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Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants

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Introduction

Throughout this paper $\mathcal{A}$ is a finite von Neumann algebra and $\mathrm{tr} : \mathcal{A} \longrightarrow \mathbb{C}$ is a finite normal faithful trace. Recall that a von Neumann algebra is finite if and only if it possesses such a trace. Let $l^2(\mathcal{A})$ be the Hilbert space completion of $\mathcal{A}$ which is viewed as a pre-Hilbert space by the inner product $\langle a, b \rangle = \mathrm{tr}(ab^*)$. A *finitely generated Hilbert $\mathcal{A}$-module V* is a Hilbert space V together with a left operation of $\mathcal{A}$ by $\mathbb{C}$-linear maps such that $\lambda \cdot 1_{\mathcal{A}}$ acts by scalar multiplication with λ on V for $\lambda \in \mathbb{C}$ and there exists a unitary $\mathcal{A}$-embedding of V in $l^2(\mathcal{A})^n = \oplus_{i=1}^n l^2(\mathcal{A})$. In the sequel $\mathcal{A}$ operates always from the left unless explicitly stated differently. A morphism of finitely generated Hilbert $\mathcal{A}$-modules is a bounded $\mathcal{A}$-operator. Let $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ be the category of finitely generated Hilbert $\mathcal{A}$-modules. This category plays an important role in the construction of L^2-invariants of finite connected CW-complexes such as L^2-Betti numbers and Novikov-Shubin invariants. For a survey on L^2-(co)homology we refer for instance to [18], [32], [44]. More information about L^2-invariants can be found for instance in [1], [5], [8], [9] [13], [14], [17], [19], [24], [25] [26], [29], [30], [31], [34], [40], [43]. These constructions of L^2-invariants use the rich functional analytic structure. However, it is a consequence of standard facts about von Neumann algebras that they can be interpreted purely algebraically. Namely, we will prove (see Theorem 2.2) if $\{\text{fin. gen. proj. } \mathcal{A}\text{-mod.}\}$ denotes the category of finitely generated projective $\mathcal{A}$-modules

Theorem 0.1 *There is an equivalence of categories compatible with the complex vector space structures on the set of morphisms and the direct sums*

$$\nu^{-1} : \{\textit{fin. gen. Hilb. } \mathcal{A}\textit{-mod.}\} \longrightarrow \{\textit{fin. gen. proj. } \mathcal{A}\textit{-mod.}\}. \quad \square$$

This is quite convenient since $\mathcal{A}$ viewed as a ring has the following nice properties (see Corollary 2.4 and Theorem 1.2). For a $\mathcal{A}$-module M define

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in \hom_R(M, R)\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \end{aligned}$$

Theorem 0.2 *1. A finite von Neumann algebra $\mathcal{A}$ is semi-hereditary, i.e. any finitely generated submodule of a projective $\mathcal{A}$-module is projective;*

2. *The category $\{$fin. pres. $\mathcal{A}$-mod.$\}$ of finitely presented $\mathcal{A}$-modules is abelian, i.e. the kernel, the image and the cokernel of a $\mathcal{A}$-linear map of finitely presented $\mathcal{A}$-modules is again finitely presented;*
3. *A $\mathcal{A}$-module is finitely presented if and only if it has a 1-dimensional finitely generated projective $\mathcal{A}$-resolution;*
4. *Let M be a finitely presented $\mathcal{A}$-module. Then $\mathbf{PT}M = \mathbf{TP}M = 0$ and M is finitely generated projective if and only if $\mathbf{T}M = 0$. In particular $\mathbf{P}M$ is finitely generated projective and the exact sequence $0 \longrightarrow \mathbf{T}M \longrightarrow M \longrightarrow \mathbf{P}M \longrightarrow 0$ splits.* □

Hence the general strategy is to read off L^2-invariants of a finitely generated Hilbert $\mathcal{A}$-chain complex C from the homology of the finitely generated projective $\mathcal{A}$-chain complex $\nu^{-1}(C)$ associated to C by Theorem 0.1. Namely, one can define for a finitely presented $\mathcal{A}$-module M invariants

$$\begin{aligned} \dim(M) &\in \mathbb{R}^{\geq 0}; \\ \alpha(M) &\in [0,\infty] \amalg \{\infty^+\}; \end{aligned}$$

and show (see Theorem 5.4)

Theorem 0.3 *Let C be a finitely generated Hilbert $\mathcal{A}$-chain complex. Then we get for the L^2-Betti numbers $b_p^{(2)}(C)$ and the Novikov-Shubin invariants $\alpha_p(C)$:*

$$\begin{aligned} b_p^{(2)}(C) &= \dim(\mathbf{P}H_p(\nu^{-1}(C))); \\ \alpha_p(C) &= \alpha(\mathbf{T}H_{p-1}(\nu^{-1}(C))). \quad \square \end{aligned}$$

We will actually consider more refined invariants than the L^2-Betti numbers and Novikov-Shubin invariants by substituting the $\mathbb{C}$-valued trace tr by the center-valued trace tr^u. We obtain in Sect. 3 for each finitely presented $\mathcal{A}$-module M and finitely generated Hilbert $\mathcal{A}$-chain complex C invariants

$$\begin{aligned} \dim^u(M) &\in Z(\mathcal{A}); \\ b_p^u(C) &\in Z(\mathcal{A}); \\ \omega^u(M) &\in \mathcal{D}(\mathcal{A}); \\ \omega^u(C) &\in \mathcal{D}(\mathcal{A}), \end{aligned}$$

where the definition of $\mathcal{D}(\mathcal{A})$ will be given in Definition 3.8. Theorem 0.3 extends to these refined invariants and the refined invariants determine the others. We use the refined invariants to show (see Corollary 3.2, Lemma 3.3 and Example 4.3)

Theorem 0.4 *1. Two finitely generated projective $\mathcal{A}$-modules P and Q are $\mathcal{A}$-isomorphic if and only if* $\dim^u(P) = \dim^u(Q)$*;*
2. If $\mathcal{A}$ is of type II_1, for instance if $\mathcal{A}$ is the von Neumann algebra of a finitely generated group which does not contain $\mathbb{Z}^n$ as subgroup of finite index, then any element in $Z(\mathcal{A})^+$ can be realized as $\dim^u(P)$ *for a finitely generated projective $\mathcal{A}$-module;*
3. Let $S \subset \mathbb{C}[\mathbb{Z}]$ be the multiplicative subset of elements in $\mathbb{C}[\mathbb{Z}]$ which are invertible in $L^\infty(S^1)$. Let M and N be finitely generated $S^{-1}\mathbb{C}[\mathbb{Z}]$-modules. Then M and N are isomorphic as $S^{-1}\mathbb{C}[\mathbb{Z}]$-modules if and only if the finitely presented $L^\infty(S^1)$-modules $M \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1)$ and $N \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1)$ are isomorphic as $L^\infty(S^1)$-modules. The $L^\infty(S^1)$-isomorphism type of $M \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1)$ resp. of $\mathbf{T}M \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1)$ is determined by its elementary ideals resp. by ω^u. □

Let Γ be a discrete group and X be a finite Γ-CW-complex with finite isotropy groups. Examples for X are universal coverings of compact CW-complexes with Γ as fundamental group and smooth manifolds with smooth cocompact proper Γ-action. We will define L^2-Betti numbers $b_p^{(2)}(X) = b_p^{(2)}(X; l^2(\Gamma))$ and Novikov-Shubin invariants $\alpha_p(X) = \alpha(X; l^2(\Gamma))$ in Sect. 6 using the cellular L^2-chain complex. In Sect. 6 we will briefly discuss singular homology, universal coefficient spectral sequences, the Leray-Serre spectral sequence, Poincaré duality, Morse inequalities, L^2-torsion, the “Zero in the spectrum” conjecture and will show (see Theorem 6.3 and Example 4.3)

Theorem 0.5 *1. Suppose that Γ is the free abelian group $\mathbb{Z}^r$ of rank r. Let $\mathbb{C}\mathbb{Z}^r_{(0)}$ be the quotient field of $\mathbb{C}\mathbb{Z}^r$. Then*

$$b_p^{(2)}(X; l^2(\mathbb{Z}^r)) \;=\; \dim_{\mathbb{C}\mathbb{Z}^r_{(0)}} \left(H_p(X; \mathbb{C}) \otimes_{\mathbb{C}\mathbb{Z}^r} \mathbb{C}\mathbb{Z}^r_{(0)}\right);$$

2. Suppose that Γ is $\mathbb{Z}$. Choose $t \in \mathbb{C}[\mathbb{Z}]$ such that the principal ideal generated by t is the annihilator of the torsion submodule of $H_p(X; \mathbb{C})$. Consider t as a polynomial in z and z^{-1}. If the torsion submodule is trivial or t has no roots on S^1, then $\alpha_p(X; l^2(\mathbb{Z})) = \infty^+$. Otherwise we get for the the highest order a of all roots of t on S^1

$$\alpha_p(X; l^2(\mathbb{Z})) = \frac{1}{a}. \quad \square$$

John Lott has given in the case $\Gamma = \mathbb{Z}^r$ an expression of the Novikov-Shubin invariants in terms of Massey products [24, pages 495 - 496].

In Sect. 7 we will generalize results of [30, Theorem 3.1]. We will prove in Sect. 7

Theorem 0.6 *Let $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_{n+1} = \Gamma$ be a nested sequence of subgroups of Γ for an integer $n \geq 0$. Suppose that Γ_1 is infinite, Γ_i is normal in Γ_{i+1}, the quotient Γ_{i+1}/Γ_i contains $\mathbb{Z}$ as subgroup for $1 \leq i \leq n$ and $B\Gamma_i$ has finite i-skeleton for $1 \leq i \leq n+1$. Then*

$$b_p^{(2)}(\Gamma) := b_p^{(2)}(E\Gamma; l^2(\Gamma)) = 0 \qquad \textit{for } p \leq n. \quad \square$$

We mention the following consequence which follows from Theorem 0.6 using the same arguments as in the proofs of [30, Theorem 4.1, Theorem 5.1 and Corollary 6.2]. Recall that the deficiency $\mathrm{def}(\Gamma)$ of a finitely presented group Γ is the maximum over all differences $g - r$ where g resp. r is the number of generators resp. relations in any presentation of Γ. It is known that this maximum always exists.

Theorem 0.7 *Let $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1$ be an exact sequence of groups such that Δ is finitely generated and infinite, Γ is finitely presented and $\mathbb{Z}$ is a subgroup of π. Then:*

1. $b_1^{(2)}(\Gamma) = 0$;
2. $\mathrm{def}(\Gamma) \leq 1$;
3. *Let M be a connected closed orientable 4-manifold with Γ as fundamental group. Then we get for its signature $\mathrm{sign}(M)$ and Euler characteristic $\chi(M)$*

$$|\mathrm{sign}(M)| \leq \chi(M). \quad \square$$

We mention that in the version of this corollary in [30] the stronger assumption was needed that Δ is finitely presented. If M is a closed orientable 4-dimensional Einstein manifold, then the sharper inequality

$$\frac{3}{2} \cdot |\mathrm{sign}(M)| \leq \chi(M)$$

holds [20].

Finally we mention the following observation about Thompson's group F. It is the group of orientation preserving dyadic PL-automorphisms of $[0, 1]$ where dyadic means that all slopes are integral powers of 2 and the break points are contained in $\mathbb{Z}[1/2]$. It has the presentation

$$F = \langle x_0, x_1, x_2, \ldots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.$$

This group has some very interesting properties. It is not elementary amenable and does not contain a subgroup which is free on two generators [3], [7]. Hence it is a very interesting question whether F is amenable. Since BF is of finite type [4], the L^2-Betti numbers $b_p^{(2)}(F)$ are defined for all $p \geq 0$. We conclude from [9, Lemma 3.1 on page 203] and $H^i(EF; \mathbb{C}) = 0$ for $i \geq 1$ that a necessary condition for F to be amenable is that $b_p^{(2)}(F)$ vanishes for all $p \geq 0$. This motivates the following result whose proof is given at the end of Sect. 7:

Theorem 0.8 *All L^2-Betti numbers $b_p^{(2)}(F)$ of Thompson's group F vanish.* $\square$

The paper of Farber [15] is related to this paper as follows. Farber constructs a category $\mathcal{E}(\mathcal{A})$ which contains the category $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ as subcategory. The main point is that it is an abelian category, it is a kind of abelian extension of $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ in the sense of [16]. An object in $\mathcal{E}(\mathcal{A})$ is a map of finitely generated Hilbert $\mathcal{A}$-modules $(\alpha : A' \longrightarrow A)$. A morphism in $\mathcal{E}(\mathcal{A})$ from $(\alpha : A' \longrightarrow A)$ to $(\beta : B' \longrightarrow B)$ is an equivalence class of maps

$f : A \longrightarrow B$ such that there exists a map $g : A' \longrightarrow B'$ with $f \circ \alpha = \beta \circ g$. Here f and f' are called equivalent if and only if $f - f' = \beta \circ h$ for some morphism $h : A \longrightarrow B'$. The embedding of {fin. gen. Hilb. $\mathscr{A}$-mod.} into $\mathscr{E}(\mathscr{A})$ is used in [15] to prove an analogue of Theorem 0.3 and to give improved L^2-Morse inequalities. The two approaches are unified by the following result which will be proved at the end of Sect. 2.

Theorem 0.9 *There is an equivalence of abelian categories*

$$\overline{\nu^{-1}} : \mathscr{E}(\mathscr{A}) \longrightarrow \{\textit{fin. pres. } \mathscr{A}\textit{-mod.}\}$$

which induces the equivalence appearing in Theorem 0.1

$$\nu^{-1} : \{\textit{fin. gen. Hilb. } \mathscr{A}\textit{-mod.}\} \longrightarrow \{\textit{fin. gen. proj. } \mathscr{A}\textit{-mod.}\}. \quad \square$$

The paper is organized as follows :

1 Semi-hereditary rings

In this section we explain some elementary properties of semi-hereditary rings and of finitely presented modules over them.

Ring will always mean associative ring with unit and R-module will mean left R-module unless explicitly stated differently. An *involution* on R is a map $* : R \longrightarrow R \quad r \mapsto r^*$ which satisfies $(r+s)^* = r^* + s^*$, $(rs)^* = s^* r^*$ and $(r^*)^* = r$ and $1^* = 1$ for all $r, s \in R$. Everything in this section does also make sense without the involution if one is careful with left and right modules. We call R *semi-hereditary* if each finitely generated ideal is projective. This property implies that each finitely generated submodule of a projective R-module is projective [6, Proposition I.6.2. on page 15]. Recall that the dual M^* of a (left) R-module is the (left) R-module $\hom_R(M, R)$ where the R-multiplication is given by $(rf)(x) = f(x)r^*$ for $f \in M^*$, $x \in M$ and $r \in R$.

Definition 1.1 *Let M be a R-submodule of N. Define the* **closure of** *M in N to be the R-submodule of N*

$$\overline{M} = \{x \in N \mid f(x) = 0 \textit{ for all } f \in N^* \textit{ with } M \subset \ker(f)\}.$$

For a R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \quad \square \end{aligned}$$

Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It can also be described as the kernel of the canonical map $i(M) : M \longrightarrow (M^*)^*$ which sends $x \in M$ to the map $M^* \longrightarrow R \quad f \mapsto f(x)^*$. Notice that $\mathbf{TP}M = 0$ and that $\mathbf{P}M = 0$ is equivalent to $M^* = 0$.

Let R be a commutative ring without zero-divisors and M be a finitely generated R-module. Then the torsion submodule of M coincides with $\mathbf{T}M$ since any torsion-free finitely generated R-module can be embedded into R^n for appropriate n [2, Proposition 3.3 in Chap. 9 on page 321]. If M is not finitely generated, this is not true in general as the example $R = \mathbb{Z}$ and $M = \mathbb{Q}$ shows because $\mathbb{Q}$ as abelian group is torsionfree and satisfies $\mathbf{T}\mathbb{Q} = \mathbb{Q}$. If we suppose additionally that R is semi-hereditary then $\mathbf{P}M$ is projective, provided M is finitely generated. If R is commutative but has zero-divisors, then in general the torsion submodule of a finitely generated R-module M does not agree with $\mathbf{T}M$. Our main example of a semi-hereditary ring will be any finite von Neumann algebra $\mathscr{A}$ (see Corollary 2.4). Notice that in general $\mathscr{A}$ has zero-divisors, is not noetherian and has finitely generated modules M which are not finitely-presented. An example is $\mathscr{A} = L^\infty(S^1)$ and $M = L^\infty(S^1)/\bigcup_{n\geq 1}(\chi_n)$ where (χ_n) is the ideal generated by the characteristic function of the subset $\{\exp(2\pi it) \mid t \in [0, 1-1/n]\}$ of S^1. Namely, $\bigcup_{n\geq 1}(\chi_n)$ cannot be finitely generated as $(\chi_n) \neq (\chi_{n+1})$ holds for all n and hence M is not finitely presented because the kernel of an epimorphism of a finitely generated module onto a finitely presented module is always finitely generated.

