly map is 1-connected and not that it is a weak equivalence, cf. Theorem 20.31 and Theorem 20.61.

Proofs of the Farrell-Jones Conjecture for prominent classes such as hyperbolic groups or finite-dimensional CAT(0)-groups are based on showing that they fall into one of the classes above. We will explain for the various classes which versions of the Farrell-Jones Conjecture is known for them.

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#### 20.2 Farrell-Hsiang Groups

The next definition is equivalent to [79, Definition 1.1], see [67, Remark.1.3.21].

**Definition 20.4 (Farrell-Hsiang group).** Let *G* be a finitely generated group and  $\mathcal{F}$  be a family of subgroups. We call *G* a *Farrell-Hsiang group with respect to*  $\mathcal{F}$  if there exists a natural number *N* such that for one (and hence all) finite set *S* of generators we can find for every  $\epsilon > 0$ :

- (i) A finite group F together with a surjective group homomorphism  $p: G \to F$ ;
- (ii) For every  $H \in \mathcal{H}(F)$  an abstract simplicial  $(G, \mathcal{F})$ -complex  $\Sigma_H$  of dimension  $\leq N$ , where  $\mathcal{H}(F)$  denotes the set of hyperelementary subgroups of F:
- (iii) For every  $H \in \mathcal{H}(F)$  a map  $f_H \colon G/p^{-1}(H) \to |\Sigma_H|$  that is  $(\epsilon, S)$ -almost *G*-equivariant, i.e., we have  $d_{L^1}(f_H(sx), sf_H(x)) \leq \epsilon$  for all  $x \in p^{-1}(H)$  and all  $s \in S$ .

The appearance of the hyperelementary subgroups in Definition 20.4 is due to the result of Swan [937, Corollary 4.2] that for a finite group F and the family  $\mathcal{H}(F)$  of hyperelementary subgroups there are elements  $\tau_H \in \text{Sw}^p(H)$  for  $H \in \mathcal{H}(F)$  satisfying

(20.5) 
$$1_{\operatorname{Sw}^{p}(F)} = \sum_{H \in \mathcal{H}} \operatorname{ind}_{H}^{F}(\tau_{H}) \in \operatorname{Sw}^{p}(F),$$

where  $Sw^{p}(F)$  denotes the Swan ring defined in Definition 12.65 and the homomorphisms  $ind_{H}^{F}$ :  $Sw^{p}(H) \rightarrow Sw^{p}(F)$  are induced by induction. This is the key ingredient in induction theorems à la Dress, see for instance [76, Section 2], and leads for instance to Theorem 13.46. There is also an *L*-theoretic version due to Dress [315, Theorem 2]

(20.6) 
$$1_{\mathrm{GW}(F)} = \sum_{H \in \mathcal{H}} \mathrm{ind}_{H}^{F}(\sigma_{H}) \in \mathrm{GW}(F)$$

for Dress' equivariant Witt ring GW(F) and elements  $\sigma_H \in GW(H)$  for  $H \in \mathcal{H}(F)$ .

It is often not so easy to check that a finitely generated group G is a Farrell-Hsiang group. The proof for  $\mathbb{Z}^2 \rtimes \mathbb{Z}/2$  can be found in [72, Lemma 3.8].

The proof of the next theorem is given in [79, Theorem 1.2]. It combines methods from controlled geometry and induction theory.

**Theorem 20.7 (Hsiang-Farrell groups and the Farrell-Jones Conjecture).** Let G be a finitely generated group G and  $\mathcal{F}$  be a family of subgroups such that G is a Hsiang-Farrell group with respect to the family  $\mathcal{F}$  in the sense of Definition 20.4.

Then the assembly maps

$$H_n^G(\mathrm{pr};;\mathbf{K}_{\mathcal{A}})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = \pi_n\big(\mathbf{K}_{\mathcal{A}}(I(G))\big)$$

and

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$$\begin{split} H_n^G(\mathrm{pr};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}) \\ & \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}(I(G))\big) \end{split}$$

are bijective for every additive *G*-category (with involution) and  $n \in \mathbb{Z}$ .

**Remark 20.8.** Definition 20.4 can be weakened if one is only interested in the *L*-theoretic Farrell–Jones conjecture. Then it suffices to consider all subgroups *H* of *F* that are either 2-hyperelementary or *p*-elementary for some odd prime *p*. In other words *p*-hyperelementary subgroups that are not *p*-elementary can be ignored for all odd primes *p*.

In the setting of higher categories one has to enlarge the class of hyperelementary groups as explained in Section 20.9.

# 20.3 Strictly Transfer Reducible Groups – Almost Equivariant Version

**Definition 20.9** (*N*-transfer space *X*). Let *N* be a natural number. An *N*-transfer space is a compact metric space *X* possessing the following property:

For any  $\delta > 0$  there exists an abstract simplicial complex *K* of dimension at most *N*, maps  $i: X \to |K|$  and  $r: |K| \to X$ , and a homotopy  $h: X \times [0, 1] \to X$  from  $r \circ i$  to  $id_X$  which is  $\delta$ -controlled, i.e., for every  $x \in X$  the diameter of the subset  $h(\{x\} \times [0, 1])$  of *X* is smaller than  $\delta$ .

**Remark 20.10** (No uniform bound on the dimensions). In Definition 20.9 and also in [67, Definition 1.3.1] it is required that there is a natural number N such that the dimensions of the simplicial complexes K appearing in Definition 20.9 is uniformly bounded by N. It turns out that this condition is not needed, cf., Remark 20.49. However, it is satisfied in all the applications, e.g., to hyperbolic groups, finite-dimensional CAT(0)-groups, or mapping class groups.

**Definition 20.11 (Strictly**  $\mathcal{F}$ **-transfer reducible group – almost equivariant version).** Let *G* be a finitely generated group, and let  $\mathcal{F}$  be a family of subgroups. We call *G strictly*  $\mathcal{F}$ *-transfer reducible* if there exists a natural number *N* such that for one (and hence all) finite set *S* of generators there exists for any given  $\epsilon > 0$ :

(i) an *N*-transfer space *X* in the sense of Definition 20.9 equipped with a *G*-action; (ii) an abstract simplicial  $(G, \mathcal{F})$ -complex  $\Sigma$  of dimension  $\leq N$ ;

(iii) a map  $f: X \to |\Sigma|$  that is  $(\epsilon, S)$ -almost G-equivariant, i.e., we have

$$d_{L^1}(f(sx), sf(x)) \le \epsilon$$

for every  $s \in S$  and every  $x \in X$ .

Note that [86, Theorem 1.2] implies that hyperbolic groups are strictly  $\mathcal{VCY}$ -transfer reducible. If there exists a group G which is strictly  $\mathcal{F}$ -transfer reducible, then  $\mathcal{F}$  must contain all cyclic subgroups of G, see [67, Remark 1.3.9].

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20.3 Strictly Transfer Reducible Groups - Almost Equivariant Version

In the sequel we denote for a family of subgroups  $\mathcal{F}$  of G by  $\mathcal{F}_2$  the family of subgroups of G consisting of those group  $H \subseteq G$  for which H or a subgroup  $H' \subseteq H$  of index [H : H'] = 2 belong to  $\mathcal{F}$ . For instance,  $(\mathcal{VC}\mathcal{Y}_I)_2 = \mathcal{VC}\mathcal{Y}$  and  $\mathcal{FIN}_2 = \mathcal{FIN}$ .

**Theorem 20.12 (Strictly transfer**  $\mathcal{F}$ -reducible groups and the Farrell-Jones **Conjecture).** Let G be a finitely generated group, and let  $\mathcal{F}$  be a family of subgroups such that G is strictly  $\mathcal{F}$ -transfer reducible in the sense of Definition 20.11.

Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{R}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{R}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{R}}) = \pi_n\big(\mathbf{K}_{\mathcal{R}}(I(G))\big);$$
  
$$H_n^G(\mathrm{pr}; \mathbf{H}_C) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_C) \to H_n^G(G/G; \mathbf{H}_C) = \pi_n\big(\mathbf{K}_C(I(G))\big),$$

are bijective for every additive *G*-category  $\mathcal{A}$ , every right exact *G*- $\infty$ -category *C*, and every  $n \in \mathbb{Z}$ , and the assembly map

$$\begin{split} H_n^G(\mathrm{pr};\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle})\colon H_n^G(E_{\mathcal{F}_2}(G);\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}) \\ & \to H_n^G(G/G;\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}) = \pi_n\big(\mathbf{L}_{\mathcal{A}}^{\langle-\infty\rangle}(I(G))\big), \end{split}$$

is bijective for every additive G-category with involution  $\mathcal{A}$  and every  $n \in \mathbb{Z}$ .

Later we will give as an illustration a proof of a special case of Theorem 20.12 in Proposition 23.24.

**Remark 20.13** (Meaning and proof of Theorem 20.12). Theorem 20.12 implies that every strictly  $\mathcal{VCY}$ -transfer reducible group satisfies both the *K*-theoretic Farrell-Jones Conjecture 13.11 and the *L*-theoretic Farrell-Jones Conjecture 13.19 with coefficients in additive *G*-categories with involution.

The *K*-theoretic part of Theorem 20.12 for additive categories is a minor reformulation of [87, Theorem 1.1], as explained in [67, Theorem A, Remark 1.3.7 and Remark 1.3.8]. So the *K*-theoretic part of the proof of Theorem 20.12 follows from [87, Theorem 1.1].

The same ideas apply also to the *L*-theoretic part, see [78, Theorem B]. Note that the passage from  $\mathcal{F}$  to  $\mathcal{F}_2$  for *L*-theory is due to [78, Lemma 9.2 and Remark 9.3]. This is consistent with the fact that Theorem 13.47 holds for the *K*-theoretic version but not for the *L*-theoretic version.

The *K*-theory part for higher categories (and hence also the one for additive categories) follows from Theorem 20.61 using Remark 20.3.

The passage to almost equivariant maps as pursued by Bartels [67] is illuminating, since it better isolates what is needed for proofs of the Farrell-Jones Conjecture, see also Remark 20.26.

**Exercise 20.14.** Let  $\Sigma$  be a finite simplicial complex such that  $|\Sigma|$  is contractible. Let *G* be a group which acts simplicially on  $\Sigma$ . Denote by  $\mathcal{F}(\Sigma)$  the family of subgroups of  $\Sigma$  which occur as subgroups of isotropy groups of  $|\Sigma|$ . Show:

- (i) The assembly maps appearing in Theorem 20.12 are isomorphisms for the family  $\mathcal{F}(\Sigma)$ ;
- (ii) Each isotropy group of  $|\Sigma|$  has finite index in *G*.

In connection with Exercise 20.14 Theorem 20.53 is interesting.

Next we prove the following lemma, which we have already used in the proof of Lemma 20.2.

**Lemma 20.15.** Let X be an N-transfer space in the sense of Definition 20.9. Then X is a compact metrizable ANR with  $\dim(X) \le N$ .

*Proof.* By definition X is a compact metric space.

Next we show that X satisfies  $\dim(X) \leq N$ . Consider an open cover  $\mathcal{U}$  of X. Since X is compact, there exists a finite subcover  $\mathcal{V} = \{V_1, \dots, V_l\}$  of  $\mathcal{U}$ . Let  $\delta > 0$  be a Lebesgue number for  $\mathcal{V}$ . Define  $V'_i = \{x \in V_i \mid \overline{B}_{\delta/2}(x) \subseteq V_i\}$  for i = 1, 2, ..., lwhere  $\overline{B}_{\delta/2}(x)$  is the closed ball of radius  $\delta/2$  around x. Then  $\mathcal{V}' = \{V'_1, \ldots, V'_l\}$  is an open cover of X. Choose an abstract simplicial complex K of dimension at most N, maps  $i: X \to |K|$  and  $r: |K| \to X$ , and a homotopy  $h: X \times [0, 1] \to X$  from  $r \circ i$  to id<sub>X</sub> which is  $\delta/2$ -controlled, i.e., for every  $x \in X$  the diameter of the subset  $h(\{x\} \times [0,1])$  of X is smaller than  $\delta/2$ . Note that this implies that for every  $x \in X$ we have  $d_X(x, r \circ i(x)) < \delta/2$ . Since the image of i is compact, it is contained in  $K_0$ for a finite subcomplex  $K_0$  of K. Hence we can assume without loss of generality that K itself is a finite abstract simplicial complex of dimension  $\leq N$ . This implies dim $(|K|) \leq N$ . Consider the open covering  $r^{-1}(\mathcal{V}') = \{r^{-1}(V'_1), \dots, r^{-1}(V'_l)\}$ . Choose an open cover  $\mathcal{W}$  of |K| which is a refinement of  $r^{-1}(\mathcal{V}')$  and satisfies dim $(W) \leq N$ . Consider the open cover  $i^{-1}(W) = \{i^{-1}(W) \mid W \in W\}$  of X. Obviously we have dim $(i^{-1}(\mathcal{W})) \leq N$  and  $i^{-1}(\mathcal{W})$  is a refinement of the open cover  $i^{-1}(r^{-1}(\mathcal{V}')) = \{(r \circ i)^{-1}(U'_1), \dots, (r \circ i)^{-1}(U'_l)\}$ . We have  $(r \circ i)^{-1}(V'_i) \subseteq V_i$  for i = 1, ..., l. Hence  $i^{-1}(\mathcal{W})$  is a refinement of  $\mathcal{V}$  and hence of  $\mathcal{U}$ .

An *N*-transfer space is an ANR by [505, Theorem 6.3 in Chapter IV on page 139] since for every  $\delta > 0$  there exists a finite simplicial complex  $\Sigma$ , maps  $i: X \to |K|$  and  $r: |K| \to X$ , and a homotopy  $h: X \times [0, 1] \to X$  from  $r \circ i$  to  $id_X$  which is  $\delta$ -controlled.

#### 20.4 Strictly Transfer Reducible Groups – Cover Version

Next we state the version of strictly transfer reducible as it appears in [87, Theorem 1.1]. and give more details about some of the claims appearing in Remark 20.13.

We begin by recalling the criterion of [87, Theorem 1.1], where only the K-theoretic version is treated. Its extension to L-theory follows from the proof of [78, Theorem B].

20.4 Strictly Transfer Reducible Groups - Cover Version

**Definition 20.16 (Strictly**  $\mathcal{F}$ **-transfer reducible group – cover version).** Let *G* be a finitely generated group. Let  $\mathcal{F}$  be a family of subgroups of *G*. Suppose:

- (i) There exists a *G*-space *X* such that the underlying space *X* is the realization of a finite-dimensional abstract simplicial complex *K*;
- (ii) There exists a *G*-space  $\overline{X}$  which contains *X* as an open *G*-subspace such that the underlying space of  $\overline{X}$  is compact, metrizable, and contractible;
- (iii) Assumption 20.17 holds;
- (iv) Assumption 20.19 holds for  $\mathcal{F}$ .

Next we give some explanations about the conditions appearing in Definition 20.16.

Assumption 20.17 (Weak Z-set condition). There exists a homotopy  $H: \overline{X} \times [0, 1] \to \overline{X}$ , such that  $H_0 = id_{\overline{X}}$  and  $H_t(\overline{X}) \subset X$  for every t > 0.

In order to state the second assumption we introduce the notion of an open  $\mathcal{F}\text{-}\mathrm{cover.}$ 

**Definition 20.18 ((Open)**  $\mathcal{F}$ -cover). Let *Y* be a *G*-space. Let  $\mathcal{F}$  be a family of subgroups of *G*. An  $\mathcal{F}$ -cover of *Y* is a collection  $\mathcal{U}$  of subsets of *Y* such that the following conditions are satisfied:

- (i)  $Y = \bigcup_{U \in \mathcal{U}} U$ ;
- (ii) For  $g \in G$  and  $U \in \mathcal{U}$ , the set  $g(U) := \{gx \mid x \in U\}$  belongs to  $\mathcal{U}$ ;
- (iii) For  $g \in G$  and  $U \in \mathcal{U}$ , we have g(U) = U or  $U \cap g(U) = \emptyset$ ;
- (iv) For every  $U \in \mathcal{U}$ , the subgroup  $\{g \in G \mid g(U) = U\}$  lies in  $\mathcal{F}$ .

We call an  $\mathcal{F}$ -cover  $\mathcal{U}$  of *Y* open if each  $U \in \mathcal{U}$  is open.

Consider an open  $\mathcal{F}$ -cover  $\mathcal{U}$ . Then its nerve Nerv( $\mathcal{U}$ ) is a simplicial complex with cell preserving simplicial *G*-action and hence a *G*-*CW*-complex. (A *G*-action on a simplicial complex is called *cell preserving* if for every simplex  $\sigma$  and element  $g \in G$  such that the intersection of the interior  $\sigma^{\circ}$  of  $\sigma$  with  $g\sigma^{\circ}$  is non-empty we have gx = x for every  $x \in \sigma$ . Note that a simplicial action is not necessarily cell preserving, but the induced simplicial action on the barycentric subdivision is cell preserving.) Moreover all isotropy groups of its geometric realization  $|\operatorname{Nerv}(\mathcal{U})|$  lie in  $\mathcal{F}$ , in other words,  $\operatorname{Nerv}(\mathcal{U})$  is a simplicial  $(G, \mathcal{F})$ -complex. Recall that by definition the dimension dim( $\mathcal{U}$ ) of an open cover is the dimension of the *CW*-complex  $|\operatorname{Nerv}(\mathcal{U})|$ .

If G is a finitely generated discrete group, then  $d_G$  denotes the word metric with respect to some chosen finite set of generators. Recall that  $d_G$  depends on the choice of the set of generators but its quasi-isometry class is independent of it.

**Assumption 20.19 (Wide open**  $\mathcal{F}$ -covers). There exists an  $N \in \mathbb{N}$ , which only depends on the *G*-space  $\overline{X}$ , such that for every  $\beta \ge 1$  there exists an open  $\mathcal{F}$ -cover  $\mathcal{U}(\beta)$  of  $G \times \overline{X}$  equipped with the diagonal *G*-action with the following two properties:

(i) For every  $g \in G$  and  $x \in \overline{X}$  there exists a  $U \in \mathcal{U}(\beta)$  such that

$$B_{\beta}(g) \times \{x\} \subset U.$$

*Here*  $B_{\beta}(g)$  *denotes the open*  $\beta$ *-ball around* g *in* G *with respect to the word metric*  $d_G$ , *i.e., the set*  $\{h \in G \mid d_G(g,h) < \beta\}$ *;* 

(ii) The dimension of the open cover  $\mathcal{U}(\beta)$  is smaller than or equal to N.

**Exercise 20.20.** Let *X* be a *G*-space. Let  $G \times_1 X$  be the topological space  $G \times X$  with the *G*-action given by  $g' \cdot (g, x) = (g'g, x)$  and let  $G \times_d X$  be the topological space  $G \times X$  with the diagonal *G*-action given by  $g' \cdot (g, x) = (g'g, g'x)$ . Show that  $G \times_1 X$  and  $G \times_d X$  are *G*-homeomorphic.

Next we describe some of the geometric constructions in [87].

Let (Z, d) be a metric space. Let  $\mathcal{U}$  be a finite-dimensional cover of Z by open sets. Recall that points in the geometric realization of the nerve  $|\operatorname{Nerv}(\mathcal{U})|$  are formal sums  $x = \sum_{U \in \mathcal{U}} x_U U$ , with  $x_U \in [0, 1]$  such that  $\sum_{U \in \mathcal{U}} x_U = 1$  and such that the intersection of all the U-s with  $x_U \neq 0$  is non-empty, i.e.,  $\{U \mid x_U \neq 0\}$  is a simplex in the nerve of  $\mathcal{U}$ . There is a well-defined map

(20.21) 
$$f = f^{\mathcal{U}} \colon Z \to |\operatorname{Nerv}(\mathcal{U})|, \quad x \mapsto \sum_{U \in \mathcal{U}} f_U(x)U$$

where

$$f_U(x) = \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)} \quad \text{with} \quad a_U(x) = d(x, Z - U) = \inf\{d(x, u) \mid u \notin U\}.$$

If Z is a G-space, d is G-invariant, and  $\mathcal{U}$  is an open  $\mathcal{F}$ -cover, then the map  $f = f^{\mathcal{U}}$  is G-equivariant.

The proof of the following proposition can be found in [87, Proposition 5.3].

**Proposition 20.22.** Let Z = (Z, d) be a metric space and let  $\beta \ge 1$ . Suppose that  $\mathcal{U}$  is an open cover of Z of dimension less than or equal to N with the property that for every  $z \in Z$  there exists a  $U \in \mathcal{U}$  such that the open ball  $B_{\beta}(z)$  of radius  $\beta$  around z lies in U.

Then the map  $f^{\mathcal{U}}: Z \to |\operatorname{Nerv}(\mathcal{U})|$  of (20.21) has the contracting property that for  $z, z' \in X$  satisfying  $d(z, z') \leq \frac{\beta}{4N}$  we get

$$d_{L^1}(f^{\mathcal{U}}(z), f^{\mathcal{U}}(z')) \le \frac{16N^2}{\beta} \cdot d(z, z').$$

Note that if  $\beta$  gets bigger, the estimate applies more often and  $f^{\mathcal{U}}$  contracts more strongly. Of course contracting maps can and will be used to gain control.

Let  $\overline{X}$  be as in Definition 20.16. Next we define a *G*-invariant metric  $d_C$  on the *G*-space  $G \times \overline{X}$ , depending on a constant C > 0. Recall that  $\overline{X}$  is assumed to be metrizable. We choose some (not necessarily *G*-invariant) metric  $d_{\overline{X}}$  on  $\overline{X}$  which generates the topology. Recall that we have already fixed some choice of a word-metric  $d_G$  on *G*.

20.4 Strictly Transfer Reducible Groups - Cover Version

**Definition 20.23.** Let C > 0. For (g, x),  $(h, y) \in G \times \overline{X}$  set

$$d_C((g,x),(h,y)) = \inf \sum_{i=1}^n C d_{\overline{X}}(g_i^{-1}x_{i-1},g_i^{-1}x_i) + d_G(g_{i-1},g_i)$$

where the infimum is taken over all finite sequences  $(g_0, x_0), \ldots, (g_n, x_n)$  with  $(g_0, x_0) = (g, x)$  and  $(g_n, x_n) = (h, y)$ .

The elementary proof of the next proposition can be found in [87, Proposition 4.3].

#### Proposition 20.24.

- (i) We obtain a *G*-invariant metric  $d_C$  on  $G \times \overline{X}$  equipped with the diagonal action by  $d_C$ ;
- (ii) We get  $d_G(g,h) \le d_C((g,x),(h,y))$  for all  $g,h \in G$  and  $x, y \in \overline{X}$ .

The next lemma illustrates Remark 20.1.

**Lemma 20.25.** Let G be a finitely generated group and  $\mathcal{F}$  be a family of subgroups. Suppose that G is strictly transfer  $\mathcal{F}$ -reducible – cover version in the sense of Definition 20.16

Then G is strictly  $\mathcal{F}$ -transfer reducible – almost equivariant version in the sense of Definition 20.11.

*Proof.* Let *N* be the number appearing in Assumption 20.19. By possibly enlarging *N* we can arrange that the dimension of the finite-dimensional abstract simplicial complex *K* whose geometric realization is *X* is less than or equal to *N*. Consider any  $\epsilon > 0$ . As *N*-transfer space, as required in Definition 20.11, we take  $\overline{X}$ . Note that  $\overline{X}$  is indeed an *N*-transfer space by Assumption 20.17, since any compact subset of *X* is a contained in |L| for a finite simplicial subcomplex *L* of *K* for which obviously dim $(L) \leq N$  holds.

Let  $\mathcal{U}$  be the open covering appearing in Assumption 20.19. Then we take  $\Sigma$  to be the simplicial complex given by the nerve of  $\mathcal{U}$  and we have by definition  $|\Sigma| = |\operatorname{Nerv}(\mathcal{U})|$ .

Fix a finite set of generators *S* and let  $d_G$  be the corresponding word metric on *G*. Fix C > 0. The function  $\overline{X} \to \mathbb{R}$  sending *x* to  $d_C((e, x), (s, sx))$  is continuous. Since  $\overline{X}$  is compact, we can find a constant *D* such that  $d_C((e, x), (s, sx)) \leq D$  holds for every  $x \in \overline{X}$  and  $s \in S$ . Choose  $\beta > 0$  satisfying  $4ND \leq \beta$  and  $\frac{16N^2}{\epsilon} < \beta$ . Then we get  $d_C((e, x), (s, sx)) \leq \frac{\beta}{4N}$  for every  $x \in \overline{X}$  and  $s \in S$ . Let  $f_{\mathcal{U}}: G \times \overline{X} \to |\operatorname{Nerv}(\mathcal{U})|$  be the *G*-map defined in (20.21). Proposition 20.22 implies that  $d_{L^1}(f_{\mathcal{U}}(e, x), f_{\mathcal{U}}(s, sx)) < \epsilon$  holds for every  $x \in \overline{X}$  and  $s \in S$ .

Define the desired map  $f: \overline{X} \to |\operatorname{Nerv}(\mathcal{U})|$  by sending x to  $f_{\mathcal{U}}(e, x)$ . Since we have

$$d_{L^{1}}(f(sx), sf(x)) = d_{L^{1}}(f_{\mathcal{U}}(e), sf_{\mathcal{U}}(e, x)) = d_{L^{1}}(f_{\mathcal{U}}(e), f_{\mathcal{U}}(s, sx)) < \epsilon,$$

the group G is strictly  $\mathcal{F}$ -transfer reducible group in the sense of Definition 20.11.  $\Box$ 

We conclude from Lemma 20.25 that Theorem 20.12 applies also to groups which are strictly transfer  $\mathcal{F}$ -reducible in the sense of Definition 20.16

In some sense one can also get the other direction of the implication appearing in Lemma 20.25 since maps from a topological space to the geometric realization of a finite-dimensional simplicial space translate to finite dimensional covers of the source, as we can pull back standard coverings of the simplicial complex.

**Remark 20.26 (Role of the compactification).** Note that in Definition 20.11 the compactification  $\overline{X}$  appearing in Definition 20.16 does not occur anymore and hence the criterion may be easier to verify. Moreover, this formulation isolates in a nice fashion what is really needed for the proof of the Farrell-Jones Conjecture. On the other hand, in many cases where the Farrell-Jones Conjecture has been proved, such as hyperbolic groups, finite-dimensional CAT(0)-groups, or mapping class groups, these compactifications  $\overline{X}$  and in particular their boundary  $\partial X = \overline{X} \setminus X$  were well-known and often played a role, leading to the necessary constructions, since the elements on the boundary correspond to geodesic rays emanating in the space X and going to infinity. It is conceivable that for future proofs for new groups Definition 20.11 may be more appropriate, but we also expect that some shadow of the notion of a compactification and its boundary and of non-positive curvature will be needed.

#### **20.5 Transfer Reducible Groups**

The strict versions of transfer reducible of the previous Sections 20.3 and 20.4 were sufficient to treat hyperbolic groups. In order to handle CAT(0) groups, one has to pass to the following generalizations of this notion, where homotopy coherent group actions come in and one has to drop *strict*.

**Definition 20.27 (Homotopy action of a (finitely presented) group on a space).** A *homotopy action* of a group G on a space X is a group homomorphism  $\rho: G \rightarrow [X, X]$  to the monoid of homotopy classes of self-maps of X.

Let *G* be finitely presented group with a finite presentation  $\langle S | R \rangle$ . A *homotopy action of the finitely presented group* ( $\varphi$ , *h*) of *G* on *X* is given by the following data:

(i) For every  $s \in S \cup S^{-1} = \{s \in G, s \text{ or } s^{-1} \text{ belongs to } S\}$ , we have a map

$$\varphi_s \colon X \to X;$$

(ii) For every word  $r = s_1 s_2 \cdots s_n \in R$  for  $s_i \in S \cup S^{-1}$ , we have a homotopy

$$h_r: \varphi_{s_1} \circ \varphi_{s_2} \circ \cdots \circ \varphi_{s_n} \simeq \mathrm{id}_X.$$

Note that a homotopy action of the finitely presented group G yields a homotopy G-action, but is a stronger notion, since the choice of the homotopies h for the relations is part of the structure.

20.5 Transfer Reducible Groups

**Definition 20.28 (Transfer**  $\mathcal{F}$ -reducible group). Let *G* be a finitely presented group and let  $\mathcal{F}$  be a family of subgroups. We call *G* transfer  $\mathcal{F}$ -reducible if for one (and hence all) finite presentation  $\langle S | R \rangle$  there exists a natural number *N* such that there is for any given  $\epsilon > 0$ :

- (i) an *N*-transfer space X in the sense of Definition 20.9 equipped with a homotopy G-action (φ, h) in the sense of Definition 20.27;
- (ii) an abstract simplicial  $(G, \mathcal{F})$ -complex  $\Sigma$  of dimension  $\leq N$ ;
- (iii) a map  $f: X \to |\Sigma|$  that is  $(\epsilon, \langle S | R \rangle)$ -almost *G*-equivariant, i.e., it satisfies:
  - (a) We have  $d_{L^1}(f(sx), sf(x)) < \epsilon$  for every  $s \in S$  and every  $x \in X$ ;
  - (b) For every x ∈ X and r ∈ R, the diameter of the subset h<sub>r</sub>({x} × [0, 1]) of X is ≤ ε.

**Remark 20.29.** Definition 20.28 is just the condition appearing in [67, Theorem B] and motivated by [78, Definition 1.8], one replaces the formulation in terms of open coverings by the formulation in terms of almost equivariant maps, in the spirit of Remark 20.1 or of Lemma 20.25 and its proof. In particular, a group which satisfies the notion of transfer reducible over  $\mathcal{F}$  in the sense of [78, Definition 1.8] is transfer  $\mathcal{F}$ -reducible group in the sense of Definition 20.28.

**Remark 20.30.** Note that [80, Main Theorem] implies that finite-dimensional CAT(0)-groups are finitely presented and transfer  $\mathcal{VCY}$ -reducible. A sketch of this proof can also be found in [67, Section 1.5]. We recall that a finite-dimensional CAT(0)-group is a group admitting a cocompact proper isometric action on a CAT(0)-space which has finite topological dimension.

**Theorem 20.31 (Transfer reducible groups and the Farrell-Jones Conjecture).** Let  $\mathcal{F}$  be a family of subgroups. Let G be a finitely presented group coming with a presentation  $\langle S | R \rangle$  such that G is transfer  $\mathcal{F}$ -reducible in the sense of Definition 20.28.

Then the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n(\mathbf{K}_{\mathcal{A}}(I(G)))$$

is bijective for  $n \le 0$  and surjective for n = 1 for every additive *G*-category, and the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)))$$

is bijective for every  $n \in \mathbb{Z}$  and every additive *G*-category with involution.

Theorem 20.31 is a reformulation of [87, Theorem 1.1] as pointed out in [67, Remarks 1.3.15 and 1.3.18] for the *K*-theory version. Its extension to *L*-theory follows from the proof of [78, Theorem B].

## **20.6 Strongly Transfer Reducible Groups**

In Theorem 20.31 we deal only with lower and middle *K*-theory. In order to treat higher algebraic *K*-theory, one needs to take higher homotopies into account.

The next definition is taken from [185, Definition 5.2], see also Wegner [992, Definition 2.1].

**Definition 20.32 (Strong homotopy action).** A *strong homotopy action*, sometimes also called a *homotopy coherent G-action*,  $(\Gamma, Z)$  of a group *G* on a topological space *Z* consists of a map

$$\Gamma \colon \prod_{k=0}^{\infty} \left( \left( \prod_{j=1}^{k} G \times [0,1] \right) \times G \times Z \right) \to Z$$

satisfying

$$\Gamma(g_k, t_k, \dots, g_1, t_1, g_0, z)$$

$$= \begin{cases} \Gamma(g_k, t_k, \dots, g_j, \Gamma(g_{j-1}, t_{j-1}, \dots, g_0, z)) & t_j = 0, 1 \le j \le k; \\ \Gamma(g_k, t_k, \dots, t_{j+1}, g_j g_{j-1}, t_{j-1}, \dots, g_0, z) & t_j = 1, 1 \le j \le k; \\ \Gamma(g_k, t_k, \dots, g_2, t_2, g_1, z) & g_0 = e; \\ \Gamma(g_k, t_k, \dots, g_{j+1}, t_{j+1} t_j, g_{j-1}, \dots, g_0, z) & g_j = e, 1 \le j \le k-2; \\ \Gamma(g_{k-1}, t_{k-1}, \dots, g_0, z) & g_k = e; \\ x & g_0 = e, k = 0. \end{cases}$$

Here we use the convention that non-existing entries are dropped, e.g.,  $g_k$ ,  $t_k$  in the first line if j = k or the entry  $t_{j-1}$  in the second line if j = 1.

Next we present the notion of *strongly transfers reducible over*  $\mathcal{F}$ , which we prefer to call *strongly*  $\mathcal{F}$ -*transfer reducible*, due to Wegner [992, Definition 3.1], where all the higher homotopies are taken into account.

Given a strong homotopy action  $\Gamma$  in the sense of Definition 20.32, we need to introduce the following notions. For  $k \in \mathbb{N}$ ,  $g \in G$ , and a subset  $S \subseteq G$  containing e and g, we define a subset of map(X, X)

(20.33)  $F_g(\Gamma, S, k)$ := { $\Gamma(g_k, t_k, \dots, g_0, ?): X \to X \mid g_i \in S, t_j \in [0, 1], g_k \dots g_0 = g$ }.

For  $(g, x) \in G \times X$ , we put

(20.34) 
$$S^0_{\Gamma,S,k}(g,x) = \{(g,x)\} \subseteq G \times X$$

and we define

(20.35) 
$$S^1_{\Gamma,S,k}(g,x) \subseteq G \times X$$

as the subset of all  $(h, y) \in G \times X$  with the property that there are  $a, b \in S$ ,  $f \in F_a(\Gamma, S, k)$ , and  $f' \in F_b(\Gamma, S, k)$  satisfying both f(x) = f'(y) and  $h = ga^{-1}b$ . For  $n \ge 2$  define inductively

(20.36) 
$$S^n_{\Gamma,S,k}(g,x) \subseteq G \times X$$

by

$$S^{n}_{\Gamma,S,k}(g,x) = \bigcup_{\{(h,y)\in S^{n-1}_{\Gamma,S,k}(g,x)\}} S^{1}_{\Gamma,S,k}(h,y).$$

**Exercise 20.37.** Let X be a G-space X. Consider the G-action as a homotopy G-action in the sense of Definition 20.32 in the obvious way. Define the subsets of G by

$$S[k] := \{ g_0 g_1 \cdots g_k \mid g_i \in S \};$$
  

$$S[k,n] := \{ a_1^{-1} b_1 \cdots a_n^{-1} b_n \mid a_1, \dots, a_n, b_1, \dots, b_n \in S[k] \}.$$

Show that then the sets  $F_g(\Gamma, S, k)$  of (20.33),  $S^1_{\Gamma,S,k}(g,x)$  of (20.35), and  $S^n_{\Gamma,S,k}(g,x)$  of (20.36) reduce to

$$F_{g}(\Gamma, S, k) = \{l_{g} \colon X \to X \mid g \in S[k]\} \subseteq \max(X, X);$$
  

$$S^{1}_{\Gamma,S,k}(g, x) = \{(gu, u^{-1}x) \mid u \in S[k, 1]\};$$
  

$$S^{n}_{\Gamma,S,k}(g, x) = \{(gv, v^{-1}x) \mid v \in S[k, n]\}.$$

**Definition 20.38 (Strongly transfer**  $\mathcal{F}$ **- reducible).** A group *G* is *strongly*  $\mathcal{F}$ *-transfer reducible* if there exists a natural number *N* with the following property: For all subsets  $S \subseteq G$  which satisfy  $S = \{g^{-1} \mid g \in S\}$  and contain the unit  $e \in G$ , and all natural numbers *n*, *k*, there are:

- an *N*-transfer space *X* in the sense of Definition 20.9;
- a strong homotopy action  $\Gamma$  on X in the sense of Definition 20.32;
- An open  $\mathcal{F}$ -cover  $\mathcal{U}$  of  $G \times X$ , where the *G*-action on  $G \times X$  is given by  $g' \cdot (g, x) = (g'g, x)$ , of dimension at most *N* such that for every  $(g, x) \in G \times X$  there exists an  $U \in \mathcal{U}$  with  $S^n_{\Gamma,S,k}(g, x) \subseteq U$ .

Hyperbolic groups are strongly transfer reducible over  $\mathcal{VCY}$  by the proof of [78, Proposition 2.1], as explained in [992, Example 3.2]. Wegner [992, Theorem 3.4] explains that finite-dimensional CAT(0)-groups are strongly transfer reducible over  $\mathcal{VCY}$  by [80, Main Theorem].

**Theorem 20.39 (Strongly transfer**  $\mathcal{F}$ -reducible groups and the Farrell-Jones Conjecture). Let G be a group and  $\mathcal{F}$  be a family of subgroups such that G is strongly  $\mathcal{F}$ -transfer reducible.

Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n \big( \mathbf{K}_{\mathcal{A}}(I(G)) \big);$$
  
$$H_n^G(\mathrm{pr}; \mathbf{H}_C) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_C) \to H_n^G(G/G; \mathbf{H}_C) = \pi_n \big( \mathbf{K}_C(I(G)) \big)$$

are bijective for every additive *G*-category  $\mathcal{A}$ , every right exact *G*-∞-category *C*, and every  $n \in \mathbb{Z}$ , and the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

*is bijective for every additive G-category with involution*  $\mathcal{A}$  *and every*  $n \in \mathbb{Z}$ *.* 

*Proof.* For the *K*-theoretic part in the setting of additive categories we refer to [992, Theorem 1.1], whose proof is based on [78, Theorem B]. The more general proof of the *K*-theory version for higher categories follows from Remark 20.3 and Theorem 20.61.

The *L*-theory part follows already from Theorem 20.31, since strongly  $\mathcal{F}$ -transfer reducible implies transfer  $\mathcal{F}$ -reducible by Lemma 20.2 (v).

We have explained in Section 19.5 why in the CAT(0)-setting one needs to consider strong homotopy G-actions instead of strict G-actions.

**Theorem 20.40 (Strongly transfer**  $\mathcal{VCY}$ -reducible groups and the Full Farrell-Jones Conjecture). Let G be a group such that G is strongly  $\mathcal{VCY}$ -transfer reducible.

Then G is a Farrell-Jones group, i.e., it satisfies the Full Farrell-Jones Conjecture 13.30.

*Proof.* Let F be a finite subgroup. Let  $\mathcal{VCY}^l$  be the family of subgroups H of  $G \wr F$ such that there is a subgroup  $H' \subseteq H$  of finite index such that H' is isomorphic to a finite product  $V_1 \times V_2 \times \cdots \times V_k$  for virtually cyclic groups  $V_i$ . Then  $G \wr F$ satisfies the Farrell-Jones Conjecture for K and L-theory with additive G-categories as coefficients, see Conjecture 13.11 and Conjecture 13.19, with respect to the family  $\mathcal{VCY}^{i}$  by [89, Theorem 5.1 (ii)]. Every element in  $\mathcal{VCY}^{i}$  is virtually abelian and hence satisfies the Farrell-Jones Conjecture for K and L-theory with additive Gcategories as coefficients, see Theorem 16.1 (ic). We conclude from the Transitivity Principle 15.13 that  $G \wr F$  satisfies the Farrell-Jones Conjecture for K and L-theory with additive G-categories as coefficients. Hence G satisfies the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products, see Conjecture 13.27 and Conjecture 13.28. It remains to show that G satisfies the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products, see Conjecture 13.29. This follows from Remark 20.3 and Theorem 20.62. П

## **20.7** Finitely *F*-Amenable Groups

Let G be a group and let  $\mathcal{F}$  be a family of subgroups. The next definition is taken from [70, Introduction], which is motivated by [87, Theorem 1.1].

**Definition 20.41 (Finitely**  $\mathcal{F}$ -amenable group action). For a natural number *N*, a *G*-action on a space *X* is called *N*- $\mathcal{F}$ -amenable if for all finite subsets *S* of *G* there exists an open  $\mathcal{F}$ -cover  $\mathcal{U}$  in the sense of Definition 20.18 of  $G \times X$  equipped with the diagonal *G*-action  $g \cdot (h, x) = (gh, gx)$  satisfying:

- The dimension of  $\mathcal{U}$  is at most N;
- The open  $\mathcal{F}$ -cover  $\mathcal{U}$  is S-long (in the group coordinate), i.e., for every  $(g, x) \in G \times X$  there is  $U \in \mathcal{U}$  with  $gS \times \{x\} \subseteq U$ .

A *G*-action on a space *X* is called *finitely*  $\mathcal{F}$ -*amenable* if it is *N*- $\mathcal{F}$ -amenable for some natural number *N*.

The proof of the next lemma can be found in [70, Lemma 4.2], whose proof is based on [449, Proposition 4.2]. It is useful for studying how N- $\mathcal{F}$ -amenability behaves under finite extensions, see [70, Section 4.1].

**Lemma 20.42.** Let G be a group G and  $\mathcal{F}$  be a family of subgroups. Then the following statements are equivalent for a compact metric space X and a G-action on it:

- (i) The G-action on X is N- $\mathcal{F}$ -amenable in the sense of Definition 20.41;
- (ii) For every finite subset S ⊆ G and every ε > 0, there exists an abstract simplicial (G, F)-complex Σ of dimension at most N together with a map f: X → |Σ| that is (ε, S)-almost G-equivariant, i.e., we have d<sub>L1</sub>(f(sx), sf(x)) ≤ ε for every s ∈ S and every x ∈ |Σ|.

**Exercise 20.43.** Suppose that *G* is finitely generated. Let  $S_1$  and  $S_2$  be two finite sets of generators. Then the second condition appearing in Lemma 20.42 holds for  $S_1$  if and only if holds for  $S_2$ .

Recall that a metric space X is an ER (= Euclidean retract) if it can be embedded in some  $\mathbb{R}^n$  as a retract. A compact metric space X is an ER if and only if it is a finite-dimensional contractible ANR, see [154, Theorem V.10.1 on page 122].

**Definition 20.44 (Finitely**  $\mathcal{F}$ -amenable group). We call a group *G* finitely  $\mathcal{F}$ -amenable if *G* admits a finitely  $\mathcal{F}$ -amenable action on a compact ER.

**Theorem 20.45 (Finitely**  $\mathcal{F}$ -amenable actions and the Farrell-Jones Conjecture). Let G be a group and  $\mathcal{F}$  be a family of subgroups. Suppose that G is finitely  $\mathcal{F}$ -amenable.

Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n \big( \mathbf{K}_{\mathcal{A}}(I(G)) \big);$$
  
$$H_n^G(\mathrm{pr}; \mathbf{H}_C) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_C) \to H_n^G(G/G; \mathbf{H}_C) = \pi_n \big( \mathbf{K}_C(I(G)) \big)$$

are bijective for every additive *G*-category  $\mathcal{A}$ , every right exact *G*-∞-category *C*, and every  $n \in \mathbb{Z}$ , and the assembly map

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$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

is bijective for every additive *G*-category with involution  $\mathcal{A}$  and every  $n \in \mathbb{Z}$ .

*Proof.* This follows from the axiomatic results in [78, Theorem 1.1] and [87, Theorem 1.1], as explained in [70, Theorem 4.8] for additive *G*-categories (with involution) as coefficients. (In [70, Theorem 4.8] it is required that  $\mathcal{F}$  is closed under passage to overgroups of finite index but this is not necessary, see also [68, Theorem 4.3]). The *K*-theoretic version for higher *G*-categories as coefficients follows from Remark 20.3 and Theorem 20.62.

Let *C* be a class of groups that is closed under isomorphisms and taking subgroups. Let ac(C) be the class of all groups *G* that admit a finitely generated C(G)-amenable action on an ER where C(G) is the family of subgroups of *G* which belong to *C*. Because of the action on a one-point-space we get  $C \subseteq ac(C)$ . Starting with  $ac^{0}(C) = C$ , we can define inductively  $ac^{n+1}(C) = ac(ac^{n}(C))$ . We set

(20.46) 
$$\operatorname{AC}(C) = \bigcup_{n=0}^{\infty} \operatorname{ac}^{n}(C).$$

Let VNIL be the class of virtually nilpotent groups and VSOLV be the class of virtually solvable groups.

**Theorem 20.47 (Groups in** AC(VSOLV) satisfy the Full Farrell-Jones Conjecture). Every group in AC(VSOLV) satisfies the Full Farrell-Jones Conjecture 13.30.

*Proof.* This follows from the Transitivity Principle, see Theorem 15.13, Theorem 16.1 (ic), and Theorem 20.45 for additive *G*-categories as coefficients as explained in [70, Corollary 4.10 and Remark 9.4]. For the setting of higher *G*-categories one needs to replace Theorem 20.45 by [185, Theorem 1.4].  $\Box$ 

The main result in [70, Lemma 9.3] says that mapping class groups belong to AC(VNIL) and hence satisfy the Full Farrell-Jones Conjecture 13.30, see [70, Theorem A and Remark 9.4].

## 20.8 Finitely Homotopy F-Amenable Groups

Next we state the version of finitely homotopy  $\mathcal{F}$ -amenable groups appearing in [185, Definition 5.4], which goes back to [78, 87, 992] and is essentially the one appearing in Bartels [69, Definition 2.11 and Theorem 2.12].

An AR (= absolute retract) is a metrizable topological space such that for every embedding  $i: X \to Y$  as a closed subspace into a metric space Y there is a retraction  $r: Y \to X$ , or, equivalently, for every metric space Z, every closed subset  $Y \subseteq Z$ ,

and every (continuous) map  $f: Y \to X$ , there exists an extension  $F: Z \to X$  of f. An ANR is an AR if and only if it is contractible, see [505, Theorem 7.1 and Proposition 7.2 in Chapter III on page 96].

**Definition 20.48 (Finitely homotopy**  $\mathcal{F}$ **-amenable group).** Let *G* be a group and let  $\mathcal{F}$  be a family of subgroups. We call *G finitely homotopy*  $\mathcal{F}$ *-amenable* if there exist:

- (i) A sequence {Γ<sub>n</sub>, Z<sub>n</sub>}<sub>n∈ℕ</sub> of homotopy coherent *G*-actions in the sense of Definition 20.32;
- (ii) A sequence {Σ<sub>n</sub>}<sub>n∈ℕ</sub> of abstract simplicial complexes coming with a simplicial *G*-action;
- (iii) A sequence  $\{f_n\}_{n \in \mathbb{N}}$  of continuous maps  $f_n \colon Z_n \to |\Sigma_n|$ ,

such that the following holds:

- (a) For every  $n \in \mathbb{N}$  the space  $Z_n$  is a compact contractible AR;
- (b) For every  $n \in \mathbb{N}$  the isotropy groups of  $|\Sigma_n|$  belong to  $\mathcal{F}$ ;
- (c) There exists a natural number N with  $\dim(\Sigma_n) \leq N$  for all  $n \in \mathbb{N}$ ;
- (d) For every  $k \in \mathbb{N}$  and elements  $g_0, g_1, \ldots, g_k$  in G we have

$$\lim_{n \to \infty} \sup_{\substack{(t_1, \dots, t_k) \in [0,1]^k, \\ z \in \mathbb{Z}_n}} d_{L^1}^{2_n} (f_n(\Gamma_n(g_k, t_k, \dots, g_1, t_1, g_0, z)), g_k \dots g_0 f_n(z)) = 0.$$

Note that a finitely homotopy  $\mathcal{F}$ -amenable group is in particular a Dress-Farrell-Hsiang-Jones group, see [185, Remark 7.2 (1)]. Hence Theorem 20.61 and Theorem 20.62 apply to homotopy  $\mathcal{F}$ -amenable groups, see also [185, Theorem 5.1].

**Remark 20.49.** The condition formulated in Definition 20.48 is slightly weaker than the assumptions in Bartels [69, Theorem 2.12] since we do not require a uniform bound on the dimension of the AR-s  $Z_n$ . In practice, however, the dimensions of the simplicial complexes  $\Sigma_n$  are usually bounded in terms of the dimensions of the spaces  $Z_n$ . In this case,  $Z_n$  is a sequence of ER-s with uniformly bounded covering dimension.

## **20.9 Dress-Farrell-Hsiang Groups**

**Definition 20.50 (Dress group).** A finite group *D* is called a *Dress group* if there exist (not necessarily distinct) prime numbers *p* and *q* and subgroups  $P \subseteq C \subseteq D$  such that *P* is normal in *C* and *C* is normal in *D*, *P* is a *p*-group, *C*/*P* is cyclic, and D/C is a *q*-group.

For *F* a finite group, we denote the family of Dress subgroups of *F* by  $\mathcal{D}(F)$ .

**Exercise 20.51.** Show for a finite group *F* that  $\mathcal{H}(F) \subseteq \mathcal{D}(F)$  holds.

The next definition is taken from [185, Definition 6.3].

**Definition 20.52 (Dress-Farrell-Hsiang group).** Let G be a finitely generated group and let  $\mathcal{F}$  be a family of subgroups. We call G a Dress-Farrell-Hsiang group over  $\mathcal{F}$  if there exist:

- (i) A sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite groups;
- (ii) A sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of surjective group homomorphism  $\alpha_n \colon G \to F_n$ ;
- (iii) A collection  $\{(\Sigma_n, D) \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ , where  $\Sigma_n$  is an abstract simplicial complex with a simplicial  $\alpha_n^{-1}(D)$ -action;
- (iv) A collection  $\{f_n \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$  of maps of sets  $f_{n,D} \colon G \to \Sigma_{n,D}$ ,

such that the following holds:

- (a) For every  $n \in \mathbb{N}$  and  $D \in \mathcal{F}(F_n)$ , the  $\alpha_n^{-1}(D)$ -isotropy groups of  $|\Sigma_{n,D}|$  belong to the family  $\mathcal{F}|_{\alpha_n^{-1}(D)} = \{H \cap \alpha_n^{-1}(D) \mid H \in \mathcal{F}\};\$ (b) There exists a natural number N with dim $(\Sigma_{n,D}) \leq N$  for all  $n \in \mathbb{N}$  and
- $D \in \mathcal{D}(F_n);$
- (c) For every  $n \in \mathbb{N}$  and  $D \in \mathcal{F}(F_n)$ , the map  $f_{n,D}$  is  $\alpha_n^{-1}(D)$ -equivariant, where  $\alpha_n^{-1}(D)$  acts on G from the left;
- (d) For every  $g \in G$  we have

$$\lim_{n\to\infty} \sup_{D\in\mathcal{D}(F_n),\gamma\in G} d_{L^1}^{\sum_{n,D}} (f_{n,D}(\gamma g), f_{n,D}(\gamma)) = 0.$$

Note that a Dress-Farrell-Hsiang group is in particular a Dress-Farrell-Hsiang-Jones group, see [185, Remark 7.2 (2)]. Hence Theorem 20.61 and Theorem 20.62 apply to Dress-Farrell-Hsiang groups, see also [185, Theorem 6.1].

The next result is due to Oliver [770, Theorem 7].

Theorem 20.53 (Fixed point free smooth actions of finite groups on disks). Let G be a finite group. Then G is not a Dress group if and only if G acts smoothly on some disk  $D^n$  such that  $(D^n)^G$  is empty.

Exercise 20.54. Let G be a finite abelian group. Show that G admits a smooth action on some disk  $D^n$  with  $(D^n)^G = \emptyset$  if and only if there are three distinct primes  $p_1$ ,  $p_2$ , and  $p_3$  such that the  $p_k$ -Sylow subgroup of G is non-cyclic for k = 1, 2, 3.

The following definition is equivalent to the one in [960, Definition 8.1].

**Definition 20.55** (A-theoretic Swan ring  $Sw^A(G)$ ). For a group G define the A-theoretic Swan ring  $Sw^A(G)$  as follows. The underlying abelian group is defined as follows. Every compact G-CW-complex X, or, equivalently, G-CW-complex X, whose underlying CW-complex is finite, or, equivalently, G-CW-complex X such that X has finitely many equivariant cells and the isotropy group of each  $x \in X$  has finite index in G, defines an element [X] in  $Sw^A(G)$ . The relations are given as follows. If X and Y are compact G-CW-complexes and there is a G-map  $f: X \to Y$ such that f is a homotopy equivalence (after forgetting the G-actions), then we require [X] = [Y]. If the compact G-CW-complex X is the union of sub G-CWcomplexes  $X_1$  and  $X_2$  and  $X_0$  is the intersection of  $X_1$  and  $X_2$ , then we require

#### 20.9 Dress-Farrell-Hsiang Groups

 $[X] = [X_1] + [X_2] - [X_0]$ . The multiplication comes from the cartesian product of two compact *G*-*CW*-complexes equipped with the diagonal *G*-action. The zero element is represented by the empty set and the unit by G/G.

The group  $Sw^A(G)$  is the A-theoretic analog of the Swan group  $Sw^p(G)$  introduced in Definition 12.65.

**Exercise 20.56.** Let G be a (not necessarily finite) group. Define for a compact G-CW-complex X the element

$$s(X) := \sum_{n \ge 0} (-1)^n \cdot [C_n^c(X)] \in \operatorname{Sw}^p(G),$$

where  $C^c_*(X)$  is the cellular  $\mathbb{Z}G$ -chain complex of X. Show that we obtain a well-defined ring homomorphism

$$s: \operatorname{Sw}^{A}(G) \to \operatorname{Sw}^{p}(G), [X] \mapsto s(X).$$

**Exercise 20.57.** Let *G* be a (not necessarily finite) group. Let A(G) be the Burnside ring defined in Example 12.63. Show that we obtain a well-defined surjective ring homomorphism  $a: A(G) \rightarrow Sw^A(G)$  by viewing a finite *G*-set as a compact 0-dimensional *G*-*CW*-complex.

**Exercise 20.58.** Let p be a prime. Then we get a sequence of homomorphisms of abelian groups

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{u} A(\mathbb{Z}/p) \xrightarrow{a} \mathrm{Sw}^{A}(\mathbb{Z}/p) \xrightarrow{s} \mathrm{Sw}^{p}(\mathbb{Z}/p) \xrightarrow{c} \mathbb{Z} \oplus \mathbb{Z}$$

where *u* sends (m, n) to  $m \cdot [\mathbb{Z}/p] + n \cdot [\{*\}]$ , the map *a* has been defined in Exercise 20.57, the map *s* has been defined in Exercise 20.56, and *c* sends [*M*] to  $(\mathrm{rk}_{\mathbb{Z}}(M), \mathrm{rk}_{\mathbb{Z}}(M^{\mathbb{Z}/p}))$ .

Show that *u* and *c* are well-defined and that the maps *u*, *a*, and  $c \circ s$  are bijective.

The appearance of the Dress subgroups in Definition 20.60 is due to the result of Ullmann-Winges [960, Theorem 8.7 ] that for a finite group *F* and the family  $\mathcal{D}(F)$  of Dress subgroup there are elements  $\mu_H \in Sw^A(H)$  for  $H \in \mathcal{D}(F)$  satisfying

(20.59) 
$$1_{\operatorname{Sw}^{A}(F)} = \sum_{H \in \mathcal{H}} \operatorname{ind}_{H}^{F}(\mu_{H}) \in \operatorname{Sw}^{A}(F),$$

where the homomorphisms  $\operatorname{ind}_{H}^{F}$ :  $\operatorname{Sw}^{A}(H) \to \operatorname{Sw}^{A}(F)$  are induced by induction. The proof of (20.59) is based on Oliver's Theorem 20.53. Note that one needs to pass to the *A*-theoretic Swan ring in the context of higher categories since  $\operatorname{Sw}^{P}(F)$ acts on  $K_{n}(RF)$  but for instance not on  $\pi_{n}(A(BG))$ .

Note that a Dress-Farrell-Hsiang group is in particular a Dress-Farrell-Hsiang-Jones group, see [185, Remark 7.2 (2)]. Hence Theorem 20.61 and Theorem 20.62 apply to Dress-Farrell-Hsiang groups.

## 20.10 Dress-Farrell-Hsiang-Jones Groups

The next definition is taken from [185, Definition 7.1]. Recall that it essentially implies all the other ones, see Remark 20.3.

**Definition 20.60 (Dress-Farrell-Hsiang-Jones group).** Let G be a finitely generated group and  $\mathcal{F}$  be a family of subgroups. We call G a *Dress-Farrell-Hsiang-Jones group* over  $\mathcal{F}$  if there exist:

- (i) A sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite groups;
- (ii) A sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of surjective group homomorphism  $\alpha_n \colon G \to F_n$ ;
- (iii) A collection  $\{(\Gamma_{n,D}, Z_{n,D}) \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$  of homotopy coherent *G*-actions in the sense of Definition 20.32;
- (iv) A collection  $\{\Sigma_{n,D} \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ , where  $\Sigma_{n,D}$  is an abstract simplicial complex with a simplicial  $\alpha_n^{-1}(D)$ -action;
- (v) A collection  $\{f_{n,D} \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$  of continuous maps  $f_{n,D} \colon G \times Z_{n,D} \to |\Sigma_{n,D}|$ ,

such that the following holds:

- (a) For every  $n \in \mathbb{N}$  and every  $D \in \mathcal{D}(F_n)$ , the topological space  $Z_{n,D}$  is a compact AR;
- (b) For every  $n \in \mathbb{N}$  and every  $D \in \mathcal{D}(F_n)$ , the  $\alpha_n^{-1}(D)$ -isotropy groups of  $|\Sigma_{n,D}|$ belong to the family  $\mathcal{F}|_{\alpha_n^{-1}(D)} = \{H \cap \alpha_n^{-1}(D) \mid H \in \mathcal{F}\};$
- (c) The exists a natural number N with dim $(\Sigma_{n,D}) \leq N$  for all  $n \in \mathbb{N}$  and  $D \in \mathcal{D}(F_n)$ ;
- (d) For every  $n \in \mathbb{N}$  and every  $D \in \mathcal{D}(F_n)$ , the map  $f_{n,D}$  is  $\alpha_n^{-1}(D)$ -equivariant, where  $\alpha_n^{-1}(D)$  acts on  $G \times Z_{n,D}$  diagonally from the left;
- (e) For every  $k \in \mathbb{N}$  and elements  $g_0, g_1, \ldots, g_k$  in G we have

$$\lim_{n \to \infty} \left( \sup_{\substack{D \in \mathcal{D}(F_n), \gamma \in G \\ (t_1, \dots, t_k) \in [0, 1]^k, \\ z \in Z_{n, D}}} u_n \right) = 0$$

for

$$u_n := d_{L^1}^{\Sigma_{n,D}}(f_{n,D}(\gamma, \Gamma_{n,D}(g_k, t_k, \dots, g_1, t_1, g_0, z)), f_{n,D}(\gamma_k g_k \dots g_0, z)).$$

The next result is taken from [185, Theorem 7.4]. We will give more details of its proof in Chapter 24.

**Theorem 20.61 (Dress-Farrell-Hsiang-Jones groups and the** *K***-theoretic Farrell-Jones Conjecture).** Let G be a finitely generated group which is a Dress-Farrell-Hsiang-Jones group over  $\mathcal{F}$  in the sense of Definition 20.60.

Then the assembly maps

$$H_n(\mathrm{pr}; \mathbf{K}_{\mathcal{A}}) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(G/G; \mathbf{K}_{\mathcal{A}}) = \pi_n \big( \mathbf{K}_{\mathcal{A}}(I(G)) \big);$$
  
$$H_n^G(\mathrm{pr}; \mathbf{H}_C) \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{H}_C) \to H_n^G(G/G; \mathbf{H}_C) = \pi_n \big( \mathbf{K}_C(I(G)) \big),$$

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are bijective for every additive G-category  $\mathcal{A}$ , every right exact G- $\infty$ -category C, and every  $n \in \mathbb{Z}$ .

**Theorem 20.62 (Dress-Farrell-Hsiang-Jones groups and the** *K***-theoretic Farrell-Jones Conjecture with finite wreath products).** Let *G* be finitely generated group which is a Dress-Farrell-Hsiang-Jones group over VCY in the sense of Definition 20.60.

Then G satisfies the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with finite wreath products, see Conjecture 13.27, and the K-theoretic Farrell-Jones Conjecture with coefficients in higher G-categories with finite wreath products, see Conjecture 13.29.

*Proof.* This follows from the last paragraph starting on page 127 in [185] and Theorem 16.1 (ic).  $\Box$ 

**Remark 20.63.** Note that we need for the proof of Theorem 20.61 and Theorem 20.62 the assumption that *G* is finitely generated. If *G* is finitely generated, then it suffices in Definitions 20.48, 20.52, and 20.60 to check the last requirement appearing there only for the elements  $g_1, g_2, \ldots, g_k$  or *g* contained in one fixed finite set *S* of generators, since then it hold automatically for any finite subset of *G* or any element of *G*, essentially because of the triangle inequality.

#### **20.11 Notes**

It is conceivable that, if G is a Dress-Farrell-Hsiang-Jones group over  $\mathcal{F}$  in the sense of Definition 20.60. the assembly map

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{F}_2}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n \big( \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G)) \big)$$

is bijective for every additive *G*-category  $\mathcal{A}$  (with involution) and every  $n \in \mathbb{Z}$ . Details of this claim have not been checked. If this claim is true, one would get that a group *G* which is a Dress-Farrell-Hsiang-Jones group over  $\mathcal{VCY}$  is a Farrell-Jones group, i.e., it satisfies the Full Farrell-Jones Conjecture 13.30.

So far the *L*-theoretic version of the Farrell-Jones Conjecture has only been established for additive categories with involution. Christoph Winges is at the time of writing working on a generalization to the setting of higher categories for all Dress-Farrell-Hsiang-Jones groups.

Sawicki [891] discusses the notion of *equivariant asymptotic dimension*, which is closely related to the notion of a transfer  $\mathcal{F}$ -reducible group, see Definition 20.28, and of finitely  $\mathcal{F}$ -amenable group, see Definition 20.44.

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There is also the notion of an *almost transfer*  $\mathcal{F}$ -*reducible group*, which is introduced in [89, Definition 5.3] and is weaker than transfer  $\mathcal{F}$ -reducible group. Both conditions are equivalent if the family  $\mathcal{F}$  is closed under the passage to overgroups of finite index, see for instance [891, Corollary 2.6].

# Chapter 21 Controlled Topology Methods

## **21.1 Introduction**

In this chapter we explain and prove in detail for any group G and any G-CWcomplex X what we have briefly discussed in Subsection 19.4.5. We will allow more
general coefficients than rings or additive G-categories, namely, categories with G-support, see Definition 21.1. This notion seems to be the most general one and
illustrates nicely what is needed to successfully establish the desired constructions
and theorems appearing in this chapter.

Given such a category with G-support  $\mathcal{B}$ , we will construct covariant functors

 $\mathcal{O}^{G}(-;\mathcal{B}), \mathcal{T}^{G}(-;\mathcal{B}), \mathcal{D}^{G}(-;\mathcal{B}): G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{ADDCAT}$ 

and for every *G*-*CW*-complex X in Theorem 21.19 the so-called TOD-sequence,

$$K(\mathcal{T}^G(X;\mathcal{B})) \to K(\mathcal{O}^G(X;\mathcal{B})) \to K(\mathcal{D}^G(X;\mathcal{B})),$$

which is a weak homotopy fibration of spectra and natural in X.

Actually, the functor  $\mathcal{D}^G(-;\mathcal{B})$  digests *G-CW*-pairs, and we will prove in Theorem 21.26 that we obtain a *G*-homology theory with values in  $\mathbb{Z}$ -modules in the sense of Definition 12.1 by the covariant functor from the category of *G-CW*-pairs to the category of  $\mathbb{Z}$ -graded abelian groups sending (X, A) to  $K_*(\mathcal{D}^G(X, A; \mathcal{B}))$ . We will analyze the coefficients of this *G*-homology theory, namely, the covariant functor

$$\mathbf{K}(\mathcal{D}^G(?;\mathcal{B})): \operatorname{Or}(G) \to \operatorname{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{D}^G(G/H;\mathcal{B}))$$

in Section 21.8.

In Lemma 21.76 (i) we will identify the assembly map appearing in the Meta-Isomorphism Conjecture 15.2 associated to the *G*-homology theory  $H^G_*(-; \mathbf{K}(\mathcal{B}(?)_{\oplus}))$  for a specific covariant Or(G)-spectrum  $\mathbf{K}(\mathcal{B}(?)_{\oplus})$  introduced in (21.68) and the family  $\mathcal{F}$ 

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{K}(\mathcal{B}(?)_{\oplus})) \to H_n^G(\{\bullet\}; \mathbf{K}(\mathcal{B}(?)_{\oplus})) = K_n(\mathcal{B}_{\oplus})$$

with the homomorphism induced by the projection  $E_{\mathcal{F}}(G) \to G/G$ 

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \to K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B})) = K_n(\mathcal{B}_{\oplus})$$

for every  $n \in \mathbb{Z}$ . Moreover we show in Lemma 21.76 (ii) that the Meta-Isomorphisms Conjecture 15.2 for the *G*-homology theory  $H^G_*(-; \mathbf{K}(\mathcal{B}(?)))$  and the family  $\mathcal{F}$  is true if and only if the spectrum  $\mathbf{K}(O^G(E_{\mathcal{F}}(G); \mathcal{B}))$  is weakly contractible.

Note that for a *G*- $\mathbb{Z}$ -category  $\mathcal{A}$  we can define the category with *G*-support  $\mathcal{A}[G]$ , see Example 21.2, and obtain an isomorphism

$$K_n(\mathcal{A}[H]_{\oplus}) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H;\mathcal{A}[G]))$$

for every  $n \in \mathbb{Z}$  and every subgroup  $H \subseteq G$ , see Remark 21.82. This boils down for a ring *R* coming with a group homomorphism  $\rho: G \to \operatorname{aut}(R)$  to an isomorphism, see Example 21.83

$$K_n(R_{\rho|H}[H]) \xrightarrow{=} K_{n+1}(\mathcal{D}^G(G/H;\underline{R}[G])).$$

So for an adequate choice of  $\mathcal{B}$ , the homomorphism  $K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \rightarrow K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B}))$  can be identified with the map appearing in the *K*-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive *G*-categories, and of course analogously for rings as coefficients. All this carries over to *L*-theory.

We also deal with a version  $\mathcal{D}_0^G(X; \mathcal{B})$  with zero-control over  $\mathbb{N}$  which also yields a *G*-homology theory, see Theorem 21.126, and is related to  $\mathcal{D}^G$  by a weak homotopy pushout, see Theorem 21.109,

These functors  $\mathcal{D}_0^G(X; \mathcal{B})$  occur in the transfer criterion for the Farrell-Jones Conjecture appearing in Theorem 23.70. The benefit of Theorem 23.70 is that it suffices to construct the transfer only on homogeneous spaces and for the functor  $\mathcal{D}_0^G$  which has the pleasant feature that it is defined with zero-control in the  $\mathbb{N}$ -direction. This has, for instance, been exploited in [81, Remarks 6.14 and 7.17].

The setup with categories with *G*-support as coefficients is too general to expect that the Farrell-Jones Conjecture holds, see Remark 21.85.

There are many different versions of the categories  $\mathcal{D}^G(X)$  constructed below and also the control conditions may vary. We have decided to concentrate in this chapter on one case, namely the setting with continuous control, as established in [74], and to use the version of the setup for totally disconnected groups, see [81], reduced to discrete groups, where it simplifies considerably. The hope is that the reader can easily understand the arguments in other but related situations if she or he has absorbed the cases presented in this chapter. Moreover, we give all the details, whereas in the literature the arguments are sometimes rather sketchy.

#### **21.2** The Definition of a Category with *G*-Support

Let *G* be a discrete group. A  $\mathbb{Z}$ -*category* is a small category  $\mathcal{A}$  enriched over the category of  $\mathbb{Z}$ -modules, i.e., for every two objects *A* and *A'* in  $\mathcal{A}$  the set of morphisms mor<sub> $\mathcal{A}$ </sub>(*A*, *A'*) has the structure of a  $\mathbb{Z}$ -module for which composition is a  $\mathbb{Z}$ -bilinear map.

**Definition 21.1 (Category with** *G***-support).** A *category with G-support* is a pair  $\mathcal{B} = (\mathcal{B}, \text{supp}_G)$  consisting of:

- A  $\mathbb{Z}$ -category  $\mathcal{B}$ ;
- A map called the *support function*

 $\operatorname{supp}_G \colon \operatorname{mor}(\mathcal{B}) \to \{ \text{finite subsets of } G \}.$ 

We require that the following axioms are satisfied for all objects *B* in  $\mathcal{B}$  and all morphisms  $u, u': B_1 \to B_2, v: B_2 \to B_3$  in  $\mathcal{B}$ :

- (i)  $\operatorname{supp}_G(u) = \emptyset \iff u = 0;$
- (ii)  $\operatorname{supp}_G(v \circ u) \subseteq \operatorname{supp}_G(v) \cdot \operatorname{supp}_G(u) := \{gg' \mid g \in \operatorname{supp}_G(v), g' \in \operatorname{supp}_G(u)\};$ (iii)  $\operatorname{supp}_G(u + u') \subseteq \operatorname{supp}_G(u) \cup \operatorname{supp}_G(u');$
- (iii)  $\operatorname{supp}_G(u + u) \cong \operatorname{supp}_G(u) \circ \operatorname{sup}_G(u)$ (iv)  $\operatorname{supp}_G(u) = \operatorname{supp}_G(-u)$ ;
- (iv) SuppG(u) = suppG(u), (v) For every chiest B in  $\mathcal{R}$  its support support
- (v) For every object B in  $\mathcal{B}$  its support supp<sub>G</sub>(B) := supp<sub>G</sub>(id<sub>B</sub>) is {e}.

**Example 21.2.** Let  $\mathcal{A}$  be a G- $\mathbb{Z}$ -category, i.e., a  $\mathbb{Z}$ -category with G action by isomorphisms of  $\mathbb{Z}$ -categories. Define the category with G-support  $\mathcal{A}[G]$  as follows. The set of objects in  $\mathcal{A}[G]$  is the set of objects in  $\mathcal{A}$ . For two objects A and A' in  $\mathcal{A}$ , a morphism  $\phi: A \to A'$  in  $\mathcal{A}[G]$  is a formal sum  $\sum_{g \in g} \phi_g \cdot g$  where  $\phi_g: gA \to A'$  is a morphism in  $\mathcal{A}$  from gA to A' and its G-support

$$\operatorname{supp}_G(\phi) := \{g \in G \mid \phi_g \neq 0\}$$

is assumed to be finite. The composite of  $\phi: A \to A'$  and  $\psi: A' \to A''$  is given by convolution, i.e.,

$$(\psi \circ \phi)_g = \sum_{\substack{g',g'' \in G \\ g = g'g''}} \psi_{g'} \circ g' \phi_{g''} \colon gA \to A''.$$

The identity of the object  $\mathcal{A}$  is given by  $\sum_{g \in g} \phi_g \cdot g$ , where  $\phi_e = id_A$  and  $\phi_g = 0$  for  $g \neq e$ . The  $\mathbb{Z}$ -structure on  $\operatorname{mor}_{\mathcal{A}[G]}(A, A')$  is given by

$$m \cdot \left(\sum_{g} \phi_{g} \cdot g\right) + n \cdot \left(\sum_{g} \psi_{g} \cdot g\right) = \sum_{g} (m \cdot \phi_{g} + n \cdot \psi_{g}) \cdot g.$$

One easily checks that  $\mathcal{A}[G]$  is a  $\mathbb{Z}$ -category and becomes with the notion of the support above a category with *G*-support.

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Given a  $\mathbb{Z}$ -category, let  $\mathcal{A}_{\oplus}$  be the associated additive category whose objects are finite tuples of objects in  $\mathcal{A}$  and whose morphisms are given by matrices of morphisms in  $\mathcal{A}$  (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [686, Section 1.3].

Let *R* be a ring. We denote by <u>*R*</u> the  $\mathbb{Z}$ -category with precisely one object whose  $\mathbb{Z}$ -module of endomorphisms is given by *R* with its  $\mathbb{Z}$ -module structure and composition is given by the multiplication in *R*. Then we can consider the additive category <u>*R*</u><sub> $\oplus$ </sub>. It can be identified with the version of <u>*R*</u><sub> $\oplus$ </sub> appearing in Section 6.6.

**Example 21.3.** Let *R* be a ring coming with a group homomorphism  $\rho: G \to \operatorname{aut}(R)$  to the group of ring automorphisms of *R*. We can consider <u>*R*</u> as a *G*- $\mathbb{Z}$ -category. We have defined the  $\mathbb{Z}$ -category <u>*R*[*G*] in Example 21.2. It yields the additive category <u>*R*[*G*]</u> $_{\oplus}$ .</u>

Denote by  $R_{\rho}[G]$  the twisted group ring. We have defined the additive category  $\underline{R_{\rho}[G]}_{\oplus}$  above. One easily checks that the additive categories  $\underline{R}[G]_{\oplus}$  and  $\underline{R_{\rho}[G]}_{\oplus}$  are isomorphic. Recall that  $\underline{R_{\rho}[G]}_{\oplus}$  is equivalent to the category  $R_{\rho}[G]$ - $\overline{\text{MOD}}_{\text{fgf}}^{\oplus}$  of finitely generated free  $R_{\rho}[G]$ -modules, see (6.42).

## **21.3** The Additive Category $O^G(X; \mathcal{B})$

## **21.3.1** The Definition of $O^G(X; \mathcal{B})$

Let X be a G-CW-complex and  $\mathcal{B}$  be a category with G-support in the sense of Definition 21.1. We define an additive category  $O^G(X; \mathcal{B})$  as follows.

**Definition 21.4** ( $O^G(X; \mathcal{B})$ ). An object in  $O^G(X; \mathcal{B})$  is a quadruple **B** = ( $S, \pi, \eta, B$ ) consisting of a set S and maps  $\pi: S \to X, \eta: S \to \mathbb{N}$ , and  $B: S \to ob(\mathcal{B})$  satisfying:

• Compact support over X

The image of  $\pi: S \to X$  is contained in a compact subset of *X*;

• Local finiteness over  $\mathbb{N}$ 

For every  $t \in \mathbb{N}$  the preimage  $\eta^{-1}(t)$  is a finite subset of *S*.

For two objects **B** =  $(S, \pi, \eta, B)$  and **B**' =  $(S', \pi', \eta', B')$ , a morphism  $\phi : \mathbf{B} \to \mathbf{B}'$ is given by a collection  $\{\phi_{s,s'} : B(s) \to B'(s') \mid s \in S, s' \in S'\}$  of morphisms in  $\mathcal{B}$ satisfying the following conditions:

• Finite G-support

There exists a finite subset  $F \subset G$  such that

$$\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$$

holds for all  $s \in S$  and  $s' \in S'$ ;

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- Bounded control over  $\mathbb{N}$

There exists a natural number *n* such that for  $s \in S$  and  $s' \in S'$  the implication

$$\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \le n$$

holds;

• Continuous control

For every  $x \in X$  and every open  $G_x$ -invariant neighborhood  $U \subseteq X$  of x, there exists an open  $G_x$ -invariant neighborhood  $U' \subseteq X$  of x satisfying  $U' \subseteq U$  and a natural number r' such that for  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$  the implications

(21.5) 
$$g\pi(s) \in U', \eta(s) \ge r' \implies \pi'(s') \in U;$$

(21.6) 
$$g^{-1}\pi'(s') \in U', \eta'(s') \ge r' \implies \pi(s) \in U,$$

hold.

Given three objects

$$\mathbf{B} = (S, \pi, \eta, B), \ \mathbf{B}' = (S', \pi', \eta', B'), \text{ and } \mathbf{B}'' = (S'', \pi'', \eta'', B'')$$

and morphisms  $\phi: \mathbf{B} \to \mathbf{B}'$  and  $\phi': \mathbf{B}' \to \mathbf{B}''$ , we define their composite  $\phi' \circ \phi: \mathbf{B} \to \mathbf{B}''$  by

$$(\phi' \circ \phi)_{s,s''} = \sum_{s' \in S'} \phi'_{s',s''} \circ \phi_{s,s'}$$

for  $s \in S$  and  $s'' \in S''$ .

Define the identity  $id_{\mathbf{B}}$  for the object  $\mathbf{B} = (S, \pi, \eta, B)$  by  $(id_{\mathbf{B}})_{s,s} = id_{B(s)}$  for  $s \in S$  and by  $(id_{\mathbf{B}})_{s,s'} = 0$  for  $s, s' \in S$  with  $s \neq s'$ .

Given two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$ , two morphisms  $\phi, \phi' : \mathbf{B} \to \mathbf{B}'$ , and  $m, n \in \mathbb{Z}$ , define the morphism  $m \cdot \phi + n \cdot \phi'$  by

$$(m \cdot \phi + n \cdot \phi')_{s,s'} = m \cdot \phi_{s,s'} + n \cdot \phi'_{s,s'}$$

for  $s \in S$  and  $s' \in S'$ .

We have to check that Definition 21.4 makes sense. The conditions *local finiteness* over  $\mathbb{N}$  and *bounded control over*  $\mathbb{N}$  ensure that the sum occurring in the definition of the composition is indeed a finite sum, namely,

$$(\phi' \circ \phi)_{s,s''} = \sum_{\substack{s' \in S' \\ \phi'_{s',s''}, \phi_{s,s'} \neq 0}} \phi'_{s',s''} \circ \phi_{s,s'}.$$

Since  $\phi$  and  $\phi'$  satisfy *finite G-support*, we can choose finite subsets F and F' of G such that  $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$  and  $\operatorname{supp}_G(\phi'_{s',s''}) \subseteq F'$  holds for  $s \in S, s' \in S'$ , and  $s'' \in S''$ . We get for  $s \in S$  and  $s'' \in S''$ 

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$$\operatorname{supp}_{G}((\phi' \circ \phi)_{s,s''}) = \operatorname{supp}_{G}\left(\sum_{s' \in S'} \phi'_{s',s''} \circ \phi_{s,s'}\right)$$

$$\subset \bigcup_{\substack{s' \in S' \\ \phi'_{s',s''}, \phi_{s,s'} \neq 0}} \operatorname{supp}_{G}(\phi'_{s',s''} \circ \phi_{s,s'})$$

$$\subset \bigcup_{\substack{s' \in S' \\ \phi'_{s',s''}, \phi_{s,s'} \neq 0}} \operatorname{supp}_{G}(\phi'_{s',s''}) \cdot \operatorname{supp}_{G}(\phi_{s,s'})$$

$$\subset \bigcup_{\substack{s' \in S' \\ \phi'_{s',s''}, \phi_{s,s'} \neq 0}} F' \cdot F$$

$$\subset F' \cdot F.$$

Since  $F' \cdot F$  is a finite subset of G, the composite  $\phi' \circ \phi$  satisfies *finite G-support*.

Since both  $\phi$  and  $\phi'$  satisfy bounded control over  $\mathbb{N}$ , there exist natural numbers n and n' such that the implications  $\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \leq n$  and  $\phi'_{s',s''} \neq 0 \implies |\eta'(s') - \eta''(s'')| \leq n'$  hold for  $s \in S$ ,  $s' \in S'$ , and  $s'' \in S''$ . Hence we have the implication  $(\phi' \circ \phi)_{s,s''} \neq 0 \implies |\eta(s) - \eta''(s'')| \leq n + n'$  for  $s \in S$ , and  $s'' \in S''$ . This shows that  $\phi' \circ \phi$  satisfies bounded control over  $\mathbb{N}$ 

Finally we show that *continuous control* is satisfied by  $\phi' \circ \phi$ . Consider  $x \in X$  and an open  $G_x$ -invariant neighborhood  $U \subseteq X$  of x. Since  $\phi'$  satisfies *continuous control*, we can find an open  $G_x$ -invariant neighborhood  $U' \subseteq X$  of x satisfying  $U' \subseteq U$  and a natural number r' such that the implication

(21.7) 
$$g'\pi'(s') \in U', \eta'(s') \ge r' \implies \pi''(s'') \in U$$

holds for all  $s' \in S'$ ,  $s'' \in S''$ , and  $g' \in \operatorname{supp}_G(\phi'_{s',s''})$ . Because of the condition *finite G*-support, there exists a finite subset  $F' \subseteq G$  with  $\operatorname{supp}_G(\phi'_{s',s''}) \subseteq F'$  for  $s' \in S'$  and  $s'' \in S''$ . Fix  $g' \in F'$ . Then  $g'^{-1}U'$  is an open  $G_{g'^{-1}x}$ -invariant neighborhood of  $g'^{-1}x$ . Since  $\phi$  satisfies *bounded control over*  $\mathbb{N}$  and *continuous control*, we can find an open  $G_{g'^{-1}x}$ -invariant neighborhood  $U''_{g'} \subseteq X$  of  $g'^{-1}x$  satisfying  $U''_{g'} \subseteq g'^{-1}U$  and a natural number  $r''_{g'}$  with  $r''_{g'} \geq r'$  such that the implication

(21.8) 
$$g\pi(s) \in U''_{g'}, \eta(s) \ge r''_{g'} \implies \pi'(s') \in g'^{-1}U', \eta'(s') \ge r'$$

holds for all  $s \in S'$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$ . Put

$$U'' := \bigcap_{g' \in F'} g' U''_{g'};$$
  
$$r'' := \max\{r''_{g'} \mid g' \in F'\}$$

Then  $U'' \subseteq X$  is an open  $G_x$ -invariant neighborhood of x. Moreover, we get for  $s \in S, s' \in S', s'' \in S'$ , and  $g \in \operatorname{supp}_G(\phi_{s,s'}), g' \in \operatorname{supp}_G(\phi'_{s',s''})$ 

21.3 The Additive Category  $O^G(X; \mathcal{B})$ 

$$g'g\pi(s) \in U'', \eta(s) \ge r'' \implies g\pi(s) \in g'^{-1}U'', \eta(s) \ge r''$$
$$\implies g\pi(s) \in U''_{g'}, \eta(s) \ge r''_{g'}$$
$$\stackrel{(21.8)}{\implies} \pi'(s') \in g'^{-1}U', \eta'(s') \ge r'$$
$$\implies g'\pi'(s') \in U', \eta'(s') \ge r'$$
$$\stackrel{(21.7)}{\implies} \phi''(s'') \in U.$$

Since  $\operatorname{supp}_G(\phi' \circ \phi)_s^{s''} \subseteq \bigcup_{s' \in S'} \operatorname{supp}_G(\phi'_{s',s''}) \cdot \operatorname{supp}_G(\phi_{s,s'})$  holds, we have shown for  $s \in S, s'' \in S''$ , and  $g'' \in \operatorname{supp}_G((\phi' \circ \phi)_{s,s'})$ 

$$g''\pi(s) \in U'', \eta(s) \ge r'' \implies \phi''(s'') \in U.$$

This finishes the proof of implication (21.5). We leave the analogous proof of the other implication (21.6) to the reader. This finishes the proof that  $\phi' \circ \phi$  satisfies the condition *continuous control* and hence the proof that the composition is well-defined.

One easily checks that the identity morphism is well-defined.

Obviously the definition of the  $\mathbb{Z}$ -structure makes sense.

Given two objects **B** =  $(S, \pi, \eta, B)$  and **B'** =  $(S', \pi', \eta', B')$ , we have to define their direct sum **B**  $\oplus$  **B'**. We put

$$\mathbf{B} \oplus \mathbf{B}' = (S \amalg S', \pi \amalg \pi', \eta \amalg \eta', B \amalg B')$$

and define the desired morphisms  $\mathbf{B} \to \mathbf{B} \oplus \mathbf{B}'$  and  $\mathbf{B}' \to \mathbf{B} \oplus \mathbf{B}'$  in the obvious way. This finishes the proof that  $O^G(X; \mathcal{B})$  is a well-defined additive category.

**Notation 21.9.** When  $\mathcal{B}$  is clear from the context, we will often omit it in the notation and write for instance  $O^G(X)$  instead of  $O^G(X; \mathcal{B})$ .

**Lemma 21.10.** (i) We can replace in Definition 21.4 the condition (21.5) by the condition

(21.11) 
$$\pi(s) \in U', \eta(s) \ge r' \implies g^{-1} \cdot \pi'(s') \in U$$

without changing  $O^G(X)$ ;

(ii) We can replace in Definition 21.4 the condition (21.6) by the condition

(21.12) 
$$\pi'(s') \in U', \eta'(s') \ge r' \implies g \cdot \pi(s) \in U$$

without changing  $O^G(X)$ ;

(iii) We can replace in Definition 21.4 simultaneously the condition (21.5) by the condition (21.11) and the condition (21.6) by the condition (21.12) without changing  $O^G(X)$ .

*Proof.* We only prove assertion (ii). The proofs of the other assertions are analogous. We first show that the condition (21.12) is automatically satisfied. Consider  $x \in X$  and an open  $G_x$ -invariant neighborhood U of x. Let  $\phi : \mathbf{B} \to \mathbf{B}'$  be a morphisms in  $O^G(X)$ . Since it satisfies *finite G-support*, we can find a finite subset  $F \subseteq G$  such

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that  $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$  holds for all  $s \in S$  and  $s' \in S'$ . Fix  $g \in F$ . We can apply condition (21.6) to the open  $G_{g^{-1}x}$ -invariant neighborhood  $g^{-1}U$  of  $g^{-1}x$ , and obtain an open  $G_{g^{-1}x}$ -invariant neighborhood  $U'_g$  of  $g^{-1}x$  with  $U'_g \subseteq g^{-1}U'$  and a natural number  $r'_g$  such that for all  $s \in S$ ,  $s' \in S'$ , and  $g_0 \in \operatorname{supp}_G(\phi_{s,s'})$  the implication

(21.13) 
$$g_0^{-1}\pi'(s') \in U'_g, \eta'(s') \ge r'_g \implies \pi(s) \in g^{-1}U$$

holds. Define

$$\begin{aligned} r' &= \max\{r'_g \mid g \in F\};\\ U' &= \bigcap_{g \in G} gU'_g. \end{aligned}$$

Then U' is an open  $G_x$ -invariant neighborhood of x with  $U' \subseteq U$  and condition (21.12) is satisfied since for  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'}) \subseteq F$  we get

$$\pi'(s') \in U', \eta'(s') \ge r' \implies g^{-1}\pi'(s') \in g^{-1}U', \eta'(s') \ge r'$$
$$\implies g^{-1}\pi'(s') \in U'_g, \eta'(s') \ge r'_g$$
$$\stackrel{(21.13)}{\implies} \pi(s) \in g^{-1}U$$
$$\implies g\pi(s) \in U.$$

The proof in the case where we replace in Definition 21.4 condition (21.6) by condition (21.12), and then show that condition (21.6) is satisfied, is analogous and left to the reader.  $\Box$ 

The next result gives a criterion when we can modify the map  $\pi$  for an object **B** = (*S*,  $\pi$ ,  $\eta$ , B) in  $O^G(X)$  without changing its isomorphism class.

**Lemma 21.14.** Consider two objects in  $O^G(X)$  of the form  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S, \pi', \eta, B)$ . Suppose that for every  $x \in X$  and open  $G_x$ -invariant neighborhood U of x there exists an open  $G_x$ -invariant neighbourhood U' of x in X with  $U' \subseteq U$  and a natural number r' such that for  $s \in S$  the implications

$$\pi(s) \in U', \eta(s) \ge r' \implies \pi'(s) \in U;$$
  
$$\pi'(s) \in U', \eta'(s) \ge r' \implies \pi(s) \in U,$$

hold.

Then **B** and **B**' are isomorphic.

*Proof.* Define mutually inverse morphisms  $\phi: \mathbf{B} \to \mathbf{B}'$  and  $\phi': \mathbf{B}' \to \mathbf{B}$  by  $\phi_{s,s} = \phi'_{s,s} = \mathrm{id}_{\mathrm{B}(s)}$  for  $s \in S$  and by  $\phi_{s,s'} = \phi'_{s',s} = 0$  for  $s, s' \in S$  with  $s \neq s'$ . One has to check that  $\phi$  and  $\phi'$  are well-defined. Note that  $\mathrm{supp}_G(\phi_{s,s'})$  and  $\mathrm{supp}_G(\phi_{s',s})$  are empty if  $s \neq s'$  and agree with  $\{e\}$  if s = s'. Hence  $\phi$  and  $\phi'$  satisfy *finite G-support* and *bounded control over*  $\mathbb{N}$  for obvious reasons and the assumptions appearing in Lemma 21.14 imply *continuous control*.

## **21.4 Functoriality of** $O^G(X; \mathcal{B})$

Consider a *G*-map  $f: X \to Y$  of *G*-*CW*-complexes. Next we show that it induces a functor of additive categories

(21.15) 
$$O^G(f): O^G(X) \to O^G(Y).$$

It sends an object  $\mathbf{B} = (S, \pi, \eta, B)$  in  $O^G(X)$  to the object  $(S, f \circ \pi, \eta, B)$  in  $O^G(Y)$ . One easily checks that the conditions *compact support over* X and *local finiteness* over  $\mathbb{N}$  are satisfied for  $(S, f \circ \pi, \eta, B)$ .

For two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  and a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  given by a collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$  in  $O^G(X)$ , define the morphism  $O^G(f)(\phi): O^G(f)(\mathbf{B}) \to O^G(f)(\mathbf{B}')$  in  $O^G(Y)$  by the same collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$ . Obviously conditions *finite G-support* and *bounded control over*  $\mathbb{N}$  are satisfied for  $O^G(f)(\phi)$ . The hard part is the proof of *continuous control*, which we will give next. We only deal with the implication (21.5), the proof of the implication (21.6) is completely analogous.

Suppose that the implication (21.5) is not satisfied for  $O^G(f)(\phi)$ . Then we can find a point  $y \in Y$  and an open  $G_y$ -invariant neighborhood U of y such that for every open  $G_y$ -invariant neighborhood U' of y with  $U' \subseteq U$  and natural number r'there exist elements  $s \in S$  and  $s' \in S$  and an element  $g \in \text{supp}_G(\phi_{s,s'})$  such that  $g\pi(s) \in U', \eta(s) \ge r'$ , and  $\pi'(s') \notin U$  hold. Since Y is a G-CW-complex, we can find a sequence of nested open  $G_y$ -invariant neighbourhoods  $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of y such that  $\bigcap_{n\ge 0} \overline{V_n} = \{y\}$ . Hence we can find a sequence of nested open  $G_y$ -invariant neighbourhoods  $U'_0 \supseteq U'_1 \supseteq U'_2 \supseteq \cdots$  of y satisfying  $\bigcap_{n\ge 0} \overline{U'_n} = \{y\}$ , a sequence of natural numbers  $r'_n$  satisfying  $\lim_{n\to\infty} r'_n = \infty$ , a sequence  $(s_n)_{n\ge 0}$  in S, a sequence  $(s'_n)_{n\ge 0}$  in S', and elements  $g \in \text{supp}(\phi_{s_n,s'_n})$  such that  $g \cdot f \circ \pi(s_n) \in U'_n$ ,  $\eta(s_n) \ge r'_n$ , and  $f \circ \pi'(s'_n) \notin U$  hold for all  $n \in \mathbb{N}$ .

