“L^2-invariants of regular coverings of compact manifolds and CW-complexes”

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Introduction

When we were asked to write this survey article for the Handbook of Geometry, we began to wonder who will read such an article. Well, certainly not someone who wants to fix his car. The possible readers could be 1.) students or more advanced mathematicians who are looking for new problems, 2.) people who are curious about this topic and want to get a first impression, 3.) others who are interested in this topic already and want to invest some time to learn more about the techniques, 4.) experts who want to see new developments and an extensive list of literature (with all their papers cited) or 5.) people who want to read and absorb something new without big effort while watching a Star Trek episode or a soccer game. We hope that any of these groups can get something from this survey article. The appeal of the topic comes from its connections to rather different fields of mathematics like spectral theory of the Laplace operator, questions about metrics with certain curvature properties on manifolds, isomorphism conjectures in algebraic K-theory and the Baum-Connes Conjecture, low-dimensional manifolds, group theory, index theory, intersection homology, representation theory of Lie groups and so on.

For the groups 1.) and 2.) we recommend to glance at section 1 and then read section 2 where the main problems are stated. Moreover, there are other conjectures and open problems stated throughout the other sections. We treat Conjecture 2.1 about the rationality of the L^2-Betti numbers, the Singer Conjecture 2.6 about the vanishing of the L^2-Betti numbers outside the middle dimension of the universal covering of an aspherical closed manifold, the Hopf Conjecture 2.7 which is a version of the Singer Conjecture for closed Riemannian manifolds with negative sectional curvature, Conjecture 8.9 about the rationality and positivity of the Novikov-Shubin invariants, Conjecture 9.10 about the triviality of the homomorphism $\Phi_\Gamma : \mathrm{Wh}(\Gamma) \longrightarrow \mathbb{R}^{>0}$ given by the Fuglede-Kadison determinant, Conjecture 9.18 about the relation of the simplicial volume and the L^2-torsion of aspherical closed manifolds, Conjecture 9.24 about the vanishing of the L^2-torsion for closed aspherical manifolds with amenable fundamental groups and the zero-in-the-spectrum Conjecture 11.1 and 11.4. None of these problems seem to be easy and have created a lot of work as discussed in sections 3, 4, 7, 8 and 9. These sections and sections 5, 6, 10 and 12 are completely independent of one another except for section 9 which uses some information from section 8, but nevertheless can be read without knowing section 8. So the reader has not to be overwhelmed by the length of this article, but pick what he is interested in. For section 11, however, we recommend to read through section 8 first. For the groups 1.), 2.) and in particular 3.) we have included some of the proofs in the text although they are very often somewhere in the literature. The motivation is sometimes that the proofs themselves are very illuminating or just nice, that the proofs are hard to find or that the proofs presented here are, hopefully, easier or better to understand than the one in the literature. It is certainly possible to read only the definitions, lemmas and theorems and skip all the proofs or additional information. We recommend this in particular to the fifth group of readers, and also the last episode of Star Trek – The Next Generation, we believe the title is "All good things" (in German TV "gestern, heute, morgen"). Although the list of references is quite long, it may well have happened that we have not cited a paper which should appear there, and we apologize for that.

This survey contains also mini-surveys on the following topics: 3-manifolds in section 3, amenable groups in section 4, residually finite groups in section 5, Kähler manifolds in section 7 and analytic Ray-Singer torsion and Reidemeister torsion in section 9. There are other survey articles on topics of this survey article or related topics, for instance [42], [109, section 8], [144], [158], [171] [203], [226], [234].

We mention that we have for simplicity restricted ourselves to the von Neumann algebra of a group. One can formulate a lot of the results also for arbitrary von Neumann algebras. Group means always discrete group. We have restricted ourselves to regular coverings, applications of L^2-cohomology to other topics are very briefly mentioned in section 12. We wish to thank the Max-Planck-Institut in Bonn for its hospitality while parts of this article were written, John Lott for a lot of discussions about this topic and Clemens Bratzler for reading through the manuscript.

1. L^2-Betti numbers for CW-complexes of finite type

In this section we introduce L^2-Betti numbers for regular coverings of CW-complexes of finite type and discuss their main properties.

Let X be a connected CW-complex of finite type. *Finite type* means that each skeleton of X is finite, but X may have infinite dimension. The *p-th Betti number* $b_p(X)$ is defined by the rank of the finitely generated abelian group $H_p(X)$ given by the cellular or, equivalently, singular homology. In algebraic topology it has turned out to be useful to improve classical invariants such as Euler characteristic and signature by passing to the universal covering and taking the action of the fundamental group into account. This leads, for instance, in the case of the Euler characteristic to the finiteness obstruction, and in the case of the signature to surgery obstructions such as symmetric signatures. We want to apply this strategy to Betti numbers.

The following naive approach does not work. One could think of applying an appropriate notion of dimension for modules over the integal group ring $\mathbb{Z}\pi$ of the fundamental group π to the homology $H_p(\widetilde{X})$ of the universal covering. Notice that the π-action on $\widetilde{X}$ induces a $\mathbb{Z}\pi$-module structure on $H_p(\widetilde{X})$. The problem is that in general $\mathbb{Z}\pi$ is not Noetherian. Hence $H_p(\widetilde{X})$ is not necessarily finitely generated although X is of finite type and therefore the p-th module $C_p(\widetilde{X})$ of the cellular $\mathbb{Z}\pi$-chain complex is finitely generated over $\mathbb{Z}\pi$. So the dimension may be infinite. Moreover, it is not at all clear that the dimension is additive.

The reason why a lot of algebraic manipulations work nicely for the complex group ring of a finite group is that this ring is semi-simple. It has this property because there are enough projections, this being a consequence of the fact that $\mathbb{C}\Gamma$ is a Hilbert space. Namely, an ideal I in $\mathbb{C}\Gamma$ is a direct summand because it has an orthogonal complement. This property does not hold for infinite groups. However, if one enlarges the complex group ring to the von Neumann algebra, all the convenient properties of the complex group ring of a finite group carry over to arbitrary groups. This motivates the following definitions.

If Γ is a group, define $l^2(\Gamma)$ by the Hilbert space of square-summable formal sums $\sum_{\gamma\in\Gamma}\lambda_\gamma\gamma$ with complex coefficients λ_γ. Square-summable, of course, means $\sum_{\gamma\in\Gamma}|\lambda_\gamma|^2<\infty$ and the inner product is given by

$$\langle\sum_{\gamma\in\Gamma}\lambda_\gamma\gamma,\sum_{\gamma\in\Gamma}\mu_\gamma\gamma\rangle \;=\; \sum_{\gamma\in\Gamma}\overline{\lambda_\gamma}\cdot\mu_\gamma.$$

The *group von Neumann algebra* of Γ is defined by the space of Γ-equivariant bounded operators from $l^2(\Gamma)$ to itself

$$\mathcal{N}(\Gamma) \;=\; \mathcal{B}(l^2(\Gamma),l^2(\Gamma))^\Gamma \tag{1.1}$$

where $l^2(\Gamma)$ is equipped with the obvious left Γ-action. This is not the standard definition, but equivalent to it. The *standard trace* of the von Neumann algebra is defined by

$$\operatorname{tr}=\operatorname{tr}_{\mathcal{N}(\Gamma)}:\mathcal{N}(\Gamma)\longrightarrow\mathbb{C}\qquad f\mapsto\langle f(e),e\rangle_{l^{(2)}(\Gamma)}$$

where $e\in\Gamma\subset l^2(\Gamma)$ is the unit element. This trace extends to matrices

$$\operatorname{tr}:M(n,n,\mathcal{N}(\Gamma)) \;\longrightarrow\; \mathbb{C} \tag{1.2}$$

by sending a matrix to the sum of the traces of the diagonal entries.

A *Hilbert $\mathcal{N}(\Gamma)$-module* is a Hilbert space V together with a left action of Γ by linear isometries such that there is a Hilbert space H and a Γ-equivariant isometric embedding of V into the tensor product of Hilbert spaces $l^2(\Gamma)\widehat{\otimes}H$. The isometric embedding is not part of the structure, only its existence is required. An example is $l^2(\Gamma)$ itself with the obvious left Γ-action. A Hilbert $\mathcal{N}(\Gamma)$-module is *finitely generated* if there is a surjective bounded Γ-equivariant operator from $l^2(\Gamma)^n=\oplus_{i=1}^n l^2(\Gamma)$ onto V for an appropriate positive integer n. This is equivalent to the existence of an isometric Γ-equivariant embedding of V into $l^2(\Gamma)^n$ for an appropriate positive integer n and to the existence of an orthogonal Γ-equivariant projection $\operatorname{pr}:l^2(\Gamma)^n\longrightarrow l^2(\Gamma)^n$ whose image is isometrically Γ-isomorphic to V for an appropriate positive integer n.

A *morphism of Hilbert $\mathcal{N}(\Gamma)$-modules* $f:U\longrightarrow V$ is a bounded Γ-equivariant operator. We call f a *weak isomorphism* if its kernel is trivial and its image is dense. The polar decomposition of a weak isomorphism f looks like $i\circ|f|$ where $|f|:U\longrightarrow U$ is a positive morphism and $i:U\longrightarrow V$ an isometric isomorphism. In particular U and V are isometrically isomorphic if there is a weak isomorphism from U to V. A sequence of Hilbert $\mathcal{N}(\Gamma)$-modules $0\to U\xrightarrow{i}V\xrightarrow{p}W\to 0$ is called *exact* if i is injective, $\operatorname{im}(i)=\ker(p)$ and p is surjective. It is called *weakly exact* if i is injective, $\operatorname{clos}(\operatorname{im}(i))=\ker(p)$ and $\operatorname{clos}(\operatorname{im}(p))=W$.

Definition 1.3 *Let V be a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module. Define its* von Neumann dimension *by*

$$\dim(V) = dim_{\mathcal{N}(\Gamma)}(V) \;=\; \mathrm{tr}(\mathrm{pr}) \qquad \in [0, \infty)$$

where $\mathrm{pr} : l^2(\Gamma)^n \longrightarrow l^2(\Gamma)^n$ *is any orthogonal Γ-equivariant projection whose image is isometrically Γ-isomorphic to V.* ■

It is not hard to check that the definition above is independent of the choice of projection. The elementary proof of the next result is left to the reader. It follows from the general properties of the universal center valued trace of a finite von Neumann algebra [131, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525 and Theorem 8.4.3 on page 532].

Lemma 1.4 *Let U,V and W be finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Then*

1. *Faithfulness*

   $\dim_{\mathcal{N}(\Gamma)}(U) = 0$ *if and only if* $U = 0$*;*

2. *Monotony*

   If $U \subset V$ then

   $$\dim_{\mathcal{N}(\Gamma)}(U) \leq \dim_{\mathcal{N}(\Gamma)}(V);$$

3. *Continuity*

   If $U_1 \supset U_2 \supset \ldots$ is a nested sequence of Hilbert $\mathcal{N}(\Gamma)$-submodules of U, then

   $$\dim_{\mathcal{N}(\Gamma)}\left(\bigcap_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} \dim_{\mathcal{N}(\Gamma)}(U_n);$$

4. *Weak exactness*

   If $0 \longrightarrow U \xrightarrow{j} V \xrightarrow{q} W \longrightarrow 0$ *is weakly exact, then*

   $$\dim_{\mathcal{N}(\Gamma)}(V) = \dim_{\mathcal{N}(\Gamma)}(U) + \dim_{\mathcal{N}(\Gamma)}(W). \qquad ■$$

We will extend this notion to arbitrary modules over $\mathcal{N}(\Gamma)$ in section 10.

Definition 1.5 *Let X be a (not necessarily connected) CW-complex of finite type. Let $p : \overline{X} \longrightarrow X$ be a regular covering of X with group of deck transformations Γ acting from the left. Define the* cellular L^2-chain complex *of $\overline{X}$ by the chain complex of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules*

$$C_*^{(2)}(\overline{X}) \;=\; l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(\overline{X}).$$

The cellular L^2-cochain complex *is defined by*

$$C^*_{(2)}(\overline{X}) \;=\; \hom_{\mathbb{Z}\Gamma}(C_*(\overline{X}), l^2(\Gamma))$$

where for $\gamma \in \Gamma$ *and* $f \in C^*_{(2)}(\overline{X})$ *the element* $\gamma \cdot f$ *sends* $x \in C_*(\overline{X})$ *to* $f(x)\gamma^{-1}$. *Define the* L^2-homology *of* $\overline{X}$ *by*

$$H_p^{(2)}(\overline{X}) \;=\; \ker(c_p^{(2)})/\operatorname{clos}(\operatorname{im}(c_{p+1}^{(2)}))$$

where $c_*^{(2)}$ *is the differential of* $C_*^{(2)}(\overline{X})$. *The definition of* L^2-cohomology *is analogous. Define the* p-th L^2-Betti number *of* $\overline{X}$ *by*

$$b_p^{(2)}(\overline{X}) \;=\; \dim_{\mathcal{N}(\Gamma)}(H_p^{(2)}(\overline{X})) \;=\; \dim_{\mathcal{N}(\Gamma)}(H^p_{(2)}(\overline{X})). \qquad \blacksquare$$

The decisive difference between L^2-homology and the Γ-equivariant homology with coefficients in $l^2(\Gamma)$ viewed as $\mathbb{Z}\Gamma$-module is that we divide by the closure of the image of the corresponding differential and not only by the image itself. The reason is that in the L^2-setting we want to keep the Hilbert space structure coming from the cellular L^2-chain complex. In order to guarantee completeness we must divide by a closed subspace. We will investigate the difference between these two homologies later when we introduce Novikov-Shubin invariants in section 8 which measure this difference. Hilbert complexes in general (not necessarily over a finite von Neumann algebra) are treated in [39].

If we deal with a smooth manifold M, then these definitions are understood for the CW-complex structure given by some smooth triangulation. We will prove in Theorem 1.7 that the choice of triangulation does not matter because two triangulations of M give homotopy equivalent CW-complexes. In the sequel we will denote by $\widetilde{X}$ the universal covering with the fundamental group $\pi_1(X)$ as group of deck transformations Γ, provided that X is connected.

Notice that both $H_p^{(2)}(\overline{X})$ and $H^p_{(2)}(\overline{X})$ are isometrically Γ-isomorphic to the kernel of the *combinatorial Laplace operator*

$$\Delta_p \;=\; (c_p^{(2)})^* \circ c_p^{(2)} + c_{p+1}^{(2)} \circ (c_{p+1}^{(2)})^* \;:\; C_p^{(2)}(\overline{X}) \longrightarrow C_p^{(2)}(\overline{X}). \tag{1.6}$$

The elementary proof can be found in [163, Theorem 3.7 on page 230]. Hence there is no difference between homology and cohomology in the L^2-setting.

Next we discuss the main properties of L^2-Betti numbers, in particular in comparison with the ones of the ordinary Betti numbers.

Theorem 1.7 *1. Homotopy invariance*

Let $\overline{X}$ *and* $\overline{Y}$ *be regular coverings of* CW*-complexes* X *and* Y *of finite type with the same group* Γ *of deck transformations. Let* $f : \overline{X} \longrightarrow \overline{Y}$ *be a* Γ*-equivariant map. If* f *is a homotopy equivalence, then*

$$b_p^{(2)}(\overline{X}) \;=\; b_p^{(2)}(\overline{Y}) \qquad \text{for } 0 \le p.$$

If f is d-connected, i.e. f induces an isomorphism on π_n for $n < d$ and an epimorphism on π_d, then

$$\begin{array}{rcll} b_p^{(2)}(\overline{X}) & = & b_p^{(2)}(\overline{Y}) & \text{for } p < d; \\ b_d^{(2)}(\overline{X}) & \geq & b_d^{(2)}(\overline{Y}). & \end{array}$$

2. *Euler-Poincaré formula*

   Let $\overline{X}$ be a regular covering of a finite CW-complex X. Let

$$\chi(X) \;=\; \sum_{p \geq 0} (-1)^p \cdot \beta_p(X) \qquad \in \mathbb{Z}$$

   be the Euler characteristic of X where $\beta_p(X)$ is the number of p-cells of X. Then

$$\chi(X) \;=\; \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\overline{X});$$

3. *Poincaré duality*

   Let $\overline{M}$ be a regular covering of the closed manifold M of dimension n. Then

$$b_p^{(2)}(\overline{M}) \;=\; b_{n-p}^{(2)}(\overline{M});$$

4. *Künneth formula*

   Let X and Y be CW-complexes of finite type. Let $\overline{X}$ and $\overline{Y}$ be regular coverings of X and Y. Then $\overline{X} \times \overline{Y}$ is a regular covering of $X \times Y$ and

$$b_n^{(2)}(\overline{X} \times \overline{Y}) \;=\; \sum_{p+q=n} b_p^{(2)}(\overline{X}) \cdot b_q^{(2)}(\overline{Y}) \qquad \text{for } n \geq 0;$$

5. *Morse inequalities*

   Let $\overline{X}$ be a regular covering of a CW-complex X of finite type. Let $\beta(X)$ be the number of p-cells in X. Then

$$\sum_{p=0}^{n} (-1)^{n-p} \cdot b_p^{(2)}(\overline{X}) \;\leq\; \sum_{p=0}^{n} (-1)^{n-p} \cdot \beta_p(X) \qquad \text{for } n \geq 0;$$

6. *L^2-Hodge-deRham decomposition*

   Let $\overline{M}$ be a covering of the oriented closed Riemannian manifold M with deck transformation group Γ. Let $\mathcal{H}_{(2)}^p(\overline{M})$ be the space of harmonic smooth L^2-p-forms on $\overline{M}$, i.e. smooth p-forms ω on $\overline{M}$ such that $\int_{\overline{M}} \omega \wedge *\omega$ is finite and ω lies in the kernel of the Laplace operator with respect to the induced Riemannian metric on $\overline{M}$. Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules

$$\mathcal{H}_{(2)}^p(\overline{M}) \longrightarrow H_{(2)}^p(\overline{M});$$

7. *Multiplicative property for finite coverings*

   Let X be a CW-complex of finite type and $p : \overline{X} \longrightarrow X$ be a regular covering with group of deck transformations Γ. Let $\Gamma_0 \subset \Gamma$ be a subgroup of Γ of finite index n. We obtain a regular covering denoted by $\overline{\overline{X}}$ by $\overline{X} \longrightarrow \overline{X}/\Gamma_0$. Notice that the coverings $\overline{\overline{X}}$ and $\overline{X}$ have the same total spaces but different groups of deck transformations. Then

$$b_p^{(2)}(\overline{\overline{X}}) \;=\; n \cdot b_p^{(2)}(\overline{X}) \qquad \textit{for } p \geq 0;$$

8. *L^2-Betti numbers for finite groups Γ*

   Let X be a CW-complex of finite type and let $p : \overline{X} \longrightarrow X$ be a regular covering with group of deck transformations Γ of finite order $|\Gamma|$. Then:

$$b_p^{(2)}(\overline{X}) \;=\; \frac{1}{|\Gamma|} \cdot b_p(\overline{X}) \qquad \textit{for } p \geq 0;$$

9. *Zero-th L^2-Betti number*

   Let X be a connected CW-complex of finite type and let $p : \overline{X} \longrightarrow X$ be a regular covering with group of deck transformations Γ and connected $\overline{X}$. Then

$$b_0^{(2)}(\overline{X}) \;=\; \begin{cases} \frac{1}{|\Gamma|} & \textit{if } |\Gamma| < \infty; \\ 0 & \textit{otherwise;} \end{cases}$$

10. *S^1-actions and L^2-Betti numbers*

    Let M be a connected closed manifold with S^1-action. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into M induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Then

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \qquad \textit{for } p \geq 0.$$

Proof : 1.) Because of the Equivariant Cellular Approximation Theorem we can assume that f is cellular. Since $\overline{X}$ and $\overline{Y}$ are free Γ-CW-complexes the map f is even a Γ-homotopy equivalence. Because of the Equivariant Cellular Approximation Theorem there is a cellular Γ-map $g : \overline{Y} \longrightarrow \overline{X}$ such that there is a cellular Γ-homotopy between $f \circ g$ (resp. $g \circ f$) and the identity. One easily checks that two cellular Γ-maps which are connected by a cellular Γ-homotopy induce the same map on L^2-homology. Now the claim for a homotopy equivalence f follows. The more general case of a d-connected Γ-map f is proven in [148, Lemma 3.3].