Theorem 1.2 *Let R be a semi-hereditary ring with involution. Then:*

1. *The following statements are equivalent for a R-module M:*
   a) *M is finitely presented;*
   b) *M has a 1-dimensional finitely generated projective R-resolution;*
   c) *M is of type FP, i.e. possesses a finite-dimensional finitely generated projective R-resolution;*
   d) *M is of type FP_∞, i.e. possesses a finitely generated projective R-resolution;*
2. *If $f : M \longrightarrow N$ is a R-map of finitely presented R-modules, then its kernel, image and cokernel are again finitely presented;*
3. *If M is a finitely presented R-module, then $\mathbf{P}M$ is a finitely generated projective R-module. If M is a finitely generated projective R-module, then $\mathbf{P}M = M$.*

Proof. 1.) Is obvious.

2.) If $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is an exact sequence of R-modules and two of the three are of type FP, then all three are of type FP [28, Theorem 11.2.c on page 212]. Hence it suffices to show for a R-map $f : M \longrightarrow N$ of finitely presented R-modules that its image is finitely presented. Let $F_1 \xrightarrow{g} F_0 \xrightarrow{p} N \longrightarrow 0$ be a finite presentation. Then $F_1 \xrightarrow{g} p^{-1}(\operatorname{im}(f)) \xrightarrow{p} \operatorname{im}(f) \longrightarrow 0$ is exact. Since $p^{-1}(\operatorname{im}(f))$ is a finitely generated submodule of a free R-module, it is projective. Let Q be a finitely generated projective R-module such that $Q \oplus p^{-1}(\operatorname{im}(f)) = F_0'$ for a finitely generated free R-module F_0'. Then one easily constructs an exact sequence of R-modules $F_1 \oplus F_0' \longrightarrow F_0' \longrightarrow \operatorname{im}(f) \longrightarrow 0$. Hence $\operatorname{im}(f)$ is finitely presented.

3.) Choose a finite presentation $R^m \xrightarrow{f} R^n \xrightarrow{p} M \longrightarrow 0$ of the R-module M. Then the sequence $0 \longrightarrow M^* \longrightarrow (R^n)^* \xrightarrow{f^*} (R^m)^*$ is exact. Since R is semi-hereditary the image of f^* and hence M^* are finitely generated projective. The projection $\mathrm{pr} : M \longrightarrow \mathbf{P}M$ induces an isomorphism $\mathrm{pr}^* : \mathbf{P}M^* \longrightarrow M^*$. Since $\mathbf{TP}M$ is trivial, the canonical map $i(\mathbf{P}M) : \mathbf{P}M \longrightarrow (\mathbf{P}M^*)^*$ is an embedding of a finitely generated R-module into a finitely generated projective R-module. Hence $\mathbf{P}M$ is projective. This finishes the proof of Theorem 1.2. □

Define for any chain complex C its *dual R-cochain complex* C^* by the R-cochain complex whose n-th R-cochain module is C_n^*. Given a R-module M, define $M^\frown$ by $\operatorname{Ext}^1_R(M, R)$. If $\mathbf{P}M = 0$ and M has a 1-dimensional projective resolution P then P^* defines a 1-dimensional projective resolution for $M^\frown$ and we get a canonical isomorphism $M \longrightarrow (M^\frown)^\frown$.

Lemma 1.3 *Let R be a semi-hereditary ring with involution. Let C be a finitely generated projective R-chain complex. Then there is a in C natural exact sequence*

$$0 \longrightarrow \big(\mathbf{T}H_{n-1}(C)\big)^\frown \longrightarrow H^n(C^*) \longrightarrow (\mathbf{P}H_n(C))^* \longrightarrow 0.$$

The sequence splits (but not naturally). In particular we obtain in C natural R-isomorphisms $\big(\mathbf{T}H_{n-1}(C)\big)^\frown \longrightarrow \mathbf{T}H^n(C^*)$ and $\mathbf{P}H^n(C^*) \longrightarrow (\mathbf{P}H_n(C))^*$.

Proof. Notice that $\big(\mathbf{T}H_{n-1}(C)\big)^\frown$ is canonically isomorphic to $\operatorname{Ext}^1_R(H_{n-1}(C), R)$ and $H_n(C)^*$ is canonically isomorphic to $(\mathbf{P}H_n(C))^*$. Under these identifications the proof for instance in [45, Theorem 13.10 on page 240] for free abelian chain complexes goes through directly. □

2 Finitely generated Hilbert $\mathscr{A}$-modules and finitely generated projective $\mathscr{A}$-modules

In this section we identify the categories of finitely generated Hilbert $\mathscr{A}$-modules and of finitely generated projective $\mathscr{A}$-modules and show that any finite von Neumann algebra is semi-hereditary.

For the rest of this paper let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathrm{tr}: \mathcal{A} \longrightarrow \mathbb{C}$ be some finite normal faithful trace. Recall that a von Neumann algebra is finite if and only if it possesses such a trace. Let $l^2(\mathcal{A})$ be the Hilbert space completion of $\mathcal{A}$ which is viewed as a pre-Hilbert space by the inner product $\langle a, b\rangle = \mathrm{tr}(ab^*)$. A *finitely generated Hilbert $\mathcal{A}$-module V* is a Hilbert space V together with a left operation of $\mathcal{A}$ by $\mathbb{C}$-linear maps such that $\lambda \cdot 1_{\mathcal{A}}$ acts by scalar multiplication with λ on V for $\lambda \in \mathbb{C}$ and there exists a unitary $\mathcal{A}$-embedding of V in $\oplus_{i=1}^n l^2(\mathcal{A})$ for some n. A morphism of finitely generated Hilbert $\mathcal{A}$-modules is a bounded $\mathcal{A}$-operator. Denote by $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ the category of finitely generated Hilbert $\mathcal{A}$-modules. For a survey on finite von Neumann algebras and Hilbert $\mathcal{A}$-modules we refer for instance to [34, Sect. 1].

A *$\mathbb{C}$-category $\mathcal{C}$* is a category such that for each two objects the set of morphisms between them carries the structure of a complex vector space for which composition of morphisms is bilinear and that $\mathcal{C}$ has a (strict) sum which is compatible with the complex vector space structures above. A *(strict) involution* on a $\mathbb{C}$-category $\mathcal{C}$ is an assignment which associates to each morphism $f: x \longrightarrow y$ a morphism $f^*: y \longrightarrow x$ and has the following properties

$$\begin{aligned}
(f^*)^* &= f; \\
(\lambda \cdot f + \mu \cdot g)^* &= \overline{\lambda} \cdot f^* + \overline{\mu} \cdot g^*; \\
(f \circ g)^* &= g^* \circ f^*; \\
(f \oplus g)^* &= f^* \oplus g^*;
\end{aligned}$$

where f,g are morphisms, λ and μ complex numbers. There is a canonical structure of a $\mathbb{C}$-category with involution on $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ where the involution is given by taking adjoint operators. We call an endomorphism resp. isomorphism f in $\mathcal{C}$ *selfadjoint* resp. *unitary* if $f = f^*$ resp. $f^* = f^{-1}$ holds. A *functor of $\mathbb{C}$-categories with involution* is a functor compatible with the complex vector space structures on the morphisms, the sums and the involutions. A natural equivalence T of functors of $\mathbb{C}$-categories with involution is called *unitary* if the evaluation of T at each object is a unitary isomorphism. An *equivalence of $\mathbb{C}$-categories with involution* is a functor of such categories such that there is a functor of such categories in the other direction with the property that both compositions are unitarily naturally equivalent to the identity.

Given a finitely generated projective $\mathcal{A}$-module P, an *inner product* on P is a map $\mu: P \times P \longrightarrow \mathcal{A}$ satisfying (cf [48, Definition 15.1.1 on page 232])

1. μ is $\mathcal{A}$-linear in the first variable;
2. μ is symmetric in the sense $\mu(x, y) = \mu(y, x)^*$;
3. μ is positive-definite in the sense that $\mu(p, p)$ is a positive element in $\mathcal{A}$, i.e. of the form a^*a for some $a \in \mathcal{A}$, and $\mu(p, p) = 0 \Longleftrightarrow p = 0$;
4. The induced map $\overline{\mu}: P \longrightarrow P^* = \hom_{\mathcal{A}}(P, \mathcal{A})$ defined by $\overline{\mu}(y)(x) = \mu(x, y)$ is bijective.

Notice that $\overline{\mu}$ is a $\mathcal{A}$-isomorphism such that the composition $P \xrightarrow{i(P)} (P^*)^* \xrightarrow{\overline{\mu}^*} P^*$ is $\overline{\mu}$. Let $\{$fin. gen. proj. $\mathcal{A}$-mod. with $\langle\ \rangle\}$ be the $\mathbb{C}$-category whose objects are finitely generated projective $\mathcal{A}$-modules with inner product (P, μ) and whose morphisms are $\mathcal{A}$-linear maps. We get an involution on it if we specify $f^* : (P_1, \mu_1) \longrightarrow (P_0, \mu_0)$ for $f : (P_0, \mu_0) \longrightarrow (P_1, \mu_1)$ by requiring $\mu_1(fx, y) = \mu_0(x, f^*(y))$ for all $x \in P_0$ and $y \in P_1$. In other words, we define $f^* := \overline{\mu_0}^{-1} \circ f^* \circ \overline{\mu_1}$ where the second f^* refers to the $\mathcal{A}$-map $f^* = \hom_{\mathcal{A}}(f, \mathrm{id}) : P_1^* \longrightarrow P_0^*$. In the sequel we will use the symbol f^* for both $f^* : P_1 \longrightarrow P_0$ and $f^* : P_1^* \longrightarrow P_0^*$.

Given a finitely generated projective $\mathcal{A}$-module (P, μ) with inner product μ, we obtain a pre-Hilbert structure on P by $\mathrm{tr} \circ \mu : P \times P \longrightarrow \mathbb{C}$. Let $\nu(P)$ be the associated Hilbert space. We will show later that this is a finitely generated Hilbert $\mathcal{A}$-module and that any $\mathcal{A}$-linear map $P_0 \longrightarrow P_1$ of finitely generated projective $\mathcal{A}$-modules P_i with inner products μ_i extends to a morphism of Hilbert $\mathcal{A}$-modules $\nu(f) : \nu(P_0, \mu_0) \longrightarrow \nu(P_1, \mu_1)$. Moreover, we will prove

Theorem 2.1 *1. The functor*

$$\nu : \{\text{fin. gen. proj. } \mathcal{A}\text{-mod. with } \langle\ \rangle\} \longrightarrow \{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$$

is an equivalence of $\mathbb{C}$-categories with involutions;

2. *Any finite generated projective $\mathcal{A}$-module has an inner product. Two finitely generated projective $\mathcal{A}$-modules with inner product are unitarily $\mathcal{A}$-isomorphic if and only if the underlying $\mathcal{A}$-modules are $\mathcal{A}$-isomorphic.*

Proof. Put $\mathcal{A}^n = \oplus_{i=1}^n \mathcal{A}$ and $l^2(\mathcal{A})^n = \oplus_{i=1}^n l^2(\mathcal{A})$. Let $\{\mathcal{A}^n\}$ resp. $\{l^2(\mathcal{A})^n\}$ be the full subcategory of $\{$fin. gen. proj. $\mathcal{A}$-mod. with $\langle\ \rangle\}$ resp. $\{$fin. gen. Hilb. $\mathcal{A}$-mod.$\}$ whose objects are $\mathcal{A}^n$ resp. $l^2(\mathcal{A})^n$ for $n = 0, 1, 2 \ldots$. Here we equip $\mathcal{A}^n$ with the standard inner product

$$\mu_{st}((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \sum_{i=1}^n a_i b_i^*.$$

Now ν as defined above yields a well-defined isomorphism of $\mathbb{C}$-categories with involutions

$$\nu : \{\mathcal{A}^n\} \longrightarrow \{l^2(\mathcal{A})^n\}$$

because the right regular representation $\mathcal{A} \longrightarrow \mathcal{B}(l^2(\mathcal{A}), l^2(\mathcal{A}))^{\mathcal{A}}$ from $\mathcal{A}$ into the space of bounded $\mathcal{A}$-operators from $l^2(\mathcal{A})$ to itself sending $a \in \mathcal{A}$ to the extension of the map $\mathcal{A} \longrightarrow \mathcal{A} \quad b \mapsto ba^*$ is known to be well-defined and bijective [12, Theorem 1 in I.5.2 on page 80, Theorem 2 in I.6.2 on page 99].

The *idempotent completion* $\mathrm{Idem}(\mathcal{C})$ of a $\mathbb{C}$-category $\mathcal{C}$ with involution has as objects (V, p) selfadjoint idempotents $p : V \longrightarrow V$. A morphism from (V_0, p_0) to (V_1, p_1) is a morphism $f : V_0 \longrightarrow V_1$ satisfying $p_1 \circ f \circ p_0 = f$. The identity on (V, p) is given by $p : (V, p) \longrightarrow (V, p)$. The idempotent completion $\mathrm{Idem}(\mathcal{C})$ inherits from $\mathcal{C}$ the structure of a $\mathbb{C}$-category with involution in the obvious way. There are functors of $\mathbb{C}$-categories with involutions

$$\begin{aligned} \mathrm{IM} : \mathrm{Idem}(\{\mathcal{A}^n\}) &\longrightarrow \{\text{fin. gen. proj. } \mathcal{A}\text{-mod. with } \langle\ \rangle\}; \\ \mathrm{IM} : \mathrm{Idem}(\{l^2(\mathcal{A})^n\}) &\longrightarrow \{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}; \end{aligned}$$

which send $(\mathcal{A}^n, p)$ resp. $(l^2(\mathcal{A})^n, p)$ to the image of p where the inner product μ_{st} on $\mathrm{im}(p)$ is given by restricting the standard inner product μ_{st} on $\mathcal{A}^n$.

Let P be a finitely generated projective $\mathcal{A}$-module. It is isomorphic to $\mathrm{im}(q)$ for some idempotent $q : \mathcal{A}^n \longrightarrow \mathcal{A}^n$. Define $p : \mathcal{A}^n \longrightarrow \mathcal{A}^n$ to be the idempotent for which $\nu(p) : l^2(\mathcal{A})^n \longrightarrow l^2(\mathcal{A})^n$ is the orthogonal projection onto the image of $\nu(q)$. Because of $\nu((\mathrm{id} - q) \circ p) = 0$ we get $(\mathrm{id} - q) \circ p = 0$. Similiarly we get $(\mathrm{id} - p) \circ q = 0$. This implies that q and p have the same image. Hence P is isomorphic to $\mathrm{im}(p)$ for some selfadjoint idempotent $p : \mathcal{A}^n \longrightarrow \mathcal{A}^n$.

Next we prove the second assertion. Because of the argument above it suffices to check the claim for $P = \mathrm{im}(p)$ for a selfadjoint idempotent $p : \mathcal{A}^n \longrightarrow \mathcal{A}^n$. The standard inner product on $\mathcal{A}^n$ induces a standard inner product $\mu_{st} : P \times P \longrightarrow \mathcal{A}$. It remains to show for a second inner product $\mu : P \times P \longrightarrow \mathcal{A}$ that there is a isometric $\mathcal{A}$-isomorphism $g : (P, \mu) \longrightarrow (P, \mu_{st})$. Let $f : P \longrightarrow P$ be the $\mathcal{A}$-automorphism uniquely determined by the property that $\mu(x, y) = \mu_{st}(f(x), y)$ holds for all $x, y \in P$. Let f^* be the adjoint of f with respect to μ_{st}. Then the following calculation shows that f is positive with respect to μ_{st}:

$$\mu_{st}(x, f(y)) = \mu_{st}(f(y), x)^* = \mu(y, x)^* = \mu(x, y) = \mu_{st}(f(x), y).$$

$$\mu_{st}(f(x), x) = \mu(x, x) \geq 0.$$

Let $g' : \mathcal{A}^n \longrightarrow \mathcal{A}^n$ be defined by the property that $\nu(g') : l^2(\mathcal{A})^n \longrightarrow l^2(\mathcal{A})^n$ is positive and $\nu(g') \circ \nu(g') = \nu(i \circ f \circ p)$ where $i : P \longrightarrow \mathcal{A}^n$ is the inclusion which is the adjoint of $p : \mathcal{A}^n \longrightarrow P$ with respect to μ_{st}. Define $g : P \longrightarrow P$ by $p \circ g' \circ i$. Then g is selfadjoint with respect to μ_{st} and $g^2 = f$. This implies

$$\mu_{st}(g(x), g(y)) = \mu_{st}(g^2(x), y) = \mu_{st}(f(x), y) = \mu(x, y).$$

This finishes the proof of the second assertion.