Since  $\phi$  satisfies *finite G-support*, we can arrange by passing to subsequences that there exists a  $g \in G$  such that  $g = g_n$  holds for all  $n \ge 0$ . Since  $\phi$  satisfies *compact support over X*, we can arrange by passing to subsequences that there exists an  $x \in X$  satisfying  $\lim_{n\to\infty} \pi(s_n) = x$ . We get  $\lim_{n\to\infty} f \circ \pi(s_n) = f(x)$ . Since  $g \cdot f \circ \pi(s_n) \in U'_n$  holds for all  $n \ge 0$ , we conclude  $\lim_{n\to\infty} g \cdot f \circ \pi(s_n) = y$ . This implies f(gx) = y. Note that  $f^{-1}(U)$  is an open  $G_{gx}$ -invariant neighborhood of gx. Since  $\phi$  satisfies *continuous control*, there exists an open  $G_{gx}$ -invariant neighborhood V'' of gx with  $V'' \subseteq f^{-1}(U)$  and a natural number r'' such that for  $s \in S$ ,  $s' \in S'$ , and  $g'' \in \text{supp}_G(\phi_{s,s'})$ , the implication

$$g''\pi(s) \in V'', \eta(s) \ge r'' \implies \pi'(s') \in f^{-1}(U)$$

holds. Hence we get for all  $n \in \mathbb{N}$  the implication

$$g\pi(s_n) \in V'', \eta(s_n) \ge r'' \implies \pi'(s'_n) \in f^{-1}(U).$$

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Since  $\lim_{n\to\infty} r'_n = \infty$ ,  $\lim_{n\to\infty} g\pi(s_n) = gx$ , and V'' is an open neighborhood of gx, we can arrange by passing to subsequences that  $g\pi(s_n) \in V''$  and  $\eta(s_n) \ge r''$  holds for all  $n \ge 0$ . Hence we get  $\pi'(s'_n) \in f^{-1}(U)$  for all  $n \ge 0$ . This implies  $f \circ \pi'(s'_n) \in U$  for all  $n \ge 0$ , a contradiction.

Obviously we get a covariant functor  $O^G(-; \mathcal{B})$  from the category of *G*-*CW*-spaces with arbitrary *G*-maps as morphisms to the category of additive categories.

## 21.5 The *TOD*-Sequence

Let X be a G-CW-complex and  $\mathcal{B}$  be a category with G-support in the sense of Definition 21.1.

**Definition 21.16** ( $\mathcal{T}^G(X)$ ). Let  $\mathcal{T}^G(X)$  be the full additive subcategory of  $O^G(X)$  consisting of those objects **B** = ( $S, \pi, \eta, B$ ) for which there exists a natural number n satisfying  $\eta(s) \le n$  for all  $s \in S$ .

**Lemma 21.17.** The inclusion  $\mathcal{T}^G(X) \to O^G(X)$  is a Karoubi filtration in the sense of Definition 8.43.

*Proof.* Consider an object  $\mathbf{B} = (S_{\mathbf{B}}, \pi_{\mathbf{B}}, \eta_{\mathbf{B}}, B_{\mathbf{B}})$  in  $O^G(X)$ , two objects  $\mathbf{U} = (S_{\mathbf{U}}, \pi_{\mathbf{U}}, \eta_{\mathbf{U}}, B_{\mathbf{U}})$  and  $\mathbf{V} = (S_{\mathbf{V}}, \pi_{\mathbf{V}}, \eta_{\mathbf{V}}, B_{\mathbf{V}})$  in  $\mathcal{T}^G(X)$ , and morphisms  $f : \mathbf{B} \to \mathbf{U}$  and  $g : \mathbf{V} \to \mathbf{B}$  in  $O^G(X)$ . By definition we can find natural numbers  $n_0$  and  $n_1$  such that  $\eta_{\mathbf{U}}(s') \le n_0$  for  $s' \in S_{\mathbf{U}}$  and  $\eta_{\mathbf{V}}(s) \le n_0$  for  $s \in S_{\mathbf{V}}$  hold and we have the implications

$$s \in S_{\mathbf{B}}, s' \in S_{\mathbf{U}}, f_{s,s'} \neq 0 \implies |\eta_{\mathbf{B}}(s) - \eta_{\mathbf{U}}(s')| \le n_1;$$
  
$$s \in S_{\mathbf{V}}, s' \in S_{\mathbf{B}}, g_{s,s'} \neq 0 \implies |\eta_{\mathbf{V}}(s) - \eta_{\mathbf{B}}(s')| \le n_1.$$

Now define objects

$$\mathbf{B}^{\mathcal{U}} = (S_{\mathbf{B}^{\mathcal{U}}}, \pi_{\mathbf{B}^{\mathcal{U}}}, \eta_{\mathbf{B}^{\mathcal{U}}}, \mathsf{B}_{\mathbf{B}^{\mathcal{U}}}) \text{ in } \mathcal{T}^{G}(X), \text{ and}$$
$$\mathbf{B}^{\perp} = (S_{\mathbf{B}^{\perp}}, \pi_{\mathbf{B}^{\perp}}, \eta_{\mathbf{B}^{\perp}}, \mathsf{B}_{\mathbf{B}^{\perp}}) \text{ in } O^{G}(X)$$

by

$$S_{\mathbf{B}^{\mathcal{U}}} := \{ s \in S_{\mathbf{B}} \mid \eta_{\mathbf{B}}(s) \le n_0 + n_1 \}; \\ S_{\mathbf{B}^{\perp}} := \{ s \in S_{\mathbf{B}} \mid \eta_{\mathbf{B}}(s) > n_0 + n_1 \},$$

and restricting the maps  $\pi_{\mathbf{B}}, \eta_{\mathbf{B}}$ , and  $\mathbb{B}_{\mathbf{B}}$ . There are obvious morphisms  $i^{\mathcal{U}} : \mathcal{B}^{\mathcal{U}} \to \mathcal{B}$ and  $i^{\perp} : \mathcal{B}^{\perp} \to \mathcal{B}$  in  $O^{G}(X)$  such that  $i^{\mathcal{U}} \oplus i^{\perp} : \mathcal{B}^{\mathcal{U}} \oplus \mathcal{B}^{\perp} \xrightarrow{\cong} \mathcal{B}$  is an isomorphism. We leave it to the reader to figure out the obvious definition of the maps  $f^{\mathcal{U}}$  and  $g^{\mathcal{U}}$ and the proof of the commutativity of the relevant diagrams. Hence the inclusion  $\mathcal{T}^{G}(X) \to O^{G}(X)$  is a Karoubi filtration.

21.6 The Definition for Pairs

**Definition 21.18** ( $\mathcal{D}^G(X)$ ). Let  $\mathcal{D}^G(X)$  be the additive category given by the quotient  $O^G(X)/\mathcal{T}^G(X)$  in the sense of Definition 8.42.

**Theorem 21.19** (TOD-sequence). The so-called TOD-sequence

$$K(\mathcal{T}^G(X)) \to K(\mathcal{O}^G(X)) \to K(\mathcal{D}^G(X))$$

is a weak homotopy fibration of spectra.

*Proof.* This follows from Lemma 21.17 and Theorem 8.46 (i).

Given a map  $f: X \to Y$  of *G-CW*-complexes, the functor of additive categories  $O^G(f): O^G(X) \to O^G(Y)$  of (21.15) induces functors of additive categories

(21.20) 
$$\mathcal{T}^{G}(f): \mathcal{T}^{G}(X) \to \mathcal{T}^{G}(Y)$$

 $\mathcal{T}^{\mathsf{G}}(f) \colon \mathcal{T}^{\mathsf{G}}(X) \to \mathcal{T}^{\mathsf{G}}(Y);$  $\mathcal{D}^{\mathsf{G}}(f) \colon \mathcal{D}^{\mathsf{G}}(X) \to \mathcal{D}^{\mathsf{G}}(Y).$ (21.21)

**Lemma 21.22.** Let  $f: X \to Y$  be a *G*-map between *G*-*CW*-complexes. Then  $\mathcal{T}^G(f): \mathcal{T}^G(X) \xrightarrow{\simeq} \mathcal{T}^G(Y)$  is an equivalence of additive categories.

*Proof.* We can assume without loss of generality that  $Y = \{\bullet\}$ .

Consider an object **B** =  $(S, \pi, \eta, B)$  in  $\mathcal{T}^G(\{\bullet\})$ . Then S is finite. Choose any map  $\pi': S \to X$  and define an object  $\mathbf{B}' = (S, \pi', \eta, B)$  in  $\mathcal{T}^G(X)$ . Since  $\mathcal{T}^G(f)(\mathbf{B}') = \mathbf{B}$ , we conclude that  $\mathcal{T}^{G}(f)$  is surjective on objects. Obviously  $\mathcal{T}^{G}(f)$  induces for two objects  $\mathbf{B}_0$  and  $\mathbf{B}_1$  in  $\mathcal{T}^G(X)$  a bijection

$$\operatorname{mor}_{\mathcal{T}^{G}(X)}(\mathbf{B}_{0},\mathbf{B}_{1}) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{T}^{G}(\{\bullet\})}(\mathcal{T}^{G}(f)(\mathbf{B}_{0}),\mathcal{T}^{G}(f)(\mathbf{B}_{1})), \quad \phi \mapsto \mathcal{T}^{G}(f)(\phi)$$

since for  $\mathcal{T}^G(X)$  the conditions finite G-support, bounded control over  $\mathbb{N}$ , and continuous control are automatically satisfied. Hence  $\mathcal{T}^{G}(f)$  is an equivalence of additive categories. 

#### **21.6 The Definition for Pairs**

Let (X, A) be a *G*-*CW*-pair. Denote by  $i: A \to X$  the inclusion.

- **Lemma 21.23.** (i) The functor  $O^G(i): O^G(A) \to O^G(X)$  of (21.15) induces an isomorphism of additive categories from  $O^G(A)$  onto its image. The image is a full additive subcategory of  $O^{G}(X)$  which is a Karoubi filtration;
- (ii) The same statement holds for the functor  $\mathcal{D}^{G}(i)$ :  $\mathcal{D}^{G}(A) \to \mathcal{D}^{G}(X)$  of (21.21).

*Proof.* (i) The image of  $O^G(i)$  can be identified with the full additive subcategory  $O^G(X)_A$  of  $O^G(X)$  whose objects  $\mathbf{B} = (S, \pi, \eta, B)$  satisfy  $\operatorname{im}(\pi) \subseteq A$ . The functor  $O^G(i; \mathcal{B})$  induces an isomorphism  $O^G(A) \xrightarrow{\cong} O^G(X)_A$ , since for every  $x \in A$  and open  $G_x$ -invariant neighbourhood U of x in A there exists an open  $G_x$ -invariant

neighbourhood V of x in X with  $U = A \cap X$ . It remains to show that the inclusion  $O^G(X)_A \subseteq O^G(X)$  is a Karoubi filtration.

Consider three objects  $\mathbf{B}_0 = (S_0, \pi_0, \eta_0, B_0)$ ,  $\mathbf{B}_1 = (S_1, \pi_1, \eta_1, B_1)$ , and  $\mathbf{B} = (S, \pi, \eta, B)$  in  $O^G(X)$  with  $\operatorname{im}(\pi_0) \subseteq A$  and  $\operatorname{im}(\pi_1) \subseteq A$ , and two morphism  $a_0: \mathbf{B} \to \mathbf{B}_0$  and  $a_1: \mathbf{B}_1 \to \mathbf{B}$  in  $O^G(X)$ . Define subsets of *S* which consists of those elements which are interacting with  $S_0$  and  $S_1$  via  $a_0$  and  $a_1$ 

$$S_0 := \{ s \in S \mid \exists s_0 \in S_0 \text{ with } (a_0)_{s,s_0} \neq 0 \};$$
  
$$\widehat{S}_1 := \{ s' \in S \setminus \widehat{S}_0 \mid \exists s_1 \in S_1 \text{ with } (a_1)_{s_1,s} \neq 0 \}.$$

Define objects  $\mathbf{B}^{\mathcal{U}} = (S^{\mathcal{U}}, \pi^{\mathcal{U}}, \eta^{\mathcal{U}}, \mathbb{B}^{\mathcal{U}})$  and  $\mathbf{B}^{\perp} = (S^{\perp}, \pi^{\perp}, \eta^{\perp}, \mathbb{B}^{\perp})$  by putting  $S^{\mathcal{U}} := \widehat{S}_0 \amalg \widehat{S}_1$  and  $S^{\perp} = S \setminus S^{\mathcal{U}}$  and defining  $\pi^{\mathcal{U}}, \eta^{\mathcal{U}}, \mathbb{B}^{\mathcal{U}}, \pi^{\perp}, \eta^{\perp},$  and  $\mathbb{B}^{\perp}$  by restricting  $\pi, \eta$ , and  $\mathbb{B}$ . There are obvious morphisms  $i^{\mathcal{U}} : \mathbf{B}^{\mathcal{U}} \to \mathbf{B}$  and  $i^{\perp} : \mathbf{B}^{\perp} \to \mathbf{B}$  in  $O^G(X)_A$  such that  $i^{\mathcal{U}} \oplus i^{\perp} : \mathbf{B}^{\mathcal{U}} \oplus \mathbf{B}^{\perp} \stackrel{\cong}{\to} \mathbf{B}$  is an isomorphism and morphisms  $a_0^{\mathcal{U}} : \mathbf{B}^{\mathcal{U}} \to \mathbf{B}_0$  and  $a_1^{\mathcal{U}} : \mathbf{B}_1^{\mathcal{U}} \to \mathbf{B}$  such that the relevant diagrams as they appear in the definition of a Karoubi filtration commute. However, we are not done since  $\mathcal{B}^{\mathcal{U}}$  is not an object in  $O^G(X)_A$ . In order to finish the proof of assertion (i) it suffices to construct an object  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbb{B}})$  in  $O^G(X)_A$  together with an isomorphism  $\phi: \widehat{\mathcal{B}} \stackrel{\cong}{\to} \mathcal{B}^{\mathcal{U}}$  in  $O^G(X)$ .

Choose functions  $u_0: \widehat{S}_0 \to S_0, g_0: \widehat{S}_0 \to G, u_1: \widehat{S}_1 \to S_1$ , and  $g_1: \widehat{S}_1 \to G$ such that  $g_0(s) \in \text{supp}((a_0)_{s,u_0(s)})$  holds for  $s \in \widehat{S}_0$  and  $g_1(s) \in \text{supp}((a_1)_{u_1(s),s})$ holds for  $s \in \widehat{S}_1$ . Define a new object  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{B})$  in  $O^G(X)_A$  by

$$\widehat{S} := \widehat{S}_0 \amalg \widehat{S}_1;$$

$$\widehat{\pi}(s) := \begin{cases} g_0(s)^{-1} \cdot \pi_0 \circ u_0(s) & \text{if } s \in \widehat{S}_0; \\ g_1(s) \cdot \pi_1 \circ u_1(s) & \text{if } s \in \widehat{S}_1; \end{cases}$$

$$\widehat{\eta}(s) := \eta(s) \quad \text{for } s \in \widehat{S};$$

$$\widehat{B}(s) := B(s) \quad \text{for } s \in \widehat{S}.$$

Recall  $S^{\mathcal{U}} = \widehat{S}_0 \amalg \widehat{S}_1 = \widehat{S}$ . In order to show that  $\widehat{\mathcal{B}}$  and  $\mathcal{B}^{\mathcal{U}}$  are isomorphic, we want to apply the criterion appearing in Lemma 21.14.

Consider an element  $x \in X$  and an open  $G_x$ -invariant neighbourhood Uof x in X. Since  $a_0$  and  $a_1$  satisfy *continuous control*, we can find an open  $G_x$ -invariant neighbourhood U' of x in X with  $U' \subseteq U$  and  $r' \in \mathbb{N}$  such that for  $s \in S$ ,  $s_0 \in S_0$ ,  $g_0 \in \text{supp}_G((a_0)_{s,s_0})$  the implication

$$g_0^{-1} \cdot \pi_0(s_0) \in U', \eta(s_0) \ge r' \implies \pi(s) \in U$$

and for  $s_1 \in S_1$ ,  $s \in S$ ,  $g_1 \in \text{supp}_G((a_1)_{s_1,s})$  the implication

$$g_1\pi_1(s_1) \in U', \eta_1(s_1) \ge r' \implies \pi(s) \in U$$

hold. This implies that for  $s \in \widehat{S}_0 \amalg \widehat{S}_1$  the implication

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$$\widehat{\pi}(s) \in U', \widehat{\eta}(s) \ge r' \implies \pi^{\mathcal{U}}(s) \in U$$

holds. The proof of the other implication

$$\pi^{\mathcal{U}}(s) \in U', \eta^{\mathcal{U}}(s) \ge r' \implies \widehat{\pi}(s) \in U$$

for  $s \in \widehat{S}_0 \amalg \widehat{S}_1$  is analogous and left to the reader. Now Lemma 21.14 implies that  $\widehat{\mathcal{B}}$  and  $\mathcal{B}^{\mathcal{U}}$  are isomorphic.

(ii) The constructions appearing in the proof of assertion (i) yield the desired result for  $\mathcal{D}^G(i)$  using Lemma 21.22.

**Definition 21.24** ( $\mathcal{D}^G(X, A)$ ). Define the additive category  $\mathcal{D}^G(X, A)$  to be the quotient of  $\mathcal{D}^G(X)$  by the image of  $\mathcal{D}^G(i) : \mathcal{D}^G(A) \to \mathcal{D}^G(X)$ .

Obviously a G-map of G-CW-pairs  $f: (X, A) \rightarrow (Y, B)$  induces a functor of additive categories

(21.25) 
$$\mathcal{D}^G(f): \mathcal{D}^G(X, A) \to \mathcal{D}^G(Y, B).$$

## 21.7 The Proof of the Axioms of a G-Homology Theory

The next theorem is the main result of this chapter. We will give its proof in detail, since it is not presented in the literature in a satisfactory way and it illustrates the main techniques needed in proofs of results and properties of controlled categories and their *K*-theory in many other cases.

**Theorem 21.26 (The algebraic** *K*-groups of  $\mathcal{D}^G(X, A)$  yield a *G*-homology theory). Let  $\mathcal{B}$  be a category with *G*-support in the sense of Definition 21.1.

Then we obtain a G-homology theory with values in  $\mathbb{Z}$ -modules in the sense of Definition 12.1 by the covariant functor from the category of G-CW-pairs to the category of  $\mathbb{Z}$ -graded abelian groups sending (X, A) to  $K_*(\mathcal{D}^G(X, A; \mathcal{B}))$ .

#### 21.7.1 The Long Exact Sequence of a Pair

**Proposition 21.27.** Given a G-CW-pair (X, A), we have the inclusions  $i: A \to X$  and  $j: X \to (X, A)$  and obtain a long exact sequence, infinite to both sides and natural in (X, A),

$$\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{D}^G(A)) \xrightarrow{K_n(\mathcal{D}(i))} K_n(\mathcal{D}^G(X)) \xrightarrow{K_n(\mathcal{D}(j))} K_n(\mathcal{D}^G(X,A)) \xrightarrow{\partial_n} K_{n-1}(\mathcal{D}^G(A)) \xrightarrow{K_{n-1}(\mathcal{D}(i))} K_{n-1}(\mathcal{D}^G(X)) \xrightarrow{K_{n-1}(\mathcal{D}(j))} K_{n-1}(\mathcal{D}^G(X,A)) \xrightarrow{\partial_{n-1}} \cdots .$$

*Proof.* This follows from Lemma 21.23 (ii) and Theorem 8.46 (i).

#### 21.7.2 Some Eilenberg Swindles on $O^{G}(X)$

Remark 21.28 (Eilenberg swindles on additive categories defined in terms of controlled topology). Sometimes we want to show that the algebraic *K*-theory of certain additive categories defined by controlled topology is weakly contractible. This is done in all cases by constructing an Eilenberg swindle. The basic strategy is illustrated for  $O^G(X)$  as follows.

One defines a functor sh:  $O^G(X) \to O^G(X)$  which shifts one position to the right over  $\mathbb{N}$ , as follows. It sends an object  $\mathbf{B} = (S, \pi, \eta, B)$  to the object  $\mathrm{sh}(\mathbf{B}) = (\mathrm{sh}(S), \mathrm{sh}(\pi), \mathrm{sh}(\eta), \mathrm{sh}(B))$ , where  $\mathrm{sh}(S) = S$ ,  $\mathrm{sh}(B) = B$ ,  $\mathrm{sh}(\pi) = \pi$ ,  $\mathrm{sh}(\eta) = \eta + 1$ . Roughly speaking, nothing is changed, only the objects are moved one position to the right in the  $\mathbb{N}$ -direction. (Sometimes one also has to vary  $\pi$ .) One easily checks that  $\mathrm{sh}(\mathbf{B})$  satisfies *compact support over* X and *local finiteness over*  $\mathbb{N}$ . The definition of  $\mathrm{sh}(\phi)$  for morphisms  $\phi \colon \mathbf{B} \to \mathbf{B}'$  is the tautological one. Again it is easy to check that  $\mathrm{sh}(\phi)$  will again satisfy *finite G*-support, bounded control over  $\mathbb{N}$ , and continuous control. Moreover there is an obvious natural equivalence  $t \colon \mathrm{id} \xrightarrow{\cong} \mathrm{sh}$  of functors of additive categories  $O^G(X) \to O^G(X)$ .

The basic idea, which works in some special cases, is to the define a functor SH:  $O^G(X) \to O^G(X)$  on objects by SH(**B**) =  $\bigoplus_{n=0}^{\infty} \operatorname{sh}^n(\mathbf{B})$ . This definition indeed makes sense since **B** satisfies *local finiteness over*  $\mathbb{N}$  and hence the set

$$\{(s,n)\in S\times\mathbb{N}\mid \eta(s)\leq n, \mathsf{B}(s)\neq 0\}=\coprod_{k=0}^n\eta^{-1}(k)$$

is finite. However, the obvious definition on morphisms will not work in general. The conditions *compact support over* X and *local finiteness over*  $\mathbb{N}$  cause no difficulties, whereas the condition *continuous control* is the problem. The reason is that in SH(**B**) the objects are moved arbitrarily far to the right in the  $\mathbb{N}$ -direction and the continuous control condition becomes more and more restrictive the larger the position with respect to  $\mathbb{N}$  is. One example where this problem does not occur is for instance the case  $X = \{\bullet\}$ , which we will handle in Lemma 21.29. If SH is well-defined, then one obtains the desired natural equivalence using  $t: \text{ id} \xrightarrow{\cong} \text{ sh by}$ 

$$\operatorname{id} \oplus \operatorname{SH} = \operatorname{sh}^0 \oplus \bigoplus_{n=0}^{\infty} \operatorname{sh}^n \xrightarrow{\cong} \operatorname{sh}^0 \oplus \operatorname{sh}\left(\bigoplus_{n=0}^{\infty} \operatorname{sh}^n\right) \cong \operatorname{sh}^0 \oplus \bigoplus_{n=1}^{\infty} \operatorname{sh}^n = \operatorname{SH}.$$

**Lemma 21.29.** If  $\mathcal{B}$  is a category with *G*-support, then  $O^G(\{\bullet\})$  is flasque. In particular,  $\mathbf{K}(O^G(\{\bullet\}))$  is weakly contractible.

*Proof.* The desired Eilenberg swindle described in Remark 21.28 is constructed in detail as follows. Next we define a functor of additive categories

SH: 
$$O^G(\{\bullet\}) \to O^G(\{\bullet\}).$$

For an object  $B = (S, \pi, \eta, B)$  in  $O^G(\{\bullet\})$ , define SH(**B**) by the quadruple (SH(S), SH( $\pi$ ), SH( $\eta$ ), SH( $\mathbb{B}$ )), where for  $s \in S$  and  $n \in \mathbb{N}$  we put

$$SH(S) = \{(s,n) \in S \times \mathbb{N} \mid \eta(s) \le n\};$$
  

$$SH(\pi)(s,n) = \pi(s);$$
  

$$SH(\eta)(s,n) = n;$$
  

$$SH(B)(s,n) = B(s).$$

Obviously SH(**B**) satisfies compact support over  $\{\bullet\}$ . Since **B** satisfies local finiteness and SH $(\eta)^{-1}(n) = \bigcup_{m=0}^{n} \eta^{-1}(m)$  holds for  $n \in \mathbb{N}$ , SH(**B**) satisfies local finiteness.

For two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  and a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  given by a collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$ , define the morphism  $SH(\phi): SH(\mathbf{B}) \to SH(\mathbf{B}')$  by the collection

$$\{\mathrm{SH}(\phi)_{(s,n),(s',n')} \colon \mathsf{B}(s) \to \mathsf{B}'(s') \mid s \in S, s' \in S', n \in \mathbb{N}, n' \in \mathbb{N}, \eta(s) \le n, \eta'(s') \le n'\}$$

for  $SH(\phi)_{(s,n),(s',n')} = \phi_{s,s'}$  if  $n - \eta(s) = n' - \eta'(s')$  and  $SH(\phi)_{(s,n),(s',n')} = 0$  otherwise.

Since  $\phi$  satisfies *finite G-support*, the same is true for SH( $\phi$ ). Since  $\phi$  satisfies *bounded control over*  $\mathbb{N}$ , we can find a natural number N such that  $\phi_{s,s'} \neq 0 \implies$  $|\eta(s) - \eta'(s')| \leq N$  holds for  $s \in S$  and  $s' \in S'$ . Now consider  $(s, n) \in$  SH(S) and  $(s', n') \in$  SH(S') with SH $(\phi)_{(s,n),(s',n')} \neq 0$ . Since then  $n - \eta(s) = n' - \eta'(s')$  and  $\phi_{s,s'} \neq 0$  hold, we get

$$|\operatorname{SH}(\eta)(s,n) - \operatorname{SH}(\eta')(s',n')| = |n - n'| = |\eta(s) - \eta'(s)| \le N.$$

Hence  $SH(\phi)$  satisfies *bounded control over*  $\mathbb{N}$ . Obviously  $SH(\phi)$  satisfies *continuous control* since we are working over  $\{\bullet\}$ . One easily checks that SH is a well-defined functor of additive categories.

It remains to construct a natural equivalence  $T: \text{ id } \oplus \text{ SH } \xrightarrow{\cong} \text{ SH } \text{ of functors of additive categories. We have to define for any object } \mathbf{B} = (S, \pi, \eta, B)$  an isomorphism  $T(\mathbf{B}): \mathbf{B} \oplus \text{SH}(\mathbf{B}) \xrightarrow{\cong} \text{SH}(\mathbf{B})$ . We obtain a bijection of sets

$$u \colon S \coprod \operatorname{SH}(S) \xrightarrow{\cong} \operatorname{SH}(S)$$

by sending  $s \in S$  to  $(s, \eta(s))$  and  $(s, n) \in SH(S)$  to (s, n + 1). Note that for  $s \in S$  we have  $B(s) = SH(B) \circ u(s)$  and for  $(s, n) \in SH(S)$  we have  $SH(B(s, n)) = B(s) = SH \circ u(s, n)$ . Now we can define  $T(\mathbf{B})_{t,t'}$  for  $t \in S \coprod SH(S)$  and  $t' \in SH(S)$  to be  $\mathrm{id}_{B(t')}$  if u(t) = t' and to be 0 if  $u(t) \neq t'$ . It is easy and left to the reader to check that  $T(\mathbf{B})$  is a well-defined isomorphism in  $O^G(\{\bullet\})$  which is natural in  $\mathcal{B}$  and hence defines the desired natural equivalence  $T: \mathrm{id} \oplus \mathrm{SH} \xrightarrow{\cong} \mathrm{SH}$ .

Thus we have defined an Eilenberg swindle (SH, *T*) on  $O^G(\{\bullet\})$ ). The weak contractibility of  $\mathbf{K}(O^G(\{\bullet\}))$  follows from Theorem 6.37 (iii).

The next result generalizes Lemma 21.29. The basic idea of the proof is the same but becomes much more complicated, since now we have to deal with the condition *continuous control*.

**Lemma 21.30.** Let X be a G-CW-complex which is G-contractible, i.e, G-homotopy equivalent to  $\{\bullet\}$ .

Then  $\mathbf{K}(O^G(X))$  is weakly contractible.

*Proof.* Denote by cone(X) the cone of X. As X is G-contractible, there are G-maps  $i: X \to cone(X)$  and  $r: cone(X) \to X$  with  $r \circ i = id_X$ . Hence the composite of maps of spectra

$$\mathbf{K}(\mathcal{O}^{G}(X)) \xrightarrow{\mathbf{K}(\mathcal{O}^{G}(i))} \mathbf{K}(\mathcal{O}^{G}(\operatorname{cone}(X))) \xrightarrow{\mathbf{K}(\mathcal{O}^{G}(r))} \mathbf{K}(\mathcal{O}^{G}(X))$$

is the identity. Therefore it suffices to show that  $\mathbf{K}(\operatorname{cone}(X))$  is weakly contractible.

We explain the basic idea of the proof before we give the details. In the construction of an Eilenberg swindle for a given object  $\mathbf{B} = (S, \pi, \eta, B)$  one assigns to  $\mathbf{B}$  a new object SH( $\mathbf{B}$ ) where one adds for  $s \in S$  a copy of B(s) at *n* for each natural number  $n \ge \eta(s)$ . The problem is to specify where this copy over *n* sits in cone(X), i.e., to define the image of this object under  $\pi^{SH}$ . The idea is to move the copies of the object B(s) with the right speed to the cone point. This has to be done fast enough so that the obvious definition of SH( $\phi$ ) for a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  still defines *continuous control* but slow enough so that the desired obvious transformation  $T(\mathbf{B}): \mathbf{B} \oplus SH(\mathbf{B}) \xrightarrow{\cong} SH(\mathbf{B})$  satisfies *continuous control*. This will lead to the properties of the function  $\rho$  below.

Recall that cone(X) is defined as the *G*-pushout

$$X \xrightarrow{i_0} X \times [0, 1]$$

$$\downarrow \qquad \qquad \downarrow^{\text{pr}}$$

$$\{\bullet\} \xrightarrow{\overline{i_0}} \operatorname{cone}(X)$$

where  $i_0: X \to X \times [0, 1]$  sends *x* to (x, 0). In the sequel we write  $[x, t] = \operatorname{pr}(x, t)$  for  $(x, t) \in X \times [0, 1]$ . For  $t' \in [0, 1]$  we define  $t' \cdot [x, t] := [x, t't]$ . Denote by \* the cone point [x, 0] for any  $x \in X$ , or, equivalently,  $* = \overline{i_0}(\{\bullet\})$ . For  $z = [x, t] \in \operatorname{cone}(X)$  we denote  $z_I$  by *t*. For  $z = [x, t] \in \operatorname{cone}(X) \setminus \{*\}$  we denote  $z_X$  by *x*. In particular,  $\operatorname{pr}(x, t)_X = x$  for  $x \in X$ ,  $t \in (0, 1]$ , and  $\operatorname{pr}(x, t)_I = t$  for  $x \in X$  and  $t \in [0, 1]$ .

Next we define a functor of additive categories

SH: 
$$O^{G}(\operatorname{cone}(X)) \to O^{G}(\operatorname{cone}(X)).$$

For this purpose we choose a function  $\rho \colon \mathbb{N} \times \mathbb{N} \to (0, 1]$  with the following three properties.

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- We have

(21.31) 
$$\lim_{m \to \infty} \rho(m, 0) = 1;$$

• For every  $m \in \mathbb{N}$ , we have

(21.32) 
$$\lim_{n \to \infty} \rho(m, n) = 0$$

• For every  $N \in \mathbb{N}$  and  $\mu > 0$ , there is an  $M \in \mathbb{N}$  such that for all  $m, m', n \in \mathbb{N}$  the implication

(21.33) 
$$m \ge M, |m - m'| \le N \implies |\rho(m, n) - \rho(m', n)| < \mu$$

holds;

• For every  $\mu > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  the implication

(21.34) 
$$n \ge N, m \le n \implies 1 - \mu \le \frac{\rho(m, n+1-m)}{\rho(m, n-m)} \le 1$$

holds.

If  $(a_k)_{k \in \mathbb{N}}$  is any sequence of elements in (0, 1] satisfying  $\lim_{k \to \infty} a_k = 0$  and  $\sum_{k=0}^{\infty} a_k = \infty$ , then we can take  $\rho(m, n) := \exp(-\sum_{k=m}^{m+n} a_k)$ . An example for  $(a_k)_{k \in \mathbb{N}}$  is  $a_k = 1/k$ .

The functor SH sends an object  $\mathbf{B} = (S, \pi, \eta, B)$  to the object  $SH(\mathbf{B}) = (SH(S), SH(\pi), SH(\eta), SH(B))$  where for  $s \in S$  we put

$$SH(S) = \{(s, n) \mid s \in S, n \in \mathbb{N}, \eta(s) \le n\}$$

and define for  $(s, n) \in SH(S)$ 

$$SH(\pi)(s,n) = \rho(\eta(s), n - \eta(s)) \cdot \pi(s);$$
  

$$SH(\eta)(s,n) = n;$$
  

$$SH(B)(s,n) = B(s).$$

Since **B** satisfies *compact support over X*, there exists a compact subset *C* of cone(*X*) with  $im(\pi) \subseteq C$ . This implies

$$im(SH(\pi)) \subseteq [0,1] \cdot C := \{t \cdot c \mid t \in [0,1], c \in C\}.$$

Since  $[0, 1] \cdot C$  is compact, SH(**B**) satisfies *compact support over X*.

Since **B** satisfies *local finiteness over*  $\mathbb{N}$  and  $SH(\eta)^{-1}(m) = \coprod_{n=0}^{m} \eta^{-1}(n)$  holds, SH(**B**) satisfies *local finiteness over*  $\mathbb{N}$ .

Consider a morphism  $\phi: \mathbf{B} = (S, \pi, \eta, B) \to \mathbf{B}' = (S', \pi', \eta, B')$  given by the collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$ . Define  $SH(\phi): SH(\mathbf{B}) \to SH(\mathbf{B}')$  by

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$$SH(\phi)_{(s,n),(s',n')} = \begin{cases} \phi_{s,s'} & \text{if } n' - \eta'(s') = n - \eta(s); \\ 0 & \text{otherwise,} \end{cases}$$

for  $(s, n) \in SH(\mathbf{B})$  and  $(s', n') \in SH(\mathbf{B}')$ .

Since  $\phi$  satisfies *finite G-support*, there exists a finite subset  $F \subseteq G$  such that  $\operatorname{supp}_G(\phi_{s,s'}) \subseteq F$  holds for every  $s \in S$ , and  $s' \in S'$ . This implies  $\operatorname{supp}_G(\operatorname{SH}(\phi)_{(s,n),(s',n')}) \subseteq F$  for every  $(s,n) \in \operatorname{SH}(S)$ , and  $(s',n') \in \operatorname{SH}(S')$ . Hence  $\operatorname{SH}(\phi)$  satisfies *finite G-support*.

Since  $\phi$  satisfies bounded control over  $\mathbb{N}$ , there exists a natural number N with  $|\eta(s) - \eta'(s')| \leq N$  for all  $s \in S$  and  $s' \in S'$  with  $\phi_{s,s'} \neq 0$ . Consider  $(s, n) \in SH(\mathbf{B})$  and  $(s', n') \in SH(\mathbf{B}')$  with  $SH(\phi)_{(s,n),(s',n')} \neq 0$ . Then  $n' - \eta'(s') = n - \eta(s)$  and  $|\eta(s) - \eta'(s')| \leq N$ . This implies

(21.35) 
$$|\operatorname{SH}(\eta)(s,n) - \operatorname{SH}(\eta')(s',n')| = |n-n'| = |\eta(s) - \eta'(s')| \le N.$$

Hence  $SH(\phi)$  satisfies bounded control over  $\mathbb{N}$ .

The hard part is to show that  $SH(\phi)$  satisfies *continuous control*. We only deal with the implication (21.5). The proof of the other implication (21.6) is completely analogous.

Consider  $[x, t] \in \operatorname{cone}(X)$  and an open  $G_{[x,t]}$ -invariant neighborhood U of [x, t]in  $\operatorname{cone}(X)$ . We have to find an open  $G_{[x,t]}$ -invariant neighborhood U' of [x, t] in  $\operatorname{cone}(X)$  satisfying  $U' \subseteq U$  and a natural number r' such that for all  $(s, n) \in \operatorname{SH}(S)$ ,  $(s', n') \in \operatorname{SH}(S')$ , and  $g \in \operatorname{supp}_G(\operatorname{SH}(\phi)_{(s,n),(s',n')})$  the implication

(21.36) 
$$g \cdot \operatorname{SH}(\pi)(s, n) \in U', \operatorname{SH}(\eta)(s, n) \ge r' \implies \operatorname{SH}(\pi')(s', n') \in U$$

holds.

We begin with the case where [x, t] is different from the cone point \*, or, equivalently  $0 < t \le 1$ . In the sequel we denote for  $t \in (0, 1]$  and  $\epsilon > 0$  by  $I_{\epsilon}(t)$  the open neighborhood of t in [0, 1] given by  $(t - \epsilon, t + \epsilon) \cap [0, 1]$ .

Choose an open  $G_x$ -invariant neighbourhood  $V_0$  of x in X and  $\epsilon > 0$  satisfying

(21.37) 
$$\operatorname{pr}(V_0 \times I_{\epsilon}(t)) \subseteq U_{\epsilon}$$

$$(21.38) \epsilon \le t/2.$$

Since  $\phi$  satisfies *continuous control*, we can find for  $t' \in [t/2, 1]$  an open  $G_x$ -invariant neighborhood V'[t'] of x, a real number  $\delta'[t'] > 0$ , and  $r'[t'] \in \mathbb{N}$  such that for  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$  the implication

(21.39) 
$$g\pi(s)_X \in V'[t'], \pi(s)_I \in I_{\delta'[t']}(t'), \eta(s) \ge r[t']$$
  
 $\implies \pi'(s')_X \in V_0, \pi'(s')_I \in I_{\epsilon/8}(t')$ 

holds. Obviously we can arrange  $0 < \delta'[t'] < \epsilon/8$ . Since [t/2, 1] is compact, we can find finitely many elements  $t'_1, t'_2, \ldots, t'_l$  in [t/2, 1] such that for each  $t' \in [t/2, 0]$  there exists an element  $i[t'] \in \{1, 2, \ldots, l\}$  satisfying  $t' \in I_{\delta'[t'_i]}(t_i)$ . Put
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$$V' = \bigcap_{i=1}^{l} V'[t'_i];$$
  

$$r'_0 = \max\{r'[t'_i] \mid i = 1, 2, \dots, l\}$$

Then V' is an open  $G_x$ -invariant neighbourhood of x in X. Moreover, for  $s \in S$ ,  $s' \in S'$ ,  $g \in \text{supp}_G(\phi_{s,s'})$ , and  $t' \in [t/2, 1]$  the implication

(21.40) 
$$g\pi(s)_X \in V', \pi(s)_I \ge t/2, \eta(s) \ge r'_0$$
  
 $\implies \pi'(s')_X \in V_0, |\pi'(s')_I - \pi(s)_I| < \epsilon/4$ 

holds, since  $\pi(s)_I \ge t/2$  implies the existence of  $i \in \{1, 2, ..., l\}$  satisfying  $\pi(s)_X \in V'[t_i]$  and  $\pi(s)_I \in I_{\delta'[t'_i]}(t_i)$ , we conclude  $\pi'(s')_X \in V_0$  and  $\pi'(s')_I \in I_{\epsilon/8}(t_i)$  from (21.39), and now one can apply the triangle inequality to  $\pi(s)_I, \pi'(s')_I$ , and  $t_i$  using  $\delta'[t'_i] + \epsilon/8 < \epsilon/8 + \epsilon/8 = \epsilon/4$ .

Let N be the number appearing in (21.35). Choose a natural number M such that (21.33) holds if we put  $\mu = \epsilon/2$ . Since  $\lim_{n\to\infty} \rho(m,n) = 0$  holds for  $m \in \{0, 1, \ldots, \max\{r'_0, M\}\}$  by (21.32), we can find a natural number r' satisfying  $r' \ge \max\{r'_0, M\}$  such that for every  $m, n \in \mathbb{N}$  the implication

(21.41) 
$$m \le \max\{r'_0, M\}, n \ge r' - \max\{r'_0, M\} \implies \rho(m, n) < t/2$$

holds. Next we show that the desired implication (21.36) holds if we put  $U' := V' \times I_{\epsilon/4}(t)$  and use the number r' above.

Consider  $(s, n) \in SH(S)$ ,  $(s', n') \in SH(S')$ , and  $g \in supp_G(\phi_{s,s'})$  satisfying  $SH(\pi)(s, n) \in U'$  and  $\eta(s, n) := n \ge r'$ . Since  $SH(\pi)(s, n) \in U'$  implies that  $SH(\pi)(s, n)_I = \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I$  belongs to  $I_{\epsilon/4}(t)$ , we get

(21.42) 
$$\rho(\eta(s), n - \eta(s)) \ge \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I \ge t - \epsilon/4 \stackrel{(21.38)}{\ge} t/2$$

We conclude from (21.41) and (21.42) that  $\eta(s) > \max\{r'_0, M\}$  holds. In particular, we get  $\eta(s) \ge r'_0$  and  $\eta(s) \ge M$ .

Since  $n' - \eta'(s') = n - \eta(s)$ , we conclude from (21.33) and (21.35)

(21.43) 
$$|\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \le \epsilon/2$$

We have  $SH(\pi)(s, n)_I = \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I \in I_{\epsilon/4}(t)$ . This implies

$$\pi(s)_I \ge \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_I \ge t - \epsilon/4 \stackrel{(21.38)}{\ge} t/2.$$

Since  $g \cdot SH(\pi)(s)_X = g\pi(s)_X \in V'$  and  $\pi(s)_I \ge t/2$  hold, we get

(21.44) 
$$\pi'(s')_X \in V_0;$$

(21.45) 
$$|\pi'(s')_I - \pi(s)_I| < \epsilon/4,$$

from (21.40). We estimate

$$\begin{split} |\operatorname{SH}(\pi')(s')_{I} - t| \\ &\leq |\operatorname{SH}(\pi')(s')_{I} - \operatorname{SH}(\pi)(s)_{I}| + |\operatorname{SH}(\pi)(s)_{I} - t| \\ &= |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s')_{I} - \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_{I}| \\ &+ |\operatorname{SH}(\pi)(s)_{I} - t| \\ &\leq |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s')_{I} - \rho(\eta'(s'), n' - \eta'(s')) \cdot \pi(s)_{I}| \\ &+ |\rho(\eta'(s'), n' - \eta'(s')) \cdot \pi(s)_{I} - \rho(\eta(s), n - \eta(s)) \cdot \pi(s)_{I}| \\ &+ |SH(\pi)(s)_{I} - t| \\ &= |\rho(\eta'(s'), n' - \eta'(s'))| \cdot |\pi'(s')_{I} - \pi(s)_{I}| \\ &+ |\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \cdot |\pi(s)_{I}| + \epsilon/4 \\ &\leq |\pi'(s')_{I} - \pi(s)_{I}| + |\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| + \epsilon/4 \\ &\leq \epsilon/4 + \epsilon/2 + \epsilon/4 \\ &= \epsilon. \end{split}$$

This implies together with (21.37) and (21.44) that  $SH(\pi')(s') \in U$  holds. This finishes the proof of the implication (21.36) in the case  $[x, t] \neq *$ .

Next we show the implication (21.36) in the case [x, t] = \*. Consider an open *G*-invariant neighborhood *U* of \*. We have to find an open *G*-invariant neighborhood *U'* of \* and a natural number r' such that for all  $(s, n) \in SH(S)$ ,  $(s', n') \in SH(S')$ , and  $g \in \text{supp}_G(SH(\phi)_{(s,n),(s',n')})$  the implication

(21.46) 
$$g \cdot \operatorname{SH}(\pi)(s, n) \in U', \operatorname{SH}(\eta)(s, n) \ge r' \implies \operatorname{SH}(\pi')(s', n') \in U$$

holds.

For  $\epsilon > 0$ , we define  $V_{\epsilon}$  to be the open *G*-invariant neighborhood of \* in cone(*X*) given by

$$V_{\epsilon} = \{ [x, t] \mid x \in X, t < \epsilon \}.$$

Since **B'** satisfies *compact support over* cone(*X*), the subset  $[0, 1] \cdot im(\pi')$  of cone(*X*) is compact. Hence there exists an  $\epsilon > 0$  satisfying

$$V_{\epsilon} \cap [0,1] \cdot \operatorname{im}(\pi') \subseteq U.$$

Since  $\operatorname{im}(\operatorname{SH}(\pi')) \subseteq [0, 1] \cdot \operatorname{im}(\pi')$  holds, it suffices to prove (21.46) in the special case  $U = V_{\epsilon}$ .

Since  $\phi$  satisfies *continuous control*, there exists an open *G*-invariant neighborhood  $U'_0$  of \* in cone(*X*) and a natural number  $r'_2$  such that for all  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$  the implication

(21.47) 
$$g\pi(s) \in U'_0, \eta(s) \ge r'_2 \implies \pi'(s') \in V_\epsilon$$

holds. Since **B** satisfies *compact support over* cone(X), there exists a  $\delta > 0$  satisfying

$$V_{\delta} \cap [0,1] \cdot \operatorname{im}(\pi) \subseteq U'_0.$$

21.7 The Proof of the Axioms of a G-Homology Theory

We get from (21.47) the implication

(21.48) 
$$\operatorname{supp}_{G}(\phi_{s,s'}) \neq \emptyset, \pi(s)_{I} < \delta, \eta(s) \ge r'_{2} \implies \pi'(s') \in V_{\epsilon}.$$

Let *N* be the number appearing in (21.35). Choose a natural number *M* such that (21.33) holds if we put  $\mu = \epsilon/2$ . Since  $\lim_{n\to\infty} \rho(m,n) = 0$  holds for  $m \in \{0, 1, ..., N + \max\{r', M\}\}$  by (21.32), we can find a natural number *r'* satisfying  $r' \ge N + \max\{r'_2, M\}$  such that for every  $m, n \in \mathbb{N}$  the implication

$$(21.49) \quad m \le N + \max\{r'_2, M\}, n \ge r' - N - \max\{r'_2, M\} \implies \rho(m, n) < \epsilon/2$$

holds.

Next we want to prove the implication (21.46) in the special case  $U = V_{\epsilon}$ , where we take r' to be the natural number above and  $U' = V_{\epsilon \delta/2}$ . Consider  $(s, n) \in SH(s)$ ,  $(s', n') \in SH(S')$ , and  $g \in \text{supp}_G(SH(\phi)_{(s,n),(s',n')})$  satisfying  $SH(\pi)(s,n)_I \leq \epsilon \delta/2$  and  $SH(\eta)(s,n) := n \geq r'$ . We have to show  $\pi'(s')_I \leq \epsilon$ .

If  $\rho(\eta'(s'), n' - \eta'(s')) < \epsilon$  holds, then we get

$$\operatorname{SH}(\pi')(s') = \rho(\eta'(s'), n' - \eta'(s')) \cdot \pi'(s') \in V_{\epsilon}.$$

Hence we can assume without loss of generality

(21.50) 
$$\rho(\eta'(s'), n' - \eta'(s')) \ge \epsilon.$$

We conclude from (21.49) and (21.50) that  $\eta'(s') > N + \max\{r'_2, M\}$  holds. In particular, we have  $\eta'(s') \ge N + r'_2$  and  $\eta'(s') \ge M$ . Since  $n' - \eta'(s') = n - \eta(s)$  holds, we conclude from (21.33) that

$$|\rho(\eta'(s'), n' - \eta'(s')) - \rho(\eta(s), n - \eta(s))| \le \epsilon/2$$

holds. This implies together with (21.50)

(21.51) 
$$\rho(\eta(s), n - \eta(s)) \ge \epsilon/2.$$

Hence we get

$$\pi(s)_{I} = \frac{\operatorname{SH}(\pi)(s)_{I}}{\rho(\eta(s), n - \eta(s))} \stackrel{(21.51)}{<} \frac{2 \cdot \operatorname{SH}(\pi)(s)_{I}}{\epsilon} \le \frac{2 \cdot \delta \cdot \epsilon/2}{\epsilon} = \delta$$

Since  $\eta'(s') \ge N + r'_2$ , we conclude  $\eta(s) \ge r'_2$  from (21.35). Finally (21.48) implies  $\pi'(s') \in V_{\epsilon}$ .

This finishes the proof that  $SH(\phi)$  is a well-defined morphism. Now one easily checks that SH is a well-defined functor of additive categories.

Next we define a natural equivalence of covariant functors of additive categories  $O^G(\operatorname{cone}(X)) \to O^G(\operatorname{cone}(X))$ 

$$T: \operatorname{id} \oplus \operatorname{SH} \xrightarrow{\cong} \operatorname{SH}.$$

We have to define for an object  $\mathbf{B} = (S, \pi, \eta, B)$  an isomorphism  $T(\mathbf{B}) : \mathbf{B} \oplus SH(\mathbf{B}) \xrightarrow{=} SH(\mathbf{B})$  in  $\mathcal{O}^G(\operatorname{cone}(X))$ . Define a bijection

$$u \colon S \bigsqcup {\mathsf{SH}}(S) \xrightarrow{\cong} {\mathsf{SH}}(S)$$

by sending  $s \in S$  to  $(s, \eta(s))$  and  $(s, n) \in SH(S)$  to (s, n+1). For  $z \in S \coprod SH(S)$  and  $(s, n) \in SH(S)$  define  $T(\mathbf{B})_{z,(s,n)}$  by  $id_{B(s)}$  for (s, n) = u(z) and by 0 otherwise. Note that  $supp_G(T(B)_{r,(s,n)})$  is empty or  $\{e\}$ . Obviously  $T(\mathbf{B})$  satisfies *finite G-support* and *bounded control over*  $\mathbb{N}$ , whereas *continuous control* is proved as follows. We only deal with the implication (21.5). The proof of the other implication (21.6) is completely analogous.

Consider an element  $[x,t] \in \operatorname{cone}(X)$  and a  $G_{[x,t]}$ -invariant neighbourhood U of [x,t] in  $\operatorname{cone}(X)$ . It remains to construct a  $G_{[x,t]}$ -invariant neighbourhood U' of [x,t] in  $\operatorname{cone}(X)$  with  $U' \subseteq U$  and a natural number r' such that for  $s \in S$  the implication

(21.52) 
$$\pi(s) \in U', \eta(s) \ge r' \implies \operatorname{SH}(\pi)(s, \eta(s)) \in U$$

and for  $(s, n) \in SH(S)$  the implication

(21.53) 
$$\operatorname{SH}(\pi)(s,n) \in U', \operatorname{SH}(\eta)(s,n) \ge r' \implies \operatorname{SH}(\pi)(s,n+1) \in U$$

hold.

Next we show that we can choose  $\mu \in (0, 1]$  and an open  $G_{[x,t]}$ -invariant neighbourhood U' of [x, t] in cone(X) satisfying

(21.54) 
$$t' \cdot U' \subseteq U \text{ for all } t' \in [1 - \mu, 1].$$

We first consider the case [x, t] = \*. Recall that pr:  $X \times [0, 1] \rightarrow \text{cone}(X)$  is the obvious projection. Let  $p: X \rightarrow X/G$  be the canonical projection. We have  $X \times \{0\} \subseteq \text{pr}^{-1}(U) \subseteq X \times [0, 1]$  as  $* \in U$ . This implies

$$X/G \times \{0\} \subseteq p(\operatorname{pr}^{-1}(U)) \subseteq X/G \times [0,1].$$

Since *X*/*G* is a *CW*-complex and hence paracompact, see [746], and  $p(\text{pr}^{-1}(U))$  is open, we can find a continuous map  $\epsilon \colon X/G \to (0, 1)$  such that  $\{(xG, t) \mid xG \in X/G, t < \epsilon(xG)\}$  is contained in  $p(\text{pr}^{-1}(U))$ . Define

$$U' = \operatorname{pr}(\{(x,t) \mid x \in X, t < \epsilon \circ p(x)\}).$$

This is an open G-invariant neighborhood of \* in cone(X) satisfying  $U' \subseteq U$  and  $[0,1] \cdot U' = U'$ . Hence we choose for  $\mu$  any value in (0,1].

Next we consider the case  $[x, t] \neq *$ , or, equivalently, t > 0. Let  $p: X \to X/G_x$ be the projection. Then  $pr^{-1}(U)$  is an open  $G_x$ -invariant neighbourhood of  $(x, t) \in X \times [0, 1]$  and  $p(pr^{-1}(U))$  is an open neighbourhood of (p(x), t) in  $X/G_x \times [0, 1]$ . Choose an open neighborhood V' of p(x) in  $X/G_x$  and  $\epsilon \in \mathbb{R}$  with  $0 < \epsilon < t/2$ such that  $V' \times (t - \epsilon, t + \epsilon)$  is contained in  $p(pr^{-1}(U))$ . Put  $V = p^{-1}(V')$ . Then V is an open  $G_x$ -invariant neighborhood of x such that  $V \times (t - \epsilon, t + \epsilon)$  is contained in pr<sup>-1</sup>(U). Choose  $\mu \in (0, 1]$  such that  $(1 - \mu) \cdot (t - \epsilon/2) > t - \epsilon$  holds. Then  $t't'' \in (t - \epsilon, t + \epsilon)$  holds for  $t' \in [1 - \mu, 1]$  and  $t'' \in (t - \epsilon/2, t + \epsilon/2)$ . Put

$$U' = \operatorname{pr}((V \times (t - \epsilon/2, t + \epsilon/2))).$$

This is an open  $G_x$ -invariant neighbourhood of [x, t] satisfying (21.54). This finishes the proof that we can choose  $\mu \in (0, 1]$  and an open a  $G_{[x,t]}$ -invariant neighbourhood U' of [x, t] in cone(X) satisfying (21.54).

Because of (21.31) and (21.34) we can choose a natural number r' such that for all  $m \in \mathbb{N}$  with  $m \ge r'$  we have

(21.55) 
$$1 - \mu \le \rho(m, 0) \le 1$$

and for all  $m, n \in \mathbb{N}$  with  $m \le n$  and  $n \ge r'$  we have

(21.56) 
$$1 - \mu \le \frac{\rho(m, n+1-m)}{\rho(m, n-m)} \le 1.$$

Now (21.52) follows from (21.54) and (21.55), since SH( $\pi$ )( $s, \eta(s)$ ) =  $\rho(\eta(s), 0) \cdot \pi(s)$  holds. Moreover, (21.53) follows from (21.54) and (21.56), since we have SH( $\eta$ )(s, n) = n and SH( $\pi$ )(s, n + 1) =  $\frac{\rho(\eta(s), n+1-\eta(s))}{\rho(\eta(s), n-\eta(s))} \cdot$ SH( $\pi$ )(s, n). One easily checks that  $T(\mathbf{B})$  is an isomorphism and the collection of the  $T(\mathbf{B})$ -s

One easily checks that  $T(\mathbf{B})$  is an isomorphism and the collection of the  $T(\mathbf{B})$ -s fit together to define the desired natural equivalence T.

Thus we have defined an Eilenberg swindle (SH, T) on  $O^G(\text{cone}(X))$ . The weak contractibility of  $\mathbf{K}(O^G(\text{cone}(X)))$  follows from Theorem 6.37 (iii). This finishes the proof of Lemma 21.30.

#### 21.7.3 Excision and G-Homotopy Invariance

**Lemma 21.57.** Let (X, A) be a G-CW-pair and let  $\mathbf{B} = (S, \pi, \eta, B)$  be an object in  $O^G(X)$ . Choose a nested sequence of open G-invariant sets

$$X \supseteq V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \supseteq A$$

together with a G-map  $\rho: V_0 \to A$  such that  $\bigcap_{n\geq 0} \overline{V_n} = A$  and  $\rho|_A = A$  hold. Fix a non-decreasing function  $\omega: \mathbb{N} \to \mathbb{N}$  with  $\lim_{n\to\infty} \omega(n) = \infty$  and a natural number  $w \in \mathbb{N}$ . Define new objects  $\mathbf{B}^{\omega,w} = (S^{\omega,w}, \pi^{\omega,w}, \eta^{\omega,w}, \mathbf{B}^{\omega,w})$  and  $\mathbf{B}^{\perp} = (S^{\perp}, \pi^{\perp}, \eta^{\perp}, \mathbf{B}^{\perp})$  by

$$S^{\omega,w} := \{ s \in S \mid \eta(s) < w \text{ or } \pi(s) \in V_{\omega \circ \eta(s)} \};$$
  
$$S^{\perp} = S \setminus S^{\omega,w},$$

and by defining  $\pi^{\omega,w}$ ,  $\eta^{\omega,w}$ ,  $B^{\omega,w}$ ,  $\pi^{\perp}$ ,  $\eta^{\perp}$ , and  $B^{\perp}$  by restricting  $\pi$ ,  $\eta$ , and B.

- (i) The desired sequences  $(V_n)_{n>0}$  and the G-map  $\rho$  exist;
- (ii) There are obvious morphisms  $i^{\omega,w} \colon \mathbf{B}^{\omega,w} \to \mathbf{B}$  and  $i^{\perp} \colon \mathbf{B}^{\perp} \to \mathbf{B}$  such that  $i^{\omega,w} \oplus i^{\perp} \colon \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \xrightarrow{\cong} \mathbf{B}$  is an isomorphism, and  $\operatorname{im}(\pi^{\perp}) \subseteq X \setminus A$ ; (iii) There is an object  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbb{B}})$  in  $O^G(A)$  such that  $\mathbf{B}^{\omega,w}$  and  $\widehat{\mathbf{B}}$  are isomor-
- phic in  $O^G(X)$ ;
- (iv) Consider an object  $\mathbf{B}' = (S', \pi', \eta', \mathbf{B}')$  in  $O^G(A)$  and a morphism  $\phi \colon \mathbf{B}' \to \mathbf{B}$ in  $O^G(X)$ . Then we can find  $(\omega, w)$  and a morphism  $\phi' : \mathbf{B}' \to \mathbf{B}^{\omega, w}$  such that  $\phi$  factorizes as

$$\phi \colon \mathbf{B}' \xrightarrow{\phi'} \mathbf{B}^{\omega, w} \xrightarrow{i^{\omega, w}} \mathbf{B}.$$

*Proof.* (i) The inclusion  $A \rightarrow X$  is a G-cofibration, or, equivalently, a G-NDR-pair. The proofs of these facts in the non-equivariant case carry over to the equivariant case. This implies assertion (i). For some information and relevant references we refer for instance to [644, Chapter 1]. The core of the proof can also be derived from the construction appearing in the proof of [687, Theorem 7.1]. The basic idea is to use the retraction  $r: D^n \setminus \{0\} \to S^{n-1}$  given by the radial projection and the continuous function  $u: D^n \to [0, 1]$  given by the Euclidean norm, which obviously satisfies  $u^{-1}(1) = S^{n-1}$  and  $u^{-1}(0, 1] = D^n \setminus \{0\}$ .

- (ii) This is obvious.
- (iii) We define  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbb{B}})$  by

$$\widehat{S} := S^{\omega, w};$$

$$\widehat{\pi}(s) := \begin{cases} a_0 & \text{if } \eta(s) < w; \\ \rho \circ \pi(s) & \text{if } \eta(s) \ge w; \end{cases}$$

$$\widehat{\eta}(s) := \eta(s);$$

$$\widehat{B}(s) := B(s),$$

where  $s \in S^{\omega,w}$  and  $a_0$  is some point in A. In order to show that  $\mathbf{B}^{\omega,w}$  and  $\widehat{\mathbf{B}}$  are isomorphic in  $O^G(X)$ , we check the criterion appearing in Lemma 21.14.

So consider  $x \in X$  and an open  $G_x$ -invariant neighborhood U of x in X. Since **B** satisfies compact support over X, we can find a compact subset  $C \subseteq X$  such that  $\operatorname{im}(\pi) \subseteq C$  holds. Choose an open  $G_x$ -invariant neighbourhood  $U'_0$  of  $x \in X$  with  $\overline{U'_0} \subseteq U$ . Next we show that there exists a natural number  $r'_0$  satisfying the implication

(21.58) 
$$y \in C, \rho(y) \in U'_0, y \in V_{\omega(r'_0)} \implies y \in U.$$

Suppose that this is not the case. Since  $V_m \subseteq V_n$  holds for  $m \ge n$ , and  $\lim_{n \to \infty} \omega(n) =$  $\infty$ , we can find a sequence  $(y_n)_{n\geq 0}$  of elements in C such that  $\rho(y_n) \in U'_0, y_n \in V_n$ , and  $y_n \notin U$  holds for  $n \ge 0$ . Since C is compact, there is a strictly monotone increasing function  $u: \mathbb{N} \to \mathbb{N}$  with  $\lim_{n \to \infty} u(n) = \infty$  and an element  $y \in C$ satisfying  $\lim_{n\to\infty} y_{u(n)} = y$ . Since for each natural number *n* we have  $y_{u(m)} \in V_{u(n)}$ for  $m \ge n$ , we get  $y \in \overline{V_{u(n)}}$  for every  $n \ge 0$ . This implies  $y \in \bigcap_{n \ge 0} \overline{V_{u(n)}} = A$  and hence  $\rho(y) = y$ . From  $\lim_{n \to \infty} y_{u(n)} = y$  we conclude  $\lim_{n \to \infty} \rho(y_{u(n)}) = \rho(y) = y$ . Since  $\rho(y_{u(n)}) \in \overline{U'_0}$  for  $n \ge 0$ , we conclude  $y \in \overline{U'_0}$  and hence  $y \in U$ . Since

 $\lim_{n\to\infty} y_{u(n)} = y$  holds, there exists a natural number  $n_0$  with  $y_{u(n)} \in U$  for  $n \ge n_0$ , a contradiction. This finishes the proof of (21.58).

Suppose that the element  $x \in X$  does not belong to A. Then we can find an open  $G_x$ -invariant neighborhood  $U'_1$  of x and a natural number  $r'_1$  satisfying the implication

(21.59) 
$$y \in C, y \in V_{\omega(r'_1)} \implies y \notin U'_1.$$

Suppose the contrary. The same ideas as in the sketch of the proof of assertion (i) lead to the construction of a sequence of open  $G_x$ -invariant sets  $X \supseteq W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq \{x\}$  with  $\bigcap_{n\geq 0} \overline{W_n} = \{x\}$ . Fix  $n \in \mathbb{N}$ . Since (21.59) does not hold for  $U'_1 = W_n, V_m \subseteq V_n$  holds for  $m \ge n$ , and  $\lim_{n\to\infty} \omega(n) = \infty$ , we can find an element  $y_n$  in X satisfying  $y_n \in C$ ,  $y_n \in V_n$ , and  $y_n \in W_n$ . Since C is compact, there is a strictly monotone increasing function  $u: \mathbb{N} \to \mathbb{N}$  and  $y \in C$  with  $\lim_{n\to\infty} y_{u(n)} = y$ . This implies  $y \in \bigcap_{n\geq 0} \overline{V_{u(n)}} = A$  and  $y \in \bigcap_{n\geq 0} \overline{W_{u(n)}} = \{x\}$ , a contradiction. This finishes the proof of the implication (21.59).

Now we define the desired open  $G_x$ -invariant neighborhood U' of x in X by  $U'_0 \cap U'_1$  and the desired natural number  $r' = \max\{r'_0, r'_1, w\}$ . We get for  $s \in S^{\omega, w}$ 

$$\begin{aligned} \widehat{\pi}(s) &\in U', \widehat{\eta}(s) \geq r' \\ &\implies \pi(s) \in C, \widehat{\pi}(s) \in U'_0, \eta(s) \geq w, \eta(s) \geq r'_0 \\ &\implies \pi(s) \in C, \rho \circ \pi(s) \in U'_0, \pi(s) \in W_{\omega \circ \eta(s)}, \eta(s) \geq r'_0 \\ &\implies \pi(s) \in C, \rho \circ \pi(s) \in U'_0, \pi(s) \in W_{\omega(r'_0)} \end{aligned}$$

$$\overset{(21.58)}{\Longrightarrow} \pi^{\omega, w}(s) = \pi(s) \in U.$$

Moreover, we have for  $s \in S^{\omega,w}$ 

$$\begin{split} \eta^{\omega,w}(s) &\geq r' \implies \pi(s) \in C, \eta(s) \geq w, \eta(s) \geq r'_1 \\ \implies \pi(s) \in C, \pi(s) \in V_{\omega \circ \eta(s)}, \eta(s) \geq r'_1 \\ \implies \pi(s) \in C, \pi(s) \in V_{r'_1} \\ \overset{(21.59)}{\implies} \pi(s) \notin U'_1 \\ \implies \pi^{\omega,w}(s) = \pi(s) \notin U'. \end{split}$$

Hence there is no  $s \in S^{\omega,w}$  satisfying  $\pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \ge r'$  and hence the implication

$$\pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \ge r' \implies \widehat{\pi}(s) \in U$$

obviously holds. This finishes the proof of assertion (iii) in the case that  $x \notin A$ .

It remains to treat the case  $x \in A$ . Then define the desired open  $G_x$ -invariant neighborhood U' of x in X by  $U'_0 \cap \rho^{-1}(U)$  and the desired natural number  $r' = \max\{r'_0, w\}$ . Then we get analogously to the argument above

$$\widehat{\pi}(s) \in U', \widehat{\eta}(s) \ge r' \implies \pi^{\omega, w}(s) \in U$$

and

$$\pi^{\omega,w}(s) \in U', \eta^{\omega,w}(s) \ge r' \implies \pi(s) \in \rho^{-1}(U), \eta(s) \ge w$$
$$\implies \rho \circ \pi(s) \in U, \eta(s) \ge w$$
$$\implies \widehat{\pi}(s) = \rho \circ \pi(s) \in U.$$

This finishes the proof of assertion (iii).

(iv) Choose a compact subset  $C \subseteq A$  satisfying  $\operatorname{im}(\pi') \subseteq C$  and a finite subset  $F \subseteq G$  such that  $\operatorname{supp}_G(\phi_{s',s}) \subseteq F$  holds for all  $s' \in S'$  and  $s \in S$ . Fix  $n \in \mathbb{N}$ . Consider  $a \in F \cdot C$ . Then  $V_n$  is an open  $G_a$ -invariant neighborhood of a in X. Since  $\phi$  satisfies *continuous control*, we can find an open  $G_a$ -invariant neighbourhood  $U_n(a)$  of a in X and a natural number  $r_n(a)$  such for  $s' \in S'$ ,  $s \in S$ , and  $g \in \operatorname{supp}_G(\phi_{s',s})$  the implication

(21.60) 
$$g \cdot \pi'(s') \in U_n(a), \eta'(s') \ge r_n(a) \implies \pi(s) \in V_n$$

holds. Since  $F \cdot C$  is compact and contained in  $\bigcup_{a \in F \cdot C} U_n(a)$ , we can find a finite subset  $\{a_1, a_2, \ldots, a_k\} \subseteq F \cdot C$  satisfying  $F \cdot C \subseteq \bigcup_{i=1}^k U_n(a_k)$ . Define a natural number

$$r_n := \max\{r_n(a_i) \mid i = 1, 2, \dots, k\}.$$

Consider  $s' \in S'$  and  $s \in S$  with  $\phi_{s',s} \neq 0$ . Then we get the implication

(21.61) 
$$\eta'(s') \ge r_n \implies \pi(s) \in V_n$$

by the following argument. Suppose  $\eta'(s') \ge r_n$ . Since  $\phi_{s',s} \ne 0$ , we can choose  $g \in \text{supp}_G(\phi_{s',s})$ . Because of  $g \cdot \pi'(s') \in F \cdot C$  we can find  $i \in \{1, 2, ..., k\}$  with  $g \cdot \pi'(s) \in U_n(a_i)$ . Since  $r_n \ge r_n(a_i)$ , we conclude from the implication (21.60) that  $\pi(s) \in V_n$  holds.

We can additionally arrange that  $r_n < r_{n+1}$  holds for  $n \in \mathbb{N}$ . Since  $\phi$  satisfies *bounded control over*  $\mathbb{N}$ , we can find a natural number N such that  $|\eta(s') - \eta(s)| \le N$  holds for all  $s' \in S'$  and  $s \in S$  with  $\phi_{s',s} \ne 0$ .

Now define a function

 $\omega\colon \mathbb{N}\to \mathbb{N}$ 

by requiring that for  $m, n \in \mathbb{N}$  with  $r_n + N \le m < r_{n+1} + N$  we have  $\omega(m) = n$  and  $\omega(m) = 0$  for  $m < r_0 + N$ . Then  $\omega$  is a non-decreasing function with  $\lim_{m\to\infty} \infty = \infty$ . Put  $w = r_0 + N$ .

Consider any  $s \in S$  such that there exists an  $s' \in S'$  with  $\phi_{s',s} \neq 0$ . Next we want to show  $s \in S^{\omega,w}$ , or, equivalently, the implication

$$\eta(s) \ge w \implies \pi(s) \in V_{\omega \circ \eta(s)}.$$

Suppose  $\eta(s) \ge w$ . Then we can choose  $n \in \mathbb{N}$  such that  $r_n + N \le \eta(s) < r_{n+1} + N$ holds. Then  $\omega \circ \eta(s) = n$  and  $\eta'(s') \ge r_n$ . We conclude  $\pi(s) \in V_{\omega \circ \eta(s)}$  from implication (21.61). Hence  $\phi$  induces the desired morphism  $\phi' : \mathbf{B}' \to \mathbf{B}^{\omega}$  by putting  $\phi'_{s',s} = \phi_{s',s}$  for  $s' \in S'$  and  $s \in S^{\omega,w}$ . This finishes the proof of Lemma 21.57.  $\Box$ 

21.7 The Proof of the Axioms of a G-Homology Theory

**Lemma 21.62.** Let X be G-CW-complex with sub G-CW-complexes  $X_0$ ,  $X_1$ , and  $X_2$  satisfying  $X = X_1 \cup X_2$  and  $X_0 = X_1 \cap X_2$ .

(i) The inclusion  $i: (X_2, X_0) \rightarrow (X, X_1)$  induces an equivalence of additive categories

$$\mathcal{D}^{G}(i) \colon \mathcal{D}^{G}(X_{2}, X_{0}) \xrightarrow{\simeq} \mathcal{D}^{G}(X, X_{1});$$

(ii) The square induced by the various inclusions

is weakly homotopy cocartesian.

*Proof.* (i) Consider an object **B** in  $O^G(X)$ . We get from Lemma 21.57 (ii) and (iii) applied to the pair  $(X, X_1)$  and the object **B** the decomposition  $\mathbf{B} = \mathbf{B}^{\omega, w} \oplus \mathbf{B}^{\perp}$  such that  $\mathbf{B}^{\omega, w}$  is isomorphic to an object in  $O^G(X_1)$  and  $\operatorname{im}(\pi^{\perp}) \subseteq X \setminus X_1$  holds. Therefore the inclusion  $\mathbf{B}^{\perp} \to \mathbf{B}$  yields an isomorphism in  $\mathcal{D}^G(X, X_1)$ . The object  $\mathbf{B}^{\perp}$  is in the image of  $\mathcal{D}^G(i)$ , since the inclusion  $X_2 \setminus X_0 \to X \setminus X_1$  is a *G*-homeomorphism. We conclude that  $\mathcal{D}^G(i)$  is surjective on the set of isomorphism classes of objects.

Consider a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  in  $O^G(X)$ . It can be written in terms of the decomposition of Lemma 21.57 (ii) applied to the pair  $(X, X_1)$  and the objects **B** and **B**' as

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{B}^{\omega, w} \oplus \mathbf{B}^{\perp} \to \mathbf{B}^{\prime \omega, w} \oplus \mathbf{B}^{\prime \perp}.$$

Define a morphism in  $O^G(X)$  by the composite

$$\psi: \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \xrightarrow{\begin{pmatrix} \mathrm{id} \ 0 \\ 0 \ b \end{pmatrix}} \mathbf{B}^{\omega,w} \oplus \mathbf{B}'^{\omega,w} \xrightarrow{\begin{pmatrix} a \ \mathrm{id} \\ c \ 0 \end{pmatrix}} \mathbf{B}'^{\omega,w} \oplus \mathbf{B}'^{\perp}$$

Then  $\mathbf{B}^{\omega,w} \oplus \mathbf{B}'^{\omega,w}$  is isomorphic to an object in the image of  $O^G(X_1) \to O^G(X)$ by Lemma 21.57 (iii), the morphism  $\phi - \psi : \mathbf{B}^{\omega,w} \oplus \mathbf{B}^{\perp} \to \mathbf{B}'^{\omega,w} \oplus \mathbf{B}'^B$  is of the shape  $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ , and  $d : \mathbf{B}^{\perp} \to \mathbf{B}'^{\perp}$  is in the image of  $O^G(X_2) \to O^G(X)$ , since  $\operatorname{im}(\pi^{\perp})$  and  $\operatorname{im}(\pi'^{\perp})$  are contained in  $X \setminus X_1 \subseteq X_2$ . This implies that the morphism in  $\mathcal{D}^G(X, X_1)$  represented by  $\phi$  is in the image of  $\mathcal{D}^G(i)$ . Hence  $\mathcal{D}^G(i)$  is full.

In order to show that  $\mathcal{D}^{G}(i)$  is an equivalence, it remains to show that  $\mathcal{D}^{G}(i)$  is faithful. This is done as follows.

Consider a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  in  $O^G(X_2)$ . Suppose that its class  $[\phi]$  in  $\mathcal{D}^G(X_2, X_0)$  is sent under  $\mathcal{D}^G(i)$  to zero. Hence there is an object  $\mathcal{B}_0$  in  $O^G(X_1)$ , an object  $\mathbf{B}_1$  in  $\mathcal{T}^G(X)$ , morphisms  $\psi: \mathcal{B} \to \mathcal{B}_0, \psi': \mathcal{B}_0 \to \mathcal{B}, \mu: \mathcal{B} \to \mathcal{B}_1$ , and

 $\mu' \colon \mathcal{B}_1 \to \mathcal{B}'$  such that  $\phi - \psi' \circ \psi$  factorizes as

$$(\phi - \psi' \circ \psi) \colon \mathcal{B} \xrightarrow{\mu} \mathcal{B}_1 \xrightarrow{\mu'} \mathcal{B}'.$$

Because of Lemma 21.22 the object **B**<sub>1</sub> is isomorphic to an object in  $\mathcal{T}^G(X_0)$ . Therefore we can replace  $\phi$  by  $\phi - \mu' \circ \mu$  without changing the element it represents in  $O^G(X_2, X_0)$ . Hence we can assume without loss of generality that  $\phi$  factorizes as

$$\phi \colon \mathcal{B} \xrightarrow{\psi} \mathcal{B}_0 \xrightarrow{\psi'} \mathcal{B}'.$$

We conclude from Lemma 21.57 (iii) and (iv) applied to the pair  $(X, X_2)$  and  $\psi$  that for appropriate  $(\omega, w) \psi \colon \mathcal{B} \to \mathcal{B}_0$  factorizes as in  $\mathcal{D}^G(X)$  as

$$\psi \colon \mathbf{B} \xrightarrow{\nu} \mathbf{B}_{0}^{\omega, w} \xrightarrow{i^{\omega, w}} \mathbf{B}_{0}$$

and there is an object  $\widehat{\mathbf{B}_0}$  in  $O^G(X_2)$  and an isomorphism  $\zeta : \mathbf{B}_0^{\omega,w} \xrightarrow{\cong} \widehat{\mathcal{B}_0}$  in  $O^G(X)$ . In the construction of  $\widehat{\mathbf{B}_0}$  an element  $a \in X_2$  and a retraction  $\rho : V_0 \to X_2$  occurs. One easily checks by going through the constructions appearing in Lemma 21.57 (i) and (iii) that we can pick  $a \in X_0$  and can arrange that  $\rho(V_0 \cap X_1) \subseteq X_0$  holds. Since  $\mathbf{B}_0$  belongs to  $X_1$ , the object  $\widehat{\mathcal{B}_0}$  is actually an object in  $O^G(X_0)$ . Hence we obtain the factorization in  $O^G(X)$ 

$$\phi\colon \mathcal{B}\xrightarrow{\zeta\circ\nu}\widehat{\mathcal{B}_0}\xrightarrow{\phi'\circ i^{\omega,w}\circ\zeta^{-1}}\mathcal{B}'.$$

Since  $O^G(X_2) \to O^G(X)$  is faithful, the factorization above can be viewed as a factorization in  $O^G(X_2)$ . Hence the class  $[\phi]$  in  $\mathcal{D}^G(X_2, X_0)$  represented by  $\phi$  is trivial.

(ii) This is a direct consequence of assertion (ii) and Proposition 21.27. This finishes the proof of Lemma 21.62

**Lemma 21.63.** The inclusion  $i: (X, A) \rightarrow (X, A) \times [0, 1]$  sending x to (x, 0) induces a weak homotopy equivalence

$$\mathbf{K}(\mathcal{D}^{G}(i)): \mathbf{K}(\mathcal{D}^{G}(X, A)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^{G}((X, A) \times [0, 1])).$$

*Proof.* Because of the Five Lemma and Proposition 21.27 it suffices to treat the case  $A = \emptyset$ .

Since we can apply Lemma 21.62 (ii) to the *G*-*CW*-complex cone(X) $\cup_X X \times [0, 1]$  with the subcomplexes cone(X),  $X \times [0, 1]$ , and X, it suffices to show that the map induced by the obvious inclusion

$$\mathbf{K}(\mathcal{D}^G(\operatorname{cone}(X))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^G(\operatorname{cone}(X) \cup_X X \times [0,1]))$$

is a weak homotopy equivalence. Because of Lemma 21.17, Lemma 21.22, and Theorem 8.46 (i), it suffices to show that the map induced by the obvious inclusion

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 $\mathbf{K}(\mathcal{O}^{G}(\operatorname{cone}(X))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{O}^{G}(\operatorname{cone}(X) \cup_{X} X \times [0,1]))$ 

is a weak homotopy equivalence. Since  $\operatorname{cone}(X)$  and  $\operatorname{cone}(X) \cup_X X \times [0, 1]$  are *G*-homeomorphic, both its source and its target are weakly contractible by Lemma 21.30. This finishes the proof of Lemma 21.63.

**Proposition 21.64.** Let  $f_0, f_1: (X, A) \to (Y, B)$  be *G*-maps of *G*-*CW*-pairs which are *G*-homotopic. Then for every  $n \in \mathbb{Z}$  the homomorphism  $K_n(\mathcal{D}^G(f_0))$  and  $K_n(\mathcal{D}^G(f_1))$  from  $K_n(\mathcal{D}^G(X, A))$  to  $K_n(\mathcal{D}^G(Y, B))$  agree.

*Proof.* Let  $i_k: (X; A) \to (X, A) \times [0, 1]$  be the map sending x to (x, k) for k = 0, 1and let pr:  $(X, A) \times [0, 1] \to (X, A)$  be the projection. Since pr  $\circ i_k = id_{(X,A)}$ holds for k = 0, 1, we conclude from Lemma 21.63 that the two homomorphisms  $K_n(\mathcal{D}^G(i_0))$  and  $K_n(\mathcal{D}^G(i_1))$  from  $K_n(\mathcal{D}^G(X, A))$  to  $K_n(\mathcal{D}^G((X, A) \times [0, 1]))$ agree. Let  $h: (X; A) \times [0, 1] \to (Y, B)$  be a *G*-homotopy between  $f_0$  and  $f_1$ . Now the claim follows from the equality  $K_n(\mathcal{D}^G(f_k)) = K_n(\mathcal{D}^G(h)) \circ K_n(\mathcal{D}^G(i_k))$  for k = 0, 1.

**Proposition 21.65.** Consider a G-CW-pair (X, A), a G-CW-complex B, and a cellular G-map  $f : A \rightarrow B$ . Put  $Y = X \cup_f B$ . Then:

- (i) The pair (Y, B) is a G-CW-pair and the canonical map  $(F, f): (X, A) \rightarrow (Y, B)$  is a cellular G-map;
- (ii) The functor  $\mathcal{D}^G(F, f)$ :  $\mathcal{D}^G(X, A) \xrightarrow{\simeq} \mathcal{D}^G(Y, B)$  is an equivalence of additive categories and induces for all  $n \in \mathbb{Z}$  an isomorphism

$$K_n(\mathcal{D}^G(F, f)): K_n(\mathcal{D}^G(X, A)) \xrightarrow{\simeq} K_n(\mathcal{D}^G(Y, B));$$

(iii) Let  $i: A \to X$  and  $j: B \to Y$  be the inclusions. We obtain a long exact Mayer-Vietoris sequence, infinite to both sides and natural in (X, A) and  $f: A \to B$ ,

$$\cdots \xrightarrow{\partial_{n+1}} K_n(\mathcal{D}^G(A)) \xrightarrow{-K_n(\mathcal{D}^G(i)) \times K_n(\mathcal{D}^G(f))} } K_n(X) \oplus K_n(B) \xrightarrow{K_n(F) \oplus K_n(j)} K_n(\mathcal{D}^G(Y)) \xrightarrow{\partial_n} K_{n-1}(\mathcal{D}^G(A))$$
$$\xrightarrow{-K_{n-1}(\mathcal{D}^G(i)) \times K_{n-1}(\mathcal{D}^G(f))} K_{n-1}(X) \oplus K_{n-1}(B)$$
$$\xrightarrow{K_{n-1}(F) \oplus K_{n-1}(j)} K_{n-1}(\mathcal{D}^G(Y)) \xrightarrow{\partial_{n-1}} \cdots$$

*Proof.* (i) This is obvious.

(ii) Apply Lemma 21.62 (i) to  $X \cup_A \operatorname{cyl}(f)$  and the *G*-subcomplexes *X*,  $\operatorname{cyl}(f)$ , and *A* and then Proposition 21.64 to the obvious *G*-homotopy equivalences  $X \cup_A \operatorname{cyl}(f) \xrightarrow{\simeq} Y$  and  $\operatorname{cyl}(f) \xrightarrow{\simeq} B$ .

(iii) This follows from assertion (ii) and Proposition 21.27.

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#### 21.7.4 The Disjoint Union Axiom

**Proposition 21.66.** Let  $\{X_i \mid i \in I\}$  be a collection of G-CW-complexes. Let  $j_i: X_i \to \coprod_{i \in I} X_i$  be the obvious inclusion for  $i \in I$ .

(i) The obvious map of additive categories

$$\bigoplus_{i \in I} \mathcal{D}^G(j_i) \colon \bigoplus_{i \in I} \mathcal{D}^G(X_i) \to \mathcal{D}^G\left(\bigsqcup_{i \in I} X_i\right)$$

is an equivalence;

(ii) The obvious map of abelian groups

$$\bigoplus_{i\in I} K_n(\mathcal{D}^G(j_i)): \bigoplus_{i\in I} K_n(\mathcal{D}^G(X_i)) \to K_n\Big(\mathcal{D}^G\Big(\coprod_{i\in I} X_i\Big)\Big)$$

is an isomorphism for every  $n \in \mathbb{Z}$ .

*Proof.* (i) In the sequel we put  $Y = \coprod_{i \in I} X_i$ . Consider an object  $\mathbf{B} = (S, \pi, \eta, B)$  in  $\mathcal{D}^G(Y)$ . Since it satisfies *compact support over* Y, there is a finite subset  $I_0 \subseteq I$  such that  $\operatorname{im}(\pi) \subseteq \coprod_{i \in I_0} Y_i$ . For  $i \in I_0$  define  $\mathbf{B}_i = (S_i, \pi_i, \eta_i, B_i)$  where  $S_i = \pi^{-1}(X_i)$  and  $\pi_i, \eta_i$ , and  $B_i$  are obtained from  $\pi, \eta$ , and B by restriction. Then **B** is the finite sum  $\bigoplus_{i \in I_0} \mathbf{B}_i$  and  $\mathbf{B}_i$  is in the image of  $\mathcal{D}^G(j_i) \colon \mathcal{D}^G(X_i) \to \mathcal{D}^G(Y)$  for  $i \in I_0$ . We leave it to the reader to check that this implies that the functor  $\bigoplus_{i \in I} \mathcal{D}^G(j_i)$  is surjective on objects, full, and faithful. Hence  $\bigoplus_{i \in I} \mathcal{D}^G(j_i)$  is an equivalence of additive categories.

(ii) This follows from assertion (i) and fact that  $K_n$  commutes with finite products, or, equivalently, with finite direct sums and is compatible with colimits over direct systems, see for instance [684, Corollary 7.2].

Now Theorem 21.26 follows from Propositions 21.27, 21.64, 21.65, and 21.66.

## **21.8** The Computation of $K_n(\mathcal{D}^G(G/H))$

In this section we analyze the coefficients  $K_n(\mathcal{D}^G(G/H))$  of the *G*-homology theory appearing in Theorem 21.26.

#### 21.8.1 Reduction to $K_n(\mathcal{B}(G/H))$

Consider a category with *G*-support  $\mathcal{B}$  in the sense of Definition 21.1. Given a *G*-set *T*, define a  $\mathbb{Z}$ -category  $\mathcal{B}(T)$  as follows. Objects are pairs (t, B) for  $t \in T$  and  $B \in ob(\mathcal{B})$ . A morphism  $\phi: (t, B) \to (t', B')$  is a morphism  $\phi: B \to B'$  in

#### 21.8 The Computation of $K_n(\mathcal{D}^G(G/H))$

 $\mathcal{B}$  satisfying  $\operatorname{supp}_G(\phi) \subseteq G_{t,t'}$  for  $G_{t,t'} := \{g \in G \mid t' = gt\}$ . Composition in  $\mathcal{B}(S)$  comes from the composition in  $\mathcal{B}$ . The identity on (t, B) is given by  $\operatorname{id}_B$ . The structure of a  $\mathbb{Z}$ -category on  $\mathcal{B}(T)$  comes from the one on  $\mathcal{B}$ . Given a map  $f: T \to T'$ , we get a functor of  $\mathbb{Z}$ -categories  $\mathcal{B}(f): \mathcal{B}(T) \to \mathcal{B}(T')$  by sending an object (t, B) to the object (f(t), B) and a morphism  $\phi: (t, B) \to (t', B')$  given by the morphism  $\phi: B \to B'$  in  $\mathcal{B}$  to the morphism  $(f(s), B) \to (f(s'), B')$  in  $\mathcal{B}(S')$  given by  $\phi: B \to B'$  again. This definition makes sense as  $G_{t,t'} \subseteq G_{f(t),f(t')}$  holds. Thus we obtain a covariant functor

$$(21.67) \qquad \qquad \mathcal{B}(?): G-SETS \to \mathbb{Z}-CAT$$

from the category of *G*-sets to the category of  $\mathbb{Z}$ -categories by sending *T* to  $\mathcal{B}(T)$ . It induces a covariant Or(G)-spectrum

(21.68) 
$$\mathbf{K}(\mathcal{B}(?)_{\oplus}) \colon \operatorname{Or}(G) \to \operatorname{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{B}(G/H)_{\oplus}).$$

We obtain another covariant Or(G)-spectrum by

(21.69) 
$$\mathbf{K}(\mathcal{D}^G(?;\mathcal{B})): \operatorname{Or}(G) \to \operatorname{SPECTRA}, \quad G/H \mapsto \mathbf{K}(\mathcal{D}^G(G/H;\mathcal{B})).$$

**Proposition 21.70.** There is a weak homotopy equivalence of covariant Or(G)-spectra

$$\mathbf{K}(\mathcal{B}(?)_{\oplus}) \xrightarrow{\simeq} \Omega \mathbf{K}(\mathcal{D}^{G}(?); \mathcal{B}).$$

In particular, we get for  $n \in \mathbb{Z}$  an isomorphism, natural in G/H,

$$K_n(\mathcal{B}(G/H)_{\oplus}) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H;\mathcal{B})).$$

*Proof.* Any  $\mathbb{Z}$ -category can be viewed as a category with *G*-support over the trivial group {1}. Hence we can consider for any *G*-set *T* the additive categories  $\mathcal{T}^{\{1\}}(\{\bullet\}; \mathcal{B}(T)), O^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$ , and  $\mathcal{D}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$ . Next we define a functor of additive categories

$$F(T): \mathcal{O}^{\{1\}}(\{\bullet\}; \mathcal{B}(T)) \to \mathcal{O}^G(T; \mathcal{B}).$$

It sends an object **B** =  $(S, \pi, \eta, B)$  to the object **B**' =  $(S', \pi', \eta', B')$ , where S' = S,  $\eta' = \eta$ , and B' and  $\pi'$  are determined by the equality  $B(s) = (\pi'(s), B'(s))$ . It induces a functor of additive categories

$$\overline{F(T)}\colon \mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))\to \mathcal{D}^{\{1\}}(T;\mathcal{B}).$$

Next we show that  $\overline{F(T)}$  is full. Consider any morphism in  $\mathcal{D}^{\{1\}}(T; \mathcal{B})$  from  $\mathbf{B} = (S, \pi, \eta, B)$  to  $\mathbf{B}' = (S', \pi', \eta', B')$ . Choose a morphism  $\phi : \mathbf{B} \to \mathbf{B}'$  in  $O^{\{1\}}(T, ; \mathcal{B})$  representing it. Since *T* is discrete and  $\phi$  satisfies *continuous control*, we can find for every  $t \in T$  a natural number r(t) such that for all  $s \in S, S' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$  the implication

$$g \cdot \pi(s) = t, \eta(s) \ge r(t) \implies \pi'(s) = t$$

holds. Since the object **B** satisfies *compact support over* T,  $\phi$  satisfies *finite* G-support, and T is discrete, there is a finite subset  $T_0 \subseteq T$  satisfying  $g \cdot \pi(s) \in T_0$  for all  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$ . Define  $r := \max\{r(t) \mid t \in T_0\}$ . Then for  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$  the implication

$$\eta(s) \ge r \implies g\pi(s) = \pi'(s')$$

is true. Since  $\phi$  satisfies *bounded control over*  $\mathbb{N}$ , we can change  $\phi$  such that  $\phi_{s,s'} = 0$  holds for  $s \in S$ ,  $s' \in S'$  satisfying  $\eta(s) < r$  and that the class represented by  $\phi$  in  $\mathcal{D}^{\{1\}}(T; \mathcal{B})$  is unchanged. Hence we can assume without loss of generality that  $g \in G_{\pi(s),\pi'(s')}$  holds for  $s \in S$ ,  $s' \in S'$ , and  $g \in \text{supp}_G(\phi_{s,s'})$ .