2.) This follows as in the classical situation, where $\chi(X)$ is expressed in terms of (ordinary) Betti numbers, from the fact that the von Neumann dimension is weakly additive (see Theorem 1.4.4).

3.) follows from the Poincaré $\mathbb{Z}\Gamma$-chain homotopy equivalence [247, Theorem 2.1 on page 23]

$$\cap[M] : C^{n-*}(\overline{M})) \longrightarrow C_*(\overline{M}).$$

4.) follows from the isomorphism of cellular chain complexes

$$C(\overline{X}) \otimes_{\mathbb{Z}} C(\overline{Y}) \longrightarrow C(\overline{X} \times \overline{Y}).$$

See also [255, Corollary 2.36 on page 181].

5.) analogous to 2.).

6.) is proven by Dodziuk [72].

7.) If V is a finitely generated $\mathcal{N}(\Gamma)$-module and $\mathrm{res}(V)$ its restriction to $\mathcal{N}(\Gamma_0)$ which is a finitely generated Hilbert $\mathcal{N}(\Gamma_0)$-module, then:

$$\dim_{\mathcal{N}(\Gamma_0)}(\mathrm{res}(V)) \;=\; n \cdot \dim_{\mathcal{N}(\Gamma)}(V).$$

8.) follows from 7.)

9.) If Γ has finite order, this follows from 8.). If Γ is infinite, this follows from the fact that $l^2(\Gamma)^\Gamma$ is trivial.

10.) is proven in [163, Theorem 3.20 on page 235]. ■

More information about Morse inequalities and L^2-invariants can be found in [89], [90], [157], [172], [173], [196], [197], [234].

Example 1.8 A good source of well understood examples is given by the special case where Γ is the free abelian group $\mathbb{Z}^r$ of rank r. On one hand everything becomes simple, on the other hand one can already see some of the important phenomenons in this special case. See also [57, section 5.], [77].

One simplification comes from Fourier transformation. Namely, we obtain a natural isometric $\mathbb{Z}^r$-equivariant isomorphism

$$l^2(\mathbb{Z}^r) \longrightarrow L^2(T^r)$$

where $L^2(T^r)$ is the Hilbert space of L^2-functions on T^r. Let $L^\infty(T^r)$ be the C^*-algebra of essentially bounded measurable functions on T^r. Then we obtain an isomorphism of C^*-algebras

$$M : L^\infty(T^r) \longrightarrow \mathcal{N}(\mathbb{Z}^r) = \mathcal{B}(L^2(T^r), L^2(T^r))^{\mathbb{Z}^r} \qquad f \mapsto M_f$$

where $M_f : L^2(T^r) \longrightarrow L^2(T^r)$ sends g to the function $f \cdot g$ which assigns to $z \in T^r$ the element $f(z) \cdot g(z)$. A morphism of Hilbert $\mathcal{N}(\mathbb{Z}^r)$-modules $L^2(T^r) \longrightarrow L^2(T^r)$ is given by M_f for some $f \in L^\infty(T^r)$. It is a weak isomorphism if and only if $f^{-1}(0)$ is a set of measure zero, and it is an isomorphism if for some $\epsilon > 0$ the set $\{z \in T^r \mid |f(z)| < \epsilon\}$ has measure zero.

In particular we see concrete examples of weak isomorphisms which are not isomorphisms. An important example is given by $M_{z_i - 1}$. The operator M_f is positive if and only if f takes values in the non-negative real numbers. In this case the spectral family $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ is given by $E_\lambda = M_{\chi_\lambda}$ where χ_λ is the characteristic function of the set $\{z \in T^r \mid f(z) \leq \lambda\}$. The von Neumann trace $\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^r)}$ becomes

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^r)} : L^\infty(T^r) \longrightarrow \mathbb{C} \qquad f \mapsto \int_{T^r} f.$$

Let $\overline{X} \longrightarrow X$ be a regular covering of a CW-complex of finite type with $\mathbb{Z}^r$ as group of deck transformations. Let $\mathbb{Z}[\mathbb{Z}^r]_{(0)}$ be the quotient field of the integral group ring of $\mathbb{Z}^r$. Let $\dim_{\mathbb{Z}[\mathbb{Z}^r]_{(0)}} \left(H_p(\overline{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^r]} \mathbb{Z}[\mathbb{Z}^r]_{(0)}\right)$ be the dimension of the finite-dimensional vector space of the quotient field. Then we get [157, Example 4.3]

$$b_p^{(2)}(\overline{X}) \;=\; \dim_{\mathbb{Z}[\mathbb{Z}^r]_{(0)}} \left(H_p(\overline{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^r]} \mathbb{Z}[\mathbb{Z}^r]_{(0)}\right). \qquad \blacksquare$$

Remark 1.9 The L^2-Hodge-deRham decomposition of Theorem 1.7.6 proves that for a regular covering $\overline{M} \longrightarrow M$ of a closed Riemannian manifold M the L^2-Betti numbers have the following analytic interpretation. Let $L^2\Omega^p(\overline{M})$ be the Hilbert space of all square-integrable $\mathbb{C}$-valued p-forms on $\overline{M}$. This is the Hilbert space completion of the space $C_0^\infty\Omega^p(\overline{M})$ of smooth $\mathbb{C}$-valued p-forms on $\overline{M}$ with compact support and the standard L^2-pre-Hilbert structure. Since $\overline{M}$ is complete (with respect to the lifted Riemannian metric), the Laplace operator Δ_p is essentially selfadjoint in $L^2\Omega^p(\overline{M})$, i.e. its closure with respect to the domain $C_0^\infty\Omega^p(\overline{M})$ is a self-adjoint operator on $L^2\Omega^p(\overline{M})$ [55]. Let $\Delta_p = \int \lambda dE_\lambda^p$ be the spectral decomposition with right-continuous spectral family $\{E_\lambda^p \mid \lambda \in \mathbb{R}\}$. Let $E_\lambda^p(\overline{x}, \overline{y})$ be the Schwartz kernel of E_λ^p. Since $E_\lambda^p(\overline{x}, \overline{x})$ is an endomorphism of a finite-dimensional complex vector space, its trace $\operatorname{tr}_{\mathbb{C}}$ is defined. Let $\mathcal{F}$ be a fundamental domain for the Γ-action on $\overline{M}$, i.e. an open subset $\mathcal{F}$ in $\overline{M}$ such that $\cup_{\gamma \in \Gamma} \gamma \operatorname{clos}(\mathcal{F}) = \overline{M}$ and $\gamma\mathcal{F} \cap \mathcal{F} = \emptyset$ for $\gamma \in \Gamma$ with $\gamma \neq 1$ [210, section 6.5]. Then the *analytic p-th spectral density function* is defined by

$$F^p(\lambda) \;=\; \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}} \left(E_\lambda^p(\overline{x}, \overline{x})\right) \, d\overline{x} \qquad \lambda \in \mathbb{R}. \tag{1.10}$$

We will later investigate this spectral density function more closely. We mention the equality

$$b_p^{(2)}(\overline{M}) \;=\; F^p(0). \tag{1.11}$$

By means of a Laplace transformation we obtain the equality

$$b_p^{(2)}(\overline{M}) \;=\; \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}} \left(e^{-t\Delta_p}(\overline{x}, \overline{x})\right) d\overline{x}. \tag{1.12}$$

where $e^{-t\Delta_p}(\overline{x}, \overline{y})$ denotes the heat kernel on $\overline{M}$, i.e. the Schwartz kernel of $e^{-t\Delta_p}$. $\blacksquare$

All the definitions and results above extend to pairs of proper cocompact Γ-CW-complexes and manifolds with boundary and proper cocompact Γ-actions [163, section 3 and 5]. The L^2-Hodge-deRham decomposition for manifolds with boundary is proven in [228].

2. Basic conjectures

In this section we state the (in our view) most important conjectures concerning L^2-Betti numbers. They motivate a lot of the work done on this subject. We also give a list of theorems which give evidence for them. Results of sections 3, 4 and 7 also prove these conjectures in special cases. More questions and conjectures will be discussed in other sections.

Conjecture 2.1 (Rationality of L^2-Betti numbers) *Let Γ be a group and A be a (m,n)-matrix with coefficients in $\mathbb{Z}\Gamma$. Let $f : l^2(\Gamma)^m \longrightarrow l^2(\Gamma)^n$ be the Γ-equivariant bounded operator induced by right multiplication with A. Then*

1. $\dim_{\mathcal{N}(\Gamma)}(\ker(f))$ *is rational;*
2. *Let k be a positive integer such that the order of any finite subgroup of Γ divides k. Then $k \cdot \dim_{\mathcal{N}(\Gamma)}(\ker(f))$ is an integer;*
3. *If Γ is torsionfree, then $\dim_{\mathcal{N}(\Gamma)}(\ker(f))$ is an integer.* ∎

Conjecture 2.1 above has the following equivalent reformulations provided that Γ is finitely presented.

Lemma 2.2 *The following statements are equivalent for a finitely presented group Γ:*

1. *Γ satisfies Conjecture 2.1.1 (resp. 2.1.2, resp. 2.1.3);*
2. *If X is a connected CW-complex of finite type with Γ as fundamental group, then $b_p^{(2)}(\widetilde{X})$ satisfies the corresponding statement of Conjecture 2.1.1 (resp. 2.1.2, resp. 2.1.3) for all $p \geq 0$;*
3. *If M is a closed manifold with Γ as fundamental group, then $b_p^{(2)}(\widetilde{M})$ satisfies the corresponding statement of Conjecture 2.1.1 (resp. 2.1.2, resp. 2.1.3) for all $p \geq 0$.*

∎

Proof : Obviously 1.) implies 2.) and 2.) implies 3.) so that it remains to prove that 3.) implies 1.). The key observation is the following. Let A be a (m,n)-matrix over $\mathbb{Z}\Gamma$ and $d \geq 3$ be an integer. Choose a finite connected 2-dimensional CW-complex X with Γ as fundamental group. By attaching n $(d-1)$-cells to X with a trivial attaching map and m d-cells one can construct a finite CW-complex Y such that the d-th differential of the cellular $\mathbb{Z}\Gamma$-chain complex $C(\widetilde{Y})$ is, for $d \geq 4$, given by A and for $d = 3$ by the map induced by A composed with the canonical inclusion $l^2(\Gamma)^n \longrightarrow l^2(\Gamma)^n \oplus l^2(\Gamma)^{n'}$ where n' is the number of 2-cells in X. Let $f : l^2(\Gamma)^m \longrightarrow l^2(\Gamma)^n$ be the bounded Γ-equivariant operator induced by A. Since Y has no $(d+1)$-cells, the kernel of f is just $H_d^{(2)}(\widetilde{Y})$. Next one can embed the finite d-dimensional CW-complex Y into $\mathbb{R}^{2d+2}$. Let M be the boundary of a regular neighborhood of X [221, chapter 3]. Then there is a $(d+1)$-connected map $M \longrightarrow X$ and because of Theorem 1.7.1

$$b_d^{(2)}(\widetilde{M}) \;=\; b_d^{(2)}(\widetilde{Y}) \;=\; \dim(\ker(f)). \qquad \blacksquare$$

Remark 2.3 If Conjecture 2.1 holds for all finitely generated groups it holds for all groups. Namely, given a (m,n) matrix A with coefficients in $\mathbb{Z}\Gamma$, it suffices to look at the subgroup Γ' generated by those elements which appear in one of the entries of A with non-zero coefficients. Namely, A can be viewed as a matrix over $\mathbb{Z}\Gamma'$ and the dimension over Γ of the associated Γ-equivariant bounded operator agrees with the one over Γ'.

Lemma 2.2 remains true if we substitute the assumption finitely presented for Γ by finitely generated and substitute the universal coverings $\widetilde{X}$ (resp. $\widetilde{M}$) by regular coverings $\overline{X} \longrightarrow X$ (resp. $\overline{M} \longrightarrow M$) for a connected finite CW-complex (resp. closed manifold M). The modification in the proof is as follows. Choose a finitely generated free group F together with an epimorphism $p : F \longrightarrow \Gamma$. Lift A to a matrix A' over $\mathbb{Z}[F]$ and construct X, Y and M as explained in the proof of Lemma 2.2 for F and A'. Then take the coverings of Y (resp. M) associated to p with Γ as group of deck transformations.

Hence Conjecture 2.1 is true for all groups Γ and matrices A if and only if for all regular coverings $\overline{M} \longrightarrow M$ of closed manifolds M with a finitely generated group of deck transformations $b_p^{(2)}(\overline{M})$ satisfies the corresponding statement. $\blacksquare$

The question of whether the third statement in Lemma 2.2 is true was asked by Atiyah [3, page 72]. Next we show that Conjecture 2.1 implies the Zero-Divisor-Conjecture. For a discussion of the Zero-Divisor-Conjecture and related conjectures we refer to [138, page 95] .

Lemma 2.4 *If the group Γ satisfies Conjecture 2.1.3, then it also satisfies the Zero-Divisor-Conjecture which says: The rational group ring $\mathbb{Q}\Gamma$ has no non-trivial zero-divisors if and only if Γ is torsionfree.* $\blacksquare$

Proof : Suppose that Γ is not torsionfree. Let $g \in \Gamma$ be an element of finite order $|g| \neq 1$. Then the norm element $N = \sum_{k=1}^{|g|} g^k$ is a zero-divisor in $\mathbb{Q}\Gamma$ since it satisfies $N \cdot (|g| - N) = 0$.

Suppose that Γ is torsionfree. Let $x \in \mathbb{Q}\Gamma$ be a zero-divisor. We have to show, under the assumption that Conjecture 2.1.3 is true, that x is trivial. By multiplying x with an appropriate integer, we can assume without loss of generality that x belongs to $\mathbb{Z}\Gamma$. Right multiplication with x induces a Γ-equivariant bounded operator $r_x : l^2(\Gamma) \longrightarrow l^2(\Gamma)$. Conjecture 2.1 implies that $\dim_{\mathcal{N}(\Gamma)}(\ker(r_x))$ is an integer. As x is a zero-divisor, the kernel of r_x is not trivial and hence $\dim_{\mathcal{N}(\Gamma)}(\ker(r_x))$ is not zero. Since $\dim_{\mathcal{N}(\Gamma)}(\ker(r_x))$ is bounded by the dimension of $l^2(\Gamma)$, which is 1, we get $\dim_{\mathcal{N}(\Gamma)}(\ker(r_x)) = 1 = \dim_{\mathcal{N}(\Gamma)}(l^2(\Gamma))$. As the kernel of r_x is a closed subspace of $l^2(\Gamma)$, this implies that r_x is the trivial map. Hence x is zero. This shows that the rational group ring has no non-trivial zero-divisors. ■

Let $\mathcal{C}$ be the smallest class of groups which i.) contains all free groups, ii.) is closed under directed unions and iii.) satisfies $G \in \mathcal{C}$ whenever G contains a normal subgroup H such that H belongs to $\mathcal{C}$ and G/H is elementary amenable. We recall that the class of *elementary amenable* groups is defined as the smallest class of groups which contains all finite and all abelian groups, and is closed under taking subgroups, forming factor groups, group extensions, and upwards directed unions.

Theorem 2.5 (Linnell [140]) *Conjecture 2.1 holds for the class $\mathcal{C}$ of groups.* ■

The key result in [140] is that for a torsionfree group Γ in the class $\mathcal{C}$ there is a division ring $D(\Gamma)$ satisfying $\mathbb{C}\Gamma \subset D(\Gamma) \subset U(\Gamma)$, where $U(\Gamma)$ is the algebra of closed densely defined operators $l^2(\Gamma) \longrightarrow l^2(\Gamma)$ which are affiliated to the group von Neumann algebra. Linnell uses the Fredholm technique developed by Connes for his proof that the reduced C^*-algebra of a free group has no non-trivial projections [62, section 7], [83], Moody's induction theorem cite[Theorem 1]Moody (1989) and Cohn's theory of universal fields of fractions [61]. This indicates that there must be a connection between Conjecture 2.1 and the Baum-Connes Conjecture for the topological K-theory of the reduced C^*-algebra of a group [13, Conjecture 3.15 on page 254] and the Isomorphisms Conjecture for the algebraic K-theory of the integral group ring of Farrell and Jones [93, 1.6 on page 257]. See [68] for a unified treatment of the last two conjectures and see [214] for more details on this connection and a general strategy based on Linnell's work how to approach Conjecture 2.1. The Baum-Connes Conjecture says that one can compute the topological K-theory of the reduced C^*-algebra of a group Γ by a complicated induction process from the topological K-theory of the complex group rings of all finite subgroups of Γ. Conjecture 2.1 can be interpreted similarly, namely all the possible dimensions of kernels of Γ-equivariant bounded operators $l^2(\Gamma)^m \longrightarrow l^2(\Gamma)^n$ which are induced by matrices over $\mathbb{Z}\Gamma$ are coming from the dimensions of kernels of Γ-equivariant bounded operators $l^2(\Gamma)^m \longrightarrow l^2(\Gamma)^n$ which are induced by matrices over $\mathbb{Z}G$ for all finite subgroups $G \subset \Gamma$. Notice that the dimension of such operators coming from a finite group

G are rational numbers which become integral when multiplied with the order of G. The missing link seems to be the not at all understood passage from finitely presented $\mathbb{Z}\Gamma$-modules to modules over the C^*-algebra of Γ. The connection between the generalization of the Euler Poincaré formula for finitely dominated CW-complexes and the Bass Conjecture [12, page 156] which is related to the Baum-Connes Conjecture and the Isomorphisms Conjecture for the algebraic K-theory is treated by Eckmann [82].

Conjecture 2.6 (Singer Conjecture) *If M is a closed aspherical Riemannian manifold of dimension n, then*

$$b_p^{(2)}(\widetilde{M}) = 0 \qquad \text{for } 2p \neq n.$$

If additionally $n = 2d$, then

$$(-1)^d \cdot \chi(M) \geq 0. \qquad \blacksquare$$

The conjecture above was originally only formulated for closed Riemannian manifolds with non-positive sectional curvature. Notice that any such manifold is aspherical by Hadamard's Theorem [99, 3.87 on page 134]. We recall that a space X is *aspherical* if its universal covering is contractible, or equivalently, all the higher homotopy groups of X are trivial. We will see that the Singer Conjecture 2.6 is true if $\pi_1(M)$ contains a normal non-trivial amenable subgroup in section 4, and if M has dimension 3 and is not exceptional in section 3. The Singer Conjecture 2.6 is true if $\widetilde{M}$ is a symmetric space of non-compact type. It is also true if M carries a non-trivial S^1-action, because then the inclusion of an orbit into M induces a map on the fundamental groups with infinite image [66, Lemma 5.1 on page 242 and Corollary 5.3 on page 243] and Theorem 1.7.10 applies.