Next we can conclude for any $\mathcal{A}$-linear map $f : (P_0, \mu_0) \longrightarrow (P_1, \mu_1)$ of finitely generated projective $\mathcal{A}$-modules that the Hilbert space completion $\nu(P_i, \mu_i)$ is a finitely generated Hilbert $\mathcal{A}$-module and f extends to a bounded $\mathcal{A}$-operator $\nu(P_0, \mu_0) \longrightarrow \nu(P_1, \mu_1)$. Because of assertion 2.) one can reduce the claim to the case $(P_i, \mu_i) = (\mathcal{A}^{n_i}, \mu_{st})$ which we have already dealt with. Hence $\nu : \{\text{fin. gen. proj. } \mathcal{A}\text{-mod. with } \langle\ \rangle\} \longrightarrow \{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$ is well-defined. It is obviously a functor of $\mathbb{C}$-categories with involution. It remains to show that it is an equivalence of $\mathbb{C}$-categories with involutions.

The following diagram commutes

$$\begin{array}{ccc} \mathrm{Idem}(\{\mathcal{A}^n\}) & \xrightarrow{\mathrm{Idem}(\nu)} & \mathrm{Idem}(\{l^2(\mathcal{A})^n\}) \\ \downarrow{\scriptstyle \mathrm{IM}} & & \downarrow{\scriptstyle \mathrm{IM}} \\ \{\text{fin. gen. proj. } \mathcal{A}\text{-mod. with } \langle\ \rangle\} & \xrightarrow[\nu]{} & \{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\} \end{array}$$

The upper horizontal arrow is induced by the isomorphism of $\mathbb{C}$-categories with involutions $\nu : \{\mathscr{A}^n\} \longrightarrow \{l^2(\mathscr{A})^n\}$. Hence it suffices to show that the vertical arrows are equivalences of $\mathbb{C}$-categories with involutions. This would follow if IM is full, i.e. induces an epimorphism on the set of unitary isomorphism classes of objects, and faithful, i.e. induces isomorphisms on the set of morphisms between to arbitrary objects onto the set of morphisms between the images of these objects (cf. [39, Theorem 1 on page 91]). Obviously IM is faithful in both cases. It is full in the second case of finitely generated Hilbert $\mathscr{A}$-modules since by definition of a finitely generated $\mathscr{A}$-module V there is a selfadjoint idempotent $p : l^2(\mathscr{A})^n \longrightarrow l^2(\mathscr{A})^n$ whose image is unitarily $\mathscr{A}$-isomorphic to V. It is full in the first case of finitely generated projective $\mathscr{A}$-modules because of the second assertion. This finishes the proof of Theorem 2.1. □

A sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of finitely generated Hilbert $\mathscr{A}$-modules is *weakly exact at* V resp. *exact at* V if $\overline{\operatorname{im}(f)} = \ker(g)$ resp. $\operatorname{im}(f) = \ker(g)$ holds. The definition for finitely generated projective $\mathscr{A}$-modules is analogous where the notion of closure of Definition 1.1 is used. In the sequel ν^{-1} is an inverse of ν which is well-defined up to unitary natural equivalence by Theorem 2.1. Theorem 2.1 implies

Theorem 2.2 *The composition of ν^{-1} with the forgetful functor induces an equivalence of $\mathbb{C}$-categories*

$$\{\textit{fin. gen. Hilb. } \mathscr{A}\textit{-mod.}\} \longrightarrow \{\textit{fin. gen. proj. } \mathscr{A}\textit{-mod.}\}. \quad \square$$

Lemma 2.3 *ν and ν^{-1} preserve weak exactness and exactness.*

Proof. A sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of finitely generated Hilbert $\mathscr{A}$-modules is weakly exact at V if and only if the following holds: $g \circ f = 0$ and for any finitely generated Hilbert $\mathscr{A}$-modules P and Q and morphisms $u : V \longrightarrow P$ and $v : Q \longrightarrow V$ with $u \circ f = 0$ and $g \circ v = 0$ we get $u \circ v = 0$. It is exact at V if and only if the following holds: $g \circ f = 0$ and for any finitely generated Hilbert $\mathscr{A}$-module P and morphism $v : P \longrightarrow V$ with $g \circ v = 0$ there is a morphism $u : P \longrightarrow U$ satisfying $f \circ u = v$. The same is true if one considers finitely generated projective $\mathscr{A}$-modules instead of finitely generated Hilbert $\mathscr{A}$-modules. Now ν and ν^{-1} obviously preserve these criterions for weak exactness and exactness and the claim follows. □

Corollary 2.4 *A finite von Neumann algebra $\mathscr{A}$ is semi-hereditary.*

Proof. Let $M \subset \mathscr{A}$ be a finitely generated ideal in $\mathscr{A}$. Choose a $\mathscr{A}$-map $f : \mathscr{A}^n \longrightarrow \mathscr{A}$ whose image is M. It suffices to show that $\ker(f)$ is a direct summand. Let $p : \mathscr{A}^n \longrightarrow \mathscr{A}^n$ be the $\mathscr{A}$-map for which $\nu(p)$ is an idempotent with $\ker(\nu(f))$ as image. Because of Lemma 2.3 p is an idempotent with $\ker(f)$ as image. □

Finally we give the promised proof of Theorem 0.9. We define

$$\overline{\nu^{-1}} : \mathscr{E}(\mathscr{A}) \longrightarrow \{\text{fin. pres. } \mathscr{A}\text{-mod.}\}$$

on objects by sending $(\alpha : A' \longrightarrow A)$ to the cokernel of $\nu^{-1}(\alpha)$. A morphism f in $\mathcal{E}(\mathcal{A})$ induces a $\mathcal{A}$-map in the obvious way. Clearly an object of the shape $0 \longrightarrow A$ is sent to $\nu^{-1}(A)$. One easily checks using standard homological algebra that $\overline{\nu^{-1}}$ is full and faithful. $\square$

3 Isomorphism invariants of finitely presented $\mathcal{A}$-modules

In this section we introduce some isomorphism invariants for finitely presented $\mathcal{A}$-modules. We will completely classify finitely generated projective $\mathcal{A}$-modules. This is a direct consequence of the following result which is taken from [21, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525 and Theorem 8.4.3. on page 532].

Theorem 3.1 *Let $\mathcal{A}$ be a finite von Neumann algebra on H. There is a map*

$$\mathrm{tr}^u = \mathrm{tr}^u_{\mathcal{A}} : \mathcal{A} \longrightarrow Z(\mathcal{A})$$

into the center $Z(\mathcal{A})$ of $\mathcal{A}$ called the center-valued trace *or* universal trace *of $\mathcal{A}$ uniquely determined by the following properties:*

1. tr^u *is $\mathbb{C}$-linear;*
2. *If $a \in \mathcal{A}$ is positive,* $\mathrm{tr}^u(a)$ *is positive;*
3. $\mathrm{tr}^u(ab) = \mathrm{tr}^u(ba)$ *for all $a, b \in \mathcal{A}$;*
4. $\mathrm{tr}^u(a) = a$ *for all $a \in Z(\mathcal{A})$.*

The map tr^u *has the following further properties:*

5. *If $a \in \mathcal{A}$ is positive and* $\mathrm{tr}^u(a) = 0$*, then $a = 0$;*
6. tr^u *is continuous in the ultraweak topology;*
7. $||\mathrm{tr}^u(a)|| \le ||a||$ *for $a \in \mathcal{A}$;*
8. $\mathrm{tr}^u(ab) = a\,\mathrm{tr}^u(b)$ *for all $a \in Z(\mathcal{A})$ and $b \in \mathcal{A}$;*
9. *Let p and q be projections in $\mathcal{A}$. Then $p \sim q$, i.e there is a partial isometry $u \in \mathcal{A}$ satisfying $p = uu^*$ and $q = u^*u$, if and only if* $\mathrm{tr}^u(p) = \mathrm{tr}^u(q)$*;*
10. *Any bounded linear functional $f : \mathcal{A} \longrightarrow \mathbb{C}$ which is central, i.e. $f(ab) = f(ba)$ for all $a, b \in \mathcal{A}$, factories as*

$$\mathcal{A} \xrightarrow{\mathrm{tr}^u} Z(\mathcal{A}) \xrightarrow{f|_{Z(\mathcal{A})}} \mathbb{C}. \quad \square$$

Define the *center-valued von Neumann dimension* of a finitely generated Hilbert $\mathcal{A}$-module V by

$$\dim^u(V) = \dim^u_{\mathcal{A}}(V) \ := \ \mathrm{tr}^u(p) \qquad \in Z(\mathcal{A})$$

where $p : l^2(\mathcal{A})^n \longrightarrow l^2(\mathcal{A})^n$ is any $\mathcal{A}$-projection whose image is isomorphic as finitely generated Hilbert $\mathcal{A}$-module to V and $\mathrm{tr}^u(p)$ is the sum of the traces

$\mathrm{tr}^u(A_{i,i})$ of the diagonal entries of the (n,n)-matrix A with entries in $\mathscr{A}$ given by p. We define for a finitely presented $\mathscr{A}$-module M

$$\dim^u(M) \ := \ \dim^u(\nu(\mathbf{P}M,\mu)) \qquad \in Z(\mathscr{A})$$

for any inner product μ on $\mathbf{P}M$. These two definitions are independent of the choices of p and μ by Theorem 2.1 and Theorem 3.1. We have $\dim^u(M) = \dim^u(\mathbf{P}M)$ by definition. Notice that the matrix algebra $M_k(\mathscr{A})$ is again a finite von Neumann algebra. We will extend this notion to arbitrary modules over finite von Neumann algebras in [33]. We get from Theorem 3.1

Corollary 3.2 *1. The following statements are equivalent for two finitely generated projective $\mathscr{A}$-modules P and Q:*

a) P and Q are $\mathscr{A}$-isomorphic;

b) P and Q are stably $\mathscr{A}$-isomorphic, i.e. $P \oplus V$ and $Q \oplus V$ are $\mathscr{A}$-isomorphic for some finitely generated projective $\mathscr{A}$-module V;

c) $\dim^u(P) = \dim^u(Q)$;

d) $[P] = [Q]$ in $K_0(\mathscr{A})$;

2. The center-valued dimension induces an injection

$$\dim^u : K_0(\mathscr{A}) \longrightarrow Z(\mathscr{A})^+ = \{a \in Z(\mathscr{A}) \ | \ a = bb^* \text{ for some } b \in \mathscr{A}\}.$$

If $\mathscr{A}$ is of type II, this map is an isomorphism (The image of $\dim^u$ is described in [21, Theorem 8.4.4 on page 533] in general). □

Notice that for a finite group π and $\mathscr{A}$ the associated von Neumann algebra which is $\mathbb{C}\pi$ in this case Corollary 3.2 reduces to the well-known fact that two finite-dimensional unitary π-representations are unitarily π-isomorphic if and only if they have the same character.

In this context we mention the computation of $K_1(\mathscr{A})$ and $K_1^w(\mathscr{A})$ (for any von Neumann algebra $\mathscr{A}$) in [35] and the following lemma. Recall that a finitely generated group Γ is virtually abelian if and only if it contains $\mathbb{Z}^r$ as normal subgroup of finite index for some $r \geq 0$. Let Γ_f be the normal subgroup of elements $\gamma \in \Gamma$ for which the set (γ) of elements conjugated to γ is finite. The definition of type I and type II_1 of a von Neumann algebra can be found in [21, Definition 6.5.1] and of type I_f means that the von Neumann algebra is finite and of type I.

Lemma 3.3 *Let Γ be a finitely generated (discrete) group. Then:*

1. The von Neumann algebra $\mathscr{N}(\Gamma)$ of Γ is a factor, i.e. its center is $\mathbb{C}$, if and only if Γ_f is trivial;

2. The von Neumann algebra $\mathscr{N}(\Gamma)$ is of type I_f if Γ is virtually abelian and of type II_1 otherwise.

Proof. 1.) follows from [12, Proposition 4 in III.7.6 on page 319].

2.) For a subgroup H of Γ define its *centralizer* by

$$C_H = \{\gamma \in \Gamma \ | \ \gamma h = h\gamma \text{ for all } h \in H\}.$$

Given elements $\gamma_1, \gamma_2, \ldots \gamma_r$ in Γ, we write $\langle \gamma_1, \gamma_2, \ldots \gamma_r \rangle$ for the subgroup of Γ generated by these elements. We abbreviate $C_\gamma = C_{\langle \gamma \rangle}$ for $\gamma \in \Gamma$. Notice that the set (γ) of elements in Γ which are conjugated to γ is finite if and only if C_γ has finite index in Γ. Obviously $C_{\delta^{-1}\gamma\delta} = \delta^{-1} C_\gamma \delta$. If $H_1 \subset H_2 \subset \Gamma$, then $C_{H_2} \subset C_{H_1}$. We have $C_{\langle \gamma_1, \gamma_2, \ldots \gamma_r \rangle} = \cap_{i=1}^r C_{\gamma_i}$.

Then Γ_f is the set of elements $\gamma \in \Gamma$ for which C_γ has finite index. This is a normal subgroup of Γ. We get from [47] that $\mathcal{N}(\Gamma)$ is type I_f if and only if Γ has a (normal) abelian subgroup of finite index and from [41] that $\mathcal{N}(\Gamma)$ is type II_1 if the index of Γ_f in Γ is infinite. See also [22] or [23]. Hence it remains to show for the finitely generated group Γ that the normal subgroup Γ_f has finite index if and only if Γ is virtually abelian.

Assume that Γ_f has finite index. A subgroup of finite index in a finitely generated group is again finitely generated. It suffices to prove this for a free group $*_r\mathbb{Z}$ of rank r. In this case, the claim follows from Schreier's Theorem [37, Proposition 3.8 and 3.9 on page 16]. Choose a set of generators $\gamma_1, \gamma_2, \ldots \gamma_r$ for Γ_f. We obtain a normal subgroup of finite index $\Gamma_f \cap C_{\Gamma_f} = \Gamma_f \cap \left(\cap_{i=1}^r C_{\gamma_i}\right)$. By definition of the centralizer, this group is abelian. Hence Γ is virtually abelian.

Assume that Γ contains $\mathbb{Z}^r$ as a normal subgroup of finite index. Then for all $\gamma \in \mathbb{Z}^r$ the centralizer C_γ contains $\mathbb{Z}^r$ and hence has finite index in Γ. This shows $\mathbb{Z}^r \subset \Gamma_f$. Hence Γ_f has finite index in Γ. □

Recall from Theorem 1.2 and Corollary 2.4 that any finitely presented $\mathcal{A}$-module M is isomorphic to $\mathbf{T}M \oplus \mathbf{P}M$ and that $\mathbf{P}M$ is finitely generated projective. In view of Corollary 3.2 we have a complete classification of finitely generated projective $\mathcal{A}$-modules. It remains to investigate finitely presented $\mathcal{A}$-modules M with $\mathbf{P}M = 0$.

A map of finitely generated Hilbert $\mathcal{A}$-modules $f : V \longrightarrow W$ is called a *weak isomorphism* if and only if it is injective and has dense image. By dimension theory this is true if and only if f is injective and V and W are isomorphic (see [9, Sect. 1]). Recall that a $\mathcal{A}$-module M of type FP, i.e. M possesses a finite-dimensional finitely generated projective $\mathcal{A}$-resolution, defines an element $[M] \in K_0(\mathcal{A})$ by $[M] = \sum_{n \geq 0} (-1)^n [P_n]$ for any choice of finite-dimensional finitely generated projective $\mathcal{A}$-resolution P of M. The next lemma is a direct consequence of the results above, Lemma 2.3 and Corollary 3.2.