Define objects  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{\mathbf{B}})$  and  $\widehat{\mathbf{B}'} = (\widehat{S'}, \widehat{\pi'}, \widehat{\eta'}, \widehat{\mathbf{B}'})$  in  $O^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$  by requiring that  $\widehat{S} = S$ ,  $\widehat{S'} = S'$ ,  $\widehat{\eta} = \eta$ , and  $\widehat{\eta'} = \eta'$  hold and we have  $\widehat{\mathbf{B}}(s) = (\pi(s), \mathbf{B}(s))$  for  $s \in S$  and  $\widehat{\mathbf{B}'}(s) = (\pi'(s'), \mathbf{B}(s'))$  for  $s' \in S$ . Then  $F(\widehat{\mathbf{B}}) = \mathbf{B}$ and  $F(\widehat{\mathbf{B}'}) = \mathbf{B'}$ . Define a morphism  $\psi: \widehat{\mathbf{B}} \to \widehat{\mathbf{B}'}$  in  $O^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$  by defining the morphisms  $\psi_{s,s'}: (\pi(s), \mathbf{B}(s)) \to (\pi'(s'), \mathbf{B}(s'))$  in  $\mathcal{B}(T)$  by the morphism  $\phi_{s,s'}: \mathbf{B}(s) \to \mathbf{B'}(s')$  in  $\mathcal{B}$ . One easily checks that  $\psi$  is well-defined and sent under F(T) to  $\phi$ . Hence the class represented by  $\psi$  in  $\mathcal{D}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$  is sent by  $\overline{F(T)}$  to the class represented by  $\phi$  in  $\mathcal{D}^{\{1\}}(T; \mathcal{B})$ . This finishes the proof that  $\overline{F(T)}$  is full.

Since F(T) is faithful, one easily checks that  $\overline{F(T)}$  is faithful. As F(T) is bijective on objects,  $\overline{F(T)}$  is bijective on objects. We conclude that  $\overline{F(T)}$ :  $\mathcal{D}^{\{1\}}(\{\bullet\}; \mathcal{B}(T)) \rightarrow \mathcal{D}^{\{1\}}(T; \mathcal{B})$  is an equivalence of additive categories. In particular, we see that the (natural in *T*) map

(21.71) 
$$\mathbf{K}(\overline{F(T)}): \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}^{\{1\}}(T; \mathcal{B}))$$

is a weak homotopy equivalence of spectra.

The canonical map

$$\mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \xrightarrow{\simeq} \operatorname{hofib} \left( \mathbf{K}(\mathcal{O}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \to \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \right)$$

is natural in *T* and is a weak homotopy equivalence by Lemma 21.17 and Theorem 8.46 (i). The projection from  $\mathbf{K}(O^{\{1\}}(\{\bullet\}; \mathcal{B}(T)))$  to the trivial spectrum is a weak homotopy equivalence by Lemma 21.29. It induces a (natural in *T*) weak homotopy equivalence

$$\operatorname{hofib}(\mathbf{K}(\mathcal{O}^{\{1\}}(\{\bullet\};\mathcal{B}(T)))\to\mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))))\xrightarrow{\simeq}\Omega\mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))).$$

The composite of the two maps above gives a weak homotopy equivalence of spectra, natural in T,

(21.72) 
$$\mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\};\mathcal{B}(T))) \xrightarrow{\cong} \Omega \mathbf{K}(\mathcal{D}^{\{1\}}(\{\bullet\};\mathcal{B}(T))).$$

#### 21.8 The Computation of $K_n(\mathcal{D}^G(G/H))$

Define the inclusion of  $\mathbb{Z}$ -categories  $I: \mathcal{B}(T) \to \mathcal{T}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$  by sending an object (t, B) to the object  $(\{*\}, \pi, \eta, B)$  given by  $\pi(*) = \{\bullet\}, \eta(*) = 0$ , and  $\pi(*) = t$ . It induces a functor of additive categories  $I_{\oplus}: \mathcal{B}(T)_{\oplus} \to \mathcal{T}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$ . Obviously  $I_{\oplus}$  is full and faithful. We leave it to the reader to show that any object in  $\mathcal{T}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))$  is isomorphic to an object in the image of  $I_{\oplus}$ . Hence  $I_{\oplus}$  is an equivalence of additive categories and induces a weak homotopy equivalence, natural in T,

(21.73) 
$$\mathbf{K}(I_{\oplus}) \colon \mathbf{K}(\mathcal{B}(T)_{\oplus}) \xrightarrow{\cong} \mathbf{K}(\mathcal{T}^{\{1\}}(\{\bullet\}; \mathcal{B}(T))).$$

Now the desired weak homotopy equivalence of covariant Or(G)-spectra from  $\mathbf{K}(\mathcal{B}(?)_{\oplus})$  to  $\Omega\mathbf{K}(\mathcal{D}^{G}(?);\mathcal{B})$  comes from the maps (21.71), (21.72), and (21.73).

We have proved in Lemma 21.29 that  $O^G(G/G)$  is flasque. The next exercise shows that this is not true in general for  $O^G(G/H)$  if  $H \neq G$ .

**Exercise 21.74.** Suppose that the category  $O^G(G/H)$  is flasque. Show that then the map  $K_n(\mathcal{B}(G/H)_{\oplus}) \to K_n(\mathcal{B}_{\oplus})$  induced by the projection  $G/H \to G/G$  and the obvious identification  $\mathcal{B}(G/G) = \mathcal{B}$  is bijective for all  $n \in \mathbb{Z}$ .

## 21.8.2 Assembly and Controlled G-homology

We have the *G*-homology theory  $K_*(\mathcal{D}^G(-;\mathcal{B}))$ , see Theorem 21.26. The covariant Or(G)-spectrum  $\mathbf{K}(\mathcal{B}(?))_{\oplus}$  of (21.68) determines a *G*-homology theory  $H^G_*(-;\mathbf{K}(\mathcal{B}(?)))$ , see Theorem 12.27.

Proposition 21.75. There is an equivalence of G-homology theories

$$T(-): K_{*+1}(\mathcal{D}^G(-;\mathcal{B})) \xrightarrow{\cong} H^G_*(-; \mathbf{K}(\mathcal{B}(?)_{\oplus})).$$

*Proof.* This follows from Corollary 18.16 and Proposition 21.70.

**Lemma 21.76.** Let  $\mathcal{B}$  be a category with G-support and let  $\mathcal{F}$  be a family of subgroups. Let pr:  $E_{\mathcal{F}}(G) \to G/G$  be the projection.

(i) The assembly map appearing in the Meta-Isomorphism Conjecture 15.2 for the *G*-homology theory  $H^G_*(-; \mathbf{K}(\mathcal{B}(?)_{\oplus}))$  and the family  $\mathcal{F}$ 

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{K}(\mathcal{B}(?)_{\oplus})) \to H_n^G(G/G; \mathbf{K}(\mathcal{B}(?)_{\oplus})) = K_n(\mathcal{B}_{\oplus})$$

*can be identified for every*  $n \in \mathbb{Z}$  *with the homomorphism induced by the projection*  $E_{\mathcal{F}}(G) \to G/G$ 

$$K_{n+1}(\mathcal{D}^G(E_{\mathcal{F}}(G);\mathcal{B})) \to K_{n+1}(\mathcal{D}^G(G/G;\mathcal{B})) = K_n(\mathcal{B}_{\oplus});$$

(ii) The Meta-Isomorphisms Conjecture 15.2 for the G-homology theory  $H^G_*(-; \mathbf{K}(\mathcal{B}(?)))$  and the family  $\mathcal{F}$  is true if and only if the spectrum  $\mathbf{K}(\mathcal{O}^G(E_{\mathcal{F}}(G); \mathcal{B}))$  is weakly contractible.

*Proof.* (i) This follows from Proposition 21.75.

(ii) This follows from assertion (i), Lemma 21.22, Lemma 21.29, and the commutative diagram of spectra

$$\mathcal{T}^{G}(E_{\mathcal{F}}(G)) \longrightarrow \mathcal{O}^{G}(E_{\mathcal{F}}(G)) \longrightarrow \mathcal{D}^{G}(E_{\mathcal{F}}(G))$$

$$\downarrow_{\mathcal{T}^{G}(\mathrm{pr})} \qquad \qquad \downarrow_{\mathcal{O}^{G}(\mathrm{pr})} \qquad \qquad \downarrow_{\mathcal{D}^{G}(\mathrm{pr})}$$

$$\mathcal{T}^{G}(G/G) \longrightarrow \mathcal{O}^{G}(G/G) \longrightarrow \mathcal{D}^{G}(G/G)$$

whose rows are weak homotopy fibrations by Theorem 21.19.

**Remark 21.77.** The benefit of Lemma 21.76 (ii) is that the proof of the Meta-Isomorphism Conjecture is reduced to the proof of the weak contractibility of the *K*-theory of the specific category  $O^G(E_{\mathcal{F}}(G); \mathcal{B})$  defined in terms of controlled topology and not just to the weak contractibility of some abstract homotopy fiber. This will allow us to use geometric tools for a proof of the Farrell-Jones Conjecture, as described in Chapter 19.

#### 21.8.3 The Definition of a Strong Category with G-Support

In this subsection we will upgrade the notion of a category with *G*-support of Definition 21.1 to the notion of a strong category with *G*-support by additionally implementing a *G*-action on  $\mathcal{B}$  and a homotopy trivialization for it.

**Definition 21.78 (Strong category with** *G*-support). A strong category with *G*-support over *G* is a triple  $\mathcal{B} = (\mathcal{B}, \text{supp}_G, \Omega)$  consisting of:

- A G- $\mathbb{Z}$ -category  $\mathcal{B}$ ;
- A map called the support function

 $\operatorname{supp}_G$ :  $\operatorname{mor}(\mathcal{B}) \to \{ \text{finite subsets of } G \};$ 

• A homotopy trivialization of the *G*-action on  $\mathcal{B}$ , i.e., a collection  $\Omega = \{\Omega_g \mid g \in G\}$ , where  $\Omega_g$  is a natural equivalence of functors of  $\mathbb{Z}$ -categories  $\mathcal{B} \to \mathcal{B}$ 

$$\Omega_g \colon \operatorname{id}_{\mathcal{B}} \xrightarrow{\cong} \Lambda_g,$$

for  $\Lambda_g: \mathcal{B} \to \mathcal{B}$  the functor given by multiplication by g such that conditions (vii) and (viii) below are satisfied.

We require that the following axioms are satisfied for all objects *B* in  $\mathcal{B}$ , all morphisms  $u, u': B_1 \to B_2, v: B_2 \to B_3$  in  $\mathcal{B}$ , and all  $g, g' \in G$ :

(i)  $\operatorname{supp}_G(u) = \emptyset \iff u = 0;$ (ii)  $\operatorname{supp}_G(v \circ u) \subseteq \operatorname{supp}_G(v) \cdot \operatorname{supp}_G(u);$ (iii)  $\operatorname{supp}_G(u + u') \subseteq \operatorname{supp}_G(u) \cup \operatorname{supp}_G(u');$ (iv)  $\operatorname{supp}_G(-u) = \operatorname{supp}_G(u);$ (v)  $\operatorname{supp}_G(B) = \{e\};$ (vi)  $\operatorname{supp}_G(gu) = g \operatorname{supp}_G(u)g^{-1};$ (vii)  $\Omega_{g'}(gB) \circ \Omega_g(B) = \Omega_{g'g}(B);$ (viii)  $\Omega_e(B) = \operatorname{id}_B;$ (ix)  $\operatorname{supp}_G(\Omega_g(B)) = \{g\}.$ 

**Remark 21.79.** In Example 21.2 we actually get the structure of a strong category with *G*-support. Namely, for  $g_0 \in G$  and object *A* in  $\mathcal{R}[G]$  which is given by an object *A* in  $\mathcal{R}$ , we define  $\Lambda_{g_0}(A)$  to be  $g_0A$  by using the given *G*-action on the objects of  $\mathcal{R}$ . For a morphism  $\phi = \sum_{g \in G} \phi_g \cdot g \colon A \to A'$  in  $\mathcal{R}[G]$ , we define  $\Lambda_{g_0}(\phi) \colon g_0A \to g'_0A'$  by  $(g_0\phi)_g = g_0 \cdot \phi_{g_0^{-1}g}$ . The desired homotopy trivialization  $\Omega$  is given by assigning to  $g_0 \in G$  the isomorphism  $\Omega_{g_0}(A) \colon A \xrightarrow{\cong} \Lambda_{g_0}(A)$  in  $\mathcal{R}[G]$  given by  $\Omega_{g_0}(A)_{g_0} = id_{g_0A}$  and  $\Omega_{g_0}(A)_{g_1} = 0$  for  $g_0 \neq g_1$ .

## **21.8.4** Reduction to $K_n(\mathcal{B}\langle H \rangle)$

Let  $\mathcal{B}$  be a strong category with *G*-support in the sense of Definition 21.78.

**Definition 21.80** ( $\mathcal{B}\langle H \rangle$ ). For a subgroup  $H \subseteq G$  define  $\mathcal{B}\langle H \rangle$  to be the  $\mathbb{Z}$ -subcategory of  $\mathcal{B}$  which has the same set of objects and for which a morphism  $\phi \colon \mathcal{B} \to \mathcal{B}'$  of  $\mathcal{B}$  belongs to  $\mathcal{B}\langle H \rangle$  if  $\operatorname{supp}_G(\phi) \subseteq H$  holds.

Define a functor  $I: \mathcal{B}\langle H \rangle \to \mathcal{B}(G/H)$  of  $\mathbb{Z}$ -categories by sending an object B to the object (eH, B) and a morphism  $\phi: B \to B'$  to the morphism  $(eH, B) \to (eH, B')$  given by  $\phi$ .

**Proposition 21.81.** The functor  $I: \mathcal{B}(H) \to \mathcal{B}(G/H)$  is an equivalence of  $\mathbb{Z}$ -categories. In particular, the homomorphism

$$K_n(I_{\oplus}): K_n(\mathcal{B}\langle H \rangle_{\oplus}) \to K_n(\mathcal{B}(G/H)_{\oplus})$$

*is bijective for all*  $n \in \mathbb{N}$ *.* 

*Proof.* Obviously *I* is full and faithful. Consider an object (gH, B) in  $\mathcal{B}(G/H)$ . Then  $\Omega_g(g^{-1}B): g^{-1}B \xrightarrow{\cong} B$  is an isomorphism in  $\mathcal{B}$  with  $\operatorname{supp}(\Omega_g(g^{-1}B)) = \{g\}$  and hence induces an isomorphism  $(e, g^{-1}B) \xrightarrow{\cong} (g, B)$  in  $\mathcal{B}(G/H)$ . This shows that any object in  $\mathcal{B}(G/H)$  is isomorphic to an object in the image of *I*. Hence *I* is an equivalence. **Remark 21.82.** Let  $\mathcal{A}$  be a G- $\mathbb{Z}$ -category. Recall from Example 21.2 and Remark 21.79 that the additive category  $\mathcal{A}[G]$  is a strong category with G-support. One easily checks for any subgroup  $H \subseteq G$ ,

$$\mathcal{A}[H] = \mathcal{A}[G]\langle H \rangle.$$

Hence we get from Proposition 21.70 and Proposition 21.81 for every  $n \in \mathbb{Z}$  an isomorphism

$$K_n(\mathcal{A}[H]_{\oplus}) \xrightarrow{=} K_{n+1}(\mathcal{D}^G(G/H;\mathcal{A}[G])).$$

**Example 21.83.** Let *R* be a ring and let  $\rho: G \to \operatorname{aut}(R)$  be a group homomorphism. We have defined the *G*- $\mathbb{Z}$ -category <u>*R*</u> in Example 21.3. Denote by  $R_{\rho|H}[H]$  the twisted group ring of  $H \subset G$  with respect to  $\rho|_H: H \to \operatorname{aut}(R)$ 

We conclude from Example 21.3 and Remark 21.82 that there is for every  $n \in \mathbb{Z}$  an isomorphism

$$K_n(R_{\rho|H}[H]) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H; \underline{R}[G])).$$

**Exercise 21.84.** Let *R* be a ring. Let  $\mathcal{B}$  be the  $\mathbb{Z}$ -linear category with one object whose endomorphism ring is the group ring  $R[\mathbb{Z}/2]$ . Let *t* be the generator of  $\mathbb{Z}/2$ . We define the support of an endomorphism  $at^0 + bt^1$  to be the subset of  $\mathbb{Z}$ 

$$\operatorname{supp}_{\mathbb{Z}}(at^{0} + bt^{1}) = \begin{cases} \emptyset & \text{if } a = b = 0; \\ \{0\} & \text{if } a \neq 0, b = 0; \\ \{0, 1\} & \text{otherwise.} \end{cases}$$

Show that the axioms of a category with Z-support are satisfied, we get isomorphisms

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \cong K_n(R);$$
  
$$H_n^{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \cong K_n(R[\mathbb{Z}/2])$$

and under this identification the assembly map  $H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \to H_n^{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}; \mathbf{K}_{\mathcal{B}})$ agrees with the map  $K_n(R) \to K_n(R[\mathbb{Z}/2])$  induced by the inclusion  $R \to R[\mathbb{Z}/2]$ .

**Remark 21.85** (Morphism Additivity). The version of the Farrell Jones Conjecture with categories with *G*-support is too general to expect that the Farrell-Jones Conjecture holds with them as coefficients, as Exercise 21.84 illustrates. It may hold for strong categories with *G*-support in the sense of Definition 21.78 if one additionally assumes

• Morphism Additivity

Let  $u: B \to B'$  be a morphism. Suppose that  $\operatorname{supp}_G(u) = L_1 \sqcup L_2$  is a disjoint union. Then we require the existence of morphisms  $u_i: B \to B'$  for i = 1, 2 satisfying  $u = u_1 + u_2$  and  $\operatorname{supp}_G(u_i) = L_i$  for i = 1, 2.

But then  $\mathcal{B}$  is already of the shape  $\mathcal{A}[G]$ , see Exercise 21.88.

#### 21.9 Induction

**Exercise 21.86.** Show that the two morphisms  $u_1$  and  $u_2$  appearing in the axiom *Morphism Additivity* stated in Remark 21.85 are unique.

**Exercise 21.87.** Let  $\mathcal{A}$  be a *G*- $\mathbb{Z}$ -category. Show that  $\mathcal{A}[G]$  defined in Example 21.2 is a strong category with *G*-support in the sense of Definition 21.78 satisfying *Morphism Additivity*.

**Exercise 21.88.** Consider a strong category  $\mathcal{B}$  with *G*-support satisfying *Morphism Additivity*. Let  $\mathcal{A}$  the *G*- $\mathbb{Z}$ -subcategory of  $\mathcal{B}$  which has the same set of objects and for which a morphism  $u: B \to B'$  in  $\mathcal{B}$  belongs to  $\mathcal{A}$  if and only if  $\text{supp}_G(u) \subseteq \{e\}$  holds for the unit element  $e \in G$ . Construct an isomorphism of *G*- $\mathbb{Z}$ -categories

$$F: \mathcal{A}[G] \xrightarrow{\cong} \mathcal{B}$$

which is compatible with the support functions.

In view of the last exercise it is superfluous to consider strong categories with G-support satisfying the axiom *Morphism Additivity* for discrete groups. This is different when one considers totally disconnected groups, see [81, Definition 3.2], where also a new condition *Support Cofinality* enters which is void for discrete groups.

**Exercise 21.89.** Show that the structure of a category with  $\mathbb{Z}$ -support on the category  $\mathcal{B}$  of Exercise 21.84 does not extend to the structure of a strong category with  $\mathbb{Z}$ -support.

## **21.9 Induction**

Let  $H \subseteq G$  be a subgroup of *G*. Let  $\mathcal{B}$  be a strong category with *G*-support in the sense of Definition 21.78. We have defined the  $\mathbb{Z}$ -category  $\mathcal{B}\langle H \rangle$  in Definition 21.80. Obviously it inherits from  $\mathcal{B}$  the structure of a strong category with *H*-support. Given an *H*-space *X*, we have denoted by ind<sub>*i*</sub>  $X = G \times_H X$  the *G*-space given by induction with the inclusion  $\iota: H \to G$ , see (12.8).

Next we construct a functor of additive categories, natural in X,

(21.90)  $\operatorname{ind}_{\iota} : O^H(X; \mathcal{B}\langle H \rangle) \to O^G(\operatorname{ind}_{\iota} X; \mathcal{B}).$ 

Let  $j: X \to \operatorname{ind}_{\iota} X$  be the  $\iota$ -equivariant map sending x to (e', x). An object  $\mathbf{B} = (S, \pi, \eta, B)$  of  $O^H(X; \mathcal{B}\langle H \rangle)$  is sent to the object  $\operatorname{ind}_{\iota} \mathbf{B} = (S, j \circ \pi, \eta, B)$  of  $O^G(\operatorname{ind}_{\iota} X; \mathcal{B})$ . Obviously  $\operatorname{ind}_{\iota}(\mathbf{B})$  satisfies *compact support over*  $\operatorname{ind}_{\iota} X$  and *local finiteness over*  $\mathbb{N}$ , since  $\mathbf{B}$  satisfies *compact support over* X and *local finiteness over*  $\mathbb{N}$ . For two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  and a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$ given by the collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$  of  $O^H(X; \mathcal{B}\langle H \rangle)$ , define the morphism  $\operatorname{ind}_{\iota}(\phi): \operatorname{ind}_{\iota}(\mathbf{B}) \to \operatorname{ind}_{\iota}(\mathbf{B}')$  of  $O^G(\operatorname{ind}_{\iota} X; \mathcal{B})$  by the same collection  $\{\phi_{s,s'}: B(s) \to B'(s') \mid s \in S, s' \in S'\}$ . Obviously the conditions *finite G*-support and bounded control over  $\mathbb{N}$  are satisfied for  $\operatorname{ind}_{\iota}(\phi)$ . Next we give

the proof of *continuous control*. We only deal with condition (21.5), the proof for condition (21.6) is analogous and left to the reader.

Consider a point (g, x) in  $\operatorname{ind}_{\iota} X$  and an open  $G_{(g,x)}$ -invariant neighborhood Uof (g, x) in  $\operatorname{ind}_{\iota} X$ . Note for the sequel that  $G_{(g,x)} = g'H_xg^{-1}$  holds and the map  $j: X \to \operatorname{ind}_{\iota} X$  is an open  $\iota$ -equivariant embedding. We have to find an open  $G_{(g,x)}$ invariant neighborhood U' of (g, x) in  $\operatorname{ind}_{\iota} X$  satisfying  $U' \subseteq U$  and a natural number r' such that for all  $s \in S$ ,  $s' \in S'$ , and  $g \in \operatorname{supp}_G((\operatorname{ind}_{\iota} \phi)_{s,s'}) = \operatorname{supp}_H(\phi_{s,s'})$  the implication

(21.91) 
$$g \cdot j \circ \pi(s) \in U', \eta(s) \ge r' \implies j \circ \pi'(s') \in U'$$

holds.

Suppose that  $(g, x) \notin \operatorname{im}(j)$ . Then  $U' = g \cdot \operatorname{im}(j)$  is an open  $G_{(g,x)}$ -invariant neighborhood of (g, x) satisfying  $U' \cap \operatorname{im}(j) = \emptyset$ . Then the implication (21.91) is satisfied for trivial reasons since  $\operatorname{supp}_G((\operatorname{ind}_\iota \phi)_{s,s'}) = \operatorname{supp}_H(\phi_{s,s'}) \subseteq H$  holds and  $h \cdot j \circ \pi(s)$  belongs to  $\operatorname{im}(j)$  and hence never belongs to U' for  $h \in H$ .

Next we treat the case  $(g, x) \in im(j)$ , or, equivalently, the case g = e. Since  $\phi$  satisfies *continuous control* and  $j^{-1}(U)$  is an open  $H_x$ -invariant neighborhood of x, we can find an open  $H_x$ -invariant neighborhood V' of x in X with  $V' \subseteq j^{-1}(U)$  such that for all  $s \in S$ ,  $s' \in S'$ , and  $h \in \text{supp}_H(\phi_{s,s'})$  the implication

$$h \cdot \pi(s) \in U', \eta(s) \ge r' \implies \pi'(s') \in j^{-1}(U)$$

holds. Put U' = j(V). Then (21.91) is satisfied for the open  $G_{j(x)}$ -invariant neighborhood U' of j(x) in ind<sub>i</sub> X and the number r' above.

One easily checks that the functor  $\operatorname{ind}_{\iota}$  of (21.90) induces for every *H*-*CW*-pair (*X*, *A*) functors of additive categories

(21.92) 
$$\operatorname{ind}_{\iota} : O^{H}(X, A; \mathcal{B}\langle H \rangle) \to O^{G}(\operatorname{ind}_{\iota} X, \operatorname{ind}_{\iota} A; \mathcal{B})$$

(21.93) 
$$\operatorname{ind}_{\iota} : \mathcal{T}^{H}(X, A; \mathcal{B}\langle H \rangle) \to O^{G}(\operatorname{ind}_{\iota} X, \operatorname{ind}_{\iota} A; \mathcal{B});$$

(21.94)  $\operatorname{ind}_{\iota} : \mathcal{D}^{H}(X, A; \mathcal{B}\langle H \rangle) \to \mathcal{D}^{G}(\operatorname{ind}_{\iota} X, \operatorname{ind}_{\iota} A; \mathcal{B}).$ 

**Proposition 21.95.** For every H-CW-pair (X, A) and every strong category with *G*-support  $\mathcal{B}$  over *G*, the functor ind<sub>i</sub> of (21.94) induces a weak homotopy equivalence

$$\mathbf{K}(\operatorname{ind}_{\iota}): \mathbf{K}(\mathcal{D}^{H}(X,A;\mathcal{B}\langle H\rangle)) \xrightarrow{-} \mathbf{K}(\mathcal{D}^{G}(\operatorname{ind}_{\iota}X,\operatorname{ind}_{\iota}A;\mathcal{B})).$$

*Proof.* We offer two proofs, a short one using basic facts about *G*-homology theories, and one direct proof which illustrates the role of the condition continuous control.

We can view the functors sending an *H*-*CW*-pair to the  $\mathbb{Z}$ -graded abelian groups  $K_*(\mathcal{D}^H(X, A; \mathcal{B}\langle H \rangle))$  and  $K_*(\mathcal{D}^G(\operatorname{ind}_\iota X, \operatorname{ind}_\iota A; \mathcal{B}))$  as *H*-homology theories. Then we get a natural transformation of *H*-homology theories by

$$\mathbf{K}_*(\mathrm{ind}_{\iota}): K_*(\mathcal{D}^H(X,A;\mathcal{B}\langle H\rangle)) \to K_*(\mathcal{D}^G(\mathrm{ind}_{\iota}X,\mathrm{ind}_{\iota}A;\mathcal{B})).$$

#### 21.9 Induction

In order to show that this is an isomorphism for every *CW*-pair (*X*, *A*), it suffices to do this in the special case X = H/K and  $A = \emptyset$  for every subgroup  $K \subseteq H$ , see Theorem 12.6. We have already constructed isomorphisms, see Proposition 21.70 and Proposition 21.81,

$$K_*(\mathcal{D}^H(H/K;\mathcal{B}\langle H\rangle)) \xrightarrow{\cong} K_{*-1}\big(((\mathcal{B}\langle H\rangle)\langle K\rangle)_{\oplus}\big) = K_{*-1}(\mathcal{B}\langle K\rangle_{\oplus}),$$

and

$$K_*(\mathcal{D}^G(\operatorname{ind}_{\iota} H/K;\mathcal{B})) = K_*(\mathcal{D}^G(G/K;\mathcal{B})) \xrightarrow{\cong} K_{*-1}(\mathcal{B}\langle K \rangle_{\oplus}).$$

Under these identifications

$$\mathbf{K}_*(\mathrm{ind}_\iota)\colon K_*(\mathcal{D}^H(H/K;\mathcal{B}\langle H\rangle))\to K_*(\mathcal{D}^G(\mathrm{ind}_\iota H/K;\mathcal{B}))$$

becomes the identity on  $K_{*-1}(\mathcal{B}\langle K \rangle_{\oplus})$ . This finishes the first proof of Proposition 21.95.

Next we present the second proof. Because of Proposition 21.27 we can assume without loss of generality  $A = \emptyset$ . It suffices to show that the functor of (21.94)

$$\operatorname{ind}_{\iota} \colon \mathcal{D}^{H}(X; \mathcal{B}\langle H \rangle) \to \mathcal{D}^{G}(\operatorname{ind}_{\iota} X; \mathcal{B}).$$

is an equivalence of additive categories.

We first show that  $\operatorname{ind}_{\iota}$  is full and faithful, in other words, that for two objects **B** =  $(S, \pi, \eta, B)$  and **B**' =  $(S', \pi, \eta', B')$  in  $\mathcal{D}^H(X; \mathcal{B}(H))$  the map induced by  $\operatorname{ind}_{\iota}$ 

(21.96) 
$$\operatorname{mor}_{\mathcal{D}^{H}(X;\mathcal{B}(H))}(\mathbf{B},\mathbf{B}') \to \operatorname{mor}_{\mathcal{D}^{G}(\operatorname{ind}_{\iota}X;\mathcal{B})}(\operatorname{ind}_{\iota}(\mathbf{B}),\operatorname{ind}_{\iota}(\mathbf{B}'))$$

is bijective. The elementary proof of injectivity is left to the reader. Surjectivity is proved a follows.

Recall  $\operatorname{ind}_{\iota}(\mathbf{B}) = (S, j \circ \pi, \eta, B)$ . Consider any element in the target of (21.96). Choose a morphism  $\phi' : (S, j \circ \pi, \eta, B) \to (S', j \circ \pi', \eta', B')$  in  $O^{G}(\operatorname{ind}_{\iota} X; \mathcal{B})$  representing it. Next we show that we can assume without loss of generality

(21.97) 
$$\operatorname{supp}_G(\phi'_{s,s'}) \subseteq H \text{ for } s \in S, s' \in S'$$

Consider  $x \in X$ . Since  $\phi'$  satisfies *continuous control* and  $\operatorname{im}(j)$  is an open  $G_{j(x)}$ -invariant neighborhood of j(x) in  $\operatorname{ind}_{\iota} X$ , we conclude from Lemma 21.10 (ii) that there are an open  $G_{j(x)}$ -invariant neighborhood  $U'_x$  of j(x) in  $\operatorname{ind}_{\iota} X$  with  $U'_x \subseteq \operatorname{im}(j)$  and a natural number  $r'_x$  such that for all  $s \in S$ ,  $s' \in S'$ , and  $g \in \operatorname{supp}_G(\phi'_{s,s'})$  the implication

$$(21.98) j \circ \pi'(s) \in U'_x, \eta'(s') \ge r'_x \implies g \cdot j \circ \pi(s) \in \operatorname{im}(j)$$

holds. Since **B'** satisfies *compact support over* X, there is a compact subset  $C \subseteq X$  with  $im(\pi) \subseteq C$ . Since  $j(C) \subseteq \bigcup_{x \in C} U'_x$  and  $j(C) \subseteq ind_t X$  is compact, there is a finite subset  $\{x_1, x_2, \ldots, x_m\} \subseteq C$  satisfying  $j(C) \subseteq \bigcup_{i=1}^m U'_{x_i}$ . Define a natural number  $r' := \max\{r'_{x_i} \mid i = 1, 2, \ldots, m\}$ . Then we get for all  $s \in S$ ,  $s' \in S'$ , and

 $g \in \text{supp}_G(\phi'_{s,s'})$  the implication

(21.99) 
$$\eta'(s') \ge r' \implies g \cdot j \circ \pi(s) \in \operatorname{im}(j)$$

since for any  $s' \in S'$  there exists an  $i \in \{1, 2, ..., m\}$  with  $j \circ \pi(s') \in U'_{x_i}$  and  $r' \ge r'_i$ and we can apply the implication (21.98). Since  $\phi'$  satisfies *bounded control over*  $\mathbb{N}$ , we can modify  $\phi'$  without changing the class which it represent in  $O^{G'}(\operatorname{ind}_{\iota} X; \mathcal{B}')$ such that for all  $s \in S$ ,  $s' \in s'$ , and  $g \in \operatorname{supp}_{G'}(\phi'_{s,s'})$  we have  $g \cdot j \circ \pi(s) \in \operatorname{im}(j)$ . Now (21.97) follows since  $g \cdot j \circ \pi(s) \in \operatorname{im}(j) \implies g \in H$ .

We conclude from (21.97) that  $\phi'_{s,s'}$  belongs to  $\mathcal{B}\langle H \rangle$ . Define a morphism  $\phi \colon \mathbf{B} \to \mathbf{B}'$  in  $O^H(X; \mathcal{B}\langle H \rangle)$  by  $\phi_{s,s'} = \phi'_{s,s'}$  for  $s \in S$  and  $s' \in S'$ . One easily checks that  $\phi$  satisfies *finite support over* H, *bounded control over*  $\mathbb{N}$ , and *continuous control* since  $\phi'$  satisfies *finite support over* G, *bounded control over*  $\mathbb{N}$ , and *continuous control*. Hence  $\phi$  is well-defined. Its class in  $\mathcal{D}^H(X; \mathcal{B}\langle H \rangle)$  is mapped by construction under the map (21.96) to the class in  $\mathcal{D}^G(\operatorname{ind}_t X; \mathcal{B})$  represented by  $\phi'$ . This shows that the map (21.96) is bijective.

It remains to show that for every object  $\mathbf{B}' = (S', \pi', \eta', \mathbf{B}')$  in  $O^G(\operatorname{ind}_t X; \mathcal{B})$ there is an object  $\mathbf{B} = (S, \pi, \eta, \mathbf{B})$  in  $O^H(X; \mathcal{B}\langle H \rangle)$  and an isomorphism  $\phi: \operatorname{ind}_t(\mathbf{B}) \xrightarrow{\cong} \mathbf{B}'$  in  $O^G(\operatorname{ind}_t X; \mathcal{B})$ . We put S = S' and  $\eta = \eta'$ . Choose functions  $\gamma: S \to G$  and  $\pi: S \to X$  such that  $\gamma(s) \cdot j \circ \pi(s) = \pi'(s)$  holds for all  $s \in S$ . Define  $\mathbf{B}: S \to \operatorname{ob}(\mathcal{B})$  by sending s to  $\gamma(s)^{-1} \cdot \mathbf{B}'(s)$ . Then we can define the desired isomorphism  $\phi$  by putting  $\phi_{s,s'} = 0$  for  $s, s' \in S$  with  $s \neq s'$  and by  $\phi_{s,s} = \Omega_{\gamma(s)}(\mathbf{B}(s)): \mathbf{B}(s) \xrightarrow{\cong} \mathbf{B}'(s)$  for  $s \in S$ . The proof that  $\phi$  is well-defined is a mild generalization of the proof of Lemma 21.14. This finishes the second proof of Proposition 21.95.

## **21.10** The Version with Zero Control over N

Next we deal with a version  $\mathcal{D}_0^G(X; \mathcal{B})$  of  $\mathcal{D}^G(X; \mathcal{B})$  where we have zero-control over  $\mathbb{N}$ .

### 21.10.1 Control Categories with Zero Control in the N-Direction

**Definition 21.100**  $(\mathcal{D}_0^G(X;\mathcal{B}))$ . Define  $\mathcal{O}_0^G(X)$  to be the additive subcategory of  $\mathcal{O}_G(X)$ , which has the same set of objects and for which a morphism  $\phi: \mathbf{B} = (S, \pi, \eta, \mathbf{B}) \to \mathbf{B}' = (S', \pi', \eta', \mathbf{B}')$  in  $\mathcal{O}^G(X)$  belongs to  $\mathcal{O}_0^G(X)$  if and only if the implication

$$\phi_{s,s'} \neq 0 \implies \eta(s) = \eta(s')$$

holds for all  $s \in S$  and  $s' \in S'$ .

21.10 The Version with Zero Control over  $\mathbb{N}$ 

Let  $\mathcal{T}_0^G(X; \mathcal{B})$  be the full subcategory of  $\mathcal{O}_G^0(X)$  consisting of those objects **B** =  $(S, \pi, \eta, B)$  for which there exists a natural number *n* such that B(s) = 0 holds for  $s \in S$  with  $\eta(s) \ge n$ .

Define  $\mathcal{D}_0^G(X)$  to be the quotient category  $\mathcal{O}_0^G(X)/\mathcal{T}_0^G(X)$  in the sense of Definition 8.42.

**Lemma 21.101.** The inclusion  $\mathcal{T}_0^G(X) \to O_0^G(X)$  is a Karoubi filtration in the sense of Definition 8.43. In particular, we get a weak homotopy fibration sequence

$$\mathcal{T}_0^G(X) \to \mathcal{O}_0^G(X) \to \mathcal{D}_0^G(X).$$

*Proof.* The proof of Lemma 21.17 carries over directly. Now apply Theorem 8.46 (i).

**Exercise 21.102.** Show for  $m \in \mathbb{Z}$ 

$$K_m(\mathcal{D}_0^{\{1\}}(\{\bullet\})) \cong \left(\prod_{n=0}^{\infty} K_m(\mathcal{B}_{\oplus})\right) / \left(\bigoplus_{n=0}^{\infty} K_m(\mathcal{B}_{\oplus})\right).$$

Let  $\rho \colon \mathbb{N} \to \mathbb{N}$  be a function which is finite-to-one, i.e., the preimage of every element in  $\mathbb{N}$  under  $\rho$  is finite. Next we construct a functor of additive categories

$$V'_{\rho}(X) \colon O_0^G(X) \to O_0^G(X)$$

which is essentially given by moving an object at the position *n* to the position  $\rho(n)$  and leaving the position in *X* fixed. More precisely,  $V_{\rho}$  sends an object **B** =  $(S, \pi, \eta, B)$  to the object  $V_{\rho}(X)(\mathbf{B}) = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{B})$  given by

$$\widehat{S} = S;$$
  

$$\widehat{\pi} = \pi;$$
  

$$\widehat{\eta} = \rho \circ \eta;$$
  

$$\widehat{B} = B.$$

Its definition on morphisms is the tautological one, i.e., a morphism  $\phi: \mathbf{B} = (S, \pi, \eta, B) \rightarrow \mathbf{B} = (S', \pi', \eta', B')$  is sent to the morphism  $V'_{\rho}(\phi)$  given by  $V'_{\rho}(X)(\phi)_{s,s'} = \phi_{s,s'}$  for  $s \in S$  and  $s' \in S$ .

We have to check that this is well-defined. Since  $\rho$  is finite-to-one, the new object  $V'_{\rho}(X)(\mathbf{B})$  satisfies the conditions *compact support over* X and *local finiteness over*  $\mathbb{N}$  as **B** does. For every natural number N, there exists a natural number N' such that the implication  $\rho(n) \ge N' \implies n \ge N$  holds for every  $n \in \mathbb{N}$ , since  $\rho$  is finite-to-one. Hence the new morphism  $V'_{\rho}(X)(\phi)$  satisfies *finite G-support* and *continuous control* as  $\phi$  does. Obviously we have for  $s \in S$ ,  $s' \in S'$ 

$$\begin{aligned} V'_{\rho}(X)(\phi)_{s,s'} \neq 0 \implies \phi_{s,s'} \neq 0 \implies \eta(s) = \eta'(s') \\ \implies \rho \circ \eta(s) = \rho \circ \eta'(s') \implies \widehat{\eta}(s) = \widehat{\eta'}(s'). \end{aligned}$$

Since  $V'_{\rho}(X)$  maps  $\mathcal{T}_0^G(X)$  to  $\mathcal{T}_0^G(X; \mathcal{B})$ , it induces a functor of additive categories

(21.103) 
$$V_{\rho}(X) \colon \mathcal{D}_0^G(X) \to \mathcal{D}_0^G(X).$$

# **21.10.2** Relating the *K*-Theory of $\mathcal{D}^G$ and $\mathcal{D}^G_0$

We have explained in Section 21.4 that  $\mathcal{D}^G(X; \mathcal{B})$  yields a covariant functor  $\mathcal{D}^G: G\text{-}CW\text{-}COM \rightarrow ADDCAT$ . One easily checks that the same construction yields a covariant functor

(21.104) 
$$\mathcal{D}_0^G \colon G\text{-}\mathsf{CW}\text{-}\mathsf{COM} \to \mathsf{ADDCAT}.$$

Composition with the functor non-connective K-theory yields the covariant functors

(21.105) 
$$\mathbf{K} \circ \mathcal{D}^G \colon \mathrm{CW}\text{-}\mathrm{COM} \to \mathrm{SPECTRA};$$

(21.106) 
$$\mathbf{K} \circ \mathcal{D}_0^G : CW \text{-} COM \to SPECTRA.$$

By precomposing with the inclusion  $Or(G) \rightarrow CW$ -COM, we get covariant Or(G)-spectra

(21.107) 
$$\mathbf{K}^{\mathcal{D}^G} : \operatorname{Or}(G) \to \operatorname{SPECTRA};$$
  
(21.108)  $\mathbf{K}^{\mathcal{D}^G} : \operatorname{Or}(G) \to \operatorname{SPECTRA}.$ 

The main result of this section is

**Theorem 21.109 (Relating the** *K***-theory of**  $\mathcal{D}^G(X)$  **and**  $\mathcal{D}^G_0$ ). *Define two functions*  $\rho_O, \rho_E \colon \mathbb{N} \to \mathbb{N}$  *by* 

$$\rho_O(n) = \begin{cases} \frac{n+2}{2} & \text{if } n \text{ is even;} \\ \frac{n+1}{2} & \text{if } n \text{ is odd;} \end{cases}$$

$$\rho_E(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Let **HPO** be the covariant functor G-CW-COM  $\rightarrow$  SPECTRA given for a G-CW-complex X by the homotopy pushout



Then there exists a zigzag of weak homotopy equivalences of covariant functors G-CW-COM  $\rightarrow$  SPECTRA from **HPO** to  $\mathbf{K} \circ \mathcal{D}^G$ .

The remainder of this section is devoted to the proof of Theorem 21.109. This needs some preparation.

For a subset  $J \subseteq \mathbb{N}$  define

$$\begin{array}{ll} (21.110) & \mathcal{O}_{J}^{G}(X) \subseteq \mathcal{O}^{G}(X); \\ (21.111) & \mathcal{D}_{J}^{G}(X) \subseteq \mathcal{D}^{G}(X), \end{array}$$

to be the full subcategory of  $O^G(X)$  and  $\mathcal{D}^G(X)$  respectively consisting of those objects **B** =  $(S, \pi, \eta, B)$  for which  $\operatorname{im}(\eta) \subseteq J$  holds.

Fix a sequence of natural numbers  $0 = i_0 < i_1 < i_2 < i_3 < \cdots$  such that  $\lim_{\to\infty} (i_j - i_{j-1}) = \infty$  holds, for instance we can take  $i_j = \frac{j(j+1)}{2}$ , since then  $i_0 = 0$  and  $i_j - i_{j-1} = j$  holds for  $j \ge 1$ . Define  $\mathbb{N}_j := \{i \in \mathbb{N} \mid i_j \le i \le i_{j+1}\}$ . Put

$$E := \bigcup_{j=0}^{\infty} \mathbb{N}_{2j};$$

$$O := \bigcup_{j=0}^{\infty} \mathbb{N}_{2j+1};$$

$$I := \{i_1, i_2, \dots\};$$

$$I_E := \{i_2, i_4, \dots\};$$

$$I_O := \{i_1, i_3, i_5, \dots\}.$$

Note that  $\mathcal{D}^G_{\mathbb{N}}(X) = \mathcal{D}^G(X)$ ,  $\mathbb{N} = E \cup O$ ,  $E \cap O = I$ ,  $I = I_O \cup I_E$ , and  $I_O \cap I_E = \emptyset$  hold.

Consider the following commutative diagram of additive categories

$$\begin{array}{ccc} (21.112) & \mathcal{D}_{I}^{G}(X) \longrightarrow \mathcal{D}_{E}^{G}(X) \\ & & & & & & \\ & & & & & \\ \mathcal{D}_{O}^{G}(X) \longrightarrow \mathcal{D}^{G}(X) \end{array}$$

whose arrows are all inclusions of full additive subcategories and which is natural in X.

#### Lemma 21.113.

(i) The following inclusions are Karoubi filtrations

$$\begin{aligned} \mathcal{D}_{I}^{G}(X) &\to \mathcal{D}_{E}^{G}(X); \\ \mathcal{D}_{I_{E}}^{G}(X) &\to \mathcal{D}_{O}^{G}(X); \\ \mathcal{D}_{I_{O}}^{G}(X) &\to \mathcal{D}_{E}^{G}(X); \\ \mathcal{D}_{O}^{G}(X) &\to \mathcal{D}^{G}(X); \end{aligned}$$

(ii) The functor induced on the Karoubi quotients

$$\mathcal{D}_{E}^{G}(X)/\mathcal{D}_{L}^{G}(X) \to \mathcal{D}^{G}(X)/\mathcal{D}_{O}^{G}(X)$$

*is an equivalence of additive categories;* (iii) *The diagram* (21.112) *is weakly homotopy cocartesian.* 

*Proof.* (i) We only show that the inclusion  $\mathcal{D}_{I}^{G}(X) \to \mathcal{D}_{E}^{G}(X)$  is a Karoubi filtration, the proof for the other inclusions is an obvious variation. Consider an object  $\mathbf{B} = (S, \pi, \eta, B)$  in  $\mathcal{O}_{E}^{G}(X)$ , objects  $\mathbf{U} = (S^{\mathbf{U}}, \pi^{\mathbf{U}}, \eta^{\mathbf{U}}, B^{\mathbf{U}})$  and  $\mathbf{V} = (S^{\mathbf{V}}, \pi^{\mathbf{V}}, \eta^{\mathbf{V}}, B^{\mathbf{V}})$  in  $\mathcal{O}_{E}^{G}(X)_{I}$ , and morphisms  $\overline{\phi} \colon \mathbf{B} \to \mathbf{U}$  and  $\overline{\psi} \colon \mathbf{V} \to \mathbf{B}$  in  $\mathcal{D}_{E}^{G}(X)$ . Let the morphisms  $\phi \colon \mathbf{B} \to \mathbf{U}$  and  $\psi \colon \mathbf{V} \to \mathbf{B}$  in  $\mathcal{O}_{E}^{G}(X)$ . Let the morphisms  $\phi \colon \mathbf{B} \to \mathbf{U}$  and  $\psi \colon \mathbf{V} \to \mathbf{B}$  in  $\mathcal{O}_{E}^{G}(X)$ . Let the morphisms  $\phi \colon \mathbf{B} \to \mathbf{U}$  and  $\psi \colon \mathbf{V} \to \mathbf{B}$  in  $\mathcal{O}_{E}^{G}(X)$ . Let the morphisms  $\phi \colon \mathbf{B} \to \mathbf{U}$  and  $\psi \colon \mathbf{V} \to \mathbf{B}$  in  $\mathcal{O}_{E}^{G}(X)$  be representatives of  $\overline{\phi}$  and  $\overline{\psi}$ . Choose a number l such that  $\phi_{s,t} = 0$  holds for  $s \in S$  and  $t \in S^{\mathbf{U}}$  with  $|\eta(s) - \eta^{\mathbf{U}}(t)| \ge l$ , and  $\psi_{r,s} = 0$  holds for  $r \in S^{\mathbf{V}}$  and  $s \in S$  with  $|\eta(s) - \eta^{\mathbf{V}}(r)| \ge l$ . Since  $\lim_{j\to\infty}(i_j - i_{j-1}) = \infty$ , we can find a natural number  $j_0 \ge 1$  such that  $(i_j - i_{j-1}) > 2l + 1$  for  $j \ge j_0$  holds. We can change the representatives  $\phi$  and  $\psi$  such that  $\phi_{s,t} = \psi_{r,s} = 0$  holds for  $s \in S$ ,  $t \in S^{\mathbf{U}}$ , and  $r \in S^{\mathbf{V}}$ , provided that  $\pi_{\mathbb{N}}(s) \le i_{j_0}$  is true. Hence we get for every natural number j the following implications for  $s \in S$ ,  $t \in S^{\mathbf{U}}$ , and  $r \in S^{\mathbf{V}}$ 

$$\eta(s) \in \mathbb{N}_{i_{2j}}, \pi^{\mathsf{U}}_{\mathbb{N}}(t) \in I, \phi_{s,t} \neq 0$$
$$\implies i_{2j} \le \eta(s) \le i_{2j} + l \text{ or } i_{2j+1} - l \le \eta(s) \le i_{2j+1};$$

$$\eta(s) \in \mathbb{N}_{i_{2j}}, \pi^{\mathbf{V}}_{\mathbb{N}}(r) \in I, \psi_{r,s} \neq 0$$
  
$$\implies i_{2j} \le \eta(s) \le i_{2j} + l \text{ or } i_{2j+1} - l \le \eta(s) \le i_{2j+1}$$

Define new objects  $\mathbf{B}^{\perp} = (S^{\perp}, \pi^{\perp}, \eta^{\perp}, B^{\perp})$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  in  $\mathcal{O}_E^G(X)$  by putting

$$S^{\perp} = \{ s \in S \mid \eta(s) < i_{2j_0} \}$$
  

$$\coprod \{ s \in S \mid i_{2j} + l < \eta(s) < i_{2j+1} - l, \text{ for some } j \in \mathbb{N} \text{ with } 2j \ge j_0 \};$$
  

$$\pi^{\perp} = \pi|_{S^{\perp}};$$
  

$$\eta^{\perp} = \eta|_{S^{\perp}};$$
  

$$B^{\perp} = B|_{S^{\perp}};$$

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$$\begin{aligned} S' &= \{ s \in S \mid i_{2j} \leq \eta(s) \leq i_{2j} + l \text{ or } i_{2j+1} - l \leq \eta(s) \leq i_{2j+1} \\ &\text{for some } j \in \mathbb{N} \text{ with } 2j \geq j_0 \}; \\ \pi' &= \pi|_{S'}; \\ \eta' &= \eta|_{S'}; \\ B' &= B|_{S'}. \end{aligned}$$

Since  $S = S' \amalg S^{\perp}$ , there are obvious morphisms  $i' \colon \mathbf{B}' \to \mathbf{B}$  and  $i^{\perp} \colon \mathbf{B}^{\perp} \to \mathbf{B}$ in  $O_F^G(X)$  given by the morphisms  $\mathrm{id}_{B'(s')}$  and  $\mathrm{id}_{B^{\perp}(s^{\perp})}$  for  $s' \in S'$  and  $s^{\perp} \in S^{\perp}$ such that  $i \oplus i^{\perp} \colon \mathbf{B}' \oplus \mathbf{B}^{\perp} \to \mathbf{B}$  is an isomorphism. Moreover, there are morphisms  $\phi': \mathbf{B}' \to \mathbf{U}$  and  $\psi': \mathbf{U} \to \mathbf{B}'$  in  $O_E^G(X)$  such that  $\phi \circ (i \oplus i^{\perp}) = \phi' \circ \mathrm{pr}'$  and  $i' \circ \psi' = \psi$  holds, where  $\mathrm{pr}': \mathbf{B}' \oplus \mathbf{B}^{\perp} \to \mathbf{B}'$  is the canonical projection. Define the object  $\mathbf{B}^{\mathcal{U}} = (S^{\mathcal{U}}, \pi^{\mathcal{U}}, \eta^{\mathcal{U}}, \mathbf{B}^{\mathcal{U}})$  in  $O^G(X)_I$  by putting for  $s' \in S'$ 

$$S^{\mathcal{U}} = S' \pi^{\mathcal{U}} = \pi'; \eta^{\mathcal{U}}(s') = \begin{cases} i_{2j} & \text{if } i_{2j} \le \eta(s') \le i_{2j} + l; \\ i_{2j+1} & \text{if } i_{2j+1} - l \le \eta(s') \le i_{2j+1}; \\ B^{\mathcal{U}} = B'. \end{cases}$$

We can consider  $\mathbf{B}^{\mathcal{U}}$  also as an object in  $O_E^G(X)$ . Since  $S^{\mathcal{U}} = S'$  and  $\mathbf{B}^{\mathcal{U}} = \mathbf{B}'$ , one easily checks that taking for  $s \in S'$  the identity  $\mathrm{id}_{\mathbf{B}'(s)}$  yields well-defined mutually inverse isomorphisms  $u: \mathbf{B}^{\mathcal{U}} \to \mathbf{B}'$  and  $v: \mathbf{B}' \to \mathbf{B}^{\mathcal{U}}$  in  $O_E^G(X)$ . Define morphisms in  $\mathcal{O}_{F}^{G}(X)$ 

$$\begin{aligned} i^{\mathcal{U}} &:= i' \circ u : \quad \mathbf{B}^{\mathcal{U}} \to \mathbf{B}; \\ \mathrm{pr}^{\mathcal{U}} &:= v \circ \mathrm{pr}' : \quad \mathbf{B} \to \mathbf{B}^{\mathcal{U}}; \\ \phi^{\mathcal{U}} &:= \phi' \circ u : \quad \mathbf{B}^{\mathcal{U}} \to \mathbf{U}; \\ \psi^{\mathcal{U}} &:= v \circ \psi' : \quad \mathbf{V} \to \mathbf{B}^{\mathcal{U}}. \end{aligned}$$

One easily checks that the images of  $i^{\mathcal{U}}$ ,  $i^{\perp}$ ,  $pr^{\mathcal{U}}$ ,  $\phi^{\mathcal{U}}$ , and  $\psi^{\mathcal{U}}$  under the projection  $\mathcal{O}_{E}^{G}(X) \to \mathcal{D}_{E}^{G}(X)$  yield the data required for a Karoubi filtration.

(ii) Next we show that for two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  in  $O_F^G(X)$  the obvious map

(21.114) 
$$\operatorname{mor}_{\mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I}^{G}(X)}(\mathbf{B},\mathbf{B}') \to \operatorname{mor}_{\mathcal{D}^{G}(X)/\mathcal{D}_{O}^{G}(X)}(\mathbf{B},\mathbf{B}')$$

is bijective.

We begin with the proof of surjectivity. It is based on the following construction. Consider a morphism  $\phi: \mathbf{B} \to \mathbf{B}'$  in  $O^G(X)$ . Since  $\mathbb{N} = E \cup O$ , one can construct objects  $\mathbf{B}^E$  and  $\mathbf{B}'^E$  in  $O^G_E(X)$  and  $\mathbf{B}^O$  and  $\mathbf{B}'^O$  in  $O^G_O(X)$  such that we get in  $O^G(X)$  identifications  $\mathbf{B}^O \oplus \mathbf{B}^E = \mathbf{B}$  and  $\mathbf{B}'^O \oplus \mathbf{B}'^E = \mathbf{B}'$ . Then  $\phi$  can be written as

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{B}^O \oplus \mathbf{B}^E \to \mathbf{B}'^O \oplus \mathbf{B}'^E.$$

Define a morphism in  $O^G(X)$  by the composite

$$\psi \colon \mathbf{B}^O \oplus \mathbf{B}^E \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & b \end{pmatrix}} \mathbf{B}^O \oplus \mathbf{B}'^O \xrightarrow{\begin{pmatrix} a & \mathrm{id} \\ c & 0 \end{pmatrix}} \mathbf{B}'^O \oplus \mathbf{B}'^E.$$

Then  $\mathbf{B}^O \oplus \mathbf{B}'^O$  is an object in  $\mathcal{D}_O^G(X)$ , the difference  $\phi - \psi$  is of the shape  $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ , and  $d: \mathbf{B}^E \to \mathbf{B}'^E$  belongs to  $\mathcal{O}_E^G(X)$ .

It remains to prove injectivity. Consider a morphism  $[\overline{\phi}]: \mathbf{B} \to \mathbf{B}'$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_I^G(X)$  whose image under (21.114) is zero. We have to show that  $[\overline{\phi}]$  itself is zero. Choose a representative  $\overline{\phi}$  in  $\mathcal{D}_E^G(X)$  of  $[\overline{\phi}]$ . By assumption there is an object  $\mathbf{U} = (S^{\mathbf{U}}, \pi^{\mathbf{U}}\eta^{\mathbf{U}}, \mathbf{B}^{\mathbf{U}})$  in  $\mathcal{O}_O^G(X)$  such that  $\overline{\nu} \circ \overline{\mu} = \overline{\phi}$  holds in  $\mathcal{D}^G(X)$  for appropriate morphisms  $\overline{\mu}: \mathbf{B} \to \mathbf{U}$  and  $\overline{\nu}: \mathbf{U} \to \mathbf{B}'$ . Choose a representative  $\phi$  in  $\mathcal{O}_E^G(X)$  of  $\overline{\phi}$ , and representatives  $\mu$  and  $\nu$  in  $\mathcal{O}_E^G(X)$  of  $\overline{\mu}$  and  $\overline{\nu}$  respectively. Fix a number l such that for  $s \in S$ ,  $s' \in S'$ , and  $t \in S^{\mathbf{U}}$  the implications

$$\begin{split} \phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \le l; \\ \mu_{s,t} \neq 0 \implies |\eta(s) - \eta^{\mathbf{U}}(t)| \le l; \\ v_{t,s'} \neq 0 \implies |\eta^{\mathbf{U}}(t) - \eta'(s')| \le l, \end{split}$$

hold. Since  $\lim_{j\to\infty}(i_j - i_{j-1}) = \infty$ , we can find a natural number  $j_0 \ge 1$  such that  $(i_j - i_{j-1}) > 2l + 1$  holds for  $j \ge j_0$ . By possibly enlarging  $j_0$  we can additionally arrange that  $\phi_{s,s'} = \sum_{t \in S^U} v_{t,s'} \circ \mu_{s,t}$  holds for  $s \in S$ ,  $s' \in S'$  with  $\eta(s), \pi'_{\mathbb{N}}(s') \ge i_{j_0}$ . Define an object  $V = (S^V, \pi^V, \eta^V, B^V)$  in  $\mathcal{D}_I^G(X)$  by putting

$$S^{\mathbf{V}} = \{t \in S^{\mathbf{U}} \mid \eta(t) \ge i_{j_0} \text{ and } \exists n \in I \text{ with } |n - \eta^{\mathbf{U}}(t)| \le l\};$$
  
$$\pi^{\mathbf{V}} = \pi^{\mathbf{U}}|_{S^{\mathbf{V}}};$$
  
$$\eta^{\mathbf{V}}(t) = n \text{ for } t \in S^{\mathbf{V}} \text{ and } n \in I \text{ with } |n - \eta^{\mathbf{U}}(t)| \le l;$$
  
$$B^{\mathbf{V}} = B^{\mathbf{U}}|_{S^{\mathbf{V}}}.$$

Define morphisms  $\alpha : \mathbf{B} \to \mathbf{V}$  and  $\beta : \mathbf{V} \to \mathbf{B}'$  in  $\mathcal{O}_E^G(X)$  by putting for  $s \in S$ ,  $s' \in S'$ , and  $t \in S^{\mathbf{V}}$ 

$$\alpha_{s,t} = \mu_{s,t};$$
  
$$\beta_{t,s'} = v_{t,s'}.$$

Then  $\phi_{s,s'} = \sum_{t \in S^U} \beta_{t,s'} \circ \alpha_{s,t}$  holds for  $s \in S$  and  $s' \in S'$  with  $\eta(s), \eta(s') \ge i_{j_0}$ . Hence we get  $\overline{\phi} = \overline{\beta} \circ \overline{\alpha}$  in  $\mathcal{D}_E^G(X)$ . Since **V** belongs to  $\mathcal{D}_I^G(X)$ , we get  $[\overline{\phi}] = 0$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_I^G(X)$ . Hence (21.114) is bijective.

It remains to construct for an object  $\mathbf{B} = (S, \pi, \eta, B)$  in  $O^G(X)$  an object  $\mathbf{B}'$  in  $O^G_E(X)$  and morphisms  $i: \mathbf{B}' \to \mathbf{B}$  and  $r: \mathbf{B} \to \mathbf{B}'$  in  $O^G(X)$  such that  $[\bar{r}] \circ [\bar{i}] = \mathrm{id}_{\mathbf{B}'}$  and  $[\bar{i}] \circ [\bar{r}] = \mathrm{id}_{\mathbf{B}}$  hold in  $\mathcal{D}^G(X)/\mathcal{D}^G_O(X)$ . We define  $\mathbf{B}' = (S', \pi', \eta', B')$  by

$$S' = \{s \in S \mid \eta(s) \in E\}; \pi' = \pi|_{S'}; \eta' = \eta|_{S'}; B' = B|_{S'},$$

and the morphisms *i* and *r* for  $s \in S$  and  $s' \in S'$  by

$$i_{s',s} = r_{s,s'} = \begin{cases} id_{B(s')} & \text{if } s = s'; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously  $r \circ i = \mathrm{id}_{\mathbf{B}'}$  holds already in  $O^G(X)$ , which implies  $[\overline{r}] \circ [\overline{i}] = \mathrm{id}_{\underline{B'}}$  in  $\mathcal{D}^G(X)/\mathcal{D}^G_O(X)$ . Define an object  $\mathbf{U} = (S^{\mathbf{U}}, \pi^{\mathbf{U}}, \eta^{\mathbf{U}}, \mathsf{B}^{\mathbf{U}})$  in  $O^G_O(X)$  by

$$S^{U} = \{s \in S \mid \eta(s) \notin E\};$$
  

$$\pi^{U} = \pi|_{s^{U}};$$
  

$$\eta^{U} = \eta|_{s^{U}};$$
  

$$B' = B|_{s^{U}}.$$

Obviously  $i \circ r - id_{\mathbf{B}} = id_{\mathbf{U}}$  holds in  $\mathcal{O}^{G}(X)$ . This implies  $[\bar{i}] \circ [\bar{r}] = id_{\mathbf{B}}$  in  $\mathcal{D}^{G}(X)/\mathcal{D}^{G}_{\mathcal{O}}(X)$ .

(iii) This follows from assertions (i) and (ii) and Theorem 8.46. This finishes the proof of Lemma 21.113.  $\hfill \Box$ 

**Lemma 21.115.** The inclusions  $\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)$  and  $\mathcal{D}_{I_E}^G(X) \to \mathcal{D}_O^G(X)$  induce weak equivalences

$$\begin{split} \mathbf{K}(\mathcal{D}_{I_O}^G(X)) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{D}_E^G(X)); \\ \mathbf{K}(\mathcal{D}_{I_E}^G(X)) &\xrightarrow{\simeq} \mathbf{K}(\mathcal{D}_O^G(X)). \end{split}$$

*Proof.* We give the proof only for the first map, the one for the second is completely analogous. We have already shown in Lemma 21.113 (i) that the inclusion  $\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)$  is a Karoubi filtration. Hence it suffices to show that  $\mathbf{K}(\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X))$  is weakly contractible. This we will do by constructing an Eilenberg swindle as follows.

We define a functor of additive categories

The idea is to move the objects one position to the right in the  $\mathbb{N}$ -direction, discarding the objects sitting at right endpoints of the intervals  $\mathbb{N}_{2j}$  since they would be moved outside the set *E*, and leaving the position in the *X*-direction fixed. Since the union of the right endpoints of the intervals  $\mathbb{N}_{2j}$  for  $j \ge 0$  is  $I_O$ , this gives a well-defined functor. Here are more details.

An object  $\mathbf{B} = (S, \pi, \eta, B)$  of  $\mathcal{D}_E^G(X) / \mathcal{D}_{I_O}^G(X)$ , which is the same as an object in  $\mathcal{O}_E^G(X)$ , is sent to the object SH( $\mathbf{B}$ ) =  $(S^{\text{SH}}, \pi^{\text{SH}}, \eta^{\text{SH}}, B^{\text{SH}})$  in  $\mathcal{O}_E^G(X)$  given by

$$S^{\text{SH}} = \{s \in S \mid \eta(s) \in E \setminus I_O\};$$
  

$$\pi^{\text{SH}} = \pi|_{S^{\text{SH}}};$$
  

$$\eta^{\text{SH}}(s) = \eta(s) + 1 \text{ for } s \in S^{\text{SH}};$$
  

$$B^{\text{SH}} = B|_{S^{\text{SH}}}.$$

Consider a morphism  $[\phi]: \mathbf{B} = (S, \pi, \eta, B) \to \mathbf{B}' = (S', \pi', \eta', B')$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ . Let  $\phi: \mathbf{B} \to \mathbf{B}'$  be a morphism in  $\mathcal{O}_E^G(X)$  representing  $[\phi]$ . Define a morphism  $\mathrm{SH}(\phi)$  in  $\mathcal{O}_E^G(X)$  by

$$SH(\phi)_{s,s'} = \phi_{s,s'}$$
 for  $s \in S^{SH}$ ,  $s' \in (S')^{SH}$ 

Define  $\mathrm{SH}(\phi)$  to be the class  $[\mathrm{SH}(\phi)]$  of  $\mathrm{SH}(\phi)$ . Note that  $\mathrm{SH}(\phi)$  depends on the choice of  $\phi \in [\phi]$ . We leave it to the reader to check that  $[\mathrm{SH}(\phi)]$  depends only on  $[\phi]$ . Moreover, let  $[\overline{\phi}] : \mathbf{B} \to \mathbf{B}'$  and  $[\psi] : \mathbf{B}' \to \mathbf{B}''$  be composable morphisms in  $\mathcal{D}_E^G(X)$ . Choose representatives  $\phi \in [\phi]$  and  $\psi \in [\psi]$ . Then it is not true that  $\mathrm{SH}(\psi) \circ \mathrm{SH}(\phi)$  and  $\mathrm{SH}(\psi \circ \phi)$  agree, but one easily checks that the classes  $[\mathrm{SH}(\psi) \circ \mathrm{SH}(\phi)] = [\mathrm{SH}(\psi \circ \phi)]$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$  agree. Therefore the functor announced in (21.116) is well-defined.

Next we construct a natural equivalence

(21.117) 
$$R_1: \operatorname{id}_{\mathcal{D}_F^G(X)/\mathcal{D}_{L_0}^G(X)} \xrightarrow{\cong} \operatorname{SH}$$

of functors  $\mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I_{O}}^{G}(X) \to \mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I_{O}}^{G}(X)$  of additive categories.

We specify for every object **B** in  $O_E^G(X)$  morphisms  $\phi: \mathbf{B} = (S, \pi, \eta, B) \rightarrow$ SH(**B**) =  $(S^{SH}, \pi^{SH}, \eta^{SH}, B^{SH})$  and  $\psi: SH(\mathbf{B}) \rightarrow \mathbf{B}$  in  $O_E^G(X)$  by putting for  $s \in S$ and  $s^{SH} \in S^{SH}$ 

$$\begin{split} \phi_{s,s^{\text{SH}}} &= \begin{cases} \text{id}_{\text{B}(s)} & \text{if } s^{\text{SH}} = s; \\ 0 & \text{otherwise;} \end{cases} \\ \psi_{s^{\text{SH}},s} &= \begin{cases} \text{id}_{\text{B}(s)} & \text{if } s = s^{\text{SH}}; \\ 0 & \text{otherwise.} \end{cases} \end{split}$$

We have  $\phi \circ \psi = \mathrm{id}_{\mathrm{SH}(\mathbf{B})}$  in  $O_E^G(X)$ . We do not have  $\psi \circ \phi = \mathrm{id}_{\mathbf{B}}$  in  $O_E^G(X)$  but  $[\psi \circ \phi] = [\mathrm{id}_{\mathbf{B}}]$  holds in  $O_E^G(X)/\mathcal{D}_{I_O}^G(X)$ . Now one easily checks that the natural equivalence  $R_1$  announced in (21.117) is well-defined.

21.10 The Version with Zero Control over  $\mathbb{N}$ 

Next we define another functor

(21.118) 
$$Z: \mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I_{O}}^{G}(X) \to \mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I_{O}}^{G}(X).$$

The informal definition is  $Z(\mathbf{B}) = \bigoplus_{m=0}^{\infty} SH^m(\mathbf{B})$  and analogously for morphisms, where  $SH^m$  is the *m*-fold composite of SH. This makes sense since over a given element in  $\mathbb{N}$  this direct sum is finite. Here are more details of this definition.

An object **B** =  $(S, \pi, \eta, B)$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ , which is the same as an object in  $\mathcal{D}_E^G(X)$ , is sent to the object  $Z(\mathbf{B}) = (S^Z, \pi^Z, \eta^Z, B^Z)$  in  $\mathcal{D}_E^G(X)$  given by

$$S^{Z} = \prod_{j \in \mathbb{N}} \prod_{n \in \mathbb{N}_{2j}} \prod_{k=i_{2j}}^{n} \eta^{-1}(k);$$
  

$$\pi^{Z} = \pi|_{S^{Z}};$$
  

$$\eta^{Z}(s) = n \quad \text{for } s \in \prod_{k=i_{2j}}^{n} \eta^{-1}(k);$$
  

$$B^{Z} = B|_{S^{Z}}.$$

Consider a morphism  $[\phi]: \mathbf{B} \to \mathbf{B}'$  in  $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ . Let the morphism  $\phi: \mathbf{B} = (S, \pi, \eta, \mathbf{B}) \to \mathbf{B}' = (S', \pi', \eta', \mathbf{B}')$  in  $\mathcal{O}_E^G(X)$  be a representative of  $[\phi]$ . Define a morphism  $Z(\phi)$  in  $\mathcal{O}_E^G(X)$  by putting for  $s \in S^Z$  and  $s' \in (S')^Z$ 

$$Z(\underline{s})_{s,s'} = \begin{cases} \phi_{s,s'} & \text{if } \exists j \in \mathbb{N}, n, n' \in \mathbb{N}_{2j} \text{ with } i_{2j} \leq \eta(s) \leq n, i_{2j} \leq \pi'(s') \leq n' \\ & \text{and } n - \eta(s) = n' - \eta'(s'); \\ 0 & \text{otherwise.} \end{cases}$$

Now define  $Z([\phi])$  to be  $[Z(\phi)]$ .

Next we construct a natural equivalence

(21.119) 
$$R_2: \operatorname{id}_{\mathcal{O}_E^G(X)} \oplus (\operatorname{SH} \circ Z) \xrightarrow{\cong} Z$$

of functors  $\mathcal{O}_E^G(X)/\mathcal{D}_{I_O}^G(X) \to \mathcal{O}_E^G(X)/\mathcal{D}_{I_O}^G(X)$  of additive categories. The idea comes from the formula

$$\mathbf{B} \oplus \operatorname{SH}(Z(\mathbf{B})) = \mathbf{B} \oplus \operatorname{SH}\left(\bigoplus_{m=0}^{\infty} \operatorname{SH}^{m}(\mathbf{B})\right)$$
$$= \mathbf{B} \oplus \bigoplus_{m=1}^{\infty} \operatorname{SH}^{m}(\mathbf{B}) = \bigoplus_{m=0}^{\infty} \operatorname{SH}^{m}(\mathbf{B}) = Z(\mathbf{B})$$

Here are some details of the construction. Note that for an object  $\mathbf{B} = (S, \pi, \eta, B)$  in  $O_E^G(X)$  the source of  $R_2(\mathbf{B})$  is given by the quadruple  $(S', \pi', \eta', B')$  and the target by the quadruple  $(S^Z, \pi^Z, \eta^Z, B^Z)$  such that

$$\begin{split} S' &= S \amalg \left( S^{Z} \right)^{\text{SH}} \\ &= S \amalg \left\{ s \in S^{Z} \mid \eta^{Z}(s) \in E \setminus I_{0} \right\} \\ &= \left( \prod_{j \in \mathbb{N}} \prod_{n=i_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left\{ s \in \prod_{j \in \mathbb{N}} \prod_{n \in \mathbb{N}_{2j}} \prod_{k=i_{2j}}^{n} \eta^{-1}(k) \mid \eta^{Z}(s) \in E \setminus I_{O} \right\} \\ &= \left( \prod_{j \in \mathbb{N}} \prod_{n=i_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left( \prod_{j \in \mathbb{N}} \prod_{\substack{n \in \mathbb{N}_{2j}, \\ n \neq i_{2j+1}}} \prod_{k=i_{2j}}^{n} \pi_{\mathbb{N}}^{-1}(k) \right) \\ &= \left( \prod_{j \in \mathbb{N}} \prod_{n \in \mathbb{N}_{2j}}^{i_{2j+1}} \eta^{-1}(n) \right) \amalg \left( \prod_{j \in \mathbb{N}} \prod_{\substack{n \in \mathbb{N}_{2j}, \\ n \neq i_{2j}}} \prod_{k=i_{2j}}^{n-1} \pi_{\mathbb{N}}^{-1}(k) \right) \\ &= \prod_{j \in \mathbb{N}} \prod_{n \in \mathbb{N}_{2j}} \prod_{k=i_{2j}}^{n} \eta^{-1}(k) \\ &= S^{Z}. \end{split}$$

Note that any element  $s \in S'$  belongs to S. Moreover, under the identification  $S' = S^Z$  above we have  $B'(s) = B^Z(s) = B(s)$  for  $s \in S'$ . So we can define an isomorphism in  $O_E^G(X)$ 

$$R'_{2}(\mathbf{B}) \colon \mathbf{B} \oplus (\mathrm{SH} \circ Z)(\mathbf{B}) = (S', \pi', \eta', \mathsf{B}') \to Z(\mathbf{B}) = (S^{Z}, \pi^{Z}, \eta^{Z}, \mathsf{B}^{Z})$$

by putting  $R'_{2}(\mathbf{B})_{s_{0},s_{1}} = \operatorname{id}_{B(s_{0})}$  if  $s_{0} = s_{1}$  and  $R_{2}(\mathbf{B})_{s_{0},s_{1}} = 0$  if  $s_{0} \neq s_{1}$  for  $s_{0} \in S'$  and  $s_{1} \in S^{Z}$ . Now define  $R_{2}(\mathbf{B})$  by  $[R'_{2}(\mathbf{B})]$ . We leave it to the reader to check that the natural equivalence announced in (21.119) is well-defined.

Putting  $R_1$  and  $R_2$  together yields a natural equivalence of functors of additive categories  $\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)$ 

$$R: \operatorname{id}_{\mathcal{D}_E^G(X)/\mathcal{D}_{I_O}^G(X)} \oplus Z \xrightarrow{\simeq} Z.$$

Theorem 6.37 (iii) implies that the spectrum  $\mathbf{K}(\mathcal{D}_{E}^{G}(X)/\mathcal{D}_{I_{O}}^{G}(X))$  is weakly contractible. This finishes the proof of Lemma 21.115.

Define injective functions  $\rho_I, \rho_{I_O}, \rho_{I_E} \colon \mathbb{N} \to \mathbb{N}$ 

$$\rho_I(j) = i_{j+1};$$
  
 $\rho_{I_E}(j) = i_{2j+2}.$ 
  
 $\rho_{I_O}(j) = i_{2j+1}.$ 

By construction they induce bijections from  $\mathbb{N}$  to I,  $I_E$ , and  $I_O$  respectively.

21.10 The Version with Zero Control over  ${\mathbb N}$ 

**Lemma 21.120.** Let J be I,  $I_O$ , or  $I_E$ . Then the functor  $V_{\rho_J}(X)$ :  $\mathcal{D}_0^G(X) \to \mathcal{D}_0^G(X)$ of (21.103) induces an isomorphism of additive categories

$$V_J(X) \colon \mathcal{D}_0^G(X) \xrightarrow{\cong} \mathcal{D}_J^G(X).$$

*Proof.* We only treat the case J = I, the other cases are completely analogous. The functor  $V_I(X)$  is bijective on the set of objects, since the function  $\mathbb{N} \to I$  sending j to  $i_{j+1}$  is a bijection. Hence it remains to show for two objects  $\mathbf{B} = (S, \pi, \eta, B)$  and  $\mathbf{B}' = (S', \pi', \eta', B')$  in  $O_0^G(X)$  that the map induced by  $V_I(X)$ 

$$\operatorname{mor}_{\mathcal{D}_{G}^{G}(X)}(\mathbf{B},\mathbf{B}') \to \operatorname{mor}_{\mathcal{D}_{G}(X)_{I}}(V_{I}(X)(\mathbf{B}), V_{I}(X)(\mathbf{B}')), \quad [\phi] \mapsto V_{I}(X)([\phi])$$

is bijective. It is obvious that it is injective. Hence we only give more details for the proof of surjectivity. Consider a morphism  $[\psi]: V_I(X)(\mathbf{B}) \to V_I(X)(\mathbf{B}')$  in  $\mathcal{D}_I^G(X)$ . Choose a representative  $\psi: V_I(X)(\mathbf{B}) \to V_I(X)(\mathbf{B}')$  in  $\mathcal{O}_I^G(X)$ . There is a natural number l such that  $\psi_{s,s'} \neq 0 \implies |\widehat{\eta}(s) - \widehat{\eta'}(s')| \leq l$  holds for  $s \in S$  and  $s' \in S'$ . Choose a natural number  $j_0 \geq 1$  such that  $i_j - i_{j-1} > l$  holds for  $j \geq j_0$ . Then the implication  $\psi_{s,s'} \neq 0 \implies \widehat{\eta}(s) = \widehat{\eta'}(s')$  holds for  $s \in S$  and  $s' \in S'$ with  $\widehat{\eta}(s), \widehat{\eta'}(s') \geq i_{j_0}$ . We can additionally arrange without changing  $[\psi]$  that  $\psi_{s,s'} = 0$  holds for  $\eta(s) \leq i_{j_0} + l$ . Then the implication  $\psi_{s,s'} \neq 0 \implies \widehat{\eta}(s) = \widehat{\eta}(s')$ holds for  $s \in S$  and  $s' \in S'$ . Since  $\widehat{\eta}(s) = \widehat{\eta}(s') \implies \eta(s) = \eta'(s')$ , we can construct a morphism  $\phi: \mathbf{B} \to \mathbf{B'}$  in  $\mathcal{O}_0^G(X)$  satisfying  $F'_I(X)(\phi) = \psi$ . Note that  $\phi$ satisfies *continuous control* as  $\psi$  satisfies *continuous control* and for every natural number N there is a natural number N' such that for all  $j \in \mathbb{N}$  the implication  $j \geq N' \implies i_j \geq N$  holds. This implies that  $[\psi]$  is in the image of the map above. This finishes the proof of Lemma 21.120.

Next we define functors of additive categories, natural in X,

(21.121) 
$$R_O(X): \mathcal{D}_O^G(X) \to \mathcal{D}_{I_F}^G(X);$$

(21.122) 
$$R_E(X): \mathcal{D}_E^G(X) \to \mathcal{D}_{L_0}^G(X)$$

satisfying

(21.123) 
$$R_O(X)|_{\mathcal{D}_{I_E}^G(X)} = \operatorname{id}_{\mathcal{D}_{I_E}^G(X)};$$
(21.124) 
$$R_E(X)|_{\mathcal{D}_{I_E}^G(X)} = \operatorname{id}_{\mathcal{D}_{I_E}^G(X)};$$

(21.124) 
$$R_E(X)|_{\mathcal{D}_{l_Q}^G(X)} = \operatorname{id}_{\mathcal{D}_{l_Q}^G(X)}$$

We only explain the construction of  $R_O(X)$ , the one for  $R_E(X)$  is completely analogous. It will be induced by the following functor of additive categories

$$R'_O(X) \colon O^G_O(X) \to O^G_{I_F}(X)$$

whose definition we describe next. An object  $\mathbf{B} = (S, \pi, \eta, B)$  is sent by  $R'_O(X)$  to the object  $\widehat{\mathbf{B}} = (\widehat{S}, \widehat{\pi}, \widehat{\eta}, \widehat{B})$  given by

$$\begin{split} \widehat{S} &= S; \\ \widehat{\pi} &= \pi; \\ \widehat{\eta}(s) &= i_{2j+2} \quad \text{if } \eta(s) \in \mathbb{N}_{2j+1}; \\ \widehat{B} &= B. \end{split}$$

The idea is to move an object with position in  $\mathbb{N}_{2j+1}$  to the right endpoint of  $\mathbb{N}_{2j+1}$ , namely to  $i_{2j+2}$ , whereas nothing is changed concerning the *X*-direction. Obviously  $\widehat{\mathbb{B}}$  satisfies the conditions *compact support over X* and *local finiteness over*  $\mathbb{N}$ , since **B** does and  $\mathbb{N}_{2j+1}$  is finite. The definition on morphisms is the tautological one. If  $\phi: \mathbf{B} = (S, \pi, \eta, \mathbf{B}) \rightarrow \mathbf{B}' = (S', \pi', \eta', \mathbf{B}')$  is given by the collection  $\{\phi_{s,s'} \mid s \in S, s' \in S'\}$ , define  $R'_O(\phi)$  by the same collection. Obviously  $R_O(\phi)$  satisfies *finite G*-support as  $\phi$  does. Since  $\phi$  satisfies *bounded control over*  $\mathbb{N}$ , we can find a natural number *n* such that for  $s \in S$  and  $s' \in S'$  the implication  $\phi_{s,s'} \neq 0 \implies$  $|\eta(s) - \eta'(s')| \leq n$  holds. Choose a natural number *m* such that  $i_{2j+1} - i_{2j} > n$ holds for  $j \geq m$ . If  $\eta(s) \in \mathbb{N}_{2j+1}$  for  $j \geq m$ , we conclude  $\eta'(s') \in \mathbb{N}_{2j+1}$  and hence  $\widehat{\eta}(s) = \widehat{\eta'}(s')$ . Put  $l = i_{2m} + n$ . Then we have for  $s \in S$  and  $s' \in S'$  the implication  $\phi_{s,s'} \neq 0 \implies |\widehat{\eta}(s) - \widehat{\eta'}(s')| \leq l$ . This shows that  $R'_O(\phi)$  satisfies *bounded control over*  $\mathbb{N}$ . Since  $\phi$  satisfies *continuous control* and for every natural number *N* there exists a natural number N' satisfying  $\widehat{\eta}(s) \geq N' \implies \eta(s) \geq N$  for  $s \in S$  and  $\widehat{\eta'}(s') \geq N' \implies \eta'(s) \geq N$  for  $s' \in S'$ , *continuous control* holds also for  $R_O(\phi)$ .

Obviously  $R'_O(X)$  induces the identity on  $O^G_{I_E}(X)$  and sends  $\mathcal{T}^G_O(X)$  to  $\mathcal{T}^G_{I_E}(X)$ . Hence  $R'_O(X)$  induces the desired functor  $R_O(X)$  announced in (21.121) and satisfying (21.123).

**Lemma 21.125.** The functors  $R_O(X)$  of (21.121) and  $R_E(X)$  of (21.122) induces weak equivalences, natural in X,

$$\mathbf{K}(R_O(X)) \colon \mathbf{K}(\mathcal{D}_O^G(X)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}_{I_E}^G(X));$$
$$\mathbf{K}(R_E(X)) \colon \mathbf{K}(\mathcal{D}_E^G(X)) \xrightarrow{\simeq} \mathbf{K}(\mathcal{D}_{I_O}^G(X)).$$

*Proof.* Because of (21.123) and (21.124) it suffices to show that the inclusions  $\mathcal{D}_{I_E}^G(X) \to \mathcal{D}_O^G(X)$  and  $\mathcal{D}_{I_O}^G(X) \to \mathcal{D}_E^G(X)$  induce weak homotopy equivalences on *K*-theory. This has already been done, see Lemma 21.115.

*Proof of Theorem* 21.109. Consider the following diagram of additive categories, natural in *X*,

where the upper two horizontal arrows are the inclusions, the functors  $V_{\rho_O}(X)$  and  $V_{\rho_E}(X)$  have been defined in (21.103), the isomorphisms of additive categories  $V_{I_E}(X)$ ,  $V_I$ , and  $V_{I_O}(X)$  come from Lemma 21.120, and the functors  $R_O(X)$  and  $R_E(X)$  have been defined in (21.121) and (21.122). If we apply the *K*-theory functor, we obtain a commutative diagram of spectra, natural in *X*,

$$\begin{split} \mathbf{K}(\mathcal{D}_{O}^{G}(X)) &\longleftarrow \mathbf{K}(\mathcal{D}_{I}^{G}(X)) &\longrightarrow \mathbf{K}(\mathcal{D}_{E}^{G}(X)) \\ & \downarrow^{\simeq} & \downarrow^{\simeq} & \downarrow^{\simeq} \\ \mathbf{K}(\mathcal{D}_{0}^{G}(X)) &\longleftarrow \mathbf{K}(\mathcal{D}_{0}^{G}(X)) & \xrightarrow{\mathbf{K}(V_{\rho_{E}}(X))} \mathbf{K}(\mathcal{D}_{0}^{G}(X)) \end{split}$$

whose vertical arrows are weak homotopy equivalences by Lemma 21.125. It induces a weak homotopy equivalence from the homotopy pushout of the upper row to the homotopy pushout **HPO**(*X*) of the lower row, natural in *X*. We have already constructed a weak homotopy equivalences from homotopy pushout of the upper row to  $\mathbf{K}(\mathcal{D}^G(X))$ , natural in *X*, in Lemma 21.113 (iii). This finishes the proof of Theorem 21.109.

# **21.11** The Proof of the Axioms of a *G*-Homology Theory for $\mathcal{D}_0^G$

Next we state the main result of this section.

**Theorem 21.126 (The algebraic** *K*-groups of  $\mathcal{D}_0^G(X, A)$  yield a *G*-homology theory). Let  $\mathcal{B}$  be a control coefficient category in the sense of Definition 21.1.

Then we obtain a G-homology theory with values in  $\mathbb{Z}$ -modules in the sense of Definition 12.1 by the covariant functor from the category of G-CW-pairs to the category of  $\mathbb{Z}$ -graded abelian groups sending (X, A) to  $K_*(\mathcal{D}_0^G(X, A; \mathcal{B}))$ .

First we start with *G*-homotopy invariance. Here the proof for  $\mathcal{D}^G$  of Lemma 21.30 does not carry over, since there we are shifting in the  $\mathbb{N}$ -direction and the construction of the natural equivalence in the relevant Eilenberg swindle cannot be done with zero-control in the  $\mathbb{N}$ -direction. Therefore we have to construct a different Eilenberg swindle where we do not move the objects in the  $\mathbb{N}$ -direction.

**Proposition 21.127.** *The inclusion*  $X \times \{0\} \rightarrow X \times [0, 1]$  *induces a weak homotopy equivalence* 

$$\mathbf{K}(\mathcal{D}_0^G(X \times \{0\})) \to \mathbf{K}(\mathcal{D}_0^G(X \times [0,1])).$$

Proof. We define a functor of additive categories

as follows.

Consider an object **B** =  $(S, \pi, \eta, B)$  in  $O_0^G(X \times [0, 1])$ . In the sequel let  $\pi_X : S \to X$ and  $\pi_{[0,1]} : S \to [0, 1]$  be the maps for which  $\pi = \pi_X \times \pi_{[0,1]}$ . We define SH(**B**) to be the object (SH(S), SH( $\pi$ ), SH( $\eta$ ), SH(B)) given by

$$SH(S) = \{(s, n) \in S \times \mathbb{N} \mid n \le \eta(s) \cdot \pi_{[0,1]}(s)\};$$
  

$$SH(\pi)(s, n) = \begin{cases} \pi(s) & \text{if } \eta(s) = 0; \\ (\pi_X(s), \pi_{[0,1]}(s) - \frac{n}{\eta(s)}) & \text{if } \eta(s) \ge 1; \end{cases}$$
  

$$SH(\eta)(s, n) = \eta(s);$$
  

$$SH(B)(s, n) = B(s).$$

The idea is to shift an object B(s) from position  $\pi_{[0,1]}(s)$  to position  $\pi_{[0,1]}(s) - \frac{1}{\eta(s)}$  if  $\eta(s) \ge 1$  and  $\pi_{[0,1]}(s) - \frac{1}{\eta(s)} \ge 0$  hold, to forget it if  $\eta(s) \ge 1$  and  $\pi_{[0,1]}(s) - \frac{1}{\eta(s)} < 0$  hold, and to leave it at  $\pi_{[0,1]}(s)$  if  $\eta(s) = 0$  holds, whereas  $\pi_X(s)$  and  $\eta(s)$  are unchanged. Then take the infinite direct sum over  $k \in \mathbb{N}$  for the *k*-fold composition. So here we are shifting in the direction of [0, 1] and not in the direction of  $\mathbb{N}$ .

We have to check that this is well-defined. Since  $im(SH(\pi)) \subseteq im(\pi_X) \times [0, 1]$ and **B** satisfies *compact support over*  $X \times [0, 1]$ ,  $SH(\mathbf{B})$  satisfies *compact support over*  $X \times [0, 1]$ . Since **B** satisfies local finiteness over  $\mathbb{N}$ , the same is true for  $SH(\mathbf{B})$ , as we get for  $m \in \mathbb{N}$ 

$$SH(\eta)^{-1}(m) = \{(s,n) \mid \eta(s) = m, n \le \eta(s) \cdot \pi_{[0,1]}(s)\}$$
$$\subseteq \bigcup_{s \in \eta^{-1}(m)} \{m \in \mathbb{N} \mid n \le m\}.$$

The definition on morphisms is the tautological one. If the morphism  $\phi: \mathbf{B} = (S, \pi, \eta, B) \rightarrow \mathbf{B}' = (S', \pi', \eta', B')$  is given by the collection  $\{\phi_{s,s'}: B(s) \rightarrow B(s') \mid s \in S, s' \in S'\}$ , then define  $SH(\phi)$  by the collection  $\{SH(\phi_{(s,n),(s',n')}) \mid (s,n) \in SH(S), (s',n') \in SH(S')\}$  where

$$SH(\phi_{(s,n),(s',n')}) = \begin{cases} \phi_{s,s'} & \text{if } n = n'; \\ 0 & \text{otherwise.} \end{cases}$$

We have to check that this is well-defined. Since  $\phi$  satisfies *finite G-support* and  $SH(\phi)_{(s,n),(s',n')}$  is zero or  $\phi_{s,s'}$ ,  $SH(\phi)$  satisfies *finite G-support*. If  $SH(\phi)_{(s,n),(s',n')} \neq 0$  holds, then  $\phi_{s,s'} \neq 0$  and hence we get  $SH(\eta)(s,n) = \eta(s) = \eta'(s') = SH(\eta')(s',n')$ . The more complicated part is to show that  $SH(\phi)$  satisfies *continuous control*, which we do next. We only deal with the implication (21.5). The proof for the other implication (21.6) is completely analogous.

Consider  $(x, t) \in X \times [0, 1]$  and an open  $G_{(x,t)}$ -invariant neighborhood U of (x, t)in  $X \times [0, 1]$ . Choose an open  $G_x$ -invariant neighborhood V of x in X and  $\epsilon > 0$ such that  $V \times I_{3\epsilon}(t) \subseteq U$  holds where  $I_{\epsilon}(t) = (t - \epsilon, t + \epsilon) \cap [0, 1]$ . Since  $\phi$  satisfies *continuous control*, we can find  $\delta(x, t, \epsilon) > 0$  with  $\delta(x, t, \epsilon) \leq \epsilon, r'(x, t, \epsilon) \in \mathbb{N}$ , and an open  $G_x$ -invariant neighborhood  $V'(x, t, \epsilon)$  of x in X such that  $V(x, t, \epsilon) \subseteq V$  and
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 $\delta(x, t, \epsilon) \leq \epsilon$  hold and we have for every  $s \in S$ ,  $s' \in S'$ , and  $g \in G$  the implication

$$(21.129) \quad g\pi_X(s) \in V'(x,t,\epsilon), \pi_{[0,1]}(s) \in I_{\delta(x,t,\epsilon)}(t), \eta(s) \ge r'(x,t,\epsilon)$$
$$\implies \pi'_X(s') \in V, \pi'_{[0,1]}(s') \in I_{\epsilon}(t).$$

Since [0, 1] is compact, we can find a finite subset  $\{t_1, t_2, \dots, t_l\} \subseteq [0, 1]$  satisfying  $\bigcup_{i=1}^l I_{\delta(x, t_i, \epsilon)}(t_i) = [0, 1]$ . Put

$$r' = \max\{r'(x, t_i, \epsilon) \mid i = 1, 2, \dots, l\};$$
$$V' = \bigcap_{i=1}^{l} V'(x, t_i, \epsilon).$$

Then r' is a natural number and V' is an open  $G_x$ -invariant neighborhood of x in X.