Conjecture 2.7 (Hopf Conjecture) *If M is a closed $2d$-dimensional Riemannian manifold of negative sectional curvature, then*

$$\begin{aligned} b_d^{(2)}(\widetilde{M}) &> 0; \\ (-1)^d \cdot \chi(M) &> 0. \qquad \blacksquare \end{aligned}$$

In the Hopf Conjecture 2.7 above, the part about the Euler characteristic goes back to Hopf. The statements about the Euler characteristic in Conjecture 2.6 and 2.7 follow from the one about the L^2-Betti numbers by the Euler -Poincaré formula $\chi(M) = \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(\widetilde{M})$ of Theorem 1.7.2.

Conjecture 2.7 has been proven by Dodziuk for a hyperbolic closed Riemannian manifold. Namely, we have already mentioned in Theorem 1.7 that there is L^2-Hodge-deRham decomposition [72]. Since the universal covering of a closed n-dimensional hyperbolic manifold is isometrically isomorphic to the n-dimensional hyperbolic space $\mathbb{H}^n$, and the von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module is zero if and only if the module is zero, the p-th L^2-Betti number of M is trivial if and only if the space of harmonic L^2-integrable p-forms on $\mathbb{H}^n$ is trivial. This space is computed in [73] using the rotational symmetry of $\mathbb{H}^n$. Donnelly and Xavier [79] have proven for a closed n-dimensional Riemannian manifold M whose sectional curvature is pinched between -1 and D_n for some real number $-1 \leq D_n < -\frac{(n-2)^2}{(n-1)^2}$ that

$$b_p^{(2)}(\widetilde{M}) = 0 \qquad \text{for } p \neq \frac{n}{2}, \frac{n \pm 1}{2}.$$

In particular, the Singer Conjecture 2.6 is true for such a manifold if n is even. This result has been improved by Jost-Yuanlong [129]. They assume that the sectional curvature of M satisfies $-a^2 \leq K \leq 0$ and the Ricci curvature is bounded from above by $-b^2$ for positive constants a, b and show that

$$b_p^{(2)}(\widetilde{M}) = 0 \qquad \text{for } p \neq \frac{n}{2}, 2pa \leq b.$$

We will see in section 7 that the Hopf Conjecture 2.7 is true if M is a Kähler manifold. Without some finiteness conditions on $\widetilde{X}$, such as being the total space of a regular covering over a finite CW-complex, Conjecture 2.6 becomes false. This follows from the work of Anderson [2] where the non-vanishing of the (reduced) L^2-cohomology of perturbations of the hyperbolic metric on the hyperbolic space is proved.

3. Low-dimensional manifolds

In this section we give information about the L^2-Betti numbers of universal coverings of compact manifolds of dimension ≤ 3. We will only consider orientable manifolds since one gets the general case by passing to the orientation covering and the multiplicative property of the L^2-Betti numbers of Theorem 1.7.7.

Example 3.1 We begin with a compact connected 1-dimensional manifold M. If M has boundary, it is $[0,1]$ and hence homotopy equivalent to a point, and its L^2-Betti numbers agree with the (ordinary) Betti numbers of the one-point-space, i.e. they are trivial except the 0-th one which is 1. If M has no boundary, then M is S^1. Since there is a covering $S^1 \longrightarrow S^1$ of degree d for $d \geq 2$, we conclude from the multiplicative property in Theorem 1.7.7

$$b_p^{(2)}(\widetilde{S^1}) \;=\; 0 \qquad \text{for } p \geq 0.$$

It is illuminating to compute $b_p^{(2)}(\widetilde{S^1})$ directly. One easily checks that the cellular L^2-chain complex is concentrated in dimension 0 and 1 and its first differential $l^2(\mathbb{Z}) \xrightarrow{t-1} l^2(\mathbb{Z})$

is given by multiplication with the element $t-1$ for t a generator of the fundamental group $\pi_1(S^1)=\mathbb{Z}$. Its Fourier transformation is the bounded $\mathbb{Z}$-equivariant operator

$$M_{z-1}: L^2(S^1) \longrightarrow L^2(S^1) \qquad f(z) \mapsto (z-1)\cdot f(z)$$

where we regard S^1 as a subset of the complex numbers $\mathbb{C}$. Obviously the kernel of M_{z-1} and hence $H_1^{(2)}(\widetilde{S^1})$ is trivial. One easily checks that $H_0^{(2)}(\widetilde{S^1})$ is the kernel of M_{z-1}^*, which is the kernel of $M_{z^{-1}-1}$ and hence trivial. ■

Example 3.2 Let F_g^d be the orientable closed surface of genus g with d embedded open 2-disks removed. From Theorem 1.7 and the fact that a compact surface with boundary is homotopy equivalent to a bouquet of circles, one derives

$$\begin{aligned}
b_0^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 1 & g=0, d=0,1 \\ 0 & \text{otherwise} \end{cases}; \\
b_1^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 0 & g=0, d=0,1 \\ d+2(g-1) & \text{otherwise} \end{cases}; \\
b_2^{(2)}(\widetilde{F_g^d}) &= \begin{cases} 1 & g=0, d=0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$ ■

In the sequel 3-manifold means a compact connected orientable 3-dimensional manifold. Notice that in dimension 3 there is no difference between topological, PL- or smooth manifolds [176], [191]. We want to compute the L^2-Betti numbers of the universal covering of a 3-manifold. For this purpose we have to recall some basic facts about such manifolds which are interesting in their own right. For more information about 3-manifolds we refer to [116],[231] and [240].

A 3-manifold M is *prime* if, given a decomposition $M_1\sharp M_2$ of M as a connected sum, M_1 or M_2 is homeomorphic to S^3. It is *irreducible* if every embedded bicollared 2-sphere bounds an embedded 3-disk. Any prime 3-manifold is irreducible or is homeomorphic to $S^1\times S^2$ [116, Lemma 3.13]. One can write a 3-manifold M as a connected sum

$$M = M_1\sharp M_2\sharp \ldots \sharp M_r$$

where each M_j is prime, and this prime decomposition is unique up to renumbering and oriented homeomorphism [116, Theorems 3.15, 3.21]. By the Sphere Theorem [116, Theorem 4.3], an irreducible 3-manifold is aspherical if and only if it is a 3-disk or has infinite fundamental group.

A properly-embedded orientable connected surface in a 3-manifold is *incompressible* if it is not a 2-sphere and the inclusion induces an injection on the fundamental groups. One says that ∂M is *incompressible in M* if and only if ∂M is empty or any component C of ∂M is incompressible in the sense above. An irreducible 3-manifold is *Haken* if it contains an embedded orientable incompressible surface. If M is irreducible, and in addition $H_1(M)$ is infinite, which is implied if ∂M contains a surface other than S^2, then M is Haken [116, Lemma 6.6 and 6.7]. (With our definitions, any properly embedded 2-disk is incompressible, and the 3-disk is Haken.)

We call a manifold *hyperbolic* if its interior admits a complete Riemannian metric whose sectional curvature is constant -1. Provided that M has no boundary, this is equivalent to the statement that the universal covering with the lifted Riemannian metric is isometrically isomorphic to the hyperbolic space of the same dimension as M. We use the definition of a Seifert 3-manifold of [231, page 429]. If $\pi_1(M)$ is infinite, M is Seifert if and only if some finite covering of M is the total space of an S^1-principal bundle over a compact 2-dimensional manifold. The work of Casson and Gabai shows that an irreducible 3-manifold with infinite fundamental group π is Seifert if and only if π contains a normal infinite cyclic subgroup [98, Corollary 2 on page 395].

Next we mention what is known about Thurston's Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental groups. Johannson [127] and Jaco and Shalen [125] have shown that, given an irreducible 3-manifold M with incompressible boundary, there is a finite family of disjoint, pairwise-nonisotopic incompressible tori in M which are not isotopic to boundary components and which split M into pieces that are Seifert manifolds or are geometrically atoroidal, meaning that they admit no embedded incompressible torus (except possibly parallel to the boundary). A minimal family of such tori is unique up to isotopy, and we will say that it gives a *toral splitting* of M. We will say that the toral splitting is *geometric* if the geometrically atoroidal pieces which do not admit a Seifert structure are hyperbolic. *Thurston's Geometrization Conjecture* for irreducible 3-manifolds with infinite fundamental groups states that such manifolds have geometric toral splittings. For completeness we mention that Thurston's Geometrization Conjecture says, for a closed 3-manifold with finite fundamental group, that its universal covering is homeomorphic to S^3, the fundamental group of M is a subgroup of $SO(4)$ and the action of it on the universal covering is conjugated by a homeomorphism to the restriction of the obvious $SO(4)$-action on S^3. This implies, in particular, the *Poincaré Conjecture* that any homotopy 3-sphere is homeomorphic to S^3.

Suppose that M is Haken. The pieces in its toral splitting are certainly Haken. Let N be a geometrically atoroidal piece. The *Torus Theorem* says that N is a special Seifert manifold or is homotopically atoroidal, i.e. any subgroup of $\pi_1(N)$ which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ is conjugate to the fundamental group of a boundary component. Thurston has shown that a homotopically atoroidal Haken manifold is a twisted I-bundle over the Klein bottle (which is Seifert) or is hyperbolic.

Thus the case in which Thurston's Geometrization Conjecture for an irreducible 3-manifold M with infinite fundamental group is still open is when M is a closed non-Haken irreducible 3-manifold with infinite fundamental group which is not Seifert. The conjecture states that such a manifold is hyperbolic.

We want to compute the L^2-Betti numbers of the universal covering of a 3-manifold under the assumption that Thurston's Geometrization Conjecture holds for the pieces in the prime decomposition with infinite fundamental group. First we deal with the case where the fundamental group of M is finite. In this case the L^2-Betti numbers are the ordinary Betti numbers of $\widetilde{M}$ divided by the order $|\pi|$ of the fundamental group π of M. If M is closed, $\widetilde{M}$ is homotopy equivalent to S^3, and hence $b_0^{(2)}(\widetilde{M}) = b_3^{(2)}(\widetilde{M}) = |\pi|^{-1}$ and $b_p^{(2)}(\widetilde{M}) = 0$ for $p \neq 0, 3$. Suppose that ∂M is non-trivial. Then M is a connected sum of homotopy spheres and k disks for some positive integer k, and hence $b_0^{(2)}(\widetilde{M}) = |\pi|^{-1}$, $b_2^{(2)}(\widetilde{M}) = (k-1) \cdot |\pi|^{-1}$ and $b_p^{(2)}(\widetilde{M}) = 0$ for $p \neq 0, 2$. Hence we only have to treat the case where $\pi_1(M)$ is infinite.

Let us say that a prime 3-manifold is *exceptional* if it is closed and no finite covering of it is homotopy equivalent to a Haken, Seifert or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known, and Thurston's Geometrization Conjecture and *Waldhausen's Conjecture* that any 3-manifold is finitely covered by a Haken manifold imply that there are none. Notice that any exceptional manifold has infinite fundamental group.

Theorem 3.3 (Lott and Lück [148], Theorem 0.1) *Let M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) non-exceptional prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then the L^2-Betti numbers of the universal covering $\widetilde{M}$ are given by*

$$\begin{aligned}
b_0^{(2)}(\widetilde{M}) &= 0; \\
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \ + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \ + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_3^{(2)}(\widetilde{M}) &= 0.
\end{aligned}$$

Proof : We give a sketch of the strategy of proof. Since the fundamental group is infinite, we get $b_0^{(2)}(\widetilde{M}) = 0$ from Theorem 1.7.9. If M is closed, we get $b_3^{(2)}(\widetilde{M}) = 0$ because of Poincaré duality 1.7.3. If M has boundary, it is homotopy equivalent to a 2-dimensional CW-complex and hence $b_3^{(2)}(\widetilde{M}) = 0$. It remains to compute the second L^2-Betti number, because the first one is then determined by the Euler-Poincaré formula 1.7.2.

It is not hard to find a general formula for the L^2-Betti numbers of a connected sum in terms of the summands. With this formula we reduce the claim to prime 3-manifolds. Since a prime 3-manifold is either irreducible or $S^1 \times S^2$, the claim is reduced to the irreducible case. If the boundary is compressible, we use the Loop Theorem [116, Theorem 4.2 on page 39] to reduce the claim to the incompressible case. By doubling M we can reduce the claim further to the case of an irreducible 3-manifold with infinite fundamental group and incompressible torus boundary. By the toral splitting and the assumptions about Thuston's

Geometrization Conjecture we can reduce further to the claim that the L^2-Betti numbers vanish if M is Seifert with infinite fundamental group or is hyperbolic with incompressible torus boundary. All these steps use the weakly exact Mayer-Vietoris sequence for L^2-(co)-homology of Cheeger and Gromov [52, Theorem 2.1 on page 10]. In the Seifert case we can assume by the multiplicative property (see Theorem 1.7.7) that M is an S^1-principal bundle over a 2-dimensional manifold. Then we use induction over the cells of the base space and the fact that the L^2-Betti numbers of S^1 vanish. The hyperbolic case is reduced to the known closed case by a careful analysis at the boundary using the fact that such a hyperbolic manifold with incompressible torus boundary has finite volume. ∎

Let $\chi_{\text{virt}}(\pi_1(M))$ be the rational-valued group Euler characteristic of the group $\pi_1(M)$ in the sense of [35, IX.7],[246]. Then the conclusion in Theorem 3.3 is equivalent to

$$\begin{aligned} b_1^{(2)}(\widetilde{M}) &= -\chi_{\text{virt}}(\pi_1(M)); \\ b_2^{(2)}(\widetilde{M}) &= \chi(M) - \chi_{\text{virt}}(\pi_1(M)). \end{aligned}$$

This is proven in [148, page 53 - 54].

Notice that Theorem 3.3 proves Conjecture 2.6 and the third of the three equivalent assertions in Lemma 2.2 for compact 3-manifolds, provided that Thurston's Geometrization Conjecture or Waldhausen's Conjecture is true. Notice that this does *not* imply Conjecture 2.1 for Γ the fundamental group of a compact 3-manifold satisfying Thurston's or Waldhausen's Conjecture.

4. Aspherical manifolds and amenability

This section is devoted to a result of Cheeger and Gromov about the vanishing of the L^2-Betti numbers of the universal covering of an aspherical CW-complex whose fundamental group contains a non-trivial normal amenable subgroup.

Let $l^\infty(\Gamma, \mathbb{R})$ be the space of bounded functions from Γ to $\mathbb{R}$ with the supremum norm. Denote by 1 the constant function with value 1. A group Γ is called *amenable* if there is a Γ-invariant linear operator $\mu : l^\infty(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ with $\mu(1) = 1$ which satisfies

$$\inf\{f(\gamma) \mid \gamma \in \Gamma\} \leq \mu(f) \leq \sup\{f(\gamma) \mid \gamma \in \Gamma\} \qquad \text{for} \quad f \in l^\infty(\Gamma).$$

The last condition is equivalent to the condition that μ is bounded and $\mu(f) \geq 0$ if $f(\gamma) \geq 0$ for all $\gamma \in \Gamma$.

We give an overview of some basic properties of this notion. The class of amenable groups satisfies the conditions appearing in the definition of elementary amenable groups in section 2, namely, it contains all finite and all abelian groups, and is closed under taking subgroups, forming factor groups, group extensions, and upwards directed unions [17, Proposition F.6.11 on page 309]. Hence any elementary amenable group is amenable. Recently Grigorchuk has constructed a finitely presented group which is amenable but not elementary amenable. Any group containing the free group on two letters $\mathbb{Z} * \mathbb{Z}$ as subgroup is not amenable [17, Proposition F.6.12 on page 310]. There are finitely generated, but not finitely presented groups, which are not amenable but do not contain $\mathbb{Z} * \mathbb{Z}$ [200]. However, no non-amenable finitely presented group is known which does not contain $\mathbb{Z}*\mathbb{Z}$. A useful geometric characterization of amenable groups is given by the Fölner criterion [17, Theorem F.6.8 on page 308] which says that a finitely presented group Γ is amenable if and only if for any positive integer n, any connected closed Riemannian manifold M with fundamental group $\pi_1(M) = \Gamma$, and $\epsilon > 0$, there is a domain $\Omega \subset \widetilde{M}$ with $(n-1)$-measurable boundary such that the $(n-1)$-measure of $\partial\Omega$ does not exceed ϵ times the measure of Ω. Such a domain can be constructed by an appropriate finite union of translations of a fundamental domain if Γ is amenable. The fundamental group of a closed connected manifold is not amenable if M admits a Riemannian metric of non-positive curvature which is not zero everywhere [9]. A group is amenable if and only if all its finitely generated subgroups are amenable [17, Proposition F.6.11 on page 309]. Any finitely generated group which is not amenable has exponential growth [17, Proposition F.6.24 on page 318]. A group Γ is amenable if and only if the canonical map from the full C^*-algebra of Γ to the reduced C^*-algebra of Γ is an isomorphism [207, Theorem 7.3.9 on page 243]. A group Γ is amenable if and only if the reduced C^*- algebra of Γ is nuclear [139]. For more information about amenable groups we refer to [206].

The next result gives a positive answer to the Singer Conjecture 2.6 for special fundamental groups.

Theorem 4.1 (Cheeger and Gromov [54]) *If X is an aspherical connected CW-complex of finite type such that its fundamental group contains a non-trivial normal amenable subgroup, then we get for the universal covering $\widetilde{X}$*

$$b_p^{(2)}(\widetilde{X}) \;=\; 0 \qquad \text{for } p \geq 0. \qquad \blacksquare$$

In this section we will explain only one of the decisive steps in the proof of Theorem 4.1 in order to illustrate the meaning of the condition about amenability. We will complete the proof in section 10.

Let X be a finite CW-complex with regular covering $\overline{X} \longrightarrow X$ with group of deck transformations Γ. Let $C^*(\overline{X})$ be the dual cochain complex with complex coefficients $\hom_{\mathbb{Z}}(C_*(\overline{X}), \mathbb{C})$ of the cellular chain complex $C_*(\overline{X})$. An element in $C^p(\overline{X})$ is a function $u : S_p(\overline{X}) \longrightarrow \mathbb{C}$ from the set of p-cells of $\overline{X}$ to the complex numbers. We call u *square-summable* if $\sum_{\overline{e}\in S_p(\overline{X})} |u(\overline{e})|^2$ is finite. The square-summable elements form a subcomplex $C^*_{(2)}(\overline{X}) \subset C^*(\overline{X})$. We equip the chain modules of $C^*_{(2)}(\overline{X})$ with the obvious Hilbert space structure. This definition agrees with the Definition 1.5. Let $\mathcal{H}^p_{(2)}(\overline{X})$ be the Hilbert submodule of $C^p_{(2)}(\overline{X})$ consisting of those elements u which satisfy $c^p_{(2)}(u) = 0$ and $(c^{p-1}_{(2)})^*(u) = 0$. There are obvious maps

$$i^p : \mathcal{H}^p_{(2)}(\overline{X}) \longrightarrow H^p_{(2)}(C^*_{(2)}(\overline{X})) = H^p_{(2)}(\overline{X}); \tag{4.2}$$

$$j^p : \mathcal{H}^p_{(2)}(\overline{X}) \longrightarrow H^p(C^*(\overline{X})) = H^p(\overline{X};\mathbb{C}). \tag{4.3}$$

The first map 4.2 is always an isomorphism by an elementary argument.