Lemma 3.4 *The following assertions are equivalent for a finitely presented $\mathcal{A}$-module M:*

1. $\mathbf{P}M = 0$;
2. $\dim^u(M) = 0$;
3. $[M] = 0$ *in* $K_0(\mathcal{A})$;
4. *If P is a 1-dimensional finitely generated projective $\mathcal{A}$-resolution, then P_0 and P_1 are $\mathcal{A}$-isomorphic;*
5. *There is an exact sequence* $0 \longrightarrow \mathcal{A}^n \xrightarrow{c_1} \mathcal{A}^n \longrightarrow M \longrightarrow 0$ *with* $c_1^* = c_1$;

6. *If* $0 \longrightarrow P_1 \xrightarrow{c_1} P_0 \longrightarrow M \longrightarrow 0$ *is a* 1*-dimensional finitely generated projective* $\mathscr{A}$*-resolution of* M*, then* $\nu(c_1)$ *is a weak isomorphism for each choice of inner products on* P_0 *and* P_1*.* □

Now we can define an invariant for finitely presented $\mathscr{A}$-modules M with $\mathbf{P}M = 0$ under the assumption that $\mathscr{A}$ is of type I_f. Let

$$\det_{\text{norm}} : M(k,k,\mathscr{A}) \longrightarrow Z(\mathscr{A})$$

be the normalized determinant defined in [35, page 521]. If $\mathscr{A}$ is abelian, this is the ordinary determinant for commutative rings. Denote by $Z(\mathscr{A})^{\text{inv}}$ the multiplicative group of units in the center of $\mathscr{A}$. Denote by $Z(\mathscr{A})^w$ the Grothendieck group of the multiplicative abelian semigroup of elements $a \in Z(\mathscr{A})$ for which multiplication with a induces an injection $Z(\mathscr{A}) \longrightarrow Z(\mathscr{A})$. If we identify $Z(\mathscr{A})$ with $L^\infty(X,\mu)$ for some measure space (X,μ), we can identify $Z(\mathscr{A})^w$ with the multiplicative group $\text{Inv}(X,\mu)$, whose elements are measurable functions from X to $\mathbb{C} \cup \{\infty\}$, for which the preimages of 0 and ∞ are zero sets. In particular, the canonical map

$$Z(\mathscr{A})^{\text{inv}} \longrightarrow Z(\mathscr{A})^w$$

is injective.

Definition 3.5 *Suppose that* $\mathscr{A}$ *is of type* I_f*. Let* M *be a finitely presented* $\mathscr{A}$*-module. For any exact sequence of* $\mathscr{A}$*-modules* $0 \longrightarrow \mathscr{A}^n \xrightarrow{f} \mathscr{A}^n \longrightarrow \mathbf{T}M \longrightarrow 0$ *define*

$$\rho^u(M) := \det_{\text{norm}}(f) \qquad \in Z(\mathscr{A})^w / Z(\mathscr{A})^{\text{inv}}. \quad \square$$

The existence of the exact sequence $0 \longrightarrow \mathscr{A}^n \xrightarrow{f} \mathscr{A}^n \longrightarrow \mathbf{T}M \longrightarrow 0$ follows from Lemma 3.4. Using [35, Sect. 2] one can show that $\det_{\text{norm}}(f)$ takes value in $Z(\mathscr{A})^w$. By definition $\rho^u(M) = \rho^u(\mathbf{T}M)$ and $\rho^u(M)$ is trivial if M is finitely generated projective. The independence of $\rho^u(M)$ from the choice of f follows from the following lemma whose elementary proof is left to the reader.

Lemma 3.6 *Let* S *be a ring and* $P_1 \xrightarrow{f} P_0$ *and and* $Q_1 \xrightarrow{g} Q_0$ *be* 1*-dimensional projective* S*-resolutions of the same* S*-module. Then there is a commutative square*

$$\begin{array}{ccc} P_1 \oplus Q_1 & \xrightarrow{f \oplus \text{id}_{Q_1}} & P_0 \oplus Q_1 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Q_1 \oplus P_1 & \xrightarrow[g \oplus \text{id}_{P_1}]{} & Q_0 \oplus P_1 \end{array}$$

whose vertical arrows are isomorphisms. □

Notice that the definition of $\rho^u(M)$ was based on the existence of the normalized determinant and its basic properties. One could try to get another invariant in the same way using any other notion of determinant which has the same properties. However, this cannot give a finer invariant in the type I_f-case and leads always to a trivial invariant in the type II_1-case because of the results in [35, Theorem 2.1 on page 521 and Theorem 3.3 on page 525].

The next result follows from Theorem 1.2 and the standard properties of the normalized determinant and center-valued trace.

Lemma 3.7 *Let* $0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$ *be an exact sequence of* $\mathscr{A}$-*modules such that two of them are finitely presented. Then:*

1. *All three are finitely presented;*
2. $\dim^u(M_1) = \dim^u(M_0) + \dim^u(M_2) \qquad \in Z(\mathscr{A})$*;*
3. $\mathbf{P}M_1 = 0 \Longleftrightarrow \mathbf{P}M_0 = \mathbf{P}M_2 = 0$;
4. *If* $\mathscr{A}$ *is of type* I_f *and* $\mathbf{P}M_1 = 0$*, then*

$$\rho^u(M_1) = \rho^u(M_0) \cdot \rho^u(M_2) \qquad \in Z(\mathscr{A})^w / Z(\mathscr{A})^{\mathrm{inv}}. \quad \square$$

Next we construct invariants which are defined for all finite von Neumann algebras. Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathscr{A}$-modules. Let $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ be the (right-continuous) family of spectral projections of the positive operator f^*f. Define the *center-valued spectral density function* of f by

$$F_f^u : [0,\infty) \longrightarrow Z(\mathscr{A})^+ \qquad \lambda \mapsto \dim^u\left(\mathrm{im}(E_{\lambda^2}^{f^*f})\right) = \mathrm{tr}^u(E_{\lambda^2}^{f^*f}).$$

Notice that F_f^u is a monotone increasing (right-continous) function. We call two monotone increasing functions $G_0, G_1 : [0,\infty) \longrightarrow Z(\mathscr{A})^+$ *dilatationally equivalent* if there are constants $\epsilon, C > 0$ satisfying

$$G_0(C^{-1} \cdot \lambda) \le G_1(\lambda) \le G_0(C \cdot \lambda) \qquad \text{for all } \lambda \in [0,\epsilon].$$

Definition 3.8 *Denote by* $\mathscr{D}(\mathscr{A})$ *the abelian semi-group of dilatational equivalence classes* $[d]$ *of monotone increasing functions* $d : [0,\infty) \longrightarrow Z(\mathscr{A})^+$ *where the addition is given by* $[d] + [e] = [d + e]$*. For a morphism of finitely generated Hilbert* $\mathscr{A}$-*modules* $f : U \longrightarrow V$ *define*

$$\omega^u(f) := [F_f^u] \qquad \in \mathscr{D}(\mathscr{A}). \quad \square$$

Lemma 3.9 *1. If* $f : U \longrightarrow V$ *is an isomorphism of finitely generated Hilbert* $\mathscr{A}$-*modules, then*

$$\omega^u(f) = 0;$$

2. $\omega^u(f \oplus g) = \omega^u(f) + \omega^u(g)$*;*
3. *If* f*,*g *and* h *are composable morphisms of finitely generated Hilbert* $\mathscr{A}$-*modules and* f *and* h *are isomorphism, then:*

$$\omega^u(f \circ g \circ h) = \omega^u(g).$$

Proof. 1.) and 2.) are obvious.

3.) The elementary proof in [26, Lemma 1.6 on page 21] for the complex-valued trace goes through word by word for the center-valued trace and shows

$$\begin{array}{rcl} F^u_g(\lambda) & \leq & F^u_{gh}(||h|| \cdot \lambda); \\ F^u_{gh}(\lambda) & \leq & F^u_g(||h^{-1}|| \cdot \lambda); \\ F^u_g(\lambda) & \leq & F^u_{fg}(||f|| \cdot \lambda); \\ F^u_{fg}(\lambda) & \leq & F^u_g(||f^{-1}|| \cdot \lambda); \end{array}$$

and the assertion follows. □

Definition 3.10 *Let M be a finitely presented $\mathscr{A}$-module. For any choice of an exact sequence $0 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$ define*

$$\omega^u(M) := \omega^u(\nu(f)) \qquad \in \mathscr{D}(\mathscr{A})$$

where $\nu(f) : \nu(P, \mu_1) \longrightarrow \nu(P_0, \mu_0)$ is the morphism of finitely generated Hilbert $\mathscr{A}$-modules defined in Sect. 2 after a choice of inner products μ_i. □

We derive from Theorem 2.1, Lemma 3.6 and Lemma 3.9 that the definition above makes sense. We get for a finitely presented $\mathscr{A}$-module M that i.) $\omega^u(M) = \omega^u(\mathbf{T}M)$, ii.) $\omega^u(M)$ is trivial if and only if M is finitely generated projective and iii.) M is trivial if and only if $\dim^u(M)$ and $\omega^u(M)$ are trivial.

Recall that we have specified a trace $\mathrm{tr} : \mathscr{A} \longrightarrow \mathbb{C}$. The definitions of von Neumann dimension and spectral density function of Sect. 3 for the universal trace make also sense for the complex valued trace tr. This yields for a finitely generated Hilbert $\mathscr{A}$-module V, a finitely presented $\mathscr{A}$-module M and a morphism $f : V \longrightarrow W$ of finitely generated Hilbert $\mathscr{A}$-modules the *von Neumann dimension (with respect to* tr*)*

$$\dim(V), \dim(M) \in \mathbb{R}^{\geq 0};$$

the *spectral density function (with respect to* tr*)*

$$F_f : [0, \infty) \longrightarrow [0, \infty] \qquad \lambda \mapsto \dim\left(\mathrm{im}(E^{f^*f}_{\lambda^2})\right) = \mathrm{tr}(E^{f^*f}_{\lambda^2});$$

and the ω-invariants with respect to tr

$$\omega(f), \omega(M) \in \mathscr{D}(\mathbb{C}).$$

We get from Theorem 3.1 that tr induces a map

$$\mathrm{tr} : \mathscr{D}(\mathscr{A}) \longrightarrow \mathscr{D}(\mathbb{C}) \qquad [d] \mapsto [\mathrm{tr} \circ d]$$

and $\dim(V)$, $\dim(M)$, $\omega(f)$ and $\omega(M)$ can be read off from $\dim^u(V)$, $\dim^u(M)$, $\omega^u(f)$ and $\omega^u(M)$ defined in Sect. 3 by

$$\begin{aligned}\dim(V) &= \operatorname{tr}(\dim^u(V));\\ \dim(M) &= \operatorname{tr}(\dim^u(M));\\ \omega(f) &= \operatorname{tr}(\omega^u(f));\\ \omega(M) &= \operatorname{tr}(\omega^u(M)).\end{aligned}$$

Of course one looses information by passing from the universal trace to the complex valued trace, but for most of the applications the invariants based on the standard complex valued trace for a von Neumann algebra of a group are sufficient mainly, because the vanishing of $b_p^{(2)}(C)$ is equivalent to the vanishing of $H_p^{(2)}(C)$. Moreover, one can assign in the complex-valued case an interesting real number to elements in $\mathscr{D}(\mathbb{C})$. The *Novikov-Shubin invariant* of $[d] \in \mathscr{D}(\mathbb{C})$ is

$$\alpha([d]) = \liminf_{\lambda\to 0^+} \frac{\ln(d(\lambda) - d(0))}{\ln(\lambda)} \qquad \in [0,\infty],$$

provided that $d(\lambda) > d(0)$ holds for all $\lambda > 0$. Otherwise, we put $\alpha([d]) = \infty^+$. Here ∞^+ is a new formal symbol which should not be confused with ∞.

Definition 3.11 *Define the* **Novikov-Shubin invariant** *of a morphism of finitely generated Hilbert $\mathscr{A}$-modules $f : U \longrightarrow V$ by*

$$\alpha(f) \;:=\; \alpha(\omega(f)) \qquad \in [0,\infty] \amalg \{\infty^+\}.$$

The **Novikov-Shubin invariant** $\alpha(M)$ *of a finitely presented $\mathscr{A}$-module M is defined by*

$$\alpha(M) \;:=\; \alpha(\omega(M)) \qquad \in [0,\infty] \amalg \{\infty^+\}. \quad \square$$

Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathscr{A}$-modules. Since f and f^*f have the same kernel we have $\dim(\ker(f)) = F_f(0)$. We have $\alpha(f) = \infty^+$ if and only if f^*f has a gap in the spectrum at zero, i.e. there exists $\epsilon > 0$ such that $F_f(\lambda) = F_f(0)$ for $0 < \lambda < \epsilon$. Moreover, f is an isomorphism if and only if f has trivial kernel, $\alpha(f) = \infty^+$ and $\dim^u(U) = \dim^u(V) \in Z(\mathscr{A})$. A finitely presented $\mathscr{A}$-module M is finitely generated projective if and only if $\alpha(M) = \infty^+$. It is trivial if and only if $\dim(M) = 0$ and $\alpha(M) = \infty^+$.

Finally we prove the following two lemmas we will need later.

Lemma 3.12 *Let C a finitely generated free $\mathscr{A}$-chain complex. For a $\mathscr{A}$-module M let $\mu(M)$ be the minimal numbers of generators. Then*

$$\dim(C_p) \geq \mu\left(H_p(C) \oplus \mathbf{T}H_{p-1}(C)\right).$$

Proof. Choose a direct sum decomposition $\ker(c_p) \oplus P_1 = C_p$. Let $P_0 \subset \ker(c_{p-1})$ be the preimage of $\mathbf{T}H_{p-1}$ under the canonical projection $\ker(c_{p-1}) \longrightarrow H_{n-1}(C)$. We get a 1-dimensional finitely generated projective resolution $0 \longrightarrow P_1 \xrightarrow{c_p|_{P_1}} P_0 \longrightarrow \mathbf{T}H_{p-1}(C) \longrightarrow 0$. From Lemma 3.4 we conclude that P_0 and P_1 are $\mathscr{A}$-isomorphic. Hence we can construct an epimorphism $C_p \longrightarrow H_p(C) \oplus \mathbf{T}H_{p-1}(C)$. $\square$

Lemma 3.13 *Let C be a $\mathscr{A}$-chain complex and d be an integer such that C_n is finitely presented with $\mathbf{P}C_n = 0$ for $n \le d$ and C_{d+1} is finitely generated. Then $H_n(C)$ is finitely presented with $\mathbf{P}H_n(C) = 0$ for $n \le d$ and $H_{d+1}(C)$ is finitely generated.*

Proof. From Lemma 3.7 we conclude that $\ker(c_n)$, $\mathrm{im}(c_n)$ and $H_{n-1}(C)$ are finitely presented and $\mathbf{P}\ker(c_n)$, $\mathbf{P}\mathrm{im}(c_n)$ and $\mathbf{P}H_{n-1}(C)$ are trivial for $n \le d$. The obvious sequence $0 \longrightarrow \mathrm{im}(c_{d+1}) \longrightarrow \ker(c_d) \longrightarrow H_d(C) \longrightarrow 0$ is exact, $\mathrm{im}(c_{d+1})$ is finitely generated and $\ker(c_d)$ is finitely presented with $\mathbf{P}\ker(c_d) = 0$. Hence $H_d(C)$ and $\mathrm{im}(c_{d+1})$ are finitely presented and $\mathbf{P}H_d(C) = 0$ by Lemma 3.7. As $0 \longrightarrow \ker(c_{d+1}) \longrightarrow C_{d+1} \xrightarrow{c_{d+1}} \mathrm{im}(c_{d+1}) \longrightarrow 0$ is exact, C_{d+1} is finitely generated and $\mathrm{im}(c_{d+1})$ is finitely presented, $\ker(c_{d+1})$ is finitely generated. This finishes the proof of Lemma 3.13 . □

4 Abelian von Neumann algebras

Next we consider the special case where $\mathscr{A}$ is abelian. Recall that any abelian von Neumann algebra $\mathscr{A}$ can be identified with $L^\infty(X,\mu)$ for an appropriate measure space (X,μ) [12, Theorem 1 in I.7.3 on page 132]. We recall the classical notion of the *k-th elementary ideal* $I_k(M)$ of a finitely presented S-module for any commutative ring S [11, Chapter VII]. Let $S^m \xrightarrow{f} S^n \longrightarrow M$ be a finite presentation and A the (m,n)-matrix describing f by $f(x) = xA$ for $x \in S^m$. A (l,l)-*minor* of A is the (ordinary) determinant (of commutative rings) of a (l,l)-submatrix of A. Define

$$I_k(M) \ := \ \begin{cases} 0 & \text{, if } n-k > m \text{ or } k < 0 \\ S & \text{, if } n-k \le 0 \\ \text{ideal generated by all} & \\ (n-k,n-k)\text{-minors} & \text{, if } 0 < n-k \le m. \end{cases}$$

The following result was stated without proof in [35, Lemma 2.2. on page 522]. For the readers convenience we include a proof here.