Now we are ready to prove the implication (21.5) for  $SH(\phi)$ . Consider  $(s, n) \in$  SH(S),  $(s', n') \in SH(S')$ , and  $g \in supp_G(SH(\phi)_{(s,n),(s',n')})$ . We want to show

$$g\operatorname{SH}(\pi)(s,n) \in V' \times I_{\epsilon}(t), \operatorname{SH}(\eta)(s,n) \geq r' \implies \operatorname{SH}(\pi')(s',n') \in U.$$

As  $\operatorname{SH}(\phi)_{(s,n),(s',n')} = \phi_{s,s'}$ , we get  $g \in \operatorname{supp}_G(\phi_{s,s'})$  and  $\operatorname{SH}(\phi)_{(n,n),(n',s')} \neq 0$ . Choose  $i \in \{1, 2, \dots, l\}$  with  $\pi_{[0,1]}(s) \in I_{\delta(x,t_i,\epsilon)}(t_i)$ . As  $V' \subseteq V'(x,t_i,\epsilon)$  and  $r'(x,t_i,\epsilon) \leq r'$  hold, we obtain  $\pi'_X(s') \in V$  and  $\pi'_{[0,1]}(s') \in I_{\epsilon}(t_i)$  from (21.129). Since

 $\delta' \leq \delta(x, t_i, \epsilon) \leq \epsilon$  holds and we have  $\pi_{[0,1]}(s) \in I_{\delta(x,t_i,\epsilon)}(t_i)$  and  $\pi'_{[0,1]}(s') \in I_{\epsilon}(t_i)$ , we conclude from the triangle inequality  $|\pi_{[0,1]}(s) - \pi'_{[0,1]}(s')| \leq 2\epsilon$ . Since  $SH(\phi)_{(s,n),(s',n')} = \phi_{s,s'} \neq 0$ , we have n = n' and  $\eta(s) = \eta'(s')$ . Hence we get for  $\eta(n) \geq 1$ 

$$| \operatorname{SH}(\pi)_{[0,1]}(s,n) - \operatorname{SH}(\pi')_{[0,1]}(s',n') |$$
  
=  $| (\pi_{[0,1]}(s) - \frac{n}{\eta(s)}) - (\pi'_{[0,1]}(s') - \frac{n'}{\eta'(s')}) |$   
=  $| \pi_{[0,1]}(s) - \pi'_{[0,1]}(s') |$   
 $\leq 2\epsilon.$ 

If  $\eta(s) = 0$ , we get  $| SH(\pi)_{[0,1]}(s, n) - SH(\pi')_{[0,1]}(s', n') | = |\pi_{[0,1]}(s) - \pi'_{[0,1]}(s')| \le 2\epsilon$ . Hence for  $(s, n) \in SH(S)$ ,  $(s', n') \in SH(S')$ , and  $g \in supp_G(SH(\phi)_{(s,n),(s',n')})$  satisfying  $g SH(\pi)(s, n) \in V' \times I_{\delta'}(t)$  and  $SH(\eta)(s, n) \ge r'$  we get

$$SH(\pi')_X(s,n') \in V;$$
  
| SH(\pi)\_{[0,1]}(s,n) - SH(\pi')\_{[0,1]}(s',n')| \le 2\\epsilon.

The latter implies using  $SH(\pi)_{[0,1]}(s,n) \in I_{\epsilon}(t)$  and the triangle inequality  $SH(\pi')_{[0,1]}(s',n') \in I_{3\epsilon}(t)$ . Hence we get

$$\operatorname{SH}(\pi')(s',n') \in V \times I_{3\epsilon}(t) \subseteq U.$$

This finishes the proof that  $SH(\phi)$  is a well-defined morphism. One easily checks that SH is a functor of additive categories.

Consider an object **B** =  $(S, \pi, \eta, B)$  in  $O_0^G(X \times [0, 1])$ . Next we define two morphisms in  $O_0^G(X \times [0, 1])$ 

$$T_0(\mathbf{B}) \colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \to \mathrm{SH}(\mathbf{B});$$
  
$$T_1(\mathbf{B}) \colon \mathrm{SH}(\mathbf{B}) \to \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}).$$

Recall that  $\mathbf{B} \oplus SH(\mathbf{B}) = (S \amalg SH(S), \pi \amalg SH(\pi), \eta \amalg SH(\eta), B \amalg SH(B))$ . For  $s \in S$  and  $(s', n') \in SH(S)$  we define

$$T_0(\mathbf{B})_{s,(s',n')} = \begin{cases} \operatorname{id}_{\mathsf{B}(s)} & \text{if } s' = s \text{ and } n' = 0; \\ 0 & \text{otherwise.} \end{cases}$$

For  $(s, n) \in SH(\mathbf{B})$  and  $(s', n') \in SH(\mathbf{B})$  we define

$$T_0(\mathbf{B})_{(s,n),(s',n')} = \begin{cases} \mathrm{id}_{\mathrm{B}(s)} & \text{if } s' = s \text{ and } n' = n+1; \\ 0 & \text{otherwise.} \end{cases}$$

We have to check that this is well-defined. Note that  $\sup_G(T_0(\mathbf{B}))$  is either empty or  $\{e\}$ . In particular, the condition *finite G-support* is satisfied. For  $s \in S$  and  $(s', n') \in SH(S)$  we have  $T_0(\mathbf{B})_{s,(s',n')} \neq 0 \implies s = s'$  and hence  $\eta(s) = \eta(s') =$  $SH(\eta')(s', n')$ . For  $(s, n) \in S$  and  $(s', n') \in SH(S)$  we have  $T_0(\mathbf{B})_{(s,n),(s',n')} \neq$  $0 \implies s = s'$  and hence  $SH(\eta)(s, n) = \eta(s) = \eta(s') = SH(\eta')(s', n')$ . It remains to show *continuous control*. We only deal with the implication (21.5). The proof for the other implication (21.6) is completely analogous.

Consider  $(x, t) \in X \times [0, 1]$  and an open  $G_{(x,t)}$ -invariant neighborhood U of (x, t)in  $X \times [0, 1]$ . Choose an open  $G_x$ -invariant neighborhood V of x in X and  $\epsilon > 0$  such that  $V \times I_{2\epsilon}(t) \subseteq U$  holds. Choose a natural number r' satisfying  $r' \ge 1/\epsilon$ . Then  $U' := V \times I_{\epsilon}(t)$  is an open  $G_{(x,t)}$ -invariant open neighborhood of (x, t) in  $X \times [0, 1]$ with  $U' \subseteq U$ . Consider  $s \in S$ ,  $(s', n') \in SH(S)$ , and  $g \in \text{supp}_G(T_0(\mathbf{B}))_{s,(s',n')}$ such that  $g\pi(s) \in U'$ . Then g = e and  $T_0(\mathbf{B})_{s,(s',n')} \neq 0$ . This implies s' = s and n' = 0 and hence  $\pi(s) = SH(\pi)(s', n')$ . We conclude  $SH(\pi)(s', n') \in U' \subseteq U$ . Consider  $(s, n) \in S$ ,  $(s', n') \in SH(S)$ , and  $g \in \text{supp}_G(T_0(\mathbf{B})_{(s,n),(s',n')})$  such that  $g SH(\pi)(s) \in U'$  and  $SH(\eta)(S, n) \ge r'$  hold. Then g = e and  $T_0(\mathbf{B})_{(s,n),(s',n')} \neq 0$ . This implies s' = s and n' = n + 1. We get  $SH(\pi)_X(s, n) = \pi_X(s) = \pi_X(s') =$  $SH(\pi)_X(s', n')$  and hence  $SH(\pi)_X(s', n') \in V$ . Moreover

$$|\operatorname{SH}(\pi)_{[0,1]}(s,n) - \operatorname{SH}(\pi')_{[0,1]}(s',n')|$$
  
=  $|(\pi_{[0,1]}(s) - \frac{n}{\eta(s)}) - (\pi_{[0,1]}(s') - \frac{n'}{\eta(s')})|$   
=  $|(\pi_{[0,1]}(s) - \frac{n}{\eta(s)}) - (\pi_{[0,1]}(s) - \frac{n+1}{\eta(s)})|$ 

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$$= \frac{1}{\eta(s)}$$
$$\leq \frac{1}{r'}$$
$$\leq \epsilon.$$

Since  $\text{SH}(\pi)_{[0,1]}(s,n) \in I_{\epsilon}(t)$ , we get  $\text{SH}(\pi)_{[0,1]}(s',n') \in I_{2\epsilon}(t)$  from the triangle inequality. Hence  $\text{SH}(\pi)(s',n') \subseteq V \times I_{2\epsilon}(t) \subseteq U'$ . This finishes the proof that  $T_0(\mathbf{B})$  is well-defined.

Next we define  $T_1(\mathbf{B})$ . For  $(s, n) \in SH(S)$  and  $s' \in S$  we define

$$T_1(\mathbf{B})_{(s,n),s'} = \begin{cases} \mathrm{id}_{\mathrm{B}(s)} & \text{if } s' = s \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

For  $(s, n) \in SH(\mathbf{B})$  and  $(s', n') \in SH(\mathbf{B})$  define

$$T_1(\mathbf{B})_{(s,n),(s',n')} = \begin{cases} id_{B(s)} & \text{if } s' = s, n \ge 1, \text{ and } n' = n-1; \\ 0 & \text{otherwise.} \end{cases}$$

We omit the proof that  $T_1(\mathbf{B})$  is well-defined, since it is very similar to the one for  $T_0(\mathbf{B})$ . Roughly speaking,  $T_0(\mathbf{B})$  shifts to the right in [0, 1], whereas  $T_1(\mathbf{B})$  shifts to the left.

Obviously  $T_0(\mathbf{B}) \circ T_1(\mathbf{B}) = id_{SH(\mathbf{B})}$ . It is not true that  $T_1(\mathbf{B}) \circ T_0(\mathbf{B}) = id_{\mathbf{B} \oplus SH(\mathbf{B})}$ . At least we can show that  $id_{\mathbf{B} \oplus SH(\mathbf{B})} - T_1(\mathbf{B}) \circ T_0(\mathbf{B})$  factorizes as a composite

(21.130) 
$$\operatorname{id}_{\mathbf{B}\oplus \operatorname{SH}(\mathbf{B})} - T_1(\mathbf{B}) \circ T_0(\mathbf{B}) \colon \mathbf{B} \oplus \operatorname{SH}(\mathbf{B}) \to \mathbf{B}'_0 \to \mathbf{B} \oplus \operatorname{SH}(\mathbf{B})$$

for an object  $\mathbf{B}'_0$  in  $\mathcal{O}^G_0(X \times \{0\})$  as follows.

We define a kind of subobject  $\mathbf{B}_0 = (S_0, \pi_0, \eta_0.B_0)$  of SH(**B**) by

$$S_{0} = \{(s,n) \in S \times \mathbb{N} \mid \eta(s) \ge 1, n \le \eta(s) \cdot \pi_{[0,1]}(s) < n+1\};$$
  

$$\pi_{0}(s,n) = (\pi_{X}(s), \pi_{[0,1]}(s) - \frac{n}{\eta(s)});$$
  

$$\eta_{0}(s,n) = \eta(s);$$
  
SH(B)(s,n) = B(s).

Note that  $S_0 \subseteq SH(\mathbf{B})$ . Actually, for a given  $s \in S$  with  $\eta(s) \ge 1$  the element of the shape  $(s, n) \in SH(S)$  belongs to  $S_0$  if and only if (s, n + 1) does not belong to SH(S) anymore. The maps  $\eta_0$  and  $B_0$  are obtained by restricting  $SH(\pi)$ ,  $SH(\eta)$ , and SH(B) to  $S_0$ . There is an obvious subobject  $\mathbf{B}^{\perp}$  of  $SH(\mathbf{P})$  such that  $\mathbf{B}_0 \oplus \mathbf{B}^{\perp} = SH(\mathbf{B})$ . Moreover, there is an obvious factorization

$$\mathrm{id}_{\mathbf{B}\oplus\mathrm{SH}(\mathbf{B})}$$
  $-T_1(\mathbf{B}) \circ T_0(\mathbf{B}) \colon \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}) \to \mathbf{B}_0 \to \mathbf{B} \oplus \mathrm{SH}(\mathbf{B}).$ 

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Hence it suffices to show that  $\mathbf{B}_0$  is isomorphic in  $O_0^G(X \times [0, 1])$  to an object  $\mathbf{B}'_0 = (S'_0, \pi'_0, \eta'_0, \mathbf{B}'_0)$  which belongs to  $O_0^G(X \times \{0\})$ . We define  $\mathbf{B}'_0$  by  $S'_0 = S_0$ ,  $\eta'_0 = \eta_0$ , and  $\mathbf{B}'_0 = \mathbf{B}_0$  and by putting  $\pi'_0(s, n) = (\pi_X(s), 0)$ . In order to show that  $\mathbf{B}_0$  and  $\mathbf{B}'_0$  are isomorphic in  $O_0^G(X \times [0, 1])$ , we verify the criterion occurring in Lemma 21.14.

Consider  $(x,t) \in X \times [0,1]$  and an open  $G_{(x,t)}$ -invariant neighborhood U of (x,t) in  $X \times [0,1]$ . Choose an open  $G_x$ -invariant neighborhood V of x in X and  $\epsilon > 0$  such that  $V \times I_{2\epsilon}(t) \subseteq U$  holds. Choose a natural number r' with  $r' \ge \frac{1}{\epsilon}$ . Put  $U' = V \times I_{\epsilon}(t)$ . Next we prove the implication for  $s \in S_0 = S'_0$ 

(21.131) 
$$\pi_0(s) \in U', \eta_0(s) \ge r' \implies \pi'_0(s) \in U.$$

From  $\pi_0(s) \in U'$  we get  $(\pi_0)_X(s) \in V$  and  $(\pi_0)_{[0,1]}(s) \in I_{\epsilon}(t)$ . By definition we have

$$(\pi_0)_{[0,1]}(s) = \pi_{[0,1]}(s) - \frac{n}{\eta(s)} > 0 = (\pi'_0)_{[0,1]}(s) > \pi_{[0,1]}(s) - \frac{n+1}{\eta(s)}.$$

This implies

$$|(\pi_0)_{[0,1]}(s) - (\pi'_0)_{[0,1]}(s)| \le \frac{1}{\eta(s)} = \frac{1}{\eta_0(s)} \le \frac{1}{r'} \le \epsilon.$$

Since  $(\pi_0)_{[0,1]}(s) \in I_{\epsilon}(t)$ , the triangle inequality implies  $(\pi'_0)_{[0,1]}(s) \in I_{2\epsilon}(t)$ . Since  $(\pi_0)_X(s) \in V$  and  $(\pi_0)_X(s) = (\pi'_0)_X(s)$ , we get  $(\pi'_0)_X(s) \in V$ . This implies  $(\pi'_0)(s) \in V \times I_{2\epsilon}(t) \subseteq U$ . This finishes the proof of (21.131). The proof of the other implication

$$\pi'_0(s) \in U', \eta_0(s) \ge r' \implies \pi_0(s) \in U$$

is completely analogous. Thus we obtain the desired factorization (21.130).

One easily checks that SH induces a functor of additive categories

$$\overline{\mathrm{SH}}\colon \mathcal{D}_0^G(X\times[0,1],X\times\{0\})\to \mathcal{D}_0^G(X\times[0,1],X\times\{0\})$$

and  $T_0(\mathbf{B})$  and  $T_1(\mathbf{B})$  induces mutually inverse isomorphisms  $\overline{T_0}(\mathbf{B}) : \mathbf{B} \oplus \overline{\mathrm{SH}}(\mathbf{B}) \xrightarrow{\cong} \overline{\mathrm{SH}}(\mathbf{B})$  and  $\overline{T_1}(\mathbf{B}) : \overline{\mathrm{SH}}(\mathbf{B}) \xrightarrow{\cong} \mathbf{B} \oplus \overline{\mathrm{SH}}(\mathbf{B})$ . The collection of the  $\overline{T_0}(\mathbf{B})$  defines a natural equivalence of functors of additive categories

$$\mathbf{T}_{0}\colon \operatorname{id}_{\mathcal{D}_{0}^{G}(X\times[0,1],X\times\{0\})} \oplus \overline{\operatorname{SH}} \xrightarrow{\cong} \overline{\operatorname{SH}}.$$

We conclude from Theorem 6.37 (iii) and Proposition 21.27 that the inclusion  $X \times \{0\} \rightarrow X \times [0, 1]$  induces a weak homotopy equivalence

$$\mathbf{K}(\mathcal{D}_0^G(X \times \{0\})) \to \mathbf{K}(\mathcal{D}_0^G(X \times [0,1])).$$

This finishes the proof of Proposition 21.127.

### 21.12 Notes

**Exercise 21.132.** Show that the proof of Proposition 21.127 about the homotopy invariance of  $\mathbf{K}(\mathcal{D}_0^G(-))$  can be modified to a new proof of the *G*-homotopy invariance for  $\mathbf{K}(\mathcal{D}^G(-))$ .

*Proof of Theorem 21.126.* The rest of the proof of Theorem 21.126 is completely analogous to the proof of Theorem 21.26, one just has to check that all constructions respect the zero-control condition appearing in the definition of  $\mathcal{D}_0^G$ .

### 21.12 Notes

We have formulated the control conditions in Definition 21.4 concretely to keep some of the arguments simple. One can also give an axiomatic approach to control conditions in terms of coarse structures, as defined by Higson-Pedersen-Roe [488, Definition 2.1], by specifying subsets of X and  $X \times X$  in which the supports of objects and of morphism have to take values in. This is explained for continuous control in [74, Section 2.3]. There are various modifications of this idea, see for instance [183, Definition 3.1], [81, Definition 4.8], and [185, Section 2.2].

### Chapter 22 Coverings and Flow Spaces

### 22.1 Introduction

In this chapter we want to give more details concerning the discussion in Section 19.5. Essentially we want to show that hyperbolic and finite-dimensional CAT(0)-groups satisfy the condition strongly transfer  $\mathcal{VCY}$ -reducible in the sense of Definition 20.38, which implies that they satisfy the Full Farrell-Jones Conjecture 13.30. Note that this concerns only input from geometric group theory; *K*-theory does not play a role at this stage. *K*-theory will enter when we show, for instance, that a strongly  $\mathcal{VCY}$ -transfer reducible group or, more generally, a Dress-Farrell-Hsiang-Jones group satisfies the Full Farrell-Jones Conjecture 13.30, see Remark 20.3, Theorem 20.39, and Theorem 20.62. The proof of Theorem 20.62 will be discussed in Chapter 23 and Chapter 24.

The basic strategy is as follows.

- Consider an appropriate metric space *X* associated to a hyperbolic group or a finite-dimensional CAT(0)-group reflecting its geometry;
- Assign to X a flow space FS(X);
- Prove for the flow space appropriate flow estimates which reflect the negative curvature or non-positive curvature condition associated to hyperbolic or CAT(0)-groups;
- Construct specific covers of the flow space, namely long and thin  $\mathcal{VCY}$ -covers with finite dimension;
- Construct an appropriate map  $\iota: G \times X \to FS(X)$  and pull back the long and thin  $\mathcal{VCY}$ -covers of FS(X) to  $G \times X$  using  $\iota$ ;
- The flow estimates will ensure that these covers on  $G \times X$  are good enough to show that *G* is strongly  $\mathcal{VCY}$ -transfer reducible.

The basic ideas are carried out for closed Riemannian manifolds with negative or non-positive sectional curvature and their fundamental groups in the seminal papers of Farrell-Jones [359, 360, 367]. The papers by Bartels-Lück [78, 80] and Bartels-Lück-Reich [86, 87] transferred these ideas to more general situations such as hyperbolic or CAT(0)-spaces and hyperbolic or finite-dimensional CAT(0)-groups, going considerably beyond the world of Riemannian manifolds and diving into geometric group theory and the theory and geometry of metric spaces. Kasprowski and Rüping [571, Theorem 6.1] simplified and unified some of the arguments, see Remark 22.46.

### 22.2 Flow Spaces

**Definition 22.1 (Flow space).** A *flow space* Y is a metric space  $(Y, d_Y)$  together with a continuous  $\mathbb{R}$ -action  $\Phi: Y \times \mathbb{R} \to Y$ .

**Notation 22.2.** We will often write  $\Phi_t : Y \to Y$  for the homeomorphism sending  $y \in Y$  to  $\Phi(t, y)$ .

For a subset  $I \subseteq \mathbb{R}$  and  $y \in Y$  we put  $\Phi_I(y) = {\Phi_t(y) | t \in I}.$ 

Note that we do *not* demand in Definition 22.1 that  $\Phi_t: Y \to Y$  is isometric.

**Definition 22.3 (Flow** *G*-space for *G*). A *flow G*-space is a flow space  $(Y, d_Y, \Phi)$  in the sense of Definition 22.1 coming with an isometric and proper *G*-action  $\rho: G \times Y \to Y$  such that  $\rho$  and  $\Phi$  commute, i.e., we have  $\Phi_t(gy) = g\Phi_t(y)$  for all  $y \in Y, g \in G$ , and  $t \in \mathbb{R}$ .

Obviously a flow *G*-space is the same as a metric space *Y* with a continuous  $G \times \mathbb{R}$ -action such that the induced action of  $G = G \times \{0\} \subseteq G \times \mathbb{R}$  on *Y* is isometric and proper.

### 22.3 The Flow Space Associated to a Metric Space

In this section we introduce the flow space FS(X) for arbitrary metric spaces following [80, Section 1]. This is the one used in the proof of the Farrell-Jones Conjecture for CAT(0)-groups, see [78, 80, 992].

**Definition 22.4.** Let *X* be a metric space. A continuous map  $c : \mathbb{R} \to X$  is called a *generalized geodesic* if there are  $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \bigsqcup \{-\infty, \infty\}$  satisfying

$$c_{-} \leq c_{+}, \quad c_{-} \neq \infty, \quad c_{+} \neq -\infty,$$

such that *c* is locally constant on the complement of the interval  $I_c := (c_-, c_+)$  and restricts to an isometry on  $I_c$ .

The numbers  $c_{-}$  and  $c_{+}$  are uniquely determined by c, provided that c is not constant.

**Definition 22.5.** Let  $(X, d_X)$  be a metric space. Let FS = FS(X) be the set of all generalized geodesics in *X*. We define a metric on FS(X) by

$$d_{\text{FS}(X)}(c,d) := \int_{\mathbb{R}} \frac{d_X(c(t),d(t))}{2e^{|t|}} dt.$$

Define a flow

 $\Phi\colon \operatorname{FS}(X)\times\mathbb{R}\to\operatorname{FS}(X)$ 

by  $\Phi_{\tau}(c)(t) = c(t + \tau)$  for  $\tau \in \mathbb{R}, c \in FS(X)$ , and  $t \in \mathbb{R}$ .

The integral  $\int_{-\infty}^{+\infty} \frac{d_X(c(t), d(t))}{2e^{|t|}} dt$  exists as  $d_X(c(t), d(t)) \le 2|t| + d_X(c(0), d(0))$  holds by the triangle inequality. Obviously  $\Phi_\tau(c)$  is a generalized geodesic with

$$\Phi_{\tau}(c)_{-} = c_{-} - \tau; \Phi_{\tau}(c)_{+} = c_{+} - \tau,$$

where  $-\infty - \tau := -\infty$  and  $\infty - \tau := \infty$ .

We note that any isometry  $(X, d_X) \rightarrow (Y, d_Y)$  induces an isometry  $FS(X) \rightarrow FS(Y)$  by composition. In particular, the isometry group of  $(X, d_X)$  acts canonically on FS(X). Moreover, this action commutes with the flow.

For a general metric space X all generalized geodesics may be constant. In the remainder of this section we will state some properties of FS(X) so that the reader can get some intuition.

**Lemma 22.6.** Let  $(X, d_X)$  be a metric space. The map  $\Phi$  is a continuous flow and we have for  $c, d \in FS(X)$  and  $\tau, \sigma \in \mathbb{R}$ 

$$d_{\mathrm{FS}(X)}(\Phi_{\tau}(c), \Phi_{\sigma}(d)) \le \mathrm{e}^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c, d) + |\sigma - \tau|.$$

Exercise 22.7. Give the proof of Lemma 22.6.

The following lemma relates distance in X to distance in FS(X).

**Lemma 22.8.** Let  $c, d: \mathbb{R} \to X$  be generalized geodesics. Consider  $t_0 \in \mathbb{R}$ .

(i)  $d_X(c(t_0), d(t_0)) \le e^{|t_0|} \cdot d_{FS}(c, d) + 2;$ (ii) If  $d_{FS(X)}(c, d) \le 2e^{-|t_0|-1}$ , then

$$d_X(c(t_0), d(t_0)) \leq \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS(X)}(c, d)}.$$

In particular,  $c \mapsto c(t_0)$  defines a uniform continuous map  $FS(X) \to X$ .

*Proof.* See [80, Lemma 1.4].

**Lemma 22.9.** Let  $(X, d_X)$  be a metric space. The maps

$$FS(X) \setminus FS(X)^{\mathbb{R}} \to \mathbb{R}, \quad c \mapsto c_{-};$$
  
$$FS(X) \setminus FS(X)^{\mathbb{R}} \to \overline{\mathbb{R}}, \quad c \mapsto c_{+},$$

are continuous.

*Proof.* See [80, Lemma 1.6].

**Proposition 22.10.** Let  $(X, d_X)$  be a metric space. Let  $(c_n)_{n \in \mathbb{N}}$  be a sequence in FS(X). Then it converges uniformly on compact subsets to  $c \in FS(X)$  if and only if it converges to c with respect to  $d_{FS(X)}$ .

*Proof.* See [80, Proposition 1.7].

**Lemma 22.11.** Let  $(X, d_X)$  be a metric space. The flow space FS(X) is sequentially closed in the space of all maps  $\mathbb{R} \to X$  with respect to the topology of uniform convergence on compact subsets.

Proof. See [80, Lemma 1.8].

**Proposition 22.12.** Let  $(X, d_X)$  be a metric space which is proper, i.e., every closed ball is compact,

Then  $(FS(X), d_{FS(X)})$  is a proper metric space.

Proof. See [80, Proposition 1.9].

**Lemma 22.13.** Let  $(X, d_X)$  be a proper metric space and  $t_0 \in \mathbb{R}$ . Then the evaluation map  $FS(X) \rightarrow X$  defined by  $c \mapsto c(t_0)$  is uniformly continuous and proper.

Proof. See [80, Lemma 1.10].

**Proposition 22.14.** Let G act isometrically and properly on the proper metric space  $(X, d_X)$ . Then the action of G on  $(FS(X), d_{FS(X)})$  is also isometric and proper. If the action of G on X is in addition cocompact, then the G-action on FS(X) is also cocompact.

Proof. See [80, Proposition 1.11].

**Lemma 22.15.** Let  $(X, d_X)$  be a metric space. Then  $FS(X)^{\mathbb{R}}$  is closed in FS(X).

Exercise 22.16. Give the proof of Lemma 22.15.

**Notation 22.17.** Let *X* be a metric space. For  $c \in FS(X)$  and  $T \in [0, \infty]$ , define  $c|_{[-T,T]} \in FS(X)$  by

$$c|_{[-T,T]}(t) := \begin{cases} c(-T) & \text{if } t \le -T; \\ c(t) & \text{if } -T \le t \le T; \\ c(T) & \text{if } t \ge T. \end{cases}$$

Obviously  $c|_{[-\infty,\infty]} = c$  and if  $c \notin FS(X)^{\mathbb{R}}$  and  $(-T,T) \cap (c_-,c_+) \neq \emptyset$  then  $(c|_{[-T,T]})_- = \max\{c_-,-T\}$  and  $(c|_{[-T,T]})_+ = \min\{c_+,T\}$ .

We denote by

$$FS(X)_f := \left\{ c \in FS(X) \setminus FS(X)^{\mathbb{R}} \mid c_- > -\infty, c_+ < \infty \right\} \cup FS(X)^{\mathbb{R}}$$

the subspace of finite geodesics.

**Lemma 22.18.** Let  $(X, d_X)$  be a metric space. The map H:  $FS(X) \times [0, 1] \rightarrow FS(X)$ defined by  $H_{\tau}(c) := c|_{[\ln(\tau), -\ln(\tau)]}$  is continuous and satisfies  $H_0 = id_{FS(X)}$  and  $H_{\tau}(c) \in FS(X)_f$  for  $\tau > 0$ .

*Proof.* See [80, Lemma 1.14].

### 22.4 The Flow Space Associated to a CAT(0)-Space

In this section we study FS(X) further in the case where X is a CAT(0)-space.

For the definition of a CAT(0)-space we refer to [165, Definition 1.1 in Chapter II.1 on page 158], namely, it is a geodesic space all of whose geodesic triangles satisfy the CAT(0)-inequality. We will follow the notation and the description of the *bordification*  $\overline{X} = X \cup \partial X$  of a CAT(0)-space X given in [165, Chapter II.8]. The definition of the topology of this bordification is briefly reviewed in Remark 22.20. In this section we will use the following convention.

- Let *X* be a complete CAT(0)-space;
- Let  $\overline{X} := X \cup \partial X$  be the bordification of X, see [165, Chapter II.8].

### 22.4.1 Evaluation of Generalized Geodesics at Infinity

**Definition 22.19.** For  $c \in FS(X)$  we set  $c(\pm \infty) := \lim_{t \to \pm \infty} c(t)$ , where the limit is taken in  $\overline{X}$ .

Since X is by assumption a CAT(0)-space, we can find for  $x_{-} \in X$  and  $x_{+} \in \overline{X}$ a generalized geodesic  $c : \mathbb{R} \to X$  with  $c(\pm \infty) = x_{\pm}$ , see [165, Proposition 8.2 in Chapter II.8 on page 261]. It is not true in general that for two different points  $x_{-}$ and  $x_{+}$  in  $\partial X$  there is a geodesic c with  $c(-\infty) = x_{-}$  and  $c(\infty) = x_{+}$ .

**Remark 22.20 (Cone topology on**  $\overline{X}$ .). A generalized geodesic ray is a generalized geodesic c that is either a constant generalized geodesic or a non-constant generalized geodesic with  $c_{-} = 0$ . Fix a base point  $x_0 \in X$ . For every  $x \in \overline{X}$ , there is a unique generalized geodesic ray  $c_x$  such that  $c(0) = x_0$  and  $c(\infty) = x$ , see [165, Proposition 8.2 in Chapter II.8 on page 261]. Define for r > 0

$$\rho_r = \rho_{r,x_0} \colon X \to B_r(x_0)$$

by  $\rho_r(x) := c_x(r)$ . The sets  $(\rho_r)^{-1}(V)$  with r > 0, V an open subset of  $\overline{B}_r(x_0)$  are a basis for the cone topology on  $\overline{X}$ , see [165, Definition 8.6 in Chapter II.8 on page 263]. A map f whose target is  $\overline{X}$  is continuous if and only if  $\rho_r \circ f$  is continuous for all r > 0. The cone topology is independent of the choice of base point, see [165, Proposition 8.8 in Chapter II.8 on page 264].

Lemma 22.21. The maps

$$FS(X) \setminus FS(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(-\infty);$$
  
$$FS(X) \setminus FS(X)^{\mathbb{R}} \to \overline{X}, \quad c \mapsto c(\infty),$$

are continuous.

Proof. See [80, Lemma 2.4].

**Proposition 22.22.** If X is proper as a metric space, then the map

$$E: \operatorname{FS}(X) \setminus \operatorname{FS}(X)^{\mathbb{R}} \to \overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$$

defined by  $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$  is injective and continuous. It is a homeomorphism onto its image.

*Proof.* See [80, Proposition 2.6].

Recall that  $FS(X)_f$  is the subspace of finite geodesics, see Notation 22.17.

**Proposition 22.23.** Assume that X is proper as a metric space. Then the map

 $E_f: \operatorname{FS}(X)_f \setminus \operatorname{FS}(X)^{\mathbb{R}} \to \mathbb{R} \times X \times X$ 

defined by  $E_f(c) = (c_-, c(-\infty), c(\infty))$  is a homeomorphism onto its image

$$\operatorname{im} E_f = \{(r, x, y) \mid x \neq y\}.$$

In particular,  $FS(X)_f \setminus FS(X)^{\mathbb{R}}$  is locally path connected.

Proof. See [80, Proposition 2.7].

22.4.2 Dimension of the Flow Space

**Lemma 22.24.** If X is proper as a metric space and its dimension dim X is  $\leq N$ , then dim  $\overline{X} \leq N$ .

Proof. See [80, Lemma 2.8].

**Proposition 22.25.** *Assume that X is proper and that* dim  $X \le N$ *. Then* 

$$\dim(\mathrm{FS}(X)\setminus\mathrm{FS}(X)^{\mathbb{R}})\leq 3N+2.$$

*Proof.* See [80, Proposition 2.9].

## 22.4.3 The Example of a Complete Riemannian Manifold with Non-Positive Sectional Curvature

Let *M* be a simply connected complete Riemannian manifold with non-positive sectional curvature. It is a CAT(0)-space with respect to the metric coming from the Riemannian metric, see [165, Theorem I.A.6 on page 173]. Let *STM* be its sphere tangent bundle. For every  $x \in M$  and  $v \in ST_xM$  there is precisely one geodesic  $c_v : \mathbb{R} \to M$  for which  $c_v(0) = x$  and  $c'_v(0) = v$  holds. Given a geodesic  $c : \mathbb{R} \to M$  in *M* and  $a_-, a_+ \in \overline{\mathbb{R}}$  with  $a_- \leq a_+$ , define the generalized geodesic

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 $c_{[a_-,a_+]}: \mathbb{R} \to M$  by sending t to  $c(a_-)$  if  $t \le a_-$ , to c(t) if  $a_- \le t \le a_+$ , and to  $c(a_+)$  if  $t \ge a_+$ . Obviously  $c_{[-\infty,\infty]} = c$ . Let  $d: \mathbb{R} \to M$  be a generalized geodesic with  $d_- < d_+$ . Then there is precisely one geodesic  $\widehat{d}: \mathbb{R} \to M$  with  $\widehat{d}_{[d_-,d_+]} = d$ . Define maps

$$\alpha: STM \times \{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\} \to FS(M), \ (v, a_i, a_+) \mapsto c_v|_{[a_-, a_+]}; \\ \beta: FS(M) \to STM \times \{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\}, \ c \mapsto (\widehat{c}'(0), c_-, c_+).$$

Then  $\alpha$  and  $\beta$  are mutually inverse homeomorphisms. They are compatible with the flow on FS(*M*) of Definition 22.5, if one uses on  $STM \times \{(a_i, a_+) \in \mathbb{R} \times \mathbb{R} \mid a_- < a_+\}$  the product flow given by the geodesic flow on *STM* and the flow on  $\overline{R}$  which is at time *t* given by the homeomorphism  $\mathbb{R} \to \mathbb{R}$  sending  $s \in \mathbb{R}$  to  $s - t, -\infty$  to  $-\infty$ , and  $\infty$  to  $\infty$ .

# 22.5 The Dynamical Properties of the Flow Space Associated to a CAT(0)-Space

In Definition 22.27 we introduce the homotopy action that we will use to show that CAT(0)-groups are strongly transfer reducible over  $\mathcal{VCY}$ . It will act on a large closed ball in *X*. (The action of *G* on the bordification  $\overline{X}$  is not suitable, because it has too large isotropy groups.) In Theorem 22.31, which is based on Proposition 22.30, we study the dynamics of the flow with respect to the homotopy action. The analog of Proposition 22.30 in the hyperbolic case is Theorem 22.34.

Throughout this section we fix the following convention:

- Let  $(X, d_X)$  be a CAT(0)-space which is proper as a metric space;
- Let  $x_0 \in X$  be a fixed base point;
- Let G be a group with a proper isometric action on  $(X, d_X)$ .

For  $x, y \in X$  and  $t \in [0, 1]$  we will denote by  $t \cdot x + (1-t) \cdot y$  the unique point z on the geodesic from x to y such that  $d_X(x, z) = td_X(x, y)$  and  $d_X(z, y) = (1-t)d_X(x, y)$ . For  $x, y \in X$  we will denote by  $c_{x,y}$  the generalized geodesic determined by  $(c_{x,y})_- = 0, c(-\infty) = x$  and  $c(\infty) = y$ . By [165, Proposition 1.4 (1) in Chapter II.1 on page 160] and Proposition 22.10,  $(x, y) \mapsto c_{x,y}$  defines a continuous map  $X \times X \to FS(X)$ . Note that  $g \cdot c_{x,y} = c_{gx,gy}$ .

### 22.5.1 The Homotopy Action on $\overline{B}_R(x)$

The next definition is a variation of some of the notions appearing in Section 20.5

**Definition 22.26 (Homotopy** *S***-action).** Let *S* be a finite subset of a group *G*. Assume that *S* contains the trivial element  $e \in G$ . Let *X* be a space.

- (i) A homotopy S-action (φ, H) on X consists of continuous maps φ<sub>g</sub>: X → X for g ∈ S and homotopies H<sub>g,h</sub>: X × [0, 1] → X for g, h ∈ S with gh ∈ S such that H<sub>g,h</sub>(-, 0) = φ<sub>g</sub> ∘ φ<sub>h</sub> and H<sub>g,h</sub>(-, 1) = φ<sub>gh</sub> holds for g, h ∈ S with gh ∈ S. Moreover, we require that H<sub>e,e</sub>(-,t) = φ<sub>e</sub> = id<sub>X</sub> for all t ∈ [0, 1];
- (ii) Let  $(\varphi, H)$  be a homotopy *S*-action on *X*. For  $g \in S$  let  $F_g(\varphi, H)$  be the set of all maps  $X \to X$  of the form  $x \mapsto H_{r,s}(x, t)$  where  $t \in [0, 1]$  and  $r, s \in S$  with rs = g;
- (iii) Let  $(\varphi, H)$  be a homotopy *S*-action on *X*. For  $(g, x) \in G \times X$  and  $n \in \mathbb{N}$ , let  $S^n_{\varphi,H}(g, x)$  be the subset of  $G \times X$  consisting of all (h, y) with the following property: There are  $x_0, \ldots, x_n \in X$ ,  $a_1, b_1, \ldots, a_n, b_n \in S$ ,  $f_1, \tilde{f}_1, \ldots, f_n, \tilde{f}_n \colon X \to X$ , such that  $x_0 = x, x_n = y$ ,  $f_i \in F_{a_i}(\varphi, H)$ ,  $\tilde{f}_i \in F_{b_i}(\varphi, H)$ ,  $f_i(x_{i-1}) = \tilde{f}_i(x_i)$  and  $h = ga_1^{-1}b_1 \cdots a_n^{-1}b_n$ ;
- (iv) Let  $(\varphi, H)$  be a homotopy *S*-action on *X* and  $\mathcal{U}$  be an open cover of  $G \times X$ . We say that  $\mathcal{U}$  is *S*-long with respect to  $(\varphi, H)$  if for every  $(g, x) \in G \times X$  there is a  $U \in \mathcal{U}$  containing  $S_{\varphi, H}^{|S|}(g, x)$ , where |S| is the cardinality of *S*.

Recall that for r > 0 and  $z \in X$  we denote by  $\rho_{r,z} \colon X \to \overline{B}_r(z)$  the canonical projection along geodesics, i.e.,  $\rho_{r,z}(x) = c_{z,x}(r)$ , see also Remark 22.20. Note that  $g \cdot \rho_{r,z}(x) = \rho_{r,gz}(gx)$  for  $x, z \in X$  and  $g \in G$ .

**Definition 22.27 (The homotopy** *S***-action on**  $\overline{B}_R(x_0)$ ). Let  $S \subseteq G$  be a finite subset of *G* with  $e \in G$  and R > 0. Define a homotopy *S*-action ( $\varphi^R, H^R$ ) on  $\overline{B}_R(x)$  in the sense of Definition 22.26 (i) as follows. For  $g \in S$ , we define the map

$$\varphi_{\varphi}^{R} \colon \overline{B}_{R}(x_{0}) \to \overline{B}_{R}(x_{0})$$

by  $\varphi_g^R(x) := \rho_{R,x_0}(gx)$ .

For  $g, h \in S$  with  $gh \in S$  we define the homotopy

**D** 

$$H^R_{g,h} \colon \varphi^R_g \circ \varphi^R_h \simeq \varphi^R_{gh}$$

by  $H_{g,h}^{R}(x,t) := \rho_{R,x_0} (t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx))).$ 

**Remark 22.28.** Note that  $H_{g,h}^R$  is indeed a homotopy from  $\varphi_g^R \circ \varphi_h^R$  to  $\varphi_{gh}$  because of

$$\begin{aligned} H_{g,h}^{R}(x,0) &= \rho_{R,x_0} \big( 0 \cdot (ghx) + 1 \cdot (g \cdot \rho_{R,x_0}(hx)) \big) \\ &= \rho_{R,x_0} \big( g \cdot \rho_{R,x_0}(hx) \big) \\ &= \varphi_g^R \circ \varphi_h^R(x), \end{aligned}$$

and

$$\begin{aligned} H_{g,h}^{R}(x,1) &= \rho_{R,x_{0}} (1 \cdot (ghx) + 0 \cdot (g \cdot \rho_{R,x_{0}}(hx))) \\ &= \rho_{R,x_{0}}(ghx) \\ &= \varphi_{gh}^{R}(x). \end{aligned}$$

22.6 The Flow Space Associated to a Hyperbolic Metric Complex

It turns out that the more obvious homotopy given by convex combination  $(x, t) \mapsto t \cdot \varphi_{ab}^R(x) + (1-t) \cdot \varphi_g^R \circ \varphi_b^R(x)$  is not appropriate for our purposes.

**Definition 22.29 (The map**  $\iota$ ). Define the map

$$\iota \colon G \times X \to \mathrm{FS}(X)$$

as follows. For  $(g, x) \in G \times X$  let  $\iota(g, x) := c_{gx_0, gx}$ .

The map  $\iota$  is *G*-equivariant for the *G*-action on  $G \times X$  defined by  $g \cdot (h, x) = (gh, x)$ .

### 22.5.2 The Flow Estimate

**Proposition 22.30.** Let  $\beta$ , L > 0. For all  $\delta > 0$ , there are T, r > 0 such that for  $x_1, x_2 \in X$  with  $d_X(x_1, x_2) \leq \beta$ ,  $x \in \overline{B}_{r+L}(x_1)$  there is a  $\tau \in [-\beta, \beta]$  satisfying

$$d_{\text{FS}(X)}(\Phi_T(c_{x_1,\rho_{r,x_1}(x)}),\Phi_{T+\tau}(c_{x_2,\rho_{r,x_2}(x)})) \le \delta$$

Proof. See [80, Proposition 3.5].

**Theorem 22.31 (Flow estimates in the CAT(0)-case).** *Let S be a finite subset of G (containing e). Then there is a*  $\beta > 0$  *such that the following holds:* 

For all  $\delta > 0$ , there are T, R > 0 such that for every  $(a, x) \in G \times \overline{B}_R(X)$ ,  $s \in S$ , and  $f \in F_s(\varphi^R, H^R)$  there is  $\tau \in [-\beta, \beta]$  satisfying

$$d_{\mathrm{FS}(X)}\left(\Phi_T(\iota(a,x)), \Phi_{T+\tau}(\iota(as^{-1},f(x)))\right) \le \delta.$$

Proof. See [80, Proposition 3.8].

### 22.6 The Flow Space Associated to a Hyperbolic Metric Complex

In the proofs of the Farrell-Jones Conjecture for hyperbolic groups, see [86, 87], a construction of a flow space FS(X) based on a construction of Mineyev [731] is used. (Note that a mistake in [731] was fixed by Mole [749].) Although one no longer needs Mineyev's construction in the proofs and one can get along with the construction presented in Section 22.3, we still briefly recall what happens in the original proofs for hyperbolic groups as an illustration for the reader and a hint how the techniques have changed over time.

If *X* is hyperbolic metric space with compactification  $\overline{X}$  and  $x_0 \in X$  a base point, there is a specific map, see [87, (8.1)],

(22.32) 
$$\iota_{x_0} \colon X \times \overline{X} \to FS(X)$$

such that the following flow estimate holds.

**Lemma 22.33.** The map  $\iota_{x_0} \colon X \times \overline{X} \to FS(X)$  from (22.32) is continuous. It is Isom(X)-equivariant with respect to the diagonal Isom(X)-action on the source where Isom(X) is the group of isometric self-homeomorphisms of X. For  $x \in X$  the map  $\iota_{x_0}(x, -) \colon \overline{X} \to FS(X)$ ,  $y \mapsto \iota_{x_0}(x, y)$  is injective.

Proof. See [86, Lemma 8.4].

**Theorem 22.34 (Flow estimate in the hyperbolic case).** Let  $\lambda \in (e^{-1}, 1)$  and  $T \in [0, \infty)$  be the constants depending only on X which appear in [86, Proposition 6.4]. Consider  $a, b \in X$  and  $c \in \overline{X}$ . Put

$$N = 2 + \frac{2}{\lambda^T \cdot (-\ln(\lambda))}$$

Then there exists a real number  $\tau_0$  such that

$$|\tau_0| \le 2 \cdot d(a,b) + 5$$

holds for the new metric  $\hat{d}$  on X defined in [731, Lemma 2.7 on page 449 and Theorem 32 on page 446] and we get for all  $\tau \in \mathbb{R}$ 

$$d_{\mathrm{FS},x_0}(\phi_{\tau} \circ \iota_{x_0}(a,c),\phi_{\tau+\tau_0} \circ \iota_{x_0}(b,c)) \leq \frac{N}{1-\ln(\lambda)^2} \cdot \lambda^{-\widehat{d}(a,b)} \cdot \lambda^{\tau}.$$

*Proof.* See [86, Theorem 8.6].

We recommend the reader to compare Theorem 22.34 with Proposition 22.30. The baby version of these two results was already discussed in Lemma 19.14.

### 22.7 Topological Dimension

Let X be a topological space. Let  $\mathcal{U}$  be an open cover. Its *dimension* dim $(\mathcal{U}) \in \{0, 1, 2, ...\} \amalg \{\infty\}$ , sometimes also called its *order*, is the infimum over all integers  $d \ge 0$  such that for any collection  $U_0, U_1, ..., U_{d+1}$  of pairwise distinct elements in  $\mathcal{U}$  the intersection  $\bigcap_{i=0}^{d+1} U_i$  is empty. An open covering  $\mathcal{V}$  is a refinement of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$  there is a  $U \in \mathcal{U}$  with  $V \subseteq U$ .

**Definition 22.35 ((Topological) dimension).** The *dimension* (sometimes also called the *topological dimension* or *covering dimension*) of a topological space *X* 

$$\dim(X) \in \{0, 1, 2, ...\} \amalg \{\infty\}$$

is the infimum over all integers  $d \ge 0$  such that any open covering  $\mathcal{U}$  possesses a refinement  $\mathcal{V}$  with dim $(\mathcal{V}) \le d$ .

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22.7 Topological Dimension

We state some basic properties of the dimension.

**Lemma 22.36.** If A is a closed subset of X, then  $\dim(A) \leq \dim(X)$ .

Exercise 22.37. Give the proof of Lemma 22.36.

**Lemma 22.38.** Let X be the union  $A_1 \cup A_2 \cup \cdots \cup A_k$  of closed subspaces  $A_i \subseteq X$ . Then

$$\dim(X) = \max\{\dim(A_i) \mid i = 1, 2, ..., k\}.$$

Proof. See [754, Corollary 9.3 on page 304].

**Lemma 22.39.** Let Z be a proper metric space. Suppose that G acts on Z isometrically and properly. Then we get for the topological dimensions of Z and  $G \setminus Z$ 

$$\dim(G\backslash Z) \le \dim(Z).$$

Proof. See [658, Lemma 3.2].

**Theorem 22.40.** Let X be a locally compact Hausdorff space having a countable basis for its topology. Suppose that every point of X has a neighborhood whose closure has topological dimension at most m. Then X has topological dimension at most m.

*Proof.* See [754, Exercise 9 on page 315].

A locally compact *CW*-complex which is second countable, i.e., has a countable basis for its topology, has the property that its topological dimension  $\dim(X)$  is the same as its dimension as a *CW*-complex. This follows from Theorem 22.40. Note that for a connected *CW*-complex, locally compact, metrizable, first countable, and locally finite are equivalent conditions, see [408, Theorem B and Proposition 2.4].

Again by Theorem 22.40, a topological m-dimensional manifold M has topological dimension m.

**Lemma 22.41.** Let n be an integer with  $n \ge 0$ . Let X be a proper metric space whose topological dimension satisfies  $\dim(X) \le n$ . Suppose that G acts properly and isometrically on X.

Then there exists a proper n-dimensional G-CW-complex Y together with a G-map  $f: X \rightarrow Y$ .

*Proof.* See [658, Lemma 3.7].

There is also the notion of a small inductive limit, see [342, Definition 1.1] or [572, Definition 3.1] which agrees with the notion of the topological dimension for second countable metric spaces, see [342, Theorem 1.7.7].

### 22.8 Long and Thin Covers

The next result is proved in [572, Theorem 1.1] based on ideas from [86, Theorem 1.4] and [80, Theorem 5.6] which in turn are motivated by [359, Proposition 7.2].

**Notation 22.42.** Let  $(X, d_X)$  be a metric space. For a subset  $A \subseteq X$  and  $\delta > 0$ , we define

$$B_{\delta}(A) := \{ y \in X \mid \exists a \in A \text{ with } d_X(y, a) < \delta \}; \\ \overline{B}_{\delta}(A) := \{ x \in X \mid \exists a \in A \text{ with } d_X(a, y) \le \delta \}.$$

Given  $x \in X$ , we write

$$B_{\delta}(x) := B_{\delta}\{x\}) = \{y \in X \mid d_X(y, x) < \delta\};$$
  
$$\overline{B}_{\delta}(x) := \overline{B}_{\delta}(\{x\}) = \{y \in X \mid d_X(y, x) \le \delta\}.$$

We call  $B_{\delta}(x)$  the open and  $\overline{B}_{\delta}(x)$  the closed ball around x of radius  $\delta$ .

Note that the open ball  $B_{\delta}(x)$  is an open subset of X, the closed ball  $B_{\delta}(x)$  is a closed subset of X, and  $\overline{B}_{\delta}(A)$  contains the closure  $\overline{B_{\delta}(A)}$  of  $B_{\delta}(A)$  in X, but  $\overline{B_{\delta}(A)}$  and  $\overline{B}_{\delta}(A)$  are not equal in general.

**Theorem 22.43 (Long and thin covers).** *Let G be a countable discrete group. Let X be a flow G-space such that the underlying topological space X is finite-dimensional, locally compact, and second countable. Let*  $\alpha > 0$  *and*  $\delta > 0$  *be real numbers.* 

Then there exists an open  $\mathcal{VCY}$ -cover  $\mathcal{U}$  of X in the sense of Definition 20.18 of dimension at most  $7 \dim(X) + 7$  which is long and thin in the following sense:

- (Long) For every point  $x \in X$  there is a  $U \in \mathcal{U}$  with  $\Phi_{[-\alpha,\alpha]}(x) \subseteq U$ ;
- (*Thin*) For every  $U \in \mathcal{U}$  there is a point  $x \in X$  with  $U \subseteq B_{\delta}(\Phi_{\mathbb{R}}(x))$ .

The long and thin covers are generalizations of the long and thin cells from [359, Proposition 7.2].

A basic strategy of the proof of Theorem 22.43 presented in [572, Theorem 1.1] consists of decomposing the flow space into three parts, a part without a short G-period, a non-periodic part with short G-period, and the periodic part with short G-period, constructing for each part an appropriate Vcyc-cover, and finally taking the union of these covers.

**Definition 22.44 (Strong contracting transfers).** A flow *G*-space *Y* admits *strong contracting transfers* if there is a natural number *N* such that, for every finite subset  $S \subseteq G$  and every natural number *k*, there is a real number  $\beta > 0$  such that for every  $\delta > 0$  there is a real number T > 0 such that there exists:

- An *N*-transfer space *X* in the sense of Definition 20.9;
- A strong homotopy action  $\Gamma$  on X in the sense of Definition 20.32;

### 22.9 Notes

• A *G*-equivariant map  $\iota: G \times X \to Y$ , where the *G*-action on  $G \times X$  is given by  $g' \cdot (g, x) = (g'g, x)$ , with the property that for every  $(g, x) \in G \times X$ , every  $s \in S$ , and every  $f \in F_g(\Gamma, S, k)$  there exists a  $\tau \in [-\beta, \beta]$  satisfying

$$d_Y(\Phi_T \circ \iota(g, x), \Phi_{\tau+T} \circ \iota(gs^{-1}, f(x))) \le \delta.$$

The next result follows from [572, Corollary 1.2] and Theorem 20.39.

**Theorem 22.45.** Let X be a flow G-space whose underlying space X is finitedimensional and the G-action on X is cocompact. Suppose that X admits strong contracting transfers.

Then G is strongly transfer  $\mathcal{VCY}$ -reducible in the sense of Definition 20.38. In particular, G satisfies the Full Farrell-Jones Conjecture 13.30.

A key ingredient in the proof that for a hyperbolic or a CAT(0)-group Theorem 22.45 applies, i.e., that the flow spaces admit strong contracting transfers, are the flow estimates as they appear for instance in Theorem 22.31 and Theorem 22.34. The basic idea of the proof of Theorem 22.45 is to pull back an appropriate  $\mathcal{F}$ -cover of the flow space *Y* coming from Theorem 22.43 back to  $G \times X$  using the map  $\iota$ .

**Remark 22.46.** Kasprowski and Rüping [571, Theorem 6.1] show using Theorem 22.45 that the *K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with finite wreath products, see Conjecture 13.27, and the *L*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories with involution with finite wreath products, see Conjecture 13.28, hold for a class of group *G* which encompasses all hyperbolic and CAT(0)-groups. It contains for instance all groups *G* which acts properly and cocompactly on a finite product of hyperbolic graphs, see [571, Theorem 1.1] Their proof also applies to the Full Farrell-Conjecture 13.30 because of Remark 20.3 and Theorem 20.62. There are groups which are neither hyperbolic nor CAT(0)-groups and belong to the class of group  $\pi_1(STF)$  of the sphere tangent bundle of a hyperbolic closed surface *F*, see [571, Section 3]. (Note that it is well-known that  $\pi_1(STF)$  satisfies the Full Farrell-Conjecture 13.30, see Theorem 16.1 (ie).)

### **22.9** Notes

Bartels developed coarse flow spaces in [68] when he was dealing with the Farrell-Jones Conjecture for relative hyperbolic groups. One advantage of his approach is that one no longer needs Mineyev's flow space, see [731]. Moreover, this point of view was essential in the proof of the Full Farrell-Jones Conjecture for mapping class groups due to Bartels-Bestvina [70], as in this setting no flow space is available. The coarse flow space still allows the construction of long and thin covers.

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The case of a reductive *p*-adic group acting on the CAT(0)-space given by its Bruhat-Tits building is analyzed in Bartels-Lück [82]. Completed Kac-Moody groups are treated in Bartels-Lück-Witzel [91].

### Chapter 23 Transfer

### 23.1 Introduction

In this chapter we give more information about the transfer which we have already mentioned in Section 19.5 and which plays a prominent role in nearly all proofs of the Farrell-Jones Conjecture. For simplicity we refine ourselves to the Whitehead group. Nevertheless, we will convey all the basic ideas which will enter in the proof of the general case.

In Section 23.2 we will consider the classical geometric transfer associated to an appropriate fibration  $F \rightarrow E \rightarrow B$  of connected finite *CW*-complexes. It is a homomorphism  $p^*$ : Wh( $\pi_1(B)$ )  $\rightarrow$  Wh( $\pi_1(E)$ ) which is given by the pullback construction with *p* applied to homotopy equivalences from finite *CW*-complexes to *B* and the notion of Whitehead torsion. We will explain its algebraic description in Section 23.3.

Section 23.4 is devoted to the down-up-formula which computes the composite

Wh $(\pi_1(B)) \xrightarrow{p^*} Wh(\pi_1(E)) \xrightarrow{p_*} Wh(\pi_1(B))$  for  $p_*$  the map induced by the group homomorphism  $\pi_1(p) : \pi_1(E) \to \pi_1(B)$ . It is given in terms of the  $\pi_1(B)$ -operation on the homology of the fiber. It implies that  $p_* \circ p^*$  is given by multiplication by the Euler characteristic  $\chi(F)$ , if p is orientable in the sense that its fiber transport is trivial. In particular,  $p^*$  is bijective if F is contractible.

In Sections 23.5 and 23.6 we will consider transfer maps in more general situations, which will be needed in the proofs of the Farrell-Jones Conjecture.

In Section 23.7 we consider a finitely generated group G and a family of subgroups  $\mathcal{F}$  such that G is strictly  $\mathcal{F}$ -transfer reducible in the sense of Definition 20.11 and show that the assembly map

$$H_1^G(\mathrm{pr}; \mathbf{K}_{\mathbb{Z}}) \colon H_1^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_1^G(G/G; \mathbf{K}_{\mathbb{Z}}) = K_1(\mathbb{Z}G)$$

is surjective. This is a special case of Theorem 20.12 and will illustrate the methods of proof for the Farrell-Jones Conjecture. Reducing to this special case avoids some formidable purely technical input, which will make the exposition much harder, but will be discussed later in Chapter 24.

We discuss a general strategy how to prove the Farrell-Jones Conjecture using transfers in Section 23.8.

### 23.2 The Geometric Transfer

Let  $F \to E \xrightarrow{p} B$  be a fibration such that F and B have the homotopy type of a connected finite *CW*-complex. Then *E* also has the homotopy type of a connected finite *CW*-complex, see for instance [373, Section 3], [640], [682, Section 1].

A simple structure  $\xi$  on a space Y of the homotopy type of a connected finite CW-complex is an equivalence class of homotopy equivalences  $u_0: Z \xrightarrow{\simeq} Y$  with some connected finite CW-complex Z as source, where we call two such homotopy equivalences  $u_k: Z_k \xrightarrow{\simeq} Y$  for k = 0, 1 equivalent if there is a simple homotopy equivalence  $v: Z_0 \xrightarrow{\simeq_s} Z_1$  such that  $u_1 \circ v$  and  $u_0$  are homotopic. Of course a connected finite CW-complex Y has a preferred simple structure given by  $id_Y$ .

Given a homotopy equivalence  $f: (Y_0, \xi_0) \xrightarrow{\cong} (Y_1, \xi_1)$  of spaces coming with simple structures, we define its Whitehead torsion  $\tau(f) \in Wh(\pi_1(Y_1))$  to be the image of the Whitehead torsion  $\tau(w) \in Wh(\pi_1(Z_1))$  of a homotopy equivalence  $w: Z_0 \xrightarrow{\cong} Z_1$  under the isomorphism  $(u_1)_*: Wh(\pi_1(Z_1)) \xrightarrow{\cong} Wh(\pi_1(Y_1))$ , where we have chosen representatives  $u_k: Z_k \xrightarrow{\cong} Y_k$  of the simple structures  $\xi_k$  for k = 0, 1and require  $u_1 \circ w \simeq f \circ u_0$ . One easily checks that this is independent of all the choices using Theorem 3.37.

Next we define a class  $\theta(p) \in H^1(B; Wh(\pi_1(E)))$ . It is given after a choice of a base point  $b \in B$  and a simple structure on the fiber  $F_b = p^{-1}(b)$  by the homomorphism  $\pi_1(B, b) \to Wh(\pi_1(E))$  which sends  $w \in \pi_1(B, b)$  to the image of Whitehead torsion of the homotopy equivalence  $F_b \to F_b$  given by the fiber transport along w under the homomorphism  $Wh(\pi_1(F_b)) \to Wh(\pi_1(E))$  coming from the inclusion  $F_b \to E$ . Recall that the *fiber transport* is a homomorphism of monoids  $\pi_1(B, b) \to [F_b, F_b]$ , see for instance [943, 15.12 on page 343], [1006, page 186]. One easily checks that this is well-defined, in particular that it is independent of the choice of base points and the simple structure on  $F_b$ .

If  $\theta(p) \in H^1(B; Wh(\pi_1(E)))$  vanishes and we have fixed simple structures  $\xi_F$ and  $\xi_B$  on *F* and *B*, then there is a preferred simple structure on  $\xi_E$  on *E*, see [373, Section 3], [640], [682, Section 1]. In the sequel we will assume that the characteristic class  $\theta(p) \in H^1(B; Wh(\pi_1(F)))$  vanishes, which is the case if *p* satisfies one of the following conditions:

- the fibration is orientable, i.e., the fiber transport  $\pi_1(B) \rightarrow [F, F]$  is trivial;
- The map  $\pi_1(F) \to \pi_1(E)$  is zero, or, equivalently,  $\pi_1(p) \colon \pi_1(E) \to \pi_1(B)$  is bijective;
- *p* is a locally trivial topological bundle with a connected finite *CW*-complex *F* as fiber;
- Wh( $\pi_1(F)$ ) or Wh( $\pi_1(E)$ ) vanishes.

Let M be a closed topological manifold. Then, by Kirby-Siebenmann [579, Essay III, Theorem 4.1 on page 118], there is a preferred simple structure

(23.1) 
$$\xi^{\text{top}}(M) \text{ on } M,$$

23.2 The Geometric Transfer

which is defined by considering any triangulation of an embedding of a closed disk bundle over M as a codimension zero submanifold into Euclidean space. The simple structure on the disk bundle obtained from the triangulation induces the preferred simple structure on M via the homotopy equivalence given by the inclusion of Minto the disk bundle. This simple structure agrees with the one obtained by any triangulation or by any handlebody decomposition (more generally what they call TOP s-decomposition) of M, whenever they exist, see [579, Essay III, Theorem 5.10 on page 131 and Theorem 5.11 on page 132]. Let  $F \to M \to B$  be a locally trivial bundle of connected closed topological manifolds. Then  $\Theta(p)$  is trivial. If we equip B and F with the simple structures  $\xi^{top}(M)$  of (23.1), then the induced simple structure on the total space  $\xi_M$  agrees with  $\xi^{top}(M)$ , see [373, Lemma 3.16].

Consider  $\alpha \in Wh(\pi_1(B))$ . Let  $f: X \to B$  be a homotopy equivalence with a connected finite *CW*-complex as source satisfying  $\tau(f) = \alpha$ . Consider the following pullback



We conclude  $\Theta(\overline{p}) = 0$  from our assumption  $\Theta(p) = 0$ . Hence there is a preferred simple structure on both  $\overline{X}$  and E and the Whitehead torsion  $\tau(\overline{f}) \in Wh(\pi_1(E))$  is defined. The *geometric transfer* 

(23.2) 
$$p^* \colon \operatorname{Wh}(\pi_1(B)) \to \operatorname{Wh}(\pi_1(E))$$

is defined by the equality  $p^*(\alpha) = \tau(\overline{f})$ . The proof that this construction is welldefined can be found in Anderson [28, 29] for locally trivial PL-bundles and in the more general setting above in [640], [373, Section 3]. The equivariant version of this construction is presented in detail in [644, Section 15].

**Example 23.3.** Let  $F \to M \to B$  be a locally trivial bundle of connected closed topological manifolds. Let W = (W, B, B', f, f') be a topological *h*-cobordism over *B*. Choose a retraction *r* of the homotopy equivalence  $B \xrightarrow{f} \partial_0 W \to W$ . Consider the pullback

$$\begin{array}{c|c} \overline{W} & \xrightarrow{\overline{r}} & E \\ \hline p & & & \downarrow p \\ W & \xrightarrow{r} & B. \end{array}$$

Then  $\overline{W}$  is a topological *h*-cobordism over *E* and the transfer homomorphism  $p^*$ : Wh $(\pi_1(B)) \rightarrow$  Wh $(\pi_1(E))$  of (23.2) sends the Whitehead torsion of *W*, see (3.48), to the Whitehead torsion of  $\overline{W}$ .

### 23.3 The Algebraic Transfer

Next we describe the algebraic version of the geometric transfer. Let R and S be rings.

**Definition 23.4 (Chain homotopy representation).** A *chain homotopy representation*  $(C_*, U)$  consists of an S-chain complex  $C_*$  and a ring homomorphism  $U_*: R \to [C_*, C_*]_S$  to the ring of S-chain homotopy classes of S-self-chain homotopy equivalences  $C_* \to C_*$ , where the multiplicative structure comes from composition.

For a matrix  $A = (a_{i,j})$  in  $GL_n(R)$ , we get a well-defined S-chain homotopy class of S-chain homotopy equivalences  $U_*(A)$ :  $\bigoplus_{i=1}^n C_* \xrightarrow{\simeq} \bigoplus_{i=1}^n C_*$  by the (n, n)matrix  $(U(a_{i,j}))_{i,j}$  of S-chain homotopy classes of S-chain maps  $U(a_{i,j})$ :  $C_* \to C_*$ . Suppose that  $C_*$  is a finite free S-chain complex. Choose a basis for  $C_*$ . Then  $\bigoplus_{i=1}^n C_*$ is a finite free S-chain complex which comes with a basis, and hence the Whitehead torsion  $\tau(U_*(A)) \in \widetilde{K}_1(S)$  of  $U_*(A)$ :  $\bigoplus_{i=1}^n C_* \xrightarrow{\simeq} \bigoplus_{i=1}^n C_*$  is defined, see (3.33). One easily checks that  $\tau(U_*(A))$  is independent of the choice of the basis on  $C_*$ . We obtain a well-defined homomorphism of abelian groups

$$(23.5) p_{II}^* \colon \widetilde{K}_1(R) \to \widetilde{K}_1(S)$$

by sending the class of [A] of A to  $\tau(U_*(A))$ . Although it is not relevant for us here, we mention that using the self-torsion of Subsection 23.7.3 one can define a map  $p_U^*: K_1(R) \to K_1(S)$  which induces the map (23.5) and is defined in the more general case where  $C_*$  is a finitely dominated projective *S*-chain complex. All of this is explained in [641, Section 4].

Given a fibration  $F \to E \xrightarrow{p} B$  such that F is a connected finite CWcomplex and B is path connected, one can assign to it using the fiber transport a chain homotopy representation U(p) for  $R = \mathbb{Z}[\pi_1(B)]$  and  $S = \mathbb{Z}[\pi_1(E)]$ whose underlying  $\mathbb{Z}[\pi_1(E)]$ -chain complex is finite free. In the special case when  $\pi_1(p): \pi_1(E) \xrightarrow{\cong} \pi_1(B)$  is bijective, it is defined as follows. Take  $C_*(F)$  to be the cellular  $\mathbb{Z}$ -chain complex of F. Put  $C_* = \mathbb{Z}[\pi_1(E)] \otimes_{\mathbb{Z}} C_*(F)$ . This is obviously a finite free  $\mathbb{Z}[\pi_1(E)]$ -chain complex. For  $w \in \pi_1(B)$  the fiber transport defines a  $\mathbb{Z}$ -chain map  $t(w)_*: C_* \to C_*$  which is well-defined up to  $\mathbb{Z}$ -chain homotopy. Choose  $\widetilde{w} \in \pi_1(E)$  whose image under  $\pi_1(p)$  is w. Define  $U(w)_*: C_* \to C_*$  by

(23.6) 
$$U(w)_*(v \otimes x) = v\widetilde{w}^{-1} \otimes t(w)_*(x).$$

This extends to a ring homomorphism  $U_*: \mathbb{Z}[\pi_1(B)] \to [C_*, C_*]_{\mathbb{Z}[\pi_1(E)]}$  by  $\mathbb{Z}$ -linearity. So the transfer of (23.5) is defined. It induces a homomorphism of abelian groups

(23.7) 
$$p_{U(p)}^* \colon \operatorname{Wh}(\mathbb{Z}[\pi_1(B)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(E)]),$$

provided that  $\Theta(p) = 0$  holds. The next theorem is taken from [641, Theorem 5.4].

23.4 The Down-Up Formula

**Theorem 23.8 (The geometric and algebraic transfer agree).** In the situation where the geometric transfer  $p^*$  of (23.2) is defined, the algebraic transfer  $p^*_{U(p)}$  of (23.7) is defined and  $p^*$  and  $p^*_{U(p)}$  agree.

In view of Theorem 23.8 we abbreviate  $p_{U(p)}^*$  by  $p^*$  in the sequel.

### 23.4 The Down-Up Formula

Consider a fibration  $F \to E \xrightarrow{p} B$  such that *F* is a connected finite *CW*-complex and *B* is path connected. The group homomorphism  $\pi_1(p) \colon \pi_1(E) \to \pi_1(B)$  induces a map  $p_* \colon Wh(\mathbb{Z}[\pi_1(E)]) \to Wh(\mathbb{Z}[\pi_1(B)])$ . Next we investigate the composite  $p_* \circ p^* \colon Wh(\mathbb{Z}[\pi_1(B)]) \to Wh(\mathbb{Z}[\pi_1(B)])$ .

For a group *G* let  $Sw^p(G)$  be the Grothendieck groups of  $\mathbb{Z}G$ -modules *M* that are finitely generated free as an abelian groups, see Definition 12.65. There is a pairing, see (12.69)

(23.9) 
$$s: \operatorname{Sw}^p(G) \otimes K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}G).$$

It induces a pairing

(23.10) 
$$s: \operatorname{Sw}^p(G) \otimes \operatorname{Wh}(G) \to \operatorname{Wh}(G).$$

Exercise 23.11. Show that the pairing (23.9) induces a well-defined pairing (23.10).

The fiber transport induces a homotopy  $\pi_1(B)$ -action on F. So  $H_n(F;\mathbb{Z})$  becomes a  $\mathbb{Z}[\pi_1(B)]$ -module. Thus we obtain a  $\mathbb{Z}[\pi_1(B)]$ -module  $H_n(F;\mathbb{Z})$  that is finitely generated as an abelian group. Define the element

(23.12) 
$$h(p) = \sum_{n \ge 0} (-1)^n \cdot [H_1(F;\mathbb{Z})] \in \operatorname{Sw}(\pi_1(B))$$

for the Swan ring Sw(G) given by  $\mathbb{Z}G$ -modules that are finitely generated as abelian groups, see Definition 12.65.

### Theorem 23.13 (Down-up formula).

(i) The composite

$$p_* \circ p^* \colon \operatorname{Wh}(\mathbb{Z}[\pi_1(B)]) \to \operatorname{Wh}(\mathbb{Z}[\pi_1(B)])$$

agrees with  $s(e^{-1}(h(p)), -)$  for the pairing s defined in (23.10), the element h(p) defined in (23.12), and the isomorphism  $e: \operatorname{Sw}^p(\pi_1(B)) \xrightarrow{\cong} \operatorname{Sw}(\pi_1(B))$  from Lemma 12.66;

(ii) If p is orientable, i.e., its fiber transport is trivial, then the composite  $p_* \circ p^*$ : Wh( $\mathbb{Z}[\pi_1(B)]$ )  $\to$  Wh( $\mathbb{Z}[\pi_1(B)]$ ) is multiplication with the Euler characteristic  $\chi(F)$ ;

- (iii) If the fiber F is contractible, then  $p_*$ : Wh $(\mathbb{Z}[\pi_1(E)]) \to$  Wh $(\mathbb{Z}[\pi_1(B)])$  is an isomorphism whose inverse is  $p^*$ : Wh $(\mathbb{Z}[\pi_1(B)]) \to$  Wh $(\mathbb{Z}[\pi_1(E)])$ .
- *Proof.* (i) See [643, Corollary 6.4].

(ii) This follows from assertion (i).

(iii) This follows from assertion (ii).

**Example 23.14.** Let *M* be a connected closed smooth manifold of dimension  $d \ge 5$ . Then we have the locally trivial bundle  $p: STM \to M$  given by the sphere bundle associated to the tangent bundle. For it the transfer  $p^*: Wh(\pi_1(B)) \to Wh(\pi_1(E))$  above is defined. Let W = (W, B, B', f, f') be an *h*-cobordism over *B*. Choose a retraction *r* of the homotopy equivalence  $B \xrightarrow{f} \partial_0 W \to W$ . As explained in Example 23.3, the pullback construction associated to *r* yields an *h*-cobordism  $\overline{W}$  over *E* and we have the equality

$$p^*(\tau(W)) = \tau(\overline{W}).$$

The map  $p_*$ : Wh( $\mathbb{Z}[\pi_1(B)]$ )  $\to$  Wh( $\mathbb{Z}[\pi_1(E)]$ ) is bijective. The composite  $p_* \circ p^*$ : Wh( $\mathbb{Z}[\pi_1(B)]$ )  $\to$  Wh( $\mathbb{Z}[\pi_1(B)]$ ) is multiplication by  $(1 + (-1)^{d-1})$  provided that *M* is orientable. If  $p_+: S_+TM \to M$  is the hemisphere bundle appearing in Section 19.5, then the transfer map

$$(p_+)^*$$
: Wh( $\mathbb{Z}[\pi_1(B)]$ )  $\rightarrow$  Wh( $\mathbb{Z}[\pi_1(S_+TM)]$ )

is an isomorphism with inverse  $(p_+)_*$ : Wh $(\mathbb{Z}[\pi_1(S_+TM)]) \to$  Wh $(\mathbb{Z}[\pi_1(B)])$ . All these claims follow from Theorem 23.13

There is also a more complicated up-down-formula in favorite cases, which computes the composite  $p^* \circ p_*$ : Wh $(\pi_1(E)) \to$  Wh $(\pi_1(E))$ , see [643, Theorem 8.2]. It leads to interesting computations of the transfer map  $p^*$ , see [643, Sections 8 and 9], but these are not needed for the purposes of this book.

### 23.5 Transfer for Finitely Dominated Z-Chain Complexes with Homotopy G-Action

Let *G* be a group and  $C_*$  be a  $\mathbb{Z}$ -chain complex  $C_*$  together with a homotopy *G*-action, i.e., a group homomorphism  $\rho: G \to [C_*, C_*]_{\mathbb{Z}}$  to the group of  $\mathbb{Z}$ -chain homotopy classes of self- $\mathbb{Z}$ -chain maps  $C_* \to C_*$ . It induces a  $\mathbb{Z}G$ - $\mathbb{Z}G$  chain homotopy representation  $U_*: \mathbb{Z}G \to [\mathbb{Z}G \otimes_{\mathbb{Z}} C_*, \mathbb{Z}G \otimes_{\mathbb{Z}} C_*]_{\mathbb{Z}G}$  in the sense of Definition 23.4 by

(23.15) 
$$U(g_0)_*(g \otimes x) = gg_0^{-1} \otimes \rho(g_0)_*(x).$$

Suppose additionally that  $C_*$  is a finite free  $\mathbb{Z}$ -chain complex. So the transfer homomorphism of (23.5) is defined. Since Wh({1}) is trivial, it induces a transfer homomorphism

(23.16) 
$$p_{C_*,\rho}^* \colon \operatorname{Wh}(G) \to \operatorname{Wh}(G).$$

An *R*-chain complex  $C_*$  is called *finitely dominated*, if there exists a finite free *R* chain complex  $D_*$  together with *R*-chain maps  $i_*: C_* \to D_*$  and  $r_*: D_* \to C_*$  satisfying  $r_* \circ i_* \simeq_R \operatorname{id}_{C_*}$ .

**Lemma 23.17.** The definition of the transfer (23.16) extends to the case where we weaken the condition that  $C_*$  is a finite free  $\mathbb{Z}$ -chain complex to the condition that  $C_*$  is finitely dominated.

*Proof.* First we consider two finite free  $\mathbb{Z}$ -chain complexes  $C_*$  and  $C'_*$ , two homotopy G-actions  $\rho: G \to [C_*, C_*]_{\mathbb{Z}}$  and  $\rho': G \to [C'_*, C'_*]_{\mathbb{Z}}$ , and a  $\mathbb{Z}$ -chain homotopy equivalence  $f_*: C_* \to C'_*$  such that  $\rho'(g) \circ f_* \simeq_{\mathbb{Z}} f_* \circ \rho(g)$  holds for all  $g \in G$  and then prove

(23.18) 
$$p_{C_{*},\rho}^{*} = p_{C'_{*},\rho'}^{*}$$

Let  $U_*$  and  $U'_*$  be the  $\mathbb{Z}G$ - $\mathbb{Z}G$ -chain representations associated to  $(C_*, \rho)$  and  $(C'_*, \rho')$ , see (23.15). From  $f_*$  we obtain a  $\mathbb{Z}G$ -chain homotopy equivalence

$$u_* := \bigoplus_{i=1}^n \operatorname{id}_{\mathbb{Z}G} \otimes_{\mathbb{Z}} f_* \colon \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C_* \xrightarrow{\simeq} \bigoplus_{i=1}^n \mathbb{Z}G \otimes_{\mathbb{Z}} C'_*$$

Given any *A* in  $GL_n(R)$ , we get a diagram of finite free  $\mathbb{Z}G$ -chain complexes

which commutes up to  $\mathbb{Z}G$ -chain homotopy and where all arrows are  $\mathbb{Z}G$ -chain homotopy equivalences. Equip  $C_*$  and  $C'_*$  with some  $\mathbb{Z}$ -basis and use in the sequel the induced  $\mathbb{Z}G$ -basis on the  $\mathbb{Z}G$ -chain complexes appearing in the diagram above. We get for the Whitehead torsion in Wh(G)

$$\tau(U_*(A)) = \tau(u) + \tau(U_*(A)) - \tau(u) = \tau(u \circ U_*(A)) - \tau(u)$$
  
=  $\tau(U'_*(A) \circ u) - \tau(u) = \tau(U'_*(A)) + \tau(u) - \tau(u) = \tau(U'_*(A)).$ 

Now (23.18) follows from the definitions.

Next we define for a  $\mathbb{Z}$ -chain complex  $C_*$  which is  $\mathbb{Z}$ -chain homotopy equivalent to some finite free  $\mathbb{Z}$ -chain complex and comes with a homotopy *G*-action its transfer map (23.16). Choose a finite free  $\mathbb{Z}$ -chain complex  $C'_*$  together with a

 $\mathbb{Z}$ -chain homotopy equivalence  $f_* \colon C'_* \xrightarrow{\simeq} C_*$ . Then there is precisely one homotopy *G*-action  $\rho' \colon G \to [C'_*, C'_*]_{\mathbb{Z}}$  such that  $f_* \circ \rho(g) \simeq \rho'(g) \circ f_*$  holds for every  $g \in G$ . Now define

$$p^*_{C_*,\rho} := p^*_{C'_*,\rho'}$$

This is independent of the choice of  $C'_*$  and  $f_*$  by (23.18).

Finally we mention that any finitely dominated  $\mathbb{Z}$ -complex  $C_*$  is  $\mathbb{Z}$ -chain homotopy equivalent to a finite projective  $\mathbb{Z}$ -chain complex, see for instance [644, Proposition 11.11 on page 222], and hence to a finite free  $\mathbb{Z}$ -chain complex, since  $\mathbb{Z}$  is a principal ideal domain.

### 23.6 Transfer for Finitely Dominated Spaces with Homotopy G-Action

Let *X* be a finitely dominated space, i.e., there is a finite *CW*-complex *Y* and maps  $i: X \to Y$  and  $r: Y \to X$  such that  $r \circ i$  is homotopic to the identity on *X*. Suppose that *X* comes with a homotopy *G*-action  $\rho: G \to [X, X]$  in the sense of Definition 20.27. Then we obtain by passing to the singular  $\mathbb{Z}$ -chain complex  $C_*^{\text{sing}}(X)$  a homotopy *G*-action  $\rho^{\text{sing}}: G \to [C_*^{\text{sing}}(X), C_*^{\text{sing}}(X)]_{\mathbb{Z}}$ . Since *X* is finitely dominated and the singular chain complex of a finite *CW*-complex *Y* is  $\mathbb{Z}$ -chain homotopy equivalent to the finite free cellular  $\mathbb{Z}$ -chain complex  $C_*^{\text{sing}}(Y)$ , see for instance [644, Proposition 13.10 on page 264] the  $\mathbb{Z}$ -chain complex  $C_*^{\text{sing}}(X)$  is finitely dominated. Hence we get from Lemma 23.17 a transfer map

(23.19) 
$$p_{X,\rho}^* \colon \operatorname{Wh}(G) \to \operatorname{Wh}(G)$$

**Remark 23.20.** One easily checks that it still satisfies the Down-Up Formula 23.13 taking into account that  $H_n(C_*^{sing}(X))$  is finitely generated as abelian group, since *X* is finitely dominated. More precisely, we get an element

(23.21) 
$$h(X;\rho) = \sum_{n\geq 0} (-1)^n \cdot [H_n(X;\mathbb{Z})] \in \operatorname{Sw}(G)$$

and the equality in Wh(G)

(23.22) 
$$p_{X,\rho}^*(u) = s(e^{-1}(h(X;\rho)), u)$$

for the pairing *s* defined in (23.10), the element  $h(X; \rho)$  defined in (23.21), and the isomorphism  $e: \operatorname{Sw}^p(G) \xrightarrow{\cong} \operatorname{Sw}(G)$  from Lemma 12.66.

Suppose additionally that X is contractible, Then  $H_n(X;\mathbb{Z}) = 0$  for  $n \ge 1$ and  $H_0(X;\mathbb{Z})$  is the  $\mathbb{Z}G$ -module given by  $\mathbb{Z}$  with the trivial *G*-action. Since  $[\mathbb{Z}] = e^{-1}(h(X;\rho))$  is the unit in  $\operatorname{Sw}^p(G)$ , the down up formula implies that  $p_{X,\rho}^*$ : Wh(*G*)  $\to$  Wh(*G*) is the identity. **Example 23.23.** Let *G* be a hyperbolic group in the sense of Gromov, see for instance [159, 165, 424, 440]. Let  $X = P_d(G)$  be the associated Rips complex for some number  $d > 16\delta + 8$  if *G* is  $\delta$ -hyperbolic space with respect to some choice *S* of a finite set of generators, see Subsection 11.6.7. Such  $\delta > 0$  exists by the definition of hyperbolic. The obvious simplicial *G*-action on *X* is cocompact and proper. The barycentric subdivision of *X* is a cocompact model for the classifying space  $E_{\mathcal{FIN}}(G)$ , see Theorem 11.29. Now take  $\overline{X} = X \cup \partial X$  to be the compactification of *X* in the sense of Gromov, see [440], [165, Section 3 in Chapter III.H]. Then  $\overline{X}$  is a contractible compact metrizable *G*-space.

Now the transfer map (23.19) is defined and yields an isomorphism

$$p^*: \operatorname{Wh}(G) \xrightarrow{\cong} \operatorname{Wh}(G).$$

A controlled version of this transfer, which works for the *K*-groups in all dimensions and is described in [87, Section 6], is a key ingredient in the proof of the *K*-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive *G*-categories for a hyperbolic group in [87, Main Theorem].

Analogously a controlled version of the transfer map above described in [78, Section 7] is a key ingredient in the proof of the *K*-theoretic Farrell-Jones Conjecture 13.11 with coefficients in additive *G*-categories for a finite-dimensional CAT(0)-group *G* in [78, Theorem B]. Here  $\overline{X}$  is a bordification defined in Bridson-Haefliger [165, Chapter II.8] of a finite-dimensional CAT(0)-space *X* on which the CAT(0)-group *G* acts properly, cocompactly, and isometrically.

### 23.7 Proof of Surjectivity of the Assembly Map in Dimension 1

In this section we give the proof of a special case of Theorem 20.12 as an illustration of the methods and results described so far. Reducing to this special case avoids some formidable purely technical input which will make the exposition much harder but will be discussed later in Chapter 24.

**Proposition 23.24.** Let G be a finitely generated group. Let  $\mathcal{F}$  be a family of subgroups such that G is strictly  $\mathcal{F}$ -transfer reducible in the sense of Definition 20.11. Then the assembly map

$$H_1^G(\operatorname{pr}; \mathbf{K}_{\mathbb{Z}}) \colon H_1^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}) \to H_1^G(G/G; \mathbf{K}_{\mathbb{Z}}) = K_1(\mathbb{Z}G)$$

is surjective.

The rest of this section is devoted to the proof of Proposition 23.24. In this section *R* will always be  $\mathbb{Z}$  for simplicity.

#### 23.7.1 Basic Strategy of the Proof of Proposition 23.24

For the remainder of this section fix an element

$$u \in K_1(\mathbb{Z}G).$$

We want to show that *u* is in the image of  $H_1^G(\text{pr}; \mathbf{K}_{\mathbb{Z}})$ .

For an element  $a = \sum_{g \in g} \lambda_g \cdot g \in \mathbb{Z}G$ , define the  $\mathbb{Z}G$ -homomorphism  $V(a): \mathbb{Z}G \to \mathbb{Z}G$  by sending x to  $\sum_{g \in G} \lambda_g \cdot xg^{-1}$ . Given a matrix  $A = (a_{i,j})_{i,j}$  in  $M_{m,n}(\mathbb{Z}G)$ , define a  $\mathbb{Z}G$ -homomorphism

(23.25) 
$$V(A): \mathbb{Z}G^m \to \mathbb{Z}G^n, \quad (x_1, x_2, \dots, x_m) \mapsto \left(\sum_{i=1}^m V(a_{i,j})(x_i)\right)_{j=1,\dots,n}.$$

One easily checks that  $V(AB) = V(A) \circ V(B)$  holds for  $A \in M_{m,n}(\mathbb{Z}G)$  and  $B \in M_{n,o}(\mathbb{Z}G)$  and  $V(I_n) = id_{\mathbb{Z}G^n}$  holds for the identity matrix  $I_n \in GL_n(\mathbb{Z}G)$ .

Choose a natural number *n* and an element  $A = (a_{i,j}) \in GL_n(\mathbb{Z}G)$  such that *u* is represented by the  $\mathbb{Z}G$ -automorphism  $V(A) \colon \mathbb{Z}G^n \xrightarrow{\cong} \mathbb{Z}G^n$  given by right multiplication with *A*. Choose a finite subset  $T \subseteq G$  such that for any  $i, j \in \{1, 2, ..., n\}$  the elements  $a_{i,j}, b_{i,j} \in \mathbb{Z}G$  are of the form  $\sum_{g \in T} \lambda_g \cdot g$  and  $e \in T$  holds. By possibly enlarging *T* we can additionally arrange that *T* is a finite set of generators of *G*.

Next let us recall what we get from the assumption that G is strictly  $\mathcal{F}$ -transfer reducible. Let N be the number appearing Definition 20.11. Then the following holds by assumption:

- We have an *N*-transfer space *X* in the sense of Definition 20.9, that is, a compact contractible metric space  $(X, d_X)$  with the property that for any  $\delta > 0$  there exists an *N*-dimensional simplicial complex *K*, maps  $i: X \to |K|$  and  $r: |K| \to X$ , and a homotopy  $h: X \times [0, 1] \to X$  from  $p \circ i$  to id<sub>X</sub> which is  $\delta$ -controlled, i.e., for every  $x \in X$  the diameter of the subset  $h(\{x\} \times [0, 1])$  of *X* is smaller than  $\delta$ ;
- The *N*-transfer space *X* comes with a *G*-action;
- For every  $\epsilon > 0$  there exists:
  - an abstract simplicial  $(G, \mathcal{F})$ -complex  $\Sigma$  of dimension  $\leq N$ ;
  - a map  $v: X \to |\Sigma|$  that is  $(\epsilon, T)$ -almost *G*-equivariant, i.e., we have  $d_{L^1}(v(gx), gv(x)) \le \epsilon$  for every  $g \in T$  and every  $x \in X$ .

Next we formulate what we need to prove Proposition 23.24. In view of the Algebraic Thin *h*-Cobordism Theorem 19.8 we have to construct for the number  $\epsilon_N$  appearing in Theorem 19.8 and the element  $u \in K_1(\mathbb{Z}G)$ 

- An abstract simplicial  $(G, \mathcal{F})$ -complex Z of dimension  $\leq N$ ;
- A free *G*-space *Y* together with a *G*-map  $w: Y \to |Z|$ ;
- An  $\epsilon_N$  automorphism  $a: M \to M$  in  $GM^G(Y)$ , i.e., an automorphism  $a: M \to M$ in  $GM^G(Y)$  such that both a and  $a^{-1}$  are  $\epsilon_N$ -controlled with respect to w and the  $L^1$ -metric  $d_{L^1}$  on |Z|. Recall that a morphism

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$$f = \{f_{x,y} \colon M_x \to N_y \mid x, y \in X\} \colon M \to N$$

in  $GM^G(Y)$  is  $\epsilon_N$ -controlled with respect to w and the  $L^1$ -metric  $d_{L^1}$  on |Z| if the implication  $x, y \in X$ ,  $f_{x,y} \neq 0 \implies d_{L^1}(w(x), w(y)) \leq \epsilon_N$  holds;

• The class  $[F^f(a)] \in K_1(\mathbb{Z}G)$  of the *RG*-automorphism  $F^f(a): F^f(M) \xrightarrow{\cong} F^f(M)$  of the finitely generated free *RG*-module  $F^f(M)$  for the functor  $F^f$  defined in (19.6) is *u*.

Put

(23.26) 
$$\epsilon = \frac{\epsilon_N}{5(72N+181)}.$$

Now make the choice of the data (X, d),  $\Sigma$ , and v described above for this choice of  $\epsilon$ , which exist by assumption.

Next we construct the desired data mentioned above. We will take for *Y* the *G*-space  $G \times X$  where the *G*-action is given by  $g' \cdot (g, x) = (g'g, x)$  for  $g' \in G$  and  $(g, x) \in Y$ . We take  $Z = \Sigma$ . Define the *G*-map  $w: Y \to |\Sigma|$  by sending (g, x) to gv(x).

So for the rest of this section we have fixed  $u \in K_1(\mathbb{Z}G)$ ,  $A \in GL_n(\mathbb{Z}G)$ , the finite subset  $T \subset G$ , numbers N and  $\epsilon_N$ , the abstract simplicial G-complex Z, the G-spaces X and Y, metrics d on X and  $d_{L^1}$  on |Z|, the map v, and the G-map  $w: Y \to |Z|$ , and we will only consider  $R = \mathbb{Z}$ . Recall that the G-action on X is not necessarily isometric, whereas the G-action on |Z| is isometric, and that v is  $(\epsilon, T)$ -almost G-equivariant.