Lemma 4.4 (Cheeger and Gromov [54], Lemma 3.1 on page 203) *The map j^p of 4.3 is injective, provided that Γ is amenable.*

Proof : Let $\overline{D} \subset \overline{X}$ be a fundamental domain for the Γ-action on X. Such a $\overline{D}$ is constructed by choosing for each closed p-cell $e \in S_p(X)$ one lift $\overline{e} \in S_p(\overline{X})$ and taking the union of these lifts for all $p \geq 0$. Because Γ is amenable, one can find a sequence of subcomplexes $\overline{X_j} \subset \overline{X}$ and natural numbers n_j and ∂n_j such that $\overline{X_j}$ is the union of n_j translates with distinct elements in $\gamma_j^k \in \Gamma$ for $k = 1, 2, \ldots n_j$ of $\overline{D}$, ∂n_j is the numbers of cells of $\overline{D}$ which meet the boundary (in the sense of point set topology) of $\overline{X_j}$ and

$$\lim_{j\to\infty} \frac{\partial n_j}{n_j} = 0. \tag{4.5}$$

Let $K^p \subset \mathcal{H}^p_{(2)}(\overline{X})$ be the kernel of j^p. We have to show that K^p is trivial. Let pr_{K^p} (resp. pr^p_j) be the orthogonal projection $C^p_{(2)}(\overline{X}) \longrightarrow C^p_{(2)}(\overline{X})$ onto K^p (resp. onto the complex subspace of square-summable cochains u which are supported on $\overline{X_j}$, i.e. $u(e) = 0$ unless e belongs to $\overline{X_j}$). Let $\chi_{\overline{e}}$ denote the characteristic function $S_p(\overline{X}) \longrightarrow \mathbb{C}$ for a given $\overline{e} \in S_p(\overline{X})$. We get

$$\begin{aligned}
\dim_{\mathcal{N}(\Gamma)}(K^p) &= \mathrm{tr}_{\mathcal{N}(\Gamma)}(\mathrm{pr}_{K^p}) \\
&= \sum_{e \in S_p(X)} \langle \mathrm{pr}_{K^p}(\chi_{\overline{e}}), \chi_{\overline{e}} \rangle \\
&= \sum_{\overline{e} \in S_p(\overline{D})} \langle \mathrm{pr}_{K^p}(\chi_{\overline{e}}), \chi_{\overline{e}} \rangle \\
&= \frac{1}{n_j} \cdot \sum_{\overline{e} \in S_p(\overline{D})} n_j \cdot \langle \mathrm{pr}_{K^p}(\chi_{\overline{e}}), \chi_{\overline{e}} \rangle \\
&= \frac{1}{n_j} \cdot \sum_{\overline{e} \in S_p(\overline{D})} \sum_{k=1}^{n_j} \langle \mathrm{pr}_{K^p}(\chi_{\gamma_j^k \overline{e}}), \chi_{\gamma_j^k \overline{e}} \rangle \\
&= \frac{1}{n_j} \cdot \sum_{\overline{e} \in S_p(\overline{X_j})} \langle \mathrm{pr}_{K^p}(\chi_{\overline{e}}), \chi_{\overline{e}} \rangle \\
&= \frac{1}{n_j} \cdot \mathrm{tr}_{\mathbb{C}}(\mathrm{pr}^p_j \circ \mathrm{pr}_{K^p}).
\end{aligned}$$

Since pr_j^p and pr_{K^p} have norm 1, we conclude

$$\dim_{\mathcal{N}(\Gamma)}(K^p) \leq \frac{1}{n_j} \cdot \dim_{\mathbb{C}}\left(\mathrm{im}(\mathrm{pr}_j^p \circ \mathrm{pr}_{K^p})\right). \tag{4.6}$$

We fix the orthogonal decomposition

$$u = u_i + u_\partial + u_e \qquad \in C^p_{(2)}(\overline{X}) \subset \mathrm{map}(S_p(\overline{X}), \mathbb{C})$$

where u_i is only supported on (closed) cells in the interior of $\overline{X_j}$, u_∂ is supported on cells which meet the boundary of $\overline{X_j}$ and u_e is supported on cells which do not meet $\overline{X_j}$ (and hence do not meet the boundary of $\overline{X_j}$).

Now consider an element $u \in K^p$ such that $u_\partial = 0$. Choose $v \in C^{p-1}(\overline{X})$ such that $c^{p-1} : C^{p-1}(\overline{X}) \longrightarrow C^p(\overline{X})$ maps v to u regarded as an element in $C^p(\overline{X})$. Notice that pr_j^p extends to a map $\mathrm{pr}_j^p : C^p(\overline{X}) \longrightarrow C^p(\overline{X})$, namely $\mathrm{pr}_j^p(y)$ for $y \in C^p(\overline{X})$ sends a cell $e \in S_p(\overline{X})$ to $y(e)$ if e belongs to $\overline{X_j}$ and to zero otherwise. From this description we conclude $\mathrm{pr}_j^p(u) = u_i$. The cochain $\mathrm{pr}_j^p \circ c^{p-1}(v)$ vanishes on cells which do not meet $\overline{X_j}$. The difference $c^{p-1} \circ \mathrm{pr}_j^{p-1}(v) - \mathrm{pr}_j^p \circ c^{p-1}(v)$ is supported on cells which meet the boundary of $\overline{X_j}$ because the boundary of a cell which does not belong to $\overline{X_j}$ cannot lie in the interior of $\overline{X_j}$. Hence we get

$$\begin{aligned}
0 &\leq \langle \mathrm{pr}_j^p(u), \mathrm{pr}_j^p(u)\rangle \\
&= \langle \mathrm{pr}_j^p \circ c^{p-1}(v), u_i\rangle \\
&= \langle c^{p-1} \circ \mathrm{pr}_j^{p-1}(v), u_i\rangle \\
&= \langle c^{p-1} \circ \mathrm{pr}_j^{p-1}(v), u\rangle \\
&= \langle \mathrm{pr}_j^{p-1}(v), (c^{p-1})^*(u)\rangle \\
&= \langle \mathrm{pr}_j^{p-1}(v), 0\rangle \\
&= 0.
\end{aligned}$$

This implies that pr_j vanishes on $K^p \cap \{u \in C^p_{(2)}(\overline{X}) \mid u_\partial = 0\}$. Since

$$\begin{aligned}
& \dim_{\mathbb{C}}\left(K^p \Big/ \left(K^p \cap \{u \in C^p_{(2)}(\overline{X}) \mid u_\partial = 0\}\right)\right) \\
\leq\ & \dim_{\mathbb{C}}\left(C^p_{(2)}(\overline{X}) \Big/ \{u \in C^p_{(2)}(\overline{X}) \mid u_\partial = 0\}\right) \\
=\ & \partial n_j,
\end{aligned}$$

we conclude

$$\dim_{\mathbb{C}}\left(\mathrm{im}(\mathrm{pr}_j^p \circ \mathrm{pr}_{K^p})\right) \leq \partial n_j. \tag{4.7}$$

We get from 4.6 and 4.7

$$\dim_{\mathcal{N}(\Gamma)}(K^p) \leq \frac{\partial n_j}{n_j}.$$

Since the limit of the right hand side is zero by 4.5, we get

$$\dim_{\mathcal{N}(\Gamma)}(K^p) = 0.$$

This finishes the proof of Lemma 4.4. ■

Notice that Lemma 4.4 proves Theorem 4.1 in the special case where the fundamental group of X is itself amenable by the following argument. In order to show that $b_p^{(2)}(\overline{X})$ vanishes, we can pass to the $(d+1)$-skeleton $\overline{Y} \subset \overline{X}$ and prove that $b_p^{(2)}(\overline{Y})$ vanishes. Since $\overline{X}$ is aspherical, we conclude

$$H^p(\overline{Y};\mathbb{C}) = H^p(\overline{X};\mathbb{C}) = 0.$$

Now Lemma 4.4 implies

$$\begin{aligned} H^p_{(2)}(\overline{Y}) &= 0; \\ b^p_{(2)}(\overline{Y}) &= 0. \end{aligned}$$

If Γ contains a normal amenable infinite group Δ such that $B\Delta$ is of finite type, then Theorem 4.1 follows by the L^2-version of the Leray-Serre spectral sequence (see, for instance, [239]) applied to the fibration $B\Delta \longrightarrow B\Gamma \longrightarrow B\Gamma/\Delta$. We will give the proof in the general case, where no assumptions about $B\Delta$ are made, in section 10.

5. Approximating L^2-Betti numbers by ordinary Betti numbers

In this section we get the L^2-Betti numbers of a regular covering $\overline{X} \longrightarrow X$ of a CW-complex of finite type as the limit of the normalized ordinary Betti numbers of a tower of finite coverings $X_m \longrightarrow X$ which converges in some sense to $\overline{X}$.

Let $\overline{X} \longrightarrow X$ be a regular covering of a CW-complex of finite type with group of deck transformations Γ. Suppose that Γ is *residually finite*, i.e. for each element $\gamma \in \Gamma$ with $\gamma \neq 1$ there is a homomorphism $\phi : \Gamma \longrightarrow G$ to a finite group with $\phi(\gamma) \neq 1$. We will assume that Γ is countable. Under this assumption Γ is residually finite if and only if there is a nested sequence of normal in Γ subgroups $\ldots \subset \Gamma_{m+1} \subset \Gamma_m \subset \ldots \Gamma_0 = \Gamma$ such that the index $[\Gamma : \Gamma_m]$ is finite for all $m \geq 0$ and the intersection $\cap_{m\geq 0}\Gamma_m$ is the trivial group. We give some information about residually finite groups at the end of this section. Consider any such sequence $(\Gamma_m)_{m\geq 0}$. Let $p_m : X_m = \Gamma_m\backslash\overline{X} \longrightarrow X$ be the covering of X associated with $\Gamma_m \subset \Gamma$. Notice that this is a finite regular covering of X, and hence X_m is again of finite type. Denote by $b_p(X_m)$ the (ordinary) p-th Betti number of X_m.

Theorem 5.1 (Lück [156]) *Under the conditions above we get for all $p \geq 0$*

$$\lim_{m\to\infty} \frac{b_p(X_m)}{[\Gamma : \Gamma_m]} \;=\; b_p^{(2)}(\overline{X}). \qquad \blacksquare$$

The inequality $\limsup_{m\to\infty} \frac{b_p(X_m)}{[\Gamma:\Gamma_m]} \;\leq\; b_p^{(2)}(\overline{X})$ for X a closed manifold is discussed by Gromov [109, 0.5.F., 8.A] and is essentially due to Kazhdan [132]. The paper of Donnelly [75] deals with the operator $f(\Delta_0)$ acting on 0-forms for a function $f \in C_0^\infty(\mathbb{R}^+)$. Theorem 5.1 is proven by Yeung [254] in the special case of a closed Kähler manifold with negative sectional curvature and by DeGeorge-Wallach [102], [103] in the special case of a closed locally symmetric space of non-compact type. In the last case all the L^2-Betti numbers vanish, so one gets

$$\lim_{m\to\infty} \frac{b_p(X_m)}{[\Gamma : \Gamma_m]} \;=\; 0.$$

This leads to the question of how fast this sequence goes to zero; in other words, one wants to know the largest ϵ for which

$$\lim_{m\to\infty} \frac{b_p(X_m)}{[\Gamma : \Gamma_m]^{1-\epsilon}} \;=\; 0.$$

is true. Such questions are treated in [227], [251], [252].

The general case of a CW-complex X of finite type is proven in [156]. There actually the entire spectral density function of the combinatorial Laplace operator on the cellular L^2-chain complex of $\overline{X}$ is approximated by the spectral density function for the combinatorial Laplace operator on the cellular chain complexes of the various X_m. The spectral density function for X_m simply encodes the eigenvalues and their multiplicities of the Laplace operator because X_m is compact, whereas the one for $\overline{X}$ is more complicated as, in general, $\overline{X}$ is not compact and the spectrum is not discrete. In some sense we try to approximate continous information by discrete data. The philosophy is that the tower of finite coverings X_m converges to $\overline{X}$.

The inequality $\limsup_{m\to\infty} \frac{b_p(X_m)}{[\Gamma:\Gamma_m]} \;\leq\; b_p^{(2)}(\overline{X})$ is the easier part of the proof of Theorem 5.1. The main trick in the proof of the other inequality is not to forget and to use essentially the fact that the combinatorial Laplace operator on the cellular chain complex already lives over the integral group ring. This allows the use of the obvious inequality $|n| \geq 1$ for an integer n different from zero. The results for the combinatorial Laplace operator on the cellular chain complexes carry over to the analytic Laplace operator acting on smooth p-forms on X provided that X is a compact smooth manifold. One can formulate Theorem 5.1 for any elliptic differential operator on a closed smooth Riemannian manifold X if one uses on each X_m and on $\overline{X}$ the lifted Riemannian metric and elliptic differential operator and substitutes the Betti numbers by the dimensions of the kernels. It seems not hard to prove the analogue of the inequality $\limsup_{m\to\infty} \frac{b_p(X_m)}{[\Gamma:\Gamma_m]} \;\leq\; b_p^{(2)}(\overline{X})$, but it is not known whether the equality holds. The question is whether the result is only true for the Laplacian because it has a cellular analogue which allows us to use the fact that everything already lives over the integral group ring. Again we see from this discussion as we have mentioned in section 2 that the passage from the integral group ring of Γ to the reduced C^*-algebra, or even the von Neumann algebra of Γ, plays a fundamental role and any understanding of it seems to give new results.

Finally we collect some basic facts about residually finite groups in order to explain how restrictive the assumption is that Γ is residually finite. For more information we refer to the survey article of Magnus [169].

The free product of two residually finite groups is again residually finite [60, page 27], [113]. A finitely generated residually finite group has a solvable word problem [183]. The automorphism group of a finitely generated residually finite group is residually finite [14]. A finitely generated residually finite group is Hopfian, i.e. any surjective endomorphism is an automorphism [168], [195, Corollary 41.44]. Let Γ be a finitely generated group possessing a faithful representation into $GL(n, F)$ for F a field. Then Γ is residually finite [168], [248, Theorem 4.2]. The fundamental group of a compact 3-manifold whose prime decomposition consists of manifolds which have finite fundamental groups, or are non-exceptional in the sense of section 3 (i.e. which are finitely covered by a manifold which is homotopy equivalent to a Haken, Seifert or hyperbolic manifold), is residually finite [117, page 380]. Let Γ be a finitely generated group. Let Γ^{rf} be the quotient of Γ by the normal subgroup which is the intersection of all normal subgroups of Γ of finite index. The group Γ^{rf} is residually finite and any finite-dimensional representation of Γ over a field factorizes over the canonical projection $\Gamma \longrightarrow \Gamma^{rf}$.

The upshot of this discussion is the slogan that the fundamental group of a geometrically interesting closed manifold is very likely to be residually finite. However, there is an infinite group Γ with four generators and four relations which has no finite quotient except the trivial one and hence satisfies $\Gamma^{rf} = \{1\}$ [119].

Meanwhile Theorem 5.1 has been generalized and put into somewhat different context in [56], [74], [92] and [229].

6. L^2-Betti numbers and groups

In this section we explain some applications of L^2-Betti numbers to group theory. Given a group Γ, we define its L^2-Betti number $b_p^{(2)}(\Gamma)$ by $b_p^{(2)}(E\Gamma)$ where $E\Gamma \longrightarrow B\Gamma$ is the universal principal Γ-bundle. This is only well-defined if the $(p+1)$-skeleton of $B\Gamma$ is finite. The definition for arbitrary groups will be given in Definition 10.9.

The next result of Cheeger and Gromov was proven in special cases by Gottlieb [104] and Rosset [212].

Theorem 6.1 (Cheeger and Gromov [54], Corollary 0.6 on page 193) *Let Γ be a group such that there exists a subgroup Γ' of finite index whose classifying space $B\Gamma'$ is a finite CW-complex. Let $\chi_{\mathrm{virt}}(\Gamma)$ be the rational-valued virtual Euler characteristic of the group Γ in the sense of [35, IX.7], [246]. Suppose that Γ contains an infinite normal amenable subgroup. Then*

$$\chi_{\mathrm{virt}}(\Gamma) = 0.$$

Proof : By definition $\chi_{\mathrm{virt}}(\Gamma) = \frac{1}{[\Gamma:\Gamma']} \cdot \chi(B\Gamma')$. We derive from the Euler-Poincaré formula of Theorem 1.7.2

$$\chi_{\mathrm{virt}}(\Gamma) \;=\; \frac{1}{[\Gamma : \Gamma']} \cdot \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(B\Gamma').$$

Now the claim follows from Theorem 4.1. ∎

We mention the following result

Theorem 6.2 (Reich [214],Corollary 9.3) *Let Γ be an infinite group which belongs to Linnell's class $\mathcal{C}$ introduced in Section 2 and has an upper bound on the orders of its finite subgroups. Suppose that there exists a subgroup Γ' of finite index whose classifying space $B\Gamma'$ is a finite CW-complex. Then*

$$\chi_{\mathrm{virt}}(\Gamma) \;\leq\; 0.$$

Next we mention the following observation about Thompson's group F. It consists of orientation preserving dyadic PL-automorphisms of $[0,1]$,where dyadic means that all slopes are integral powers of 2 and the break points are contained in $\mathbb{Z}[1/2]$. It has the presentation

$$F = \langle x_0, x_1, x_2, \ldots \;\mid\; x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.$$

This group has some very interesting properties. It is not elementary amenable and does not contain a subgroup which is free on two generators [33], [45]. Hence it is a very interesting question whether F is amenable. Since BF is of finite type [36], the L^2-Betti numbers $b_p^{(2)}(F)$ are defined for all $p \geq 0$. We conclude from Theorem 4.1 of Cheeger and Gromov that a necessary condition for F to be amenable is that $b_p^{(2)}(F)$ vanishes for all $p \geq 0$. This motivates the following result.

Theorem 6.3 (Lück [157], Theorem 0.8) *All the L^2-Betti numbers $b_p^{(2)}(F)$ of Thompson's group F vanish.* ∎

For the proof of Theorem 6.3 we need the next result.

Given a selfmap $f : F \longrightarrow F$, its *mapping cylinder* M_f is obtained by gluing the bottom of the cylinder $F \times [0,1]$ to F by the identification $(x,0) = f(x)$. Its *mapping torus* T_f is obtained from the mapping cylinder by identifying the top and the bottom by the identity. If f is a homotopy equivalence, T_f is homotopy equivalent to the total space of a fibration over S^1 with fiber F. Conversely, the total space of such a fibration is homotopy equivalent to the mapping torus of the self homotopy equivalence of F given by the fiber transport with a generator of $\pi_1(S^1)$. The homotopy type of T_f depends only on the homotopy class of f. There is an obvious map from T_f to S^1 which induces an epimorphism $\mu : \pi_1(T_f) \longrightarrow \mathbb{Z}$.