Lemma 4.1 *Let $\mathscr{A} = L^\infty(X,\mu)$ be an abelian von Neumann algebra and $t : \mathscr{A}^n \longrightarrow \mathscr{A}^n$ be a normal morphism, i.e. t and t^* commute. Then there exists a unitary isomorphism $u : \mathscr{A}^n \longrightarrow \mathscr{A}^n$ such that $u^* \circ t \circ u$ is diagonal.*

Proof. We first construct a measurable map

$$\lambda : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

with the property that $\lambda(A)$ is an eigenvalue of A for all $A \in M_n(\mathbb{C})$. Set-theoretically, the map is defined as follows. Let $\lambda(A)$ be the eigenvalue of A with the following property. Let $\lambda(A) = r \cdot e^{2\pi i t}$ for $r \ge 0$ and $t \in [0,1[$ be the polar decomposition of $\lambda(A)$. If λ' is any other eigenvalue of A with polar decomposition $\lambda' = r' \cdot e^{2\pi t'}$, then either $r' < r$, or $r' = r$ and $t \le t'$ holds. In other words, the norm of $\lambda(A)$ is maximal and among those eigenvalues with the same norm as $\lambda(A)$, the angle between the real axis and $\lambda(A)$ with respect to the origin in the anticlockwise direction is minimal. The map λ is of course not continuous but, as we will see, measurable. Define the following sets for $1 \leq k \leq n$

$$\begin{array}{lcl} S & := & \{A \in M_n(\mathbb{C}) \mid \lambda(A) \in \mathbb{R}\}; \\ S_k^c & := & \{A \in S^c \mid \text{ there are } k \text{ eigenvalues of } A \text{ satisfying } |\lambda'| = |\lambda(A)|\}; \end{array}$$

where we count the eigenvalues with multiplicity and S^c is the complement of S in $M_n(\mathbb{C})$. The eigenvalues of A are the roots of its characteristic polynomial and the map sending A to its characteristic polynomial is continuous. The roots of a polynomial depend continuously on the coefficients in the following sense. Given a polynomial p of degree n and $\epsilon > 0$, there is $\delta > 0$ such that for any polynomial q of degree n with the property that the difference of the i-th coefficients of p and q have norm less than or equal to δ for all i, there are numerations of the roots $\lambda_i(p)$ of p and of the roots $\lambda_i(q)$ of q, satisfying $|\lambda_i(p) - \lambda_i(q)| \leq \epsilon$ for all i. This implies that λ is continuous on each set S_k^c and that the disjoint union $\coprod_{k=1}^l S_k^c$ is an open set for all $1 \leq l \leq n$. The set S is closed. Hence each of the sets S_k^c and the set S are measurable and the restrictions of λ to these sets is measurable. Hence the map λ is measurable.

We leave it to the reader to verify that the function $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$, sending A to the orthogonal projection $\mathrm{pr}_{\mathrm{im}(A)}$ onto the image of A, is measurable.

Next, we construct a measurable map $v_1 : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^n$ such that $\|v_1(A)\| = 1$ and $Av_1(A) = \lambda(A)v_1(A)$ holds for all A in $M_n(\mathbb{C})$. Namely, define

$$v_1(A) = \frac{\mathrm{pr}_{\ker(A-\lambda(A)I)}(e_i)}{\|\mathrm{pr}_{\ker(A-\lambda(A)I)}(e_i)\|}$$

if $\mathrm{pr}_{\ker(A-\lambda(A)I)}(e_{i-1}) = 0$ and $\mathrm{pr}_{\ker(A-\lambda(A)I)}(e_i) \neq 0$, where e_1, e_2 , $\ldots e_n$ are the standard basis vectors. Given v_1, one easily constructs measurable maps $v_2, \ldots, v_n$ from $M_n(\mathbb{C})$ to $\mathbb{C}^n$ such that $\{v_1(A), v_2(A), \ldots, v_n(A)\}$ is an orthonormal basis for all A. Let $U(A)$ be the unitary map having this orthonormal basis as columns. Provided that A is normal, the matrix $U(A)^{-1}AU(A)$ is a diagonal block matrix.

$$U(A)^{-1}AU(A) = \begin{pmatrix} \lambda(A) & 0 \\ 0 & B(A) \end{pmatrix}$$

and the $(n-1, n-1)$-matrix $B(A)$ is normal again. Iterating this process, we can even assume that $U(A)^{-1}AU(A)$ is a diagonal matrix. Since the composition of measurable functions is again measurable, we get for normal $T \in M_n(L^\infty(X, \mu))$ the desired unitary element $U \in M_n(L^\infty(X, \mu))$ by putting $U(x) = U(T(x))$. This finishes the proof of Lemma 4.1. □

Example 4.2 Let $\mathscr{A} = L^\infty(X, \mu)$ be an abelian von Neumann algebra and let M be any finitely presented $\mathscr{A}$-module M. Because of the polar decomposition and Lemma 4.1 there are elements t_1, t_2, $\ldots$, t_l in $\mathscr{A}$ such that

$$M \cong \oplus_{i=1}^l \mathscr{A}/(t_i).$$

Let $\mathrm{Zero}(t_i)$ be $\{x \in X \mid t_i(x) = 0\}$ and let χ_i be the characteristic function of $\mathrm{Zero}(t_i)$. Then

$$\begin{aligned}
\mathbf{P}M &= \oplus_{i=1}^{l}(\chi_i);\\
\mathbf{T}M &= \oplus_{i=1}^{l}\mathscr{A}/(\chi_i + t_i);\\
\dim^u(M) &= \sum_{i=1}^{l}\chi_i;\\
\rho^u(M) &= \prod_{i=1}^{l}(\chi_i + t_i);\\
I_k(M) &= \left(\prod_{i\in I} t_i \;\middle|\; I \subset \{1,2,\ldots,n\}, |I| = l-k\right) \qquad \text{for } 0 \le k \le l.
\end{aligned}$$

In particular $\mathscr{A}/(t)$ and $\mathscr{A}/(s)$ are $\mathscr{A}$-isomorphic if and only if $\mathrm{Zero}(t) = \mathrm{Zero}(s)$ (up to sets of measure zero) and t/s and s/t are essentially bounded outside $\mathrm{Zero}(t) = \mathrm{Zero}(s)$. $\square$

Example 4.3 Next we consider the von Neumann algebra $\mathscr{N}(\mathbb{Z}) = L^\infty(S^1)$ with its standard trace $\mathrm{tr} : L^\infty(S^1) \longrightarrow \mathbb{C}$ given by integration and want to classify $L^\infty(S^1)$-modules of the shape

$$\oplus_{i=1}^{n} L^\infty(S^1)/(p_i)$$

where $p_i(z)$ is a polynomial in z. Since $\mathbb{C}[\mathbb{Z}]$ is a principal ideal domain [2, Proposition V.5.8 on page 151 and Corollary V.8.7 on page 162] any such $L^\infty(S^1)$-module is of the shape

$$N \otimes_{\mathbb{C}[\mathbb{Z}]} L^\infty(S^1)$$

for some finitely generated $\mathbb{C}[\mathbb{Z}]$-module N. Let $S \subset \mathbb{C}[\mathbb{Z}]$ be the multiplicative subset of elements in $\mathbb{C}[\mathbb{Z}]$ which become invertible in $L^\infty(S^1)$. This is the set of elements which can be written as finite products of non-zero complex numbers and elements of the form $z - a$ with $|a| \neq 1$. The localization $S^{-1}\mathbb{C}[\mathbb{Z}]$ is again a principal ideal domain [2, Corollary V.8.7 on page 162] and we have

$$N \otimes_{\mathbb{C}[\mathbb{Z}]} L^\infty(S^1) \;\cong\; S^{-1}N \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1).$$

Hence it suffices to investigate for a finitely generated $S^{-1}\mathbb{C}[\mathbb{Z}]$-module M the $L^\infty(S^1)$-module

$$M^{(2)} \;:=\; M \otimes_{S^{-1}\mathbb{C}[\mathbb{Z}]} L^\infty(S^1).$$

From [2, Theorem X.5.7 on page 370] we get non-negative integers n, l and elements $t_1, t_2, \ldots, t_l$ in $S^{-1}\mathbb{C}[\mathbb{Z}]$ such that no t_i is zero or a unit, t_i divides t_{i+1} and

$$M \;=\; S^{-1}\mathbb{C}[\mathbb{Z}]^n \oplus \oplus_{i=1}^{l} S^{-1}\mathbb{C}[\mathbb{Z}]/(t_i)$$

and n, l and the ideals (t_i) are uniquely determined by this property. We can also write M in the form

$$M = S^{-1}\mathbb{C}[\mathbb{Z}]^n \oplus \oplus_{i=1}^k S^{-1}\mathbb{C}[\mathbb{Z}]/((z-a_i)^{r_i})$$

where n,k,r_i are integers with $n,k \geq 0$ and $r_i \geq 1$ and $a_i \in S^1$. For $a = \exp(i\psi) \in S^1$ and $r \in \mathbb{Z}, r \geq 1$ define

$$f_{a,r} : [0,\infty) \longrightarrow L^\infty(S^1)^+ \qquad \lambda \mapsto \chi_{\{\cos(\phi)+i\sin(\phi) \;|\; \lambda \geq |\phi-\psi|^r\}}.$$

We claim

$$\begin{aligned}
\dim^u(M) &= n \cdot 1_{L^\infty(S^1)};\\
\rho^u(M) &= \prod_{i=1}^{l} t_i;\\
I_k(M) &= 0 \qquad \text{for } k < n\\
I_k(M) &= (\prod_{i=1}^{n+l-k} t_i) \qquad \text{for } n \leq k;\\
\omega^u(M) &= \sum_{i=1}^{k} [f_{a_i,r_i}];\\
\alpha(M) &= \begin{cases} \min\{\frac{1}{r_i} \;|\; i = 1,\ldots k\} & \text{, if } k > 0\\ \infty^+ & \text{, if } k = 0.\end{cases}
\end{aligned}$$

The first four equations follow directly from Example 4.2. We get from Lemma 3.9

$$\omega^u(M) = \sum_{i=1}^{k} \omega^u(L^\infty(S^1)/((z-a_i)^{r_i})).$$

Hence it suffices to check for the fourth equation the special case for $b \in S^1$ and $r \in \mathbb{Z}, r \geq 1$

$$\omega^u(L^\infty(S^1)/((z-b)^r)) = [f_{b,r}].$$

Since the group of isometries on S^1 acts transitively on S^1, it suffices to treat the case $b = 1$. The spectral density function of the morphism $L^\infty(S^1) \longrightarrow L^\infty(S^1)$ given by multiplication with $(z-1)^r$ assigns to $\lambda \in [0,\infty)$ the characteristic function of the set

$$\{z \in S^1 \;|\; \lambda \geq |z-1|^r\} = \{\cos(\phi) + i\sin(\phi) \;|\; \lambda \geq |2 - 2\cos(\phi)|^{r/2}\}$$

This proves the fifth equation. Since

$$\lim_{\phi \to 0} \frac{2-2\cos(\phi)}{\phi^2} = 1$$

the sixth equation follows.

The last equation follows from the fourth using [26, Lemma 1.10 on page 23]

$$\begin{aligned}
\alpha([d]+[e]) &= \min\{\alpha([d]),\alpha([e])\} && \text{for } [d],[e]\in\mathscr{D}(\mathbb{C});\\
\operatorname{tr}([f_{a,r}]) &= [\lambda^{1/r}] \qquad \in\mathscr{D}(\mathbb{C});\\
\alpha([\lambda^{1/r}]) &= \frac{1}{r}.
\end{aligned}$$

We mention the consequence of these equations that the elementary ideals $I_k(M^{(2)})$ for $k\geq 0$ determine the isomorphism type of both M and $M^{(2)}$ and that $\omega^u(M^{(2)})$ determines the isomorphism type of both $\operatorname{Tors}(M)$ and $\mathbf{T}M^{(2)}$.

Namely, the computations above show that the elementary ideals $I_k(M^{(2)})$ determine both n and $\prod_{i=1}^k t_i$ up to multiplication with units in $L^\infty(S^1)$. Hence n and $\prod_{i=1}^k t_i$ for all k are up to multiplication with units in $S^{-1}\mathbb{C}[\mathbb{Z}]$ uniquely determined by the isomorphism type of $M^{(2)}$. This implies that the isomorphism type of $M^{(2)}$ determines n and the $S^{-1}\mathbb{C}[\mathbb{Z}]$-ideals (t_i) and hence the $S^{-1}\mathbb{C}[\mathbb{Z}]$-isomorphism type of M itself.

In order to show that $\omega^u(M^{(2)})$ determines the isomorphism type of $\operatorname{Tors}(M)$, we must show the following. Given non-negative integers n and q, pairwise disjoint elements $a_i\in S^1$ for $i=1,2,\ldots,q$ and non-negative integers $m_{i,j}$ for $i=1,2,\ldots,q$ and $j\in\mathbb{Z}$, $j\geq 1$ for which only finitely many are different from zero, the class

$$\sum_{i=1}^q\sum_{j\geq 1} m_{i,j}\cdot[f_{a_i,j}] \qquad \in\mathscr{D}(\mathscr{A})$$

determines q, a_i and $m_{i,j}$ up to permutation of the indices i. We have already introduced an (additive) semi-abelian group structure on $\mathscr{D}(\mathscr{A})$ by $[d+e]=[d]+[e]$. Analogously multiplication defines a (multiplicative) structure of an abelian semi-group by

$$[d]\cdot[e] \;:=\; [d\cdot e].$$

Let $\chi_{i_0,\epsilon}$ be the characteristic function for the set $\{\cos(\phi)+i\sin(\phi)\mid\epsilon\geq|\phi-\psi|\}$ if we write $a_{i_0}=\cos(\psi)+i\sin(\psi)$. Denote the corresponding constant function $[0,\infty)\longrightarrow L^\infty(S^1)$ in the same way. Then one easily checks for small enough ϵ:

$$\left(\sum_{i=1}^q\sum_{j\geq 1} m_{i,j}\cdot[f_{a_i,j}]\right)\cdot[\chi_{i_0,\epsilon}] \;=\; \sum_{j\geq 1} m_{i_0,j}\cdot[f_{a_{i_0},j}].$$

Since the group of isometries on S^1 acts transitively on S^1, it suffices to show for a sequence of non-negative integers m_j for $j=1,2,\ldots$ for which only finite many are different from zero that

$$\sum_{j\geq 1} m_j\cdot[f_{1,j}] \qquad \in\mathscr{D}(\mathscr{A})$$

determines the integers m_j. It suffices to show for all positive integers j_0 that the sums $\sum_{j\geq j_0} m_j$ are determined. For $[d],[e]\in\mathscr{D}(\mathscr{A})$ we write

$$[d] \leq [e]$$

if and only if there are representatives d and e and constants $C, \epsilon > 0$ with the property that $d(\lambda) \leq e(C \cdot \lambda)$ holds for all $\lambda \in [0, \epsilon)$. It suffices to prove

$$\sum_{j \geq j_0} m_j \;=\; \max \left\{ k \in \mathbb{Z} \;\middle|\; k \geq 0 \text{ and } k \cdot [f_{1,j_0}] \leq \sum_{j \geq 1} m_j \cdot [f_{1,j}] \right\}.$$

Consider $k \in \mathbb{Z}$ with $k \geq 0$, $0 < \epsilon < 1$ and $C > 1$ such that for $\lambda \in [0, \epsilon]$

$$k \cdot f_{1,j_0}(\lambda) \;\leq\; \sum_{j \geq 1} m_j \cdot f_{1,j}(C \cdot \lambda)$$

holds. Fix $\lambda \in [0, \epsilon]$ satisfying $\lambda^{(j_0)^{-1}}/2 \geq (C \cdot \lambda)^{j^{-1}}$ for all $j < j_0$. Let χ be the characteristic function of the set $\{\cos(\phi) + i \sin(\phi) \mid \phi \in [\lambda^{j_0}/2, \lambda^{j_0}]\}$. Multiplying the inequality of elements in $L^\infty(S^1)^+$ above with χ yields

$$k \cdot \chi \leq \sum_{j \geq j_0} m_j \cdot \chi.$$

This implies $k \leq \sum_{j \geq j_0} m_j$. Since $[f_{1,j_0}] \leq [f_{1,j}]$ for $j \geq j_0$, the claim follows. $\square$

5 L^2-Betti numbers and Novikov-Shubin invariants for chain complexes

In this section we introduce L^2-invariants such as L^2-homology, L^2-Betti numbers and Novikov-Shubin invariants for finitely generated Hilbert $\mathscr{A}$-chain complexes and express them in terms of the homology of finitely generated projective $\mathscr{A}$-chain complexes which is associated to it by Theorem 2.1. In the sequel ν^{-1} is an inverse of ν which is well-defined up to unitary equivalence as described in Theorem 2.1.