Note that so far we have not used the *G*-action on *X* which will enter in the construction of the desired  $\epsilon_N$ -controlled automorphism  $a: M \to M$  in  $GM^G(Y)$  satisfying  $u = [F^f(a)]$ . The only thing that remains to be done is the construction of *a*, which will occupy the rest of this section.

### 23.7.2 The Width Function

**Definition 23.27 (Width function).** Let  $\mathcal{A}$  be an additive category. A *width function* wd = wd\_{\mathcal{A}} on  $\mathcal{A}$  is a function

wd: mor(
$$\mathcal{A}$$
)  $\rightarrow \mathbb{R}^{\geq 0} \amalg \{-\infty, \infty\}$ 

satisfying the following axioms.

(i) Consider finitely many objects  $A_1, \ldots, A_m$  and  $B_1, \ldots, B_n$  and morphisms  $f_{i,j}: A_i \to B_j$  for  $i = 1, \ldots, m$  and  $j = 1, \ldots, n$  in  $\mathcal{A}$ . Let  $f: \bigoplus_{i=1}^m A_i \to \bigoplus_{i=1}^n B_j$  be the morphism given by the collection of the  $f_{i,j}$ -s. Then

$$wd(f) \le max\{wd(f_{i,j}) \mid i = 1, ..., m, j = 1, ..., n\};$$

(ii) Consider morphisms  $f: A \to B$  and  $g: B \to C$  in  $\mathcal{A}$ . Then we get

$$\operatorname{wd}(g \circ f) \le \operatorname{wd}(f) + \operatorname{wd}(g);$$

(iii) Consider morphisms  $f, g: A \to B$  and  $\lambda, \mu \in \mathbb{Z}$ . Then

$$wd(\lambda \cdot f + \mu \cdot g) \le max\{wd(f), wd(g)\};$$

(iv) We have  $f = 0 \iff wd(f) = -\infty$  for every morphism  $f: M \to N$  in  $\mathcal{A}$ .

We define the width wd(A) of an object to be the width  $wd(id_A)$  of the identity on *A*.

We call wd *trivial on objects* if for every object A we have wd(A) = 0.

**Remark 23.28 (Passage to idempotent completion).** Let  $\mathcal{A}$  be an additive category with width function wd. Then its idempotent completion inherits a width function  $\widehat{wd}$  which assigns to a morphism  $f: (A, p) \to (B, q)$  in Idem( $\mathcal{A}$ ) the width  $wd_{\mathcal{A}}(f)$  of the underlying morphism  $f: A \to B$  in  $\mathcal{A}$ . Note that the identity of an object (A, P) in Idem( $\mathcal{A}$ ) is given by  $p: A \to A$  and hence  $\widehat{wd}(A, p) = wd(p)$ . So, even if wd is trivial on objects,  $\widehat{wd}$  is not necessarily trivial on objects.

Next we present our main example of a width function.

**Example 23.29 (Width function on**  $GM^G(Y)$ ). We define a width function on the additive category  $GM^G(Y)$  from Definition 19.4 as follows, where we use the data fixed in Subsection 23.7.1.

Given two objects  $M = \{M_x \mid x \in Y\}$  and  $N = \{N_y \mid y \in Y\}$  and a morphism  $f: M \to N$  in  $GM^G(Y)$  which consists of a collection of morphisms  $f = \{f_{x,y}: M_x \to N_y \mid x, y \in Y\}$  in  $\mathcal{F}^{\kappa}(\mathbb{Z})$ , we define the *width* 

$$\operatorname{wd}_Z(f) \in \mathbb{R}^{\geq 0} \amalg \{-\infty\}$$

to be the supremum of the set  $\{d_{L^1}(w(x), w(y)) \mid x, y \in X, f_{x,y} \neq 0\}$  if *f* is not the zero homomorphism and to be  $-\infty$  otherwise. Note that this width function is trivial on objects.

Note that for  $\epsilon \ge 0$  a morphism  $f: M \to N$  in  $GM^G(Y)$  is  $\epsilon$ -controlled in the sense of Subsection 19.4.3 if and only if  $wd_Z(f) \le \epsilon$  holds.

**Exercise 23.30.** Show that the axioms of a width function which is trivial on objects are satisfied in Example 23.29.

**Example 23.31 (Width function on** GM(X)). Let (X, d) be any metric space. Let GM(X) be  $GM^G(X)$  for  $G = \{1\}$ . We will equip it the following width function  $wd_X$ .

Given two objects  $M = \{M_x \mid x \in X\}$  and  $N = \{N_y \mid y \in X\}$  and a morphism  $f: M \to N$  in GM(X), which consists of a collection of morphisms  $f = \{f_{x,y}: M_x \to N_y \mid x, y \in X\}$  in  $\mathcal{F}^{\kappa}(\mathbb{Z})$ , we define the *width* 

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$$\operatorname{wd}(f) \in \mathbb{R}^{\geq 0} \amalg \{-\infty\}$$

to be the supremum of the set  $\{d(x, y) \mid x, y \in X, f_{x,y} \neq 0\}$  if f is not the zero homomorphism and to be  $-\infty$  otherwise. Note that this width function is trivial on objects.

### 23.7.3 Self-Torsion

Let  $\mathcal{A}$  be an additive category. Let  $C_* = (C_*, c_*)$  be a *bounded*  $\mathcal{A}$ -chain complex i.e., a sequence of morphisms in  $\mathcal{A}$ 

$$\cdots \xrightarrow{c_{n+2}} C_{n+2} \xrightarrow{c_{n+1}} C_{n+1} \xrightarrow{c_n} C_n \xrightarrow{c_{n-1}} C_{n-1} \xrightarrow{c_{n-2}} \cdots$$

such that  $c_{n+1} \circ c_n = 0$  holds for  $n \in \mathbb{Z}$  and there exists a natural number N with  $C_n = 0$  for  $n \in \mathbb{Z}$  with  $|n| \ge N$ . There are obvious notions of a chain map, a chain homotopy, and a chain contraction. Let  $f_* \colon C_* \xrightarrow{\simeq} D_*$  be a chain homotopy equivalence of bounded  $\mathcal{A}$ -chain complexes. Denote by  $\operatorname{cone}(f_*)$  its mapping cone whose *n*-th differential  $e_n$  is given by

(23.32) 
$$e_n \colon C_{n-1} \oplus D_n \xrightarrow{\begin{pmatrix} -c_{n-1} & 0\\ f_{n-1} & d_n \end{pmatrix}} C_{n-2} \oplus D_{n-1}$$

Given an  $\mathcal{A}$ -chain map  $g_*: D_* \to C_*$  and  $\mathcal{A}$ -chain homotopies  $h_*: g_* \circ f_* \simeq \operatorname{id}_{C_*}$ and  $k_*: f_* \circ g_* \simeq \operatorname{id}_{D_*}$ , define an  $\mathcal{A}$ -chain isomorphism  $u_*: \operatorname{cone}(f_*) \xrightarrow{\cong} \operatorname{cone}(f_*)$ by  $u_n = \begin{pmatrix} \operatorname{id}_{C_{n-1}} & 0\\ f_n \circ h_{n-1} - k_{n-1} \circ f_{n-1} \operatorname{id}_{D_n} \end{pmatrix}$  and an  $\mathcal{A}$ -chain homotopy  $\delta_*: u_* \simeq 0_*$  by  $\delta_n = \begin{pmatrix} h_{n-1} & g_n\\ 0 & -k_n \end{pmatrix}$ . Then we obtain a chain contraction  $\gamma_*$  of  $\operatorname{cone}(f_*)$  by

(23.33) 
$$\gamma_n = u_{n+1}^{-1} \circ \delta_n$$

Now consider a self-chain homotopy equivalence  $f_*: C_* \xrightarrow{\simeq} C_*$  of the bounded  $\mathcal{A}$ -chain complex  $C_*$ . Define objects in  $\mathcal{A}$  by

$$C_{\text{all}} = \bigoplus_{n \in \mathbb{Z}} C_n;$$
  

$$\operatorname{cone}(f)_{\text{odd}} = \bigoplus_{n \in \mathbb{Z}} \operatorname{cone}(f_*)_{2n+1};$$
  

$$\operatorname{cone}(f)_{\text{ev}} = \bigoplus_{n \in \mathbb{Z}} \operatorname{cone}(f_*)_{2n}.$$

23 Transfer

We obtain isomorphisms

(23.34) 
$$(e + \gamma)_{odd}$$
:  $\operatorname{cone}(f_*)_{odd} \xrightarrow{=} \operatorname{cone}(f_*)_{ev};$   
(23.35)  $(e + \gamma)_{ev}$ :  $\operatorname{cone}(f_*)_{ev} \xrightarrow{\cong} \operatorname{cone}(f_*)_{odd},$ 

satisfying

(23.36) 
$$(e + \gamma)_{ev} \circ (e + \gamma)_{odd} = (id + \gamma \circ \gamma);$$
  
(23.37) 
$$(e + \gamma)_{odd} \circ (e + \gamma)_{ev} = (id + \gamma \circ \gamma).$$

Let

$$I_{\text{odd}}: \operatorname{cone}(f_*)_{\text{odd}} \xrightarrow{\cong} C_{\text{all}};$$
$$I_{\text{ev}}: \operatorname{cone}(f_*)_{\text{ev}} \xrightarrow{\cong} C_{\text{all}},$$

be the obvious isomorphisms coming from  $\operatorname{cone}(f_*)_n = C_{n-1} \oplus C_n$ .

Thus we obtain an automorphism

$$I_{\mathrm{ev}} \circ (e + \gamma) \circ I_{\mathrm{odd}}^{-1} \colon C_{\mathrm{all}} \xrightarrow{\cong} C_{\mathrm{all}}.$$

Its class

(23.38) 
$$t(f_*) := [I_{\text{ev}} \circ (e+\gamma) \circ I_{\text{odd}}^{-1}] \in K_1(\mathcal{A})$$

is called the *self-torsion* of  $f_*$ . The proof that this element is well-defined and has the following properties in [644, Section 12] for  $R\Gamma$ -modules carries directly over to additive categories.

### Lemma 23.39.

(i) Let  $f_*, g_*: C_* \xrightarrow{\simeq} C_*$  be self-chain homotopy equivalences of the bounded  $\mathcal{A}$ -chain complex  $C_*$ . If they are chain homotopic, then

$$t(g_*) = t(f_*);$$

(ii) Consider a commutative diagram of bounded A-chain complexes with self-chain homotopy equivalences as vertical arrows

$$0 \longrightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \longrightarrow 0$$
$$\simeq \left| f_* \qquad \simeq \right| g_* \qquad \simeq \left| h_* \right|$$
$$0 \longrightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \longrightarrow 0$$

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where for each  $n \in \mathbb{Z}$  the sequence  $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{p_n} E_n \to 0$  is split exact, i.e., there exists a morphism  $s_n \colon E_n \to D_n$  such that  $p_n \circ s_n = \mathrm{id}_{E_n}$  holds and  $i_n \oplus s_n \colon C_n \oplus E_n \xrightarrow{\cong} D_n$  is an isomorphism. Then we get

$$t(g_*) = t(f_*) + t(h_*);$$

(iii) Let  $f_*, g_*: C_* \xrightarrow{\simeq} C_*$  be self-chain homotopy equivalences of the bounded  $\mathcal{A}$ -chain complex  $C_*$ . Then we get

$$t(g_* \circ f_*) = t(f_*) + t(g_*).$$

**Exercise 23.40.** Let  $f_*: C_* \xrightarrow{\cong} C_*$  be a chain automorphism of a bounded  $\mathcal{A}$ -chain complex. Show

$$t(f_*) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot [f_n] \in K_1(\mathcal{A})$$

### 23.7.4 Self-Torsion and Width Functions

Let  $\mathcal{A}$  be an additive category coming with a width function wd in the sense of Definition 23.27. We define the *width* wd( $C_*$ ) of a bounded  $\mathcal{A}$ -chain complex  $C_* = (C_*, c_*)$  to be

(23.41) 
$$\operatorname{wd}(C_*) = \max\{\operatorname{wd}(C_n), \operatorname{wd}(c_n) \mid n \in \mathbb{Z}\}.$$

We define the *width* wd( $f_*$ ) of an  $\mathcal{A}$ -chain map  $f_*: C_* \to D_*$  of bounded  $\mathcal{A}$ -chain complexes to be

(23.42) 
$$\operatorname{wd}(f_*) = \max\{\operatorname{wd}(f_n) \mid n \in \mathbb{Z}\},\$$

and the *width* wd( $h_*$ ) of an  $\mathcal{A}$ -chain homotopy  $h_*: C_* \to D_{*+1}$  of bounded  $\mathcal{A}$ -chain complexes to be

(23.43) 
$$\operatorname{wd}(h_*) = \max\{\operatorname{wd}(h_n) \mid n \in \mathbb{Z}\}.$$

**Notation 23.44.** For  $\epsilon > 0$  and two  $\mathcal{A}$ -chain maps  $f_*, g_* \colon C_* \to D_*$ , we write

$$f_* \simeq_{\epsilon} g_*$$

if there exists an  $\mathcal{A}$ -chain homotopy  $h_*: f_* \simeq g_*$  with  $wd(h_*) \le \epsilon$ .

**Definition 23.45** ( $\epsilon$ -controlled isomorphism). An  $\epsilon$ -controlled isomorphism  $f: A \xrightarrow{\cong} B$  in  $\mathcal{A}$  is an isomorphism  $f: A \xrightarrow{\cong} B$  satisfying

$$wd(A), wd(B), wd(f), wd(f^{-1}) \le \epsilon.$$

If A = B, we talk of an  $\epsilon$ -controlled automorphism.

**Exercise 23.46.** Let  $f: A \to B$  be an  $\epsilon$ -controlled  $\mathcal{A}$ -isomorphism and  $g: B \to C$  be a  $\delta$ -controlled  $\mathcal{A}$ -isomorphism.

Show that  $g \circ f \colon A \to C$  is an  $(\epsilon + \delta)$ -controlled  $\mathcal{A}$ -isomorphism.

**Definition 23.47** ( $\epsilon$ -controlled chain homotopy equivalence). Consider  $\epsilon > 0$  and an  $\mathcal{A}$ -chain map  $f_*: C_* \to D_*$ . We call  $f_*$  an  $\epsilon$ -controlled  $\mathcal{A}$ -chain homotopy equivalence if there is an  $\mathcal{A}$ -chain map  $g_*: D_* \to C_*$  satisfying

$$wd(C_*), wd(D_*), wd(f_*), wd(g_*) \le \epsilon$$

and

$$g_* \circ f_* \simeq_{\epsilon} \operatorname{id}_{C_*};$$
  
$$f_* \circ g_* \simeq_{\epsilon} \operatorname{id}_{D_*}.$$

If  $C_* = D_*$ , we talk of an  $\epsilon$ -controlled  $\mathcal{A}$ -self-chain homotopy equivalence.

The next lemma is a direct consequence of the axioms appearing in Definition 23.27.

**Lemma 23.48.** Consider  $\delta, \epsilon > 0$ .

(i) Let f<sub>\*</sub>, g<sub>\*</sub>, h<sub>\*</sub>: C<sub>\*</sub> → D<sub>\*</sub> be A-chain maps of bounded A-chain complexes and λ, μ ∈ Z. Then

$$wd(\lambda \cdot f_* + \mu \cdot g_*) \le \max\{wd(f_*), wd(g_*)\}$$

and

$$f_* \simeq_{\epsilon} g_*, g_* \simeq_{\epsilon} h_* \implies f_* \simeq_{\epsilon} h_*$$

(ii) Let f<sub>\*</sub>, f'<sub>\*</sub>: D<sub>\*</sub> → E<sub>\*</sub>, u<sub>\*</sub>: C<sub>\*</sub> → D<sub>\*</sub>, and v<sub>\*</sub>: E<sub>\*</sub> → F<sub>\*</sub> be A-chain maps of bounded A-chain complexes satisfying f<sub>\*</sub> ≃<sub>ε</sub> f'<sub>\*</sub>, wd(u<sub>\*</sub>) ≤ δ, and wd(v<sub>\*</sub>) ≤ δ. Then

$$v_* \circ f_* \simeq_{\delta+\epsilon} v_* \circ f'_*;$$
  
$$f_* \circ u_* \simeq_{\delta+\epsilon} f'_* \circ u_*;$$

(iii) Let  $f_*: C_* \to D_*$  and  $g_*: D_* \to E_*$  be  $\epsilon$ -controlled  $\mathcal{A}$ -chain homotopy equivalences of bounded  $\mathcal{A}$ -chain complexes.

Then  $g_* \circ f_* \colon C_* \to E_*$  is a  $3\epsilon$ -controlled  $\mathcal{A}$ -chain homotopy equivalence of bounded  $\mathcal{A}$ -chain complexes.
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**Exercise 23.49.** Give the proof of Lemma 23.48.

**Proposition 23.50.** Let  $\mathcal{A}$  be an additive category coming with a width function wd. Consider  $\epsilon > 0$ . Let  $f_* \colon C_* \xrightarrow{\simeq} C_*$  be an Idem $(\mathcal{A})$ -self-chain homotopy equivalence which is  $\epsilon$ -controlled.

Then there is a 5 $\epsilon$ -controlled  $\mathcal{A}$ -automorphism  $a: A \xrightarrow{\cong} A$  such that the selftorsion  $t(F_*) \in K_1(\mathcal{A}) = K_1(\text{Idem}(\mathcal{A}))$  satisfies

$$t(f_*) = [a].$$

*Proof.* Recall that we have defined a width function  $\widehat{\mathrm{vd}}$  on Idem( $\mathcal{A}$ ) in Remark 23.28. By assumption we have Idem( $\mathcal{A}$ )-chain homotopy equivalences  $f_*: C_* \to C_*$  and  $g_*: C_* \to C_*$  and Idem( $\mathcal{A}$ )-chain homotopies  $h_*: f_* \circ g_* \simeq \mathrm{id}_{C_*}$  and  $k_*: g_* \circ f_* \simeq \mathrm{id}_{C_*}$  such that  $\widehat{\mathrm{vd}}(C_*)$ ,  $\widehat{\mathrm{vd}}(f_*)$ ,  $\widehat{\mathrm{vd}}(g_*)$ ,  $\widehat{\mathrm{vd}}(h_*)$ , and  $\widehat{\mathrm{vd}}(k_*)$  are less than or equal to  $\epsilon$ . Let  $\mathrm{cone}(f_*)$  be the mapping cone of  $f_*$ , see (23.32). In the sequel we will apply over and over again the axioms appearing Definition 23.27 and Lemma 23.48. One easily checks  $\widehat{\mathrm{vd}}(\mathrm{cone}(f_*)) \leq \epsilon$ . We have constructed a chain contraction  $\gamma$  for  $\mathrm{cone}(f_*)$  in (23.33). We get  $\widehat{\mathrm{vd}}(\gamma_*) \leq 3\epsilon$ . We conclude from (23.36) and (23.37) that the Idem( $\mathcal{A}$ )-automorphism  $I_{\mathrm{ev}} \circ (e + \gamma) \circ I_{\mathrm{odd}}^{-1}$ :  $C_{\mathrm{all}} \xrightarrow{\cong} C_{\mathrm{all}}$  is  $3\epsilon$ -controlled. Its class in  $K_1(\mathrm{Idem}(\mathcal{A}))$  is by definition  $t(f_*)$ .

For each object  $C_n = (A_n, p_n)$  in Idem $(\mathcal{A})$  we can consider the object  $C_n^{\perp} = (A_n, \operatorname{id} - p_n)$  in Idem $(\mathcal{A})$ . Obviously we have  $\widehat{\operatorname{wd}}(C_n) = \widehat{\operatorname{wd}}(C_n^{\perp}) = \widehat{\operatorname{wd}}(p_n) = \widehat{\operatorname{wd}}(\operatorname{id}_{A_n} - p_n)$ . The Idem $(\mathcal{A})$ -isomorphisms

$$a_n = p_n \oplus (\mathrm{id}_{A_n} - p_n) \colon C_n \oplus C_n^{\perp} \xrightarrow{\cong} A_n = (A_n, \mathrm{id}_{A_n})$$

and

$$b_n = (p_n \oplus (\mathrm{id}_{A_n} - p_n)) \colon A_n = (A_n, \mathrm{id}_{A_n}) \xrightarrow{\cong} C_n \oplus C_n^{\perp}$$

are inverse to one another and satisfy  $\operatorname{wd}(a_n)$ ,  $\operatorname{wd}(b_n) \leq \epsilon$ . Put  $A_{\operatorname{all}} = \bigoplus_{n \in \mathbb{Z}} A_n$  and  $C_{\operatorname{all}}^{\perp} = \bigoplus_{n \in \mathbb{Z}} C_n^{\perp}$ . The collection of the isomorphisms  $a_n$ -s and  $b_n$ -s yields mutually inverse Idem( $\mathcal{A}$ )-isomorphisms  $a_{\operatorname{all}} \colon C_{\operatorname{all}} \oplus C_{\operatorname{all}}^{\perp} \xrightarrow{\cong} A_{\operatorname{all}}$  and  $b_{\operatorname{all}} \colon A_{\operatorname{all}} \xrightarrow{\cong} C_{\operatorname{all}} \oplus C_{\operatorname{all}}^{\perp}$  with  $\operatorname{wd}(a_{\operatorname{all}})$ ,  $\operatorname{wd}(b_{\operatorname{all}}) \leq \epsilon$ . Define the  $\mathcal{A}$ -automorphism

$$a \colon A_{\text{all}} \xrightarrow{b_{\text{all}}} C_{\text{all}} \oplus C_{\text{all}}^{\perp} \xrightarrow{(I_{\text{ev}} \circ (e+\gamma) \circ I_{\text{odd}}^{-1}) \oplus \text{id}_{C_{\text{all}}^{\perp}}} C_{\text{all}} \oplus C_{\text{all}}^{\perp} \xrightarrow{a_{\text{all}}} A_{\text{all}}$$

One easily checks that  $t(f_*) = [a]$  in  $K_1(\mathcal{A})$  and a is a  $5\epsilon$ -controlled  $\mathcal{A}$ -automorphism.

## 23.7.5 Finite Domination

Consider a full and faithful inclusion  $\mathcal{A} \to \mathcal{B}$  of additive categories, e.g., the inclusion of the category of finitely generated free *R*-modules into the category of free *R*-modules for a ring *R*. Let  $C_*$  be a (not necessarily finite-dimensional) positive  $\mathcal{B}$ -chain complex. Consider a *finite domination*  $(D_*, i_*, r_*.h_*)$  of  $C_*$  over  $\mathcal{A}$ , i.e., a finite-dimensional positive  $\mathcal{A}$ -chain complex  $D_*$ ,  $\mathcal{B}$ -chain maps  $i_*: C_* \to D_*$  and  $r_*: D_* \to C_*$ , and a  $\mathcal{B}$ -chain homotopy  $h_*: r_* \circ i_* \simeq id_{D_*}$ . From these data we construct an explicit finite-dimensional positive chain complex  $P_*$  over Idem $(\mathcal{A})$  with dim $(P_*) = \dim(D_*)$  together with Idem $(\mathcal{B})$ -chain homotopy equivalences  $f_*: C_* \to P_*$  and  $g_*: P_* \to C_*$  and Idem $(\mathcal{B})$ -chain homotopies  $k_*: f_* \circ g_* \simeq id_{P_*}$  and  $l_*: g_* \circ f_* \simeq id_{C_*}$  following [838] and [78, Remark 8.3].

Define the chain complex C' over  $\mathcal{A}$  by defining its *m*-th chain object to be

$$C'_m = \bigoplus_{j=0}^m D_j$$

and its *m*-th differential to be

$$c'_m \colon C'_m = \bigoplus_{j=0}^m D_j \to C'_{m-1} = \bigoplus_{k=0}^{m-1} D_k$$

where the (j, k)-entry  $(c'_m)_{j,k}: D_j \to D_k$  for  $j \in \{0, 1, 2, \dots, m\}$  and  $k \in \{0, 1, 2, \dots, m-1\}$  is given by

$$(c'_{m})_{j,k} := \begin{cases} 0 & \text{if } j \ge k+2; \\ (-1)^{m+k} \cdot d_{j} & \text{if } j = k+1; \\ \text{id} - r_{j} \circ i_{j} & \text{if } j = k, j \equiv m \mod 2; \\ r_{j} \circ i_{j} & \text{if } j = k, j \equiv m+1 \mod 2; \\ (-1)^{m+k+1} \cdot i_{k} \circ h_{k-1} \circ \ldots \circ h_{j} \circ r_{j} & \text{if } j \le k-1. \end{cases}$$

Define chain maps  $f'_*: C_* \to C'_*$  and  $g'_*: C'_* \to C_*$  by

$$f'_m: C_m \to C'_m = D_0 \oplus D_1 \oplus \dots \oplus D_m, \quad x \mapsto (0, 0, \dots, i_m(x))$$

and

$$g'_m \colon C'_m = D_0 \oplus \cdots \oplus D_m \to C_m, \quad (x_0, x_1, \dots, x_m) \mapsto \sum_{j=0}^m h_{m-1} \circ \cdots \circ h_j \circ r_j(x_j).$$

We have  $g'_* \circ f'_* = r_* \circ i_*$  and hence  $h_*$  is a chain homotopy  $g'_* \circ f'_* \simeq id_{C_*}$ . We obtain a chain homotopy  $k'_* \colon f'_* \circ g'_* \simeq id_{C'_*}$  if we define

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$$k'_m: C'_m = D_0 \oplus D_1 \oplus \cdots \oplus D_m \to C'_{m+1} = D_0 \oplus D_1 \oplus \cdots \oplus D_m \oplus D_{m+1}$$

to be the obvious inclusion.

Let N be the dimension of  $D_*$ . Thus we get  $C'_m = C'_N$  for  $m \ge N$  and  $c'_{m+1} = id - c'_m$  for  $m \ge N+1$ . Since  $c'_{m+1} \circ c'_m = 0$  holds for all m, we conclude  $c'_m \circ c'_m = c'_m$  for  $m \ge N+1$ . Hence C' has the form

$$\cdots \to C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{\operatorname{id} - c'_{N+1}} C'_N \xrightarrow{c'_{N+1}} C'_N \xrightarrow{c'_N} C'_{N-1}$$
$$\xrightarrow{c'_{N-1}} \cdots \xrightarrow{c'_1} C'_0 \to 0 \to \cdots .$$

Define the desired *N*-dimensional chain complex  $P_*$  over Idem( $\mathcal{A}$ ) by

$$0 \to 0 \to (C'_N, \mathrm{id} - c'_{N+1}) \xrightarrow{c'_N \circ i} C'_{N-1} \xrightarrow{c'_{N-1}} \cdots \xrightarrow{c'_1} C'_0 \to 0 \to \cdots$$

where  $i: (C'_N, \text{id} - c'_{N+1}) \to C'_N$  is the morphism in  $\text{Idem}(\mathcal{A})$  given by  $\text{id} - c'_{N+1}: C'_N \to C'_N$ . Let

$$u_*\colon P_*\to C'$$

be the Idem( $\mathcal{A}$ )-chain map for which  $u_m$  is the identity for  $m \leq N - 1$ ,  $u_N$  is  $i: (C'_N, id - c'_{N+1}) \to C'_N$ , and  $u_m: 0 \to C_m$  is the canonical map for  $m \geq N + 1$ . Let

$$v_* \colon C'_* \to P_*$$

be the Idem( $\mathcal{A}$ )-chain map which is given by the identity for  $m \leq N-1$ , by the canonical projection  $C'_m \to 0$  for  $m \geq N+1$  and for m = N by the morphism  $C_N \to (C'_N, \mathrm{id} - c'_{N+1})$  defined by  $\mathrm{id} - c'_{N+1} \colon C'_N \to C'_N$ . Obviously  $v_* \circ u_* = \mathrm{id}_{P_*}$ . We obtain a chain homotopy  $l'_* \colon \mathrm{id}_{C'_*} \sim u_* \circ v_*$  if we take  $l'_m = 0$  for  $m \leq N$ ,  $l'_m = c'_{N+1}$  for  $m \geq N, m - N \equiv 0 \mod 2$ , and  $l'_m = 1 - c'_{N+1}$  for  $m \geq N, m - N \equiv 1 \mod 2$ .

Define the desired  $Idem(\mathcal{B})$ -chain map

$$f_*\colon C_*\to P_*$$

to be the composite  $v_* \circ f'_*$  and the desired Idem( $\mathcal{B}$ )-chain map

$$g_*: P_* \to C_*$$

to be the composite  $g'_* \circ u_*$ . We obtain the desired Idem( $\mathcal{B}$ )-chain homotopies by putting

$$k_* = v_* \circ h_* \circ u_* \colon f_* \circ g_* \simeq \mathrm{id}_{P_*}$$

and

$$l_* = -g'_* \circ l'_* \circ f'_* + h_* \colon g_* \circ f_* \simeq \mathrm{id}_{C_*} \,.$$

## 23.7.6 Finite Domination and Width Functions

Consider a full and faithful inclusion  $\mathcal{A} \to \mathcal{B}$  of additive categories. Suppose that  $\mathcal{B}$  comes with a width function wd. Consider a  $\mathcal{B}$ -chain complex  $C_*$  together with a *finite domination*  $(D_*, i_*, r_*.h_*)$  of  $C_*$  over  $\mathcal{A}$ . For  $\epsilon > 0$  we call it  $\epsilon$ -controlled if wd $(i_*)$ , wd $(r_*)$ , wd $(h_*) \leq \epsilon$  hold.

# **Proposition 23.51.** *Fix a natural number N.*

Then, for every  $\epsilon > 0$  and every  $\epsilon$ -controlled domination of  $(D_*, i_*, r_*.h_*)$  of a  $\mathcal{B}$ -chain complex  $C_*$  over  $\mathcal{A}$  with dim $(D_*) \leq N$  and wd $(C_*)$ , wd $(D_*) \leq \epsilon$ , there is an N-dimensional Idem $(\mathcal{A})$ -chain complex  $P_*$  with wd $(P_*) \leq (N+2)\epsilon$  together with an  $(2N+5)\epsilon$ -controlled Idem $(\mathcal{B})$ -chain homotopy equivalence  $f_* \colon P_* \to C_*$ .

*Proof.* This follows from the explicit constructions of the *N*-dimensional  $\mathcal{A}$ -chain complex  $P_*$ , the Idem( $\mathcal{B}$ )-chain maps  $f_*: C_* \to P_*$  and  $g_*: P_* \to C_*$  and the Idem( $\mathcal{B}$ )-chain homotopies  $k_*: f_* \circ g_* \simeq \operatorname{id}_{P_*}$  and  $l_*: g_* \circ f_* \simeq \operatorname{id}_{C_*}$  of Subsection 23.7.5, the axioms appearing Definition 23.27, and Lemma 23.48.

# 23.7.7 Comparing Singular and Simplicial Chain Complexes

Let X = (X, d) be a metric space. As before we denote the singular chain complex of X by  $C_*^{\text{sing}}(X)$ . For  $\delta > 0$  we define

$$C^{\operatorname{sing},\,\delta}_*(X) \subset C^{\operatorname{sing}}_*(X)$$

to be the chain subcomplex generated by all singular *n*-simplices  $\sigma \colon \Delta_n \to X$  for  $n \ge 0$  for which the diameter of  $\sigma(\Delta_n)$  is less than or equal to  $\delta$ , i.e., for all  $y, z \in \Delta_n$  we have  $d(\sigma(y), \sigma(z)) \le \delta$ .

We have defined the additive category GM(X) and its width function  $wd_X$  in Example 23.31. The  $\mathbb{Z}$ -chain complex  $C_*^{\operatorname{sing},\delta}(X)$  can be considered as a GM(X)-chain complex, denoted again by  $C_*^{\operatorname{sing},\delta}(X)$ , via the barycenter map, i.e., for  $x \in X$  the module  $C_n^{\operatorname{sing},\delta}(X)_x$  is generated by all singular *n*-simplices which satisfy the condition above and map the barycenter of  $\Delta_n$  to *x*. Obviously  $wd_X(C_*^{\operatorname{sing},\delta}(X)) \leq \delta$  holds. Note that the image of the GM(X)-chain complex  $C_*^{\operatorname{sing},\delta}(X)$  under the functor *F* of (19.5) can be identified with the  $\mathbb{Z}$ -chain complex  $C_*^{\operatorname{sing},\delta}(X)$ .

The proof of the next result can be found in [87, Lemma 6.7].

**Lemma 23.52.** Let X = (X, d) be a metric space.

(i) For  $\delta' > \delta > 0$  the inclusion

$$\operatorname{inc}^{\delta,\delta'}_*: C^{\operatorname{sing},\delta}_*(X,d) \to C^{\operatorname{sing},\delta'}_*(X,d)$$

is a  $\delta'$ -controlled GM(X)-chain homotopy equivalence;

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- (ii) For every  $\delta > 0$  the inclusion

$$i: C_*^{\operatorname{sing},\delta}(X,d) \to C_*^{\operatorname{sing}}(X)$$

is a GM(X)-chain homotopy equivalence;

(iii) Suppose X = |L| for the geometric realization |L| of an abstract simplicial complex L. Let  $C_*(L)$  denote the simplicial chain complex considered as a GM(X)-chain complex via the barycenters. Suppose all simplices of L have diameter smaller than  $\delta$  with respect to the metric d on X.

Then  $\operatorname{wd}_X(C_*(L)) \leq \delta$  and realization defines a  $\operatorname{GM}(X)$ -chain map

$$C_*(L) \to C^{\operatorname{sing}, o}_*(X)$$

which is a  $\delta$ -controlled GM(X)-chain homotopy equivalence.

The next result is proved in [78, Lemma 8.5].

**Lemma 23.53.** Let X = (X, d) be a metric space. Consider  $\mu, \nu > 0$ . Let  $\varphi, \varphi' : X \to X$  be maps satisfying

$$d(x, y) \le \mu \implies d(\varphi(x), \varphi(y)), d(\varphi'(x), \varphi'(y)) \le v$$

for all  $x, y \in X$ . Let  $h: \varphi \simeq \varphi'$  be a homotopy.

Then there is a GM(X)-chain homotopy  $H_*: C_*^{\operatorname{sing},\mu,\nu}(\varphi)_* \simeq C_*^{\operatorname{sing},\mu,\nu}(\varphi')_*$  of GM(X)-chain maps  $C_*^{\operatorname{sing},\mu}(X) \to C_*^{\operatorname{sing},\nu}(X)$  satisfying

supp 
$$H_*$$
 ⊆ {( $h_t(x), y$ ) |  $t \in [0, 1], d(x, y) \le \mu$ }.

**Proposition 23.54.** Consider a natural number N and an N-transfer space X = (X, d) in the sense of Definition 20.9.

Then for every  $\epsilon > 0$  there is an N-dimensional Idem $(GM(X)^f)$ -chain complex  $P_*$  with  $wd_X(P_*) \leq (12N + 24)\epsilon$  together with a  $(24N + 60)\epsilon$ -controlled Idem(GM(X))-chain homotopy equivalence  $f_* \colon P_* \to C_*^{sing,\epsilon}(X)$ .

*Proof.* Fix  $\epsilon > 0$ . We can choose an *N*-dimensional abstract simplicial complex *K*, maps  $i: X \to |K|$  and  $r: |K| \to X$ , and a homotopy  $h: X \times [0,1] \to X$  from  $r \circ i$  to  $id_X$  which is  $\epsilon$ -controlled, i.e., for every  $x \in X$  the diameter of the subset  $h(\{x\} \times [0,1])$  of *X* is smaller than  $\epsilon$ . By subdividing *K* we can arrange that for any simplex  $\sigma \in K$  the diameter of the subset  $r(|\sigma|)$  of *X* is less or equal to  $\epsilon$ . This implies wd<sub>*X*</sub>( $C_*(K)$ )  $\leq \epsilon$ , where we consider  $C_*(K)$  as a GM(X)-chain complex using the image of the barycenters of simplices under *r*. Analogously we can consider  $C_*^{sing,3\epsilon}(|K|)$  as a GM(X)-chain complex with wd<sub>*X*</sub>( $C_*^{sing,3\epsilon}(|K|)$ )  $\leq 3\epsilon$ . We get for any  $x, y \in X$ 

$$d(r \circ i(x), r \circ i(y)) = d(h_0(x), h_0(y))$$
  

$$\leq d(h_0(x), h_1(x)) + d(h_1(x), h_1(y)) + d(h_1(y), h_0(y))$$
  

$$\leq \epsilon + d(x, y) + \epsilon = 2\epsilon + d(x, y).$$

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Hence  $i: X \to K$  induces a GM(X)-chain map

$$C^{\operatorname{sing},\epsilon,3\epsilon}_{*}(i) \colon C^{\operatorname{sing},\epsilon}_{*}(X) \to C^{\operatorname{sing},3\epsilon}_{*}(|K|).$$

Obviously r induces a GM(X)-chain map

$$C^{\operatorname{sing},3\epsilon}_*(r)\colon C^{\operatorname{sing},3\epsilon}_*(|K|)\to C^{\operatorname{sing},3\epsilon}_*(X).$$

Let

$$\operatorname{inc}^{\epsilon,3\epsilon}: C^{\operatorname{sing},\epsilon}_*(X) \to C^{\operatorname{sing},3\epsilon}_*(X)$$

be the inclusion. We conclude from Lemma 23.53 applied in the case  $\mu = \epsilon$  and  $\nu = 3\epsilon$  to  $h: r \circ i \simeq id_X$  that there is a GM(X)-chain homotopy of GM(X)-chain maps from  $C_*^{\text{sing},\epsilon}(X)$  to  $C_*^{\text{sing},3\epsilon}(X)$ 

$$H_*: C_*^{\operatorname{sing},\epsilon}(r) \circ C_*^{\operatorname{sing},\epsilon,3\epsilon}(i) \simeq \operatorname{inc}^{\epsilon,3\epsilon}$$

with  $\operatorname{wd}_X(H_*) \leq 2\epsilon$ , since for  $t \in [0, 1]$  and  $x, y \in X$  with  $d(x, y) \leq \epsilon$  we get

$$\begin{aligned} d(h_t(x), y) &\leq d(h_t(x), h_1(x)) + d(h_1(x), y) \\ &= d(h_t(x), h_1(x)) + d(x, y) \leq \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence we get

$$C^{\operatorname{sing},\epsilon}_*(r) \circ C^{\operatorname{sing},\epsilon,3\epsilon}_*(i) \simeq_{3\epsilon} \operatorname{inc}^{\epsilon,3\epsilon}$$

From Lemma 23.52 (iii) we get  $3\epsilon$ -controlled GM(X)-chain homotopy equivalences

$$\begin{aligned} a_* \colon C_*(K) &\to C^{\mathrm{sing}, 3\epsilon}_*(|K|);\\ b_* \colon C^{\mathrm{sing}, 3\epsilon}_*(|K|) &\to C_*(K), \end{aligned}$$

satisfying

$$\begin{split} b_* \circ a_* \simeq_{3\epsilon} \operatorname{id}_{C_*(K)}; \\ a_* \circ b_* \simeq_{3\epsilon} \operatorname{id}_{C_*^{\operatorname{sing}, 3\epsilon}(|K|)}. \end{split}$$

We conclude from Lemma 23.52 (i) that  $\operatorname{inc}^{\epsilon,3\epsilon} : C_*^{\operatorname{sing},\epsilon}(X) \to C_*^{\operatorname{sing},3\epsilon}(X)$  is a  $3\epsilon$ -controlled GM(X)-chain homotopy equivalence and we can choose a  $3\epsilon$ -controlled GM(X)-chain homotopy equivalence  $\operatorname{inc}_{\epsilon,3\epsilon}^{-1} : C_*^{\operatorname{sing},3\epsilon}(X) \to C_*^{\operatorname{sing},\epsilon}(X)$  satisfying

$$\begin{split} & \operatorname{inc}_{\epsilon,3\epsilon}^{-1} \circ \operatorname{inc}_{\epsilon,3\epsilon} \simeq_{3\epsilon} \operatorname{id}_{C^{\operatorname{sing},\epsilon}_*(X)}; \\ & \operatorname{inc}_{\epsilon,3\epsilon}^{-1} \circ \operatorname{inc}_{\epsilon,3\epsilon}^{-1} \simeq_{3\epsilon} \operatorname{id}_{C^{\operatorname{sing},3\epsilon}_*(X)}. \end{split}$$

Now define GM(X)-chain maps

$$j_* = b_* \circ C_*^{\operatorname{sing}, \epsilon, 3\epsilon}(i) \colon C_*^{\operatorname{sing}, \epsilon}(X) \to C_*(K)$$

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and

$$p_* = \operatorname{inc}_{\epsilon,3\epsilon}^{-1} \circ C_*^{\operatorname{sing},3\epsilon}(r) \circ a_* \colon C_*(K) \to C_*^{\operatorname{sing},\epsilon}(X)$$

We conclude using Lemma 23.48

$$p_* \circ j_* \simeq_{12\epsilon} \operatorname{id}_{C^{\operatorname{sing}},\epsilon}(X)$$
.

Now we get from Proposition 23.51 applied to the inclusion  $\operatorname{Idem}(\operatorname{GM}(X)^f) \to \operatorname{Idem}(\operatorname{GM}(X))$  and the domination of the  $\operatorname{Idem}(\operatorname{GM}(X))$ -chain complex  $C_*^{\operatorname{sing},\epsilon}(X)$  by the  $\operatorname{GM}(X)^f$ -chain complex  $C_*(K)$  above an *N*-dimensional  $\operatorname{Idem}(\operatorname{GM}(X))^f$ -chain complex  $P_*$  with  $\operatorname{wd}_X(P_*) \leq (12N + 16)\epsilon$  together with a  $(16N + 40)\epsilon$ -controlled  $\operatorname{Idem}(\operatorname{GM}(X))$ -chain homotopy equivalence  $f_* \colon P_* \to C_*^{\operatorname{sing},\epsilon}(X)$ .

# 23.7.8 Taking the Group Action on X into Account

Consider a natural number N and an N-transfer space X = (X, d) in the sense of Definition 20.9. Suppose that X comes with a (not necessarily isometric) *G*-action. Let  $T \subseteq G$  be a finite subset. Fix  $\epsilon > 0$ .

Since *T* is finite and *X* is compact, there exists a real number  $\delta$  with  $0 < \delta < \epsilon$ such that the implication  $d(x, y) \le \delta \implies d(gx, gy) \le \epsilon$  holds for all  $g \in T, x \in X$ and  $y \in X$ . By the same argument applied to  $\delta$  instead of  $\epsilon$  there exists a real number  $\gamma$  with  $0 < \gamma < \delta$  such that the implication  $d(x, y) \le \gamma \implies d(gx, gy) \le \delta$  holds for all  $g \in T, x \in X$ , and  $y \in X$ . Now we get from Proposition 23.54 an *N*-dimensional Idem(GM(X)<sup>f</sup>)-chain complex  $P_*$  together with Idem(GM(X))-chain homotopy equivalences  $f_* \colon P_* \to C_*^{\text{sing},\gamma}(X)$  and  $g_* \colon C_*^{\text{sing},\gamma}(X) \to P_*$  and Idem(GM(X))chain homotopies  $h_* \colon g_* \circ f_* \simeq \text{id}_{P_*}$  and  $k_* \colon f_* \circ g_* \simeq \text{id}_{C_*^{\text{sing},\gamma}(X)}$  such that wd<sub>X</sub>( $P_*$ ), wd<sub>X</sub>( $f_*$ ), wd<sub>X</sub>( $g_*$ ), wd<sub>X</sub>( $h_*$ ), and wd<sub>X</sub>( $k_*$ ) are less than or equal to 24(N + 60) $\gamma$ .

Define the finite subset  $T^2$  of G by

$$T^{2} = \{g \in G \mid \exists g_{1}, g_{2} \in T, g = g_{1}, g_{2}\}.$$

Since every  $g \in T^2$  satisfies  $d(x, y) \le \gamma \implies d(gx, gy) \le \epsilon$ , the map  $l_g : X \to X$  sending x to gx induces a GM(X)-chain map

$$C^{\operatorname{sing},\gamma,\epsilon}_*(l_{\varrho}): C^{\operatorname{sing},\gamma}_*(X) \to C^{\operatorname{sing},\epsilon}_*(X).$$

Now define for  $g \in T^2$  an Idem $(GM(X)^f)$ -chain homotopy equivalence

(23.55) 
$$\varphi[g]_* \colon P_* \to P_*$$

by the composite

$$P_* \xrightarrow{f_*} C_*^{\operatorname{sing},\gamma}(X) \xrightarrow{C_*^{\operatorname{sung},\gamma,\epsilon}(l_g)} C_*^{\operatorname{sing},\epsilon}(X) \xrightarrow{\operatorname{inc}_{\gamma,\epsilon}^{-1}} C_*^{\operatorname{sing},\gamma}(X) \xrightarrow{g_*} P_*$$

where  $\operatorname{inc}_{\gamma,\epsilon}^{-1}$  is an  $\epsilon$ -controlled GM(X)-chain homotopy equivalence coming from Lemma 23.52 (i) which is up to  $\epsilon$ -controlled homotopy an chain homotopy inverse of the  $\epsilon$ -controlled GM(X)-chain homotopy equivalence  $\operatorname{inc}_{\gamma,\epsilon}$ .

Recall that  $\operatorname{inc}_*^{\gamma} : C_*^{\operatorname{sing},\gamma} \to C_*^{\operatorname{sing}}(X)$  is the inclusion of  $\operatorname{Idem}(\operatorname{GM}(X))$ -chain complexes.

Proposition 23.56. We get with the choices above:

(i) We obtain for every  $g \in T^2$ 

$$(x, y) \in \operatorname{supp}(\varphi_g) \implies d(gx, y) \le (48N + 121)\epsilon;$$

(ii) For  $g, h \in T$  there exists an Idem(GM(X))-chain homotopy

$$\Phi[g,h]_* \colon \varphi[gh]_* \simeq \varphi[g]_* \circ \varphi[h]_*$$

satisfying

$$(x, y) \in \operatorname{supp}(\varphi_g) \implies d(ghx, y) \le (72N + 181)\epsilon;$$

(iii) We obtain for every  $g \in T^2$  an up to Idem(GM(X))-chain homotopy commutative diagram whose vertical arrows are Idem(GM(X))-chain homotopy equivalences



*Proof.* (i) We get  $wd_X(f_*) \le (24N + 60)\gamma$  and  $wd_X(g_*) \le (24N + 60)\gamma$  from Proposition 23.54 and  $wd_X(\operatorname{inc}_{\gamma,\epsilon}^{-1}) \le \epsilon$  from Lemma 23.52 (i). Obviously we have

(23.57) 
$$(x, y) \in \operatorname{supp}(C^{\operatorname{sung},\gamma,\epsilon}_*(l_g)) \Longrightarrow y = gx.$$

One easily checks for  $(x, y) \in \text{supp}(\varphi[g])$ 

$$d(gx, y) \le \operatorname{wd}_X(f_*) + \operatorname{wd}_X(\operatorname{inc}_{\gamma, \epsilon}^{-1}) + \operatorname{wd}_X(g_*)$$
$$\le (24N + 60)\gamma + \epsilon + (24N + 60)\gamma$$
$$\le (24N + 60)\epsilon + \epsilon + (24N + 60)\gamma\epsilon$$
$$= (48N + 121)\epsilon.$$

(ii) The desired homotopy  $\Phi[gh]_*$  is given by the composite of the following homotopies and identities

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$$\begin{split} \varphi[gh] \\ &= g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_{gh}) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \delta, \epsilon}(l_g) \circ C_*^{\operatorname{sing}, \gamma, \delta}(l_h) \circ f_* \\ \overset{(1)}{\cong} g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \delta, \epsilon}(l_g) \circ \operatorname{inc}^{\gamma, \delta} \circ (\operatorname{inc}^{\gamma, \delta})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \delta}(l_h) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_g) \circ (\operatorname{inc}^{\gamma, \delta})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \delta}(l_h) \circ f_* \\ \overset{(2)}{\cong} g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_g) \circ (\operatorname{inc}^{\gamma, \delta})^{-1} \circ (\operatorname{inc}^{\delta, \epsilon})^{-1} \circ \operatorname{inc}^{\delta, \epsilon} \\ &\circ C_*^{\operatorname{sing}, \gamma, \delta}(l_h) \circ f_* \\ &= g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_g) \circ (\operatorname{inc}^{\gamma, \delta})^{-1} \circ (\operatorname{inc}^{\delta, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_h) \circ f_* \\ \overset{(3)}{\cong} g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_g) \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_h) \circ f_* \\ \overset{(4)}{\cong} g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_g) \circ f_* \circ g_* \circ (\operatorname{inc}^{\gamma, \epsilon})^{-1} \circ C_*^{\operatorname{sing}, \gamma, \epsilon}(l_h) \circ f_* \\ &= \varphi[g] \circ \varphi[h]. \end{split}$$

In the sequel we will apply (an obvious variation of) Lemma 23.48 (ii) over and over again. Here the homotopy (1) comes from the  $\delta$ -controlled homotopy id  $\simeq \operatorname{inc}^{\gamma,\delta} \circ (\operatorname{inc}^{\gamma,\delta})^{-1}$  of Lemma 23.52 (i). The homotopy (2) comes from the  $\epsilon$ -controlled homotopy id  $\simeq \operatorname{inc}^{\delta,\epsilon} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1}$  of Lemma 23.52 (i). The homotopy (3) comes from the sequence homotopies each of which comes from Lemma 23.52 (i)

$$(\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1} \simeq (\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1} \circ \operatorname{inc}^{\gamma,\epsilon} \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} = (\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1} \circ \operatorname{inc}^{\delta,\epsilon} \circ \operatorname{inc}^{\gamma,\delta} \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \simeq (\operatorname{inc}^{\gamma,\epsilon})^{-1} \circ \operatorname{inc}^{\gamma,\delta} \circ (\operatorname{inc}^{\gamma,\epsilon})^{-1} \simeq (\operatorname{inc}^{\gamma,\epsilon})^{-1}.$$

One easily checks that the latter chain homotopy from  $(\operatorname{inc}^{\gamma,\delta})^{-1} \circ (\operatorname{inc}^{\delta,\epsilon})^{-1}$  to  $(\operatorname{inc}^{\gamma,\epsilon})^{-1}$  is 9 $\epsilon$ -controlled. The homotopy (4) comes from Proposition 23.54. We get wd<sub>X</sub>( $f_*$ )  $\leq (24N + 60)\gamma$  and wd<sub>X</sub>( $g_*$ )  $\leq (24N + 60)\gamma$  from Proposition 23.54 and wd<sub>X</sub>( $\operatorname{inc}^{\gamma,\epsilon})^{-1} \leq \epsilon$ , wd<sub>X</sub>( $(\operatorname{inc}^{\gamma,\delta})^{-1}$ )  $\leq \delta$ , and wd<sub>X</sub>( $(\operatorname{inc}^{\epsilon,\delta})^{-1}$ )  $\leq \delta$  from Lemma 23.52 (i). We have  $\gamma \leq \delta \leq \epsilon$ . Recall the implication (23.57), which we can apply to  $l_g$  and  $l_h$  and  $l_{gh}$ . One easily checks that for all  $(x, y) \in \operatorname{supp}(\Phi[g, h])$  we have

$$d(ghx, y) \le (72N + 181)\epsilon.$$

(iii). The chain map  $\operatorname{inc}_*^{\gamma} : C_*^{\operatorname{sing},\gamma} \to C_*^{\operatorname{sing}}(X)$  is a  $\operatorname{GM}(X)$ -chain homotopy equivalence by Lemma 23.52 (ii). The  $\operatorname{GM}(X)$ -chain maps  $f_* : P_* \to C_*^{\operatorname{sing},\gamma}(X)$  and  $g_* : P_* \to C_*^{\operatorname{sing},\gamma}(X)$  are chain homotopy inverses of one another. We have  $\operatorname{inc}_*^{\gamma,\epsilon} \circ \operatorname{inc}_*^{\epsilon} = \operatorname{inc}_*^{\gamma}$  and  $(\operatorname{inc}^{\gamma,\epsilon})^{-1}$  is a  $\operatorname{GM}(X)$  chain homotopy inverse of  $\operatorname{inc}^{\gamma,\epsilon}$ . Obviously  $\operatorname{inc}_*^{\epsilon} \circ C_*^{\operatorname{sing},\gamma,\epsilon}(l_g)$  and  $C_*^{\operatorname{sing}}(l_g) \circ \operatorname{inc}_*^{\gamma}$  agree. Now assertion (iii) follows. This finishes the proof of Proposition 23.56.

## **23.7.9** Passing to $Y = G \times X$

Now we consider the data we have fixed in Subsection 23.7.1. Recall that  $Y = G \times X$  with the *G*-action given by g'(g, x) = (g'g, x). We define a functor of additive categories

(23.58) 
$$\operatorname{ind}: \operatorname{GM}(X) \to \operatorname{GM}^G(Y)$$

as follows. An object  $M = \{M_x \mid x \in X\}$  is sent to the object

$$ind(M) = \{ind(M)_{(g,x)} \mid (g,x) \in Y\}$$

given by  $\operatorname{ind}(M)_{(g,x)} = M_x$ . A morphism  $f = \{f_{x,y} \colon M_x \to N_y \mid x, y \in X\}$  from  $M = \{M_x \mid x \in X\}$  to  $N = \{N_y \mid y \in X\}$  is sent to the morphism

$$ind(f) = \{ind(f)_{(g,x),(h,y)} \colon M_x \to N_y \mid (g,x), (h,y) \in Y\}$$

given by  $\operatorname{ind}(f)_{(g,x),(h,y)} = f_{x,y}$  if g = h and by  $\operatorname{ind}(f)_{(g,x),(h,y)} = 0$  if  $g \neq h$ . In the sequel we consider on  $\operatorname{GM}^G(Y)$  the width function  $\operatorname{wd}_Z$  of Example 23.29 with respect to the map  $w: G \times X \to Z$  that we have defined by w(g, x) = gv(x) for the given  $(\epsilon, T)$ -almost *G*-equivariant map  $v: X \to Z = |\Sigma|$  in Subsection 23.7.1. Obviously ind induces a functor of additive categories

(23.59) 
$$\operatorname{Idem}(\operatorname{ind}): \operatorname{Idem}(\operatorname{GM}(X)) \to \operatorname{Idem}(\operatorname{GM}^G(Y)).$$

Fix  $\epsilon > 0$ . Since  $v: X \to |Z|$  has a compact metric space as source, we can find a real number  $\xi > 0$  such that the implication

$$d(x,y) \leq (12N+24)\xi \implies d_{L^1}(v(x),v(y)) \leq \epsilon$$

holds for  $x, y \in X$ . From Proposition 23.54 we get Idem(GM(X))-chain maps  $f_*: P_* \to C^{sing, \xi}_*(X)$  and  $g_*: C^{sing, \xi}_*(X) \to P_*$ , and Idem(GM(X))-chain homotopies  $h_*: g_* \circ f_* \simeq id_{P_*}$  and  $k_*: f_* \circ g_* \simeq id_{C^{sing, \xi}(X)}$  such that

$$\mathrm{wd}_X(P_*), \mathrm{wd}_X(f_*), \mathrm{wd}_X(g_*), \mathrm{wd}_X(h_*), \mathrm{wd}_X(k_*) \le (12N + 24)\xi$$

holds, where  $wd_X$  is understood to be over the metric space X. They induce Idem(GM(X))-chain maps

 $Idem(ind)(f_*): Idem(ind)(P_*) \to Idem(ind)(C_*^{\epsilon}(X));$  $Idem(ind)(g_*): Idem(ind)(C_*^{\epsilon}(X)) \to Idem(ind)(P_*),$ 

and Idem(GM(X))-chain homotopies

 $Idem(ind)(h_*): Idem(ind)(g_*) \circ Idem(ind)(f_*) \simeq id_{Idem(ind)(P_*)};$  $Idem(ind)(k_*): Idem(ind)(f_*) \circ Idem(ind)(g_*) \simeq id_{Idem(ind)(C_*^{sing,\epsilon}(X))},$ 

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such that

wd(Idem(ind)(
$$P_*$$
)), wd(Idem(ind)( $f_*$ )), wd(Idem(ind)( $g_*$ )),  
wd(Idem(ind)( $h_*$ )), wd(Idem(ind)( $k_*$ ))  $\leq \epsilon$ 

holds. We give the proof for the width of  $Idem(ind)(f_*)$ , the proofs for the other terms are analogous. Consider  $(g, x), (h, y) \in supp(Idem(ind)(f_*))$ . Then we have g = h and  $x, y \in supp(f_*)$ . The latter implies  $d(x, y) \leq (12N + 24)\xi$  and hence  $d_{L^1}(v(x), v(y)) \leq \epsilon$ . We compute

$$\begin{aligned} d_{L^1}(w(g,x),w(h,y)) &= d_{L^1}(w(g,x),w(g,y)) = d_{L^1}(gv(x),gv(y)) \\ &= d_{L^1}(v(x),v(y)) \le \epsilon. \end{aligned}$$

Given  $g \in T$ , we define an Idem $(GM^G(Y))$ -chain map

$$U[g]_*$$
: Idem(ind)( $P_*$ )  $\rightarrow$  Idem(ind)( $P_*$ )

by putting  $(U(g)_n)_{(g_1,x_1),(g_1,x_2)}$ :  $(P_n)_{x_1} \to (P_n)_{x_2}$  to be equal to  $(\varphi[g]_n)_{x_1,x_2}$  if  $g_2 = g_1 g^{-1}$  and zero otherwise, where  $\varphi[g]_*$  has been introduced in (23.55). For  $g, h \in T$  we define Idem(GM<sup>G</sup>(G × X))-chain homotopies

$$H[g,h]_*: U(gh) \simeq U(g) \circ U(h)$$

by putting  $(H[g,h]_n)_{(g_1,x_1,g_2,x_2)}$ : Idem(ind) $(P_n)_{x_1} \rightarrow$  Idem(ind) $(P_{n+1})_{x_2}$  to be equal to  $(\Phi[g,h]_n)_{x_1,x_2}$  if  $g_2 = g_1(gh)^{-1}$  and zero otherwise, where  $\Phi[g,h]$  has been defined in the proof Proposition 23.56 (ii). Proposition 23.56 implies for  $g, h \in T$ 

(23.60) 
$$\operatorname{wd}(U[g]_*) \le (48N + 121)\epsilon;$$

(23.61) 
$$\operatorname{wd}(H[g,h]) \le (72N+181)\epsilon$$

For  $a = \sum_{g \in T} \lambda_g \cdot g \in \mathbb{Z}G$ , we define an Idem $(GM^G(G \times X))$ -chain map

$$U[a]_* = \sum_{g \in T} \lambda_g \cdot U[g]_* \colon \operatorname{Idem}(\operatorname{ind})(P_*) \to \operatorname{Idem}(\operatorname{ind})(P_*)$$

For elements  $a = \sum_{g \in T} \lambda_g \cdot g$  and  $b = \sum_{g \in T} \mu_h \cdot h$  in *RG*, we define a Idem(GM<sup>G</sup>(Y))-chain homotopy

$$H[a.b]_*: U[ab]_* \simeq U[a]_* \circ U[b]_*$$

by  $H[a.b]_* = \sum_{g,h\in T} \lambda_g \mu_h \cdot \Phi[g,h]_*$ : Idem(ind) $(P_*) \to$  Idem(ind) $(P_{*+1})$ . For the matrix  $A = (a_{i,j}) \in GL_n(\mathbb{Z}G)$  and its inverse  $B = (b_{i,j})$ , we define Idem( $GM^G(Y)$ )-chain maps for Idem(ind) $(P_*)^n = \bigoplus_{i=1}^n Idem(ind)(P_*)$ 

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$$U[A]_* = (U(a_{i,j}))_{i,j}: \operatorname{Idem}(\operatorname{ind})(P_*)^n \to \operatorname{Idem}(\operatorname{ind})(P_*)^n;$$
  
$$U[B]_* = (U(a_{i,j}))_{i,j}: \operatorname{Idem}(\operatorname{ind})(P_*)^n \to \operatorname{Idem}(\operatorname{ind})(P_*)^n.$$

Define  $Idem(GM^G(Y))$ -chain homotopies

$$\begin{split} K_* \colon U(A)_* \circ U(B)_* &\simeq \mathrm{id}_{\mathrm{Idem}(\mathrm{ind})(P_*)^n}; \\ L_* \colon U(B)_* \circ U(A)_* &\simeq \mathrm{id}_{\mathrm{Idem}(\mathrm{ind})(P_*)^n}, \end{split}$$

by  $(K_n)_{i,k} = \sum_{j=1}^n H(a_{i,j}b_{j,k})$  and  $(L_n)_{i,k} = \sum_{j=1}^n H(b_{i,j}a_{j,k})$  for  $i, k \in \{1, 2, ..., n\}$ . We conclude from the axioms appearing in Definition 23.27 and the inequalities (23.60) and (23.61)

(23.62) 
$$\operatorname{wd}(U[A]_*) \le (48N + 121)\epsilon;$$

(23.63) 
$$\operatorname{wd}(U[B]_*) \le (48N + 121)\epsilon;$$

(23.64) 
$$\operatorname{wd}(K) \le (72N + 181)\epsilon;$$

(23.65) 
$$\operatorname{wd}(L) \le (72N + 181)\epsilon$$

There is an obvious identification of  $GM^G(Y)$ -chain complexes

$$\operatorname{ind}(C_*^{\operatorname{sing}}(X)) = C_*^{\operatorname{sing}}(Y)$$

since *G* is discrete. Under this identification the two  $GM^G(Y)$ -chain maps  $ind(C_*^{sing}(l_g))$  and  $C_*^{sing}(L_g)$  agree, where  $L_g: Y \to Y$  sends (h, x) to  $(hg^{-1}, gx)$ . Under this identification we conclude from Proposition 23.56 (iii) that we obtain an up to chain homotopy commutative diagram of chain homotopy equivalences of  $Idem(GM^G(X))$ -chain complexes

$$\begin{array}{c} \operatorname{ind}(P_*) \xrightarrow{\operatorname{ind}(\operatorname{inc}^{\epsilon}_* \circ f_*)} & \subset C^{\operatorname{sing}}_*(Y) \\ & \bigvee U[g]_* & & \bigvee C^{\operatorname{sing}}_*(L_g) \\ \operatorname{ind}(P_*) \xrightarrow{\operatorname{ind}(\operatorname{inc}^{\epsilon}_* \circ f_*)} & \subset C^{\operatorname{sing}}_*(Y). \end{array}$$

If we apply the functor *F* of (19.5) to the diagram above and use the identification of  $F(C_*^{sing}(G \times X))$  with the singular  $\mathbb{Z}$ -chain complex  $C_*^{sing}(G \times X)$ , we obtain an up to  $\mathbb{Z}G$ -chain homotopy commutative diagram of  $\mathbb{Z}G$ -chain homotopy equivalences

$$F(\operatorname{ind}(P_*)) \xrightarrow{F(\operatorname{ind}(\operatorname{inc}_*^{\epsilon} \circ f_*))} C_*^{\operatorname{sing}}(Y)$$

$$\downarrow^{F(U[g]_*)} \qquad \qquad \downarrow^{C_*^{\operatorname{sing}}(L_g)}$$

$$F(\operatorname{ind}(P_*)) \xrightarrow{F(\operatorname{ind}(\operatorname{inc}_*^{\epsilon} \circ f_*))} C_*^{\operatorname{sing}}(Y).$$

Since *X* is contractible and  $Y = G \times X$  is equipped with the *G*-action  $g' \cdot (g, x) = (g'g, x)$ , the projection pr:  $Y = G \times X \to G$  is a *G*-homotopy equivalence and induces a  $\mathbb{Z}G$ -chain homotopy equivalence  $C_*^{\text{sing}}(Y) \to C_*^{\text{sing}}(G)$ . There is an obvious  $\mathbb{Z}G$ -chain homotopy equivalence  $a_*: C_*^{\text{sing}}(G) \to 0[\mathbb{Z}G]_*$  onto the  $\mathbb{Z}$ -chain complex concentrated in dimension 0 whose 0-th chain module is  $\mathbb{Z}G$ . We obtain a  $\mathbb{Z}G$ -chain homotopy equivalence

$$b_*: \operatorname{ind}(P_*) \xrightarrow{\operatorname{ind}(\operatorname{inc}_*^{\epsilon} \circ f_*)} C_*^{\operatorname{sing}}(Y) \xrightarrow{C_*^{\operatorname{sing}}(\operatorname{pr})} C_*^{\operatorname{sing}}(G) \xrightarrow{a_*} 0[\mathbb{Z}G]_*$$

such that the following diagram of finite free  $\mathbb{Z}G$ -chain complexes commutes up to  $\mathbb{Z}G$ -chain homotopy for every  $g \in T$ , where  $r_{g^{-1}} \colon G \to G$  sends g' to  $g'g^{-1}$ 

$$\begin{split} F(\operatorname{ind}(P_*)) & \xrightarrow{b_*} & 0[\mathbb{Z}G]_* \\ & \downarrow^{F(U[g]_*)} & \downarrow^{0[r_{g^{-1}}]} \\ F(\operatorname{ind}(P_*)) & \xrightarrow{b_*} & 0[\mathbb{Z}G]_*. \end{split}$$

One easily checks that following diagram of finite free  $\mathbb{Z}G$ -chain complexes commutes up to  $\mathbb{Z}G$ -chain homotopy, where V(A) has been defined in (23.25)

$$F(\operatorname{ind}(P_*))^n \xrightarrow{b_*} 0[\mathbb{Z}G^n]_*$$

$$\downarrow^{F(U[A]_*)} \qquad \qquad \downarrow^{0[V(A)]_*}$$

$$F(\operatorname{ind}(P_*))^n \xrightarrow{b_*} 0[\mathbb{Z}G^n]_*.$$

We conclude from Lemma 23.48

(23.66) 
$$u = [V(A)] = t(0[V(A)]_*) = t(F(U(A)_*)) \in K_1(\mathbb{Z}G).$$

(Note that (23.66) is closely related to up-down-formula, see Remark 23.20.) Recall that  $U(A)_*$  is a  $(72N + 181)\epsilon$ -controlled Idem $(GM^G(Y))$ -chain homotopy equivalence, see (23.62), (23.63), (23.64), and (23.65). Hence Proposition 23.50 together with (23.66) implies

$$u = [F^f(a)] \in K_1(\mathbb{Z}G)$$

for some  $5(72N + 181)\epsilon$ -controlled automorphism *a* in  $GM^G(Y)$ . By our choice  $\epsilon = \frac{\epsilon_N}{5(48N+181)}$ , see (23.26), we have  $5(72N + 181)\epsilon = \epsilon_N$ . This finishes the proof of Proposition 23.24.

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# 23.8 The Strategy Theorem

Consider a covariant functor

$$E: G-CW-COM \rightarrow SPECTRA.$$

Given a G-CW-complex space Z, we obtain from E a new covariant functor

(23.67) 
$$\mathbf{E}_Z : G \text{-} \mathbf{CW} \text{-} \mathbf{COM} \to \mathsf{SPECTRA}, \quad X \mapsto \mathbf{E}(X \times Z)$$

The canonical projection  $q: X \times Z \to X$  yields a transformation of covariant functors *G*-CW-COM  $\to$  SPECTRA.

$$(23.68) pr: \mathbf{E}_Z \to \mathbf{E}.$$

Let  $L: Or(G) \rightarrow G$ -CW-COM be the obvious inclusion.

Theorem 23.69 (Strategy Theorem). Suppose that the following conditions hold:

(i) The covariant functor

**E**: 
$$G$$
-CW-COM  $\rightarrow$  SPECTRA

is excisive;

(ii) There exists a map of covariant Or(G)-spectra

$$trf: L^*E \rightarrow L^*E_Z$$

such that the composite  $L^*\mathbf{pr} \circ \mathbf{trf} : L^*\mathbf{E} \to L^*\mathbf{E}$  is a weak homotopy equivalence of covariant Or(G)-spectra;

(iii) The projection onto the second factor  $pr_2: Z \times Z \rightarrow Z$  is a homotopy equivalence of *G*-*CW*-complexes.

Then

$$H_n^G(\operatorname{pr}; L^*\mathbf{E}) \colon H_n^G(Z; L^*\mathbf{E}) \to H_n^G(\{\bullet\}, L^*\mathbf{E})$$

is bijective for all  $n \in \mathbb{Z}$ , where pr:  $Z \to \{\bullet\}$  is the projection. Moreover, we obtain for all  $n \in \mathbb{Z}$  a commutative diagram of isomorphisms

$$\begin{array}{c|c} H_n^G(Z; L^* \mathbf{E}) & \xrightarrow{H_n^G(\mathrm{pr}; L^* \mathbf{E})} & \xrightarrow{H_n^G(\{\bullet\}; L^* \mathbf{E})} \\ & \cong & \downarrow & \cong & \downarrow \\ & & & \cong & \downarrow \\ & & & & \pi_n(\mathbf{E}(Z)) & \xrightarrow{\cong} & \pi_n(\mathbf{E}(\{\bullet\})). \end{array}$$

#### 23.8 The Strategy Theorem

*Proof.* The desired commutative diagram appearing in assertion (iii) comes from Theorem 18.11 applied to E. Moreover, by Theorem 18.11 the vertical arrows are bijective for all  $n \in \mathbb{Z}$ . It remains to prove the bijectivity of  $H_n^G(\text{pr}; L^*\mathbf{E})$ :  $H_n^G(Z; L^*\mathbf{E}) \to H_n^G(\{\bullet\}, L^*\mathbf{E})$  for all  $n \in \mathbb{Z}$ . We have the following commutative diagram

$$\begin{aligned} H_n^G(Z; L^* \mathbf{E}) & \xrightarrow{H_n^G(\mathrm{pr}; L^* \mathbf{E})} & \xrightarrow{H_n^G(\{\mathbf{e}\}; L^* \mathbf{E})} \\ H_n(Z; \mathrm{trf}) & & \downarrow H_n(\{\mathbf{e}\}; \mathrm{trf}) \\ H_n^G(Z; L^* \mathbf{E}_Z) & \xrightarrow{H_n^G(\mathrm{pr}; L^* \mathbf{E}_Z)} & \xrightarrow{H_n^G(\{\mathbf{e}\}; L^* \mathbf{E}_Z)} \\ H_n(Z; L^* \mathrm{pr}) & & \downarrow H_n(\{\mathbf{e}\}; L^* \mathrm{pr}) \\ H_n^G(Z; L^* \mathbf{E}) & \xrightarrow{H_n^G(\mathrm{pr}; L^* \mathbf{E})} & \xrightarrow{H_n^G(\{\mathbf{e}\}; L^* \mathbf{E})} \end{aligned}$$

for which the composites of the vertical arrows are in both cases isomorphisms by Lemma 12.6. Hence it suffices to show that  $H_n^G(\text{pr}; L^*\mathbf{E}_Z)$  is bijective for all  $n \in \mathbb{Z}$ . From Theorem 18.11 applied to  $\mathbf{E}_Z$ , we obtain a commutative diagram

whose vertical arrows are bijective. Since  $pr_2$  is by assumption a G-homotopy equivalence,  $H_n^G(\text{pr}; L^* \mathbf{E}_Z)$  is bijective for all  $n \in \mathbb{Z}$ . П

Let  $\mathcal{A}$  be any additive *G*-category. We have defined the additive category  $\mathcal{A}[G]$ in Example 21.2 and explained in Remark 21.79 that it comes with the structure of a strong category with G-support in the sense of Definition 21.78. So we can consider the covariant Or(G)-spectra  $\mathbf{K}^{\mathcal{D}^G}$  of (21.107) and  $\mathbf{K}^{\mathcal{D}^G_0}$  of (21.108). We get another covariant Or(G)-spectrum  $\mathbf{K}^{\mathcal{D}^G_0}_{E_{VCY}(G)}$  by sending an object G/H to  $\mathbf{K}(\mathcal{D}^G_0(G/H \times E_{VCY}(G)))$ , see (23.67).

Theorem 23.70 (Transfer criterion for the Farrell-Jones Conjecture). Suppose that there is a map of covariant Or(G)-spectra

$$\mathrm{trf} \colon \mathrm{K}^{\mathcal{D}_0^G} \to \mathrm{K}^{\mathcal{D}_0^G}_{E_{\mathcal{VC}\mathcal{Y}}(G)}$$

such that  $\mathbf{pr} \circ \mathbf{trf}$  is a weak homotopy equivalence of covariant Or(G)-spectra, where **pr** has been defined in (23.68). Then the K-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories 13.11 holds for G.

The analogous statement holds for the L-theoretic Farrell-Jones Conjecture with coefficients in additive G-categories with involution 13.19.

*Proof.* We give the proof for *K*-theory only, the one for *L*-theory is completely analogous.

The projection onto the second factor  $\text{pr}_2: E_{\mathcal{VCY}}(G) \times E_{\mathcal{VCY}}(G) \rightarrow E_{\mathcal{VCY}}(G)$ is a *G*-homotopy equivalence by Theorem 11.19. The functor  $\mathbf{K} \circ \mathcal{D}_0^G: \text{CW-COM} \rightarrow$ SPECTRA of (21.106) is excisive by Theorem 21.126. We conclude from Theorem 23.69 applied to it that

$$H_n^G(\mathrm{pr}; \mathbf{K}^{\mathcal{D}_0^G}) \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}^{\mathcal{D}_0^G}) \to H_n^G(\{\bullet\}, \mathbf{K}^{\mathcal{D}_0^G})$$

is bijective for all  $n \in \mathbb{Z}$ . Now Theorem 21.109 using Mayer-Vietoris sequences and the Five Lemma implies that

$$H_n^G(\mathrm{pr};\mathbf{K}^{\mathcal{D}^G})\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}^{\mathcal{D}^G})\to H_n^G(\{\bullet\},\mathbf{K}^{\mathcal{D}^G})$$

is bijective for all  $n \in \mathbb{Z}$ . We conclude from Lemma 21.76 (i), Remark 21.82, and Lemma 12.6 that the assembly map appearing in the *K*-theoretic Farrell-Jones Conjecture with coefficients in additive *G*-categories 13.11 is bijective for all  $n \in \mathbb{Z}$ .

The benefit of Theorem 23.70 is that it suffices to construct the transfer only on homogeneous spaces and for the functor  $\mathcal{D}_0^G$  which has the pleasant feature that it is defined with zero-control in the  $\mathbb{N}$ -direction. This has for instance been exploited in [81, Remarks 6.14 and 7.17].

# **23.9 Notes**

There seems to be no construction of a transfer in the Baum-Connes setting. That is the reason why some of the spectacular results about the validity of the Farrell-Jones Conjecture for certain groups, for instance all lattices in second countable locally compact Hausdorff groups, do not carry over to the Baum-Connes Conjecture. This might be different if one replaces the group  $C^*$ -algebra by the group Fréchet algebra.

# Chapter 24 Higher Categories as Coefficients

# 24.1 Introduction

In this chapter we give more information and details about the proofs of Theorem 20.61 and Theorem 20.62 following [185]. We have already seen in Lemma 20.2 that these are the most general results about the (*K*-theoretic) Farrell-Jones Conjecture, which imply and unify all the ones proved so far. Some basic strategies and some of the history of their proofs have already been discussed in special cases in Chapter 19 and Chapter 23. Roughly speaking, the main achievement in [185] is to generalize the formulations and proofs of the *K*-theoretic Farrell-Jones Conjecture as they appear for additive categories, for instance in [72, 78, 79, 86, 543, 992, 993], to higher categories, where a first step in this direction was already taken for *A*-theory in [344, 960].

Recall the general setup for assembly maps with higher categories as coefficients from Section 8.5. Starting with a right-exact G- $\infty$ -category  $C: I(G) \rightarrow CATREX$ , left Kan extension along the fully faithful functor  $I(G) \rightarrow Or(G)$  yields a functor  $E_C \circ O: Or(G) \rightarrow CATREX$  which sends G/H to  $\operatorname{colim}_{I(H)} C$ , see also Proposition 8.37. As explained in Sections 8.5.3, 8.5.4, and 8.5.5, this is the appropriate analog of the formation of group rings in this setting. Composing this functor with any finitary localizing invariant F (e.g., algebraic K-theory), we obtain a spectrumvalued functor

(24.1) 
$$F_C := F \circ E_C \circ O \colon \operatorname{Or}(G) \to \operatorname{Sp}$$

Applying the construction from Section 8.5.2, we obtain a *G*-homology theory in the guise of a colimit-preserving functor

$$\mathbf{H}^{G}(-; F_{C})$$
: Fun(Or(G)<sup>op</sup>, Spc)  $\rightarrow$  Sp,

and thus also an assembly map associated to any family  $\mathcal{F}$  of subgroups.

We are going to give an argument why this assembly map is an equivalence for Dress-Farrell-Hsiang-Jones groups over  $\mathcal{F}$ . There is a slightly more general version of this argument which allows F to take values in an arbitrary stable  $\infty$ -category. For the sake of concreteness and to avoid some ultimately irrelevant technicalities, we will focus on the case of spectrum-valued localizing invariants. See [185] for the general version.

# 24.2 Strategy of the Proof

In the following, we will refer to objects of  $Fun(Or(G)^{op}, Spc)$  as *G*-spaces. The reader may prefer to think of these as *G*-*CW*-complexes, see Section 8.5.2 for further explanations.

The general strategy of proof is very close to the argument explained in Section 23.8: we want to use some sort of "transfer" to exhibit the assembly map as a retract of another assembly map whose "coefficients" have a form which makes it more feasible to prove that it is an equivalence. However, the concrete implementation of this idea is somewhat different.

The "new" assembly map does not come about by considering products with another topological space. Instead, we introduce a new variable to the homology theory  $\mathbf{H}^G(-; F_C)$  which digests the more geometric input data provided by the Dress-Farrell-Hsiang-Jones condition. Since that condition does not provide us with a single space, but rather with a sequence  $(M_n)_n$  of spaces, (think of the spaces  $Z_{n,D}$ or  $\Sigma_{n,D}$  in the Definition 20.60 of a Dress-Farrell-Hsiang-Jones group), the new variable will come in the form of a sequence of spaces. In order to retain symmetry, the extended functor also takes sequences of *G*-spaces as input in the first variable. Let us write  $\widetilde{\mathbf{H}}^G(-; F_C)$  for this extended functor. For the sake of brevity, we are intentionally vague about the type of the second input variable. Precise statements can be found in Section 24.3 below.

For this extended functor, it becomes unlikely to find a retraction back to the original assembly map. As a replacement, we will consider the functor  $\overline{\Delta} \colon \text{Sp} \to \text{Sp}$  which sends a spectrum **E** to  $\overline{\Delta}(\mathbf{E}) := \text{cofib}(\bigoplus_{\mathbb{N}} \mathbf{E} \to \prod_{\mathbb{N}} \mathbf{E})$ . This functor comes equipped with a natural transformation  $\delta \colon \text{id} \Rightarrow \overline{\Delta}$  which is objectwise given by the composite

$$\mathbf{E} \xrightarrow{\text{diag.}} \prod_{\mathbb{N}} \mathbf{E} \to \overline{\Delta}(\mathbf{E}).$$

That is, we will produce a commutative diagram of the following shape: (24.2)



Assuming the middle horizontal arrow is an equivalence, it is a matter of diagram chasing to see that the assembly map at the top is an equivalence as well, since  $\pi_n(\overline{\Delta}(\mathbf{E})) \cong \prod_{\mathbb{N}} \pi_n(\mathbf{E}) / \bigoplus_{\mathbb{N}} \pi_n(\mathbf{E})$  holds. This presents us with the following tasks:

## Task 1

We have to explain what the new geometric variable is and how we extend  $\mathbf{H}^{G}(-; F_{C})$  to a functor  $\widetilde{\mathbf{H}}^{G}(-; F_{C})$  in two variables. Moreover, we have to identify a class of inputs  $(\Sigma_{n})_{n}$  for the second variable which ensures that the associated assembly map

$$\widetilde{\mathbf{H}}^{G}((E_{\mathcal{F}}(G))_{n}, (\Sigma_{n})_{n}; F_{C}) \to \widetilde{\mathbf{H}}^{G}((G/G)_{n}, (\Sigma_{n})_{n}; F_{C})$$

is an equivalence, where  $(E_{\mathcal{F}}(G))_n$  and  $(G/G)_n$  indicate the constant sequence on  $E_{\mathcal{F}}(G)$  and G/G, respectively. This is the subject of Section 24.3. The punchline is that we may plug in any sequence  $(\Sigma_n)_n$  of G-simplicial complexes whose dimension is uniformly bounded and whose stabilizers lie in the family  $\mathcal{F}$ .

• Task 2

We have to factor the transformation  $\delta$  as shown in the above diagram. It will turn out that the lower vertical maps in diagram 24.2 are always defined, while the existence of the maps labeled *t* will rely on the data provided by the Dress-Farrell-Hsiang-Jones condition. This will be explained in Section 24.4.

In general, we will only give an outline of the arguments, focusing rather on ideas and heuristics. The reader interested in full details is referred to [185].

# 24.3 Introducing the Geometric Variable

Before giving the construction of the extended functor  $\widetilde{\mathbf{H}}^G(-; F_C)$ , let us formulate a number of desiderata this extension is supposed to satisfy. Generalizing the functor  $\overline{\Delta}$ , let us define for any sequence  $(\mathbf{E}_n)_n$  of spectra their *reduced product* as

(24.3) 
$$\prod_{n\in\mathbb{N}}^{\operatorname{red}}\mathbf{E}_{n} := \operatorname{cofib}\Big(\bigoplus_{n\in\mathbb{N}}\mathbf{E}_{n} \to \prod_{n\in\mathbb{N}}\mathbf{E}_{n}\Big).$$

(i) Since  $\delta$  is supposed to factor over the extension  $\widetilde{\mathbf{H}}^G(-; F_C)$ , it appears sensible to require for every sequence of *G*-spaces  $(X_n)_n$  that there are natural maps

$$\widetilde{\mathbf{H}}^G((X_n)_n, (*)_n; F_C) \to \prod_{n \in \mathbb{N}}^{\text{red}} \mathbf{H}^G(X_n; F_C)$$

if we plug in the constant sequence on the point in the second variable. These maps should feature in constructing the lower vertical maps in (24.2);

(ii) We have to identify a reason why the map induced by the projection  $E_{\mathcal{F}}(G) \rightarrow G/G$ 

(24.4) 
$$\widetilde{\mathbf{H}}^G((E_{\mathcal{F}}(G))_n, (\Sigma_n)_n; F_C) \to \widetilde{\mathbf{H}}^G((G/G)_n, (\Sigma_n)_n; F_C)$$

is an equivalence for a sequence  $(\Sigma_n)_n$  of *G*-simplicial complexes whose dimension is uniformly bounded and whose stabilizers are contained in  $\mathcal{F}$ . If the second variable is suitably excisive, the pushout squares describing the attachment of the top-dimensional equivariant simplices should induce a pullback square of spectra

$$\begin{split} \widetilde{\mathbf{H}}^{G}((E_{\mathcal{F}}(G))_{n},(S_{n}\times\partial\Delta^{d})_{n};F_{C}) &\longrightarrow \widetilde{\mathbf{H}}^{G}((E_{\mathcal{F}}(G))_{n},(\Sigma_{n}^{(d-1)})_{n};F_{C}) \\ & \downarrow \\ & \downarrow \\ \widetilde{\mathbf{H}}^{G}((E_{\mathcal{F}}(G))_{n},(S_{n}\times\Delta^{d})_{n};F_{C}) &\longrightarrow \widetilde{\mathbf{H}}^{G}((E_{\mathcal{F}}(G))_{n},(\Sigma_{n})_{n};F_{C}). \end{split}$$

The projection map  $E_{\mathcal{F}}(G) \to G/G$  induces a transformation from this square to the analogous square with G/G replacing  $E_{\mathcal{F}}G$ . By induction, we can assume that this transformation is an equivalence on the upper two corners of the square. If the second variable of the extended functor is also equivariantly homotopy invariant, the map between the bottom left corners simplifies to

(24.5) 
$$\widetilde{\mathbf{H}}^G((E_{\mathcal{F}}(G))_n, (S_n)_n; F_C) \to \widetilde{\mathbf{H}}^G((G/G)_n, (S_n)_n; F_C).$$

It remains to explain why this map (24.5) is an equivalence, since then the map (24.4) is an equivalence.

(iii) We would like to arrange things so that there are natural equivalences

$$\widetilde{\mathbf{H}}^G((X_n)_n, (S_n)_n; F_{\mathcal{C}}) \xrightarrow{\simeq} \widetilde{\mathbf{H}}^G((X_n \times S_n)_n, (*)_n; F_{\mathcal{C}}).$$

----

Hence the map (24.5) is an equivalence if and only if the map

(24.6) 
$$\widetilde{\mathbf{H}}^G((E_{\mathcal{F}}(G) \times S_n)_n, (*)_n; F_{\mathcal{C}}) \to \widetilde{\mathbf{H}}^G((G/G \times S_n)_n, (*)_n; F_{\mathcal{C}})$$

is an equivalence.

\_\_\_\_

The projection maps  $E_{\mathcal{F}}(G) \times S_n \to S_n$  are *G*-homotopy equivalences because each  $S_n$  is a *G*-set with isotropy in  $\mathcal{F}$ , i.e., these maps are equivalences in Fun(Or(*G*)<sup>op</sup>, Spc). Hence the map (24.4) is an equivalence. This will take care of Task 1 appearing in Section 24.2.

This argument simultaneously takes care of the 0-dimensional case, providing the start of the inductive argument, see Theorem 24.13 (iv).

These desiderata alone do not tell us what the "correct" construction of the extension  $\widetilde{\mathbf{H}}^G(-; F_C)$  is. Ultimately, we want to be able to make use of certain distance estimates in metric spaces like those that appear in the Dress-Farrell-Hsiang-Jones condition 20.60, so the extension should make substantial use of geometric information. This prompts us to take a cue from Chapter 21, which shows that the methods of controlled algebra allow us to construct a homology theory by considering controlled modules with continuous control, or, more generally controlled objects over additive categories. Obviously, we will have to generalize these methods to work for right-

#### 24.3 Introducing the Geometric Variable

exact  $\infty$ -categories instead of additive categories. Our treatment here constitutes an attempt to work as much in analogy to Chapter 21 as possible. In particular, we will only concern ourselves with objects satisfying a version of continuous control. This allows us to avoid introducing a slew of additional terminology at the expense of somewhat ad-hoc looking definitions. For a more systematic treatment of controlled methods over right-exact  $\infty$ -categories, see [180] and [185], which use the language of equivariant *bornological coarse spaces* developed in [181].

For the reader who is trying to keep track of the parallels between the definitions we are about to present and the treatment in Chapter 21, let us point out that the definitions in this chapter are analogous to the version of continuous control with zero control over  $\mathbb{N}$  from Section 21.10.

In what follows, let  $(M_n)_n$  be a sequence of metric spaces.

One central example of a right-exact  $\infty$ -category is the  $\infty$ -category of pointed finite spaces  $\operatorname{Spc}_*^f$ . This  $\infty$ -category can be characterized as the smallest full subcategory of  $\operatorname{Spc}_*$  which contains the 0-sphere  $S^0$  and is closed under finite colimits. A more concrete description of  $\operatorname{Spc}_*^f$  is as the Dwyer-Kan localization of the category of pointed finite *CW*-complexes  $\operatorname{CW}_*^f$  at the collection of (pointed) homotopy equivalences. In order to provide some motivation for our definition of controlled objects, we will first introduce a category of controlled *CW*-complexes, which gives a 1-categorical model for the category of controlled objects over  $\operatorname{Spc}_*^f$ , and then proceed to the general construction from there. Such categories originate in the work of Vogell [967], but our concrete implementation is closer to the treatment by Weiss [1000].

The definition of a controlled *CW*-complex is relatively straightforward if one keeps in mind that passage from the *CW*-complex to its cellular chain complex should yield a chain complex of controlled abelian groups: each chain group should be a geometric module, and the differentials should be controlled morphisms. Moreover, a morphism of controlled *CW*-complexes is expected to induce a morphism of chain complexes of controlled abelian groups.

**Definition 24.7.** Let *Q* be a pointed *CW*-complex, i.e., a relative *CW*-complex with (-1)-skeleton  $Q^{(-1)} = *$ .

(i) Denote by

$$c_d(Q) := \pi_0(Q^{(d)} \setminus Q^{(d-1)})$$

the set of open *d*-cells of *Q* and set  $c(Q) := \bigcup_{d \ge 0} c_d(Q)$ ; (ii) A *labeling* on *Q* is a function

$$\ell \colon c(Q) \to \coprod_{n \in \mathbb{N}} M_n.$$

We call the pair  $(Q, \ell)$  a *labeled CW-complex*.

For any subset  $A \subseteq Q$  of a *CW*-complex Q, denote by  $\langle A \rangle$  the smallest subcomplex of Q containing A. In the following, set

$$M:=\coprod_{n\in\mathbb{N}}M_n,$$

where we use the convention that the distance of two points in different  $M_n$ -s is  $\infty$ .

**Definition 24.8.** Let  $(Q, \ell)$  and  $(Q', \ell')$  be labeled *CW*-complexes and let  $f: Q \to Q'$  be a cellular map.

- (i) We call (Q, ℓ) locally finite if ℓ<sup>-1</sup>(B) is finite for every metrically bounded subset B of M;
- (ii) We call *f* continuously controlled if the following conditions are satisfied:

(a) if 
$$\ell(z) \in M_n$$
, then  $\ell(c(\langle f(z) \rangle)) \subseteq M_n$ ;  
(b)  $\sup \left\{ d(\ell(z), \ell(z')) \mid \ell(z) \in M_n, \ z' \in c(\langle f(z) \rangle) \right\} \xrightarrow{n \to \infty} 0$ ;

(iii) We call  $(Q, \ell)$  continuously controlled if the identity map on Q is continuously controlled.

Denote the category of locally finite, continuously controlled *CW*-complexes and continuously controlled cellular maps by  $CW^{lf}((M_n)_n)$ .

Definition 24.8 is basically dictated by the requirement that taking cellular chain complexes should define a functor from controlled *CW*-complexes to chain complexes of geometric modules. If this definition is indicative of what controlled objects over a right-exact  $\infty$ -category are, we have to find a way to express the data encoded in a locally finite, continuously controlled *CW*-complex  $(Q, \ell)$  without referring to the cells of Q. Thinking of Spc<sub>\*</sub> as a localization of the category of pointed *CW*complexes, we can try to describe  $(Q, \ell)$  in terms of its subcomplexes: the labeling of Q induces a functor

$$C_O: \mathcal{P}(M) \to \mathrm{CW}_*$$

from the power set of M, considered as a poset, to the category of pointed CWcomplexes sending a subset B to the maximal subcomplex  $Q_B \subseteq Q$  which satisfies  $\ell(c(Q_B)) \subseteq B$ .