Theorem 6.4 (Lück [155], Theorem 2.1. on page 207) *Let F be a connected CW-complex of finite type and $f : F \longrightarrow F$ be a selfmap. Let*

$$\mu : \pi_1(T_f) \xrightarrow{\phi} \Gamma \xrightarrow{\psi} \mathbb{Z}$$

be a factorization of μ into epimorphisms. Let $\overline{T_f} \longrightarrow T_f$ be the regular covering of T_f with Γ as group of deck transformations which is associated to ϕ. Then

$$b_p^{(2)}(\overline{T_f}) = 0 \qquad \textit{for } p \geq 0.$$

Proof : For simplicity we give here only the proof in the case where f is a homotopy equivalence and ϕ is the identity on $\pi_1(T_f)$, i.e. $\overline{T_f}$ is the universal covering $\widetilde{T_f}$. Consider any positive integer d. Let Γ_d be the preimage of $d\mathbb{Z}$ under $\mu : \pi_1(T_f) \longrightarrow \mathbb{Z}$. Let $\overline{\overline{T_f}}$ be the covering $\widetilde{T_f} \longrightarrow \Gamma_d \backslash \widetilde{T_f}$ with Γ_d as group of deck transformations. We get from the multiplicative property of Theorem 1.7.7

$$b_p^{(2)}(\widetilde{T_f}) \;=\; \frac{1}{d} \cdot b_p^{(2)}(\overline{\overline{T_f}}).$$

One easily checks that $\Gamma_d \backslash \widetilde{T_f}$ is homotopy equivalent to T_{f^d}. On T_{f^d} there is a CW-complex structure whose number of p-cells is bounded by the number C which is the sum of the numbers of p-cells and the number of $(p-1)$-cells in F. Notice that C is independent of d. Since the p-th chain module in the cellular L^2-chain complex of $\overline{\overline{T_{f^d}}}$ is $\oplus_{i=1}^{C} l^2(\Gamma_d)$, we conclude

$$b_p^{(2)}(\overline{\overline{T_{f^d}}}) = \dim_{\mathcal{N}(\Gamma_d)}(H_p^{(2)}(\overline{\overline{T_{f^d}}})) \leq \dim_{\mathcal{N}(\Gamma_d)}(C_p^{(2)}(\overline{\overline{T_{f^d}}})) = C.$$

Hence we have shown that for all $d \geq 1$

$$0 \leq b_p^{(2)}(\widetilde{T_f}) \leq \frac{C}{d}.$$

Taking the limit for $d \to \infty$ finishes the proof of Theorem 6.4. ■

Next we give the proof of Theorem 6.3. There is a subgroup $F_1 \subset F$ together with a monomorphism $\Phi : F_1 \longrightarrow F_1$ such that F_1 is isomorphic to F and F is the HNN-extension of F_1 with respect to Φ with one stable letter [36, Proposition 1.7 on page 370]. From the topological description of HNN-extensions [166, page 180] we conclude that F is the fundamental group of the mapping torus $T_{B\Phi}$ of the map $B\Phi : BF_1 \longrightarrow BF_1$ induced by Φ. The inclusion $BF_1 \longrightarrow BF$ induces on the fundamental groups the inclusion of F_1 in F. The calculation in [155, page 207] shows that the cellular $\mathbb{Z}F$-chain complex of the universal covering $\widetilde{T_{B\Phi}}$ of $T_{B\Phi}$ is the mapping cone of a certain $\mathbb{Z}F$-chain map from $\mathbb{Z}F \otimes_{\mathbb{Z}F_1} C(EF_1)$ to itself. Since $\mathbb{Z}F$ is free over $\mathbb{Z}F_1$, we conclude for $p \geq 1$

$$H_p(\mathbb{Z}F \otimes_{\mathbb{Z}F_1} C(EF_1)) = \mathbb{Z}F \otimes_{\mathbb{Z}F_1} H_p(C(EF_1)) = 0.$$

This implies $H_p(\widetilde{T_{B\Phi}}; \mathbb{Z}) = 0$ for $p \geq 2$. Hence $T_{B\Phi}$ is a model for BF. Now Theorem 6.3 follows from Theorem 6.4. ■

Next we treat the notion of deficiency. Let Γ be a finitely presented group. Its *deficiency* is the maximum over all differences $g - r$,where g resp. r is the number of generators resp. relations of a presentation of Γ. One can show that the maximum does exist. Sometimes the deficiency of a group is what one would guess from an obvious presentation as in the following cases

group	presentation	deficiency
$*_{i=1}^{g}\mathbb{Z}$	$\langle s_1, \ldots s_g \mid \emptyset \rangle$	g
$\mathbb{Z}/n, n \geq 2$	$\langle s \mid s^n = 1 \rangle$	0
$\mathbb{Z}/n \times \mathbb{Z}/n, n \geq 2$	$\langle s, t \mid s^n = t^n = [s,t] = 1 \rangle$	-1

On the other hand, the group $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

and one may think that its deficiency is -2. However, it turns out that its deficiency is -1. For instance, there is the following presentation, which looks on the first glance to be the presentation above with one relation missing

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

The following calculation shows that, from the five relations appearing in the presentation above, the relation $t_0^2 = 1$ follows which shows that the presentation above indeed is a presentation of $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$.

We start with proving inductively for $k = 1, 2, \ldots$ the relation $s_i^k t_i s_i^{-k} = t_i^{r_i^k}$ for $i = 0, 1$ where $r_0 = 3$ and $r_1 = 4$. The beginning of the induction is obvious, the induction step follows from the calculation

$$s_i^{k+1} t_i s_i^{-(k+1)} = s_i s_i^k t_i s_i^{-k} s_i^{-1} = s_i t_i^{r_i^k} s_i^{-1} = \left(s_i t_i s_i^{-1}\right)^{r_i^k} = (t_i^{r_i})^{r_i^k} = t_i^{r_i^{k+1}}.$$

This implies, for $k = 2$, $i = 0$ and $k = 3$,$i = 1$

$$\begin{aligned} t_0 &= t_0^{3^2}; \\ t_1 &= t_1^{4^3}. \end{aligned}$$

Since $t_0^2 = t_1^3$, we conclude

$$\begin{aligned}(t_0^2)^4 &= 1;\\ (t_0^2)^{21} &= 1.\end{aligned}$$

As 4 and 21 are prime, we get $t_0^2 = 1$ and the claim follows.

The example above is a special case of a family of examples described by Hog-Ancheloni, Lustig and Metzler [122]. The example shows that the deficiency is not additive under free products in general. However, we believe that this is true for torsionfree finitely presented groups. The example above plays a fundamental role in the counterexample, up to homotopy, of the Kneser Conjecture in dimension 4 [137].

The link between the deficiency and the L^2-Betti numbers of a group is the following elementary lemma.

Lemma 6.5 *1. Let Γ be a finitely presented group. Then*

$$\operatorname{def}(\Gamma) \leq 1 - b_0^{(2)}(\Gamma) + b_1^{(2)}(\Gamma) - b_2^{(2)}(\Gamma).$$

2. If M is a closed oriented 4-manifold, then we get for its signature

$$|\operatorname{sign}(M)| \leq b_2^{(2)}(\widetilde{M}).$$

Proof : 1.) Given a presentation with r relations and g generators, let X be the corresponding connected 2-dimensional CW-complex with fundamental group isomorphic to Γ which has precisely one cell of dimension 0, g cells of dimension 1 and r cells of dimension 2. Since the classifying map $X \longrightarrow B\Gamma$ is 2-connected, we conclude from Theorem 1.7

$$1 - g + r = \chi(X) = b_0^{(2)}(\widetilde{X}) - b_1^{(2)}(\widetilde{X}) + b_2^{(2)}(\widetilde{X}) \geq b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(\Gamma).$$

2.) According to the L^2-signature theorem [3, page 71], the signature $\sigma(M)$ is the difference of the von Neumann dimensions of two complementary subspaces of the second L^2-cohomology $H^2_{(2)}(\widetilde{M})$. This implies

$$|\operatorname{sign}(M)| \;\leq\; \dim_{\mathcal{N}(\pi_1(M))}(H^2_{(2)}(\widetilde{M})) \;=\; b_2^{(2)}(\widetilde{M}). \qquad \blacksquare$$

Lemma 6.5 has an analogous formulation if one uses ordinary L^2-Betti numbers with coefficients in any field. The values of the deficiencies of the groups in the list above follow from Lemma 6.5. In particular, one sees that the deficiency is defined as a natural number, i.e. there is an upper bound on the possible values $g - r$ appearing in the definition of deficiency. One also rediscovers the well-known fact that the deficiency of a finite group is less than or equal to zero.

If Γ is a torsion-free one-relator group, the 2-dimensional CW-complex associated with the presentation is aspherical and hence $B\Gamma$ is 2-dimensional [166, chapter III §§9 -11]. If Γ has a presentation with g generators and one (non-trivial) relation, its deficiency is $g-1$. We conjecture that for a torsion-free group having a presentation with $g \geq 2$ generators and one non-trivial relation $b_2^{(2)}(\Gamma) = 0$ and $b_1^{(2)}(\Gamma) = \mathrm{def}(\Gamma) - 1 = g - 2$ holds. This would follow from Conjecture 2.1. Namely, the kernel of the second differential of the L^2-chain complex of $\widetilde{E\Gamma}$ is a submodule of $l^2(\Gamma)$ so that its dimension $b_2^{(2)}(\Gamma)$ is less or equal to the dimension of $l^2(\Gamma)$ which is 1. Since Γ is by assumption torsionfree, the dimension of the kernel is an integer by Conjecture 2.1. The second differential in the cellular $\mathbb{Z}\Gamma$-chain complex cannot be trivial because the relation in Γ is assumed to be non-trivial. Hence the dimension of kernel of the second differential of the L^2-chain complex of $E\Gamma$ is trivial. This shows $b_2^{(2)}(\Gamma) = 0$. The Euler Poincaré formula of Theorem 1.7.2 and Theorem 1.7.9 imply $b_0^{(2)}(\Gamma) = 0$ and $b_1^{(2)}(\Gamma) = g - 2$. We get from Lemma 6.5 that $\mathrm{def}(\Gamma) = g - 1$.

The following is a direct consequence of [87, Theorem 2.5]. Let M be a compact 3-manifold with fundamental group Γ and prime decomposition $M = M_1 \sharp M_2 \sharp \ldots \sharp M_r$. Let $s(M)$ be the number of prime factors M_i with non-empty boundary and $t(M)$ be the number of prime factors which are S^2-bundles over S^1. Denote by $\chi(M)$ the Euler characteristic. Then

$$\begin{aligned} \mathrm{def}(\pi_1(M)) &= \dim_{\mathbb{Z}/2}(H_1(\pi;\mathbb{Z}/2)) - \dim_{\mathbb{Z}/2}(H_2(\pi;\mathbb{Z}/2)) \\ &= s(M) + t(M) - \chi(M). \end{aligned}$$

Theorem 6.6 (Lück [157], Theorem 0.7) *Let $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1$ be an exact sequence of groups such that Δ is finitely generated and infinite, Γ is finitely presented and $\mathbb{Z}$ is a subgroup of π. Then*

1. $b_1^{(2)}(\Gamma) = 0$;
2. $\mathrm{def}(\Gamma) \leq 1$;
3. *Let M be a connected closed orientable 4-manifold with Γ as fundamental group. Then we get for its signature $\mathrm{sign}(M)$ and Euler characteristic $\chi(M)$*

$$|\,\mathrm{sign}(M)| \leq \chi(M).$$

Proof : Assertion 1.) follows from Theorem 6.7 below applied to $B\Delta \longrightarrow B\Gamma \longrightarrow B\pi$. The other two assertions follow from the first one by Lemma 6.5. ■

The structure of groups of deficiency greater than or equal to 2 is examined in [15]. Theorem 6.6 generalizes results of [80], [81], [120] [128] and [250]. The key ingredient in the proof of Theorem 6.6 is the next result.

Theorem 6.7 (Lück [157], Theorem 7.1) *Let $d \geq 0$ be an integer. Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of spaces such that F (resp. E) has the homotopy type of a connected CW-complex with finite d-skeleton (resp. $(d+1)$-skeleton). Suppose that the image of $\pi_1(F) \longrightarrow \pi_1(E)$ is infinite and $\mathbb{Z}$ is a subgroup of $\pi_1(B)$. Then*

$$b_1^{(2)}(\widetilde{E}) \ = \ 0.$$

Proof : We give only a sketch of the proof. The idea is to consider the L^2-version of the Leray-Serre type spectral sequence associated to the fibration which is explained in [157] (see also [239]). It suffices to show that its E^2-term vanishes on the p-axis and on the q-axis. The vanishing on the p-axis is essentially a consequence of Theorem 1.7.9 and the assumption that the image of $\pi_1(F) \longrightarrow \pi_1(E)$ is infinite. Since $\mathbb{Z}$ is a subgroup of $\pi_1(B)$, there is a map $f : S^1 \longrightarrow B$ inducing an injection on the fundamental groups. Let $p_0 : E_0 \longrightarrow S^1$ be the pull back of $p : E \longrightarrow B$ with f. A spectral sequence comparison argument shows that it suffices to prove the vanishing of the E^2-term of p_0 on the q-axis. Since S^1 is 1-dimensional, the E^2-term is the E^∞-term, so that this is equivalent to the vanishing of the L^2-homology of $\widetilde{E_0}$. This follows from Theorem 6.4. ∎

More information about groups with vanishing first L^2-Betti number can be found in [16]. See also [121], [147].

7. Kähler hyperbolic manifolds

In this section we explain Gromov's notion of Kähler-hyperbolic manifolds and his computations of the L^2-Betti numbers of the universal coverings of closed Kähler hyperbolic manifolds. In particular, we prove the Hopf Conjecture 2.7 for closed Kähler manifolds. In this section all manifolds have no boundary and come with a complex structure.

We recall some basic facts about Kähler manifolds which are standard in the compact case and extend to the not necessarily compact but complete situation. More details can be found for instance in [249, chapter V]. Let M be a (complex) manifold. Let h be a Hermitian metric on M. In particular, we have for each $x \in M$ a Hermitian form $h_x : T_xM \times T_xM \longrightarrow \mathbb{C}$. This induces a Riemannian metric g on M and a 2-form called *fundamental 2-form* ω defined on M by

$$\begin{aligned} g_x = \Re(h_x) :& \quad T_xM \times T_xM \quad \longrightarrow \mathbb{R}; \\ \omega_x = -\frac{1}{2} \cdot \Im(h_x) :& \quad T_xM \times T_xM \quad \longrightarrow \mathbb{R}. \end{aligned}$$

Definition 7.1 *A* Kähler manifold *M is a complex manifold M with Hermitian metric h such that (M,g) is complete and ω is closed, i.e. $d\omega = 0$. In this context ω is called the* Kähler form*.* ■

Let M be a Kähler manifold of complex dimension $m = dim_{\mathbb{C}}(M)$ and real dimension $n = \dim_{\mathbb{R}}(M) = 2m$. Next we deal with its Hodge theory. We introduce the following notations and identifications:

$$\begin{aligned}
T_xM \otimes_{\mathbb{R}} \mathbb{C} &= \operatorname{res}_{\mathbb{R}} T_xM \otimes_{\mathbb{R}} \mathbb{C};\\
\operatorname{Alt}^p(T_xM \otimes_{\mathbb{R}} \mathbb{C}) &= \{\text{alternating p-forms over the complex vector space } T_xM \otimes_{\mathbb{R}} \mathbb{C}\};\\
\operatorname{Alt}^p_{\mathbb{R}}(\operatorname{res}_{\mathbb{R}} T_xM, \operatorname{res}_{\mathbb{R}} \mathbb{C}) &= \{\text{real alternating p-forms on} \operatorname{res}_{\mathbb{R}} T_xM \text{ with values in } \operatorname{res}_{\mathbb{R}} \mathbb{C}\};\\
\operatorname{Alt}^p(T_xM \otimes_{\mathbb{R}} \mathbb{C}) &= \operatorname{Alt}^p_{\mathbb{R}}(\operatorname{res}_{\mathbb{R}} T_xM, \operatorname{res}_{\mathbb{R}} \mathbb{C});\\
\Omega^p(M) &= C^\infty(\operatorname{Alt}^p(TM \otimes_{\mathbb{R}} \mathbb{C})) = \{\text{smooth p-forms on the smooth}\\
&\qquad \text{manifold } M \text{ with values in } \operatorname{res}_{\mathbb{R}} \mathbb{C}\}.
\end{aligned}$$

Let $J_x : T_xM \longrightarrow T_xM$ be multiplication with i and $(J_x \otimes_{\mathbb{R}} id) : T_xM \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow T_xM \otimes_{\mathbb{R}} \mathbb{C}$ be the induced map. Let $(T_xM \otimes_{\mathbb{R}} \mathbb{C})'$ resp. $(T_xM \otimes_{\mathbb{R}} \mathbb{C})''$ be the eigenspace of $(J_x \otimes_{\mathbb{R}} id)$ for the eigenvalue i (resp. $-i$). We obtain identifications (resp. decompositions):

$$\begin{aligned}
T_xM \otimes_{\mathbb{R}} \mathbb{C} &= (T_xM \otimes_{\mathbb{R}} \mathbb{C})' \oplus (T_xM \otimes_{\mathbb{R}} \mathbb{C})'';\\
T_xM &= (T_xM \otimes_{\mathbb{R}} \mathbb{C})';\\
\operatorname{Alt}^{p,q}(T_xM \otimes_{\mathbb{R}} \mathbb{C}) &= \operatorname{Alt}^p((T_xM \otimes_{\mathbb{R}} \mathbb{C})') \otimes_{\mathbb{C}} \operatorname{Alt}^q((T_xM \otimes_{\mathbb{R}} \mathbb{C})'')\\
\operatorname{Alt}^r(T_xM \otimes_{\mathbb{R}} \mathbb{C}) &= \oplus_{p+q=r} \operatorname{Alt}^{p,q}(T_xM \otimes_{\mathbb{R}} \mathbb{C});\\
\Omega^{p,q}(M) &= C^\infty(\operatorname{Alt}^{p,q}(TM \otimes_{\mathbb{R}} \mathbb{C}))\\
\Omega^r(M) &= \oplus_{p+q=r} \Omega^{p,q}(M).
\end{aligned}$$

We denote the map induced by complex conjugation by

$$^- : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q}(M)$$

and the *Hodge star operator* by

$$* : \Omega^{p,q}(M) \longrightarrow \Omega^{m-p,m-q}(M).$$

Let $L^2\Omega^p(M)$ be the *Hilbert space of square-integrable p-forms on M* with respect to the inner product

$$\langle \omega, \eta \rangle \;=\; \omega \wedge *\overline{\eta} \;=\; \int \langle \omega, \eta \rangle_x \, \mathrm{dvol}\,.$$

Let $d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$ be the *exterior differential.* Its *adjoint*

$$d^* = (-1)^{(n+1)r+1} \overline{*} d \overline{*} : \Omega^{r+1}(M) \longrightarrow \Omega^r(M)$$

satisfies $\langle d\omega, \eta\rangle = \langle \omega, d^*\eta\rangle$. Define

$$\begin{array}{rcl} \partial : \Omega^{p,q}(M) & \longrightarrow & \Omega^{p+1,q}(M); \\ \overline{\partial} : \Omega^{p,q}(M) & \longrightarrow & \Omega^{p,q+1}(M) \end{array}$$

as the composition

$$\Omega^{p,q}(M) \hookrightarrow \Omega^r(M) \xrightarrow{d} \Omega^{r+1}(M) \xrightarrow{\text{pr}} \left\{ \begin{array}{l} \Omega^{p+1,q}(M) \\ \Omega^{p,q+1}(M) \end{array} \right. .$$

Define *Laplace operators*

$$\begin{array}{rll} \Delta = & dd^* + d^*d : \Omega^r(M) & \longrightarrow \Omega^r(M); \\ \Box = & \partial\partial^* + \partial^*\partial : \Omega^{p,q}(M) & \longrightarrow \Omega^{p,q}(M); \\ \overline{\Box} = & \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \Omega^{p,q}(M) & \longrightarrow \Omega^{p,q}(M). \end{array}$$

These operators are related as follows.