Definition 5.1 *If C is a chain complex of finitely generated Hilbert $\mathscr{A}$-chain modules, define its* p**-th** $L^{(2)}$**-homology** *to be the finitely generated Hilbert $\mathscr{A}$-module*

$$H_p^{(2)}(C) = \ker(c_p)/\overline{\operatorname{im}(c_{p+1})},$$

its **center-valued** p**-th** L^2**-Betti number** *by*

$$b_p^u(C) := \dim^u(H_p^{(2)}(C)) \qquad \in Z(\mathscr{A}),$$

its p**-th** L^2**-Betti number** *by*

$$b_p^{(2)}(C) := \dim(H_p^{(2)}(C)) \qquad \in \mathbb{R}^{\geq 0},$$

its p**-th** ω**-invariant** *by*

$$\omega_p^u(C) \;:=\; \omega(c_p : C_p \longrightarrow C_{p-1}) \qquad \in \mathscr{D}(\mathscr{A}),$$

and its p**-th Novikov-Shubin invariant** *by*

$$\alpha_p(C) \;:= \alpha(c_p : C_p \longrightarrow C_{p-1}) \qquad \in [0, \infty] \amalg \{\infty^+\}. \quad \square$$

Lemma 5.2 *Let C be a finitely generated Hilbert $\mathscr{A}$-chain complex. Then there is a in C natural isomorphism*

$$h(C) : \nu^{-1}(H_p^{(2)}(C)) \longrightarrow \mathbf{P}H_p(\nu^{-1}(C)).$$

Proof. We define $h(C)$ by the following commutative diagram whose columns are exact and whose middle and lower vertical arrows are isomorphisms by Lemma 2.3

$$\begin{tikzcd}
0 & 0 \\
\nu^{-1}(H_p^{(2)}(C)) \arrow[u] \arrow[r, "h(C)"] & \mathbf{P}H_p(\nu^{-1}(C)) \arrow[u] \\
\nu^{-1}(\ker(c_p)) \arrow[u, "\nu^{-1}(q)"] \arrow[r, "\nu^{-1}(i)"] & \ker(\nu^{-1}(c_p)) \arrow[u, "r"'] \\
\nu^{-1}(\overline{\mathrm{im}(c_{p+1})}) \arrow[u, "\nu^{-1}(j)"] \arrow[r, "\nu^{-1}(k)"] & \overline{\mathrm{im}(\nu^{-1}(c_{p+1}))} \arrow[u, "l"'] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where i, j, k and l are the obvious inclusions and q and r the obvious projections. Then $h(C)$ is an isomorphism by the five-lemma. $\square$

Lemma 5.3 *If $f : C \longrightarrow D$ is a chain homotopy equivalence of finitely generated Hilbert $\mathscr{A}$-chain complexes, then we get for all p*

$$\omega_p^u(C) = \omega_p^u(D).$$

Proof. We will need the following fact for an exact sequence of finitely generated Hilbert $\mathscr{A}$-chain complexes $0 \longrightarrow C \longrightarrow D \xrightarrow{p} E \longrightarrow 0$: If E is contractible then there is a chain map $s : E \longrightarrow D$ with $p \circ s = \mathrm{id}_E$. Namely, choose for any $n \geq 0$ a morphism $\sigma_n : E_n \longrightarrow D_n$ with $p_n \circ \sigma_n = \mathrm{id}_{E_n}$. If γ is a chain contraction for E, define

$$s_n = d_{n+1} \circ \sigma_{n+1} \circ \gamma_n + \sigma_n \circ \gamma_{n-1} \circ e_n.$$

If $\mathrm{cone}(f)$ resp. $\mathrm{cyl}(f)$ is the mapping cone resp. mapping cylinder of $f : C \longrightarrow D$ (see [28, page 213]), there are canonical exact sequences

$$0 \longrightarrow C \longrightarrow \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(f) \longrightarrow 0$$

and

$$0 \longrightarrow D \longrightarrow \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(C) \longrightarrow 0.$$

In both sequence the third chain complex is contractible. Hence we obtain from the fact above an isomorphism of $\mathscr{A}$-chain complexes

$$C \oplus \operatorname{cone}(f) \cong D \oplus \operatorname{cone}(C).$$

The ω-invariants are invariant under isomorphisms of $\mathscr{A}$-chain complexes by Lemma 3.9. Hence it suffices to show for finitely generated projective $\mathscr{A}$-chain complexes C and D that C and $C \oplus D$ have in all dimensions the same ω-invariants, provided that D is contractible. A $\mathscr{A}$-chain complex is elementary if for some n it is concentrated in two consecutive dimensions n and $n-1$ and the n-th differential is an isomorphism. Since D is contractible, D is a sum of elementary contractible finitely generated Hilbert $\mathscr{A}$-chain complexes. Hence we can assume without loss of generality that D is elementary. But then the claim follows Lemma 3.9. $\square$

The next theorem enables us to read of the center-valued L^2-Betti numbers and ω-invariants of a finitely generated Hilbert $\mathscr{A}$-chain complex C from the homology of the $\mathscr{A}$-chain complex $H_p(\nu^{-1}(C))$.

Theorem 5.4 *Let C be a finitely generated Hilbert $\mathscr{A}$-chain complex. Then:*

$$\begin{aligned} b_p^u(C) &= \dim^u(\mathbf{P}H_p(\nu^{-1}(C))); \\ b_p^{(2)}(C) &= \dim(\mathbf{P}H_p(\nu^{-1}(C))); \\ \omega_p^u(C) &= \omega^u(\mathbf{T}H_{p-1}(\nu^{-1}(C))); \\ \alpha_p(C) &= \alpha(\mathbf{T}H_{p-1}(\nu^{-1}(C))). \quad \square \end{aligned}$$

Proof. It suffices to prove the claim for the invariants based on the center-valued trace since they determine the others. The assertion about the center-valued L^2-Betti numbers follows from Lemma 5.2. We will now and later need the following general observation which follows from the fact that a chain map of projective chain complexes is a homotopy equivalence if and only if it induces an isomorphism on homology. Let Σ^p denote the p-fold suspension $\Sigma\Sigma\ldots\Sigma$.

Lemma 5.5 *Let S be a ring and C be a projective S-chain complex. Suppose that for each p there is a 1-dimensional projective S-resolution $P[p]$ of $H_p(C)$. Then there is a S-chain map $j[p] : \Sigma^p P[p] \longrightarrow C$ which induces the identity on the p-th homology. The S-chain map*

$$\oplus_{p\geq 0} j[p] : \oplus_{p\geq 0}\Sigma^p P[p] \longrightarrow C$$

is a S-chain homotopy equivalence. $\square$

Because of Theorem 1.2 and Lemma 3.4 we can find a 1-dimensional finitely generated projective $\mathscr{A}$-resolution $P[p]$ of $\mathbf{T}H_p(\nu^{-1}(C))$ and $\mathbf{P}H_p(\nu^{-1}(C))$ is finitely generated projective. Let $p[\mathbf{P}H_p(\nu^{-1})]$ be the obvious $\mathscr{A}$-chain complex concentrated in dimension p. From Lemma 5.5 we get a chain homotopy equivalence

$$\oplus_{p\geq 0}\Sigma^p P[p] \oplus p[\mathbf{P}H_p(\nu^{-1}(C))] \longrightarrow \nu^{-1}(C).$$

From Lemma 5.3 we conclude

$$\begin{aligned}
\omega^u(\mathbf{T}H_{p-1}(\nu^{-1}(C))) &= \omega_p^u(\Sigma^{p-1}\nu(P[p-1])) \\
&= \omega_p^u(\Sigma^{p-1}\nu(P[p-1]) \oplus (p-1)[\nu(\mathbf{P}H_{p-1}(\nu^{-1}(C)))]) \\
&= \omega^u(\oplus_{p\geq 0}\Sigma^p\nu(P[p]) \oplus p[\nu(\mathbf{P}H_p(\nu^{-1}(C)))]) \\
&= \omega_p^u(\nu(\nu^{-1}(C))) \\
&= \omega_p^u(C)
\end{aligned}$$

This finishes the proof of Theorem 5.4. $\square$

6 L^2-Betti numbers and Novikov-Shubin invariants for spaces

In this section we will extend the notions and results of Sect. 4 for $\mathcal{A}$-Hilbert chain complexes to proper Γ-CW-complexes of finite type and discuss applications.

Let Γ be a discrete group and X be a proper Γ-CW-complex of finite type. Finite type means that all its skeleta are finite. Recall that a Γ-CW-complex is finite resp. of finite type if and only if $\Gamma\backslash X$ is finite resp. of finite type, and is proper if and only if all isotropy groups of the Γ-action are finite [28, Theorem 1.23 on page 18]. An example of a proper Γ-CW-complex which is finite resp. of finite type is the universal covering of a CW-complex which is finite resp. of finite type with fundamental group Γ. Let $\mathcal{A}$ be a finite von Neumann algebra and V be a finitely generated Hilbert $\mathcal{A}$-module together with a unitary representation $\mu : \Gamma \longrightarrow U_{\mathcal{A}}(V)^{\mathrm{op}}$ into the (opposite of the) group of unitary $\mathcal{A}$-automorphism of V. In particular V is a $\mathcal{A}$-$\mathbb{Z}\Gamma$-bimodule. Then the *cellular L^2-chain complex of X with coefficients in V* is the finitely generated Hilbert $\mathcal{A}$-chain complex

$$C^{(2)}(X;V) \;=\; V \otimes_{\mathbb{Z}\Gamma} C^c(X)$$

where $C^c(X)$ is the cellular $\mathbb{Z}\Gamma$-chain complex. Its L^2-homology is the *L^2-homology of X with coefficients in V* and denoted by $H_p^{(2)}(X;V)$. For details about Γ-CW-complexes and the Hilbert $\mathcal{A}$-module structure on $C^{(2)}(X;V)$ we refer to [34, Sect. 3]. If we specify an inverse ν^{-1} as described in Theorem 2.1, then V and the unitary representation μ determine a $\mathcal{A}$-$\mathbb{Z}\Gamma$-bimodule Q which is finitely generated projective over $\mathcal{A}$. The *cellular $\mathcal{A}$-chain complex of X with coefficients in Q* is the finitely generated $\mathcal{A}$-chain complex

$$C^{\Gamma}(X;Q) \;= Q \otimes_{\mathbb{Z}\Gamma} C^c(X).$$

Its homology is the *homology of X with coefficients in Q* and denoted by $H_p^{\Gamma}(X;Q)$. The center-valued L^2-Betti number $b_p^u(X;V)$, the L^2-Betti number $b_p^{(2)}(X;V)$, the ω-invariant $\omega_p^u(C)$ and the Novikov-Shubin invariant $\alpha_p(X;V)$ are defined as the corresponding invariants for $C^{(2)}(X;V)$ (see Definition 5.1). Everything above has also a cohomological analogue which gives the same information [34, Lemma 3.10 on page 231]. Now Theorem 5.4 applied to this situation gives:

Theorem 6.1 *Under the conditions above we get*

$$\begin{aligned} b_p^u(X;V) &= \dim^u(\mathbf{P}H_p^{\Gamma}(X,Q)); \\ b_p^{(2)}(X;V) &= \dim(\mathbf{P}H_p^{\Gamma}(X,Q)); \\ \omega_p^u(C) &= \omega^u(\mathbf{T}H_{p-1}^{\Gamma}(X,Q)); \\ \alpha_p(C) &= \alpha(\mathbf{T}H_{p-1}^{\Gamma}(X,Q)). \quad \square \end{aligned}$$

The advantage of this theorem is that it reduces the computations of L^2-invariants of the L^2-chain complex of X to the study of the homology of X with coefficents in a $\mathscr{A}$-$\mathbb{Z}\Gamma$- bimodule Q and that $\mathscr{A}$ is semi-hereditary. In particular all standard results for homology with coefficients over a semi-hereditary ring apply to $H^{\Gamma}(X;Q)$. Next we give a list of these tools and their applications.

1. Singular homology

If we want to compute $H^{\Gamma}(X,Q)$ we can also use the singular chain complex $C^s(X)$ of X which is (up to $\mathbb{Z}\Gamma$-chain homotopy naturally) $\mathbb{Z}\Gamma$-homotopy equivalent to the cellular one [28, Proposition 13.10 on page 264]. There are problems in the L^2-setting. For instance, for an arbitrary free Γ-CW-complex the differentials in the L^2-chain complex need not to be bounded operators. Notice that using the singular chain complex one can define $H^{\Gamma}(X;Q)$ for all Γ-spaces X. Sometimes it can be useful for the computation of the homology of a proper Γ-CW-complex of finite type to consider also the homology of more general spaces. We will make use of this for instance in Sect. 7. We will explain in a different paper how to use this approach to give a convenient reformulation of the notion of singular L^2-cohomology in [9, Sect. 2].

2. Universal coefficient spectral sequences

If one knows the homology with complex coefficients $H_n(X;\mathbb{C})$ of X, then there are spectral sequences computing the homology resp. cohomology of X with coefficients in Q (see [38, Theorem 12.1 on page 400], [42, Theorem 3.14 on page 73]). Namely, there is a spectral sequence converging to $H_{p+q}^{\Gamma}(X;Q)$ whose E^2-term is

$$E_{p,q}^2 = \operatorname{Tor}_p^{\mathbb{C}\Gamma}(Q,H_q(X;\mathbb{C}))$$

and a spectral sequence converging to $H_{\Gamma}^{p+q}(X;Q)$ whose E_2-term is

$$E_2^{p,q} = \operatorname{Ext}_{\mathbb{C}\Gamma}^p(H_q(X;\mathbb{C});Q).$$

3. Leray-Serre spectral sequence

Let $F \longrightarrow E \longrightarrow B$ be a fibration such that B is a CW-complex. Assume for simplicity that F, E and B are connected and possess universal coverings. We abbreviate $\Gamma = \pi_1(E)$ and $\pi = \pi_1(B)$. Let S be a ring and Q be a S-$\mathbb{Z}\Gamma$-bimodule. Let $\widetilde{E}$ and $\widetilde{B}$ be the universal coverings of E and B and let $\overline{F}$ be the covering of F associated to the map $\pi_1(F) \longrightarrow \Gamma$ induced by the inclusion. The composition $q : \widetilde{E} \longrightarrow B$ of p with the universal covering of E is a Γ-equivariant fibration with fiber $\overline{F}$ if we equip B with the trivial Γ-action. The equivariant fiber transport yields a homomorphism $\pi \longrightarrow \left([\overline{F}, \overline{F}]^{\Gamma}\right)^{\mathrm{op}}$ into the opposite of the monoid of Γ-homotopy classes of Γ-maps from $\overline{F}$ to itself. Thus we get the structure of a S-$\mathbb{Z}\pi$-bimodule on $H_q^{\Gamma}(\overline{F}; Q)$. The Leray-Serre spectral sequence converges to $H_{p+q}^{\Gamma}(\widetilde{E}; Q)$ and has as E^2-term

$$E_{p,q}^2 \;=\; H_p^{\pi}(\widetilde{B}; H_q^{\Gamma}(\overline{F}; Q)).$$

Of course there is also a cohomological version. The fiber transport of a Γ-fibration is explained in [27, Sect. 1 and Theorem 6.1]. The construction of the Leray-Serre spectral sequence and the identification of its E^2-term is for instance given in [42, Sect. 5.1] in the language of local coefficient systems, provided that Γ acts trivially on Q. We will expain in a different paper the rather straightforward extension of the proof to the general case if one uses [27, Sect. 7]. The L^2-Leray-Serre spectral sequence for a fibration of CW-complexes of finite type was constructed in [46].