The data encoded in the functor  $C_Q$  are quite redundant. To make this more precise, observe that  $C_Q$  admits a filtration

$$C_Q^{(-1)} \subseteq C_Q^{(0)} \subseteq C_Q^{(1)} \subseteq \cdots \subseteq C_Q,$$

where  $C_Q^{(d)}(B) := C_Q(B)^{(d)}$ . Note that  $\operatorname{colim}_d C_Q^{(d)} \cong C_Q$ . This filtration exhibits  $C_Q$  as a cell complex over its indexing category  $\mathcal{P}(M)$  in the sense that for each *d* there exists a pushout

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In particular,  $C_Q$  is a cofibrant object in the projective model structure on Fun( $\mathcal{P}(M)$ , Top<sub>\*</sub>).

This observation can be refined. Consider the collection  $\mathcal{U}$  of subsets  $U \subseteq \prod_{n \in \mathbb{N}} (M_n \times M_n)$  satisfying

$$\sup\left\{d(x,x')\mid (x,x')\in U\cap (M_n\times M_n)\right\}\xrightarrow{n\to\infty} 0,$$

so the continuous control condition on  $(Q, \ell)$  translates into

$$\left\{d(\ell(z),\ell(z')) \mid z \in c(Q), \ z' \in c(\langle z \rangle)\right\} \in \mathcal{U}.$$

In particular, there exists some  $U \in \mathcal{U}$  such that for all  $z \in c(K)$  we have

$$\ell(\langle z \rangle) \times \ell(\langle z \rangle) \subseteq U.$$

Then a relative version of our observation about the cell structure on  $C_Q$  implies that the functor  $L \circ C_Q : \mathcal{P}(M) \to \operatorname{Spc}_*$  obtained by composing with the localization functor  $L : \operatorname{Top}_* \to \operatorname{Spc}_*$  (which is the pointed version of the localization functor in (8.1)) is left Kan extended from the sub-poset

$$\mathcal{P}^{U,T}(M) := \{ B \in \mathcal{P}(S) \mid B \times B \subseteq U, \ B \subseteq T \}$$

for some  $U \in \mathcal{U}$  and some locally finite subset  $T \subseteq M$ , i.e., a subset T whose intersection with each metrically bounded subset of M is finite. See [185, Proposition 4.8] for a detailed argument. Moreover, every element of  $\mathcal{P}^{U,T}(M)$  is a metrically bounded subset of M, so  $Q_B$  is a finite complex for every  $B \in \mathcal{P}^{U,T}(M)$ . Hence  $(L \circ C_O)(B) \in \text{Spc}_*^f$  for such subsets.

Our new description of controlled *CW*-complexes is entirely in terms of (finite) *CW*-complexes. This prompts the following definition, in which Ind(C) denotes the free filtered colimit-completion of a right-exact  $\infty$ -category *C*.

**Definition 24.9.** Let *C* be a right-exact  $\infty$ -category. A functor  $C : \mathcal{P}(M) \to \text{Ind}(C)$  is a  $\mathcal{U}$ -controlled object if there exist  $U \in \mathcal{U}$  and a locally finite subset  $T \subseteq M$  such that the following holds:

(i) the functor *C* is left Kan extended from  $\mathcal{P}^{U,T}(M)$ ;

(ii) for every  $B \in \mathcal{P}^{U,T}(M)$ , the value C(B) lies in C.

While we have succeeded in describing controlled *CW*-complexes without any mention of cells, the full subcategory of  $\mathcal{U}$ -controlled objects in Fun( $\mathcal{P}(M)$ , Spc<sup>f</sup><sub>\*</sub>) fails to capture the notion of a controlled morphism because a continuously controlled

morphism  $f: (Q, \ell) \to (Q', \ell')$  does not induce a natural transformation  $C_Q \Rightarrow C_{Q'}$ .

This can be remedied as follows. For any  $U \in \mathcal{U}$  and subset *B* of *M*, there exists the *U*-thickening

$$U[B] := \{x \in M \mid \exists b \in B \text{ satisfying } (x, b) \in U\}.$$

This construction defines a functor  $U[-]: \mathcal{P}(M) \to \mathcal{P}(M)$ . While a continuously controlled morphism  $f: (Q, \ell) \to (Q', \ell')$  does not induce a natural transformation  $F_Q \Rightarrow F_{Q'}$ , it does induce a natural transformation  $C_Q \Rightarrow C_{Q'} \circ U[-]$  for sufficiently large  $U \in \mathcal{U}$ .

One problem with the functor  $C_{Q'} \circ U[-]$  is that it is not a  $\mathcal{U}$ -controlled object anymore. However, the functor U[-] admits a right adjoint functor U(-), so natural transformations  $C_Q \Rightarrow C_{Q'} \circ U[-]$  are the same as natural transformations  $C_Q \circ U(-) \Rightarrow C_{Q'}$ , and one can check that precomposition with U(-) preserves  $\mathcal{U}$ -controlled objects, see [180, Corollary 3.3.29]. The choice of  $U \in \mathcal{U}$  is not canonical, and replacing U by a larger set  $U' \in \mathcal{U}$  should determine the same map, so we are looking for a category whose objects are  $\mathcal{U}$ -controlled objects and whose mapping spaces are given by

$$\operatorname{mor}(C, C') \simeq \operatorname{colim}_{U \in \mathcal{U}} \operatorname{nat}(C \circ U(-), C').$$

If we were working with ordinary categories, such a category could be constructed by hand, but it is also determined by a universal property: using the methods of [412], one can identify the result as the localization of the category of  $\mathcal{U}$ -controlled objects at the collection of morphisms  $C \circ U(-) \Rightarrow C$ . As long as U contains the diagonal of M, such a comparison map exists, since then  $U(B) \subseteq B$  for all subsets B of M. The sub-poset of  $\mathcal{U}$  given by the sets containing the diagonal is cofinal in  $\mathcal{U}$ , so the above colimit does not change if we restrict to this sub-poset.

There are analogous tools to understand localizations of  $\infty$ -categories, see for example [242, Chapter 7.2]. Letting  $\tilde{C}((M_n)_n)$  denote the full subcategory of Fun( $\mathcal{P}(M)$ , Ind(C)) spanned by the  $\mathcal{U}$ -controlled objects, define

$$C((M_n)_n) := \overline{C}((M_n)_n)[\{C \circ U(-) \Rightarrow C\}_{C,U \in \mathcal{U}}^{-1}].$$

Then one can show that mapping spaces in this localization are precisely given by the desired colimit formula, and  $C((M_n)_n)$  is a right-exact  $\infty$ -category, see [180, Proposition 3.5.9]. With a bit of additional effort, see [185, Corollary 4.9], it also follows that the assignment  $(Q, \ell) \mapsto C_Q$  refines to a functor

(24.10) 
$$\operatorname{CW}^{\mathrm{lf}}((M_n)_n) \to \operatorname{Spc}^{\mathrm{f}}_*((M_n)_n).$$

We obtain a version of the  $\infty$ -category  $C((M_n)_n)$  for a sequence of metric spaces  $(M_n)_n$  with isometric *G*-action by very formal means. The construction of the  $\infty$ -category  $C((M_n)_n)$  is sufficiently functorial to carry an induced *G*-action. In fact,

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we can also allow for a *G*-action on *C*; then the *G*-action is induced by conjugation on the functor category  $Fun(\mathcal{P}(M), C)$ .

**Definition 24.11.** For a sequence  $(M_n)_n$  of metric spaces with isometric *G*-action, define

$$C((M_n)_n)_G := \operatorname{colim}_{I(G)} C((M_n)_n).$$

Taking stock of what we have accomplished so far, the  $\infty$ -category  $C((M_n)_n)_G$  is an analog of the additive category  $O_0^G(X)$  from Section 21.10, where instead of a product  $X \times \mathbb{N}$  we are considering the space  $M = \prod_{n \in \mathbb{N}} M_n$ .

Up to this point, we have only considered the "geometric" input variable, and our constructions have no relation to the homology theory  $\mathbf{H}^G(-; F_C)$  yet. To address this point, let  $\operatorname{Met}^G$  denote the category whose objects are sequences of metric spaces  $(M_n)_n$  with isometric *G*-action, and whose morphisms are sequences of *G*-equivariant functions  $(f_n \colon M_n \to M'_n)_n$  such that there exist a, b > 0 with the property that  $d_{M'_n}(f_n(x), f_n(y)) \leq a \cdot d_{M_n}(x, y) + b$  for all *n* and all  $x, y \in M_n$ . If *S* is a *G*-set, we can artificially turn it into a metric space with *G*-action by declaring each pair of distinct points to have distance 1. The precise value of the distance function is irrelevant, but we do want the underlying space to be discrete and metrically bounded (so any finite, positive number will do). This allows us to define the functor

$$\prod_{\mathbb{N}} G\operatorname{-SETS} \times \operatorname{Met}^G \to \operatorname{Met}^G, \quad ((S_n)_n, (M_n)_n) \mapsto (S_n \times M_n)_n,$$

where we equip each product  $S_n \times M_n$  with the sum of the individual metrics. This functor can be composed with the formation of controlled objects over a fixed right-exact  $\infty$ -category with *G*-action *C* to obtain the functor

$$C^{\Pi} \colon \prod_{\mathbb{N}} G\text{-SETS} \times \operatorname{Met}^{G} \to \operatorname{CATREX}, \quad ((S_{n})_{n}, (M_{n})_{n}) \mapsto C((S_{n} \times M_{n})_{n})_{G}.$$

In turn, this functor can be composed with any finitary localizing invariant  $F: CATREX \rightarrow Sp$ .

Regarding G-sets as discrete G-spaces induces a fully faithful functor

$$G$$
-SETS  $\rightarrow$  Fun(Or( $G$ )<sup>op</sup>, Spc))

Therefore, we obtain an extension of  $F \circ C^{\prod}$  to a functor

$$\widehat{\mathbf{H}}^{G}(-; F_{\mathcal{C}}) \colon \prod_{\mathbb{N}} \operatorname{Fun}(\operatorname{Or}(G)^{\operatorname{op}}, \operatorname{Spc}) \times \operatorname{Met}^{G} \to \operatorname{Sp}$$

by taking the left Kan extension along the fully faithful functor

$$\prod_{\mathbb{N}} G\operatorname{-SETS} \times \operatorname{Met}^{G} \to \prod_{\mathbb{N}} \operatorname{Fun}(\operatorname{Or}(G)^{\operatorname{op}}, \operatorname{Spc}) \times \operatorname{Met}^{G}.$$

As the notation indicates, this functor is supposed to be closely related to the extension of  $\mathbf{H}^G(-; F_C)$  we are looking for. Note that

$$\mathbf{H}^{G}((G/G)_{n}, (M_{n})_{n}; F_{C}) \simeq F(C((M_{n})_{n})_{G}),$$

which we identified as an analog of  $O_0^G(X)$  from Section 21.10. However, it was the quotient category  $\mathcal{D}_0^G(X)$  of  $O_0^G(X)$  that gave rise to a homology theory because the continuous control condition only imposes restrictions for large natural numbers, and this is what forces excision in the end. Consequently, the extension  $\widetilde{\mathbf{H}}^G(-; F_C)$  we are looking for should be a similar quotient of  $\widehat{\mathbf{H}}^G(-; F_C)$ .

In fact, for each sequence of metric spaces  $(M_n)_n$  and each  $k \in \mathbb{N}$ , we have the family  $\delta_{M_k}$  which is given by  $M_k$  at the *k*-th component and by the empty space at all other components. Then the evident map  $\delta_{M_k} \to (M_n)_n$  induces a functor  $C^{\prod}((S_n)_n, \delta_{M_k}) \to C^{\prod}((S_n)_n, (M_n)_n)$ .

# Definition 24.12. Define

$$\widetilde{\mathbf{H}}^{G}(-; F_{\mathcal{C}}): \prod_{\mathbb{N}} \operatorname{Fun}(\operatorname{Or}(G)^{\operatorname{op}}, \operatorname{Spc}) \times \operatorname{Met}^{G} \to \operatorname{Sp}$$

as the evident functor which satisfies

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$$\begin{split} \widetilde{\mathbf{H}}^G((S_n)_n, (M_n)_n; F_C) \\ \simeq \operatorname{cofib} \Bigl(\bigoplus_n \widehat{\mathbf{H}}^G((S_n)_n, \delta_{M_n}; F_C) \to \widehat{\mathbf{H}}^G((S_n)_n, (M_n)_n; F_C) \Bigr). \end{split}$$

We claim that this functor satisfies the first two desiderata formulated at the beginning of this section.

**Theorem 24.13 (Properties of**  $\widetilde{\mathbf{H}}^G(-; F_C)$ ). Let  $(X_n)_n$  be a sequence of objects in Fun( $Or(G)^{op}$ , Spc).

(i) If  $(\Sigma_n)_n$  is a sequence of d-dimensional G-simplicial complexes equipped with the  $L^1$ -metric, then the attaching squares for the equivariant d-simplices induce a pullback square

$$\begin{split} \widetilde{\mathbf{H}}^{G}((X_{n})_{n},(S_{n}\times\partial\Delta^{d})_{n};F_{C}) &\longrightarrow \widetilde{\mathbf{H}}^{G}((X_{n})_{n},(\Sigma_{n}^{(d-1)})_{n};F_{C}) \\ & \downarrow \\ & \downarrow \\ \widetilde{\mathbf{H}}^{G}((X_{n})_{n},(S_{n}\times\Delta^{d})_{n};F_{C}) &\longrightarrow \widetilde{\mathbf{H}}^{G}((X_{n})_{n},(\Sigma_{n})_{n};F_{C}); \end{split}$$

(ii) The projection  $\Delta^d \rightarrow *$  induces an equivalence

$$\widetilde{\mathbf{H}}^{G}((X_{n})_{n}, (S_{n} \times \Delta^{d})_{n}; F_{C}) \xrightarrow{\sim} \widetilde{\mathbf{H}}^{G}((X_{n})_{n}, (S_{n})_{n}; F_{C});$$

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- (iii) There are natural equivalences

$$\mathbf{H}^{G}((X_{n})_{n}, (S_{n})_{n}; F_{\mathcal{C}}) \simeq \mathbf{H}^{G}((X_{n} \times S_{n})_{n}, (*)_{n}; F_{\mathcal{C}})$$

for every sequence of G-sets  $(S_n)_n$ ; (iv) In particular, the map

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$$\widetilde{\mathbf{H}}^G((E_{\mathcal{F}}(G))_n, (\Sigma_n)_n; F_C) \to \widetilde{\mathbf{H}}^G((*)_n, (\Sigma_n)_n; F_C)$$

is an equivalence for every sequence of G-simplicial complexes  $(\Sigma_n)_n$  with uniformly bounded dimension and stabilizers in  $\mathcal{F}$ .

*Idea of proof.* We have stressed the analogy between  $\mathbf{\tilde{H}}^G((X_n)_n, (\mathbf{M}_n)_n; F_C)$  and  $\mathbf{K}(\mathcal{D}_0^G(X))$  from Chapter 21. Hopefully, this lends some credibility to the claim that the arguments presented in Section 21.11 can be adapted to prove the excision and homotopy invariance statements of the theorem. See [185, Section 2.5] for detailed arguments.

For the third point, it is easy to see that the functors on both sides are left Kan extended along  $\prod_{\mathbb{N}} G$ -SETS  $\rightarrow \prod_{\mathbb{N}} \text{Fun}(Or(G)^{\text{op}}, \text{Spc})$ . If  $(T_n)_n$  is another sequence of *G*-sets, we have a natural identification

$$\widetilde{\mathbf{H}}^G((T_n)_n, (S_n)_n; F_C) \simeq \widetilde{\mathbf{H}}^G((T_n \times S_n)_n, (*)_n; F_C)$$

by definition.

As explained at the beginning of the section, assertion (iv) follows by induction on the dimension bound for the sequence  $(\Sigma_n)_n$ .

Theorem 24.13 indicates that we are on the right track to carry out the proof of the isomorphism conjecture sketched out at the beginning of this section. What we have not talked about yet is the construction of the vertical maps in diagram (24.2). The construction of the upper vertical maps (the "transfer maps") is the subject of the next section.

For the lower vertical maps, we proceed as follows. If each metric space  $M_n$  is bounded, i.e., there exists a number C such that the distance between any two points is bounded by C, (which happens for example in the case when  $M_n$  is a simplicial complex equipped with the  $L^1$ -metric), the projection maps  $M_n \rightarrow *$  induce a map

$$\widetilde{\mathbf{H}}^G((X_n)_n, (M_n)_n; F_C) \to \widetilde{\mathbf{H}}^G((X_n)_n, (*)_n; F_C).$$

As explained in our list of desiderata, we want to show that the right-hand term admits a transformation to  $\prod_{n \in \mathbb{N}}^{\text{red}} \mathbf{H}^G(X_n; F_C)$ . Since  $\widetilde{\mathbf{H}}^G(-; F_C)$  is left Kan extended, it suffices to construct this transformation in the case when each  $X_n$  is a *G*-set.

Recall the collection  $\mathcal{U}$  of subsets of  $\prod_{n \in \mathbb{N}} S_n \times S_n$  which encodes the continuous control condition. Fix an element  $U \in \mathcal{U}$ , and suppose additionally that U contains the diagonal and is *G*-invariant as a set (the collection of such sets is a cofinal subposet of  $\mathcal{U}$ ). Interpreting U as a relation on  $\prod_{n \in \mathbb{N}} S_n$ , it makes sense to compose U with itself. In particular, all powers  $U^k$  by natural numbers are defined. Then we

can define a variant  $C_U((S_n)_n)$  of  $C((S_n)_n)$  in which we consider only objects that are  $U^k$ -controlled for some  $k \ge 0$  and localize these at the collection of morphisms  $C \circ U(-) \Rightarrow C$ . Through some straightforward cofinality arguments, one finds that

$$C((S_n)_n) \simeq \operatorname{colim}_{U \in \mathcal{U}} C_U((S_n)_n)$$

This colimit is filtered, so we can commute it past any application of a finitary localizing invariant.

For a given  $U \in \mathcal{U}$ , the continuous control condition forces that the intersection  $U \cap (S_n \times S_n)$  is simply the diagonal of  $S_n \times S_n$  for sufficiently large *n*. Together with the condition that objects in  $C_U((S_n)_n)$  are left Kan extended from locally finite subsets, we obtain a splitting

$$C_U((S_n)_n) \simeq \mathcal{D} \oplus \prod_{n \ge N} \bigoplus_{S_n} C.$$

After taking G-orbits (i.e., applying  $\operatorname{colim}_{I(G)}$ ), we obtain a map

$$\operatorname{colim}_{I(G)} C_U((S_n)_n) \simeq (\operatorname{colim}_{I(G)} \mathcal{D}) \oplus \operatorname{colim}_{I(G)} \left(\prod_{n \ge N} \bigoplus_{S_n} C\right) \to \prod_n^{\operatorname{red}} \operatorname{colim}_{I(G)} \bigoplus_{S_n} C.$$

Decomposing each  $S_n$  into its orbits, we have

$$\operatorname{colim}_{I(G)} \bigoplus_{S_n} C \simeq \bigoplus_{T \in G \setminus S_n} \operatorname{colim}_{I(G)} \bigoplus_T C.$$

These identifications can be performed in a way that allows us to identify  $\operatorname{colim}_{I(G)} \bigoplus_T C$  with the value of the Kan extension of C along  $I(G) \to \operatorname{Or}(G)$  at T: for intuition, G acts by permutation on  $\bigoplus_T C$ , so after a choice of basepoint in T, we can think of  $\bigoplus_T C$  as the induction of the  $\infty$ -category with H-action C to an  $\infty$ -category with G-action  $\bigoplus_{G/H} C$ ; taking the colimit then yields  $\operatorname{colim}_{I(G)} \bigoplus_{G/H} C \approx \operatorname{colim}_{I(H)} C$ . In particular, it follows that

$$F\left(\bigoplus_{T\in G\setminus S_n} \operatorname{colim}_{I(G)} \bigoplus_T C\right) \simeq \bigoplus_{T\in G\setminus S_n} \mathbf{H}^G(T; F_C) \simeq \mathbf{H}^G(S_n; F_C).$$

In total, this provides us with a transformation

$$F\left(\operatorname{colim}_{I(G)} C_U((S_n)_n)\right) \to \prod_{n \in \mathbb{N}}^{\operatorname{red}} \mathbf{H}^G(S_n; F_C),$$

which can be checked to induce a transformation

$$\widehat{\mathbf{H}}^{G}((S_{n})_{n},(*)_{n};F_{C})\xrightarrow{\simeq} F(C((S_{n})_{n})_{G}) \to \prod_{n\in\mathbb{N}}^{\mathrm{red}}\mathbf{H}^{G}(S_{n};F_{C})$$

by taking the filtered colimit over all U. Since we are mapping to the reduced product, this map further factors via a map

$$\widetilde{\mathbf{H}}^G((S_n)_n,(*)_n;F_C)\to\prod_{n\in\mathbb{N}}^{\mathrm{red}}\mathbf{H}^G(S_n;F_C),$$

and this is the transformation we were looking for.

# 24.4 The Transfer Map

In this section, we explain the final ingredient for the construction of diagram (24.2), namely the maps labeled *t*. These are variants of the transfer maps introduced in Chapter 23, in particular of the transfer for geometric modules discussed in Section 23.7. In the  $\infty$ -categorical setting, it is not feasible to construct such maps by hand, so we will have to rely on suitable machinery.

Section 23.4 interpreted the transfer in terms of the Swan group action on *K*-theory. This is the line of thought we are going to continue in our construction of the transformation *t*. We should begin by characterizing the involved terms in  $\infty$ -categorically meaningful terms. For modules over the group ring of *G*, we have already done this in terms of the homotopy orbit construction, and we exploited this in the definition of  $\widetilde{\mathbf{H}}^G(-; F_C)$  in the preceding section. We have introduced the Swan ring  $\mathrm{Sw}^p(G)$  in terms of of *G*-representations on finitely generated projective  $\mathbb{Z}$ -modules and the abelian group  $\mathrm{Sw}(G)$  in terms of finitely generated  $\mathbb{Z}$ -modules in Definition 12.65 and constructed an isomorphism of abelian groups  $\mathrm{Sw}^p(G) \xrightarrow{\cong} \mathrm{Sw}(G)$  in Lemma 12.66. Equivalently,  $\mathrm{Sw}^p(G)$  is  $K_0$  of the exact category  $\mathrm{Fun}(I(G), \mathbb{Z}\text{-MOD}_{\mathrm{fgp}})$  in which exactness is detected by the forgetful functor to  $\mathbb{Z}\text{-MOD}_{\mathrm{fgp}}$ . Example 12.67 exhibited an action of  $\mathrm{Sw}^p(G)$  on the *K*-theory of any group ring *RG*.

On the level of the associated exact categories, this action admits the following description. Every additive category  $\mathcal{A}$  admits a canonical action by the symmetric monoidal category  $\mathbb{Z}$ -MOD<sub>fgp</sub>; this essentially boils down to the fact that there is a sensible interpretation of the term  $\mathbb{Z}^k \otimes A$  for every object  $A \in \mathcal{A}$  because  $\mathcal{A}$  has direct sums. Now one observes that Fun(I(G),  $\mathbb{Z}$ -MOD<sub>fgp</sub>) is a model for the *G*-fixed points of the trivial *G*-action on  $\mathbb{Z}$ -MOD<sub>fgp</sub> (in the 2-categorical sense), while the "group ring" of  $\mathcal{A}$  is given by the *G*-orbits (in the 2-categorical sense) of whichever action we are considering on  $\mathcal{A}$ . In ordinary algebra, it is a straightforward observation that a *G*-action on a module *M* over an algebra *R* (with *G* acting on both *M* and *R*) induces an  $R^G$ -module structure on the coinvariants  $M_G$ . This observation categorifies, and in the case at hand it yields an action of the symmetric monoidal exact category Fun(I(G),  $\mathbb{Z}$ -MOD<sub>fgp</sub>) on the coinvariants  $\mathcal{A}_{hG}$ . Since *K*-theory refines to a lax symmetric monoidal functor (we will say more about this later), this action finally induces the action of the Swan group via the map

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$$Sw(G) \otimes K_n(\mathcal{A}_{hG}) \cong K_0(Fun(I(G), \mathbb{Z}\text{-}MOD_{fgp})) \otimes K_n(\mathcal{A}_{hG})$$
$$\rightarrow K_n(Fun(I(G), \mathbb{Z}\text{-}MOD_{fgp}) \otimes \mathcal{A}_{hG})$$
$$\xrightarrow{act} K_n(\mathcal{A}_{hG}).$$

This discussion adapts to the setting of right-exact  $\infty$ -categories in relatively direct fashion. The first important ingredient is the existence of a tensor product for right-exact  $\infty$ -categories. We have silently pretended in the preceding paragraph that something like this exists for exact categories; since all of that was merely to motivate what follows next, we choose not to elaborate on this issue.

**Proposition 24.14.** *The*  $\infty$ *-category* CATREX *admits a symmetric monoidal structure such that the following holds:* 

- (i) the unit object of CATREX is the  $\infty$ -category Spc<sup>f</sup><sub>\*</sub> of pointed finite spaces;
- (ii) the tensor product C ⊗ C' of two right-exact ∞-categories comes equipped with a functor m: C × C' → C ⊗ C' which preserves finite colimits in each variable separately such that for every right-exact ∞-category D the restriction functor

 $m^*$ : Fun<sup>ex</sup>( $C \otimes C', \mathcal{D}$ )  $\rightarrow$  Fun( $C \times C', \mathcal{D}$ )

is an equivalence onto the full subcategory of functors which preserve finite colimits in each variable separately; i.e. right-exact functors out of the tensor product classify biexact functors.

*Proof.* This follows from the universal property of  $\text{Spc}_*^f$  and [689, Corollary 4.8.1.4] by unwinding [689, Corollary 4.8.1.2].

From the symmetric monoidal structure on CATREX, one obtains an induced symmetric monoidal structure on CATST in the following way.

**Proposition 24.15.** The Spanier-Whitehead stabilization SW of 8.24 is naturally equivalent to the functor  $- \otimes \text{Sp}^{\omega}$ . It follows that CATST carries a symmetric monoidal structure with the following properties:

- (i) the unit object of CATST is the  $\infty$ -category Sp<sup> $\omega$ </sup> of compact spectra;
- (ii) the tensor product C ⊗<sub>st</sub> C' of two stable ∞-categories comes equipped with a functor m: C × C' → C ⊗<sub>st</sub> C' which is exact in each variable separately such that for every stable ∞-category D the restriction functor

 $m^*$ : Fun<sup>ex</sup>( $C \otimes_{st} C', \mathcal{D}$ )  $\rightarrow$  Fun( $C \times C', \mathcal{D}$ )

is an equivalence onto the full subcategory of functors which are exact in each variable separately;

- (iii) the functor SW: CATREX  $\rightarrow$  CATST admits a symmetric monoidal refinement;
- (iv) the inclusion functor CATST  $\rightarrow$  CATREX admits a lax symmetric monoidal refinement; the functor  $Sp_*^f \rightarrow Sp^{\omega}$  is the unit of the adjunction, while the structure maps  $C \otimes C' \rightarrow C \otimes_{st} C'$  are equivalences for any two stable  $\infty$ -categories C and C'.

*Proof.* This is explained for example in [195, Construction 5.1.1].

These results allow one to find statements analogous to the fact that  $\mathbb{Z}$ -MOD<sub>fgp</sub> acts on every additive category.

## Corollary 24.16.

- (i) The ∞-category Spc<sup>t</sup><sub>\*</sub> admits a symmetric monoidal structure refining the smash product of pointed spaces, and every right-exact ∞-category carries an action by the symmetric monoidal ∞-category Spc<sup>t</sup><sub>\*</sub>;
- (ii) the ∞-category Sp<sup>ω</sup> admits a symmetric monoidal structure, and every stable ∞-category carries an action by the symmetric monoidal ∞-category Sp<sup>ω</sup>.

*Proof.* The tensor unit in a symmetric monoidal  $\infty$ -category refines uniquely to a commutative algebra object by [689, Corollary 3.2.1.9], and every object in a symmetric monoidal  $\infty$ -category carries a module structure over this commutative algebra [689, Proposition 3.4.2.1]. Note that a commutative algebra structure on a right-exact  $\infty$ -category *C* is precisely a symmetric monoidal structure on *C* whose tensor product preserves finite colimits in each variable.

The category of pointed finite CW-complexes  $CW_*^f$  carries a symmetric monoidal structure given by the smash product. Localizing at the pointed homotopy equivalences, one obtains a symmetric monoidal structure on  $Spc_*^f$  which turns it into a commutative algebra object in CATREX. We have just argued that there exists only one such commutative algebra structure.

# Corollary 24.17.

- (i) The right-exact ∞-category Fun(I(G), Spc<sup>f</sup><sub>\*</sub>) admits a symmetric monoidal structure such that the forgetful functor Fun(I(G), Spc<sup>f</sup><sub>\*</sub>) → Spc<sup>f</sup><sub>\*</sub> is symmetric monoidal;
- (ii) Let C be a right-exact G- $\infty$ -category. Then  $\operatorname{colim}_{I(G)} C$  carries an action by the symmetric monoidal right-exact  $\infty$ -category  $\operatorname{Fun}(I(G), \operatorname{Spc}^{\mathrm{f}}_{*})$ .

*Proof.* The first statement is merely the assertion that there exists a pointwise symmetric monoidal structure on functor categories, see [689, Remark 2.1.3.4]. The second assertion follows for example from the arguments in [185, Section 3.4].  $\Box$ 

**Remark 24.18.** Corollary 24.17 remains true if we replace  $\text{Spc}_*^f$  by  $\text{Sp}^{\omega}$  and assume that *C* is stable.

This provides almost all ingredients for an analog of the Swan group action on *K*-theory in this setting. The final input is the following.

**Theorem 24.19 (Properties of** Fun<sup>floc</sup>(CATST, Sp)). The  $\infty$ -category Fun<sup>floc</sup>(CATST, Sp) of finitary localizing invariants admits a symmetric monoidal structure such that commutative algebras in this  $\infty$ -category are identified with lax symmetric monoidal finitary localizing invariants CATST  $\rightarrow$  Sp.

Moreover, the tensor unit in  $\operatorname{Fun}^{\operatorname{floc}}(\operatorname{CATST}, \operatorname{Sp})$  is the non-connective algebraic *K*-theory functor **K**. In particular, **K** admits a unique lax symmetric monoidal refinement, and every Sp-valued, finitary localizing invariant is a module over **K**.

*Proof.* See [144, Theorem 1.5 & Corollary 1.7].

As in the proof of Corollary 24.16, the fact that **K** is the tensor unit in the  $\infty$ -category of finitary localizing invariants implies that every finitary localizing invariant *F* refines to a module over the lax symmetric monoidal functor **K**. So  $F(\operatorname{colim}_{I(G)} C)$  becomes a module over the ring spectrum

$$\mathbf{Sw}(G) := \mathbf{K}(\operatorname{Fun}(I(G), \operatorname{Spc}^{\mathrm{I}}_{*}))$$

for every right-exact G- $\infty$ -category C. In particular, this structure includes the action maps

$$\mathbf{Sw}(G) \otimes F(\operatorname{colim}_{I(G)} C) \to F(\operatorname{Fun}(I(G), \operatorname{Spc}^{\operatorname{I}}_{*}) \otimes (\operatorname{colim}_{I(G)} C)) \\ \to F(\operatorname{colim}_{I(G)} C)$$

coming from Corollary 24.17.

**Remark 24.20.** Recall that  $\text{Spc}_*^f$  is the Dwyer-Kan localization of the category of pointed finite *CW*-complexes  $\text{CW}_*^f$  at the pointed homotopy equivalences. The localization functor has the property that it sends pushouts along subcomplex inclusions to pushouts in  $\text{Spc}_*^f$ . In particular, the localization functor induces a ring homomorphism

$$Sw^A(G) \to \pi_0 Sw(G)$$

where  $Sw^A(G)$  denotes the *A*-theoretic Swan group from 20.55.

Before we proceed to generalize this discussion to categories of controlled objects and relate it to the construction of diagram (24.2), let us briefly return to the induction statement for the *A*-theoretic Swan group mentioned in Section 20.9.

If H is a finite index subgroup of G, the restriction functor

$$\operatorname{res}_{H}^{G}$$
:  $\operatorname{Fun}(I(G), \operatorname{Spc}_{*}^{t}) \to \operatorname{Fun}(I(H), \operatorname{Spc}_{*}^{t})$ 

admits a left adjoint

$$\operatorname{ind}_{H}^{G}$$
:  $\operatorname{Fun}(I(H), \operatorname{Spc}_{*}^{t}) \to \operatorname{Fun}(I(G), \operatorname{Spc}_{*}^{t}),$ 

because the colimit appearing in the formula for the left Kan extension is indexed by a category equivalent to G/H, considered as a discrete category. As a left adjoint functor,  $\operatorname{ind}_{H}^{G}$  preserves colimits, so it induces a map on *K*-theory which we denote by  $\operatorname{ind}_{H}^{G}$  as well.

**Proposition 24.21.** *Let G be a finite group. Then the image of the sum of induction maps* 

$$\sum_{H \in \mathcal{D}(G)} \operatorname{ind}_{H}^{G} \colon \bigoplus_{H \in \mathcal{D}(G)} \pi_{0} \mathbf{Sw}(H) \to \pi_{0} \mathbf{Sw}(G)$$

contains the unit element, where  $\mathcal{D}(G)$  denotes the family of Dress subgroups of G, see Definition 20.50.

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*Proof.* As explained in Section 20.9, the results of [770] imply that the sum of induction maps

$$\sum_{H\in \mathcal{D}(G)} \mathrm{ind}_{H}^{G} \colon \bigoplus_{H\in \mathcal{D}(G)} \mathrm{Sw}^{A}(H) \to \mathrm{Sw}^{A}(G)$$

is surjective. We claim that for each subgroup H of G there exists a commutative square

$$\begin{array}{c|c} \operatorname{Fun}(I(H),\operatorname{CW}^{\mathrm{f}}_{*}) \xrightarrow{\operatorname{ind}_{H}^{G}} \operatorname{Fun}(I(G),\operatorname{CW}^{\mathrm{f}}_{*}) \\ & & \downarrow l_{H} \\ & & \downarrow l_{G} \\ \operatorname{Fun}(I(H),\operatorname{Spc}^{\mathrm{f}}_{*}) \xrightarrow{\operatorname{ind}_{H}^{G}} \operatorname{Fun}(I(G),\operatorname{Spc}^{\mathrm{f}}_{*}) \end{array}$$

in which the vertical maps are those described in Remark 24.20. If we replace the horizontal functors in this square by their right adjoints, the resulting square obviously commutes. Then the respective unit and counit transformation induce the associated Beck-Chevalley transformation

$$\mathrm{ind}_{H}^{G} \circ l_{H} \to \mathrm{ind}_{H}^{G} \circ l_{H} \circ \mathrm{res}_{H}^{G} \circ \mathrm{ind}_{H}^{G} \xrightarrow{\simeq} \mathrm{ind}_{H}^{G} \circ \mathrm{res}_{H}^{G} \circ l_{G} \circ \mathrm{ind}_{H}^{G} \to l_{G} \circ \mathrm{ind}_{H}^{G}.$$

This transformation can be checked to be an equivalence using the pointwise formula for the left Kan extension and the fact that the localization functor  $CW_*^f \rightarrow Spc_*^f$  preserves finite coproducts.

The unit element in  $\pi_0 \mathbf{Sw}(G)$  is represented by  $S^0$  equipped with the trivial *G*-action, which lies in the image of  $l_G$ . Therefore, the image of the sum of induction maps for  $\pi_0 \mathbf{Sw}$  contains the unit element.

**Remark 24.22.** With very small additional effort, one can prove the projection formula

$$\operatorname{ind}_{H}^{G}(x) \cdot y = \operatorname{ind}_{H}^{G}(x \cdot \operatorname{res}_{H}^{G}(y))$$

for all  $x \in \pi_0 \mathbf{Sw}(H)$  and  $y \in \mathbf{Sw}(G)$ . It then follows that the sum of induction maps in Proposition 24.21 is surjective.

**Remark 24.23.** Let *G* be a finite group. It is possible to upgrade the association  $H \mapsto Sw(H)$  to a Sp-valued Green functor *S*. Similarly, the association  $H \mapsto \mathbf{K}(\operatorname{colim}_{I(H)} C)$  can be upgraded to a Sp-valued Mackey functor *M* which becomes a module over *S*; see [100, 101, 244] for details on how to make these statements precise.

The general theory of Mackey functors implies that the assembly map

$$\operatorname{colim}_{\operatorname{Or}_{\mathcal{D}(G)}(G)} M \to M(G) \simeq \mathbf{K}(\operatorname{colim}_{I(G)} C)$$

is an equivalence for every finite group *G*. We omit the details of this argument since the assertion about the assembly map also follows from Theorem 20.61: a finite group *G* can been seen to be a Dress-Farrell-Hsiang-Jones group over  $\mathcal{D}(G)$  by setting  $\alpha_n := \operatorname{id}_G$  and  $Z_{n,D} := * =: \Sigma_{n,D}$  for all *n* and *D* in Definition 20.60.

After this digression, we return to the question of how the maps *t* in diagram (24.2) are constructed. The short answer is that they arise through a many-objects version of the action of the Swan group. Note that the category Met<sup>G</sup> admits a symmetric monoidal structure in which  $(M_n)_n \otimes (M'_n)_n := (M_n \times M'_n)_n$ .

Proposition 24.24. The functor

$$Met^{G} \times Fun(I(G), CATREX) \to Fun(I(G), CATREX)$$
$$((M_{n})_{n}, C) \mapsto C((M_{n})_{n})$$

admits a lax symmetric monoidal refinement.

*Proof.* This is shown in [185, Section 3.3]. The basic idea is that if  $C \in C((M_n)_n)$  and  $C' \in C'((M'_n)_n)$  are controlled objects, one obtains a controlled object in  $(C \otimes C')((M_n \times M'_n)_n)$  by left Kan extending the composite functor

$$\mathcal{P}(M) \times \mathcal{P}(M') \xrightarrow{C \times C'} \operatorname{Ind}(C) \times \operatorname{Ind}(C') \to \operatorname{Ind}(C \otimes C')$$

along the functor  $\mathcal{P}(M) \times \mathcal{P}(M') \to \mathcal{P}(\coprod_{n \in \mathbb{N}} M_n \times M'_n)$  taking the fiberwise product. In other words, the lax symmetric monoidal structure is given by a version of the Day convolution product.

Proposition 24.24 allows us to introduce a controlled version of Swan *K*-theory. Since  $\text{Spc}_*^f$  refines to a commutative algebra object in CATREX by Proposition 24.14 and since  $\lim_{I(G)}$  is canonically a lax symmetric monoidal functor (being right adjoint to a symmetric monoidal functor), we obtain a lax symmetric monoidal functor

$$\operatorname{Met}^G \to \operatorname{CATREX}, \quad (M_n)_n \mapsto \operatorname{Spc}^{\mathrm{f}}_*((M_n)_n)^G := \lim_{I(G)} \left( \operatorname{Spc}^{\mathrm{f}}_*((M_n)_n) \right)$$

whose *K*-theory is our version of controlled Swan theory. Generalizing Corollary 24.17, the arguments in [185, Section 3.4] show that the functor  $C(-)_G$  from Definition 24.11 becomes a module over the lax symmetric monoidal functor  $\operatorname{Spc}_*^f(-)^G$  for every right-exact G-∞-category C. To see that this is a many-objects version of Corollary 24.17, it is helpful to remember that lax symmetric monoidal functors from one symmetric monoidal category to another can be identified with commutative algebra objects with respect to the Day convolution symmetric monoidal structure on the functor category, see [425].

More explicitly, this module structure encodes in particular for any two sequences  $(M_n)_n$  and  $(N_n)_n$  of metric spaces with isometric *G*-action multiplication maps

$$\operatorname{Spc}^{\mathrm{f}}_{*}((M_{n})_{n})^{G} \otimes C((N_{n})_{n})_{G} \to C((M_{n} \times N_{n})_{n})_{G}.$$

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These multiplication maps are of particular interest to us when some sequences are the constant sequence on a point.

In the case that  $N_n = *$  for all *n*, we obtain multiplication maps

$$\operatorname{Spc}^{\mathrm{f}}_{*}((M_{n})_{n})^{G} \otimes C^{\prod}((S_{n})_{n}, (*)_{n})_{G} \to C^{\prod}((S_{n})_{n}, (M_{n})_{n})_{G}$$

After left Kan extending from sequences of *G*-sets to sequences of *G*-spaces and composing with *K*-theory, it follows that each object in  $\text{Spc}_*^f((M_n)_n)^G$  induces a transformation

$$\mathbf{H}^{G}(-,(*)_{n};F_{C}) \rightarrow \mathbf{H}^{G}(-,(M_{n})_{n};F_{C})$$

for any finitary localizing invariant F, and consequently also a transformation

$$\widetilde{\mathbf{H}}^{G}(-, (*)_{n}; F_{C}) \rightarrow \widetilde{\mathbf{H}}^{G}(-, (M_{n})_{n}; F_{C})$$

between the "reduced" versions of these functors. As a functor on  $\prod_{\mathbb{N}} G$ -SETS, the left-hand side is given on a sequence of G-sets  $(S_n)_n$  by

$$F\Big(\operatorname{colim}_N\prod_{n\geq N}\bigoplus_{T\in G\setminus S_n}\operatorname{colim}_{\mathcal{G}^G(T)}C\Big),$$

where the first colimit is taken along the obvious projection maps, see the discussion at the end of Section 24.3. Specializing to a constant sequence  $(S)_n$ , this functor receives a natural map

$$\mathbf{H}^{G}(S; F_{\mathcal{C}}) \simeq F\left(\bigoplus_{T \in G \setminus S} \operatorname{colim}_{\mathcal{G}^{G}(T)} \mathcal{C}\right) \to F\left(\operatorname{colim}_{N} \prod_{n \geq N} \bigoplus_{T \in G \setminus S_{n}} \operatorname{colim}_{\mathcal{G}^{G}(T)} \mathcal{C}\right)$$

induced by the diagonal functor. Left Kan extending again, this induces natural maps

$$t: \mathbf{H}^{G}(X; \mathbf{K}_{\mathcal{C}}) \to \widetilde{\mathbf{H}}^{G}((X)_{n}, (*)_{n}; F_{\mathcal{C}}) \to \widetilde{\mathbf{H}}^{G}((X)_{n}, (M_{n})_{n}; F_{\mathcal{C}}),$$

which define the transformation t appearing in diagram (24.2).

Note that

$$\operatorname{Spc}^{\mathrm{f}}_{*}((*)_{n})^{G} \simeq \prod_{n \in \mathbb{N}} \operatorname{Fun}(I(G), \operatorname{Spc}^{\mathrm{f}}_{*}),$$

so we can consider an object T in this category as a sequence of objects  $(T_n)_n$ . The following proposition summarizes the important properties of the transformation t.

**Proposition 24.25.** Let  $(M_n)_n$  be a sequence of metric spaces with isometric *G*-action and consider an object

$$T \in \operatorname{Spc}^{\mathrm{f}}_{*}((M_{n})_{n})^{G}.$$

(i) Multiplication by the K-theory class determined by T induces a natural transformation

$$t: \mathbf{H}^G(-; F_C) \to \widetilde{\mathbf{H}}^G(-, (M_n)_n; F_C);$$

(ii) Suppose that M<sub>n</sub> is bounded for all n and denote by (T<sub>n</sub>)<sub>n</sub> the sequence of objects corresponding to the image of T under the maps induced by the projection (M<sub>n</sub>)<sub>n</sub> → (\*)<sub>n</sub>.

If  $[T_n] = [S^0] \in K_0(\operatorname{Fun}(I(G), \operatorname{Spc}^f_*))$  for all n, then the composite map

$$\mathbf{H}^{G}(-;F_{C}) \xrightarrow{t} \widetilde{\mathbf{H}}^{G}(-,(M_{n})_{n};F_{C}) \to \overline{\Delta}(\mathbf{H}^{G}(-;F_{C}))$$

is equivalent to the map

$$\delta \colon \mathbf{H}^{G}(-; F_{C}) \to \overline{\Delta}(\mathbf{H}^{G}(-; F_{C})).$$

*Proof.* This follows from the arguments in [185, Section 2.4]. While *loc. cit.* only considers lax symmetric monoidal finitary localizing invariants, this can be easily generalized using that F is a module over **K** by Theorem 24.19.

See also the comments in Section 24.5.

Proposition 24.25 is our tool to obtain the maps t in diagram (24.2) and to prove that the resulting diagram commutes. Explicitly, this means that we have to produce suitable objects T such that

$$[T_n] = [S^0] \in K_0(\operatorname{Fun}(I(G), \operatorname{Spc}^{\mathrm{f}}_*)).$$

What makes the construction of such objects possible is the existence of the transformation

$$CW^{lf}((M_n)_n) \rightarrow Spc^{f}_*((M_n)_n)$$

from (24.10), which allows us to import point-set data into the  $\infty$ -categorical setting. However, we still have to enhance this transformation in a way which actually produces objects in the fixed point  $\infty$ -category  $\operatorname{Spc}_*^f((M_n)_n)$ . Since everything is sufficiently functorial, it is enough to construct objects in  $\lim_{I(G)} \operatorname{CW}^{\operatorname{lf}}((M_n)_n)$ , where the limit still has to be interpreted in the  $\infty$ -categorical sense. Since  $\operatorname{CW}^{\operatorname{lf}}((M_n)_n)$ itself is an ordinary category with a strict *G*-action, it is still possible to write down an explicit model for this limit.

**Definition 24.26.** Let X be an ordinary category with a strict *G*-action. Define its *homotopy fixed points*  $X^{hG}$  as the following category:

- (i) objects in  $X^{hG}$  are objects  $X \in X$  together with a collection of isomorphisms  $\{n_a : X \xrightarrow{\cong} gX\}_{a \in G}$  such that  $n_{ha} = gn_h \circ n_a$  for all  $g, h \in G$ :
- {η<sub>g</sub>: X → gX}<sub>g∈G</sub> such that η<sub>hg</sub> = gη<sub>h</sub> ∘ η<sub>g</sub> for all g, h ∈ G;
  (ii) morphisms f: (X, {η<sub>g</sub>}<sub>g</sub>) → (X', {η'<sub>g</sub>}<sub>g</sub>) are morphisms f: X → X' in X such that the square



commutes for all  $g \in G$ .
**Lemma 24.27.** Let X be an ordinary category with a strict G-action. Then

$$\mathcal{X}^{\mathrm{h}G} \simeq \lim_{I(G)} \mathcal{X}.$$

*Proof.* This follows directly from the existence of the Rezk model structure on ordinary categories. See [185, Lemma 5.7] for an argument. 

#### 24.4.1 Discrete Transfer Spaces

Let us first consider the case of Dress-Farrell-Hsiang groups. This is the instance of the Dress-Farrell-Hsiang-Jones condition in which  $Z_{n,D} = *$  for all *n* and *D*. For convenience, let us repeat what the given data are, see Definition 20.52. For a fixed family  $\mathcal{F}$ , we have

- (i) A sequence  $\{F_n\}_{n \in \mathbb{N}}$  of finite groups;
- (ii) A sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  of surjective group homomorphisms  $\alpha_n \colon G \to F_n$ ;
- (iii) A collection  $\{(\Sigma_{n,D}) \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$ , where  $\Sigma_n$  is an abstract simplicial complex with a simplicial  $\alpha_n^{-1}(D)$ -action;
- (iv) A collection  $\{f_n \mid n \in \mathbb{N}, D \in \mathcal{D}(F_n)\}$  of maps of sets  $f_{n,D} \colon G \to \Sigma_{n,D}$ ,

such that the following holds:

- (a) For every n ∈ N and D ∈ F(F<sub>n</sub>), the α<sub>n</sub><sup>-1</sup>(D)-isotropy groups of |Σ<sub>n,D</sub>| belong to the family F|<sub>α<sub>n</sub><sup>-1</sup>(D)</sub> = {H ∩ α<sub>n</sub><sup>-1</sup>(D) | H ∈ F};
  (b) There exists a natural number N with dim(Σ<sub>n,D</sub>) ≤ N for all n ∈ N and
- $D \in \mathcal{D}(F_n);$
- (c) For every  $n \in \mathbb{N}$  and  $D \in \mathcal{F}(F_n)$ , the map  $f_{n,D}$  is  $\alpha_n^{-1}(D)$ -equivariant, where  $\alpha_n^{-1}(D)$  acts on G from the left;
- (d) For every  $g \in G$  we have

$$\lim_{n\to\infty} \sup_{D\in\mathcal{D}(F_n),\gamma\in G} d_{L^1}^{\sum_{n,D}} (f_{n,D}(\gamma g), f_{n,D}(\gamma)) = 0.$$

From these data, we want to construct an object in the category

$$\operatorname{CW}^{\operatorname{lf}}((\bigsqcup_{D\in\mathcal{D}(F_n)}G\times_{\alpha_n^{-1}(D)}\Sigma_{n,D})_n)^{\operatorname{h} G}.$$

Let us write  $\overline{D} := \alpha_n^{-1}(D)$  in the sequel. To understand the construction of this object, let us work our way backwards. After projecting to the constant sequence on a point, each component of the resulting object in

$$\operatorname{CW}^{\operatorname{lf}}((*)_n)^{\operatorname{h} G} \simeq \prod_{\mathbb{N}} \operatorname{Fun}(I(G), \operatorname{CW}^{\operatorname{f}}_*)$$

should represent the multiplicative unit in  $Sw^A(G)$ . By an induction along the skeletal filtration of a representative, we see that any element in  $Sw^A(G)$  is a sum of suspensions of *G*-sets. Proposition 24.21 shows that for each *n*, we may represent the multiplicative unit of  $Sw^A(G)$  by

$$\sum_{i=1}^{r_n} \left[ S^{d_i} \wedge (G/H_i)_+ \right] = \left[ \bigvee_{i=1}^{r_n} S^{d_i} \wedge (G/H_i)_+ \right]$$

such that each  $H_i$  is of the form  $\overline{D}$  for some  $D \in \mathcal{D}(F_n)$ . Hence, our goal is to turn

$$Q := \bigvee_{n \in \mathbb{N}} \bigvee_{i=1}^{r_n} S^{d_i} \wedge (G/H_i)_+$$

into a controlled *CW*-complex over  $(\coprod_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D})_n$ . For simplicity, we equip each sphere with the based CW-structure containing exactly one cell. Define a labeling on Q as the following composite:

$$(24.28) \qquad \ell \colon c(Q) \cong \coprod_{n \in \mathbb{N}} \coprod_{i=1}^{r_n} c(S^{d_i} \wedge (G/H_i)_+) \cong \coprod_{n \in \mathbb{N}} \coprod_{i=1}^{r_n} G/H_i \rightarrow \coprod_{n \in \mathbb{N}} \coprod_{D \in \mathcal{D}(F_n)} G \times G/\overline{D} \rightarrow \coprod_{n \in \mathbb{N}} \coprod_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D}.$$

The first map comes from the fact that each  $H_i$  is of the form  $\overline{D}$  for some  $D \in \mathcal{D}(F_n)$ , and the map includes  $G/H_i$  as the component  $\{e\} \times G/H_i$ . The second map is given on each summand by

$$G \times G/\overline{D} \to G \times_{\overline{D}} \Sigma_{n,D}, \quad (g, \gamma \overline{D} \mapsto (\gamma, f_{n,D}(\gamma^{-1}g)).$$

**Lemma 24.29.** The labeled CW-complex  $(Q, \ell)$  is an object in the category  $CW^{lf}((\coprod_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D})_n).$ 

*Proof.* The *n*-th summand  $Q_n := \bigvee_{i=1}^{r_n} S^{d_i} \wedge (G/H_i)_+$  of Q is a finite CW-complex, so  $(Q, \ell)$  is locally finite. Since these summands are themselves wedge sums of suspensions, we find that

$$\sup \left\{ d(\ell(z), \ell(z')) \mid z \in c(Q_n), z' \in c(\langle z \rangle) \right\} = 0$$

for all *n*. So  $(Q, \ell)$  is continuously controlled as well.

In the next step, we have to promote  $(Q, \ell)$  to an object in the homotopy fixed point category  $CW^{lf}((\coprod_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D})_n)^{hG}$ . Note that Q is a G-CW-complex. Since the action of the group G on the category only changes the labeling of a CWcomplex, it makes sense to claim that the action by an element  $g \in G$  on Q induces 24.4 The Transfer Map

an isomorphism

$$g \colon (Q, \ell) \to g(Q, \ell) = (Q, (g \cdot -) \cdot \ell).$$

Suppose that  $\gamma \overline{D}$  represents an open cell in  $Q_n$  under the bijections in (24.28). Then

$$g \cdot \ell(\gamma \overline{D}) = (g\gamma, gf_{n,D}(\gamma^{-1})),$$

while

$$\ell(g\gamma D) = (g\gamma, f_{n,D}(\gamma^{-1}g)).$$

These are not equal, but assumption (d) from above implies that the multiplication map by g does define an isomorphism in the category of continuously controlled *CW*-complexes. It is straightforward to check that these multiplication maps compose in the required fashion, so we finally have an object

$$((Q, \ell), \{g \cdot -\}_{g \in G}) \in \mathrm{CW}^{\mathrm{lf}} \Big( \Big( \bigsqcup_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D} \Big)_n \Big)^{\mathrm{hG}}$$

**Remark 24.30.** This step profits crucially from the use of controlled methods. Note that (ignoring the question which metrics to choose) any *G-CW*-complex can be considered as a labeled complex over itself by picking a point in each equivariant open cell. Such a labeling is an equivariant function, so the multiplication maps induced by group elements are isomorphisms of labeled complexes, and no mention of control is required. What the preceding argument witnesses is that the controlled categories we are considering enjoy some functoriality with respect to non-equivariant maps, as long as the failure to be equivariant becomes asymptotically zero (in the sense of condition (d)).

Lemma 24.27 combined with the transformation from (24.10) allows us to interpret this object as an object of

$$T \in \operatorname{Spc}_*^{\mathsf{f}} \left( \left( \bigsqcup_{D \in \mathcal{D}(F_n)} G \times_{\overline{D}} \Sigma_{n,D} \right)_n \right)^G.$$

Proposition 24.25 implies that we obtain a commutative diagram as in (24.2), and from the existence of this diagram we conclude that the assembly map

$$\operatorname{colim}_{\operatorname{Or}_{\mathcal{F}}(G)} F_{\mathcal{C}} \to F(\operatorname{colim}_{I(G)} \mathcal{C})$$

is an equivalence for every finitary localizing invariant F.

#### 24.4.2 Transfer Spaces with Homotopy Coherent Actions

In this section, we sketch the construction of the transfer for finitely homotopy  $\mathcal{F}$ -amenable groups, see Definition 20.48, which is the special case of the Dress-Farrell-Hsiang-Jones condition in which all the finite groups  $F_n$  are trivial. The

actual proof requires a rather significant amount of notation and bookkeeping which we sweep under the rug in favor of spelling out the overarching ideas. As before, let us remind ourselves of the available data. There exist:

- (i) A sequence {Γ<sub>n</sub>, Z<sub>n</sub>}<sub>n∈ℕ</sub> of homotopy coherent *G*-actions in the sense of Definition 20.32;
- (ii) A sequence  $\{\Sigma_n\}_{n \in \mathbb{N}}$  of abstract simplicial complexes with a simplicial *G*-action;
- (iii) A sequence  $\{f_n\}_{n \in \mathbb{N}}$  of continuous maps  $f_n \colon Z_n \to |\Sigma_n|$ ,

such that the following holds:

- (a) For every  $n \in \mathbb{N}$  the space  $Z_n$  is a compact contractible AR;
- (b) For every  $n \in \mathbb{N}$  the isotropy groups of  $|\Sigma_n|$  belong to  $\mathcal{F}$ ;
- (c) There exists a natural number N with dim $(\Sigma_n) \leq N$  for all  $n \in \mathbb{N}$ ;
- (d) For every  $k \in \mathbb{N}$  and elements  $g_0, g_1, \ldots, g_k$  in G we have

$$\lim_{n \to \infty} \sup_{\substack{(t_1, \dots, t_k) \in [0,1]^k, \\ z \in Z_n}} d_{L^1}^{\Sigma_n} (f_n(\Gamma_n(g_k, t_k, \dots, g_1, t_1, g_0, z)), g_k \dots g_0 f_n(z)) = 0.$$

Conditions (b) and (c) are required for the arguments from Section 24.3. We want to see how conditions (a) and (d) allow us to construct a suitable object in  $\operatorname{Spc}_*^f((|\Sigma_n|)_n)^G$ . Morally, this object should be given by considering the sequence  $(Z_n)_n$  as a controlled object over  $(|\Sigma_n|)_n$ , but there are at least two reasons why this does not work directly:

- (i) We only have a way to convert controlled *CW*-complexes into objects in  $\operatorname{Spc}_*^f((|\Sigma_n|)_n)$ , and  $Z_n$  is not a *CW*-complex;
- (ii) Our approach to constructing objects in  $\text{Spc}^{f}_{*}((|\Sigma_{n}|)_{n})^{G}$  is to build objects in the homotopy fixed points  $\text{CW}^{\text{lf}}((|\Sigma_{n}|)_{n})^{\text{h}G}$ , but the definition of the homotopy fixed points involves only strict *G*-actions, not homotopy coherent ones.

We address these problems in reverse order.

A construction by Vogt [969] provides strictifications of the homotopy coherent G-actions  $\Gamma_n$ : by systematic gluing of mapping cylinders, one can define a space  $X_n$  with a strict G-action which contains  $Z_n$  as a deformation retract such that, for example, the action by  $g \in G$  on  $X_n$  is homotopic to the map  $\Gamma_n(g)$  through a preferred choice of deformation retraction. We denote the preferred choice of retraction by  $r_n: X_n \to Z_n$ . The resulting topological spaces  $X_n$  are still not CW-complexes, but this can now be fixed by a functorial CW-approximation. We decide to replace each  $X_n$  by the realization  $|\text{Sing}(X_n)|$  of its singular complex.

Each *CW*-complex  $|\text{Sing}(X_n)|_+$  can now be equipped with a labeling  $\ell_n: c(|\text{Sing}(X_n)|_+) \to |\Sigma_n|$ . Recalling that each cell *z* corresponds to a singular simplex  $\Delta^{\dim(z)} \xrightarrow{z} X_n$ , the labeling is given by

$$\ell_n(z) := f_n(r_n(z(b_{\dim(z)}))),$$

where  $b_{\dim(z)}$  denotes the barycenter of the standard dim(z)-simplex.

#### 24.4 The Transfer Map

Since we allow arbitrary singular simplices, the induced labeled *CW*-complex  $(Q := \bigvee_n |\text{Sing}(X_n)|_+, \coprod_{n \in \mathbb{N}} \ell_n)$  has no chance of being continuously controlled. However, we can identify continuously controlled subcomplexes of Q. Let  $(\delta_n)_n$  be a sequence of positive real numbers converging to zero. Letting  $\text{Sing}^{\delta_n}(X_n)$  denote the subcomplex consisting of simplices z in  $X_n$  such that the diameter of  $f_n \circ r_n \circ z$  is smaller than  $\delta_n$ , the subcomplex

$$Q((\delta_n)_n) := \bigvee_n (|\operatorname{Sing}^{\delta_n}(X_n))|_+$$

is continuously controlled. Remembering that the only global property of  $(\operatorname{Sing}^{\delta_n}(X_n))_+$  that we actually care about is that  $|\operatorname{Sing}^{\delta_n}(X_n)|$  is contractible. This is unproblematic: the simplicial set  $\operatorname{Sing}^{\delta_n}(X_n)$  is the subcomplex of  $\operatorname{Sing}(X)$  of those simplices whose image is contained in a member of the open cover of  $X_n$  obtained by pulling back the open cover of  $\delta_n$ -balls via  $f_n \circ r_n$ ; it follows from excision that this does not change the homotopy type.

By putting a restriction on the size of simplices, we have introduced a new issue: the *CW*-complexes  $|\text{Sing}^{\delta_n}(X_n)|$  do not carry a *G*-action anymore. Nevertheless, if we let an individual element  $g \in G$  act on  $Q((\delta_n)_n)$  through the action on the ambient complex Q, one can find another sequence  $(\delta'_n)_n$  of positive reals converging to zero such that  $\delta'_n > \delta_n$  and multiplication by g defines a map  $g : : Q((\delta_n)_n) \to Q((\delta'_n)_n)$ . One can show that the inclusion map  $Q((\delta_n)_n) \to Q((\delta'_n)_n)$  is sent to an equivalence in  $\text{Spc}^{f}_{*}((|\Sigma_n|)_n)$  [185, Lemma 4.14 & 4.25], so through the eyes of the target category this multiplication map is just as good as an automorphism.

This prompts the following definition. Consider the category  $\widetilde{CW}^{lf}((|\Sigma_n|)_n)$  in which objects are sequential diagrams

$$(Q_1, \ell_1) \to (Q_2, \ell_2) \to \cdots$$

in  $CW^{lf}((|\Sigma_n|)_n)$  such that each connecting map is sent to an equivalence by the functor  $CW^{lf}((|\Sigma_n|)_n) \rightarrow Spc_*^f((|\Sigma_n|)_n)$ , and whose morphisms are natural transformations of the form

for some *s* which depends on the morphism. Through some careful estimates, which rely crucially on the precise definition of the strictification and condition (d) on the sizes of the traces of the homotopy coherent *G*-actions, one can show that there is a choice of positive real numbers  $(\delta_n^k)_{k,n}$ , with  $\delta_n^k < \delta_n^{k+1}$  and  $(\delta_n^k)_n$  converging to zero for each *k*, such that the sequence of inclusion maps

$$Q((\delta_n^1)_n) \to Q((\delta_n^2)_n) \to \cdots$$

refines to an object in  $\widetilde{CW}^{\text{lf}}((|\Sigma_n|)_n)^{\text{h}G}$ . The action map associated to  $g \in G$  on this object is induced by the *G*-action on *Q*.

A final stumbling block that we have tacitly ignored up to this point is that the labeled *CW*-complex  $(Q(\delta_n)_n), \ell)$  is not locally finite (so we have actually been lying by considering it as an object of  $CW^{lf}((|\Sigma_n|)_n)$ ). This is where the assumption on  $X_n$  being a compact ANR enters the picture. It is essentially a characterization of compact ANRs that these are retracts up to homotopy of finite *CW*-complexes in such a way that the traces of the witnessing homotopy can be made arbitrarily small [505, Corollary IV.6.2]. This can be used to show that, despite not being locally finite, the labeled *CW*-complex  $(Q(\delta_n)_n), \ell)$  does represent an object in the idempotent completion of  $\operatorname{Spc}_*^f((|\Sigma_n|)_n)$  [185, Proposition 4.17 & Lemma 5.16]. Since localizing invariants are invariant under passage to idempotent completions, this does not cause any further problems.

#### 24.4.3 The General Transfer

The construction of the transfer in the case of a general Dress-Farrell-Hsiang-Jones group is even heavier on notation, but does not require any new ideas beyond the input provided by Sections 24.4.1 and 24.4.2. One first considers a *G*-*CW*-complex Q as in 24.4.1. As we saw in (24.28), this complex admits a labeling over the *G*-set  $\coprod_{n \in \mathbb{N}} \coprod_{D \in \mathcal{D}(F_n)} G \times G/\overline{D}$ . Then one performs a fiberwise version of the transfer construction from Section 24.4.2 to obtain the actual transfer object *T*. Combining the arguments presented in the preceding two subsections, this again allows for the construction of a diagram as in (24.2), which then proves Theorem 20.61. See [185, Section 7] for details.

#### 24.5 Notes

We have sketched an argument which proves the isomorphism conjecture for Dress-Farrell-Hsiang-Jones groups with respect to any finitary localizing invariant with values in Sp. Using that the  $\infty$ -category Sp acts on any presentable, stable  $\infty$ -category, this argument generalizes to the case of an arbitrary finitary localizing invariant with values in a presentable, stable  $\infty$ -category. The main theorem of [185] asserts the isomorphism conjecture only for lax symmetric monoidal finitary localizing invariants, but the proof can be easily modified, following the outline given here, by letting controlled Swan *K*-theory act to define the transfer.

An alternative argument to derive the case of arbitrary finitary localizing invariants which uses the theory of localizing motives has been given by Reis [850].

# Chapter 25 Analytic Methods

### 25.1 Introduction

The methods of proofs for the Farrell-Jones Conjecture and the Baum-Conjecture are rather different. But both use controlled methods, see Section 19.4, Chapter 21, and [488]. In the Farrell-Jones setting transfers were a key ingredient, see Section 19.5 and Chapter 23, which do not seem to exist in the Baum-Connes setting. In the Baum-Connes setting *KK*-theory, see Section 10.5, is the main tool, which does not work out in the Farrell-Jones setting. This has for instance the consequence that the Full Farrell-Jones Conjecture is known for all (not necessarily cocompact) lattices in path connected second countable locally compact Hausdorff groups, whereas the Baum-Connes Conjecture is not known for  $SL_n(\mathbb{Z})$  for  $n \ge 3$ . On the other hand the Baum-Connes Conjecture with coefficients is known for a-T-menable groups, whereas the Farrell-Jones Conjecture has not been proved for elementary amenable groups. We have given status reports for the Farrell-Jones Conjecture and the Baum-Conjecture in Sections 16.2 and 16.4 and discussed open cases in Section 16.9. We have linked these two conjectures in Subsection 15.14.4.

We give only a very brief survey of the methods used in the Baum-Connes Conjecture. More information can be found for instance in the survey articles [426, 742, 963].

# 25.2 The Dirac-Dual Dirac Method

Next we briefly discuss the *Dirac-dual Dirac method*, which is the key strategy in many of the proofs of the Baum-Connes Conjecture 14.9 or the Baum-Connes Conjecture 14.11 with coefficients, see for instance [486, Theorem 7.1].

A *G*-*C*\*-algebra *A* is called *proper* if there exists a locally compact proper *G*-space *X* and a *G*-homomorphism  $\sigma: C_0(X) \to \mathcal{B}(A), f \mapsto \sigma_f$  satisfying  $\sigma_f(ab) = a\sigma_f(b) = \sigma_f(a)b$  for  $f \in C_0(X), a, b \in A$ , and for every net  $\{f_i \mid i \in I\}$ , which converges to 1 uniformly on compact subsets of *X*, we have  $\lim_{i \in I} \| \sigma_{f_i}(a) - a \| = 0$  for all  $a \in A$ . A locally compact *G*-space *X* is proper if and only if  $C_0(X)$  is proper as a *G*-*C*\*-algebra.

The following result is proved in Tu [956], extending results of Kasparov-Skandalis [565, 557].

**Theorem 25.1 (The Baum-Connes Conjecture with coefficients for proper** G-C\*-algebras). The Baum-Connes Conjecture 14.11 with coefficients holds for proper G-C\*-algebras.

**Theorem 25.2 (Dirac-dual Dirac method).** Let G be a countable (discrete) group. Let F be  $\mathbb{R}$  or  $\mathbb{C}$ . Suppose that there exist a proper G-C\*-algebra A, elements  $\alpha \in KK_i^G(A, F)$ , called the Dirac element, and  $\beta \in KK_i^G(F, A)$ , called the dual Dirac element, satisfying

$$\beta \otimes_A \alpha = 1 \in KK_0^G(F, F).$$

Then the Baum-Connes Conjecture 14.9 over F is true for G.

*Proof.* We only treat the case  $F = \mathbb{C}$  and the case of trivial coefficients. The assembly map appearing in Theorem 14.9 is a retract of the bijective assembly map from Theorem 25.1. This follows from the following commutative diagram for any cocompact *G*-*CW*-subcomplex  $C \subseteq \underline{E}G$ 

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective.

The reader should note the formal similarity between the proof of Theorem 25.2 and the proof of the Strategy Theorem 23.69.

In order to give a glimpse of the basic ideas from operator theory, we briefly describe how to define the Dirac element  $\alpha$  in the case where *G* acts on a complete Riemannian manifold *M*. Let  $T_{\mathbb{C}}M$  be the complexified tangent bundle and let  $\operatorname{Cliff}(T_{\mathbb{C}}M)$  be the associated Clifford bundle. Let *A* be the proper *G*-*C*\*-algebra given by the sections of  $\operatorname{Cliff}(T_{\mathbb{C}}M)$  which vanish at infinity. Let *H* be the Hilbert space  $L^2(\wedge^*T_{\mathbb{C}}^*M)$  of  $L^2$ -integrable differential forms on  $T_{\mathbb{C}}M$  with the obvious  $\mathbb{Z}/2$ -grading coming from even and odd forms. Let *U* be the obvious *G*-representation on *H* coming from the *G*-action on *M*. For a 1-form  $\omega$  on *M* and  $u \in H$ , define a homomorphism of  $C^*$ -algebras  $\rho: A \to \mathcal{B}(H)$  by

$$\rho_{\omega}(u) := \omega \wedge u + i_{\omega}(u).$$

Now  $D = (d + d^*)$  is a symmetric densely defined operator  $H \to H$  and defines a bounded self-adjoint operator  $F: H \to H$  by putting  $F = \frac{D}{\sqrt{1+D^2}}$ . Then  $(U, \rho, F)$ is an even cocycle and defines an element  $\alpha \in K_0^G(M) = KK_0^G(C_0(M), \mathbb{C})$ . More details of this construction and the construction of the dual Dirac element  $\beta$ , under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [963, Chapter 9]. 25.4 Notes

# 25.3 Banach KK-Theory

Skandalis [917] showed that the Dirac-dual Dirac method cannot work for all groups as long as one works with *KK*-theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [536]. This problem can be circumvented if one drops the condition unitary and works with a variant of *KK*-theory for Banach algebras, as worked out by Lafforgue [599, 601, 603].

## **25.4 Notes**

Nishikawa [760] describes a variation of the Dirac-dual-Dirac method which was used by Brodzki-Guentner-Higson-Nishikawa [168] to give a new proof the Baum-Connes Conjecture for groups which act properly and cocompactly on a finite-dimensional CAT(0)-cubical complex with bounded geometry.

# Chapter 26 Solutions of the Exercises

## **Chapter 2**

**2.7.** Check that the homomorphism  $\psi : K_0(R) \to K'_0(R)$ ,  $[P] \to [P]$  is well-defined using the fact that every exact sequence  $0 \to P_0 \to P_1 \to P_2 \to 0$  of finitely generated projective *R*-modules splits. Obviously  $\psi$  is the inverse of  $\phi$ .

**2.11.** Show that the *R*-*R*-bimodule  $(_RR^n{}_{M_n(R)}) \otimes_{M_n(R)} (_{M_n(R)}R^n{}_R)$  is isomorphic as an *R*-*R*-bimodule to *R* and that the  $M_n(R)$ - $M_n(R)$ -bimodule  $(_{M_n(R)}R^n{}_R) \otimes_R (_RR^n{}_{M_n(R)})$  is isomorphic as an  $M_n(R)$ - $M_n(R)$ -bimodule to  $M_n(R)$ .

**2.16.** See [860, Theorem 1.2.3 on page 8].

**2.29.** There exists a nowhere vanishing vector field on  $S^n$  if and only if there exist *F*-subbundles  $\xi$  and  $\eta$  in  $TS^n$  such that  $TS^n = \xi \oplus \eta$  and  $\xi$  is a 1-dimensional trivial *F*-vector bundle. Now apply Theorem 2.27.

**2.32.** Let  $\xi$  be a vector bundle over Y. It suffices to construct a  $C^0(X)$ -isomorphism

$$\alpha(\xi) \colon C^0(X) \otimes_{C^0(Y)} C^0(\xi) \xrightarrow{\cong} C^0(f^*\xi).$$

Given  $\phi \in C^0(X)$  and  $s \in C^0(\xi)$ , define  $\alpha(\xi)(\phi \otimes s)$  to be the section of  $f^*\xi$  which sends  $x \in X$  to  $\phi(x) \cdot s \circ f(x) \in \xi_{f(x)} = (f^*\xi)_x$ . Since  $\alpha(\xi \oplus \eta)$  can be identified with  $\alpha(\xi) \oplus \alpha(\eta)$  and  $\alpha(\underline{F})$  is obviously bijective,  $\alpha(\xi)$  is bijective for all *F*-vector bundles  $\xi$  over *Y*.

**2.33.** Because of the identification (2.31) and the homotopy invariance of the functor  $K^0(X)$  we get

$$K_0(C(D^n)) \cong K^0(D^n) \cong K^0(\{\bullet\}) \cong \mathbb{Z}$$

**2.40.** This follows from the fact that  $\mathbb{Z} \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$  is isomorphic as a  $\mathbb{Z}$ -chain complex to  $C_*(X)$ .

**2.49.** We can assume without loss of generality that X is connected, otherwise treat any component of X separately. Put  $\pi = \pi_1(X)$ . For i = 0, 1, 2 let  $\overline{X_i} \to X_i$  be the  $\pi$ -covering obtained from the universal covering  $\widetilde{X} \to X$  by the pull back construction associated to  $j_i: X_i \to X$ . Since  $X_i$  is finitely dominated, we conclude from Lemma 2.48 that  $C_*(\overline{X_i})$  is finitely dominated as a  $\mathbb{Z}\pi$ -chain complex and

directly from the definitions that  $(j_i)_*(o(X_i)) = o(C_*(\widetilde{X}_i))$  holds in  $K_0(\mathbb{Z}[\pi])$  for i = 0, 1, 2. There is an exact sequence of  $\mathbb{Z}\pi$ -chain complexes

$$0 \to C_*(\overline{X_0}) \to C_*(\overline{X_1}) \oplus C_*(\overline{X_2}) \to C_*(\overline{X}) \to 0.$$

Since  $\pi_1(C)$  is finitely presented for each  $C \in \pi_0(X_i)$  and  $i \in \{0, 1, 2\}, \pi_1(X)$  is finitely presented. This follows essentially from the Seifert-van Kampen Theorem. We conclude from Lemma 2.36 (ii) and Lemma 2.48 that  $C_*(\widetilde{X})$  is finitely dominated and hence *X* is finitely dominated and that we get in  $K_0(\mathbb{Z}[\pi])$ 

$$\begin{split} o(X) &= o(C_*(\widetilde{X})) = o(C_*(\overline{X_1})) + o(C_*(\overline{X_2})) - o(C_*(\overline{X_0})) \\ &= (j_1)_*(o(X_1)) + (j_2)_*(o(X_2)) - (j_0)_*(o(X_1)). \end{split}$$

**2.50.** Recall that we have chosen a finite domination (Z, i, r) of *X*. Construct an extension  $g: \operatorname{cyl}(r) \cup_Z \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \to X$  of  $\operatorname{id}_X \coprod F \cup_X F: X \coprod \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i) \to X$  and a homotopy equivalence  $h: Z \to \operatorname{cyl}(r) \cup_Z \operatorname{cyl}(i) \cup_X \operatorname{cyl}(i)$ . Now the claim follows from the commutative diagram



where *j* is the inclusion.

**2.56.** Let (B, b) be a functorial additive invariant for finite *CW*-complexes. Define a natural transformation T(X):  $\bigoplus_{C \in \pi_0(C)} \mathbb{Z} \to B(X)$  by sending  $\{n_C \mid C \in \pi_0(X)\}$  to  $\sum_{C \in \pi_0(X)} n_c \cdot A(i_C)(a(\{\bullet\}))$ , where  $i_C : \{\bullet\} \to X$  is any map whose image is contained in *C*. Obviously it is the only possible natural transformation satisfying  $T(\{\bullet\})(\chi(\{\bullet\})) = b(\{\bullet\})$ . Using additivity and homotopy invariance one proves by induction over the number of cells for a finite *CW*-complex *X* that

$$T(X)(\{\chi(C) \mid C \in \pi_0(X)\}) = b(X)$$

holds. More details can be found in [642, Theorem 4.1].

**2.58.** (i) Fix a finitely dominated *CW*-complex *Y*. Define a functor *A* from finitely dominated *CW*-complexes to abelian groups by  $A(X) := U(X \times Y)$ . Define  $a(X) \in A(X)$  to be  $u(X \times Y)$ . Check that (A, a) is a functorial additive invariant for finitely dominated *CW*-complexes. Hence there exists a unique transformation  $T_Y: U(?) \rightarrow U(? \times Y)$  sending u(X) to  $u(X \times Y)$ . Define B(Y) as the abelian group of

transformations  $U(?) \rightarrow U(? \times Y)$  and  $b(Y) := T_Y$ . Show that (B, b) is a functorial additive invariant for finitely dominated *CW*-complexes. Hence there is a natural transformation  $S: U \rightarrow B$  satisfying S(Y)(u(Y)) = b(Y) for all finitely dominated *CW*-complexes *Y*. This *S* gives the desired natural bilinear pairing P(X, Y).

(ii) If *Y* is a finite *CW*-complex with  $\chi(C) = 0$  for all  $C \in \pi_0(Y)$ , then o(C) = 0 for every  $C \in \pi_0(C)$  by Lemma 2.18 and Theorem 2.39. Theorem 2.57 implies u(Y) = 0. We conclude from (i)

$$u(X \times Y) = P(X, Y)(u(X), u(Y)) = P(X, Y)(u(X), 0) = 0.$$

Hence  $X \times Y$  is homotopy equivalent to a finite *CW*-complex by Theorem 2.39 and Theorem 2.57.

**2.66.** We define a functor  $F: \operatorname{Or}_{\mathcal{F}}(G) \to \operatorname{Sub}_{\mathcal{F}}(G)$  as follows. It sends an object G/H to the subgroup H. Consider a G-map  $f: G/H \to G/K$ . Choose  $g \in G$  with f(1H) = gK. Since f is a G-map, we get hgK = hf(1H) = f(hH) = f(1H) = gK and hence  $g^{-1}hg \in K$  for all  $h \in H$ . Hence we can define F(f) to be the class of the homomorphism  $c(g^{-1}): H \to K, h \mapsto g^{-1}hg$ . The morphism F(f) does not depend on the choice of g, since any other choice of g is of the form gk for some  $k \in K$  and we have  $c((gk)^{-1}) = c(k^{-1}) \circ c(g^{-1})$  and  $c(k^{-1}) \in \operatorname{inn}(K)$ .

Obviously F is bijective on objects and surjective on morphisms.

**2.76.** Let  $t \in \mathbb{Z}/2$  be the generator. Let  $a + bt \in R[\mathbb{Z}/2]$  be an idempotent. Since  $(a+bt)^2 = (a^2+b^2) + (ab+ba)t$  holds, we conclude  $a^2 + b^2 = a$  and ab + ba = b. This implies

$$(a + b)^{2} = a^{2} + ab + ba + b^{2} = a^{2} + b^{2} + ab + ba = a + b.$$

Since by assumption 0 and 1 are the only idempotents in *R*, we get a + b = 0 or a + b = 1.

Suppose that b = -a. Then we get

$$(2a)^{2} = 4a^{2} = 2(a^{2} + (-a)^{2}) = 2(a^{2} + b^{2}) = 2a.$$

Hence 2a = 0 or 2a = 1. Since 2 is not a unit in R we conclude 2a = 0. Hence we get

$$a = a^{2} + b^{2} = a^{2} + a^{2} = 2a^{2} = (2a)a = 0a = 0.$$

This implies a + bt = 0.

It remains to treat the case b = 1 - a. Then we get

$$(a-1) \cdot (2a-1) = 2a^2 - 3a + 1 = a^2 + (1-a)^2 - a = a^2 + b^2 - a = a - a = 0.$$

Because *R* is an integral domain, we get either (a - 1) = 0 or (2a - 1) = 0. Since 2 is not invertible in *R*, the case (2a - 1) = 0 cannot occur and hence a + bt = 1.

**2.77.** (i) The ring homomorphism  $\epsilon \colon \mathbb{Z}[x] \to \mathbb{Z}$  sending x to 1 induces a ring homomorphism  $\overline{\epsilon} \colon R \to \mathbb{Z}$ . Since  $2 = \overline{\epsilon}(\overline{2})$  is not a unit in  $\mathbb{Z}$ , the element  $\overline{2} = 2 \cdot \overline{1}$  is not invertible in *R*.

(ii) Let  $u \in \mathbb{Z}[x]$  be an element such that  $\overline{u} \in R$  is a non-trivial idempotent. We can write  $u = \sum_{i=0} a_i \cdot x^i$  for some natural number *n* and integers  $a_0, a_1, \ldots, a_n$  satisfying  $a_n \neq 0$ .

Next we show that  $n \le 1$  holds. Suppose the contrary. We can arrange without changing  $\overline{u}$  that  $a_n = \pm 1$  holds. The leading coefficient of  $u^2$  is in degree 2n and given by  $a_n^2$ . Since  $\overline{u}^2 = \overline{u}$ , the difference  $u^2 - u$  must be in the ideal generated by  $2x^2 - 3x + 1$ . Since  $2n \ge 3$ , the leading coefficient of  $u^2 - u$  is in degree n and given by  $a_n^2$ . This implies that  $a_n^2$  is a multiple of 2. This contradicts  $a_n = \pm 1$ . Hence u must be of the form  $a_0 + a_1x$  for  $a_0, a_1 \in \mathbb{Z}$ .

The image of  $\overline{u}$  under the ring homomorphism  $\overline{\epsilon}: R \to \mathbb{Z}$  is  $a_0 + a_1$ . The image  $\overline{u}$  under the ring homomorphism  $\overline{\delta}: R \to \mathbb{Z}$ , which is induced by the ring homomorphism  $\delta: \mathbb{Z}[x] \to \mathbb{Z}[1/2]$  sending x to 1/2, is  $a_0 + a_1/2$ . Since 0 and 1 are the only idempotents in  $\mathbb{Z}$  and  $\mathbb{Z}[1/2]$ , we get  $(a_0 + a_1), (a_0 + a_1/2) \in \{0, 1\}$ . From  $a_0 = (a_0 + 2a_1) - (a_0 + a_1)$  we conclude  $a_0 \in \{-1, 0, 1, 2\}$ . Hence only the following four cases can occur for  $(a_0, a_1)$ , namely (0, 0), (1, 0), (2, -2), and (-1, 2). Since  $\overline{u}$  is supposed to be a non-trivial idempotent, we must have  $(a_0, a_1) = (2, -2)$  or  $(a_0, a_1) = (-1, 2)$ . The elements  $v_1 = \overline{2 - 2x}$  and  $v_2 = -1 + 2x$  in R are different and indeed non-trivial idempotents, since we get in both cases  $v_i^2 - v_i = \overline{2 - 6x + 4x^2} = \overline{2 \cdot (x^2 - 3x + 1)} = 0$ .

(iii) The element  $\overline{x} + (1 - \overline{x}) \cdot t$  in  $R[\mathbb{Z}/2]$  is an idempotent by the following computation

$$(\overline{x} + (1 - \overline{x}) \cdot t)^2 = \overline{x}^2 + (1 - \overline{x})^2 + 2 \cdot (\overline{x} \cdot (1 - \overline{x})) \cdot t$$
  
=  $\overline{x^2 + (1 - x)^2} + \overline{2 \cdot (x \cdot (1 - x))} \cdot t$   
=  $\overline{2x^2 - 2x + 1} + \overline{-2x^2 + 2x} \cdot t$   
=  $\overline{x} + (1 - \overline{x}) \cdot t$ .

**2.91.** Choose an integer  $n \ge 0$  and a matrix  $A \in M_n(FH)$  such that  $A^2 = A$  and  $\operatorname{im}(r_A: FH^n \to FH^n) \cong_{FH} V$ . We compute for  $h \in G$  if  $l_h: V \to V$  is given by left multiplication by h

$$\chi_F(V)(h^{-1}) = \operatorname{tr}_F(l_{h^{-1}} \colon V \to V)$$
  
=  $\operatorname{tr}_F(l_{h^{-1}} \circ r_A \colon FH^n \to FH^n)$   
=  $\sum_{i=1}^n \operatorname{tr}_F(FH \to FH, \ u \mapsto h^{-1}ua_{i,i})$   
=  $\operatorname{tr}_F(FH \to FH, \ u \mapsto h^{-1}u\left(\sum_{i=1}^n a_{i,i}\right))$ 

Write  $\sum_{i=1}^{n} a_{i,i} = \sum_{k \in H} \lambda_k \cdot k$ . Then we get

$$\chi_F(V)(h^{-1}) := \operatorname{tr}_F\left(FH \to FH, \ u \mapsto h^{-1}u\left(\sum_{k \in H} \lambda_k \cdot k\right)\right)$$
$$= \sum_{k \in H} \lambda_k \cdot \operatorname{tr}_F\left(FH \to FH, \ u \mapsto h^{-1}uk\right)$$
$$= \sum_{k \in H} \lambda_k \cdot \left|\{u \in H \mid u = h^{-1}uk\}\right|$$
$$= \sum_{k \in (h)} \lambda_k \cdot \left|\{u \in H \mid h = uku^{-1}\}\right|$$
$$= \sum_{k \in (h)} \lambda_k \cdot |C_H\langle h\rangle|$$
$$= |C_H\langle h\rangle| \cdot \sum_{k \in (h)} \lambda_k$$
$$= |C_H\langle h\rangle| \cdot \operatorname{HSFH}(V)(h).$$

**2.95.** Suppose that  $\widetilde{K}_0(FG)$  is a torsion group. This is equivalent to the statement that  $\widetilde{K}_0(FG) \otimes_{\mathbb{Z}} F$  is trivial. Lemma 2.18 and Lemma 2.93 imply that  $\operatorname{class}_F(G)_f \cong_F F$  and hence  $\operatorname{con}_F(G)_f$  consists only of one element. Hence every element in *G* of finite order is trivial.

**2.97.** Because of the commutative diagram appearing in the proof of Lemma 2.93, it suffices to prove the claim in the case when *G* is finite. In this case one computes that HS(P) evaluated at the unit  $e \in G$  is  $\frac{\dim_F(P)}{|H|}$ .

**2.100.** Show  $\sum_{g \in G} \operatorname{HS}_{\mathbb{Z}G}(P)(g) = \operatorname{HS}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$ 

**2.114.** The list of finite groups of order  $\leq 9$  consists of the cyclic groups  $\mathbb{Z}/n$  for n = 1, 2, 3, ..., 9, the abelian non-cyclic groups  $\mathbb{Z}/2 \times \mathbb{Z}/2$ ,  $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ ,  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , and  $\mathbb{Z}/3 \times \mathbb{Z}/3$ , and the following non-abelian groups  $S_3 = D_6$ ,  $D_8$ , and  $Q_8$ . Now inspecting Theorem 2.113 gives the answer:

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2, \ \mathbb{Z}/3 \times \mathbb{Z}/3, \ Q_8.$$

**2.116.** Theorem 2.115 implies that  $K_0(FD_8)$  is  $\mathbb{Z}^n$  for some *n*. We conclude from Theorem 2.89 that  $n = |\operatorname{con}_F(D_8)_f|$ . A presentation for  $D_8$  is  $\langle x, y | x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$ . In particular,  $D_8$  is a semidirect product  $\mathbb{Z}/4 \rtimes \mathbb{Z}/2$  if  $\mathbb{Z}/4$  is the group generated by *x* and  $\mathbb{Z}/2$  the subgroup generated by *y*. The elements  $x^2$ , *y*, *xy*,  $x^2y$ , and  $x^3y$  have order 2, the elements *x* and  $x^{-1}$  have order four. We have one conjugacy class of elements of order 4, namely (*x*) and three conjugacy classes of elements of order two, namely  $(x^2)$ , (y), and (yx). As we also have the conjugacy class of the unit, we see  $|\operatorname{con}_{\mathbb{C}}(D_8)_f| = 5$ . Since *x* is conjugate to  $x^{-1}$ , we conclude

 $|\operatorname{con}_{\mathbb{R}}(D_8)_f| = 5$ . Since every cyclic subgroup of order 4 is conjugate to  $\langle x \rangle$ , we get  $|\operatorname{con}_{\mathbb{Q}}(D_8)_f| = 5$ . This shows

$$K_0(FD_8) \cong \mathbb{Z}^5$$
 for  $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

**2.119.** Recall that a hyperbolic group does not contain  $\mathbb{Z}^2$  as a subgroup. Because of Remark 2.118 it suffices to show for a torsionfree hyperbolic group *G* that it is cyclic if there exists an element *g* different from the unit element with finite (g). The finiteness of (g) is equivalent to the condition that the centralizer  $C_g \langle g \rangle = \{h \in G \mid hg = gh\}$  has finite index in *G*. Since  $\langle g \rangle$  is infinite, hyperbolic implies that  $C_G \langle g \rangle$  is virtually cyclic. Hence *G* is a torsionfree virtually cyclic group and therefore cyclic.

**2.124.** Write  $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$  for distinct primes  $p_1, p_2, \dots, p_r$  and integers  $n_i \ge 1$ . Then Lemma 2.12 implies

$$K_0(\mathbb{Z}/n) = \prod_{i=1}^r K_0(\mathbb{Z}/p_i^{n_i}).$$

Since  $\mathbb{Z}/p_i^{n_i}$  is local, the claim follows from Theorem 2.123.

**2.127.** A counterexample is given by  $G_1 = G_2 = \mathbb{Z}/3$ , since  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3]) = \{0\}$  and  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}/3]) \neq \{0\}$  by Theorem 2.113.

## **Chapter 3**

**3.3.** Let  $f: \mathbb{R}^n \to \mathbb{R}^n$  be an R-automorphism. This is the same as a K-linear isomorphism  $V^n \to V^n$ . Since V is a K-vector space with infinite countable basis, we can choose a K-isomorphism  $\alpha: \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} V$ . Let  $a: \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} V^n$  be the K-isomorphism given by  $\bigoplus_{k=0}^{\infty} f$ . Let  $\gamma: V^n \oplus \bigoplus_{k=0}^{\infty} V^n \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} V^n$  be the K-isomorphism which sends  $v \oplus (v_0, v_1, v_2, ...)$  to  $(v, v_0, v_1, v_2, ...)$ . One easily checks  $\gamma^{-1} \circ a \circ \gamma = f \oplus a$  Define an K-automorphism  $b: V \to V$  by  $\alpha \circ a \circ \alpha^{-1}$ . This is the same as an R-automorphism  $b: R \to R$ . Now one computes  $[f] + [b] = [f \oplus b] = [b]$  in  $K_1(R)$  using the fact that conjugate automorphisms define the same element in  $K_1(R)$ . This implies [f] = 0.