Lemma 7.2 *If M is a Kähler manifold, then*

$$\begin{array}{rcl} \partial \circ \partial & = & 0; \\ \overline{\partial} \circ \overline{\partial} & = & 0; \\ d & = & \partial + \overline{\partial}; \\ \Delta & = & 2 \cdot \Box \;=\; 2 \cdot \overline{\Box}. \end{array} \qquad \blacksquare$$

Definition 7.3 *Define the* space of harmonic L^2-forms *by*

$$\begin{array}{rcl} \mathcal{H}^{p,q}_{(2)}(M) & = & \left\{ \omega \in \Omega^{p,q}(M) \mid \overline{\Box}(\omega) = 0, \int \omega \wedge *\omega < \infty \right\}; \\ \mathcal{H}^{r}_{(2)}(M) & = & \left\{ \omega \in \Omega^{r}(M) \mid \Delta(\omega) = 0, \int \omega \wedge *\omega < \infty \right\}. \end{array} \qquad \blacksquare$$

Theorem 7.4 (L^2-Hodge-deRham decomposition in the Kaehler case) *If M is Kähler, then*

$$\begin{array}{rcl} L^2\Omega^r(M) & = & \mathcal{H}^r_{(2)}(M) \oplus \overline{d(L^2\Omega^{r-1}(M))} \oplus \overline{d^*(L^2\Omega^{r+1}(M))}; \\ \mathcal{H}^r_{(2)}(M) & = & \oplus_{p+q=r} \mathcal{H}^{p,q}_{(2)}(M). \end{array} \qquad \blacksquare$$

Theorem 7.5 (L^2-Lefschetz Theorem) *Let M be Kähler with Kähler form ω and real dimension $n = 2m$. Define*

$$L^k : \Omega^r(M) \longrightarrow \Omega^{r+2k}(M) \qquad \phi \mapsto \phi \wedge \omega^k.$$

Then

1. *L^k commutes with d, d^* and Δ;*

2. *L^k induces bounded operators*

$$\begin{array}{rcl} L^k : L^2\Omega^r(M) & \longrightarrow & L^2\Omega^{r+2k}(M); \\ L^k : \mathcal{H}^r(M) & \longrightarrow & \mathcal{H}^{r+2k}(M); \end{array}$$

3. *These operators are quasi-isometries, i.e. $C^{-1} \cdot \|\phi\| \leq \|L^k(\phi)\| \leq C \cdot \|\phi\|$ for appropriate $C > 0$, and in particular injective for $2r + 2k \leq n$ and surjective for $2r + 2k \geq n$.*

Proof : We give a sketch of the proof. The Kähler condition implies that $\omega_x^m \neq 0$ for $x \in M$ and ω is parallel with respect to the Levi-Civita connection of (M, g). If $w : I \longrightarrow M$ is a path from x to y and T_w is the induced isometric parallel transport, then $T_w^* \omega_y = \omega_x$ and the following diagram commutes

$$\begin{array}{ccc} \mathrm{Alt}^r(T_xM \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow{L_x^k} & \mathrm{Alt}^{r+2k}(T_xM \otimes_{\mathbb{R}} \mathbb{C}) \\ {\scriptstyle T_w^*}\downarrow & & \downarrow{\scriptstyle T_w^*} \\ \mathrm{Alt}^r(T_yM \otimes_{\mathbb{R}} \mathbb{C}) & \xrightarrow[L_y^k]{} & \mathrm{Alt}^{r+2k}(T_yM \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

Linear algebra shows that $L_x^k : \mathrm{Alt}^r(T_xM \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \mathrm{Alt}^{r+2k}(T_xM \otimes_{\mathbb{R}} \mathbb{C})$ is injective for $2r+2k \leq n$ and surjective for $2r+2k \geq n$. Now everything follows except for the surjectivity statement for L^k. It suffices to prove surjectivity in the case $2r + 2k = n$ because of the factorization

$$L^{l+k} : \Omega^{r-2l}(M) \xrightarrow{L^l} \Omega^r(M) \xrightarrow{L^k} \Omega^{r+2k}(M).$$

If M is compact, the claim follows for $\mathcal{H}^r(M)$ as the Hodge star operator yields an isomorphism $\mathcal{H}^r(M) \cong \mathcal{H}^{r+2k}(M)$ and L^k is an injective map $\mathcal{H}^r(M) \longrightarrow \mathcal{H}^{r+2k}(M)$ of finite dimensional complex vector spaces of the same dimension. In general, one argues as follows. Consider the adjoint of L_x^k

$$K_x^k : \mathrm{Alt}^{r+2k}(T_xM \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \mathrm{Alt}^r(T_xM \otimes_{\mathbb{R}} \mathbb{C}).$$

As L_x^k is surjective, each K_x^k is injective and we get a quasi-isometry

$$K : L^2\Omega^{r+2k}(M) \longrightarrow L^2\Omega^r(M)$$

which is the adjoint of L. As K is injective, L has dense image. As L is a quasi-isometry, L has closed image. Hence L^k is surjective. The proof for $\mathcal{H}^r(M)$ is analogous. ■

Corollary 7.6 *Let M be a compact Kähler manifold with $n = dim_{\mathbb{R}}(M)$, $m = dim_{\mathbb{C}}(M)$. Put $b_r(M) = dim_{\mathbb{Q}}(H_r(M, \mathbb{Q})) = dim_{\mathbb{C}}(\mathcal{H}^r(M))$ and $h_{p,q}(M) = dim_{\mathbb{C}}(\mathcal{H}^{p,q}(M))$. Then:*

1. $b_r(M) = \sum_{p+q=r} h_{p,q}(M)$;
2. $b_r(M) = b_{n-r}(M)$, $h_{p,q}(M) = h_{m-p,m-q}(M)$;
3. $h_{p,q}(M) = h_{q,p}(M)$;
4. $b_r(M)$ *is even for* r *odd;*
5. $h_{1,0}(M) = \frac{1}{2} \cdot b_1(M)$ *depends only on* $\pi_1(M)$;
6. $b_r(M) \leq b_{r+2}(M)$ *for* $r \leq m$. ■

The following definitions are taken from [108].

Definition 7.7 *A p-form* η *is* bounded *if*

$$\|\eta\|_\infty = sup\{\|\eta_x\| \mid x \in M\} < \infty.$$

A p-form ω *is* d(bounded) *if* $\omega = d\eta$ *for a bounded* $(p-1)$*-form* η. *We call* ω $\widetilde{d}$(bounded) *if its lift* $\widetilde{\omega}$ *to* $\widetilde{M}$ *is d(bounded).* ■

Definition 7.8 *A* Kähler hyperbolic manifold M *is a compact Kähler manifold* M *whose Kähler form is* $\widetilde{d}$*(bounded).* ■

Theorem 7.9 (Gromov [108], Main Theorem 2.5 on page 283) *Let* M *be a Kähler hyperbolic manifold with* $n = 2m = dim_{\mathbb{R}}(M)$ *and universal covering* $\widetilde{M}$. *Then*

1. *We get outside the middle dimension:*

$$\begin{aligned} \mathcal{H}^{p,q}_{(2)}(\widetilde{M}) &= 0 \quad \textit{for } p+q \neq m; \\ b^{(2)}_r(\widetilde{M}) &= 0 \quad \textit{for } r \neq m; \end{aligned}$$

2. *We get for the middle dimension:*

$$\begin{aligned} \mathcal{H}^{p;q}_{(2)}(\widetilde{M}) &\neq 0 \quad \textit{for } p+q = m; \\ b^{(2)}_m(\widetilde{M}) &\neq 0; \end{aligned}$$

3. $(-1)^m \cdot \chi(M) > 0$.

Proof : We only give the proof of the easy part 1.). Notice that 3.) follows from 1.) and 2.) by the Euler-Poincaré formula 1.7.2. Because of Poincaré duality and the L^2-Hodge-deRham decomposition 1.7, it suffices to prove

$$\mathcal{H}^r_{(2)}(\widetilde{M}) \;=\; 0 \qquad \text{for } r < m.$$

Choose k with $2r + 2k = n$. Let $\widetilde{\omega}$ be the lift of the Kähler form ω on M to the universal covering $\widetilde{M}$. Then by the L^2-Lefschetz Theorem 7.5

$$L^k : \mathcal{H}^r_{(2)}(\widetilde{M}) \longrightarrow \mathcal{H}^{r+2k}_{(2)}(\widetilde{M}) \qquad \phi \mapsto \phi \wedge \widetilde{\omega}^k$$

is bijective. Put $\mu = \phi \wedge \widetilde{\omega}^{k-1} \wedge \eta$ for $\widetilde{\omega} = d\eta$ with $\|\eta\|_\infty < \infty$. As $\|\omega^{k-1}\|_\infty < \infty$, μ is an L^2-form. Obviously $L^k(\phi) = d(\mu)$. By the L^2-Hodge-deRham decomposition Theorem 7.4, $L^k(\phi) = 0$ and hence $\phi = 0$. ■

Theorem 7.9 has been generalized by Jost and Zuo [130]. Further information about L^2-cohomology and Kähler manifolds can be found in [1].

Next we make some comments about the notion of Kähler hyperbolicity and applications of Theorem 7.9. Here is a list of examples of Kähler hyperbolic manifolds (see [108, section 0]):

- M is a compact Kähler manifold and homotopy equivalent to a compact Riemannian manifold of negative sectional curvature;
- M is a compact Kähler manifold, $\pi_1(M)$ is hyperbolic in Gromov's sense [107] and $\pi_2(M) = 0$;
- $\widetilde{M}$ is a symmetric Hermitian space of non-compact type with no Euclidean factor;
- M is a submanifold of a Kähler hyperbolic manifold;
- M is a product of two Kähler hyperbolic manifolds.

Next we give two compact Kähler manifolds which are not Kähler hyperbolic. Equip $\mathbb{C}P^n$ with the Fubini-study metric which is, up to a positive constant, uniquely determined by the property that it is $U(n+1)$-invariant. Then the fundamental 2-form is the first Chern class $c_1(L)$. This is closed but not exact. Hence $\mathbb{C}P^n$ is Kähler but not Kähler hyperbolic. The torus T^{2n} with a complex structure is Kähler but cannot be Kähler hyperbolic because $\chi(T^{2n}) = 0$.

Theorem 7.10 *Let M be a compact Kähler manifold. Then the following assertions are equivalent and they are true if M is Kähler hyperbolic:*

1. *M is Moishezon, i.e. the transcendental degree of the field of meromorphic functions on M is $\dim_{\mathbb{C}}(M)$;*

2. *M is Hodge, i.e. the Kähler form represents an integral cohomology class, i.e. it represents an element in the image of $H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{C})$;*

3. *M can be holomorphically embedded in $\mathbb{C}P^n$;*

4. *M is a projective algebraic variety.*

Proof : The equivalence of these statements is due to Kodaira [134], [249, Chapter VI] and Moishezon [177]. The first assertion is a consequence of Theorem 7.9 and proven in [108, section 3] using a version of the L^2-index theorem. ■

8. Novikov-Shubin invariants

In this section we introduce and study spectral density functions and Novikov-Shubin invariants. They were introduced analytically by Novikov and Shubin in [196].

Definition 8.1 *Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Let $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ denote the (right-continuous) family of spectral projections of the positive operator f^*f. Define the* spectral density function of f *by*

$$F(f,\lambda) : \mathbb{R} \longrightarrow [0,\infty) \qquad \lambda \mapsto \dim_{\mathcal{N}(\Gamma)}\left(\mathrm{im}(E_{\lambda^2}^{f^*f})\right). \qquad ■$$

The spectral density function is monotone and right-continuous. It takes values in $[0,\|f\|]$. Here, and in the sequel, $|x|$ denotes the norm of an element x of a Hilbert $\mathcal{N}(\Gamma)$-module and $\|f\|$ the operator norm of a morphism. Since f and f^*f have the same kernel, $\dim_{\mathcal{N}(\Gamma)}(\ker(f)) = F(f,0)$. Given two morphisms f and g, we call their spectral density functions *dilatationally equivalent* if there are constants $C>0$ and $\epsilon > 0$ such that

$$F(f,C^{-1}\cdot\lambda) \leq F(g,\lambda) \leq F(f,C\cdot\lambda) \qquad \text{for } \lambda \leq \epsilon$$

holds.

Example 8.2 Suppose that Γ is finite. Then a morphism $f : U \longrightarrow V$ of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules is just a linear Γ-equivariant map of (over $\mathbb{C}$ finite-dimensional) unitary Γ-representations. Let $0 \leq \lambda_0 < \ldots < \lambda_r$ be the eigenvalues of f^*f and μ_i be the multiplicity of λ_i, i.e. the dimension of the eigenspace of λ_i. Then the spectral density function is a right continuous step function which is zero for $\lambda < 0$ and has a step of height $\frac{\mu_i}{|\Gamma|}$ at each $\sqrt{\lambda_i}$. Given two such maps f and g, their spectral density functions are dilatationally equivalent if and only if the kernels of f and g have the same complex dimension. ■

These notions become much more interesting in the case when Γ is infinite, because then the spectrum of f^*f is in general not discrete anymore.

Lemma 8.3 *1. Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Let $\mathcal{L}(f,\lambda)$ denote the set of all Hilbert $\mathcal{N}(\Gamma)$-submodules L of U with the property that $|f(x)| \leq \lambda \cdot |x|$ holds for $x \in L$. Then*

$$F(f,\lambda) = \sup\left\{\dim_{\mathcal{N}(\Gamma)}(L) \mid L \in \mathcal{L}(f,\lambda)\right\};$$

2. Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be morphisms of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Then

$$F(f,\lambda) \leq F(gf, \|g\| \cdot \lambda).$$

If additionally f has dense image, then

$$F(g,\lambda) \leq F(gf, \|f\| \cdot \lambda);$$

3. Let $u : U \longrightarrow V$, $f : V \longrightarrow W$ and $i : W \longrightarrow X$ be morphisms of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Suppose that u is an isomorphism and i is injective with closed image. Then the spectral density functions of f and $i \circ f \circ u$ are dilatationally equivalent.

4. Let $f_i : U_i \longrightarrow V_i$ be morphisms of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules for $i = 0,1$. Then

$$F(f_0 \oplus f_1, \lambda) = F(f_0,\lambda) + F(f_1,\lambda).$$

Proof : 1.) If $E^{f^*f}_{\lambda^2}(x) = x$,

$$|f(x)|^2 = \left|\int_0^\infty \mu \; d\langle E^{f^*f}_\mu(x), x\rangle\right| \leq \lambda^2 \cdot \left|\int_0^\infty 1 \; d\langle E^{f^*f}_\mu(x), x\rangle\right| \leq \lambda^2 \cdot |x|^2.$$

Hence the image of $E^{f^*f}_{\lambda^2}$ belongs to $\mathcal{L}(f,\lambda)$. This shows

$$F(f,\lambda) \leq \sup\left\{\dim_{\mathcal{N}(\Gamma)}(L) \mid L \in \mathcal{L}(f,\lambda)\right\}.$$

It remains to prove that $\dim_{\mathcal{N}(\Gamma)}(L) \leq \dim_{\mathcal{N}(\Gamma)}(\mathrm{im}(E^{f^*f}_{\lambda^2}))$ holds for all $L \in \mathcal{L}(f,\lambda)$. If $\lambda \geq 0$ and $x \in U$ satisfies $E^{f^*f}_{\lambda^2}(x) = 0$ and $x \neq 0$, then $|f(x)| > \lambda \cdot |x|$. Hence $E^{f^*f}_{\lambda^2}$ induces a weak isomorphism from L to $\mathrm{clos}(E^{f^*f}_{\lambda^2}(L))$ and the claim follows from Lemma 1.4.

2.) Consider $L \in \mathcal{L}(f,\lambda)$. For all $x \in L$ we get $|gf(x)| \leq \|g\| \cdot |f(x)| \leq \|g\| \cdot \lambda \cdot |x|$. This implies that $L \in \mathcal{L}(gf, \|g\| \cdot \lambda)$, and the first equation follows.

Now suppose that f has dense image. Consider $L \in \mathcal{L}(g,\lambda)$. For all $x \in f^{-1}(L)$, we have

$$|gf(x)| \leq \lambda \cdot |f(x)| \leq \lambda \cdot \|f\| \cdot |x|.$$

This implies $f^{-1}(L) \in \mathcal{L}(gf, \|f\| \cdot \lambda)$. It remains to show $\dim_{\mathcal{N}(\Gamma)}(L) \leq \dim_{\mathcal{N}(\Gamma)}(f^{-1}(L))$. Let $p : U \longrightarrow U/\ker f$ be the projection and let $\overline{f} : U/\ker(f) \longrightarrow V$ be the map induced by f. Since p is surjective, we get from Lemma 1.4

$$\dim_{\mathcal{N}(\Gamma)} \left(f^{-1}(L)\right) \geq \dim_{\mathcal{N}(\Gamma)} \left(p(f^{-1}(L))\right) = \dim_{\mathcal{N}(\Gamma)} \left(\overline{f}^{-1}(L)\right).$$

Next we show that the weak isomorphism $\overline{f}$ induces a weak isomorphism from $\overline{f}^{-1}(L)$ to L. Notice that then $\dim_{\mathcal{N}(\Gamma)}(\overline{f}^{-1}(L)) = \dim_{\mathcal{N}(\Gamma)}(L)$ because of Lemma 1.4, and the second equation will follow.

Because of the Polar Decomposition Theorem applied to $\overline{f}$, it suffices to prove for a positive weak isomorphism $h : V \longrightarrow V$ and a Hilbert $\mathcal{N}(\Gamma)$-submodule $L \subset V$ that $h(h^{-1}(L))$ is dense in L. Now L has an orthogonal decomposition of the form $L = \operatorname{clos}(h(h^{-1}(L))) \oplus M$, where M is a $\mathcal{N}(\Gamma)$-submodule of L. If we can show $\dim_{\mathcal{N}(\Gamma)} M = 0$, then Lemma 1.4.1 will imply $M = 0$ and we will be done. As $h(h^{-1}(M)) \subset M$ and $h(h^{-1}(M)) \subset h(h^{-1}(L))$, it follows that $h(h^{-1}(M)) = 0$. Therefore $M \cap \operatorname{im}(h) = 0$. For $\lambda > 0$, consider the map $\pi_\lambda : M \longrightarrow E^h_\lambda(V)$ given by $\pi_\lambda(m) = E^h_\lambda(m)$. If $m \in \ker(\pi_\lambda)$, then the spectral theorem shows that $m \in \operatorname{im}(h)$. Therefore $\ker(\pi_\lambda) = 0$, and Lemma 1.4 implies

$$\dim_{\mathcal{N}(\Gamma)} M \leq \dim_{\mathcal{N}(\Gamma)}(E^h_\lambda(V)).$$

As h is injective, continuity of the dimension 1.4.3 and the right-continuity of the spectral family implies

$$\lim_{\lambda \to 0^+} \dim_{\mathcal{N}(\Gamma)}(E^h_\lambda(V)) \;=\; \dim_{\mathcal{N}(\Gamma)}(E^h_0(V)) \;=\; \dim_{\mathcal{N}(\Gamma)}(\ker(h)) \;=\; 0.$$

Hence $\dim_{\mathcal{N}(\Gamma)} M = 0$ and assertion 2.) is proven.