4. Poincaré duality

Suppose that X is a smooth orientable manifold of dimension m with a smooth proper Γ-action such that $\Gamma\backslash M$ is compact. Then there are smooth Γ-equivariant triangulations so that we get a finite proper Γ-CW-complex structure and we can talk about the cellular $\mathbb{Z}\Gamma$-chain complex $C^c(X)$. There is a fundamental class $[X]$ in $H_m(X, \partial X; \mathbb{Q}^w)$ where w is the homomorphism $w : \Gamma \longrightarrow \{\pm 1\}$ sending $\gamma \in \Gamma$ to 1 resp. -1 if γ acts orientation preserving resp. reversing and $\mathbb{Q}^w$ is the rational numbers with the right Γ-action given by $r\gamma = w(\gamma)r$ for $r \in \mathbb{Q}$ and $\gamma \in \Gamma$. The $\mathscr{A}$-chain complex $C^{m-p}(X, \partial X; {}^wQ)$ has as p-th-chain module $\hom_{\mathbb{Z}\pi}(C_{m-p}^c(X), {}^wQ)$ where ${}^wQ$ is the left $\mathbb{Z}\Gamma$-module obtained from Q by $\gamma q = w(\gamma) q \gamma^{-1}$ for $\gamma \in \Gamma$ and $q \in Q$ and carries the same left $\mathscr{A}$-operation as Q. Then we obtain a $\mathscr{A}$-chain homotopy equivalence unique up to $\mathscr{A}$-chain homotopy

$$\cap[X] : C^{m-p}(X, \partial X; {}^wQ) \longrightarrow C_p(X; Q).$$

Details of these facts above can be found in [34, Sect. 5]. Let Q^w be the $\mathscr{A}$-$\mathbb{Z}\Gamma$-bimodule obtained from the $\mathscr{A}$-$\mathbb{Z}\Gamma$-bimodule Q by changing the Γ-action to $q\gamma = w(\gamma) q \gamma$ for $\gamma \in \Gamma$ and $q \in Q$. We conclude from Lemma 1.3 that for all p there are natural identifications

$$\begin{aligned} H^{m-p}(X,\partial X;{}^wQ) &= H_p(X;Q);\\ \big(\mathbf{P}H_{m-p}(X,\partial X;Q^w)\big)^* &= \mathbf{P}H_p(X;Q);\\ \big(\mathbf{T}H_{m-1-p}(X,\partial X;Q^w)\big)\widehat{\ } &= \mathbf{T}H_p(X;Q). \end{aligned}$$

Notice for any finitely presented $\mathscr{A}$-module M that $(\mathbf{P}M)^*$ resp. $(\mathbf{T}M)\widehat{\ }$ and $\mathbf{P}M$ resp. $\mathbf{T}M$ are (not canonically) isomorphic. In particular we get back the Poincaré duality assertions for the L^2-Betti numbers and Novikov-Shubin invariants [26, Proposition 3.2 on page 33].

5. Morse inequalities

Assume that Γ acts freely on X so that $\Gamma\backslash X$ is a CW-complex of finite type. Let $i_X(p)$ be the number of p-cells in $\Gamma\backslash X$. Then we get from Lemma 3.12 (see also [15, Sect. 8])

$$i_X(p) \geq \mu(H_p^\Gamma(X;Q) \oplus \mathbf{T}H_{p-1}^\Gamma(X;Q)).$$

Since the left side of the equation is equal or greater than the p-th L^2-Betti number $b_p^{(2)}(X;V)$, this improves the Morse inequalities of [43]. In particular one gets if $\alpha_p(X;V) \neq \infty^+$ the inequality

$$i_X(p) > b_p^{(2)}(X).$$

6. Deficiency of groups

It is shown in [30, Theorem 6.1 on page 212] for a group group Γ whose classifying space $B\Gamma = \Gamma\backslash E\Gamma$ has finite 3-skeleton that the deficiency of Γ satisfies

$$\operatorname{def}(\Gamma) \leq 1 - b_0^{(2)}(E\Gamma;V) + b_1^{(2)}(E\Gamma;V) - b_2^{(2)}(E\Gamma;V).$$

Suppose that $\alpha_3(E\Gamma;V) \neq \infty^+$ what is equivalent to $\mathbf{T}H_2^\Gamma(E\Gamma;Q) \neq 0$. Then this inequality must be a strict inequality. This follows from an elementary modification of the proof in [30, Theorem 6.1 on page 212] which is based on the following observation. If X is the universal covering of a finite 2-dimensional CW-complex with fundamental group Γ, then the classifying map is 2-connected and induces an epimorphism $H_2^\Gamma(X;Q) \longrightarrow H_2^\Gamma(E\Gamma;Q)$ and $\mathbf{T}H_2^\Gamma(X;Q) = 0$. If $\mathbf{T}H_2^\Gamma(E\Gamma;Q)$ is not trivial, we conclude:

$$b_2^{(2)}(X;V) > b_2^{(2)}(E\Gamma;V).$$

7. L^2-torsion

Using for instance [5] or [34] one can also translate the definition of L^2-torsion for finite-dimensional finitely generated Hilbert $\mathscr{A}$-chain complexes to the setting of finite-dimensional finitely generated projective $\mathscr{A}$-chain complexes using the same pattern as above for the L^2-Betti numbers and Novikov-Shubin invariants. This allows for instance to carry over the results in [36] about torsion invariants and fibrations from ordinary torsion to L^2-torsion. Analogously to the extension of L^2-Betti numbers and Novikov-Shubin invariants one can try to use the center-valued trace to define a refined Fuglede-Kadison determinant and L^2-torsion taking values in $K_1^w(\mathscr{A})$ or $Z(\mathscr{A})^w$. This has been done in [34]. However, in the II_1-case this requires a suitable center-valued version of the condition necessary for the definition of the real-valued L^2-torsion that the Novikov-Shubin invariants are positive. The details have so far not yet been carried out. Without such additional conditions one would always get a trivial invariant in the type II_1-case because of the computations of $K_1^w(\mathscr{A})$ in [35, Theorem 2.1 on page 521 and Theorem 3.3 on page 525].

8. The "Zero in the spectrum" conjecture

The "Zero in the spectrum" conjecture says that there is no contractible closed Riemannian manifold M with an isometric proper cocompact action of a discrete group Γ such that for all p zero is not in the spectrum of the Laplace operator in dimension p [18, page 238], [25]. The author does not even know an example of a finite proper Γ-CW-complex X such that for all p zero is not in the spectrum of the combinatorial Laplace operator in dimension p. Notice that the last condition on X is equivalent to the purely algebraic statement that $H_p^{\Gamma}(X;\mathscr{N}(\Gamma))$ is trivial for all p, or equivalently $C^{\Gamma}(X;\mathscr{N}(\Gamma))$ is contractible.

9. Homological computations

In some special situations $H^{(2)}(X;V)$ depends only on the homology of X with complex coefficients $H(X;\mathbb{C})$. Namely, suppose that $H_p(X;\mathbb{C})$ has a 1-dimensional finitely generated projective $\mathbb{C}\Gamma$-resolution $P[p]$ for all $p \geq 0$. Let Q be the $\mathscr{A}$-$\mathbb{Z}\Gamma$-bimodule associated to V as explained in Sect. 4. Then we get from Lemma 5.5 a $\mathbb{C}\Gamma$-chain homotopy equivalence

$$\oplus_{p\geq 0} j[p] : \oplus_{p\geq 0} \Sigma^p P[p] \longrightarrow C^c(X;\mathbb{C}).$$

It induces a chain homotopy equivalence of $\mathscr{A}$-chain complexes

$$\oplus_{p\geq 0} Q \otimes_{\mathbb{C}\Gamma} \Sigma^p P[p] \longrightarrow C^{\Gamma}(X;Q).$$

We get isomorphisms of $\mathscr{A}$-modules

$$Q \otimes_{\mathbb{C}\Gamma} H_p(X;\mathbb{C}) \oplus \operatorname{Tor}_1^{\mathbb{C}\Gamma}(Q, H_{p-1}(X;\mathbb{C})) \longrightarrow H_p^{\Gamma}(X;V).$$

Notice that $\mathrm{Tor}_1^{\mathbb{C}\Gamma}(Q, H_{p-1}(X;\mathbb{C}))$ is finitely generated projective as $\mathscr{A}$ is semi-hereditary. Hence we get

Lemma 6.2 *Suppose that $H_p(X;\mathbb{C})$ has a 1-dimensional finitely generated projective $\mathbb{C}\Gamma$-resolution for all $p \geq 0$. Then we obtain isomorphisms of $\mathscr{A}$-modules*

$$\begin{aligned} \mathbf{P}\left(Q \otimes H_p(X;\mathbb{C})\right) \oplus \mathrm{Tor}_1^{\mathbb{C}\Gamma}(Q, H_{p-1}(X;\mathbb{C})) &\longrightarrow \mathbf{P}H_p^{\Gamma}(X;V); \\ \mathbf{T}\left(Q \otimes H_p(X;\mathbb{C})\right) &\longrightarrow \mathbf{T}H_p^{\Gamma}(X;V). \end{aligned}$$

In particular $H_p(X;\mathbb{C})$ determines the L^2-Betti numbers and Novikov-Shubin invariants. □

Notice that the assumption on $H_p(X)$ is always satisfied if Γ is a finitely generated free group. Namely, the complex group ring of a finitely generated free group is a fir, i.e free ideal ring [10, Corollary 3 on page 68]. This phenomenon was already observed in [15].

One can make computations more explicite in the following case.

Theorem 6.3 *Suppose that Γ is the free abelian group $\mathbb{Z}^r$ of rank r. Let $\mathbb{C}\mathbb{Z}^r_{(0)}$ be the quotient field of $\mathbb{C}\mathbb{Z}^r$. Then*

$$b_p^{(2)}(X; l^2(\mathbb{Z}^r)) \;=\; \dim_{\mathbb{C}\mathbb{Z}^r_{(0)}}\left(H_p(X;\mathbb{C}) \otimes_{\mathbb{C}\mathbb{Z}^r} \mathbb{C}\mathbb{Z}^r_{(0)}\right).$$

Proof. We abbreviate $C = \mathbb{C} \otimes_{\mathbb{Z}} C^c(X)$, $C^{(2)} = l^2(\mathbb{Z}^r) \otimes_{\mathbb{C}\mathbb{Z}^r} C$ and $C_{(0)} = \mathbb{C}\mathbb{Z}^r_{(0)} \otimes_{\mathbb{C}\mathbb{Z}^r} C$. We first treat the case where $C_{(0)}$ has trivial homology. Then we can find a $\mathbb{C}[\mathbb{Z}^r]_{(0)}$-chain contraction γ'. Choose $u \in \mathbb{C}\mathbb{Z}^r$ with $u \neq 0$ and maps $\gamma_n : C_n \longrightarrow C_{n+1}$ such that $l_u \circ \gamma_n' = (\gamma_n)_{(0)}$ holds for all n where l_u is multiplication with u. Then γ is a chain homotopy of $\mathbb{C}[\mathbb{Z}^r]$-chain maps $l_u \simeq 0 : C \longrightarrow C$. This induces a chain homotopy of chain maps of finite Hilbert $\mathscr{N}(\mathbb{Z}^r)$-chain complexes $l_u \simeq 0 : C^{(2)} \longrightarrow C^{(2)}$. Hence multiplication with u induces the zero map on the L^2-homology of $C^{(2)}$. This is only possible if the L^2-homology is trivial and hence all L^2-Betti numbers of $C^{(2)}$ vanish.

Next we treat the general case. Put $b_n = \dim_{\mathbb{C}[\mathbb{Z}^r]_{(0)}}(H_n(C_{(0)}))$. Then there is a $\mathbb{C}[\mathbb{Z}^r]_{(0)}$-isomorphism

$$\oplus_{i=1}^{b_n} \mathbb{C}[\mathbb{Z}^r]_{(0)} \longrightarrow H_n(C_{(0)}) = H_n(C) \otimes_{\mathbb{C}[\mathbb{Z}^r]} \mathbb{C}[\mathbb{Z}^r]_{(0)}.$$

By composing it with a map given by multiplication with a suitable element in $\mathbb{C}[\mathbb{Z}^r]$ one can construct a $\mathbb{C}[\mathbb{Z}^r]$-map

$$i_n : \oplus_{i=1}^{b_n} \mathbb{C}[\mathbb{Z}^r] \longrightarrow H_n(C)$$

such that $(i_n)_{(0)}$ is a $\mathbb{C}[\mathbb{Z}^r]_{(0)}$-isomorphism. Let D be the finite free $\mathbb{C}[\mathbb{Z}^r]$-chain complex whose n-th chain module is $D_n = \oplus_{i=1}^{b_n} \mathbb{C}[\mathbb{Z}^r]$ and whose differentials are all trivial. Choose a $\mathbb{C}[\mathbb{Z}^r]$-chain map $j : D \longrightarrow C$ which induces on the n-th homology the map i_n. Let $\mathrm{cone}(j)$ be its mapping cone. There is a canonical exact sequence

$$0 \longrightarrow C \longrightarrow \mathrm{cone}(j) \longrightarrow \Sigma D \longrightarrow 0.$$

It remains exact under the passage from C to $C_{(0)}$ or $C^{(2)}$. We conclude from the long exact homology sequence that $\text{cone}(j)_{(0)}$ is acyclic. Hence the L^2-homology of $\text{cone}(j)^{(2)}$ is trivial by the first step. We conclude from the long weakly exact L^2-homology sequence [8, Theorem 2.1 on page 10]

$$b_n^{(2)}(C^{(2)}) = b_n^{(2)}(D^{(2)}) = \dim_{\mathcal{N}(\mathbb{Z}^r)}(D_n^{(2)}) = b_n = \dim_{\mathbb{C}\mathbb{Z}^r_{(0)}}(H_n(C_{(0)}))$$

and the claim follows. $\square$

7 L^2-Betti numbers, fibrations and deficiency of groups

In this section we generalize [30, Theorem 3.1] using the algebraic description of L^2-homology as described in this paper. We mention that the proof presented here seems to be more conceptual than the one in [30]. We will show

Theorem 7.1 *Let $d \geq 0$ be an integer and let $F \longrightarrow E \longrightarrow B$ be a fibration of spaces such that F resp. E has the homotopy type of a connected CW-complex with finite d-skeleton resp. $d+1$-skeleton. Let $\overline{F}$ be the covering of F associated to the map $\pi_1(F) \longrightarrow \pi_1(E)$ induced by the inclusion. Suppose that $b_p^{(2)}(\overline{F}; l^2(\pi_1(E))) = 0$ for $p \leq d-1$ and $\pi_1(B)$ contains an element of infinite order. Then we get*

$$b_p^{(2)}(\widetilde{E}) := b_p^{(2)}(\widetilde{E}; l^2(\pi_1(E)) = 0 \qquad \text{for } p \leq d. \quad \square$$

Before we give the proof of Theorem 7.1 we prove the next lemma which is probably well-known but for whose second assertion we could not find a reference.

Lemma 7.2 *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of path-connected spaces and let $d \geq 0$ be an integer. Then:*

1. *If both F and B are homotopy equivalent to finite CW-complexes resp. finite-dimensional CW-complexes resp. CW-complexes of finite type resp. CW-complexes, then the same is true for E.*
2. *Suppose that F is homotopy equivalent to a CW-complex with finite d-skeleton resp. CW-complex of finite type, E is homotopy equivalent to a CW-complex with finite $d+1$-skeleton resp. CW-complex of finite type and B has the homotopy type of a CW-complex. Then B is homotopy equivalent to a CW-complex with finite $d+1$-skeleton resp. CW-complex of finite type.*

Proof. 1.) This is done by induction over the skeletons of B and the fact that any fibration over D^n is fiber homotopy equivalent to the trivial fibration. More details can be found for instance in [36, Sect. 1].

2.) We only treat the case where F has finite d-skeleton and E has finite $d+1$-skeleton. If $d = 0$, the claim follows from the fact that a connected space X of the homotopy type of a CW-complex is homotopy equivalent to a CW-complex with finite 1-skeleton if and only if $\pi_1(X)$ is finitely generated. Hence it remains to treat the case $d \geq 1$.

We construct inductively for $n \geq 1$ a $\max\{2, n\}$-dimensional CW-complex X_n together with a n-connected map $f_n : X_n \longrightarrow B$ such that $\pi_1(f_n)$ is an isomorphisms, X_1 is finite, f_{n+1} extends f_n and, provided that $n \leq d$, the space X_{n+1} is obtained from X_n by attaching finitely many cells of dimension $n + 1$. Then the direct limit yields a CW-complex X with finite $d + 1$-skeleton and a homotopy equivalence $f : X \longrightarrow B$.