**3.7.** We get from Theorem 3.6 an isomorphism  $i: \mathbb{H}^{\times}/[\mathbb{H}^{\times}, \mathbb{H}^{\times}] \xrightarrow{\cong} K_1(\mathbb{H})$ . Obviously the collection of maps  $\mu_n$  defines a homomorphism  $\mu: K_1(\mathbb{H}) \to \mathbb{R}$ . The norm of a quaternion z = a + bi + cj + dk is defined by  $N(z) := \sqrt{a^2 + b^2 + c^2 + d^2}$ . Let  $N: \mathbb{H}^{\times}/[\mathbb{H}^{\times}, \mathbb{H}^{\times}] \to \mathbb{R}^{>0}$  be the induced homomorphism of abelian groups. Its restriction to  $\mathbb{R}^{>0} \subseteq \mathbb{H}$  is the identity. Since  $\mu_1(z) = |z|^4$  for  $z \in \mathbb{H}$ , it remains to prove  $N^{-1}(1) \subseteq [\mathbb{H}^{\times}, \mathbb{H}^{\times}]$ .

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Since  $ie^{j\theta}i^{-1} = e^{-j\theta}$  holds for  $\theta \in \mathbb{R}$  and similarly with *i*, *j*, and *k* cyclically permuted,  $e^{2i\theta_1}$ ,  $e^{2j\theta_2}$ , and  $e^{2k\theta_3}$  are all commutators. These generate an open neighborhood of 1 in  $S^3 = N^{-1}(1)$ . Since  $S^3$  is connected, the claim follows.

**3.18.** Take the norm on  $\mathbb{Z}[i]$  sending a + bi to  $\sqrt{a^2 + b^2}$ . It yields a Euclidean algorithm. A direct calculation shows  $\mathbb{Z}[i]^{\times} = \{1, -1, i, -i\}$ . Now apply Theorem 3.17.

**3.22.** This follows from Theorem 3.20 and Theorem 3.21.

**3.25.** The map  $\phi$  is induced by the composite

$$K_1(\mathbb{Z}[\mathbb{Z}/5]) \xrightarrow{f_*} K_1(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^{\times} \xrightarrow{||} \mathbb{R}^{>0}$$

Since  $(1 - t - t^{-1}) \cdot (1 - t^2 - t^3) = 1$ , the element  $1 - t - t^{-1}$  is a unit in  $\mathbb{Z}[\mathbb{Z}/5]$  and defines an element in Wh( $\mathbb{Z}/5$ ). Its image under  $\phi$  is  $(1 - 2 \cdot \cos(2\pi/5))$  and hence different from 1.

**3.36.** Let  $\epsilon_*$  be a chain contraction for  $E_*$ . Choose for any  $n \in \mathbb{Z}$  an *R*-homomorphism  $\sigma_n : E_n \to D_n$  satisfying  $p_n \circ \sigma_n = id_{E_n}$ . Define  $s_n : E_n \to D_n$  by  $d_{n+1} \circ \sigma_{n+1} \circ \epsilon_n + \sigma_n \circ \epsilon_{n-1} \circ e_n$ .

There are examples of short exact sequences of *R*-chain complexes whose boundary operator in the associated long homology sequence is not trivial and hence for which  $H_n(p_*)$  is not surjective for all  $n \in \mathbb{Z}$ .

**3.41.** This is done by the following sequence of elementary collapses. We describe the simplicial complexes obtained after each step:

- (i) The standard 2-simplex spanned by  $v_0$ ,  $v_1$ ,  $v_2$ ;
- (ii) Three vertices  $v_0$ ,  $v_1$ ,  $v_2$  and two edges  $\{v_0, v_1\}$  and  $\{v_0, v_2\}$ ;
- (iii) The standard 1-simplex spanned by  $v_0, v_1$ ;
- (iv) The standard 0-simplex given by  $v_0$ .

**3.46.** This follows from  $\widetilde{K}_0(\mathbb{Z}) = 0$ , see Example 2.4, and Wh({1}) = 0, see Theorem 3.17, together with Theorem 2.39 and Theorem 3.45.

**3.49.** Choose a non-trivial element in Wh( $\mathbb{Z}/5$ ), see Exercise 3.25. By Theorem 3.47 we can find an *h*-cobordism ( $W, M_0, M_1$ ) whose Whitehead torsion is *x*. Hence it is non-trivial. In order to show that ( $W \times S^3$ ;  $M_0 \times S^3$ ,  $M_1 \times S^3$ ) is trivial, we have to show  $\tau(i_0 \times id_{S^3}) = 0$  for  $i_0: M_0 \to W$  the inclusion. This follows from Theorem 3.37 (iv), since both  $\tau(id_{S^3})$  and  $\chi(S^3)$  vanish.

**3.57.** By definition  $\mathbb{RP}^3$  is the lens space L(V) for the cyclic group  $\mathbb{Z}/2$ , where V has as underlying unitary vector space  $\mathbb{C}^2$  and the generator *s* of  $\mathbb{Z}/2$  acts on V by – id. The cellular  $\mathbb{Z}[\mathbb{Z}/2]$ -chain complex  $C_*(SV)$  is concentrated in dimensions 0, 1, 2, 3 and is given by

$$\cdots \to 0 \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s-1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s+1} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{s-1} \mathbb{Z}[\mathbb{Z}/2] \to 0 \to \cdots$$

Hence  $\mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_*(SV)$  is the  $\mathbb{R}$ -chain complex

$$\cdots \to 0 \to \mathbb{R} \xrightarrow{2 \cdot \mathrm{id}} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{2 \cdot \mathrm{id}} \mathbb{R} \to 0 \to \cdots$$

It is contractible, a chain contraction  $\gamma_*$  is given by  $\gamma_0 = \gamma_2 = 1/2 \cdot \text{id}$  and  $\gamma_n = 0$  for  $n \neq 0, 2$ . Hence  $(c + \gamma)_{\text{odd}} \colon \mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_{\text{odd}}(SV) \to \mathbb{R}^- \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_{\text{odd}}(SV)$  is given by

$$\begin{pmatrix} 2 & 1/2 \\ 0 & 2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

This implies  $\rho(\mathbb{RP}^3; V) = 4$ .

**3.77.** We use induction over  $n \ge 0$ . The case n = 0, i.e., the trivial group, follows from Example 2.4. The induction step from n to n + 1 is a direct consequence of Theorem 3.76 (i), since  $R[\mathbb{Z}^n][\mathbb{Z}]$  is isomorphic to  $R[\mathbb{Z}^{n+1}]$ .

**3.82.** Because of Theorem 3.80 (ii) the ring  $\mathbb{Z}[\mathbb{Z}^n]$  is regular. Hence we get from Exercise 3.77 and Lemma 3.85 that  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^n]) = 0$ .

To show Wh( $\mathbb{Z}^n$ ) = 0, we use induction over  $n \ge 0$ . The case n = 0, i.e., the trivial group, follows from Example 2.4 and Theorem 3.17. The induction step from n to n + 1 follows from Theorem 3.81, since  $\mathbb{Z}[\mathbb{Z}^n][\mathbb{Z}]$  is isomorphic to  $\mathbb{Z}[\mathbb{Z}^{n+1}]$ .

**3.92.** Obviously (2)  $\xrightarrow{\cong} (N_{\mathbb{Z}/2})$  sending 2 to  $N_{\mathbb{Z}/2}$  is an isomorphism of rings without unit.

Theorem 3.89 together with Lemma 3.91 yields exact sequences

$$K_1(\mathbb{Z}) \to K_1(\mathbb{Z}/n) \to K_0((n)) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/n);$$
  

$$K_1(\mathbb{Z}[\mathbb{Z}/2]) \to K_1(\mathbb{Z}) \to K_0((N_{\mathbb{Z}/2})) \to K_0(\mathbb{Z}[\mathbb{Z}/2]) \to K_0(\mathbb{Z}),$$

since the ring homomorphism  $\mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}$  sending a + bt to a - b induces an isomorphism of rings  $\mathbb{Z}[\mathbb{Z}/2]/(N_{\mathbb{Z}/2}) \xrightarrow{\cong} \mathbb{Z}$ . Because of Theorem 3.6 and Theorem 3.17 the determinant induces isomorphisms

det: 
$$K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\};$$
  
det:  $K_1(\mathbb{Z}/n) \xrightarrow{\cong} \mathbb{Z}/n^{\times}.$ 

The map  $K_k(\mathbb{Z}[\mathbb{Z}/2]) \to K_k(\mathbb{Z})$  is surjective for k = 0, 1, as its composite with  $K_k(\mathbb{Z}) \to K_k(\mathbb{Z}[\mathbb{Z}/2])$  is the identity. The map  $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/n)$  is injective, since its composite with the map  $K_0(\mathbb{Z}/n) \to \mathbb{Z}$ ,  $[P] \mapsto |P|$  is injective by Theorem 2.4. This implies

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$$K_0((n)) \cong \begin{cases} 0 & \text{if } n = 2; \\ (\mathbb{Z}/n)^{\times}/\{\pm 1\} & \text{if } n \ge 3; \end{cases}$$
$$K_0((N_{\mathbb{Z}/2})) = \{0\}; \\ \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) = \{0\}. \end{cases}$$

**3.96.** Because of Remark 3.95 it suffices to show for each two-sided ideal  $I \subseteq F$  that E(F, I) = SL(F, I). This is trivial if I = 0. If I = F, this follows from Theorem 3.17.

**3.101.** Consider  $k \in \mathbb{Z}$  with (k, |G|) = 1. Choose  $l \in \mathbb{Z}$  with  $kl = 1 \mod |G|$ . Choose a generator  $t \in G$ . Define elements  $u, v \in \mathbb{Z}G$ .

$$u = 1 + t + t^{2} + \dots + t^{k-1};$$
  

$$v = 1 + t^{k} + t^{2k} + \dots + t^{(l-1)k}$$

Then  $(t-1) \cdot (t^k - 1) \cdot (uv - 1) = 0$  holds in  $\mathbb{Z}G$ . One easily checks that  $(t-1) \cdot (t^k - 1) \cdot w = 0 \iff w \in (N_G)$  for  $w \in \mathbb{Z}G$ . Hence  $\overline{u} \in \mathbb{Z}G/(N_G)$  is a unit and maps to  $\overline{k}$  under the map  $j_1: \mathbb{Z}G/(N_G) \to \mathbb{Z}/|G|$ . Now the claim follows from the Mayer-Vietoris sequence associated to the diagram (3.97).

**3.103.** Since  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/2]) = 0$ , see Theorem 2.113 (i), we can assume without loss of generality  $|G| \ge 3$ .

Suppose d = 1. Then  $G \setminus X$  is a connected finitely dominated 1-dimensional *CW*-complex. Its homology is finitely generated and it is homotopy equivalent to a 1-dimensional *CW*-complex *Y* with precisely one 0-cell. Hence the *CW*-complex *Y* is finite.

Suppose that  $d \ge 2$ . Then *d* is odd by Theorem 3.102 (i). The unit sphere *S* in  $\mathbb{C}^{(d+1)/2}$  with the *G*-action for which the generator acts by multiplication by  $\exp(2\pi i/|G|)$  is a free *d*-dimensional *G*-homotopy representation such that  $G \setminus S$  is compact and hence finite. By elementary obstruction theory there exists a *G*-map  $X \to S$ . Now apply Theorem 2.39 (i), Lemma 3.102 (ii), and Exercise 3.101.

**3.111.** Obviously the image of the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G \times \mathbb{Z})} K_1(RH) \to K_1(R[G \times \mathbb{Z}])$$

is contained in the image of the map  $K_1(RG) \to K_1(R[G \times \mathbb{Z}])$ . Theorem 3.72 implies  $K_0(\mathbb{Z}G) = \{0\}$  if  $K_1(RG) \to K_1(R[G \times \mathbb{Z}])$  is surjective.

If *R* is a commutative integral domain,  $K_0(R)$  and hence  $K_0(RG)$  cannot be zero. Namely, if *F* is its quotient field, the homomorphism

$$K_0(R) \to \mathbb{Z}, \ [P] \mapsto \dim_F(F \otimes_R P)$$

is a well-defined surjective map.

**3.118.** This follows from Theorem 3.115 and Theorem 3.116 (iv).

**3.119.** Theorem 3.115 implies for a finite group *G* that Wh(*G*) is non-trivial and finite if and only if  $SK_1(\mathbb{Z}G)$  is non-trivial and  $r_{\mathbb{R}}(G) = r_{\mathbb{Q}}(G)$  holds. Because of Theorem 3.116 (ii) the smallest order abelian group *G* with-non-trivial  $SK_1(\mathbb{Z}G)$  is  $G = \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/3$ . One easily checks  $r_{\mathbb{R}}(G) = r_{\mathbb{Q}}(G)$  for  $G = \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/3$ .

**3.124.** A counterexample is given by  $G_1 = G_2 = \mathbb{Z}/3$ , since Wh( $\mathbb{Z}/3$ ) = {0} and Wh( $\mathbb{Z}/3 \times \mathbb{Z}/3$ )  $\neq$  {0} by Theorem 3.116.

# **Chapter 4**

**4.5.** Apply Remark 4.4 to the obvious pullback of rings



Or, if one does not like the ring  $\{0\}$  consisting of one element, use Lemma 3.9 and the Bass-Heller-Swan decomposition 4.3.

**4.9.** This follows by induction over k using Theorem 4.7.

**4.11.** It suffices to show  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$  for  $k \ge 0$  since then all other claims follow from Theorem 4.7.

The exact Mayer-Vietoris sequence appearing in Example 4.10 yields for a prime p the exact sequence

$$K_1(\mathbb{Z}[\mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\exp(2\pi i/p)][\mathbb{Z}^k]) \to K_1(\mathbb{F}_p[\mathbb{Z}^k]) \\ \to \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}^k]) \to \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \to 0.$$

We have  $\widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/3)]) = \{0\}$  by Theorem 2.106 and Example 2.107. Hence it suffices to show that the map  $K_1(\mathbb{Z}[\exp(2\pi i/3)][\mathbb{Z}^k]) \to K_1(\mathbb{F}_3[\mathbb{Z}^k])$  is surjective. Because of Theorem 2.23, Theorem 3.80, and the Bass-Heller-Swan decomposition for lower and middle *K*-theory for regular rings, see Theorem 4.7, it suffices to prove the surjectivity of  $K_i(\mathbb{Z}[\exp(2\pi i/3)]) \to K_i(\mathbb{F}_3)$  for i = 0, 1. The case i = 0follows from the fact that  $K_0(\mathbb{F}_3)$  is generated by  $[\mathbb{F}_3]$ . It remains to treat i =1. Let  $f: \mathbb{Z}[\exp(2\pi i/3)] \to \mathbb{F}_3$  be the ring homomorphism which is uniquely determined by the property that it sends  $\exp(2\pi i/3)$  to 1. Because of Theorem 3.17 it suffices to show that for every unit u in  $\mathbb{F}_3$  we can find a unit u' in  $\mathbb{Z}[\exp(2\pi i/3)]$ which is mapped to u under f. Since  $\pm \exp(2\pi i/3)$  is a unit in  $\mathbb{Z}[\exp(2\pi i/3)]$  and  $f(\pm \exp(2\pi i/3)) = \pm 1$ , we conclude  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = 0$  for  $k \ge 0$ .

**4.13.** The pullback of rings appearing in Example 4.12 yields a pullback of rings

where  $j_2 = j_1$ . Put  $j := j_1 = j_2$ . We obtain from Remark 4.4 the exact sequence

$$\begin{split} K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) &\xrightarrow{j_* \oplus j_*} K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) \\ & \to K_0(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) \to K_0(\mathbb{Z}[\mathbb{Z}/3] \times \mathbb{Z}^k) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \\ & \xrightarrow{j_* \oplus j_*} K_0((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) \to K_{-1}(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k]) \to \cdots \end{split}$$

The following facts are consequences of Theorem 3.80 (i), Exercise 4.5, Exercise 4.11, and Theorem 4.7. We have  $K_n(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = K_n((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k]) = \{0\}$  for  $n \leq -1$  and  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) = \{0\}$ . We can identify the map

$$j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_0((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$$

with the map  $j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4)$  which in turn can be identified the map  $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$  sending (a, b) to (a + b, a + b). The map

$$j_* \oplus j_* \colon K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$$

can be identified with the direct sum of the map  $j_* \oplus j_* \colon K_1(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_1(\mathbb{Z}[\mathbb{Z}/3]) \to K_1(\mathbb{F}_2 \times \mathbb{F}_4)$  with the *k*-fold direct sum of copies of the map  $j_* \oplus j_* \colon K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \to K_0(\mathbb{F}_2 \times \mathbb{F}_4)$ . In order to prove

$$K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) \cong \begin{cases} \mathbb{Z}^{k+1} & \text{for } n = 0; \\ \mathbb{Z} & \text{for } n = -1; \\ 0 & \text{for } n \le -2, \end{cases}$$

it remains to show that the map  $j_*: K_1(\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}^k]) \to K_1((\mathbb{F}_2 \times \mathbb{F}_4)[\mathbb{Z}^k])$  is surjective.

Recall that we established an identification of rings  $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times \mathbb{F}_4$  in Example 4.12. Because of Lemma 3.9 and Theorem 3.17 the determinant induces an isomorphism  $K_1(\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]) \xrightarrow{\cong} (\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3])^{\times}$ . Hence it suffices to show that for every unit *u* in  $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$  we can find a unit *u'* in  $\mathbb{Z}[\mathbb{Z}/3]$  which is mapped under the obvious projection pr:  $\mathbb{Z}[\mathbb{Z}/3] \to \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3]$  to *u*. There are three units in  $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{F}_2 \times \mathbb{F}_4$ , namely,  $1 \otimes 1$ ,  $1 \otimes t$ , and  $1 \otimes t^2$ . Obviously they are images of units under pr.

We conclude  $N^p K_n(\mathbb{Z}[\mathbb{Z}/6 \times \mathbb{Z}^k])$  for  $n \le 0, p \ge 1$ , and  $k \ge 0$  from Theorem 4.3, since  $K_0(\mathbb{Z}[\mathbb{Z}/6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/6])^{k-1} \cong \mathbb{Z}^k \cong K_0(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]), K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong K_{-1}(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6])$ , and  $K_n(\mathbb{Z}[\mathbb{Z}^k \times \mathbb{Z}/6]) = \{0\}$  for  $n \le -1$  holds.

**4.15.** This follows from Lemma 4.14, since the assumptions imply that  $K_m(R) \to K_m(R[\mathbb{Z}^n])$  induced by the inclusion  $R \to R[\mathbb{Z}^n]$  is bijective.

**4.19.** Because of Theorem 4.7. it suffices to prove that *RG* is regular, provided that *R* is regular, *G* is a finite group, and the order |G| of *G* is invertible in *R*. Since *R* is Noetherian and *G* is finite, *RG* is Noetherian. Let *M* be any finitely generated *RG*-module. Then the *RG*-module *M* is a direct summand in the *RG*-module *M'* :=  $RG \otimes_R M$ , where *g* acts on  $x \otimes m$  by  $gx \otimes m$ . So *M'* does not see the *G*-action on *M*. The injection  $M \to M'$  is given by  $m \mapsto \frac{1}{|G|} \cdot \sum_{g \in G} g \otimes g^{-1}m$  and the retraction  $M' \to M$  by  $g \otimes m \mapsto gm$ . Let  $P_*$  be a finite projective *R*-resolution of the finitely generated *R*-module *M*. Then  $P'_*$  is a finite *RG*-resolution of *M'*. Since *M* is a direct *RG*-summand in *M'*, it possesses a finite projective *RG*-resolution as well.

## Chapter 5

**5.6.** This follows from Lemma 3.11, Theorem 3.12, and Definition 5.1.

**5.8.** This follows from Theorem 5.7 and the fact that  $H_2(E(R))$  is the kernel of the universal central extensions  $\phi^R \colon St(R) \to E(R)$  of the perfect group E(R).

**5.17.** Obviously the matrices  $d_{1,2}(u)$  and  $d_{1,3}(v)$  represent the trivial element in  $K_1(R)$ . Hence they belong to E(R) by Lemma 3.11 and Theorem 3.12. Let  $\tilde{d}_{1,2}(u)$  and  $\tilde{d}_{1,3}(v)$  be fixed preimages of  $d_{1,2}(u)$  and  $d_{1,3}(v)$  under the canonical map  $\phi^R$ : St(R)  $\rightarrow E(R)$ . Then any other lifts are of the form  $\tilde{d}_{1,2}(u) \cdot x$  and  $\tilde{d}_{1,3}(v) \cdot y$  for elements in the center of St(R). One easily checks  $[\tilde{d}_{1,2}(u), \tilde{d}_{1,3}(v)] =$  $[\tilde{d}_{1,2}(u) \cdot x, \tilde{d}_{1,3}(v) \cdot y]$ .

**5.20.** We get  $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$  with generator  $\{-1, -1\}$  from Theorem 5.18 (vi).

**5.23.** We obtain Wh<sub>2</sub>( $\mathbb{Z}/n$ ) = 0 for n = 1, 2, 3, 4 from Section 5.8. By Theorem 3.115 the Whitehead group Wh( $\mathbb{Z}/n$ ) vanishes if and only if n = 1, 2, 3, 4, 6. Theorem 2.113 (i) implies  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/n]) = 0$  for n = 1, 2, 3, 4. We conclude  $K_i(\mathbb{Z}[\mathbb{Z}/n]) = 0$  for n = 1, 2, 3, 4 and all  $i \leq -1$  from Theorem 4.22 (i) and (v) We conclude  $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \neq 0$  from Example 4.12. Hence the answer is n = 1, 2, 3, 4.

#### Chapter 6

**6.1.** Let *Z* be acyclic. Since  $H_0(Z)$  is the free abelian group with  $\pi_0(Z)$  as  $\mathbb{Z}$ -basis, *Z* is path connected. Since the classifying map  $f: Z \to B\pi$  for  $\pi = \pi_1(Z)$  is 2-connected, it induces by the Hurewicz Theorem an isomorphism  $H_1(Z) \to H_1(\pi)$  and an epimorphism  $H_2(Z) \to H_2(\pi)$ .

**6.7.** If  $P_1$  and  $P_2$  are two perfect subgroups of G, then the subgroup  $\langle P_1, P_2 \rangle$  generated by  $P_1 \cup P_2$  is again a perfect subgroup of G.

**6.8.** Recall that E(R) = [GL(R), GL(R)] by Lemma 3.11. We know already because of Theorem 5.7 that E(R) = [GL(R), GL(R)] is perfect, since only a perfect group possesses a universal central extension. Since the image of a perfect subgroup under an epimorphism of groups is perfect and the only perfect subgroup of the abelian group GL(R)/[GL(R), GL(R)] is the trivial group, every perfect subgroup of GL(R) is contained in E(R).

**6.10.** Since BGL(R) and hence  $BGL(R)^+$  is path connected, this follows directly from the definitions in the case n = 0. If n = 1, this follows from Theorem 3.12, Theorem 6.5 (iv), and Exercise 6.8.

**6.22.** This follows by induction over k from Theorem 4.3, Theorem 4.22 (i), and Theorem 6.21.

**6.26.** We conclude from Example 2.4 and Theorem 3.17 that the sequence looks like

$$\{\pm 1\} \xrightarrow{j_1} \mathbb{Q}^{\times} \xrightarrow{\partial_1} \bigoplus_p \mathbb{Z} \xrightarrow{i_0} \mathbb{Z} \xrightarrow{j_0} \mathbb{Z} \to 0$$

where *p* runs through all prime numbers,  $j_1$  is the inclusion, and  $j_0$  is the identity. Hence the map  $i_0$  is the zero map. The map  $\partial_1$  sends a rational number of the shape  $\pm p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$  for pairwise distinct primes  $p_1, p_2, \ldots, p_k$  and integers  $n_1, n_2, \ldots, n_k$  to the element  $(n_p)_p$ , whose entry for  $p = p_i$  is  $n_i$  for  $i = 1, 2, \ldots, k$  and is 0 for any prime *p* which is not contained in  $\{p_1, p_2, \ldots, p_k\}$ .

**6.27.** We get from Corollary 6.25 the exact sequence for  $n \ge 1$ ,

$$\bigoplus_{p} K_{n}(\mathbb{F}_{p}) \to K_{n}(\mathbb{Z}) \to K_{n}(\mathbb{Q}) \to \bigoplus_{p} K_{n-1}(\mathbb{F}_{p}).$$

By Theorem 6.23  $K_n(\mathbb{F}_p) = 0$  holds for n = 2k for  $k \ge 1$ , and  $K_n(\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$  holds for all  $n \ge 1$ .

**6.31.** From the analog of the sequence (6.28) for  $K^{\text{TOP}}$  and the assumption that *k* is odd, we conclude

$$K_n^{\text{TOP}}(\mathbb{R}; \mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \equiv 0 \mod 4; \\ \{0\} & n = 1, 2, 3 \mod 4. \end{cases}$$

We know  $K_n(\mathbb{R}) = \{0\}$  for  $n \le -1$  from Theorem 4.7. Now the sequence (6.28) and Theorem 6.30 imply

$$K_n(\mathbb{R}; \mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \ge 0 \text{ and } n \equiv 0 \mod 4; \\ \{0\} & n \ge 0 \text{ and } n = 1, 2, 3 \mod 4; \\ \{0\} & n \le -1. \end{cases}$$

**6.38.** Generators for  $K_0(\mathcal{A})$  are isomorphism classes of objects. Relations are  $[P_1] + [P_2] = [P_1 \oplus P_2]$  for any objects  $P_1, P_2$ .

The generators of  $K_1(\mathcal{A})$  are conjugacy classes of automorphisms of objects of  $\mathcal{A}$ . Relations are  $[g \circ f] = [g] + [f]$  for any automorphisms f, g of the same object and  $\begin{bmatrix} f_1 & f_0 \\ 0 & f_2 \end{bmatrix} = [f_1] + [f_2]$  for any automorphisms  $f_i \colon P_i \to P_i$  for i = 1, 2 and any morphism  $f_0 \colon P_2 \to P_1$ .

The functor *S* induces homomorphisms  $S_i : K_i(\mathcal{A}) \to K_i(\mathcal{A})$  for i = 1, 2. The existence of the natural transformation *T* implies that the two homomorphisms  $S_i + id_{K_i(\mathcal{A})}$  and  $S_i$  coincide. Hence  $id_{K_i(\mathcal{A})}$  is the zero-homomorphism, which means  $K_i(\mathcal{A}) = 0$ .

**6.39.** Let  $\mathcal{A}$  be the additive category of countably generated projective *R*-modules. Let *S* be the functor sending an object *P* to  $(P \oplus P \oplus \cdots)$ . Then we obtain a natural transformation *T*: id  $\oplus S \to S$  by rebracketing, i.e.  $(P \oplus P \oplus \cdots) = P \oplus (P \oplus P \oplus \cdots)$ . Hence  $\mathcal{A}$  is flasque and we can apply Theorem 6.37 (iii).

**6.46.** This follows directly from the Resolution Theorem 6.45.

**6.51.** Since the rings  $\mathbb{Z}$ ,  $\mathbb{Z}[1/p]$ , and  $\mathbb{F}_p$  are regular, this follows from Example 6.50 using Theorem 4.7 and Exercise 6.46.

**6.55.** Because of Conjecture 6.53 it suffices to construct the corresponding sequence for  $H_*(-; \mathbf{K})$ 

$$\cdots \to H_n(BG_0; \mathbf{K}(R)) \to H_n(BG_1; \mathbf{K}(R)) \oplus H_n(BG_2; \mathbf{K}(R))$$
$$\to H_n(BG; \mathbf{K}(R)) \to H_{n-1}(BG_0; \mathbf{K}(R))$$
$$\to H_{n-1}(BG_1; \mathbf{K}(R)) \oplus H_{n-1}(BG_2; \mathbf{K}(R)) \to \cdots$$

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One can arrange that  $BG_i$  is a sub CW-complex of BG and  $BG = BG_1 \cup BG_2$  and  $BG_0 = BG_1 \cap BG_2$ . Now the desired sequence above is the associated Mayer-Vietoris sequence.

**6.56.** Because of Conjecture 6.53 it suffices to construct the corresponding sequence for  $H_*(-; \mathbf{K})$ 

$$\cdots \to H_n(BG; \mathbf{K}(R)) \xrightarrow{\mathrm{id} - \phi_*} H_n(BG; \mathbf{K}(R)) \to H_n(B(G \rtimes_{\phi} \mathbb{Z}); \mathbf{K}(R))$$
$$\to H_{n-1}(BG; \mathbf{K}(R)) \xrightarrow{\mathrm{id} - \phi_*} H_{n-1}(BG; \mathbf{K}(R)) \to \cdots$$

The automorphism  $\phi$  induces a homotopy equivalence  $B\phi: BG \to BG$ . The mapping torus of  $B\phi$  is a model for  $B(G \rtimes_{\phi} \mathbb{Z})$ . Now the desired long exact sequence comes from the Wang sequence associated to the fibration  $BG \to B(G \rtimes_{\phi} \mathbb{Z}) \to S^1$ .

**6.60.** If *R* is regular, then R[t] is regular. There is an obvious identification (R[t])G = (RG)[t]. Hence we obtain a commutative diagram

where the vertical arrows are induced by the canonical inclusions  $R \to R[t]$  and  $RG \to (RG)[t]$ . The horizontal arrows are bijective by assumption. Since *R* is regular, the left vertical arrow is bijective because of Theorem 6.16 (ii) and the Atiyah-Hirzebruch spectral sequence. Hence also the right vertical arrow is bijective. This implies  $NK_n(RG) = 0$  for all  $n \in \mathbb{Z}$ .

**6.63.** Consider the following commutative diagram, where Nil<sub>n</sub> stands for the *n*-homotopy group of Nil( $RG_0$ ;  $RG_1$ ,  $RG_2$ ),  $NK_n$  stands for  $NK_n(RG_0; RG_1, RG_2)$ , and the letters  $\iota$  and  $\pi$  denote obvious inclusions or projections. The maps  $i_*$ ,  $l_*$ , and  $f_*$  are induced by the map of spectra **i**, **l**, and **f**. The middle column is the long exact sequence associated to the homotopy cartesian square appearing in Theorem 6.61 (i) with boundary operator  $\partial_n$ . Theorem 6.61 implies that the two horizontal short sequences are (split) exact and the diagram (without the dashed arrows) commutes.



Now an easy diagram chase shows that there exist the dotted arrows uniquely determined by the property that the diagram remains commutative.

Define the desired long exact Mayer-Vietoris sequence by the homomorphism  $\alpha' : K_0(RG_1) \oplus K_n(RG_2) \rightarrow \ker(p_n)$  which is the restriction of  $\alpha$ , the homomorphism  $\beta$ , and the homomorphism  $(j_1)_* \oplus (j_2)_* : K_n(RG) \rightarrow K_n(RG_1) \oplus K_n(RG_2)$ . We leave it to the reader to check using the diagram above that this sequence is indeed exact.

**6.73.** There is an obvious projection pr:  $R \to R_0$ . Since pr  $\circ i = id_{R_0}$  for the inclusion  $i: R_0 \to R$ , it suffices to prove that  $KH_n(i) \circ KH_n(\text{pr}): KH_n(R) \to KH_n(R)$  is surjective. Define a map  $\varphi: R \to R[t]$  by sending  $r_n \in R_n$  to  $r_n \cdot t^n$ . For k = 0, 1 let  $ev_k: R[t] \to R$  be the ring homomorphism given by putting t = 0 for k = 0 and t = 1 for k = 1. Then  $ev_1 \circ \varphi = id_R$  and  $KH_n(ev_k)$  is bijective for k = 0, 1 and  $n \in \mathbb{Z}$  by homotopy invariance. Hence  $KH_n(ev_0)$  and  $KH_n(\varphi)$  are isomorphisms. Since  $ev_0 \circ \varphi$  agrees with  $i \circ pr$ , the claim follows.

#### Chapter 7

**7.7.** The composite of two cofibrations is again a cofibration. The same is true for weak equivalences. Hence coC and wC are indeed subcategories of C.

Axioms (i), (ii) and (iv) appearing in Definition 7.5 are obviously satisfied.

Consider chain maps  $i_* \colon A_* \to B_*$  and  $f_* \colon A_* \to C_*$  of finite projective *R*-chain complexes such that  $i_n \colon A_n \to B_n$  is split injective for all  $n \in \mathbb{Z}$ . Define  $D_*$  to be the cokernel of the chain map  $i_* \oplus f_* \colon A_* \to B_* \oplus C_*$ . Then we obtain a short exact sequence of finite projective *R*-chain complexes  $0 \to A_* \xrightarrow{i_* \oplus f_*} B_* \oplus C_* \xrightarrow{pr_*} D_* \to 0$ ,

since for every  $n \ge 0$  the map  $i_n$  is split injective and hence the sequence of *R*-modules  $0 \to A_n \xrightarrow{i_n \oplus f_n} B_n \oplus C_n \xrightarrow{\text{pr}_n} D_n \to 0$  is split exact. One easily checks that we obtain a pushout of finite projective *R*-chain complexes



such that the lower horizontal arrow is a cofibration. Hence axiom (iii) is satisfied.

Axiom (v) follows from the long exact homology sequences associated to a short exact sequence of *R*-chain complexes and the Five Lemma.

**7.19.** This follows from the property of the map **i** of (7.14) that  $\pi_n(\mathbf{i})$  is bijective for  $n \ge 1$ , from Remark 7.15, and from Theorem 7.18, since  $K_n(\mathbb{Z})$  vanishes for  $n \le -1$  and is  $\mathbb{Z}$  for n = 0.

**7.23.** Since  $Wh_2(\mathbb{Z})$  is trivial, one easily checks that under the isomorphism (7.22) the kernel of  $L_2(S^1)$  is isomorphic to  $NA_2(\{\bullet\}) \oplus NA_2(\{\bullet\})$  and hence non-trivial.

**7.28.** We obtain from the fibration (7.24) the exact sequence

$$\pi_1(BG_+ \land A(\{\bullet\})) \to \pi_1(A(BG)) \to \pi_1(Wh(BG))$$
$$\to \pi_0(BG_+ \land A(\{\bullet\})) \to \pi_0(A(BG)).$$

Since  $A(\{\bullet\})$  is connected, the Atiyah-Hirzebruch spectral sequence shows that  $\pi_0(\{\bullet\}_+ \land A(\{\bullet\})) \to \pi_0(BG_+ \land A(\{\bullet\}))$  is bijective. Since the homomorphism  $\pi_0(\{\bullet\}_+ \land A(\{\bullet\})) \to \pi_0(BG_+ \land A(\{\bullet\}))$  is split injective, the map

$$\pi_0(BG_+ \wedge A(\{\bullet\})) \to \pi_0(A(BG))$$

is injective. Using diagram (7.25), we obtain a short exact sequence

$$H_1(B\pi_1(BG); \mathbf{K}(\mathbb{Z})) \to K_1(\mathbb{Z}\pi_1(BG)) \to \pi_1(Wh(BG)) \to 0.$$

Again by the Atiyah-Hirzebruch spectral sequence we obtain an isomorphism  $H_1(B\pi_1(BG); \mathbf{K}(\mathbb{Z})) \cong G/[G, G] \times \{\pm 1\}$ . Hence the image of the map

$$H_1(B\pi_1(BG); \mathbf{K}(\mathbb{Z})) \to K_1(\mathbb{Z}\pi_1(BG))$$

is the subgroup of  $K_1(\mathbb{Z}G) = K_1(\mathbb{Z}\pi_1(BG))$  given by the trivial units  $\{\pm g \mid g \in G\}$ . This implies  $Wh(G) \cong \pi_1(Wh(BG))$ . **7.33.** Suppose such M exists. The long exact homotopy sequence of the fibration (7.24) looks like

$$\cdot \to \pi_n(M_+ \land A(\{\bullet\})) \to A_n(M) \to Wh_n(M) \to \cdots$$

The splitting (7.31) yields isomorphisms

$$A_n(M) \cong \operatorname{Wh}_n^{\operatorname{DIFF}}(M) \oplus \pi_n(\Sigma^{\infty} M).$$

Rationally the Atiyah-Hirzebruch sequence always collapses. Hence we obtain from Theorem 6.24 and Theorem 7.18 isomorphisms

$$\pi_n(M_+ \wedge A(\{\bullet\})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(M; \mathbb{Q}) \oplus \bigoplus_{k \ge 1} H_{n-4k-1}(M; \mathbb{Q}).$$

Hence we obtain the long exact sequence of Q-modules

$$\cdots \to \operatorname{Wh}_{n+1}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n(M; \mathbb{Q}) \oplus \bigoplus_{k \ge 1} H_{n-4k-1}(M; \mathbb{Q})$$
$$\to H_n(M; \mathbb{Q}) \oplus \operatorname{Wh}_n^{\mathsf{DIFF}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Wh}_n(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \cdots$$

Since by assumption the map  $Wh_n^{D\,\text{IFF}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to Wh_n(M) \otimes_{\mathbb{Z}} \mathbb{Q}$  is bijective for  $n \ge 0$ , we obtain for every  $n \ge 0$  isomorphisms

$$H_n(M;\mathbb{Q}) \oplus \bigoplus_{k\geq 1} H_{n-4k-1}(M;\mathbb{Q}) \cong H_n(M;\mathbb{Q}).$$

This implies for every  $n \ge 0$  and  $k \ge 1$  that  $H_{n-4k-1}(M; \mathbb{Q}) = 0$ , a contradiction to  $H_0(M; \mathbb{Q}) = \mathbb{Q}$ .

### **Chapter 9**

**9.6.** It is straightforward to check that e(P) is a well-defined *R*-homomorphism, compatible with direct sums and natural. It remains to show that it is bijective for a finitely generated projective *R*-module *P*. Let *Q* be another finitely generated projective *R*-module. Since  $e(P \oplus Q)$  is up to isomorphism  $e(P) \oplus e(Q)$ , the map  $e(P \oplus Q)$  is bijective if and only if both e(P) and e(Q) are bijective. Since we can find *Q* such that  $P \oplus Q \cong R^n$ , it suffices to consider the case P = R, which follows from a direct computation.

**9.14.** Let  $b_i(M) := \dim_{\mathbb{R}}(H_i(M;\mathbb{R}))$  be the *i*-th-Betti number. Poincaré duality implies  $b_i(M) = b_{4k-i}(M)$  for all  $i \ge 0$ . We conclude directly from the definition of the signature that  $\operatorname{sign}(M) \equiv b_{2k}(M) \mod 2$ . We get modulo 2

$$\begin{split} \chi(M) &\equiv \sum_{i=0}^{4k} (-1)^i \cdot b_i(M) \\ &\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=2k+1}^{4k} (-1)^i \cdot b_i(M) \\ &\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=2k+1}^{4k} (-1)^i \cdot b_{4k-i}(M) \\ &\equiv \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) + b_{2k}(M) + \sum_{i=0}^{2k-1} (-1)^i \cdot b_i(M) \\ &\equiv b_{2k}(M) \\ &\equiv \text{sign}(M). \end{split}$$

#### 9.16.

- (i) If *n* is odd, then dim( $\mathbb{CP}^n$ ) is not divisible by four and hence sign( $\mathbb{CP}^n$ ) = 0. If *n* is even, then the intersection pairing of  $\mathbb{CP}^n$  looks like  $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$  (*a*, *b*)  $\mapsto$  *ab* and hence sign( $\mathbb{CP}^n$ ) = 1.
- (ii) Since *STM* is the boundary of the total space *DTM* of the disk tangent bundle, Theorem 9.15 (i) implies sign(STM) = 0.
- (iii) We get sign(M) = 0 from assertions (v) and (vi) of Theorem 9.15.

**9.23.** Note in the situation under consideration that  $\epsilon = 1$  and the involution on  $\mathbb{Z}$  is the trivial involution. Hence the projection pr:  $R \to Q_{\epsilon}(R)$  is the identity. We conclude from Remark 9.21 that  $(P, \lambda)$  admits a quadratic refinement if and only if there exists a map  $\mu: P \to \mathbb{Z}$  such that  $\mu(nx) = n^2 \mu(x)$  holds for all  $n \in \mathbb{Z}$  and  $x \in P$ ,  $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y)$  is true for all  $x, y \in P$ , and  $\lambda(x, x) = 2 \cdot \mu(x)$  is valid for all  $x \in P$ . Obviously the existence of  $\mu$  implies  $\lambda(x, x)$  is even for all  $x \in P$ . Suppose that  $\lambda(x, x)$  is even for all  $x \in P$ . Then we can define  $\mu(x) := \lambda(x, x)/2$  and  $\mu$  has all the desired properties.

**9.27.** Show that the diagonal in  $P \oplus P$  is a Lagrangian for the non-degenerate  $\epsilon$ -quadratic form  $(P \oplus P, \psi \oplus -\psi)$  and then apply Lemma 9.26.

**9.28.** This follows from Lemma 9.11, Remark 9.24, and Lemma 9.26.

**9.31.** A non-degenerate quadratic form on *V* is a map  $\mu: V \to \mathbb{F}_2$  such that  $\mu(0) = 0$  holds, we get a non-degenerate symmetric pairing  $\lambda: V \times V \to \mathbb{F}_2$  by  $\lambda(p,q) = \mu(p+q) + \mu(p) + \mu(q)$  and  $\lambda(p,p) = 0$  holds for all  $p \in V$ , see Remark 9.21. Fix a basis  $\{e_1, e_2\}$  for *V*. Then  $\lambda(e_i, e_j) = 1$  for  $i \neq j$ , since  $\lambda$  is non-degenerate and we already know  $\lambda(e_1, e_1) = \lambda(e_2, e_2) = \lambda(e_1 + e_2, e_1 + e_2) = 0$  and  $\lambda(e_1, e_2) = \lambda(e_2, e_1)$ . This implies that either  $\mu(e_1) = \mu(e_2) = \mu(e_1 + e_2) = 1$  or that precisely one of the elements  $\mu(e_1), \mu(e_2), \mu(e_1 + e_2)$  is 1. By possibly replacing

the basis  $\{e_1, e_2\}$  by the basis  $\{e_1, e_1 + e_2\}$  or  $\{e_2, e_1 + e_2\}$ , we can arrange that either  $\mu(e_1) = \mu(e_2) = \mu(e_1 + e_2) = 1$  or that  $\mu(e_1) = \mu(e_2) = 0$  and  $\mu(e_1 + e_2) = 1$ . The first one has Arf invariant 1, the second 0. Hence there are up to isomorphism precisely two non-degenerate quadratic forms on *V*.

**9.44.** By the definition of the self-intersection number it suffices to show  $\mu(f) \neq 0$  in  $Q_{\epsilon}(\mathbb{Z}\pi)$ . The map  $\mathbb{Z}\pi \to \mathbb{Z}/2$  sending  $\sum_{g \in \pi} n_g \cdot g$  to  $\sum_{g \in G} \overline{n_g}$  induces a map of abelian groups  $Q_{\epsilon}(\mathbb{Z}\pi) \to \mathbb{Z}/2$ . Since the set of double points consists of precisely one element, it sends  $\mu(f)$  to  $\overline{1}$  and hence  $\mu(f) \neq 0$ .

**9.45.** Consider the inclusion  $i: S^1 \to S^1 \times S^1$  onto the first factor. One easily changes it locally to an immersion  $j: S^1 \to S^1 \times S^1$  in general position with exactly one double point such that *i* and *j* are homotopic. We conclude from Exercise 9.44 that *i* and *j* are not regularly homotopic.

**9.52.** Denote by  $C^{n-*}(\widetilde{X})_{untw}$  the  $\mathbb{Z}\pi$ -chain complex which is analogously defined as  $C^{n-*}(\widetilde{X})$ , but now with respect to the untwisted involution. Its *n*-th homology  $H_n(C^{n-*}(\widetilde{X})_{untw})$  depends only on the homotopy type of *X*. If *X* carries the structure of a Poincaré complex with respect to  $w: \pi_1(X) \to \{\pm 1\}$ , then the Poincaré  $\mathbb{Z}\pi$ chain homotopy equivalence induces a  $\mathbb{Z}\pi$ -isomorphism  $H_n(C^{n-*}(\widetilde{X})_{untw}) \cong \mathbb{Z}^w$ . Thus we rediscover *w* from  $H_n(C^{n-*}(\widetilde{X})_{untw})$ .

**9.60.** This follows from the fact that two embeddings  $M \to \mathbb{R}^{n+m}$  for large enough *m* are diffeotopic.

**9.69.** It suffices to show that *f* is *l*-connected for l = k + 1, k + 2, ... By assumption this holds for l = k + 1. In the induction step *f* is *l*-connected for some  $l \ge k + 1$  and we have to show that *f* is (l + 1)-connected, i.e.,  $\pi_{l+1}(f) = 0$ . By Lemma 9.64 (ii), which applies also to the case where *M* is only a finite Poincaré complex, it suffices to show that  $K_l(\widetilde{M}) = 0$ . By Lemma 9.64 (i), which applies also to the case where *M* is only a finite Poincaré complex, it suffices to show that  $K_l(\widetilde{M}) = 0$ . By Lemma 9.64 (i), which applies also to the case where *M* is only a finite Poincaré complex, it suffices to show  $K_{n-l}(\widetilde{M}) = 0$ . Since *f* is (k + 1)-connected and  $n - l \le k$ ,  $K_{n-l}(\widetilde{M}) = 0$  vanishes by Lemma 9.64 (ii).

**9.75.** Let  $f: M \to S^{4k+2}$  be any map of degree one. Choose an embedding  $i: M \to \mathbb{R}^{4k+2+m}$  for large enough m. Then the given stable trivialization of the tangent bundle defines a trivialization of the normal bundle. It can be viewed as bundle map  $\overline{f}: \mu(i) \to \mathbb{R}^m$  covering f. Thus we obtain a normal map of degree one  $(\overline{f}, f)$ . It defines a surgery obstruction  $\sigma(\overline{f}, f) \in L_{4k+2}(\mathbb{Z})$ . Since  $L_{4k+2}(\mathbb{Z})$  is isomorphic to  $\mathbb{Z}/2$ , this is the same as an element  $\alpha(M) \in \mathbb{Z}/2$ . It is independent by the choice of f and  $\overline{f}$  and depends only on the stably framed bordism class of M, since by a theorem due to Hopf the homotopy class of f is uniquely determined by its degree and the surgery obstruction is an invariant under normal bordism.

**9.100.** Because of Theorem 9.99 (ii) we can assume that  $F: W \to X \times [0, 1]$  is a simple homotopy equivalence. We conclude from Theorem 3.37 (iii) that both inclusions  $M \to W$  and  $N \to W$  are simple homotopy equivalences. By Theorem 3.47 there exists a diffeomorphism  $W \to M \times [0, 1]$ . Hence the restriction of this diffeomorphism to N is a diffeomorphism  $N \to M \times \{1\} = M$ .

**9.107.** This follows from the various Rothenberg sequences, since the  $\mathbb{Z}/2$ -Tate cohomology of any  $\mathbb{Z}[\mathbb{Z}/2]$ -module is annihilated by multiplication with 2.

**9.112.** Since Wh( $\mathbb{Z}$ ),  $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}])$ , and  $K_n(\mathbb{Z}[\mathbb{Z}])$  for  $n \leq -1$  vanish, see Example 2.4, Theorem 3.17, and Theorem 4.7, the decoration does not matter by Theorem 9.106. We conclude from (9.109) and the computations of  $L_n(\mathbb{Z})$  in Theorem 9.29, Theorem 9.32, and Theorem 9.82:

$$L_n(\mathbb{Z}[\mathbb{Z}]) \cong L_{n-1}(\mathbb{Z}) \oplus L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \equiv 0, 1 \mod 4; \\ \mathbb{Z}/2 & n \equiv 2, 3 \mod 4. \end{cases}$$

**9.115.** We conclude from Conjectures 3.110, 4.18, and 9.114 and from Theorem 9.106 that the decoration does not matter. If g = 0,  $\pi_1(F_g)$  is trivial and hence  $L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]) = L_n^{\langle -\infty \rangle}(\mathbb{Z})$ . Suppose  $g \ge 1$ . Then  $F_g$  itself is a model for  $B\pi_1(F_g)$ . Because of Conjecture 9.114 we get

$$H_n(F_g; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]).$$

Next we compute  $H_n(F_g; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))$  using the Atiyah-Hirzebruch spectral sequence. This is easy, since  $F_g$  is 2-dimensional, the edge homomorphism which describes  $H_n(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}) \rightarrow H_n(F_g; \mathbf{L}^{\langle -\infty \rangle})$  is split injective, and  $L_n^{\langle -\infty \rangle}(\mathbb{Z})$  is  $\mathbb{Z}$  if  $n \equiv 0 \mod 4$ ,  $\mathbb{Z}/2$  if  $n \equiv 0 \mod 4$ , and  $\{0\}$  otherwise. The result is

$$L_n^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(F_g)]) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0 \mod 4; \\ \mathbb{Z}^{2g} & n \equiv 1 \mod 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 2 \mod 4; \\ (\mathbb{Z}/2)^{2g} & n \equiv 3 \mod 4. \end{cases}$$

**9.142.** By of Poincaré duality it suffices to show  $f_*(\mathcal{L}(M) \cap [M]_{\mathbb{Q}}) = \mathcal{L}(N) \cap [N]_{\mathbb{Q}}$ . But this follows from the Novikov Conjecture 9.137 because of Remark 9.141, since we can put N = BG.

9.149. This follows from the long exact homotopy sequence associated to a fibration.

**9.150.** Let  $C \subseteq \pi_1(X)$  be any finite cyclic subgroup. Since the universal covering  $\widetilde{X}$  is a model for  $E\pi_1(X)$ , it is also a model for EC after restricting the group action. Hence  $C \setminus \widetilde{X}$  is a finite-dimensional *CW*-model for *BC*. This implies that the group homology  $H_n(C)$  of *C* is trivial in dimensions  $n > \dim(X)$ . It is known that the

homology of *C* is *C* in all odd dimensions. Hence *C* must be trivial. This shows that  $\pi_1(X)$  is torsionfree.

**9.152.** The top homology group  $H_n(M; \mathbb{F}_2)$  with  $\mathbb{F}_2$ -coefficients of any closed *n*-dimensional manifold *M* is known to be isomorphic to  $\mathbb{F}_2$ . If *M* is simply connected and aspherical it is homotopy equivalent to the one-point-space  $\{\bullet\}$ . This implies n = 0 and hence  $M = \{\bullet\}$  for a simply connected aspherical manifold.

9.153. See [660, Lemma 3.2].

**9.166.** See Example 3.62.

**9.167.** Let k and n be natural numbers such at least one of them is even. Then  $S^k$  and  $S^n$  are topologically rigid but  $S^k \times S^n$  is not. See Remark 9.165.

**9.178.** Let  $G_k$  be an  $n_k$ -dimensional Poincaré duality group for k = 0, 1. Let  $P_*^k$  be an  $n_k$ -dimensional finite projective  $\mathbb{Z}[G_k]$ -resolution of the trivial  $\mathbb{Z}[G_k]$ -module  $\mathbb{Z}$ . Then  $P_*^0 \otimes_{\mathbb{Z}} P_*^1$  is an  $(n_0 + n_1)$ -dimensional finite projective  $\mathbb{Z}[G_0 \times G_1]$ -resolution of the trivial  $\mathbb{Z}[G_0 \times G_1]$ -module  $\mathbb{Z}$ . The obvious chain map given by the tensor product over  $\mathbb{Z}$  and the obvious identification  $\mathbb{Z}[G_0] \otimes_{\mathbb{Z}} \mathbb{Z}[G_1] = \mathbb{Z}[G_0 \times G_1]$ 

$$\begin{split} \hom_{\mathbb{Z}[G_0]}(P^0_*, \mathbb{Z}[G_0]) \otimes_{\mathbb{Z}} \hom_{\mathbb{Z}[G_1]}(P^1_*, \mathbb{Z}[G_1]) \\ \xrightarrow{\cong} \hom_{\mathbb{Z}[G_0 \times G_1]}(P^0_* \otimes_{\mathbb{Z}} P^1_*, \mathbb{Z}[G_0 \times G_1]) \end{split}$$

is an isomorphism of  $\mathbb{Z}$ -cochain complexes. Since  $\hom_{\mathbb{Z}[G_0]}(P^0_*, \mathbb{Z}[G_0])$  is a free  $\mathbb{Z}$ -cochain complex whose cohomology is concentrated in dimension  $n_k$  and given there by  $\mathbb{Z}$ , there exists a  $\mathbb{Z}$ -chain homotopy equivalence from  $[n_k](\mathbb{Z})$ , which is the  $\mathbb{Z}$ -chain complex concentrated in dimension  $n_k$  and having  $\mathbb{Z}$  as  $n_k$ -th chain module, to  $\hom_{\mathbb{Z}[G_0]}(P^0_*, \mathbb{Z}[G_0])$ . Hence  $\hom_{\mathbb{Z}[G_0]}(P^0_*, \mathbb{Z}[G_0]) \otimes_{\mathbb{Z}} \hom_{\mathbb{Z}[G_1]}(P^1_*, \mathbb{Z}[G_1])$  is  $\mathbb{Z}$ -chain homotopy equivalent to  $[n_0](\mathbb{Z}) \otimes [n_1](\mathbb{Z}) \cong [n_0+n_1](\mathbb{Z})$ . This implies that  $H^n(\hom_{\mathbb{Z}[G_0 \times G_1]}(P^0_* \otimes_{\mathbb{Z}} P^1_*, \mathbb{Z}[G_0 \times G_1]))$  is  $\mathbb{Z}$  in dimension  $(n_0 + n_1)$  and trivial otherwise. Hence  $G_0 \times G_1$  is an  $(n_0 + n_1)$ -dimensional Poincaré duality group.

9.193. This follows from Theorem 9.171, Theorem 13.32 (iv), and Theorem 16.1 (ia).

**9.200.** The map  $\alpha_0: \pi_0(G(M)) \to \operatorname{Out}(\pi)$  is bijective by Remark 9.147. The map  $\pi_0(\operatorname{Top}(M)) \to \pi_0(G(M))$  is surjective as *M* is topologically rigid.

10.3. Since

#### Chapter 10

$$\chi(X) = \sum_{n \ge 0} (-1)^n \cdot \dim_{\mathbb{Q}}(H_n(X; \mathbb{Q})) = \sum_{n \ge 0} (-1)^n \cdot \dim_{\mathbb{Q}}(H^n(X; \mathbb{Q}))$$

holds, this follows directly from (10.1) and (10.2).

**10.17.** We obtain from (10.16) an isomorphism

$$K^*_G(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{C \in \mathcal{C}(G)} H^*(X^C/G; \mathbb{Q}(\zeta_C)).$$

Now use the fact  $\dim_{\mathbb{Q}}(\mathbb{Q}(\zeta_C)) = \varphi(|C|)$ .

**10.21.** Without loss of generality we can assume that  $K_j^H(Y, B)$  is torsionfree for all  $j \in \mathbb{Z}$ . Now check that we obtain two *G*-homology theories on pairs of finite proper *G*-*CW*-complexes (X, A) by putting

$$\begin{split} \mathcal{H}^*_G(X,A) &:= \bigoplus_{i+j=n} K^G_i(X,A) \otimes_{\mathbb{Z}} K^H_j(Y,B); \\ \mathcal{K}^*_H(X,A) &:= K^*_{G \times H}((X,A) \times (Y,B)). \end{split}$$

(When one wants to check the exactness of the long exact sequence of a pair for  $\mathcal{H}_G^*$ , we need the assumption that  $K_i^H(Y, B)$  is torsionfree and hence the functor  $- \otimes_{\mathbb{Z}} K_j^H(Y, B)$  is exact for all  $j \in \mathbb{Z}$ .) The external multiplication defines a natural transformation  $T_G^* : \mathcal{H}_G^* \to \mathcal{K}_G^*$  of *G*-cohomology theories for pairs of finite proper *G*-*CW*-complexes. One checks that  $T_G^*(G/H) : \mathcal{H}_G^*(G/H) \to \mathcal{K}_G^*(G/H)$  is bijective for all finite subgroups  $H \subseteq G$ . Now prove by induction over the number of equivariant cells using the Five Lemma, the long exact sequence of a pair, excision, and *G*-homotopy invariance that  $T_G^n(X, A)$  is bijective for all pairs of finite proper *G*-*CW*-complexes (X, A) and all  $n \in \mathbb{Z}$ .

**10.23.** This follows from the long exact sequence of the pair (DE, SE), the Thom isomorphisms (10.22), and the commutativity of the following diagram, which is a consequence of the naturality of the product

$$\begin{array}{c} K_{G}^{*}(X) \xrightarrow{-\cup e(p)} & K_{G}^{*}(X) \\ K_{G}^{*}(p_{DE}) & \cong & \bigvee K_{G}^{*}(p_{DE}) \\ K_{G}^{*}(DE) \xrightarrow{-\cup K_{G}^{0}(j)(\lambda_{E})} & K_{G}^{*}(DE) \\ & & & \downarrow K_{G}^{*}(DE) \\ & & & \downarrow K_{G}^{*}(DE) \\ & & & \downarrow K_{G}^{*}(j) \\ K_{G}^{*}(DE) \xrightarrow{-\cup \lambda_{E}} & K_{G}^{*}(DE, SE) \end{array}$$

**10.29.** If *G* contains an element *g* of order  $\ge 3$ , then show  $||xx^*|| \ne ||x||^2$  for  $x = g + 1 - g^{-1}$ . If *G* contains an element *g* of order 2, then show  $||xx^*|| \ne ||x||^2$  for  $x = g + i \in L^1(G, \mathbb{C})$ . Finally one checks directly that  $L^1(G, F)$  is a  $C^*$ -algebra if *G* is trivial or if *G* has order 2 and  $F = \mathbb{R}$ .

**10.35.** Since  $\mathcal{K}$  is the colimit colim<sub> $n\to\infty$ </sub>  $M_n(\mathbb{C})$ , we conclude from Morita equivalence and the compatibility with colimits over directed systems that the obvious inclusion of  $C^*$ -algebras  $\mathbb{C} \to \mathcal{K}$ , induces an isomorphism  $K_n(\mathbb{C}) \xrightarrow{\cong} K_*(\mathcal{K})$ . To finish the calculation, one directly proves that  $K_n(\mathbb{C})$  is  $\mathbb{Z}$  for n = 0 and trivial for n = 1 and applies Bott periodicity.

**10.45.** Since *G* is by assumption finite,  $H_n(BG; \mathbb{Q})$  is  $\mathbb{Q}$  if n = 0 and is trivial for  $n \neq 0$ . We conclude from the Chern characters (10.1) and (10.7) that  $\dim_{\mathbb{Q}}(K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(KO_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$ . We have  $K_0(C_r^*(G)) \cong \operatorname{Rep}_{\mathbb{C}}(G)$  and  $KO_0(C_r^*(G)) \cong \operatorname{Rep}_{\mathbb{R}}(G)$ . Now use the obvious fact that  $\dim_{\mathbb{Q}}(\operatorname{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1 \iff G = \{1\}$  holds.

**10.47.** Let  $c: S^1 \to S^1$  be the automorphism of  $S^1$  sending  $z \in S^1$  to  $z^{-1}$ . Let  $T_c$  be the mapping torus. One easily checks that  $T_c$  is a model for *BG*. Elementary considerations about homology theories lead to the so-called Wang sequence

$$\cdots \xrightarrow{\partial_{n+1}} K_n(S^1) \xrightarrow{\operatorname{id} - K_n(c)} K_n(S^1) \xrightarrow{K_n(i)} K_n(T_c) \xrightarrow{\partial_n} K_{n-1}(S^1) \xrightarrow{\operatorname{id} - K_{n-1}(c)} K_{n-1}(S^1) \xrightarrow{K_{n-1}(i)} \cdots$$

We know that  $K_n(S^1) \cong \mathbb{Z}$  for all  $n \in \mathbb{Z}$ . Elementary considerations about homology theories imply that  $K_n(c) = -\operatorname{id}_{K_n(S^1)}$  for odd n and  $K_n(c) = \operatorname{id}_{K_0(S^1)}$  for even n. Hence the Wang sequence reduces to

$$\cdots \to \mathbb{Z} \xrightarrow{2 \cdot \mathrm{id}} \mathbb{Z} \xrightarrow{K_1(i)} K_1(T_c) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{K_0(i)} K_0(T_c) \to \mathbb{Z} \xrightarrow{2 \cdot \mathrm{id}} \mathbb{Z} \to \cdots$$

This implies

$$K_n(C_r^*(G)) \cong K_n(T_c) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd.} \end{cases}$$

**10.62.** Obviously  $\hom_{\{1\}}(F, i^*F) \cong F$ . Since  $i_*F = C_0(G, F)$ , all homomorphisms of *G*-*C*<sup>\*</sup>-algebras from  $i_*F$  to *F* are zero and hence  $\hom_G(i_*F, F)$  vanishes.

**10.71.** Put  $G = \mathbb{Z}/p$ . Since *p* is an odd prime, we have  $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V^G) \equiv 0 \mod 2$  and hence  $\dim(SV) = \dim(SV^G) \equiv 0 \mod 2$ . Since  $\dim(SV) = d - 1$ , we get  $K_n(SV) = K_n(SV^G) = K_n(S^{d-1})$  for all  $n \in \mathbb{Z}$ . Since  $\operatorname{Rep}_{\mathbb{C}}(G) \cong \mathbb{Z}^p$ , we get  $\operatorname{im}(\theta_G) = \mathbb{Z}[1/p]$  and  $\operatorname{im}(\theta_{\{1\}}) \cong \mathbb{Z}[1/p]^{p-1}$ . We conclude from Theorem 10.69

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV)$$
  

$$\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(SV^G/C_G G) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/C_G\{1\})$$
  

$$\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/G).$$

The Atiyah-Hirzebruch spectral sequence converges to  $K_n(SV/G)$  and has as  $E^2$ -term  $E_{r,s}^2 = H_r(SV/G; K_s(\{\bullet\}))$ . Since |G| is a *p*-power, we get a  $\mathbb{Z}[1/p]$ -isomorphism  $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} H_r(SV/G) \cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} [1/p]_G H_r(SV)$ . Since *p* is odd, the *G*-operation on  $H_i(SV)$  is trivial. Hence we get a  $\mathbb{Z}[1/p]$ -isomorphism

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E_{r,s}^2 \cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} H_r(SV; K_s(\{\bullet\}))$$
$$\cong \begin{cases} \mathbb{Z}[1/p] & \text{if } r = 0, d-1 \text{ and } s \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E_{r,s}^2$  is a finitely generated free  $\mathbb{Z}[1/p]$ -module for each (r, s) and we conclude from the isomorphism (10.1) for each  $n \in \mathbb{Z}$ 

$$\sum_{r+s=n} \operatorname{rk}_{\mathbb{Z}[1/p]} \left( \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} E_{r,s}^2 \right) = \operatorname{rk}_{\mathbb{Z}[1/p]} \left( \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(SV/G) \right).$$

This implies that all differentials in the Atiyah-Hirzebruch spectral sequence are trivial after inverting p and we get

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(SV/G) \cong \begin{cases} \mathbb{Z}[1/p] & \text{if } d \text{ is even;} \\ \mathbb{Z}[1/p]^2 & \text{if } d \text{ is odd and } n \text{ is even;} \\ 0 & \text{if } d \text{ is odd and } n \text{ is odd.} \end{cases}$$
$$\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}).$$

Now the claim follows from

$$\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n^{\mathbb{Z}/p}(SV) \cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(SV/G)$$
$$\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} K_n(S^{d-1}) \oplus \mathbb{Z}[1/p]^{p-1} \otimes_{\mathbb{Z}} K_n(S^{d-1})$$
$$\cong \mathbb{Z}[1/p]^p \otimes_{\mathbb{Z}} K_n(S^{d-1}).$$

**10.74.** The abelian group  $K_1^{\text{ALG}}(C(X, F))$  is not finitely generated because of Theorem 3.121, whereas  $K_1^{\text{TOP}}(C(X, F))$ , which is  $K^1(X)$  or  $KO^1(X)$ , is finitely generated.

## Chapter 11

**11.6.** Define the *n*-skeleton of  $\widetilde{X}$  to be  $p^{-1}(X_n)$ . Use the facts that a covering over a contractible space such as  $D^n$  is trivial and a covering is a local homeomorphism.

**11.7.** The Euler characteristic of a compact *CW*-complex can be computed by counting cells. Each equivariant cell in  $X - X^{\mathbb{Z}/p}$  contributes *p* (non-equivariant) cells.

**11.12.** Choose an irrational number  $\theta$ . Let  $\phi: S^1 \to S^1$  be the homeomorphism given by multiplication by the complex number  $\exp(2\pi i\theta)$ . The space  $S^1$  with the associated  $\mathbb{Z}$ -action is free but not proper.

**11.14.** Suppose that there is a free smooth  $\mathbb{Z}/p$ -action on  $S^{2n}$ . By Remark 11.13 we obtain a free  $\mathbb{Z}/p$ -*CW*-structure on  $S^{2n}$ . By a previous exercise we get the contradiction

$$0 \equiv \chi(\emptyset) \equiv \chi((S^{2n})^{\mathbb{Z}/p}) \equiv \chi(S^{2n}) \equiv 2 \mod p.$$

**11.17.** This follows from Theorem 11.16 (i).

**11.20.** Suppose that  $E_{\mathcal{F}}(G)$  has a zero-dimensional model. Hence it is a disjoint union of spaces of the shape G/H. Since  $E_{\mathcal{F}}(G)$  is path connected, it must be G/G. This implies  $G \in \mathcal{F}$ .

If  $G \in \mathcal{F}$  holds, G/G is a 0-dimensional G-CW-model for  $E_{\mathcal{F}}(G)$ . An example for L is  $\mathbb{R}$ .

**11.35.** We obtain from Subsection 11.6.13 that there is a *G*-*CW*-model for  $\underline{E}G$  which is obtained from G/M by attaching free cells of dimensions  $\leq 2$ . Let  $i: G/M \to \underline{E}G$  be the inclusion. Consider the map

$$j = \mathrm{id}_{EG} \times_G i \colon EG \times_G G/M \to EG \times_G EG.$$

Since for a space *Y* the canonical projection  $EG \times_G (G \times Y) \to Y$  is a homotopy equivalence, we conclude by a Mayer-Vietoris argument that  $H_n(j)$  is bijective for  $n \ge 3$ . The canonical projections  $EG \times_G G/M \to EG/M = BM$  and  $EG \times_G \underline{E}G \to EG \times_G \{\bullet\} = BG$  are homotopy equivalences, since  $\underline{E}G$  is (after forgetting the group action) contractible.

**11.36.** Since hyperbolic groups, arithmetic groups, mapping class groups,  $Out(F_n)$ , and one-relator groups have a finite-dimensional model for <u>E</u>G by Subsections 11.6.7, 11.6.8, 11.6.9, 11.6.10, and 11.6.13, it suffices to show for a group G with a *d*-dimensional model for EG that  $H_k(BG; \mathbb{Q}) = 0$  holds for k > d.

The cellular  $\mathbb{Q}$ -chain complex  $C_*(X)$  of a proper *G*-*CW*-complex *X* consists of projective  $\mathbb{Q}G$ -modules, since for any finite subgroup  $H \subseteq G$  the  $\mathbb{Q}G$ -module  $\mathbb{Q}[G/H]$  is projective. Since <u>*E*</u>*G* is contractible (after forgetting the group action), its cellular  $\mathbb{Q}$ -chain complex yields a dim(<u>*E*</u>*G*)-dimensional projective  $\mathbb{Q}G$ -resolution of the trivial  $\mathbb{Q}G$ -module  $\mathbb{Q}$ .

**11.48.** Since *H* is infinite and countable, its cardinality is  $\aleph_0$ . We conclude  $\underline{gd}(H) = 1$  from Remark 11.47. Theorem 11.46 implies that  $\underline{gd}(H \rtimes \mathbb{Z}) \leq 2$ . Since  $H \rtimes \mathbb{Z}$  is finitely generated and does not contain a finitely generated free group of finite index, we cannot have  $\underline{gd}(H \rtimes \mathbb{Z}) \leq 1$ .
**11.55.** The universal covering on *M* is the hyperbolic space and hence contractible. Therefore *M* is a model for *BG*. Since  $H_n(BG; \mathbb{Z}^w) \cong \mathbb{Z}$  for  $w: G = \pi_1(N) \to \{\pm\}$  given by the first Stiefel-Whitney class of *N*, we conclude  $cd(G) = dim(N) = \underline{gd}(G)$ . As  $S^1$  is not hyperbolic, we have  $dim(M) \ge 2$ . Since *G* is hyperbolic and hence satisfies conditions ( $\underline{M}$ ) and ( $\underline{NM}$ ), we conclude  $\underline{gd}(G) = \underline{gd}(G)$  from Theorem 11.54 (iii).

**11.61.** This follows from Theorem 11.60.

**11.64.** Apply Theorem 11.63 to *X* and take  $Y = H \setminus \underline{E}G$  for a torsionfree subgroup *H* of *G* with [G : H] = 2.

#### Chapter 12

**12.4.** The desired  $\mathbb{Z}/2$ -pushout for the 1-skeleton is obvious and for the 2-skeleton given by

where pr is the projection. Now one easily checks that  $C_*(S^2) \otimes_{Or(\mathbb{Z}/2)} R_{\mathbb{C}}$  is given by the  $\mathbb{Z}$ -chain complex concentrated in dimensions 0,1, and 2

$$\cdots \to \{0\} \to \{0\} \to R_{\mathbb{C}}(\{1\}) \xrightarrow{c_2} R_{\mathbb{C}}(\mathbb{Z}/2) \xrightarrow{0} R_{\mathbb{C}}(\mathbb{Z}/2) \to \{0\} \to \cdots$$

where  $c_2$  is induction with the inclusion  $\{1\} \rightarrow \mathbb{Z}/2$ . This implies

$$H_n^{\mathbb{Z}/2}(S^2; R_{\mathbb{C}}) \cong \begin{cases} \mathbb{Z}^2 & n = 0; \\ \mathbb{Z} & n = 1; \\ \{0\} & \text{otherwise.} \end{cases}$$

**12.7.** By applying Lemma 12.5 to the skeletal filtration  $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X = \bigcup_{n \ge 0} X_n$ , the claim can be reduced to finite-dimensional pairs. Using the axioms of a *G*-homology theory, the Five Lemma, and induction over the dimension, one reduces the proof to the special case  $(X, A) = (G/H, \emptyset)$ .

**12.11.** This follows directly from the axiom about the compatibility with conjugation applied in the case  $X = \{\bullet\}$ .

**12.16.** The real line  $\mathbb{R}$  with the obvious action of  $D_{\infty} = \mathbb{Z}/2 * \mathbb{Z}/2 = \mathbb{Z} \rtimes \mathbb{Z}/2$  is a  $D_{\infty}$ -*CW*-model for  $\underline{E}D_{\infty}$ , see Theorem 11.25. Up to conjugacy there are two subgroups  $H_0$  and  $H_1$  of order two in  $D_{\infty}$ . One obtains a  $D_{\infty}$ -pushout



where  $pr_0$  and  $pr_1$  are the obvious projections and *i* is the obvious inclusion. Hence the associated long exact Mayer-Vietoris sequence reduces to

$$0 \to K_1^{D_{\infty}}(\underline{E}D_{\infty}) \to R_{\mathbb{C}}(\{1\}) \oplus R_{\mathbb{C}}(\{1\})$$
$$\xrightarrow{f} R_{\mathbb{C}}(\{1\}) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \oplus R_{\mathbb{C}}(\mathbb{Z}/2) \to K_0^{D_{\infty}}(\underline{E}D_{\infty}) \to 0$$

where f sends (v, w) to  $(v+w, i_*(v), -i_*(w))$  for  $i_*$  the map induced by the inclusion  $i: \{1\} \rightarrow \mathbb{Z}/2$ . This implies

$$K_n^{D_{\infty}}(\underline{E}D_{\infty}) \cong \begin{cases} \mathbb{Z}^3 & n \text{ even}; \\ \{0\} & n \text{ odd.} \end{cases}$$

**12.36.** One easily checks that for a given group *G* and every subgroup  $H \subseteq G$  and every  $n \in \mathbb{Z}$  the map  $H_n^G(G/H; \mathbf{t}) : H_n^G(G/H; \mathbf{E}) \to H_n^G(G/H; \mathbf{F})$  can be identified with  $\pi_n(\mathbf{t}(t^G(G/H))) : \pi_n(\mathbf{E}(t^G(G/H))) \to \pi_n(\mathbf{F}(t^G(G/H)))$  and hence is bijective by assumption. Now apply Lemma 12.6.

**12.47.** The argument appearing in the solution of Exercise 12.16 yields a long exact Mayer- Vietoris sequence

$$\cdots \to K_0(R) \oplus K_0(R) \to K_0(R) \oplus K_0(R[\mathbb{Z}/2]) \oplus K_0(R[\mathbb{Z}/2])$$
$$\to H_0^{D_{\infty}}(\underline{E}D_{\infty}; \mathbf{K}_R) \to K_{-1}(R) \oplus K_{-1}(R)$$
$$\to K_{-1}(R) \oplus K_{-1}(R[\mathbb{Z}/2]) \oplus K_{-1}(R[\mathbb{Z}/2]) \to \cdots$$

Since the obvious map  $K_n(R) \to K_n(R[\mathbb{Z}/2])$  is split injective, we obtain for  $n \in \mathbb{Z}$  isomorphisms

$$K_n(R[\mathbb{Z}/2]) \oplus \operatorname{coker}(K_n(R) \to K_n(R[\mathbb{Z}/2])) \cong H_n^{D_\infty}(\underline{E}D_\infty; \mathbf{K}_R).$$

If  $n \leq -1$ , then  $K_n(R[\mathbb{Z}/2]) = 0$  for  $R = \mathbb{Z}, \mathbb{C}$  by Theorem 4.16 and Theorem 4.22. Hence

$$H_n^{D_{\infty}}(\underline{E}D_{\infty}; \mathbf{K}_R) \cong \{0\} \text{ for } n \leq -1.$$

The map  $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\mathbb{Z}/2])$  is bijective by Example 2.107 and  $K_0(\mathbb{Z}) = \mathbb{Z}$  by Example 2.4. Hence

$$H_0^{D_\infty}(\underline{E}D_\infty;\mathbf{K}_\mathbb{Z})\cong\mathbb{Z}.$$

Since  $K_0(\mathbb{C}H) \cong R_{\mathbb{C}}(H)$  for a finite group H, one easily checks

$$H_0^{D_{\infty}}(\underline{E}D_{\infty};\mathbf{K}_{\mathbb{C}})\cong\mathbb{Z}^3.$$

**12.49.** Since X/G has no odd-dimensional cells, X has no odd-dimensional equivariant cells. Moreover, if X/G is finite, then X has only finitely many equivariant cells. We conclude for any coefficient system M that the Bredon homology  $H_p(X; M)$  vanishes if p is odd or if p is larger than the dimension of X. If X has only finitely many equivariant cells and M(G/H) is a finitely generated free abelian group for any finite subgroup  $H \subseteq G$ , then  $H_p(X; M)$  is finitely generated free abelian for all  $p \in \mathbb{Z}$ . Since  $K_q^G(G/H) = 0$  for odd q and is a finitely generated free abelian group for even q for every finite subgroup  $H \subseteq G$ , and all isotropy groups of X are by assumption finite, we conclude for the  $E^2$ -terms of the equivariant Atiyah-Hirzebruch spectral sequence of Theorem 12.48 that  $E_{p,q}^2 = 0$  if p + q is odd. If X has only finitely many equivariant cells, then  $E_{p,q}^2$  is finitely generated free if p + q is even and vanishes for large enough q. Now the claim follows from this spectral sequence.

**12.54.** Consider the long exact sequence of the pair  $(EG \times_G X, EG \times_G X^G)$  and of the pair  $(X/G, X^G/G) = (X/G, X^G)$  and the map between them induced by the projection  $(EG \times_G X, EG \times_G X^G) \rightarrow (X/G, X^G/G)$ , and use the fact that  $(X, X^G)$  is relatively free and hence  $\mathcal{H}_n(EG \times_G X, EG \times_G X^G) \rightarrow \mathcal{H}_n(X/G, X^G/G)$  is bijective.

**12.59.** From Theorem 12.58 we get a natural isomorphism of spectral sequences from the equivariant Atiyah-Hirzebruch spectral sequence converging to  $\mathcal{BH}^G(X)$  to the equivariant Atiyah-Hirzebruch spectral sequence converging to  $\mathcal{H}^G_*(X)$ . One easily checks that all the differentials in the equivariant Atiyah-Hirzebruch spectral sequence converging to  $\mathcal{BH}^G(X)$  vanish.

**12.60.** For every finite group  $H \subseteq G$  the group  $N_GH/H \cdot C_GH$  is finite and hence  $\mathbb{Q}[N_GH/H \cdot C_GH]$  is semisimple. Therefore every  $\mathbb{Q}[N_GH/H \cdot C_GH]$ -module is flat. Because of Theorem 12.58 it suffices to show that for every finite subgroup  $H \subseteq G$  and every  $n \in \mathbb{Z}$  the map

$$H_p(C_GH \setminus \iota^H_{\mathcal{F} \subset G}; \mathbb{Q}) \colon H_p(C_GH \setminus E_{\mathcal{F}}(G)^H; \mathbb{Q}) \to H_p(C_GH \setminus E_{\mathcal{G}}(G)^H; \mathbb{Q})$$

is bijective. This is obviously true if  $H \notin \mathcal{F}$ . Suppose  $H \in \mathcal{F}$ . Then the claim follows from fact that both  $C_*(E_{\mathcal{F}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $C_*(E_{\mathcal{G}}(G)^H) \otimes_{\mathbb{Z}} \mathbb{Q}$  are projective  $\mathbb{Q}[C_G H]$ -resolutions of the trivial  $\mathbb{Q}[C_G H]$ -module  $\mathbb{Q}$ , which implies that

$$C_*(\iota^H_{\mathcal{F}\subseteq \mathcal{G}})\otimes_{\mathbb{Z}}\mathbb{Q}\to C_*(E_{\mathcal{F}}(G)^H)\otimes_{\mathbb{Z}}\mathbb{Q}\to C_*(E_{\mathcal{G}}(G)^H)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is a  $\mathbb{Q}[C_G H]$ -chain homotopy equivalence and hence induces after applying  $\mathbb{Q} \otimes_{\mathbb{Q}[C_G H]} - a \mathbb{Q}$ -chain homotopy equivalence.

**12.64.** The desired pairing is given by

$$A(G) \times M(G) \to M(G), \quad ([G/H], x) \mapsto \operatorname{ind}_{H}^{G} \circ \operatorname{res}_{G}^{H}(x).$$

**12.81.** This follows from Remark 12.80 using [910, Chapter IV, §4], as explained in [793].

**12.88.** Every subgroup  $F \subseteq SL_2(\mathbb{Z})$  is conjugate to one of the groups  $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$  with generators given by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

So we shall restrict from now on to the study of the actions of  $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$  given by the actions of the above described generators. The case  $\mathbb{Z}/4$  has been carried out in Example 12.87. The computations for the other cases is analogous. We get in all cases that  $G \setminus \underline{E}G$  is homeomorphic to  $S^2$ . There are up to conjugacy four non-trivial finite subgroups in *G* in the case  $F = \mathbb{Z}/2$ . These are all isomorphic to  $\mathbb{Z}/2$ . There are up to conjugacy three non-trivial finite subgroups in *G* in the case  $F = \mathbb{Z}/3$ . These are all isomorphic to  $\mathbb{Z}/3$ . In the case  $F = \mathbb{Z}/6$  there are up to conjugacy three non-trivial finite subgroups. The first is isomorphic to  $\mathbb{Z}/2$ , the second to  $\mathbb{Z}/3$ , and the third to  $\mathbb{Z}/6$ . Hence we get in all cases  $K_1^G(\underline{E}G) = 0$  and

$$\begin{split} & K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/2}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6; \\ & K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/3}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_3) \cong \mathbb{Z}^8; \\ & K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/4}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_4) \cong \mathbb{Z}^9; \\ & K_0^{\mathbb{Z}^2 \rtimes \mathbb{Z}/6}(\underline{E}\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \cong \mathbb{Z}^{10}. \end{split}$$

**12.96.** Since *G* is finite, an easy spectral sequence argument shows that there is an isomorphism  $u: \mathbb{Q} \otimes_{\mathbb{Z}G} K_n(R) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{H}_n^{G,\xi}(EG; \mathbf{K}_R)$ , where *G* acts on  $K_n(R)$  via  $\alpha \circ \xi$  and trivially on  $\mathbb{Q}$ . Moreover, there is a commutative diagram

where the lower horizontal arrow *N* is the norm map and the right vertical arrow res is restriction with the inclusion  $R \to R_{\alpha \circ \xi} G$ . Since the lower vertical arrow *N* is an isomorphism, the upper horizontal arrow is injective.

## Chapter 13

**13.3.** If we replace in Conjecture 13.1 the family  $\mathcal{VCY}$  by  $\mathcal{FIN}$ , then the Conjecture 13.2 for  $\mathbb{Z}$  reduces to the statement that for any ring *R* the map induced by the projection  $E\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$ 

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) \to H_n^G(\mathbb{Z}/\mathbb{Z};\mathbf{K}_R) = K_n(R\mathbb{Z})$$

is an isomorphism. Since  $\mathbb{Z}$  acts freely on  $\mathbb{Z}$  and  $(E\mathbb{Z})/\mathbb{Z} = S^1$ , we get an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) = H^{\{1\}}(S^1;\mathbf{K}_R) = K_n(R) \oplus K_{n-1}(R).$$

Under this identification the assembly map above becomes the restriction of the Bass-Heller-Swan isomorphism of Theorem 6.16

$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\equiv} K_n(R\mathbb{Z})$$

to  $K_n(RG) \oplus K_{n-1}(RG)$ . This implies that  $NK_n(R)$  vanishes for all  $n \in \mathbb{Z}$  and all rings *R*, a contradiction by Example 3.69.

**13.5.** If we replace in Conjecture 13.4 the family  $\mathcal{VCY}$  by  $\mathcal{FIN}$ , then Conjecture 13.7 for  $\mathbb{Z}$  reduces to the statement that for any ring *R* with involution the map induced by the projection  $E\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$ 

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{L}_R^{\langle -\infty\rangle})\to H_n^G(\mathbb{Z}/\mathbb{Z};\mathbf{L}_R^{\langle -\infty\rangle})=L_n^{\langle -\infty\rangle}(R\mathbb{Z})$$

is an isomorphism. Since  $\mathbb{Z}$  acts freely on  $\mathbb{Z}$  and  $(E\mathbb{Z})/\mathbb{Z} = S^1$ , we get an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(R\mathbb{Z}) \oplus L_{n-1}^{\langle -\infty\rangle}(R\mathbb{Z}).$$

Under this identification the assembly map above can be identified with the isomorphism appearing in the Shaneson splitting (9.109).

**13.15.** Let  $G = \mathbb{Z}$ ,  $R = \mathbb{Q}[\mathbb{Z}/3]$ , and  $\alpha : G \to \operatorname{aut}(R)$  be the group homomorphism which sends the generator of *G* to the automorphism of  $\mathbb{Q}[\mathbb{Z}/3]$  induced by the group automorphism  $-\operatorname{id}: \mathbb{Z}/3 \xrightarrow{\cong} \mathbb{Z}/3$ .

**13.16.** The structure of an abelian group on each set of morphisms comes from the obvious structure of an abelian group on  $M_{m,n}(R)$ . The direct sum of [m] and [n] is [m + n]. The direct sum on morphisms is given by taking block matrices. The zero object is [0]. We obtain a natural equivalence from  $\underline{R}_{\oplus}$  to the additive category of finitely generated free *R*-modules by sending [m] to  $R^m$  and a morphism  $[m] \to [n]$  given by a (m, n)-matrix *A* to the *R*-linear map  $R^m \to R^n$  given by right multiplication by *A*.

**13.21.** We only present the proof of the harder implication. Suppose that for every two objects *A* and *B* in  $\mathcal{A}$  the induced map  $\operatorname{mor}_{\mathcal{A}}(A_0, A_1) \to \operatorname{mor}_{\mathcal{B}}(F(A_0), F(A_1))$  sending *f* to F(f) is bijective and for each object *B* in  $\mathcal{B}$  there exists an object *A* in  $\mathcal{A}$  such that F(A) and *B* are isomorphic in  $\mathcal{B}$ . Choose for any object  $B \in \mathcal{B}$  an object  $A(B) \in \mathcal{A}$  and an isomorphism  $u(B) \colon B \xrightarrow{\cong} F(A(B))$  in  $\mathcal{B}$ . Next we define a functor  $F' \colon \mathcal{B} \to \mathcal{A}$  of additive categories. It sends an object B to A(B). A morphism  $f \colon B_0 \to B_1$  is sent to the morphism  $F'(f) \colon A(B_0) \to A(B_1)$  which is uniquely determined by the property that  $F(F'(f)) = u(B_1) \circ f \circ u(B_0)^{-1}$ . One easily checks that  $F'(g \circ f) = F'(g) \circ F'(f)$  and  $F'(f_0 + f_1) = F'(f_0) + F'(f_1)$  holds. Consider two objects  $B_0$  and  $B_1$ . We have to show that for the natural inclusions  $j_i \colon B_i \to B_0 \oplus B_1$  is an isomorphism. This follows from the diagram below, which commutes by the definition of F' and whose lower left vertical arrow is an isomorphism, since *F* is compatible with direct sums:

$$B_{0} \oplus B_{1} \xrightarrow{j_{0} \oplus j_{1} = \mathrm{id}} B_{0} \oplus B_{1}$$

$$\cong \left| u(B_{0}) \oplus u(B_{1}) \qquad u(B_{0} \oplus B_{1}) \right| \cong$$

$$F(A(B_{0})) \oplus F(A(B_{1})) \xrightarrow{F(F'(j_{0})) \oplus F(F'(j_{1}))} F(A(B_{0} \oplus B_{1}))$$

$$\downarrow \cong \qquad \mathrm{id} \downarrow \cong$$

$$F(A(B_{0}) \oplus A(B_{1})) \xrightarrow{F(F'(j_{0}) \oplus F'(j_{1}))} F(A(B_{0} \oplus B_{1})).$$

Hence F' is a functor of additive categories. Natural transformations of functors of additive categories  $S: F \circ F' \to id_{\mathcal{B}}$  and  $T: F' \circ F \to id_{\mathcal{A}}$  are determined by S(B) = u(B) and F(T(A)) = u(F(A)).

**13.33.** This follows from Theorem 13.32 (v), since G is virtually cyclic if Q is virtually cyclic.

**13.34.** Let *G* be a group. It is the directed union of its finitely generated subgroups. Hence by Theorem 13.32 (vi) the Full Farrell-Jones Conjecture 13.30 holds for all groups if and only if it holds for all finitely generated groups. Any finitely generated group can be written as a colimit over a directed set of finitely presented groups. Hence by Theorem 13.32 (vi) the Full Farrell-Jones Conjecture 13.30 holds for all finitely generated groups if and only if holds for all finitely presented groups. Finally note that a group is finitely presented if and only if it occurs as the fundamental group of a connected orientable closed 4-manifold.

**13.43.** This follows from Lemma 13.42 by the following argument. Since  $K_W$  is finite and the image of  $\phi$  is by assumption infinite, the composite  $p_W \circ \phi \colon V \to Q_W$  has infinite image. Since  $Q_W$  is isomorphic to  $\mathbb{Z}$  or  $D_{\infty}$ , the same is true for the image of  $p_W \circ \phi \colon V \to Q_W$ . By assertion (v) of Lemma 13.42 the kernel of  $p_W \circ \phi \colon V \to Q_W$  is  $K_V$ . Hence  $\phi(K_V) \subseteq K_W$  and  $\phi$  induces maps  $\phi_K$  and  $\phi_Q$  making the diagram of

interest commutative. Since the image of  $p_W \circ \phi \colon V \to Q_W$  is infinite,  $\phi_Q(Q_V)$  is infinite. This implies that  $\phi_Q$  is injective, since both  $Q_V$  and  $Q_W$  are isomorphic to  $D_{\infty}$  or  $\mathbb{Z}$ .

**13.44.** Suppose that *G* admits a proper cocompact isometric action on  $\mathbb{R}$ . Since the action is cocompact and  $\mathbb{R}$  is not compact, the group *G* must be infinite. Let *K* be the kernel of the homomorphism  $\rho: G \to \operatorname{aut}(\mathbb{R})$  coming from the *G*-action. Since the action is proper, *K* must be finite. Let  $Q \subseteq \operatorname{aut}(\mathbb{R})$  be the image of  $\rho$ . The group of isometries of  $\mathbb{R}$  is  $\mathbb{R} \rtimes \mathbb{Z}/2$ , where  $\mathbb{Z}/2$  corresponds to  $\{\pm \operatorname{id}\}$  and  $\mathbb{R}$  to translations  $l_r: \mathbb{R} \to \mathbb{R}$  with elements  $r \in \mathbb{R}$ . Let  $r_0 := \inf\{r \in \mathbb{R} \mid r > 0, l_r \in Q\}$ . Since *Q* acts properly, we have  $r_0 > 0$  and  $Q \cap \mathbb{R} \subseteq \mathbb{R}$  is the infinite cyclic group generated by  $r_0$ . Now one easily checks that *Q* is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z} \rtimes \mathbb{Z}/2$ . Hence *G* is virtually cyclic.

If *G* is virtually cyclic, then it admits an epimorphism with finite kernel onto  $\mathbb{Z}$  or  $\mathbb{Z} \rtimes \mathbb{Z}/2$  by Lemma 13.42 (i). These two groups and hence *G* admit proper cocompact isometric actions on  $\mathbb{R}$ .

**13.49.** Suppose that *H* is infinite and belongs to  $\mathcal{HE}_p \cap \mathcal{VCY}_I$ . Then there are exact sequences  $1 \to \mathbb{Z} \to H \xrightarrow{q} Q \to 1$  and  $1 \to P \xrightarrow{i} H \to \mathbb{Z} \to 1$  where *i*:  $P \to H$  is the inclusion of a finite normal subgroup *P*, and *Q* is a finite *p*-group. The restriction  $q|_P: P \to Q$  is injective, since *P* is a finite subgroup of *H* and the kernel of *q* is infinite cyclic. Hence *P* is a finite *p*-group. Fix an element  $t \in H$  whose image under the epimorphism  $H \to \mathbb{Z}$  is a generator. Then  $t \in N_G P$ . Let  $p^m$  be the order of *Q*. Consider any  $x \in P$ . We have  $q(t^{p^m}xt^{-p^m}) = q(t)^{p^m}q(x)q(t)^{-p^m} = q(x)$ . Since  $q|_P: P \to Q$  is injective, we get  $t^{p^m}xt^{-p^m} = x$ . In particular,  $H \cong P \rtimes_{\phi} \mathbb{Z}$  for the automorphism  $\phi: P \xrightarrow{\cong} P$  of *p*-power order given by conjugation with *t*.