3.) By the Open Mapping Theorem there is a constant $D > 0$ such that

$$D^{-1} \cdot |x| \leq |i(x)| \leq D \cdot |x|$$

holds for all $x \in W$. Hence $F(i \circ f \circ u)$ and $F(f \circ u)$ are dilatationally equivalent by the first assertion. By the second assertion, $F(f \circ u)$ and $F(f)$ are dilatationally equivalent.

4.) This follows from additivity of the dimension 1.4.4. ■

Definition 8.4 *Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Define its* Novikov-Shubin invariant *by*

$$\alpha(f) \;=\; \liminf_{\lambda \to 0^+} \frac{\ln(F(f,\lambda) - F(f,0))}{\ln(\lambda)} \;\in [0,\infty],$$

provided that $F(f,\lambda) > F(f,0)$ holds for all $\lambda > 0$. Otherwise, we put

$$\alpha(f) \;=\; \infty^+.$$

Here ∞^+ is a new formal symbol which should not be confused with ∞.

Let $\overline{X} \longrightarrow X$ be a regular covering over the CW-complex X of finite type with Γ as group of deck transformations. Define its p-th Novikov-Shubin invariant

$$\alpha_p(\overline{X}) \;=\; \alpha(c_p^{(2)}) \qquad \in [0,\infty] \amalg \{\infty^+\}$$

where $c_p^{(2)}$ is the p-th differential in the L^2-chain complex of $\overline{X}$ introduced in Definition 1.5. ■

Example 8.5 We consider the example of the universal covering of S^1 with $\mathbb{Z}$ as group of deck transformations using the notation and results of Example 1.8 and Example 3.1. The spectral family of the first differential $c_1^{(2)}$ of the L^2-chain complex has as projection for λ the operator given by multiplication with the characteristic function of the set $\{z \in S^1 \mid |z-1| \leq \lambda\}$. Hence for small $\lambda > 0$ we get for the spectral density function

$$F(c_1^{(2)},\lambda) \;=\; \operatorname{vol}\{z \in S^1 \mid |z-1| \leq \lambda\} \;=\; \operatorname{vol}\{\cos(\phi) + i\cdot\sin(\phi) \mid \lambda \geq |2-2\cos(\phi)|\}.$$

Because of

$$\lim_{\phi \to 0} \frac{2-2\cos(\phi)}{\phi^2} \;=\; 1$$

$F(c_1^{(2)},\Lambda)$ and λ are dilatationally equivalent, and hence we get

$$\alpha_1(\widetilde{S^1}) \;=\; 1. \qquad ■$$

Notice that $\alpha(f) = \infty^+$ precisely if and only if f^*f has a gap in the spectrum above 0. Moreover, a morphism $f: U \longrightarrow V$ of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules is an isomorphism if and only if $\dim_{\mathcal{N}(\Gamma)}(\ker(f)) = 0$, $\dim_{\mathcal{N}(\Gamma)}(U) = \dim_{\mathcal{N}(\Gamma)}(V)$ and $\alpha(f) = \infty^+$. If there is no gap in the spectrum above 0, then $\alpha(f)$ is the supremum over all non-negative numbers β for which there is an $\epsilon > 0$ such that

$$F(f,\lambda) - F(f,0) \leq t\lambda^\beta \qquad \text{for } 0 \leq \lambda \leq \epsilon.$$

The invariant $\alpha(f)$ measures how fast $F(f,\lambda)$ approaches $F(f,0)$ for $\lambda \to 0+$.

Remark 8.6 Notice that the Novikov-Shubin invariant $\alpha_p(\overline{X})$ is ∞^+ if and only if the image of the p-th differential $c_p^{(2)}$ of the L^2-chain complex of $\overline{X}$ is closed. Hence $\alpha_p(\overline{X})$ measures the difference between the L^2-homology of $\overline{X}$, which is $\ker(c_p^{(2)})/\operatorname{clos}(\operatorname{im}(c_{p+1}^{(2)}))$, and the Γ-equivariant homology of $\overline{X}$ with coefficients in the $\mathbb{Z}\Gamma$-module $l^2(\Gamma)$, which is $\ker(c_p^{(2)})/\operatorname{im}(c_{p+1}^{(2)})$. We will continue this discussion in section 10. ■

Before we give the main properties of the Novikov-Shubin invariants, we recall some notions and facts from group theory. A finitely generated group Γ is *nilpotent* if Γ possesses a finite lower central series

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_s = \{1\} \qquad\qquad \Gamma_{k+1} = [\Gamma, \Gamma_k].$$

If $\overline{\Gamma}$ contains a nilpotent subgroup Γ of finite index, then $\overline{\Gamma}$ is said to be *virtually nilpotent.* Let d_i be the rank of the quotient Γ_i/Γ_{i+1} and let d be the integer $\sum_{i\geq 1} id_i$. Then $\overline{\Gamma}$ has polynomial growth of degree d [11]. Note that a group has polynomial growth if and only if it is virtually nilpotent [105].

Theorem 8.7 1. *Homotopy invariance*

Let $\overline{X}$ and $\overline{Y}$ be regular coverings of CW-complexes X and Y of finite type with the same group Γ of deck transformations. Let $f : \overline{X} \longrightarrow \overline{Y}$ be a Γ-equivariant map. If f is a homotopy equivalence, then the spectral density functions of $c_p^{(2)}$ for $\overline{X}$ and $\overline{Y}$ are dilatationally equivalent for all p, and in particular

$$\alpha_p(\overline{X}) \;=\; \alpha_p(\overline{Y}) \qquad \text{for } 0 \leq p.$$

If f is d-connected, i.e. f induces an isomorphism on π_n for $n < d$ and an epimorphism on π_d, then the spectral density functions of $c_p^{(2)}$ for $\overline{X}$ and $\overline{Y}$ are dilatationally equivalent for $p \leq d$, and in particular

$$\alpha_p(\overline{X}) \;=\; \alpha_p(\overline{Y}) \qquad \text{for } p \leq d;$$

2. *Equality of analytic and combinatorial version*

   Let $\overline{M}$ be a covering of the oriented closed Riemannian manifold M with deck transformation group Γ. We can define the spectral density function (see Example 1.9) and Novikov-Shubin invariants analytically in terms of the Laplace operator acting on differential forms on $\overline{M}$. Then the analytic and the combinatorial spectral density function are dilatationally equivalent. In particular, the analytically defined Novikov-Shubin invariants and the combinatorial Novikov-Shubin invariants agree.

3. *Poincaré duality*

   Let $\overline{M}$ be a regular covering of the closed manifold M of dimension n. Then

$$\alpha_p(\overline{M}) \;=\; \alpha_{n+1-p}(\overline{M});$$

4. *Dependency on the fundamental group*

   The Novikov-Shubin invariants of the universal covering of a connected CW-complex of finite type $\alpha_p(\widetilde{X})$ for $p \leq 2$ depend only on $\pi_1(X)$. If M is a closed n-dimensional manifold with $n \leq 4$, then $\alpha_p(\widetilde{M})$ depends only on $\pi_1(M)$ for all p;

5. *Invariance under finite coverings*

   Let X be a CW-complex of finite type and $p : \overline{X} \longrightarrow X$ be a regular covering with group of deck transformations Γ. Let $\Gamma_0 \subset \Gamma$ be a subgroup of Γ of finite index n. We obtain a regular covering denoted by $\overline{\overline{X}}$ by $\overline{X} \longrightarrow \overline{X}/\Gamma_0$. Notice that the coverings $\overline{\overline{X}}$ and $\overline{X}$ have the same total spaces but different groups of deck transformations. Then

$$\alpha_p(\overline{\overline{X}}) = \alpha_p(\overline{X}) \qquad \text{for } p \geq 0;$$

6. *First Novikov-Shubin invariant*

   Let X be a connected CW-complex of finite type with fundamental group π and universal covering $\widetilde{X}$. Then

   (a) *$\alpha_1(\widetilde{X})$ is finite if and only if π is infinite and virtually nilpotent. In this case $\alpha_1(\widetilde{X})$ is the growth rate of π;*

   (b) *$\alpha_1(\widetilde{X})$ is ∞^+ if and only if π is finite or nonamenable;*

   (c) *$\alpha_1(\widetilde{X})$ is ∞ if and only if π is amenable and not virtually nilpotent;*

7. *S^1-actions and Novikov-Shubin invariants*

   Let M be a connected closed manifold with S^1-action. Suppose that for one orbit S^1/H (and hence all orbits) the inclusion into M induces a map on π_1 with infinite image. (In particular, the S^1-action has no fixed points.) Then

$$\alpha_p(\widetilde{M}) \geq 1 \qquad \text{for all } p;$$

8. *Positivity of the Novikov-Shubin invariants for 3-manifolds*

   Let M be a 3-manifold which has finite fundamental group or satisfies the assumptions of Theorem 3.3. Then we get for the Novikov-Shubin invariants of the universal covering

$$\alpha_p(\widetilde{M}) > 0 \qquad \text{for all } p;$$

9. *Novikov-Shubin invariants for $\mathbb{Z}$ as group of deck transformations*

   Let $\overline{X} \longrightarrow X$ be a regular covering of the CW-complex X of finite type with $\mathbb{Z}$ as group of deck transformations. Since $\mathbb{C}[\mathbb{Z}]$ is a principal ideal domain, one can write the $\mathbb{C}[\mathbb{Z}]$-module

$$H_{p-1}(\overline{X}, \mathbb{C}) = \mathbb{C}[\mathbb{Z}]^n \oplus \oplus_{i=1}^k \mathbb{C}[\mathbb{Z}]/((z - a_i)^{r_i})$$

   for integers n, k and r_i with $n, k \geq 0$ and $r_i \geq 1$ and $a_i \in \mathbb{C}$ with $a_i \neq 0$ and $a_i \neq a_j$ for $i \neq j$, where $z \in \mathbb{Z}$ is the generator. Then

$$\alpha_p(\overline{X}) = \min \left\{ \frac{1}{r_i} \;\middle|\; i = 1, 2 \ldots, r, a_i \in S^1 \right\}$$

   if $k \geq 1$ and there is at least one a_i with $a_i \in S^1$, and

$$\alpha_p(\overline{X}) = \infty^+$$

   otherwise;

10. *Hyperbolic manifolds*

If M is a hyperbolic closed manifold of dimension n, then

$$\begin{array}{rcll} \alpha_p(\overline{M}) & = & 1 & \text{if } n \text{ is odd and } p = \frac{n\pm 1}{2}, \text{and} \\ \alpha_p(\overline{M}) & = & \infty^+ & \text{otherwise;} \end{array}$$

11. *Kähler hyperbolic manifolds*

If M is Kähler hyperbolic in the sense of Definition 7.8, then

$$\alpha_p(\widetilde{M}) = \infty^+ \qquad \text{for } p \geq 0.$$

Proof : 1.) As in the proof of Theorem 1.7.1, one justifies the assumption that f is a cellular Γ-homotopy equivalence. Hence it suffices to show, for a chain map $f : C \longrightarrow D$ of chain complexes of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules, that the spectral density functions of the p-th differentials c_p and d_p are dilatationally equivalent for all p.

We begin with the case $D = 0$. This means that C is contractible and we must prove $\alpha_p(C) = \infty^+$. Let γ_* be a chain contraction for C, i.e. a collection of morphisms $\gamma_p : C_p \longrightarrow C_{p+1}$ satisfying $c_{p+1} \circ \gamma_p + \gamma_{p-1} \circ c_p = \mathrm{id}$. Using c_p and γ_{p-1}, we can construct morphisms $\overline{c_p} : C_p/\operatorname{clos}(\operatorname{im}(c_{p+1})) \longrightarrow C_{p-1}$ and $\overline{\gamma_{p-1}} : C_{p-1} \longrightarrow C_p/\operatorname{clos}(\operatorname{im}(c_{p+1}))$ such that $\overline{\gamma_{p-1}} \circ \overline{c_p} = \mathrm{id}$. Hence $\overline{c_p}$ induces an invertible operator onto its image. Lemma 8.3.3 implies $\alpha(c_p) = \alpha(\overline{c_p}) = \infty^+$.

The case where f is an isomorphism of chain complexes follows from Lemma 8.3.3. Now we can treat the general case.

There are exact sequences of chain complexes $0 \longrightarrow C \longrightarrow \operatorname{cyl}(f) \longrightarrow \operatorname{cone}(f) \longrightarrow 0$ and $0 \longrightarrow D \longrightarrow \operatorname{cyl}(f) \longrightarrow \operatorname{cone}(C) \longrightarrow 0$, where cone denotes the mapping cone. The chain complexes $\operatorname{cone}(f)$ and $\operatorname{cone}(C)$ are contractible as chain complexes of Hilbert $\mathcal{N}(\Gamma)$-modules since f is a homotopy equivalence by assumption. We obtain chain isomorphisms $C \oplus \operatorname{cone}(f) \longrightarrow \operatorname{cyl}(f)$ and $D \oplus \operatorname{cone}(C) \longrightarrow \operatorname{cyl}(f)$ by the following general construction for an exact sequence $0 \longrightarrow C \xrightarrow{j} D \xrightarrow{q} E \longrightarrow 0$ with contractible E: Choose a chain contraction ϵ for E, and for each p a morphism $t_p : E_p \longrightarrow D_p$ such that $q_p \circ t_p = \mathrm{id}$. Put

$$s_p = d_{p+1} \circ t_{p+1} \circ \epsilon_p + t_p \circ \epsilon_{p-1} \circ e_p.$$

This defines a chain map $s : E \longrightarrow D$ such that $q \circ s = \mathrm{id}$. Define a chain map $u : D \to C$ by mapping $x \in D_p$ to $y = u_p(x)$ which is the unique element $y \in C_p$ such that $x = s_p q_p(x) + j_p(y)$. Then $j + s$ is a chain isomorphism $C \oplus E \longrightarrow D$ with inverse $u \oplus q$. Since $C \oplus \operatorname{cone}(f)$ and $D \oplus \operatorname{cone}(C)$ are isomorphic and $\operatorname{cone}(f)$ and $\operatorname{cone}(C)$ are contractible, the claim follows from the special cases which we have already proven and Lemma 8.3.4.

The more general case of a d-connected map is proven in [148, Lemma 3.3 on page 33]. The homotopy invariance of the analytically defined Novikov-Shubin invariants is proven by Gromov and Shubin [110].

2.) This is proven by Efremov [84], [85].

3.) This follows from homotopy invariance as in the proof of Theorem 1.7.3.

4.) This follows from 1.) applied to the classifying map $\widetilde{X} \longrightarrow E\Gamma$ and Poincaré duality.

5.) This is similar to the proof of Theorem 8.3.7.

6.) This is proven in [148, Lemma 3.5 on page 34] using [34] and [241].

7.) This is proven in [148, Theorem 3.1].

8.) This is proven in [148, Theorem 0.1].

9.) This is proven in [157, Example 4.3].

10.) This is proven in [142, Proposition 46 on page 499] and [76].

11.) This is proven in [108, Theorem 1.4.A on page 274]. ■

Remark 8.8 We have already mentioned in Remark 1.9 that the L^2-Betti numbers are invariants of the asymptotic large time behaviour of the heat kernel on the regular covering $\overline{M} \longrightarrow M$ of a closed Riemannian manifold M (see 1.12). The Novikov-Shubin invariants measure the speed of convergence of the limit as $t \to \infty$ in the analytic definition of the L^2-Betti numbers. Namely, one gets

$$\min\{\alpha_p(\overline{M}), \alpha_{p-1}(\overline{M})\} \;=\; \sup\left\{\beta_p \in [0,\infty) \;\middle|\; \lim_{t\to\infty} \frac{\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(e^{-t\Delta_p}(\overline{x},\overline{x})\right) d\overline{x} - b_p^{(2)}(\overline{M})}{t^{\beta_p/2}} = 0\right\}$$

in $[0,\infty]$ where we do not distinguish between ∞ and ∞^+ here. The supremum of the right is the same as the supremum over all numbers $\beta_p \geq 0$ for which there is a $K > 0$ such that

$$\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(e^{-t\Delta_p}(\overline{x},\overline{x})\right) d\overline{x} - b_p^{(2)}(\overline{M}) \leq t^{\beta_p/2} \qquad \text{for } K \leq t.$$

See the discussion in [110, appendix] about decay exponents and Laplace transforms. ■

We mention that more explicit calculations of the Novikov-Shubin invariants for 3-manifolds can be found in [148]. The connection for abelian fundamental groups with Massey products and information about locally symmetric spaces can be found in [142, section VI and VII]. Finally we mention the following conjecture [148, Conjecture 7.1 on page 56].

Conjecture 8.9 (Rationality and Positivity of Novikov-Shubin invariants)
The Novikov-Shubin invariants of the universal covering $\widetilde{M}$ of a closed Riemannian manifold M are all positive and rational. ■

Conjecture 8.9 is true if M is hyperbolic or Kähler hyperbolic, or if the fundamental group of M is abelian or free. The positivity of the Novikov-Shubin invariants plays a role in the definition of L^2-torsion of Section 9 and is proven in [148, Theorem 0.1 on page 16] for all 3-manifolds satisfying the asumption of Theorem 3.3 that none of its prime factors is exceptional. A similar discussion as for Conjecture 2.1, Lemma 2.2 and Remark 2.3 can be made for the Conjecture 8.9 above [148, section 7]. Novikov-Shubin invariants for arbitrary spaces with Γ-action (without any finiteness assumptions) are constructed in [162]. Further references on Novikov-Shubin invariants are [89], [90], [111], [157].

9. L^2-torsion

In this section we introduce and study L^2-torsion. This is the L^2-version of the classical combinatorially defined Reidemeister torsion and the analytically defined Ray-Singer torsion. We will restrict ourselves in this section to the case of the universal covering $\widetilde{X} \longrightarrow X$ of a connected finite CW-complex , for simplicity and due the fact that this is the most important case for applications. The general case of a regular covering is not much harder.

We begin with recalling the notions of combinatorial Reidemeister torsion and analytic Ray-Singer torsion in order to motivate the L^2-versions we will introduce later. A reader who is familiar with these concepts may skip this part.