The induction begin $n = 1$ is done as follows. Since $\pi_0(F)$ is trivial, $\pi_1(F)$ is finitely generated and $\pi_1(E)$ is finitely presented, we conclude from the long homotopy sequence of the fibration that $\pi_1(B)$ is finitely presented. Let X_1 be the finite 2-dimensional CW-complex associated to some finite presentation of $\pi_1(B)$. Obviously there is a map $f_1 : X_1 \longrightarrow B$ which induces an isomorphism on the fundamental groups.

The induction step from $n \geq 1$ to $n+1$ is done as follows. Let $p_n : Y_n \longrightarrow X_n$ be the pull back fibration of p with respect to f_n

$$\begin{array}{ccc} Y_n & \xrightarrow{g_n} & E \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p} \\ X_n & \xrightarrow[f_n]{} & B \end{array}$$

Let $\widetilde{B} \longrightarrow B$ be the universal covering of B with $\pi = \pi_1(B)$ as group of deck transformations. We obtain the following π-equivariant pull back by pulling back the universal covering of B using the square above

$$\begin{array}{ccc} \overline{Y_n} & \xrightarrow{\overline{g_n}} & \overline{E} \\ {\scriptstyle \overline{p_n}}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ \widetilde{X_n} & \xrightarrow[\widetilde{f_n}]{} & \widetilde{B} \end{array}$$

Notice that $\widetilde{X_n}$ is the universal covering of X_n, as $\pi_1(f_n)$ is bijective. The coverings $\overline{Y_n}$ and $\overline{E}$ are not-necessarily the universal coverings but they have connected total spaces as F is path-connected. The following diagram commutes for $k \geq 1$ where the horizontal maps are Hurewicz homomorphisms and the vertical maps are induced by the square above

$$\begin{array}{ccc} \pi_k(\overline{g_n}) & \longrightarrow & H_k(\overline{g_n}) \\ \downarrow & & \downarrow \\ \pi_k(\widetilde{f_n}) & \longrightarrow & H_k(\widetilde{f_n}) \end{array}$$

The left vertical arrow is an isomorphism for $k \geq 1$ by [49, Corollary 8.8 on page 187]. In particular $\overline{g_n}$ is n-connected because $\widetilde{f_n}$ is n-connected. Since $\overline{Y_n}$ and $\overline{E}$ are path-connected, we conclude from [49, Corollary 7.9 on page 180] that $H_k(\overline{g_n})$ is zero for $k \leq n$. The lower horizontal arrow is an isomorphism for $k \leq n+1$ by the Hurewicz isomorphism since $\widetilde{X_n}$ and $\widetilde{B}$ are simply-connected and $\widetilde{f_n}$ is n-connected [49, Corollary 7.10 on page 181]). Hence the right vertical arrow is surjective for $k = n+1$.

Next we show that $H_{n+1}(\overline{g_n})$ is a finitely generated $\mathbb{Z}\pi$-module provided $n \leq d$. As F and X_n have finite n-skeletons up to homotopy, the same is true for Y_n by the first assertion. By assumption E has finite $n+1$-skeleton up to homotopy. Hence we can assume without loss of generality, that the cellular $\mathbb{Z}\pi$-chain complexes of $\overline{Y_n}$ resp. $\overline{E}$ have finitely generated free $\mathbb{Z}\pi$-modules in dimensions $\leq n$ resp. $\leq n+1$. Hence the mapping cone D of the $\mathbb{Z}\pi$-chain map $C(\overline{g_n})$ has finitely generated free $\mathbb{Z}\pi$-modules as chain modules in dimensions $\leq n+1$. By definition $H_k(D)$ and $H_k(\overline{g_n})$ agree so that $H_k(D)$ is trivial for $k \leq n$ as shown above. Hence there is a long exact sequence of $\mathbb{Z}\pi$-modules

$$\{0\} \longrightarrow \ker(d_{n+1}) \longrightarrow D_{n+1} \xrightarrow{d_{n+1}} D_n \xrightarrow{d_n} \ldots \xrightarrow{d_1} D_0 \longrightarrow \{0\}$$

Since D_k is finitely generated free for all $k \leq n+1$, we conclude that $\ker d_{n+1}$ is a finitely generated projective $\mathbb{Z}\pi$-module. This implies that $H_{n+1}(D) = H_{n+1}(\overline{g_n})$ is finitely generated over $\mathbb{Z}\pi$. Since $H_{n+1}(\widetilde{f_n})$ is a quotient of $H_{n+1}(\overline{g_n})$, it is finitely generated over $\mathbb{Z}\pi$. This shows that $\pi_{n+1}(\widetilde{f_n})$ is finitely generated over $\mathbb{Z}\pi$ provided $n \leq d$.

Given a set of generators of the $\mathbb{Z}\pi$-module $\pi_n(f_n) \cong \pi_n(\widetilde{f_n})$, there is the standard procedure of attaching cells to X_n, one for each generator, to obtain X_{n+1} such that f_n extends to a $(n+1)$-connected map $f_{n+1} : X_{n+1} \longrightarrow B$. This finishes the proof of Lemma 7.2. □

The second assertion 2.) in Lemma 7.2 becomes false if one substitutes of finite type by homotopy equivalent to a finite CW-complex or by homotopy equivalent to a finite-dimensional CW-complex. Here is a counterexample. Realize the exact sequence of abelian groups $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ by a fibration of Eilenberg-MacLane spaces of type 1. Notice that $K(\mathbb{Z}, 1)$ is homotopy equivalent to a circle and $K(\mathbb{Z}/2, 1)$ is not-homotopy equivalent to a finite-dimensional CW-complex because its cohomology ring with $\mathbb{Z}/2$-coefficients is a free polynomial algebra in one generator. One can say nothing about the fiber if one has only information about the total and the base space. For instance the homotopy fiber of the map $S^1 \vee S^1$ to S^1 which is the identity on each component is an infinite wedge of S^1-s and hence not homotopy equivalent to a CW-complex of finite type. Other examples come from path fibrations $\Omega X \longrightarrow PX \longrightarrow X$ over finite CW-complexes since PX is contractible and ΩX has often homology in all dimensions.

Now we are ready to give the proof of Theorem 7.1. Because of Lemma 7.2 we can assume that both E and B are connected CW-complexes with finite $d+1$-skeletons and F is a connected CW-complex with finite d-skeleton. We abbreviate $\Gamma = \pi_1(E)$ and $\pi = \pi_1(B)$. Recall from Sect. 4 that the E^2-term of the Leray-Serre spectral sequence which converges to $H^{\Gamma}_{p+q}(\widetilde{E}; \mathcal{N}(\Gamma))$ is

$$E_{p.q}^2 \;=\; H_p^{\pi}(\widetilde{B}; H_q^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))).$$

Next we show for all $r \geq 2$ that $E_{p,q}^r$ is finitely presented with $\mathbf{P}E_{p,q}^r = 0$ if $p + q \leq d$ and $q \neq d$ and and is finitely generated if $p + q = d + 1$ and $q \leq d - r$. Notice that then the same is true for $r = \infty$ since the Leray-Serre spectral sequence is a first quadrant spectral sequence.

Since F has finite d-skeleton and $b_q^{(2)}(\overline{F}; \mathcal{N}(\Gamma)) = 0$ for $q \leq d - 1$, the $\mathcal{N}(\Gamma)$-module $H_q(\overline{F}; \mathcal{N}(\Gamma))$ is finitely presented with $\mathbf{P}H_q(\overline{F}; \mathcal{N}(\Gamma)) = 0$ for $q \leq d - 1$ because of Lemma 3.7 and Theorem 6.1. Hence $E_{p,q}^2$ is finitely presented with $\mathbf{P}E_{p,q}^2 = 0$ for $p \leq d$ and $q \leq d - 1$ and is finitely generated for $p = d + 1$ and $q \leq d - 1$ since B has finite $d + 1$-skeleton. This finishes the induction begin $r = 2$. The induction step follows from an iterated application of Lemma 3.13 using the fact that the E^{r+1}-term is the homology of the E^r-term and the r-th differentials are $d_{p,q}^r : E_{p.,q}^r \longrightarrow E_{p-r,q+r-1}^r$.

We get from Lemma 3.7 and the Leray-Serre spectral sequence $\mathbf{P}H_n^{\Gamma}(\widetilde{E}; \mathcal{N}(\Gamma)) = 0$ for $n \leq d - 1$ and that there is an exact sequence of $\mathcal{N}(\Gamma)$-modules

$$H_0^{\pi}(\widetilde{B}; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) \longrightarrow H_d^{\Gamma}(\widetilde{E}; \mathcal{N}(\Gamma)) \longrightarrow M \longrightarrow 0$$

where $H_d^{\Gamma}(\widetilde{E}; \mathcal{N}(\Gamma))$ and M are finitely presented $\mathcal{N}(\Gamma)$-modules and $\mathbf{P}M = 0$. Recall from Theorem 6.1 that $b_p^{(2)}(\widetilde{E}) := b_p^{(2)}(\widetilde{E}; l^2(\Gamma))$ vanishes for all $p \leq d$ if and only if $\mathbf{P}H_p^{\Gamma}(\widetilde{E}; \mathcal{N}(\Gamma))$ vanishes for all $p \leq d$. Hence it suffices because of Lemma 3.7 to construct a finitely presented $\mathcal{N}(\Gamma)$-module N with $\mathbf{P}N = 0$ such that there is an epimorphism from N onto $H_0^{\pi}(\widetilde{B}; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma)))$.

Let $f : S^1 \longrightarrow B$ be a map which induces an injection on the fundamental groups. Let $F \longrightarrow E_0 \xrightarrow{p_0} S^1$ and $f_0 : E_0 \longrightarrow E$ be given by the pull back construction. The Leray-Serre spectral sequence for p_0 yields an exact sequence

$$0 \longrightarrow H_0^{\mathbb{Z}}(S^1; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) \longrightarrow H_d^{\Gamma}(\overline{E_0}; \mathcal{N}(\Gamma))$$
$$\longrightarrow H_1^{\mathbb{Z}}(S^1; H_{d-1}^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) \longrightarrow 0$$

where we identify $\pi_1(S^1) = \mathbb{Z}$ and $\overline{E_0} \longrightarrow E_0$ is the covering given by $\pi_1(f_0) : \pi_1(E_0) \longrightarrow \Gamma$. Let $g : F \longrightarrow F$ be a cellular map whose homotopy class is given by the fiber transport for p_0 with a generator in $\mathbb{Z}$. Let T_g be the mapping torus of g which is obtained from the cylinder over F by identifying the bottom and the top by g. There is a homotopy equivalence $h : T_g \longrightarrow E_0$ such that its composition with p_0 is homotopic to the canonical projection $T_g \longrightarrow S^1$. Let $g_d : F_d \longrightarrow F_d$ be the restriction of g to the d-skeleton of F. Denote by $j : T_{g_d} \longrightarrow T_g$ the inclusion. Let $\overline{T_{g_d}}$ and $\overline{T_g}$ be the coverings associated to the homomorphisms from $\pi_1(T_{g_d})$ and $\pi_1(T_g))$ to Γ which are induced by j, h and f_0. Since T_g is obtained from T_{g_d} by attaching cells of dimensions $\geq d + 1$, the inclusion j induces an epimorphism

$$j_* : H_d^{\Gamma}(\overline{T_{g_d}}; \mathcal{N}(\Gamma)) \longrightarrow H_d^{\Gamma}(\overline{T_g}; \mathcal{N}(\Gamma)).$$

Let Δ be the image of $\pi_1(f_0) : \pi_1(E_0) \longrightarrow \Gamma$. Then the map $\pi_1(T_{g_d}) \longrightarrow \mathbb{Z}$ induced by the canonical projection $T_{g_d} \longrightarrow S^1$ factorizes over the epimorphism

$$\pi_1(T_{g_d}) \xrightarrow{\pi_1(j)} \pi_1(T_g) \xrightarrow{\pi_1(h)} \pi_1(E_0) \xrightarrow{\pi_1(f_0)} \Delta.$$

Let $\overline{\overline{T_{g_d}}}$ be the covering of T_{g_d} associated to the map $\pi_1(T_{g_d}) \longrightarrow \Delta$ above. Since T_{g_d} is finite, we conclude from [30, Lemma 1.2.3 on page 205 and Theorem 2.1. on page 207]

$$b_d^{(2)}(\overline{T_{g_d}}; \mathcal{N}(\Gamma)) \;=\; b_d^{(2)}(\overline{\overline{T_{g_d}}}; \mathcal{N}(\Delta)) \;=\; 0.$$

Hence $H_d^{\Gamma}(\overline{T_{g_d}}; \mathcal{N}(\Gamma))$ is finitely presented and $\mathbf{P}H_d^{\Gamma}(\overline{T_{g_d}}; \mathcal{N}(\Gamma))$ is trivial. Since the $\mathcal{N}(\Gamma)$-module $H_1^{\mathbb{Z}}(S^1; H_{d-1}^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma)))$ is finitely presented and $\mathbf{P}H_1^{\mathbb{Z}}(S^1; H_{d-1}^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) = 0$, the kernel N of the composition

$$\begin{aligned} H_d^{\Gamma}(\overline{T_{g_d}}; \mathcal{N}(\Gamma)) \xrightarrow{j_*} H_d^{\Gamma}(\overline{T_g}; \mathcal{N}(\Gamma)) \xrightarrow{h_*} H_d^{\Gamma}(\overline{E_0}; \mathcal{N}(\Gamma)) \\ \longrightarrow H_1^{\mathbb{Z}}(S^1, H_{d-1}^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) \end{aligned}$$

is finitely generated and satisfies $\mathbf{P}N = 0$ because of Lemma 3.7. The epimorphism $h_* \circ j_*$ induces an epimorphism

$$N \longrightarrow H_0^{\mathbb{Z}}(S^1; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))).$$

The map f induces a surjective homomorphism

$$H_0^{\mathbb{Z}}(S^1; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))) \longrightarrow H_0^{\pi}(B; H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma)))$$

as it can be identified with the projection $\mathbb{C} \otimes_{\mathbb{C}\mathbb{Z}} H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma)) \longrightarrow \mathbb{C} \otimes_{\mathbb{C}\pi} H_d^{\Gamma}(\overline{F}; \mathcal{N}(\Gamma))$ where $\mathbb{Z}$ and π act trivially on $\mathbb{C}$. This finishes the proof Theorem 7.1. □

Next we give the promised proof of Theorem 0.6 and Theorem 0.8. Theorem 0.6 follows by induction over n from Theorem 7.1 applied to the fibrations $B\Gamma_i \longrightarrow B\Gamma_{i+1} \longrightarrow B\Gamma_{i+1}/\Gamma_i$ since $b_p^{(2)}(\Gamma_i) = b_p^{(2)}(\overline{B\Gamma_i}; l^2(\Gamma_{i+1}))$ holds for the covering $\overline{B\Gamma_i}$ associated to the inclusion $\Gamma_i \longrightarrow \Gamma_{i+1}$ [30, Lemma 1.2.3 on page 205]. The induction begin follows from $b_0^{(2)}(\Gamma) = |\Gamma|^{-1}$ [30, Lemma 1.2.5 on page 205]. □

Next we prove Theorem 0.8. There is a subgroup $F_1 \subset F$ together with a monomorphism $\Phi : F_1 \longrightarrow F_1$ such that F_1 is isomorphic to F and F is the HNN-extension of F_1 with respect to Φ with one stable letter [4, Proposition 1.7 on page 370]. From the topological description of HNN-extensions [37, page 180] we conclude that F is the fundamental group of the mapping torus $T_{B\Phi}$ of the map $B\Phi : BF_1 \longrightarrow BF_1$ induced by Φ. The inclusion $BF_1 \longrightarrow BF$ induces on the fundamental groups the inclusion of F_1 in F. The argument in [30, page 207] shows that the cellular $\mathbb{Z}F$-chain complex of the universal covering $\widetilde{T_{B\Phi}}$ of $T_{B\Phi}$ is the mapping cone of a certain $\mathbb{Z}F$-chain map from $\mathbb{Z}F \otimes_{\mathbb{Z}F_1} C(EF_1)$ to itself. Since $\mathbb{Z}F$ is free over $\mathbb{Z}F_1$, we conclude for $p \geq 1$

$$H_p(\mathbb{Z}F \otimes_{\mathbb{Z}F_1} C(EF_1)) = \mathbb{Z}F \otimes_{\mathbb{Z}F_1} H_p(C(EF_1)) = 0.$$

This implies $H_p(\widetilde{T_{B\Phi}}; \mathbb{Z}) = 0$ for $p \geq 2$. Hence $T_{B\Phi}$ is a model for BF. Now the claim follows from [30, Theorem 2.1. on page 207]. □

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