Suppose *H* is isomorphic to  $P \rtimes_{\phi} \mathbb{Z}$  for some finite *p*-group *P* and automorphism  $\phi: P \to P$  whose order is  $p^m$  for some natural number *m*. Then obviously *H* belongs to  $\mathcal{VCY}_I$ . The exact sequence  $1 \to \mathbb{Z} \xrightarrow{p^m \cdot \mathrm{id}} \mathbb{Z} \to \mathbb{Z}/p^m \to 1$  induces an exact sequence  $1 \to \mathbb{Z} \to P \rtimes_{\phi} \mathbb{Z} \to P \rtimes_{\phi} \mathbb{Z}/p^m \to 1$ . Since  $P \rtimes_{\phi} \mathbb{Z}/p^m$  is a finite *p*-group, *H* belongs to  $\mathcal{HE}_p$ .

**13.50.** Because of Exercise 13.49 there exists a finite *p*-group *P* and an automorphism  $\phi: P \to P$ , whose order is a *p*-power, such that *G* is isomorphic to  $P \rtimes_{\phi} \mathbb{Z}$ . Note that a model for  $E_{\mathcal{FIN}}(G)$  is  $E\mathbb{Z}$  considered as a *G*-*CW*-complex by restriction with the canonical epimorphism  $G \to \mathbb{Z}$ . We conclude from Theorem 6.64 and Remark 6.65 that

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(G/G; \mathbf{K}_R) = K_n(RG)$$

is bijective after applying  $- \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$  for all  $n \in \mathbb{Z}$  if and only if we have  $N_{\pm}K_n(RP; \phi)[1/p] = 0$  for all  $n \in \mathbb{Z}$ . This follows from Theorem 6.66.

**13.52.** This follows directly from Theorem 13.51, since  $\mathcal{P}(G, R)$  is empty if  $\mathbb{Q} \subseteq R$  holds or if *G* is torsionfree.

**13.61.** The group *G* satisfies the Full Farrell Jones Conjecture 13.30 by Theorem 13.32 (iv) and (v). Since every virtually cyclic subgroup of *G* is of type *I*, Theorem 13.60 implies that the projection pr induces for every additive *G*-category with involution  $\mathcal{A}$  and all  $n \in \mathbb{Z}$  an isomorphism

$$H_n^G(\mathrm{pr}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(G/G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = \pi_n(\mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}(I(G))).$$

Hence we get from Remark 13.20 that the projection pr induces for all  $n \in \mathbb{Z}$  an isomorphism

$$H_n^G(\mathrm{pr};\mathbf{L}_{\mathbb{Z}}^{\langle-\infty\rangle})\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{Z}}^{\langle-\infty\rangle}) \to H_n^G(G/G;\mathbf{L}_{\mathbb{Z}}^{\langle-\infty\rangle}) = L_n^{\langle-\infty\rangle}(\mathbb{Z}G).$$

Recall that we have an extension  $1 \to F \to G \xrightarrow{f} \mathbb{Z}^d \to 1$  for a finite group *F*. Hence the restriction  $f^* \mathbb{Z}^d$  with *f* of  $\mathbb{Z}^{\mathbb{Z}^d}$  is a model for  $E_{\mathcal{FIN}}(G)$ . Hence it suffices to construct for any free  $\mathbb{Z}^n$ -*CW*-complex *X* an appropriate spectral sequence converging to  $H_n^G(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ . Since the assignment sending *X* to  $H_n^G(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$  is a  $\mathbb{Z}^d$ -homology theory in the sense of Definition 12.1 and *X* is assumed to be a free  $\mathbb{Z}^d$ -*CW*-complex, the equivariant Atiyah-Hirzebruch spectral sequence of Theorem 12.48 converges to  $H_n^G(f^*X; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$  and has as  $\mathbb{E}^2$ -term  $H_p(C_*(X) \otimes_{\mathbb{Z}^d} H_q^G(G/F; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}))$ . Using the induction structure on  $H_*^?(-; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$  and Lemma 12.12, one can identify the  $\mathbb{Z}^d$ -modules  $H_q^G(G/F; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ and  $L_a^{\langle -\infty \rangle}(\mathbb{Z}F)$ .

**13.69.** Induction with  $i: H \to G$  and restriction with  $f: H \to \mathbb{Z}$  induces homomorphisms  $i_*: G_0(\mathbb{C}H) \to G_0(\mathbb{C}G)$  and  $f^*: G_0(\mathbb{C}\mathbb{Z}) \to G_0(\mathbb{C}H)$ . The class  $[\mathbb{C}]$  of the trivial  $\mathbb{C}\mathbb{Z}$ -module  $\mathbb{C}$  is sent under  $i_* \circ f^*$  to the class of  $\mathbb{C}[G/H]$ . Since there exists a short exact sequence  $0 \to \mathbb{C}\mathbb{Z} \to \mathbb{C}\mathbb{Z} \to \mathbb{C} \to 0$ , we have  $[\mathbb{C}] = 0$  in  $G_0(\mathbb{C}\mathbb{Z})$ .

#### Chapter 14

**14.10.** Equip  $\mathbb{R}$  with the  $G = \mathbb{Z} \times \mathbb{Z}/k$ -action where  $\mathbb{Z}$  acts by translation and  $\mathbb{Z}/k$  acts trivial. There is a *G*-pushout

$$\mathbb{Z} \times \{0, 1\} \xrightarrow{j} \mathbb{Z}$$

$$\downarrow^{i} \qquad \qquad \downarrow^{j}$$

$$\mathbb{Z} \times [0, 1] \longrightarrow \mathbb{R}$$

where we think of  $\mathbb{Z}$  as the *G*-space  $G/(\mathbb{Z}/k)$ , the map *i* is the inclusion and *j* sends (n, 0) to *n* and (n, 1) to n+1. Hence  $\mathbb{R}$  is a *G*-*CW*-complex whose isotropy groups are all finite and whose *H*-fixed point set is contractible for every finite subgroup  $H \subseteq G$ .

We conclude that  $\mathbb{R}$  is a model for  $\underline{E}G$ . The Mayer-Vietoris sequence associated to the *G*-pushout looks like

$$\cdots \to K_n^G(\mathbb{Z}) \oplus K_n^G(\mathbb{Z}) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} K_n^G(\mathbb{Z}) \oplus K_n^G(\mathbb{Z}) \to K_n^G(\underline{E}G) = K_n^G(\mathbb{R})$$
$$\to K_{n-1}^G(\mathbb{Z}) \oplus K_{n-1}^G(\mathbb{Z}) \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} K_{n-1}^G(\mathbb{Z}) \oplus K_{n-1}^G(\mathbb{Z}) \to \cdots$$

where we identify  $K_n^G(\mathbb{Z} \times [0,1]) \cong K_n^G(\mathbb{Z})$  via the isomorphism induced by the projection  $\mathbb{Z} \times [0,1] \to \mathbb{Z}$ . Since  $K_n^G(\mathbb{Z}) \cong K_n^G(G/(\mathbb{Z}/k))$  is  $\operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/k)$  for *n* even and zero for *n* odd, we conclude for all  $n \in \mathbb{Z}$ 

$$K_n^G(\underline{E}G) \cong \operatorname{Rep}_{\mathbb{C}}(\mathbb{Z}/k) \cong \mathbb{Z}^k.$$

**14.15.** We only treat the case  $F = \mathbb{C}$ , the case  $F = \mathbb{R}$  is analogous. Since H and G are torsionfree and satisfy the Baum-Connes Conjecture 14.9, they also satisfy the Baum-Connes Conjecture for torsionfree groups 10.44 by Remark 14.14. Hence it suffices to show that the homomorphism  $K_n(Bf): K_n(BH) \to K_n(BG)$  is bijective for all  $n \in \mathbb{Z}$ . This follows from the Atiyah-Hirzebruch spectral sequence converging to  $K_n(BH)$  and  $K_n(BG)$ , since  $H_n(Bf;\mathbb{Z}): H_n(BH;\mathbb{Z}) \to H_n(BG;\mathbb{Z})$  is bijective for all  $n \in \mathbb{Z}$  by assumption and hence f induces isomorphisms between the  $E^2$ -pages.

**14.28.** Take  $G = \mathbb{Z}/2$ . Consider the Atiyah-Hirzebruch spectral sequence converging to  $K_{p+q}(BG)$  with  $E^2$ -term  $E_{p,q}^2 = H_p(BG; K_q(\{\bullet\}))$ . Its  $E^2$ -term looks like

:	:	:		:		
$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	
0	0	0	0	0	0	
$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	
0	0	0	0	0	0	
$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	
0	0	0	0	0	0	
:	:	:	:	:	:	

Because of the checkerboard pattern and by a standard edge argument applied to the split injection  $K_*(\{\bullet\}) \to K_n(BG)$  coming from the inclusion  $\{\bullet\} \to BG$ , all differentials are trivial and the  $E^2$ -term is the  $E^\infty$ -term. Hence  $K_1(BG)$  is non-trivial. (Actually it is  $\mathbb{Z}/2^\infty \cong \mathbb{Z}[1/2]/\mathbb{Z}$ .) On the other hand  $K_1(C_r^*(\mathbb{Z}/2))$  is trivial.

**14.32.** Since any group is the directed union of its finitely generated subgroups, it suffices to consider finitely generated free abelian groups and finitely generated free groups by Theorem 14.31 (iv). Since a finitely generated abelian group is the direct product of finitely many copies of  $\mathbb{Z}$  and of a finite group, Theorem 14.31 (iii), (vi), and (vi) imply that it suffices to prove the claim for  $\mathbb{Z}$  and any finite group. The Baum-Connes Conjecture 14.11 with coefficients obviously holds for finite groups. It holds for  $\mathbb{Z}$  by Theorem 14.31 (vii).

**14.33.** Let  $F_g$  be the surface of genus  $g \ge 1$ . Let  $\overline{F} \to F$  be the covering associated to the epimorphism  $\pi_1(F_g) \to H_1(F_g)$ . Then  $\overline{F_g}$  is a non-compact 2-manifold and hence homotopy equivalent to a 1-dimensional *CW*-complex. Hence  $\pi_1(\overline{F_g})$  is free,  $H_1(F_g)$  is a finitely generated free abelian group and we have the exact sequence  $1 \to \pi_1(\overline{F_g}) \to G \to H_1(F_g) \to 1$ . We conclude from Theorem 14.31 that *G* satisfies the Baum-Connes Conjecture 14.9. Since  $F_g$  itself is a model for *BG*, we get  $K_n(F_g) \cong K_n(C_r^*(G))$ . Now an easy application of the Atiyah-Hirzebruch spectral sequence yields the claim.

**14.43.** Note that  $K_0(C_r^*(G)) = R_{\mathbb{C}}(G)$ . One easily checks by inspecting the definition of (10.48) of the trace for a finite-dimensional complex representation *V* that  $\operatorname{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$  sends the class of [V] to  $|G|^{-1} \cdot \dim_{\mathbb{C}}(V)$ .

**14.55.** Suppose that *F* is  $S^2$ . Since *M* is spin and hence in particular orientable and any orientation preserving self-diffeomorphism of  $S^2$  is isotopic to the identity, *M* must be  $S^1 \times S^2$  and hence carries a Riemannian metric of positive scalar curvature.

Suppose that *F* is not  $S^2$ . Then *F* and hence *M* are aspherical. Hence it suffices to show by Lemma 14.54 that the Baum-Connes Conjecture 14.11 with coefficients holds for  $\pi_1(M)$ . The Baum-Connes Conjecture 14.11 with coefficients holds for all finitely generated free groups and for  $\mathbb{Z}$  by Theorem 14.31 (v). Hence it holds for every free group and every finitely generated abelian group by Theorem 14.31 (iii) and (iv). Let  $\overline{F} \to F$  be the covering associated to the epimorphism  $\pi_1(F) \to H_1(F)$ . Then  $\overline{F}$  is a non-compact 2-manifold and hence homotopy equivalent to a 1-dimensional *CW*-complex. Hence  $\pi_1(\overline{F})$  is free. Now apply Theorem 14.31 (ii) to the short exact sequences  $1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(S^1) \to 1$  and  $1 \to \pi_1(\overline{F}) \to \pi_1(F) \to$  $H_1(F) \to 1$ .

## Chapter 15

**15.3.** Let  $\alpha: H \to G$  be a group homomorphism. Then  $\alpha_* E_{C(H)}(H)$  is a *G*-*CW*-complex whose *G*-isotropy groups are of the shape  $\alpha(L)$  for  $L \in C(H)$  and hence all belong to C(G). This implies that there is up to *G*-homotopy precisely one *G*-map  $f: \alpha_* E_{C(H)}(H) \to E_{C(G)}(G)$ . The following diagram commutes

Now apply  $\mathcal{H}^G_*$  to this diagram and combine it with the following commutative diagram coming from the induction structure applied to  $\alpha$ 

**15.7.** The restriction of a *G*-*CW*-complex *X* to *K* with  $\phi$  is a *K*-*CW*-complex  $\phi^*X$  by Remark 11.3. For a point  $x \in X$  the *K*-isotropy group  $K_x$  of  $\phi^*X$  is  $\phi^{-1}(G_x)$ , where  $G_x$  is the *G*-isotropy group of *x*. In particular, we get  $\phi(K_x) = G_x$  and hence every *K*-isotropy group of  $\phi^*X$  belongs to  $\phi^*\mathcal{F}$ . Consider a subgroup  $H \subset K$ . Then  $(\phi^*X)^H = X^{\phi(H)}$ . Now apply these assertions to  $X = E_{\mathcal{F}}(G)$ .

**15.15.** Let *G* be any group. Denote by pr:  $G \times \mathbb{Z} \to \mathbb{Z}$  the projection. The Fibered Meta-Isomorphism Conjecture 15.8 predicts that the assembly map

$$H_n(\mathrm{pr}; \mathbf{K}_R) \colon H_n^{G \times \mathbb{Z}}(E_{\mathrm{pr}^* \mathcal{FIN}}(G \times \mathbb{Z}); \mathbf{K}_R) \to H_n^{G \times \mathbb{Z}}(G \times \mathbb{Z}/G \times \mathbb{Z}; \mathbf{K}_R) = K_n(R[G \times \mathbb{Z}])$$

is bijective for all  $n \in \mathbb{Z}$ . A model for  $E_{pr^* \mathcal{FIN}}(G \times \mathbb{Z})$  is  $pr^* \mathbb{EZ}$ . Since  $\mathbb{Z}$  acts freely on  $\mathbb{EZ}$  and  $(\mathbb{EZ})/\mathbb{Z} = S^1$ , the left side of the map above can be identified with

$$H_n^{G \times \mathbb{Z}}(E_{\mathrm{pr}^* \mathcal{FIN}}(G \times \mathbb{Z}); \mathbf{K}_R) = H_n^{G \times \mathbb{Z}}(\mathrm{pr}^* E\mathbb{Z}; \mathbf{K}_R)$$
$$= H_n^G(G/G \times S^1)$$
$$= H_n(G/G; \mathbf{K}_R) \oplus H_{n-1}(G/G; \mathbf{K}_R)$$
$$= K_n(RG) \oplus K_{n-1}(RG).$$

Under this identification the assembly map above becomes the restriction of the Bass-Heller-Swan isomorphism of Theorem 6.16

$$K_n(RG) \oplus K_{n-1}(RG) \oplus NK_n(RG) \oplus NK_n(RG) \xrightarrow{=} K_n(RG[t, t^{-1}])$$

to  $K_n(RG) \oplus K_{n-1}(RG)$ . Hence the Fibered Meta-Isomorphism Conjecture 15.8 implies that for every group *G* and  $n \in \mathbb{Z}$  we have  $NK_n(RG) = 0$ .

**15.17.** This follows from Lemma 15.16 applied to the inclusion  $i: H \to G$ , since  $C(H) = i^*C(G)$  holds.

**15.35.** Put  $\Gamma = G \times_{\phi} \mathbb{Z}$ . The proof is completely analogous to the one in Example 15.30 but now applied to a 1-dimensional  $\Gamma$ -*CW*-complex *T* which is a tree and whose 1-skeleton is obtained from the 0-skeleton by the  $\Gamma$ -pushout



Here q is the disjoint union of the identity id:  $\Gamma/G \to \Gamma/G$  and the  $\Gamma$ -map  $\Gamma/G \to \Gamma/G$  sending  $\gamma G$  to  $\gamma t G$  for  $t \in \Gamma$  a lift of the generator in  $\mathbb{Z}$ .

**15.42.** (i) Put  $\pi = \pi_1(X)$ . Conjecture 15.41 yields a weak homotopy equivalence  $E\pi_+ \wedge_{\pi} \mathbf{S}(\widetilde{X}) \to \mathbf{S}(X)$  because of the identifications  $X = \pi \setminus \widetilde{X}$  and  $E_{\mathcal{TR}}(\pi)_+ \wedge_{\operatorname{Or}(\pi)} \mathbf{S}_{\widetilde{X}}^{\pi} = E\pi_+ \wedge_{\pi} \mathbf{S}(\widetilde{X})$ .

(ii) Suppose that **S** is of the shape  $X \mapsto X_+ \wedge \mathbf{H}_{\mathbb{Z}}$  for  $\mathbf{H}_{\mathbb{Z}}$  the Eilenberg-spectrum of  $\mathbb{Z}$ . Recall that the homology theory associated to  $\mathbf{H}_{\mathbb{Z}}$  is singular homology  $H_*$ . Then

$$\pi_n((E\pi)_+ \wedge_\pi \mathbf{S}(\bar{X})) \cong H_n(E\pi \times_\pi \bar{X});$$
  
$$\pi_n((B\pi)_+ \wedge_\pi \mathbf{S}(\{\bullet\})) \cong H_n(B\pi),$$

and  $H_n(E\pi \times_{\pi} \widetilde{X})$  and  $H_n(B\pi)$  are not isomorphic in general.

(iii) Suppose that  $\widetilde{X}$  is contractible or **S** is of the shape  $Y \mapsto \mathbf{T}(\Pi(Y))$  for some covariant functor **T**: GROUPOIDS  $\rightarrow$  SPECTRA. Then the projection  $\widetilde{X} \rightarrow \{\bullet\}$  induces a  $\pi$ -map  $\mathbf{f}: \mathbf{S}(\widetilde{X}) \rightarrow \mathbf{S}(\{\bullet\})$  such that, after forgetting the group action, **f** is a weak homotopy equivalence. Hence we obtain a weak homotopy equivalence

$$E\pi_+ \wedge_{\pi} \mathbf{S}(\widetilde{X}) \xrightarrow{(\mathrm{id}_{E\pi})_+ \wedge_{\pi} \mathbf{f}} E\pi_+ \wedge \mathbf{S}(\{\bullet\}) = B\pi_+ \wedge \mathbf{S}(\{\bullet\}).$$

**15.48.** If Conjecture 15.41 is true for  $(*_{i \in I}G_i, C(*_{i \in I}G_i))$ , it is also true for  $(G_i, C(G_i))$  for every  $i \in I$  by Theorem 15.47 (i).

Suppose that Conjecture 15.41 holds for  $(G_i, C(G_i))$  for every  $i \in I$ . We conclude from the assumptions that for two groups  $H_1, H_2 \in C$  Conjecture 15.41 holds for  $(H_1 \times H_2, C(H_1 \times H_2))$ . Hence Theorem 15.47 (iii) applies. By assumption Theorem 15.47 (iv) also applies. Now we can proceed as in the proof of assertion (vii) of Theorem 13.32 using Theorem 15.47 (ii), (iii), and (iv) to show that Conjecture 15.41 holds for  $(*_{i \in I}G_i, C(*_{i \in I}G_i))$ .

**15.53.** The key ingredient is to construct for a group homomorphism  $\phi: K \to G$  and a subgroup  $H \subseteq K$  a natural weak homotopy equivalence of spaces

$$E\mathcal{G}^{K}(K/H) \times_{\mathcal{G}^{K}(K/H)} p^{*} \phi^{*} Z \xrightarrow{-} K/H \times_{K} (EK \times \phi^{*} Z)$$

where  $p: \mathcal{G}^K(K/H) \to \mathcal{G}^K(K/K) = I(K)$  is induced by the projection  $K/H \to K/K$ . Because of the third isomorphism appearing in [280, Lemma 1.9], it suffices to construct a map

$$u: p_* E \mathcal{G}^K(K/H) \times_K \phi^* Z \xrightarrow{\simeq} K/H \times_K (EK \times \phi^* Z)$$

where here and in the sequel we consider a *K*-space as a  $\mathcal{G}^{K}(K/K) = I(K)$ -space and vice versa in the obvious way. Since  $(K/H \times EK) \times_{K} \phi^{*}Z = K/H \times_{K} (EK \times \phi^{*}Z)$ , it suffices to construct for every *K*-set *S* a natural *K*-homotopy equivalence

$$v: p_* E \mathcal{G}^K(S) \xrightarrow{\simeq} S \times E K,$$

since we then can define  $u = v \times_K id_{\phi^*Z}$  for S = K/H. Unravelling the definition we see that the source of v is

$$p_*E\mathcal{G}^K(S) = \coprod_{s \in S} K \times E\mathcal{G}^K(S)(s) / \sim$$

for the equivalence relation  $\sim$  given by

$$(k,x) \sim (k(k')^{-1}, E\mathcal{G}^K(S)(k': s \rightarrow k's)(u)).$$

Define a *K*-map  $\coprod_{s \in S} K \times E\mathcal{G}^K(S)(s) \to S \times E\mathcal{G}^K(K/K)$  by sending the element (k, x) in the summand  $K \times E\mathcal{G}^K(S)(s)$  belonging to  $s \in S$  to the element  $(ks, E\mathcal{K}^K(k \cdot K/K \to K/K)(u))$ . One easily checks that it is compatible with ~ and induces the desired *K*-map

$$v: p_* E \mathcal{G}^K(S) = \bigsqcup_{s \in S} K \times E \mathcal{G}^K(S)(s) / \rightarrow S \times E K.$$

It remains to show that v is a *K*-homotopy equivalence. Since the source and target of v are free *K*-*CW*-complexes, it suffices to show that v is a homotopy equivalence (after forgetting the *K*-action). We obtain a (non-equivariant) homeomorphism

$$\bigsqcup_{s \in S} E\mathcal{G}^{K}(S)(s) \xrightarrow{\cong} \bigsqcup_{s \in S} K \times E\mathcal{G}^{K}(S)(s) / \sim$$

by sending the element  $x \in E\mathcal{G}^{K}(S)(s)$  belonging to the summand of  $s \in S$  to the element represented by  $(1, x) \in K \times E\mathcal{G}^{K}(S)(s)$ . Hence both the source and the target of *v* have the property that each path component is contractible. Since *v* is a bijection on the path components, it is a homotopy equivalence.

**15.62.** We get  $\pi_n(\mathbf{A}(\{\bullet\})) \cong K_n(\mathbb{Z}) \cong 0$  for  $n \leq -1$  and  $\pi_0(\mathbf{A}(\{\bullet\})) \cong K_0(\mathbb{Z}) \cong \mathbb{Z}$  from Example 2.4, Theorem 3.17, and Theorem 7.18 (i). Now apply the Atiyah-Hirzebruch spectral sequence to  $H_n(BG; \mathbf{A}(\{\bullet\}))$  for  $n \leq 0$ .

**15.70.** This follows from the *p*-chain spectral sequence, see Subsection 12.6.2, and Theorem 15.69 by an inspection of the resulting long exact sequence. See also [676, Proposition 1.2].

**15.100.** Consider the commutative diagram appearing in Remark 15.98. The two left vertical arrows are bijective, as explained in Remark 15.98. The upper horizontal arrow is bijective by assumption. The lower horizontal arrow is bijective by Theorem 15.97. Hence the right vertical arrow is bijective.

**15.101.** Since  $R[G \times \mathbb{Z}] = R[\mathbb{Z}][G]$  and  $R[\mathbb{Z}]$  is regular, Conjecture 15.99 is true for *G* and  $G \times \mathbb{Z}$ . We obtain from the Bass-Heller Swan decompositions for *K*-theory, see Theorem 6.16, and homotopy *K*-theory, see Theorem 15.76, the commutative diagram with isomorphisms as horizontal arrows

where the maps denoted by *h* are induced by the canonical map  $\mathbf{K} \to \mathbf{K}\mathbf{H}$  and are bijective.

**15.104.** (P)  $\implies$  (I): Let  $d: G \rightarrow \prod_{i \in I} G$  be the diagonal embedding. Then  $(\prod_{i \in I} G, \prod_{i \in I} \mathcal{F}_i)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8 because of (P). Hence  $(G, d^* \prod_{i \in I} \mathcal{F}_i)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8 by Lemma 15.16. One easily checks  $d^* \prod_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i$ .

(I)  $\implies$  (P): Consider the projection  $\operatorname{pr}_j: \prod_{i \in I} G_i \to G_j$  for  $j \in I$ . We conclude from Lemma 15.16 that  $(\prod_{i \in I} G_i, \operatorname{pr}_j^* \mathcal{F}_j)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8 for every  $j \in I$ . Hence  $(\prod_{i \in I} G_i, \bigcap_{j \in I} \operatorname{pr}_j^* \mathcal{F}_j)$  satisfies the Fibered Meta Isomorphism Conjecture 15.8 because of (I). One easily checks  $\prod_{i \in I} \mathcal{F}_i = \bigcap_{j \in I} \operatorname{pr}_j^* \mathcal{F}_j$ .

## Chapter 16

**16.2.** We know that L/K is a smooth manifold, which is diffeomorphic to  $\mathbb{R}^{\dim(L/K)}$ , and, equipped with the obvious left *G* action, it is a model for the classifying space for proper *G*-actions, see Theorem 11.24. Hence  $G \setminus L/K$  is an aspherical closed smooth manifold of dimension  $\geq 5$ . Since *G* satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (id), the claim follows from Theorem 9.171.

**16.3.** Let *G* be a group. It is the directed union of its finitely generated subgroups. Hence by Theorem 16.1 (iif) the Full Farrell-Jones Conjecture 13.30 holds for all groups if and only if it holds for all finitely generated groups. Any finitely generated group can be written as a directed colimit of finitely presented groups. Hence by Theorem 16.1 (iif) the Full Farrell-Jones Conjecture 13.30 holds for all finitely generated groups if and only if holds for all finitely presented groups. Now the claim follow from Theorem 16.1 (iia), since every finitely presented group is a subgroup of U.

**16.4.** We obtain an embedding of rings  $R \to \text{end}_S(R)$  by sending  $r \in R$  to the *S*-homomorphism of right *S*-modules  $l_r \colon R \to R$ ,  $r' \mapsto rr'$ . Since *R* is finitely generated free as a right *S*-module, we obtain for some natural number *k* an identification of rings  $\text{end}_S(R) = M_k(S)$ . The inclusion of rings  $R \to M_k(S)$  yields an inclusion of rings  $M_n(R) \to M_n(M_k(S)) = M_{kn}(S)$ . By passing to units we obtain an inclusion of groups  $\text{GL}_n(R) \to \text{GL}_{kn}(S)$ . Now the claim follows from Theorem 16.1 (iia).

**16.6.** This follows from the commutative diagram

whose horizontal arrows are assembly maps and whose vertical arrows are change of theory maps. Moreover, the left vertical arrow is bijective, since  $K_n(R) \rightarrow KH_n(R)$  is bijective for all  $n \in \mathbb{Z}$  and all regular rings R, and the lower horizontal arrow is bijective because of Theorem 16.5 (i).

**16.8.** This follows directly from Theorem 16.7 (iic).

**16.9.** This follows from Theorem 2.81, Lemma 10.51, Lemma 10.53, Theorem 13.65, Theorem 16.1, and Theorem 16.7.

**16.16.** By Lemma 15.23 (ii) it suffices to prove the injectivity for any finitely generated subgroup of G, since G is the directed union of its finitely generated subgroups. The relevant equivariant homology theories are (strongly) continuous by [71, Lemma 6.2].

**16.22.** We have  $G/[G,G] = H_1(G) \cong \mathbb{Z}$ , and the projection pr:  $G \to H_1(G)$ induces an isomorphisms on the group homology  $H_n(G) \to H_n(G/[G,G])$  for all  $n \in \mathbb{Z}$ . This follows from Alexander-Lefschetz duality. The Atiyah-Hirzebruch spectral sequence implies that  $H_n(\text{pr}; \mathbf{K}(R)) : H_n(G; \mathbf{K}(R)) \to H_n(G/[G,G]; \mathbf{K}(R))$ is bijective for all  $n \in \mathbb{Z}$ . Since *G* satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1 (ie) and hence the *K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings 6.53 by Theorem 13.65 (ii), the map  $K_n(RG) \to K_n(R[G/[G,G]])$  induced by pr is a bijection. Since  $G/[G,G] \cong \mathbb{Z}$ , we get  $K_n(R[G/[G,G]]) \cong K_n(R) \oplus K_{n-1}(R)$  from the Bass-Heller-Swan decomposition for algebraic *K*-theory, see Theorem 6.16.

The *L*-theory case is treated analogously, but now replacing Theorem 6.16 by (9.109).

**16.23.** One shows by induction over i = d, (d - 1), ..., 0 that  $G/G_i$  satisfies the Full Farrell-Jones Conjecture 13.30. The induction beginning i = d is trivial. The induction step from *i* to (i - 1) follows from Theorem 16.1 (iid) and the short exact sequence  $1 \rightarrow G_i/G_{i-1} \rightarrow G/G_{i-1} \rightarrow G/G_i \rightarrow 1$ .

**16.26.** Show by induction over i = 0, 1, 2, ..., d that  $G_i$  is torsionfree and satisfies the Baum-Connes Conjecture 14.11 with coefficients using Theorem 16.7 (if) and (iic).

**16.28.** We want to apply Theorem 16.1 (iic). So we need to show that *K* satisfies the Full Farrell-Jones Conjecture 13.30 and that for any extension  $1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  the group *G* satisfies the Full Farrell-Jones Conjecture 13.30. Since *K* is either the fundamental group of a closed surface or a countable free group, see [1010, Lemma 2.1], both *K* and *G* are strongly poly-surface groups or normally poly-free groups and hence satisfy the Full Farrell-Jones Conjecture 13.30 by Theorem 16.25 or Theorem 16.27.

**16.36.** We only treat the *K*-theory case, the argument for *L*-theory is completely analogous. Let *G* be a group with a finite model for *BG* and let *R* be a regular ring. Choose *M*, *i*, and *r* as they appear in Theorem 16.35. We obtain a commutative diagram

The horizontal arrows are assembly maps. The composite of the two vertical arrows of the left column and the right column are the identity. Since the middle horizontal arrow is bijective, the same is true for the upper horizontal arrow.

**16.40.** We conclude from Theorem 3.115 that for a natural number *n* the vanishing of  $\mathbb{Q} \otimes_{\mathbb{Z}} Wh(\mathbb{Z}/n) = 0$  implies n = 1, 2, 3, 4, 6. Now apply Theorem 16.39.

## Chapter 17

**17.2.** This follows from Theorem 12.79 and Theorem 13.36.

**17.6.** Since *G* is elementary amenable, it satisfies the *L*-theoretic Farrell-Jones Conjecture 13.8 with coefficients in rings with involution after inverting 2, see [475, Theorem 5.2.1]. So we can apply Theorem 17.5.

For every non-trivial finite cyclic subgroup  $C \subseteq G$  we have  $C \subseteq C_G C \subseteq \bigoplus_{\mathbb{Z}} F$ and hence  $H_p(C_G C; \mathbb{Q}) = 0$  for  $p \neq 0$  and  $H_0(C_G C; \mathbb{Q}) \cong \mathbb{Q}$ . Hence we get from Theorem 17.5 for all  $n \in \mathbb{Z}$  an isomorphism

$$\bigoplus_{p+q=n} H_p(G; \mathbb{Q}) \otimes_{\mathbb{Z}} L_q(\mathbb{Z}) \oplus \bigoplus_{(C) \in J, C \neq \{1\}} \mathbb{Q} \otimes_{\mathbb{Q}[N_G C/C_G C]} \Theta_C \cdot \left(\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}C)\right)$$
$$\xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G).$$

We get  $L_n^{\langle j \rangle}(\mathbb{Z}C) = 0$  for odd *n* from Theorem 9.204 (iv), since *F* and hence *C* has odd order. From the Lyndon-Serre spectral sequence applied to the group extension to  $1 \to \bigoplus_{\mathbb{Z}} F \to G \to \mathbb{Z} \to 1$ , we conclude  $H_*(G; \mathbb{Q}) \cong H_*(\mathbb{Z}; \mathbb{Q})$ . Hence we obtain for odd *n* an isomorphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}) \oplus \mathbb{Q} \otimes_{\mathbb{Z}} L_{n-1}^{\langle j \rangle}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G).$$

This implies

$$\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(\mathbb{Z}G) \cong \begin{cases} \mathbb{Q} & n \equiv 1 \mod 4; \\ \{0\} & n \equiv 3 \mod 4. \end{cases}$$

**17.24.** We use the presentation  $D_{\infty} = \langle s, t | s^2 = 1, sts = t^{-1} \rangle$ . Consider the split exact sequence  $1 \to \mathbb{Z} \xrightarrow{i} D_{\infty} \xrightarrow{p} \mathbb{Z}/2 \to 1$  where *i* sends  $1 \in \mathbb{Z}$  to *t* and *p* sends *t* to  $\overline{0}$  and *s* to the generator  $\overline{1}$ . The set con( $\mathbb{Z}/2$ ) consists of  $\mathbb{Z}/2$  itself. Note that we view  $\mathbb{Z}$  as an  $\mathbb{Z}[\mathbb{Z}/2]$ -module using the action of  $\mathbb{Z}/2$  given by  $-\operatorname{id}_{\mathbb{Z}} : \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$ . Obviously we get:

$$\begin{split} \overline{C}_{\mathbb{Z}/2}(\overline{0}) &= \mathbb{Z}/2; \\ \overline{C}_{\mathbb{Z}/2}(\overline{1}) &= \{0\}; \\ H^{1}(\langle \overline{0} \rangle; \mathbb{Z}) &= \{0\}; \\ \overline{C}_{\mathbb{Z}/2}(\overline{0}) \setminus H^{1}(\langle \overline{0} \rangle; \mathbb{Z}) &= \{0\}; \\ H^{1}(\langle \overline{1} \rangle; \mathbb{Z}) &= \mathbb{Z}/2; \\ \overline{C}_{\mathbb{Z}/2}(\overline{1}) \setminus H^{1}(\langle \overline{1} \rangle; \mathbb{Z}) &= \mathbb{Z}/2; \\ \mathbb{Z}^{\langle \overline{0} \rangle} &= \mathbb{Z} \\ \mathbb{Z}^{\langle \overline{1} \rangle} &= \{0\} \\ H_{m}(\mathbb{Z}^{\langle \overline{0} \rangle}; \mathbb{Q}) &= \begin{cases} \mathbb{Q} & \text{if } m = 0, 1; \\ \{0\} & \text{otherwise}; \end{cases} \\ H_{m}(\mathbb{Z}^{\langle \overline{0} \rangle}; \mathbb{Q}) &= \begin{cases} \mathbb{Q} & m = 0; \\ \{0\} & \text{otherwise}. \end{cases} \\ (\overline{C}_{\mathbb{Z}/2}\langle \overline{0} \rangle)_{y} &= \overline{C}_{\mathbb{Z}/2}\langle \overline{0} \rangle \text{ for } y \in H^{1}(\langle \overline{0} \rangle; \mathbb{Z}); \end{cases} \\ H_{m}(\mathbb{Z}^{\langle \overline{0} \rangle}; \mathbb{Q})^{\langle \overline{C}_{\mathbb{Z}/2}\langle \overline{0} \rangle)_{y} &= \begin{cases} \mathbb{Q} & \text{if } m = 0; \\ \{0\} & \text{if } m \neq 0; \end{cases} \text{ for } y \in H^{1}(\langle 0 \rangle; \mathbb{Z}); \\ (\overline{C}_{\mathbb{Z}/2}\langle \overline{1} \rangle)_{z} &= \overline{C}_{\mathbb{Z}/2}\langle \overline{1} \rangle \text{ for } z \in H^{1}(\langle \overline{1} \rangle; \mathbb{Z}); \end{cases} \\ H_{m}(\mathbb{Z}^{\langle \overline{1} \rangle}; \mathbb{Q})^{\langle C_{\mathbb{Z}/2}\langle \overline{1} \rangle\rangle_{z}} &= \begin{cases} \mathbb{Q} & \text{if } m = 0; \\ \{0\} & \text{otherwise}; \end{cases} \text{ for } z \in H^{1}(\langle \overline{1} \rangle; \mathbb{Z}). \end{cases} \end{split}$$

So we get from Lemma 17.19

$$con_f(p)^{-1}((0)) = \{(e_{D_\infty})\};con_f(p)^{-1}((\overline{1})) = \{(s), (st)\};con_f(D_\infty) = \{(e_{D_\infty}), (s), (st)\}.$$

Theorem 17.21 implies

$$\operatorname{rk}_{\mathbb{Z}}(K_n(C_r^*(D_\infty))) = \begin{cases} 3 & n \text{ even}; \\ 0 & n \text{ odd.} \end{cases}$$

**17.31.** Since  $H_1(G)$  is the abelianization of G, we obtain a short exact sequence  $\mathbb{Z} \xrightarrow{D} \bigoplus_{i=1}^{n} \mathbb{Z} \to H_1(G) \to 0$ , where D sends  $x \in \mathbb{Z}$  to  $(d_1x, d_2x, \dots, d_nx)$ .

**17.32.** One easily checks that *G* is torsionfree and the word  $s_1s_2s_1s_2^{-1}s_1^{-2} \in F$  is a commutator. Put  $R = \mathbb{C}[\mathbb{Z}/m]$ . Then  $R \cong \prod_{i=1}^m \mathbb{C}$  is semisimple and in particular regular, and we obtain from Lemma 17.30 (i) an isomorphism for  $n \in \mathbb{Z}$ 

$$\begin{split} K_n(\mathbb{C}[\mathbb{Z}/m \times G]) &\cong K_n(\mathbb{C}[\mathbb{Z}/m][G]) \\ &\cong K_n(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-1}(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-1}(\mathbb{C}[\mathbb{Z}/m]) \oplus K_{n-2}(\mathbb{C}[\mathbb{Z}/m]). \end{split}$$

We get from Example 2.4, Lemma 2.12, Theorem 3.6, Lemma 3.9, and Theorem 4.7

$$K_n(\mathbb{C}[\mathbb{Z}/m]) = \begin{cases} \mathbb{C}[\mathbb{Z}/m]^{\times} & n = 1; \\ \mathbb{Z}^m & n = 0; \\ 0 & n \leq -1. \end{cases}$$

**17.33.** The group *G* is solvable and torsionfree and hence satisfies Conjecture 3.110, Conjecture 4.18, and the Farrell-Jones Conjecture 9.114 for torsionfree groups for *L*-theory. We conclude from Theorem 9.106 that  $L_n^s(\mathbb{Z}[G]) = L_n^{\langle -\infty \rangle}(\mathbb{Z}[G])$ . The group *G* is a one-relator group with presentation  $\langle s_1, s_2 | s_1 s_2 s_1^{-1} s_2 \rangle$ . The word  $s_1 s_2 s_1^{-1} s_2 \in F$  is not a commutator. Hence we get from Lemma 17.30 (ii) a short exact sequence

$$0 \to H_1(BG) \otimes_{\mathbb{Z}} L_{n-1}^{\langle -\infty \rangle}(\mathbb{Z}) \to H_n(BG, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z}))$$
$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H_1(BG); L_{n-2}^{\langle -\infty \rangle}(\mathbb{Z})) \to 0$$

and an isomorphism

$$L_n^s(\mathbb{Z}G) \cong L_n^{\langle -\infty \rangle}(\mathbb{Z}) \oplus H_n(BG, \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})).$$

We have  $H_1(BG) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$  and  $L_n^{\langle -\infty \rangle}(\mathbb{Z}) \cong \mathbb{Z}, 0, \mathbb{Z}/2, 0$  for  $n \equiv 0, 1, 2, 3 \mod 4$ . Hence we get

$$L_n^s(\mathbb{Z}G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0 \mod 4 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 1 \mod 4 \\ \mathbb{Z}/2 & n \equiv 2 \mod 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 3 \mod 4 \end{cases}$$

17.40. This follows from Theorem 3.115, Theorem 3.116 (iv), and Theorem 17.39 (ii).

**17.45.** This follows from Theorem 9.106 and the Shaneson splitting, see Theorem 9.108, if we can construct an orientable closed aspherical smooth 3-manifold N such that  $L_i^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(M)])$  contains *p*-torsion for at least one  $i \in \mathbb{Z}$ . Namely, then we can take  $M = N \times T^{n-3}$ .

If p = 2, take  $N = T^3$ . If p is odd, use Example 17.44.

**17.49.** The  $\mathbb{Z}/3$ -action given by  $\phi$  on  $\mathbb{Z}^2$  is free outside the origin. Now apply Theorem 17.47 (iii) together with (17.48).

**17.54.** We get from Theorem 10.79 (i)

$$K_n(C_r^*(\mathbb{Z}/m,\mathbb{C})) \cong \begin{cases} \mathbb{Z}^m & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Since  $K_n(C_r^*(\mathbb{Z}/2, \mathbb{C})) \to K_n(C_r^*(\mathbb{Z}/6, \mathbb{C}))$  is split injective, the computation for  $K_n(C_r^*(SL_2(\mathbb{Z}), \mathbb{C}))$  follows.

We have  $C_r^*(\mathbb{Z}/2, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}, C_r^*(\mathbb{Z}/3, \mathbb{R}) \cong \mathbb{R} \times \mathbb{C}$ , and  $C_r^*(\mathbb{Z}/6, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}$ . We get from Theorem 10.79 (ii)

$$\begin{aligned} &KO_n(C_r^*(\mathbb{Z}/2,\mathbb{R})) \cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R});\\ &KO_n(C_r^*(\mathbb{Z}/4,\mathbb{R})) \cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R}) \oplus K_n(\mathbb{C});\\ &KO_n(C_r^*(\mathbb{Z}/6,\mathbb{R})) \cong KO_n(\mathbb{R}) \oplus KO_n(\mathbb{R}) \oplus K_n(\mathbb{C}) \oplus K_n(\mathbb{C}). \end{aligned}$$

Since  $KO_n(C_r^*(\mathbb{Z}/2,\mathbb{R})) \to K_n(C_r^*(\mathbb{Z}/6,\mathbb{R}))$  is split injective, the computation for  $KO_n(C_r^*(SL_2(\mathbb{Z}),\mathbb{R}))$  follows by inspecting the values of  $KO_n(\mathbb{R})$  and  $K_n(\mathbb{C})$ .

**17.55.** The group *G* contains a subgroup of finite index which is finitely generated free. Hence it satisfies the Full Farrell-Jones Conjecture 13.30 by Theorem 16.1. It satisfies the Baum-Connes 14.11 with coefficients by Theorem 16.7. We conclude from Theorem 13.51 that the assembly maps

$$\begin{aligned} H_0^G(\underline{E}G;\mathbf{K}_{\mathbb{C}}) &\xrightarrow{\cong} K_0(\mathbb{C}G); \\ K_n^G(\underline{E}G) &\xrightarrow{\cong} K_n(C^r_*(G)) \end{aligned}$$

are isomorphisms. Since for a finite group *H* we have  $K_0(\mathbb{C}H) = K_0(C_r^*(H)) = \operatorname{Rep}_{\mathbb{C}}(H)$  and  $K_{-1}(\mathbb{C}H) = K_1(C_r^*(H)) = \{0\}$ , we get from Example 15.30 exact sequences

$$\operatorname{Rep}_{\mathbb{C}}(C) \xrightarrow{i_* \oplus i_*} \operatorname{Rep}_{\mathbb{C}}(D_8) \oplus \operatorname{Rep}_{\mathbb{C}}(D_8) \to K_0(\mathbb{C}G) \to 0$$

and

$$0 \to K_1(C_r^*(G)) \to \operatorname{Rep}_{\mathbb{C}}(C) \xrightarrow{i_* \oplus i_*} \operatorname{Rep}_{\mathbb{C}}(D_8) \oplus \operatorname{Rep}_{\mathbb{C}}(D_8) \to K_0(C_r^*(G)) \to 0,$$

where  $i: C \to D_8$  is the inclusion. The group *C* has two irreducible complex representations, the trivial 1-dimensional complex representation  $\mathbb{C}$  and the non-trivial 1-dimensional complex representation  $\mathbb{C}^-$ . The group  $D_8$  has four 1-dimensional irreducible complex representations and one 2-dimensional irreducible complex representation. The homomorphism  $i_*: \operatorname{Rep}_{\mathbb{C}}(C) \to \operatorname{Rep}_{\mathbb{C}}(D_8)$  sends the class of  $\mathbb{C}$ to the class of the sum of the four 1-dimensional irreducible representations and  $\mathbb{C}^$ to the sum of two copies of the 2-dimensional irreducible representation, see [908, Subsections 3.3, 5.1 and 5.3]. Hence  $i_*: \operatorname{Rep}_{\mathbb{C}}(C) \to \operatorname{Rep}_{\mathbb{C}}(D_8)$  looks like

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^{10}.$$

We conclude that  $i_*$  is injective and its cokernel is isomorphic to  $\mathbb{Z}^8 \oplus \mathbb{Z}/2$ .

**17.61.** Note that *G* is the right-angled Artin group associated to the simplicial graph *X* consisting of three vertices  $e_0, e_1$ , and  $e_2$  and two edges  $[e_0, e_1]$  and  $[e_1, e_2]$ . Note that  $\Sigma = X$  in this case. Hence we get  $r_{-1} = 1$ ,  $r_0 = 3$ , and  $r_1 = 2$ . We conclude from (17.58) and Theorem 17.60

$$H_n(G) \cong \begin{cases} \mathbb{Z} & n = 0; \\ \mathbb{Z}^3 & n = 1; \\ \mathbb{Z}^2 & n = 2; \\ \{0\} & n \ge 3; \end{cases}$$
$$K_n(C_r^*(G)) \cong \mathbb{Z}^3 \text{ for } n \in \mathbb{Z},$$

and

$$KO_n(C_r^*(G,\mathbb{R})) \cong \begin{cases} \mathbb{Z} & n \equiv 0 \mod (8); \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2 & n \equiv 1 \mod (8); \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^4 & n \equiv 2 \mod (8); \\ (\mathbb{Z}/2)^5 & n \equiv 3 \mod (8); \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & n \equiv 4 \mod (8); \\ \mathbb{Z}^3 & n \equiv 5 \mod (8); \\ \mathbb{Z}^2 & n \equiv 6 \mod (8); \\ \{0\} & n \equiv 7 \mod (8). \end{cases}$$

**17.63.** We can arrange without changing the isomorphism type of  $(\mathbb{Z}/2)^3 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2$  that the inclusions of  $\mathbb{Z}/2$  into  $(\mathbb{Z}/2)^2$  and  $(\mathbb{Z}/2)^3$  are given by sending *x* to (x, 0) and (x, 0, 0). Hence *G* is isomorphic to the right-angled Coxeter group associated to the simplicial graph with vertices  $e_0, e_1, e_2, e_3$  and edges  $[e_0, e_1], [e_0, e_2], [e_1, e_2],$  and  $[e_2, e_3]$ . Then the associated flag complex  $\Sigma$  is obtained from *X* by adding the 2-simplex  $[e_0, e_1, e_2]$ . Hence the number of the simplices of *X* is r = 10. Now apply Theorem 17.62.

**17.65.** Recall from the proof of Theorem 17.64 that M is aspherical. In particular  $\pi$  is torsionfree and we get for any abelian group A using Poincaré duality and the Universal Coefficient Theorem

$$H_n(B\pi; A) \cong H_n(M; A); \text{ for } n \ge 0;$$
  

$$H_1(B\pi; A) \cong \pi/[\pi, \pi] \otimes_{\mathbb{Z}} A;$$
  

$$H_2(B\pi; A) \cong \hom_{\mathbb{Z}}(\pi, A);$$
  

$$H_3(B\pi; A) \cong A;$$
  

$$H_n(B\pi; A) \cong \{0\} \text{ for } n \notin \{0, 1, 2, 3\}.$$

The independence of  $L_n^{\langle i \rangle}(\mathbb{Z}\pi)$  from the decoration follows from Theorem 9.106 and from Conjectures 3.110 and 4.18, which hold for  $\pi$  by Theorem 13.65 (xii). We obtain from Theorem 17.64 (ii) an isomorphism

$$H_n(B\pi; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(\mathbb{Z}\pi).$$

Next we apply the Atyiah-Hirzebruch spectral sequence to  $H_n(B\pi; \mathbf{L}_{\mathbb{Z}}^{\langle -\infty \rangle})$ . Recall that  $L_n(\mathbb{Z})$  is  $\mathbb{Z}, \{0\}, \mathbb{Z}/2, \{0\}$  for  $n \equiv 0, 1, 2, 3 \mod 4$ , see Theorem 9.204 (i). Since the composite  $L_n(\mathbb{Z}) \to L_n(\mathbb{Z}\pi) \to L_n(\mathbb{Z})$  is the identity, all differentials in the Atiyah-Hirzebruch spectral sequence are trivial. Hence we obtain isomorphisms

$$L_0(\mathbb{Z}\pi) \cong H_0(B\pi; L_0(\mathbb{Z})) \oplus H_2(B\pi; L_2(\mathbb{Z}/2));$$
  
$$L_2(\mathbb{Z}\pi) \cong H_0(B\pi; L_2(\mathbb{Z})) \oplus H_2(B\pi; L_0(\mathbb{Z}/2)),$$

and two short exact sequences

$$0 \to H_1(B\pi; L_2(\mathbb{Z})) \to L_3(\mathbb{Z}\pi) \to H_3(B\pi; L_0(\mathbb{Z})) \to 0;$$
  
$$0 \to H_1(B\pi; L_0(\mathbb{Z})) \to L_1(\mathbb{Z}\pi) \to H_3(B\pi; L_2(\mathbb{Z})) \to 0.$$

The first one splits because of  $H_3(B\pi; L_0(\mathbb{Z})) \cong \mathbb{Z}$ . In order to show that the second one splits, it suffices to show that it splits after localization at 2 since  $H_3(B\pi; L_2(\mathbb{Z})) \cong \mathbb{Z}/2$ . This follows from Lemma 9.116 (i).

## **Chapter 18**

**18.9.** This follows from the following facts. We have  $\emptyset/G = \emptyset$ . If  $f: X \to Y$  is a *G*-homotopy equivalence,  $f/G: X/G \to Y/G$  is a homotopy equivalence. If the *G*-*CW*-complex *X* is the union of *G*-*CW*-subcomplexes  $X_1$  and  $X_2$  with intersection  $X_0$ , then the *CW*-complex X/G is the union of *CW*-subcomplexes  $X_1/G$  and  $X_2/G$  with intersection  $X_0/G$ . If  $\{X_i \mid i \in I\}$  is a collection of *G*-*CW*-complexes, then the canonical map  $(\coprod_{i \in I} X_i)/G \to \coprod_{i \in I} X_i/G$  is a homeomorphism.

**18.12.** Suppose that **E** is weakly  $\mathcal{F}$ -excisive. Theorem 18.11 (ii) and (iv) imply that the assignment sending (X, A) to coker $(\pi_n(\emptyset_+) \to \pi_n(\mathbf{E}(X/A)))$  is a *G*-homology theory.

Suppose that the assignment sending (X, A) to coker  $(\pi_n(\mathbf{E}(\emptyset_+)) \to \pi_n(\mathbf{E}(X/A)))$  is a *G*-homology theory. Then we get from Theorem 18.11 (ii) and (iv) and from Lemma 12.6 applied to  $\mathbf{E}^{\mathbb{V}_0} \to \mathbf{E}$  that  $\mathbf{E}$  is weakly  $\mathcal{F}$ -excisive.

**18.13.** We use induction over the dimension  $d = \dim(X)$  of *X*. The induction beginning d = 0 follows from the fact that *X* is a finite union of homogenous spaces  $\coprod_{i=1}^{k} G/H_i$  and hence we get an isomorphism  $\bigoplus_{i=1}^{k} \pi_n(\mathbf{E}(G/H_i)) \xrightarrow{\cong} \pi_n(\mathbf{E}(X))$ . The induction step from (d-1) to  $d \ge 1$  is done as follows. Choose a *G*-pushout



Because of the associated Mayer-Vietoris sequence, it suffices to show for all  $n \in \mathbb{Z}$ and  $i \in \{1, 2, ..., l\}$  that  $\pi_n(G/H_i \times S^{d-1})$ ,  $\pi_n(G/H_i \times D^d)$ , and  $\pi_n(X_{d-1})$  are finitely generated. This follows from the induction hypothesis and the fact that  $\pi_n(G/H_i \times D^d) \cong \pi_n(G/H_i)$  holds by weak *G*-homotopy invariance.

**18.17.** Since the projection  $E_{\mathcal{F}}(G) \to G/G$  induces a homotopy equivalence  $EG \times_G E_{\mathcal{F}}(G) \to EG \times_G G/G$ , the map induced by the projection  $E_{\mathcal{F}}(G) \to G/G$  induces a weak homotopy equivalence  $\mathbf{E}(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathbf{E}(G/G)$ . Obviously we get a weak equivalence of Or(G) spectra from  $\mathbf{K}_R \circ \mathcal{G}^G$  for  $\mathcal{G}^G : Or(G) \to GROUPOIDS$  defined in (12.29) to  $\mathbf{E}|_{Or(G)}$ , since there is an equivalence of groupoids  $\mathcal{G}^G(G/H) \xrightarrow{\cong} \Pi(EG \times_G G/H)$ , which is natural in G/H. Now apply Lemma 12.6 and Corollary 18.16.

#### Chapter 19

**19.7.** One easily checks that F and  $F^f$  are compatible with the structures of an additive category, is fully faithful, and every object of RG-MOD<sub>fgf</sub> is isomorphic to some object in the image of F.

**19.12.** Define a functor of additive categories  $F: \mathcal{T}(X) \to \mathrm{GM}^{\{1\}}(X)$  by sending an object  $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$  to the object  $F(M) = \{F(M)_x \mid x \in X\}$  given by  $F(M)_x = \bigoplus_{s \in \mathbb{N}} M_{(x,s)}$  and a morphism  $f = \{f_{(x,s),(y,t)} \mid (x,s), (y,t) \in X \times \mathbb{N}\}$  from  $M = \{M_{(x,s)} \mid (x,s) \in X \times \mathbb{N}\}$  to  $N = \{N_{(y,t)} \mid (y,t) \in X \times \mathbb{N}\}$  to the morphism  $F(f): F(M) \to F(N)$  which is defined for  $x, y \in X$  by the morphism  $\bigoplus_{s \in \mathbb{N}} M_{(x,s)} \to \bigoplus_{t \in \mathbb{N}} N_{(y,t)}$  given by the collection of *R*-homomorphisms  $\{f_{(x,s),(y,s)}: M_{(x,s)} \to N_{(y,t)} \mid s,t \in \mathbb{N}\}$ . Obviously  $F \circ I$  is the identity on  $GM^{\{1\}}(X)$ . It remains to show that  $I \circ F$  is naturally equivalent to the identity on  $\mathcal{T}(X)$ . For this purpose we have to construct for every object M in  $\mathcal{T}(X)$  a natural isomorphism  $u: I \circ F(M) \xrightarrow{\cong} M$  in  $\mathcal{T}(X)$ . For (x, s) and (y, t) in  $X \times \mathbb{N}$  we define  $u_{(x,s),(y,t)}: I \circ F(M)_{(x,s)} \to M_{(y,t)}$  to be the projection  $\bigoplus_{s \in \mathbb{N}} M_{(x,s)} \to M_{(y,t)}$  to the summand belonging to (y, t) if s = 0 and x = y, and to be zero otherwise. For (x, s) and (y, t) in  $X \times \mathbb{N}$  we define  $(u^{-1})_{(x,s),(y,t)}: M_{(x,s)} \to (I \circ F(M))_{(y,t)}$  to be the inclusion  $M_{(x,s)} \to \bigoplus_{t \in \mathbb{N}} M_{(y,t)}$  of the summand belonging to (x, s) if x = y, and to be zero otherwise.

**19.13.** The hyperbolic metric is given by

$$d_{\text{hyp}}((x_1, y_1), (x_2, y_2)) = 2 \cdot \ln\left(\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}}\right).$$

Hence  $\gamma_{(x_k,y_k)}$ :  $[0,\infty) \to \mathbb{H}^2$  sends *t* to  $(x_k, \exp(t) \cdot y_k)$ , since for  $t, s \in \mathbb{R}$  we get  $d_{\text{hyp}}(\gamma_{(x_k,y_k)}(t), \gamma_{(x_k,y_k)}(s)) = |t-s|$ . We compute for  $t \ge 0$ 

$$\begin{split} &\lim_{t \to \infty} d_{hyp}(\gamma_{(x_1,y_1)}(t), \gamma_{x_2,y_2}(t)) \\ &= \lim_{t \to \infty} d_{hyp}((x_1, \exp(t) \cdot y_1), (x_2, \exp(t) \cdot y_2)) \\ &= \lim_{t \to \infty} 2 \cdot \ln \left( \frac{\sqrt{(x_2 - x_1)^2 + (\exp(t) \cdot y_2 - \exp(t) \cdot y_1)^2}}{2\sqrt{\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2}} \right. \\ &+ \frac{\sqrt{(x_2 - x_1)^2 + (\exp(t) \cdot y_2 + \exp(t) \cdot y_1)^2}}{2\sqrt{\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2}} \\ &= \lim_{t \to \infty} 2 \cdot \ln \left( \sqrt{\frac{(x_2 - x_1)^2}{4\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2}} + \frac{(\exp(t) \cdot y_2 - \exp(t) \cdot y_1)^2}{4\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2}} \right. \\ &+ \sqrt{\frac{(x_2 - x_1)^2}{4\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2}} + \frac{(\exp(t) \cdot y_2 + \exp(t) \cdot y_1)^2}{4\exp(t) \cdot y_1 \cdot \exp(t) \cdot y_2} \\ &= \lim_{t \to \infty} 2 \cdot \ln \left( \sqrt{\frac{(x_2 - x_1)^2}{4y_1y_2 \cdot \exp(2t)}} + \frac{(y_2 - y_1)^2}{4y_1y_2}}{4y_1y_2} \right) \\ &= 2 \cdot \ln \left( \sqrt{\frac{(y_2 - y_1)^2}{4y_1y_2}} + \sqrt{\frac{(y_2 + y_1)^2}{4y_1y_2}} \right) \end{split}$$

$$= 2 \cdot \ln\left(\frac{|y_2 - y_1| + (y_2 + y_1)}{\sqrt{4y_1y_2}}\right)$$
  
=  $2 \cdot \ln\left(\frac{2\max\{y_1, y_2\}}{\sqrt{4y_1y_2}}\right)$   
=  $2 \cdot \left(\ln(\max\{y_1, y_2\}) - \frac{\ln(y_1)}{2} - \frac{\ln(y_2)}{2}\right)$   
=  $|\ln(y_1) - \ln(y_2)|.$ 

**19.15.** Let  $t \in [T - r', T + r']$ . We have  $|\tau| \le d(x_1, x_2) \le 2\beta$ . From  $T - r' = r'' > 2\beta$  we conclude  $t, t + \tau > 0$ . If  $t \ge d(x, x_1)$ , then  $t + \tau \ge d(x, x_2)$  holds and hence we get  $c_{x_1,x}(t) = x = c_{x_2,x}(t + \tau)$  so that the assertion follows in this case. Hence we can assume without loss of generality  $0 < t < d(x, x_1)$ . One easily checks that  $0 < t + \tau < d(x, x_2)$  and  $d(c_{x_1,x}(t), x) = d(x, x_1) - t = d(x, x_2) - (t + \tau) = d(c_{x_2,x}(t+\tau), x)$  hold. We can suppose without loss of generality  $d(x, x_1) \le d(x, x_2)$ , the proof in the other case is analogous, interchanging the role of  $x_1$  and  $x_2$ . We have  $d(x, x_1) = d(x, c_{x_2,x}(d(x, x_2) - d(x, x_1)))$ . The Intercept Theorem implies

$$d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) = \frac{d(x_1, c_{x_2,x}(d(x, x_2) - d(x, x_1))) \cdot (d(x, x_1) - t)}{d(x, x_1)}.$$

We have  $d(x_2, c_{x_2,x}(d(x, x_2) - d(x, x_1))) = d(x, x_1)$ . Hence the triangle inequality implies

$$\begin{aligned} d(x_1, c_{x_2,x}(d(x, x_2) - d(x, x_1))) &\leq d(x_1, x_2) - d(x_2, c_{x_2,x}(d(x, x_2) - d(x, x_1))) \\ &= d(x_1, x_2) - d(x, x_1) \\ &\leq d(x_1, x_2) - (d(x, x_2) - d(x_1, x_2)) \\ &= 2d(x_1, x_2) - d(x, x_2) \\ &\leq 2d(x_1, x_2) \\ &\leq 4\beta. \end{aligned}$$

Hence we get

$$d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \le \frac{4 \cdot \beta \cdot (d(x,x_1) - t)}{d(x,x_1)}$$

Since we have  $r'' = T - r' \le t < d(x, x_1)$  and

$$\begin{aligned} d(x,x_1) - t &\leq d(x,x_0) + d(x_0,x_1) - t \leq r' + r'' + L + \beta - (T - r') \\ &= r' + r'' + \beta + L - r'' = r' + \beta + L, \end{aligned}$$

the asserted inequality  $d(c_{x_1,x}(t), c_{x_2,x}(t+\tau)) \leq \frac{4\cdot\beta\cdot(r'+\beta+L)}{r''}$  follows.

We have  $t \leq T + r' = 2r' + r''$ . We have already shown  $|\tau| \leq 2\beta$  and hence  $t + \tau \leq 2r' + r'' + 2\beta$ . Since this implies  $t, t + \tau \in [0, 2r' + r'' + 2\beta]$ , we get  $c_{x_1,x}(t) \in B_{2r'+r''+2\beta}(x_1)$  and  $c_{x_1,x}(t+\tau) \in B_{2r'+r''+2\beta}(x_2)$ .

#### Chapter 20

**20.14.** The assertion follows from Theorem 20.12 applied to  $N = \dim(\Sigma)$ ,  $X = |\Sigma|$ ,  $f = id_{|\Sigma|} E$ , and any  $\epsilon > 0$ . Since  $\Sigma$  is finite, the group of simplicial automorphisms of  $\Sigma$  is also finite. Therefore *G* contains a normal subgroup of finite index which acts trivially on  $\Sigma$  and hence on  $|\Sigma|$ .

**20.20.** Define a *G*-homeomorphism  $f: G \times_1 X \xrightarrow{\cong} G \times_d X$  by sending (g, x) to (g, gx).

**20.37.** The assertion for  $F_g(\Gamma, S, k)$  is a consequence of the equality

$$\Gamma(g_k, t_k, \dots, g_1, t_1, g_0, z) = g_k \cdots g_0 z_1$$

The assertion for  $S^1_{\Gamma,S,k}(g,x)$  is proved as follows. Consider  $(h, y) \in G \times X$  with the property that there are  $a, b \in S$ ,  $f \in F_a(\Gamma, S, k)$ , and  $f' \in F_b(\Gamma, S, k)$  satisfying both f(x) = f'(y) and  $h = ga^{-1}b$ . We conclude from the assertion for  $F_g(\Gamma, S, k)$  that this is equivalent to the condition that for some a, b in S we have ax = by and  $h = ga^{-1}b$ .

The claim for  $S_{\Gamma,S,k}^n(g,x)$  follows by induction on *n*, since  $S[k,n] = \{u_1 \cdots u_n \mid u_i \in S[k,1]\}$  holds.

**20.43.** Suppose that the condition is satisfied for  $S_1$  and  $\epsilon > 0$ . Choose a natural number k such that each element in  $S_2$  can be written as a word in the generators of  $S_2$  consisting of at most k elements. Then the conditions is satisfied for  $S_2$  and  $k \cdot \epsilon > 0$ , since for an element g which can be written as a word in l-elements of  $S_1$  we conclude from the triangle inequality and the G-invariance of the  $L^1$ -metric that  $d_{L^1}(f(gx), gf(x)) \le l \cdot \epsilon$  holds.

**20.51.** Consider  $H \in \mathcal{D}(H)$ . Choose a prime *q* and a normal subgroup  $H \subseteq F$  such that *H* is cyclic and *F*/*H* is a *q*-group. Now take *p* to be any prime,  $P = \{1\}, C = H$ , and D = F in the Definition 20.50.

**20.54.** Because of Theorem 20.53 we have to show for any finite abelian group G that it is a Dress group if and only if the set of primes for which the *p*-Sylow group is non-cyclic consists of at most two elements. This follows from the fact that G is the direct product of its *p*-Sylow subgroups and G is cyclic if and only if all its *p*-Sylow subgroups are cyclic.

**20.56.** Obviously  $C_n(X)$  is a  $\mathbb{Z}G$ -module whose underlying abelian group is finitely generated free. Hence s(X) is well-defined.

Suppose that  $f: X \to Y$  is a *G*-map which is (after forgetting the *G*-action) a homotopy equivalence. Then we obtain an exact sequence of finite  $\mathbb{Z}G$ -chain complexes  $0 \to C_*(Y) \to \operatorname{cone}(C_*(f)) \to \Sigma C_*(X) \to 0$  such that each chain module is finitely generated free as an abelian group and  $\operatorname{cone}(C_*(f))$  has trivial homology. Define for a finite  $\mathbb{Z}G$ -chain complex  $E_*$ , whose chain modules are finitely generated free as abelian groups, the element  $s(E_*) = \sum_{n \ge 0} (-1)^n \cdot [E_n]$  in  $\operatorname{Sw}^p(G)$ . Now one easily checks

$$\begin{split} s(X) &= s(C^{c}_{*}(X));\\ s(Y) &= s(C^{c}_{*}(Y));\\ s(\operatorname{cone}(C_{*}(f))) &= 0;\\ s(\Sigma C_{*}(X)) &= -s(C_{*}(X));\\ s(\operatorname{cone}(C_{*}(f))) &= s(C_{*}(Y)) + s(\Sigma C_{*}(X)). \end{split}$$

This implies s(X) = s(Y).

Suppose that the compact *G*-*CW*-complex *X* is the union of sub *G*-*CW*-complexes  $X_1$  and  $X_2$  and  $X_0$  is the intersection of  $X_1$  and  $X_1$ . Then we conclude from the short exact sequence of  $\mathbb{Z}G$ -chain complexes

$$0 \to C_*(X_0) \to C_*(X_1) \oplus C_*(X_2) \to C_*(X) \to 0$$

that  $s(X) = s(X_1) + s(X_2) - s(X_0)$  holds. Hence the map  $s: \operatorname{Sw}^A(G) \to \operatorname{Sw}^p(G)$ sending [X] to s(X) is a well-defined map of abelian groups. It is compatible with the multiplication, since there is a  $\mathbb{Z}G$ -chain isomorphism  $C_*(X) \otimes C_*(Y) \xrightarrow{\cong} C_*(X \times Y)$ for any two compact *G*-*CW*-complexes *X* and *Y*.

**20.57.** Since for two finite *G*-sets *S* and *S'* we can view *S* II *S'* as the *G*-pushout of  $S \leftarrow \emptyset \rightarrow S'$ , we get a well-defined homomorphism of abelian groups  $a: A(G) \rightarrow Sw^A(G)$  by sending [*S*] to [*S*]. It is compatible with the multiplication, since it is defined on A(G) and  $Sw^A(G)$  by the cartesian product equipped with the diagonal *G*-action.

In order to show that the homomorphism *a* is surjective, we show by induction over d = 0, 1, 2, ... that for any cocompact finite *G*-*CW*-complex *X* of dimension  $\leq d$  the class [X] is in the image of *a*. The induction beginning d = 0 is obvious since a cocompact 0-dimension *G*-*CW*-complex is the same as a finite *G*-set. The induction step from (d - 1) to  $d \geq 1$  is done as follows. We can write X as a *G*-pushout

$$\begin{array}{c|c} \coprod_{i \in I_d} G/H_i \times S^{d-1} \xrightarrow{q} X_{d-1} \\ & \downarrow \\ & \downarrow \\ & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d \longrightarrow X \end{array}$$

for a finite set *I* and subgroups  $H_i \subseteq G$  of finite index. Since we can replace *q* by inclusion of  $\coprod_{i \in I_d} G/H_i \times S^{d-1}$  into the mapping cylinder cyl(*q*) and the projections cyl(*q*)  $\rightarrow X_{d-1}$  and  $G/H_i \times D^d \rightarrow G/H_i$  are *G*-homotopy equivalences, we obtain in Sw<sup>A</sup>(*G*)

$$[X] = \sum_{i \in I_d} [G/H_i] + [X_{d-1}] - \sum_{i \in I_d} [G/H_i \times S^{d-1}].$$

Since by induction hypothesis  $[X_{d-1}]$ ,  $[G/H_i]$ , and  $[G/H_i \times S^{d-1}]$  lie in the image of *a*, the same is true for [X].

**20.58.** The map *u* is obviously well-defined and an isomorphism of abelian groups. The map *c* is well-defined, since for an exact sequence of  $\mathbb{Z}[\mathbb{Z}/p]$ -modules  $0 \to M_0 \to M_1 \to M_2 \to 0$  the sequence of  $\mathbb{Q}$ -modules  $0 \to \mathbb{Q} \otimes_{\mathbb{Z}} M_0^{\mathbb{Z}/p} \to \mathbb{Q} \otimes_{\mathbb{Z}} M_1^{\mathbb{Z}/p} \to \mathbb{Q} \otimes_{\mathbb{Z}} M_2^{\mathbb{Z}/p} \to 0$  is exact. The composite  $c \circ s \circ a \circ u$  sends (m, n) to (m, m+n) and hence is bijective. This implies that *a* is injective. Since *a* is surjective by Exercise 20.57, it is bijective. Hence all three maps *u*, *a*, are  $c \circ s$  are bijective.

## **Chapter 21**

**21.74.** Since  $O^G(G/H)$  is flasque, we get  $K_n(O^G(G/H)) = 0$  for all  $n \in \mathbb{Z}$  from Lemma 6.37 (iii). We conclude from the  $\mathcal{T}O\mathcal{D}$ -sequence of Theorem 21.19 that the canonical map  $K_{n+1}(\mathcal{D}^G(G/H)) \xrightarrow{\cong} K_n(\mathcal{T}^G(G/H))$  is an isomorphism for all  $n \in \mathbb{Z}$ . We have already constructed a natural isomorphism  $K_n(\mathcal{B}(G/H)_{\oplus}) \xrightarrow{\cong} K_{n+1}(\mathcal{D}^G(G/H))$  for every  $n \in \mathbb{Z}$  in Proposition 21.70. We get a canonical isomorphism  $K_n(\mathcal{T}^G(G/H)) \xrightarrow{\cong} K_n(\mathcal{T}^G(G/G))$  from Lemma 21.22. These three isomorphisms can be combined to an isomorphism  $K_n(\mathcal{T}^G(G/G)) \xrightarrow{\cong} K_n(\mathcal{B}(G/H)_{\oplus})$ . There is an obvious identification  $\mathcal{T}^G(G/G) = \mathcal{B}_{\oplus}$ . Under it we get an isomorphism  $K_n(\mathcal{B}[G/H]_{\oplus}) \xrightarrow{\cong} K_n(\mathcal{B}_{\oplus})$  which comes from the obvious projection  $G/H \to G/G$ and the obvious identification  $\mathcal{B}(G/G) = \mathcal{B}$ .

**21.84.** We leave the elementary proof that  $(\mathcal{B}, \operatorname{supp}_{\mathbb{Z}})$  satisfies the axioms appearing in Definition 21.1 to the reader.

The category  $\mathcal{B}(\mathbb{Z}/\mathbb{Z}) \cong \mathcal{B}$  is  $R[\mathbb{Z}/2]$  and hence  $H_n^{\mathbb{Z}}(\mathbb{Z}/\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \cong K_n(R[\mathbb{Z}/2])$ .

The category  $\mathcal{B}(\mathbb{Z})$  can be identified with the category  $\coprod_{n \in \mathbb{Z}} \underline{R}$ . Hence the obvious functor of additive categories  $\bigoplus_{n \in \mathbb{Z}} \underline{R}_{\oplus} \to \mathcal{B}(\mathbb{Z})_{\oplus}$  is an equivalence. Thus we get an isomorphism

$$\alpha \colon \bigoplus_{n \in \mathbb{Z}} K_n(R) \xrightarrow{\cong} K_n(\mathcal{B}(\mathbb{Z}))$$

since the algebraic *K*-theory of additive categories is compatible with direct sums over arbitrary index sets. Let  $s: \mathbb{Z} \to \mathbb{Z}$  be the automorphism sending *n* to n + 1 and sh:  $\bigoplus_{n \in \mathbb{Z}} K_n(R) \to \bigoplus_{n \in \mathbb{Z}} K_n(R)$  be the shift automorphism sending  $(x_n)_{n \in \mathbb{N}}$  to  $(x_{n-1})_{n \in \mathbb{N}}$ . Then the following diagram commutes

$$\bigoplus_{n \in \mathbb{Z}} K_n(R) \xrightarrow{\alpha} K_n(\mathcal{B}[\mathbb{Z}]_{\oplus})$$
sh 
$$\downarrow \qquad \cong \downarrow K_n(\mathcal{B}(s)_{\oplus})$$

$$\bigoplus_{n \in \mathbb{Z}} K_n(R) \xrightarrow{\cong} K_n(\mathcal{B}(\mathbb{Z})_{\oplus}).$$

The following sequence of abelian group is exact

$$0 \to \bigoplus_{n \in \mathbb{Z}} K_n(R) \xrightarrow{\text{id-sh}} \bigoplus_{n \in \mathbb{Z}} K_n(R) \xrightarrow{\epsilon} K_n(R) \to 0$$

where  $\epsilon$  sends  $(x_n)_{n\geq 0}$  to  $\sum_{n\in\mathbb{Z}} x_n$ . Since a model for  $E\mathbb{Z}$  is  $\mathbb{R}$  with the standard  $\mathbb{Z}$ -action, we obtain a long exact sequence

$$\cdots \to K_n(\mathcal{B}(\mathbb{Z})_{\oplus}) \xrightarrow{\operatorname{id} - K_n(\mathcal{B}(s)_{\oplus})} K_n(\mathcal{B}(\mathbb{Z})_{\oplus}) \to H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \to K_{n-1}(\mathcal{B}(\mathbb{Z})_{\oplus}) \xrightarrow{\operatorname{id} - K_{n-1}(\mathcal{B}(s)_{\oplus})} K_{n-1}(\mathcal{B}(\mathbb{Z})_{\oplus}) \to \cdots .$$

Hence we obtain an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_{\mathcal{B}}) \cong K_n(R).$$

We leave the elementary proof of the claim about the identification of the assembly map to the reader.

**21.86.** Suppose that we have two other morphisms  $u'_i: B \to B'$  for i = 1, 2 satisfying  $u = u'_1 + u'_2$  and  $\operatorname{supp}_G(u'_i) = L_i$  for i = 1, 2. Put  $v_i := u_i - u'_i$  for i = 1, 2. Then we have  $0 = v_1 + v_2$  and hence  $\operatorname{supp}_G(v_1 + v_2) = \emptyset$ . We conclude

$$supp_G(v_1) = supp_G((v_1 + v_2) + (-v_2)) \subseteq supp_G(v_1 + v_2) \cup supp_G(-v_2)$$
$$= \emptyset + supp_G(v_2) = supp_G(u_2 + (-u'_2)) \subseteq supp_G(u_2) \cup supp_G(-u'_2)$$
$$\subseteq supp_G(u_2) \cup supp_G(u'_2) \subseteq L_2 \cup L_2 = L_2$$

and

$$\operatorname{supp}_{G}(v_{1}) = \operatorname{supp}_{G}(u_{1} + (-u_{1}')) \subseteq \operatorname{supp}_{G}(u_{1}) \cup \operatorname{supp}_{G}(-u_{1}')$$
$$\subseteq \operatorname{supp}_{G}(u_{1}) \cup \operatorname{supp}_{G}(u_{1}') \subseteq L_{1} \cup L_{1} = L_{1}.$$

Since  $L_1 \cap L_2 = \emptyset$ , we conclude  $\operatorname{supp}_G(v_1) = \emptyset$  and hence  $v_1 = 0$ . This implies  $u_1 = u'_1$ . Analogously one shows  $u_2 = u'_2$ .

**21.87.** Obviously  $\mathcal{A}[G]$  with the support defined in Example 21.2 is a category with *G*-support. It is a strong category with *G*-support, since  $\mathcal{A}$  comes with a *G*-action and we can define the desired homotopy trivialization  $\Omega_g \colon \operatorname{id}_{\mathcal{B}} \xrightarrow{\cong} \Lambda_g$ 

by the isomorphisms  $id_{gA} \cdot g \colon A \to gA$  in  $\mathcal{R}[G]$  for  $A \in \mathcal{R}$ . Morphism Additivity obviously holds.

**21.88.** Define a functor of G- $\mathbb{Z}$ -categories

$$F: \mathcal{A}[G] \xrightarrow{\cong} \mathcal{B}$$

by requiring that *F* is the identity on the set of objects and sends a morphism  $\sum_{g \in G} (f_g : gA \to A) \cdot g$  from *A* to *A'* in  $\mathcal{R}[G]$  to the morphism  $\sum_{g \in G} (f_g \circ \Omega_g(A))$  from *A* to *A'* in  $\mathcal{R}[G]$ . Using Exercise 21.86 one easily checks that *F* is full and faithful and hence an isomorphism of *G*- $\mathbb{Z}$ -categories. Obviously *F* is compatible with the support functions.

**21.89.** Suppose such an extension to the structure of a strong category with  $\mathbb{Z}$ -support exists. The natural transformation  $\Omega_1$  for the generator  $1 \in \mathbb{Z}$  is an isomorphism in  $\mathcal{B}$  with support {1}. This is a contradiction since no morphism in  $\mathcal{B}$  has support {1}.

21.102. Because of Theorem 8.46 (i) and Lemma 21.101 its suffices to show

$$K_m(\mathcal{T}_0^{\{1\}}(\{\bullet\})) \cong \bigoplus_{n=0}^{\infty} K_m(\mathcal{B}_{\oplus});$$
  
$$K_m(\mathcal{O}_0^{\{1\}}(\{\bullet\})) \cong \prod_{n=0}^{\infty} K_m(\mathcal{B}_{\oplus}).$$

Non-connective *K*-theory is compatible with infinite direct products of additive categories, by [212], see also [573, Theorem 1.2]. It is also compatible with directed unions, see for instance [684, Corollary 7.2], and hence with infinite direct sums. Since the obvious functors

$$\begin{split} \bigoplus_{n=0}^{\infty} \mathcal{B}_{\oplus} \xrightarrow{\simeq} \mathcal{T}_{0}^{\{1\}}(\{\bullet\}); \\ \mathcal{O}_{0}^{\{1\}}(\{\bullet\}) \xrightarrow{\simeq} \prod_{n=0}^{\infty} \mathcal{B}_{\oplus}, \end{split}$$

are equivalences of additive categories, the claim follows.

**21.132.** The key observation is the following. Given a morphism  $\phi: \mathbf{B} = (S, \pi, \eta, B) \rightarrow \mathbf{B}' = (S', \pi', \eta', B')$ , there exists, because of *bounded control* over  $\mathbb{N}$ , a natural number *n* such that for  $s \in S$  and  $s' \in S'$  the implication  $\phi_{s,s'} \neq 0 \implies |\eta(s) - \eta'(s')| \le n$  holds. Hence for any natural number *r* with r > n we conclude that

$$\left|\frac{1}{\eta(s)} - \frac{1}{\eta'(s')}\right| = \left|\frac{\eta(s) - \eta'(s')}{\eta(s) \cdot \eta(s')}\right| \le \frac{n}{r \cdot (r-n)}$$

holds for  $s \in S$  and  $s' \in S'$  with  $\phi_{s,s'} \neq 0$ , provided that  $\eta(s) \geq r$  or  $\eta(s') \geq r$ . Obviously we have  $\lim_{r\to\infty} \frac{n}{r\cdot(r-n)} = 0$ .

# Chapter 22

**22.7.** Obviously  $\Phi_{\tau} \circ \Phi_{\sigma} = \Phi_{\tau+\sigma}$  for  $\tau, \sigma \in \mathbb{R}$  and  $\Phi_0 = id_{FS(X)}$ . We estimate for  $c \in FS(X)$  and  $\tau \in \mathbb{R}$ 

$$d_{\mathrm{FS}(X)}(c, \Phi_{\tau}(c)) = \int_{\mathbb{R}} \frac{d_X(c(t), c(t+\tau))}{2e^{|t|}} dt$$
$$\leq \int_{\mathbb{R}} \frac{|\tau|}{2e^{|t|}} dt$$
$$= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|t|}} dt$$
$$= |\tau|.$$

We estimate for  $c, d \in FS(X)$  and  $\tau \in \mathbb{R}$ 

$$\begin{split} d_{\mathrm{FS}(X)} \left( \Phi_{\tau}(c), \Phi_{\tau}(d) \right) &= \int_{\mathbb{R}} \frac{d_X \left( c(t+\tau), d(t+\tau) \right)}{2 \mathrm{e}^{|t|}} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{d_X \left( c(t), d(t) \right)}{2 \mathrm{e}^{|t-\tau|}} \, \mathrm{d}t \\ &\leq \int_{\mathbb{R}} \frac{d_X \left( c(t), d(t) \right)}{2 \mathrm{e}^{|t| - |\tau|}} \, \mathrm{d}t \\ &= \mathrm{e}^{|\tau|} \cdot \int_{\mathbb{R}} \frac{d_X \left( c(t), d(t) \right)}{2 \mathrm{e}^{|t|}} \, \mathrm{d}t \\ &= \mathrm{e}^{|\tau|} \cdot d_{\mathrm{FS}(X)} \left( c, d \right). \end{split}$$

The two inequalities above together with the triangle inequality imply for  $c, d \in FS(X)$  and  $\tau, \sigma \in \mathbb{R}$ 

$$\begin{split} &d_{\mathrm{FS}(X)} \left( \Phi_{\tau}(c), \Phi_{\sigma}(d) \right) \\ &= d_{\mathrm{FS}(X)} \left( \Phi_{\tau}(c), \Phi_{\sigma-\tau} \circ \Phi_{\tau}(d) \right) \\ &\leq d_{\mathrm{FS}(X)} \left( \Phi_{\tau}(c), \Phi_{\tau}(d) \right) + d_{\mathrm{FS}(X)} \left( \Phi_{\tau}(d), \Phi_{\sigma-\tau} \circ \Phi_{\tau}(d) \right) \\ &\leq \mathrm{e}^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c, d) + |\sigma - \tau|. \end{split}$$

This implies that  $\Phi$  is continuous at  $(c, \tau)$ .

**22.16.** Note that  $FS(X)^{\mathbb{R}}$  is the space of constant generalized geodesics. Let  $c \in FS(X) - FS(X)^{\mathbb{R}}$ . Pick  $t_0, t_1 \in \mathbb{R}$  such that  $c(t_0) \neq c(t_1)$ . Set  $\delta := d_X(c(t_0), c(t_1))/2$ . Consider any  $x \in X$ . Then we have  $d_X(x, c(t_0)) \ge \delta$  or  $d_X(x, c(t_1)) \ge \delta$ . Denote by  $c_x$  the constant generalized geodesic at x. Suppose  $d_X(x, c(t_0)) \ge \delta$ . Then

 $d_X(x, c(t)) \ge \delta/2$  if  $t \in [t_0 - \delta/2, t_0 + \delta/2]$ . Thus in this case we get

$$d_{\mathrm{FS}(X)}(c_x,c) \ge \int_{t_0-\delta/2}^{t_0+\delta/2} \frac{\delta/2}{2\mathrm{e}^{|t|}} \,\mathrm{d}t =: \epsilon_0.$$

Similarly, we get in the case  $d_X(x, c(t_1)) \ge \delta$ 

$$d_{\mathrm{FS}(X)}(c_x,c) \ge \int_{t_1-\delta/2}^{t_1+\delta/2} \frac{\delta/2}{2\mathrm{e}^{|t|}} \,\mathrm{d}t =: \epsilon_1.$$

Hence  $B_{\epsilon}(c) \cap FS(X)^{\mathbb{R}} = \emptyset$  if we set  $\epsilon := \min\{\epsilon_0/2, \epsilon_1/2\}.$ 

**22.37.** If  $\dim(X) = \infty$ , the claim is obviously true. So we can assume in the sequel that  $\dim(X)$  is a natural number.

Let  $\mathcal{U}$  be an open covering of A. For  $U \in \mathcal{U}$  choose an open subset  $U' \subseteq X$ satisfying  $U = A \cap U'$ . Then  $\mathcal{U}' = \{U' \mid U \in \mathcal{U}\} \amalg \{X \setminus A\}$  is an open covering of X. Let  $\mathcal{V}'$  be a refinement of  $\mathcal{U}'$  with dim $(\mathcal{V}) \leq \dim(X)$ . Then  $\mathcal{V} = \{V' \cap A \mid V' \in \mathcal{V}'\}$ is an open covering of A with dim $(\mathcal{V}) \leq \dim(\mathcal{V}') \leq \dim(X)$  which is a refinement of  $\mathcal{U}$ . This implies dim $(A) \leq \dim(X)$ .

#### Chapter 23

**23.11.** We have to show for  $\epsilon \in \{\pm 1\}$  and  $g \in G$  that, for any  $\mathbb{Z}G$ -module M which is finitely generated free as an abelian group,  $s([M], (\epsilon \cdot g))$  lies in the kernel of the projection  $K_1(\mathbb{Z}G) \to Wh(G)$  for the element  $(\epsilon \cdot g) \in K_1(\mathbb{Z}G)$  represented by the trivial unit  $\epsilon \cdot g \in \mathbb{Z}G^{\times}$  and the element  $[M] \in Sw(G)$  represented by M. It is not hard to check that  $s([M], (\epsilon g))$  is represented by the composite of the automorphisms  $(\epsilon \cdot l_g) \otimes_{\mathbb{Z}} id_{\mathbb{Z}G}$  and  $id_M \otimes r_g$  of  $M \otimes_2 \mathbb{Z}G$ , where  $l_g \colon M \to M$  is left multiplication and  $r_g \colon \mathbb{Z}G \to \mathbb{Z}G$  is right multiplication. One easily checks that the class of  $(\epsilon \cdot l_g) \otimes_{\mathbb{Z}} id_{\mathbb{Z}G}$  in  $K_1(\mathbb{Z}G)$  lies in the image of  $K_1(\mathbb{Z}) \to K_1(\mathbb{Z}G)$  and the class of  $id_M \otimes r_g$  in  $K_1(\mathbb{Z}G)$  is  $rk_{\mathbb{Z}}(M) \cdot (g)$ .

**23.30.** (i) For  $x, y \in Z$  with  $f_{x,y} \neq 0$  there exist  $i \in \{1, \ldots, m\}$  and  $j \in \{1, \ldots, n\}$  with  $(f_{i,j})_{x,y} \neq 0$  which implies  $d^{L^1}(w(x), w(y)) \leq wd(f_{i,j})$ .

(ii) For  $x, y \in Z$  with  $(g \circ f)_{x,y} \neq 0$  there exists a  $z \in Z$  with  $f_{x,z} \neq 0$  and  $g_{z,y} \neq 0$  and hence we get

$$d^{L^{1}}(w(x), w(y)) \le d^{L^{1}}(w(x), w(z)) + d^{L^{1}}(w(z), w(y)) \le \operatorname{wd}(f) + \operatorname{wd}(g).$$

(iii) Suppose that  $(\lambda \cdot f + \mu \cdot g)_{x,y} \neq 0$  holds. Then  $f_{x,y} \neq 0$  or  $g_{x,y} \neq 0$  holds. This implies  $d^{L^1}(w(x), w(y)) \leq wd(f)$  or  $d^{L^1}(w(x), w(y)) \leq wd(g)$ .

(iv) This follows from the definition of the width.

It is trivial on objects since we have  $(id_M)_{x,y} \neq 0 \implies x = y \implies d^{L^1}(w(x), w(y)) = 0.$ 

**23.40.** Define for a bounded  $\mathcal{A}$ -chain complex  $C_*$  the number  $d(C_*)$  to be the minimum over those numbers d for which there exist integers a and b such that  $a \leq b$  holds, we have  $C_n = 0$  for n < a and n > b, and d = b - a holds. Then we use induction over  $d(C_*)$ . In the induction beginning  $d(C_*) = 0$  the  $\mathcal{A}$ -chain complex  $C_*$  is concentrated in one dimension and the claim follows directly from the definition. The induction step follows from Additivity.

**23.46.** The inverse of  $g \circ f$  is  $f^{-1} \circ g^{-1}$ . One easily checks using the axioms appearing in Definition 23.27 that wd $(g \circ f)$ , wd $((g \circ f)^{-1}) \leq \epsilon + \delta$  holds.

**23.49.** In the sequel we will apply the axioms appearing in Definition 23.27 over and over again.

(i) The equality  $wd(\lambda \cdot f_* + \mu \cdot g_*) \le max\{wd(f_*), wd(g_*)\}$  follows directly from these axioms. If  $h_*: f_* \simeq g_*$  and  $k_*: g_* \simeq h_*$  are  $\mathcal{A}$ -chain homotopies, then  $h_* + k_*$  is an  $\mathcal{A}$ -chain homotopy  $f_* \simeq h_*$ .

(ii) If  $h_*$ :  $f_* \simeq f'_*$  is an  $\mathcal{A}$ -chain homotopy, then we obtain  $\mathcal{A}$ -chain homotopies  $v_{*+1} \circ h_*$ :  $v_* \circ f_* \simeq v_* \circ f'_*$  and  $h_* \circ u_*$ :  $f_* \circ u_* \simeq f'_* \circ u_*$ 

(iii) Choose  $\mathcal{A}$ -chain maps  $u_* \colon D_* \to C_*$  and  $v_* \colon E_* \to D_*$  satisfying

$$\begin{aligned} & \operatorname{wd}(u_*) \leq \epsilon; \\ & \operatorname{wd}(v_*) \leq \epsilon; \\ & u_* \circ f_* \simeq_{\epsilon} \operatorname{id}_{C_*}; \\ & f_* \circ u_* \simeq_{\epsilon} \operatorname{id}_{D_*}; \\ & v_* \circ g_* \simeq_{\epsilon} \operatorname{id}_{D_*}; \\ & g_* \circ v_* \simeq_{\epsilon} \operatorname{id}_{E_*}. \end{aligned}$$

Now assertions (i) and (ii) imply

$$\begin{aligned} & \operatorname{wd}(g_* \circ f_*) \leq 2\epsilon; \\ & \operatorname{wd}(u_* \circ v_*) \leq 2\epsilon; \\ & (u_* \circ v_*) \circ (g_* \circ f_*) \simeq_{3\epsilon} \operatorname{id}_{C_*}; \\ & (g_* \circ f_*) \circ (u_* \circ v_*) \simeq_{3\epsilon} \operatorname{id}_{E_*}. \end{aligned}$$

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