Definition 9.1 *Let $\widetilde{X} \longrightarrow X$ be the universal covering of a connected finite CW-complex with group of deck transformations $\Gamma = \pi_1(X)$. Let V be a unitary (finite-dimensional) Γ-representation. Let $C_*(\widetilde{X})$ be the cellular $\mathbb{Z}\Gamma$-chain complex. We obtain a $\mathbb{C}$-chain complex $V \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{X})$. Its chain modules inherit a Hilbert space structure from the cellular $\mathbb{Z}\Gamma$-basis and the inner product on V. Suppose that $H_p(V \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{X}))$ is trivial for all $p \geq 0$. Define the combinatorial Laplace operator of $\widetilde{X}$ with coefficients in V by*

$$\Delta_p \;=\; c_{p+1} \circ c_p^* + c_{p-1}^* \circ c_p : V \otimes_{\mathbb{Z}\Gamma} C_p(\widetilde{X}) \longrightarrow V \otimes_{\mathbb{Z}\Gamma} C_p(\widetilde{X}).$$

Then the Reidemeister torsion *of $\widetilde{X}$ with coefficients in V is defined as*

$$\rho(\widetilde{X}; V) \;=\; -\sum_{p \geq 0} (-1)^p \cdot p \cdot \ln\left(\det_{\mathbb{C}}(\Delta_p)\right). \qquad \blacksquare$$

The original definition uses chain contractions of $V \otimes_{\mathbb{Z}\Gamma} C(\widetilde{X})$. The calculation in the proof of [163, Lemma 7.12 on page 257] shows that the definition above agrees with the logarithm of the classical one. We use the logarithm because then the combinatorial version will coincide with the analytic one.

Reidemeister [215] introduced this invariant to classify lens spaces up to PL-homeomorphism (and up to diffeomorphism) using the work of Franz [94] (see also [59, chapter V], [175, section 3]). This classification of lens spaces was generalized by deRham [216] who proved that two orthogonal G-representations V and W are isometrically $\mathbb{R}G$-isomorphic if and only if their unit spheres are G-diffeomorphic (see also [149, Proposition 3.2 on page 478], [153, page 317], [220, section 4]).

The result of deRham does not hold in the topological category. Namely, there are non-linearly isomorphic G-representations V and W whose unit spheres are G-homeomorphic, by the work of Cappell-Shaneson [46] (see also [114]). However, if G has odd order, G-homeomorphic implies G-diffeomorphic for unit spheres in G-representations as shown by Hsiang-Pardon [124] and Madsen-Rothenberg [167].

Let M be a closed Riemannian manifold with $\Gamma = \pi_1(M)$. Then one defines $\rho(\widetilde{M};V)$ to be $\rho(\widetilde{X};V)$ for any smooth triangulation X of M, provided that $H_p(V \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{X}))$ vanishes for all $p \geq 0$. The last condition ensures that the definition is independent of the choice of triangulation and depends only on the simple homotopy type of M and, in particular, only on the homeomorphism type of M. If this condition is not satisfied, one modifies the definition by a term which is essentially given by the torsion of the deRham isomorphism and which relates this homology (which is canonically isomorphic to the cohomology) to the finite-dimensional Hilbert space given by harmonic forms on M with coefficients in V. Then $\rho(\widetilde{M};V)$ is independent of the choice of triangulation but depends on the Riemannian metric.

Ray-Singer [211] defined the analytic counterpart of Reidemeister torsion using a regularization of the zeta-function as follows. Let M be a closed Riemannian manifold with $\Gamma = \pi_1(M)$. Let $\Delta_p : \Omega^p(M;V) \longrightarrow \Omega^p(M;V)$ be the Laplace operator acting on smooth p-forms on M with coefficients in the unitary (finite-dimensional) Γ-representation V. This is an essentially selfadjoint operator with discrete spectrum as M is compact. The *zeta-function* is defined by

$$\zeta_p(s) \;=\; \sum_{\lambda > 0} \lambda^{-s}$$

where λ runs through the positive eigenvalues of Δ_p listed with multiplicity. The zeta-function is holomorphic for $\Re(s) > \dim(M)/2$ and has a meromorphic extension to $\mathbb{C}$ with no poles in 0 [232]. So its derivative for $s = 0$ is defined. Now the *Ray-Singer torsion* of M is defined by [211, Definition 1.6 on page 149] (our definition is the old one multiplied by the factor 2)

$$\rho(M;V) \;=\; \sum_{p \geq 0} (-1)^p \cdot p \cdot \frac{d}{ds}\, \zeta_p(s)\big|_{s=0}.$$

The basic idea is that $\frac{d}{ds}\zeta_p(s)|_{s=0}$ is a generalization of the ordinary determinant $\det_{\mathbb{C}}$. Namely, if $f : V \longrightarrow V$ is a positive linear automorphism of the finite-dimensional complex vector space V and λ_1, λ_2, ..., λ_r are the eigenvalues of f listed with multiplicity, then we get

$$\begin{aligned}
\frac{d}{ds}\,\zeta_p(s)\big|_{s=0} &= \frac{d}{ds}\sum_{i=1}^{r}\lambda^{-s}\Big|_{s=0}\\
&= \sum_{i=1}^{r}\left(-\ln(\lambda)\cdot\lambda^{-s}\right)\big|_{s=0}\\
&= -\ln\left(\prod_{i=1}^{r}\lambda_i\right)\\
&= -\ln\left(\det_{\mathbb{C}}(f)\right).
\end{aligned}$$

Ray and Singer conjectured that the analytic and combinatorial versions agree. This conjecture was proven independently by Cheeger [50] and Müller [184]. Manifolds with boundary and manifolds with symmetries, sum (= glueing) formulas and fibration formulas are treated in [67], [149], [153], [165], [242], [243], [244]. Non-unitary coefficient systems are studied in [25], [26], [186]. Further references are [18], [19], [20], [21], [22], [23], [24], [32], [40], [69], [88], [95], [96], [101], [135], [136], [143], [181], [187], [208], [213], [245].

The definition of combinatorial L^2-torsion is based on the notion of the determinant which we treat next.

Definition 9.2 *Let $f : U \longrightarrow V$ be a morphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Let $F(f,\lambda)$ be the spectral density function of Definition 8.1 which is a monotone non-decreasing right-continuous function. Let dF be the unique measure on the Borel σ-algebra on $\mathbb{R}$ which satisfies $dF(]a,b]) = F(b) - F(a)$ for $a < b$. Then define the* (generalized) Kadison-Fuglede determinant

$$\det_{\mathcal{N}(\Gamma)}(f) \in [0,\infty)$$

by the positive real number

$$\det_{\mathcal{N}(\Gamma)}(f) = \exp\left(\int_{0+}^{\infty}\ln(\lambda)\; dF\right)$$

if the Lebesgue integral $\int_{0+}^{\infty}\ln(\lambda)\; dF$ converges to a real number and by 0 otherwise. ■

Example 9.3 To illustrate this definition, we look at the example where Γ is finite. We essentially get the ordinary determinant $\det_{\mathbb{C}}$. Namely, we have computed the spectral density function for finite Γ in Example 8.2. Let λ_1, λ_2, ..., λ_r be the non-zero eigenvalues of f^*f with multiplicity μ_i. Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f,

$$\begin{aligned}
\det_{\mathcal{N}(\Gamma)}(f) &= \exp\left(\sum_{i=1}^{r} \frac{\mu_i}{|\Gamma|} \cdot \ln(\sqrt{\lambda_i})\right) \\
&= \prod_{i=1}^{r} \lambda_i^{\frac{\mu_i}{2\cdot|\Gamma|}} \\
&= \det_{\mathbb{C}}(\overline{f^*f})^{\frac{1}{2\cdot|\Gamma|}}.
\end{aligned}$$

If f is an isomorphism we get

$$\det_{\mathcal{N}(\Gamma)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|\Gamma|}}. \qquad \blacksquare$$

If f is an isomorphism, then Definition 9.2 of $\det_{\mathcal{N}(\Gamma)}(f)$ reduces to the classical notion due to Fuglede and Kadison [97]. The proof of the next two lemmas can be found in [154, Lemma 4.1 on page 94 and Lemma 4.2 on page 97].

Lemma 9.4 *Let $f : M \longrightarrow N$ be a morphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Then*

1. *We have for $0 < \epsilon \leq a$:*

$$\begin{aligned}
\int_{\epsilon}^{a} \ln(\lambda)\, dF &= -\int_{\epsilon}^{a} \frac{1}{\lambda} \cdot (F(\lambda) - F(0))\, d\lambda \\
&\quad + \ln(a) \cdot (F(a) - F(0)) - \ln(\epsilon) \cdot (F(\epsilon) - F(0)); \\
\int_{0+}^{a} \ln(\lambda)\, dF &= \lim_{\epsilon \to 0+} \int_{\epsilon}^{a} \ln(\lambda)\, dF; \\
\int_{0+}^{a} \frac{1}{\lambda} \cdot (F(\lambda) - F(0))\, d\lambda &= \lim_{\epsilon \to 0+} \int_{\epsilon}^{a} \frac{1}{\lambda} \cdot (F(\lambda) - F(0))\, d\lambda;
\end{aligned}$$

2. *If the Novikov-Shubin invariant satisfies $\alpha(f) > 0$ and $a \geq \|f\|$, then the integrals*

$$\int_{0+}^{\infty} \ln(\lambda)\, dF$$

and

$$\ln(a) \cdot (F(a) - F(0)) - \int_{0+}^{a} \frac{1}{\lambda} \cdot (F(\lambda) - F(0))\, d\lambda$$

do converge to the same real number and we have

$$\det_{\mathcal{N}(\Gamma)}(f) > 0. \qquad \blacksquare$$

Lemma 9.5 *Let M and N be finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. Let $s, t : M \longrightarrow M$, $u : M \longrightarrow N$ and $v : N \longrightarrow N$ be morphisms. Suppose that s, t and v have trivial kernel. Then we get*

1. $\det_{\mathcal{N}(\Gamma)}(s) = \det_{\mathcal{N}(\Gamma)}(s^*) = \sqrt{\det_{\mathcal{N}(\Gamma)}(s^*s)} = \sqrt{\det_{\mathcal{N}(\Gamma)}(ss^*)}$;
2. *If $0 \leq s$ (i.e. s is a positive operator), then*
$$\lim_{\epsilon \to 0+} \det_{\mathcal{N}(\Gamma)}(s + \epsilon \cdot \mathrm{id}) = \det_{\mathcal{N}(\Gamma)}(s);$$
3. *If $0 \leq s \leq t$, then*
$$\det_{\mathcal{N}(\Gamma)}(s) \leq \det_{\mathcal{N}(\Gamma)}(t);$$
4. $\det_{\mathcal{N}(\Gamma)}(st) = \det_{\mathcal{N}(\Gamma)}(s) \cdot \det_{\mathcal{N}(\Gamma)}(t)$;
5. $\det_{\mathcal{N}(\Gamma)} \begin{pmatrix} s & u \\ 0 & v \end{pmatrix} = \det_{\mathcal{N}(\Gamma)}(s) \cdot \det_{\mathcal{N}(\Gamma)}(v)$. ■

The notation of determinant class below is taken from [43].

Definition 9.6 *Let $\widetilde{X} \longrightarrow X$ be the universal covering of a finite CW-complex X with Γ as group of deck transformations. We call $\widetilde{X}$ of* determinant class *if $\det(\Delta_p) > 0$ holds for all $p \geq 0$ where Δ_p is the combinatorial Laplace operator introduced in 1.6. If $\widetilde{X}$ is of determinant class, we define its L^2-*torsion *as*

$$\rho^{(2)}(\widetilde{X}) \;=\; -\sum_{p \geq 0} (-1)^p \cdot p \cdot \ln\left(\det_{\mathcal{N}(\Gamma)}(\Delta_p)\right) \;\in \mathbb{R}. \qquad ■$$

Schick [229] has introduced the class $\mathcal{G}$ of groups which has the following properties. All amenable groups belong to $\mathcal{G}$. Moreover, if $1 \to \Delta \to \Gamma \to \pi \to 1$ is an extension, $\Delta \in \mathcal{G}$ and π is amenable, then $\Gamma \in \mathcal{G}$. If Γ is the direct limit or the inverse limit of a directed system $\{\Gamma_i \mid i \in I\}$ of groups with $\Gamma_i \in \mathcal{G}$ for all $i \in I$, then $\Gamma \in \mathcal{G}$. The class $\mathcal{G}$ is closed under free products. It is residually closed and, in particular, contains all residually finite groups.

Lemma 9.7
1. *Let X be a connected finite CW-complex. If $\alpha_p(\widetilde{X}) > 0$ for all $p \geq 0$ or if $\pi_1(X)$ belongs to $\mathcal{G}$, then $\widetilde{X}$ is of determinant class;*
2. *Let $f : X \longrightarrow Y$ be a homotopy equivalence of finite connected CW-complexes. If $\widetilde{X}$ or $\widetilde{Y}$ is of determinant class, then both $\widetilde{X}$ and $\widetilde{Y}$ are of determinant class.*

Proof : 1.) If $\alpha_p(\widetilde{X}) > 0$, this follows from Lemma 9.4.2. If $\Gamma \in \mathcal{G}$, the claim is proven in [229, Theorem 1.14] (see also [41, appendix], [56], [156, Theorem 3.4.2 on page 476].

2.) We get from Theorem 8.7 that the spectral density functions of the p-th differential, and hence of the combinatorial Laplace operator Δ_p on the L^2-chain complex of $\widetilde{X}$ and of $\widetilde{Y}$, are dilatationally equivalent. Lemma 9.4.1 implies that Δ_p for $\widetilde{X}$ is of determinant class if and only if the one for $\widetilde{Y}$ is. ■

Next we describe the favorite situation for L^2-torsion.

Definition 9.8 *Let $\widetilde{X} \longrightarrow X$ be the universal covering of a connected finite CW-complex X with $\Gamma = \pi_1(X)$. We call $\widetilde{X}$ of* acyclic determinant class *if $\widetilde{X}$ is of determinant class and $b_p^{(2)}(\widetilde{X}) = 0$ holds for all $p \geq 0$. We call $\widetilde{X}$* admissible *if $\alpha_p(\widetilde{X}) > 0$ and $b_p^{(2)}(\widetilde{X}) = 0$ holds for all $p \geq 0$.* ■

Of course admissible implies of acyclic determinant class because of Lemma 9.4.2. If X is not connected, we mean by the phrase that $\widetilde{X}$ is of acyclic determinant class (resp. admissible) that the universal covering of each component of X has this property, and we write $\rho^{(2)}(\widetilde{X})$ for the sum of the L^2-torsions of the universal coverings of the components of X. This remark is relevant for the sum formula appearing in the next theorem.

The L^2-torsion for universal coverings of finite CW-complexes of acyclic determinant class behaves like a multiplicative Euler characteristic, as the following result illuminates.

Theorem 9.9

1. *Homotopy invariance*

   Let $f : X \longrightarrow Y$ be a homotopy equivalence of connected finite CW-complexes with $\Gamma = \pi_1(X) = \pi_1(Y)$. Suppose that $\widetilde{X}$ or $\widetilde{Y}$ is of acyclic determinant class (resp. admissible). Let

   $$\Phi_\Gamma : \mathrm{Wh}(\Gamma) \longrightarrow \mathbb{R}^{>0}$$

   be the map from the Whitehead group of Γ (see [59, §11], [175, page 373]) to the multiplicative group of positive real numbers which assigns to the class of an invertible (n,n)-matrix A over $\mathbb{Z}\Gamma$ the determinant $\det_{\mathcal{N}(\Gamma)}(f)$ of the isomorphism $f : l^2(\Gamma)^n \longrightarrow l^2(\Gamma)^n$ which is induced by right multiplication with A. Let $\tau(f) \in \mathrm{Wh}(\Gamma)$ be the Whitehead torsion [59, chapter IV], [175, page 377]. Then both $\widetilde{X}$ and $\widetilde{Y}$ are of acyclic determinant class (resp. admissible) and we get

   $$\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X}) = \ln(\Phi_\Gamma(\tau(f)));$$

2. *Fundamental groups belonging to $\mathcal{G}$*

Let $f : X \longrightarrow Y$ be a homotopy equivalence of connected finite CW-complexes. Suppose that $\Gamma = \pi_1(X) = \pi_1(Y) \in \mathcal{G}$ and that $b_p^{(2)}(\widetilde{X})$ or $b_p^{(2)}(\widetilde{Y})$ is trivial for all $p \geq 0$. Then both $\widetilde{X}$ and $\widetilde{Y}$ are of acyclic determinant class and

$$\rho^{(2)}(\widetilde{X}) \;=\; \rho^{(2)}(\widetilde{Y});$$

3. *Sum formula*

   Consider the pushout of finite CW-complexes such that j_1 is an inclusion of CW-complexes and j_2 is cellular

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ X_2 & \xrightarrow[i_2]{} & X \end{array}$$

   Assume that $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are of acyclic determinant class (resp. admissible) and that for $i = 0, 1, 2$ the map $\pi_1(X_i) \longrightarrow \pi_1(X)$ induced by the inclusion is injective for all base points in X_i. Then $\widetilde{X}$ is of acyclic determinant class (resp. admissible) and we get

$$\rho^{(2)}(\widetilde{X}) \;=\; \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0});$$

4. *Fibration Formula*

   Let $F \longrightarrow E \longrightarrow B$ be a fibration of connected finite CW-complexes. Suppose that $\widetilde{F}$ is of acyclic determinant class (resp. admissible) and the inclusion induces an injection $\pi_1(F) \longrightarrow \pi_1(E)$. Then $\widetilde{E}$ is of acyclic determinant class (resp. admissible) and we get

$$\rho^{(2)}(\widetilde{E}) \;=\; \chi(B) \cdot \rho^{(2)}(\widetilde{F})$$

   where $\chi(B)$ is the Euler characteristic of B;

5. *Product formula*

   Let X and Y be connected finite CW-complexes. Suppose that $\widetilde{X}$ is of acyclic determinant class (resp. admissible). Then $\widetilde{X} \times \widetilde{Y}$ is of acyclic determinant class (resp. admissible) and we get

$$\rho^{(2)}\left(\widetilde{X} \times \widetilde{Y}\right) \;=\; \chi(Y) \cdot \rho^{(2)}(\widetilde{X});$$

6. *Poincaré duality*

   Let M be a closed manifold of even dimension. Suppose $\widetilde{M}$ is of acyclic determinant class. Then we get

$$\rho^{(2)}(\widetilde{M}) = 0;$$

7. *Multiplicative property for finite coverings*

   Let X be a connected finite CW-complex and $p : Y \longrightarrow X$ be a finite d-sheeted covering. Suppose that $\widetilde{Y}$ or $\widetilde{X}$ is of acyclic determinant class (resp. admissible). Then both $\widetilde{Y}$ and $\widetilde{X}$ are of acyclic determinant class (resp. admissible) and we get

$$\rho^{(2)}(\widetilde{Y}) \;=\; d \cdot \rho^{(2)}(\widetilde{X});$$

8. *S^1-actions*

   Let M be a connected closed manifold with S^1-action. Suppose that for one orbit S^1/H (and hence all orbits) the inclusion into M induces a map on π_1 with infinite image. (In particular, the S^1-action has no fixed points.) Then $\widetilde{M}$ is admissible, and in particular, of acyclic determinant class and

$$\rho^{(2)}(\widetilde{M}) \;=\; 0.$$

Proof : The proof is given for admissible CW-complexes in [156]. The case of determinant class follows analogously using the following result. Let $0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$ be a short exact sequence of Hilbert $\mathcal{N}(\Gamma)$-chain complexes which are finite-dimensional and whose chain modules are finitely generated. Suppose for two of the chain complexes that they are L^2-acyclic and the associated Laplace operators are of determinant class in each dimension. Then the third chain complex has the same property. The statement about the acyclicity follows from the long weakly exact homology sequence of Cheeger and Gromov [52, Theorem 2.1 on page 10]. The strategy of proof for the determinant class is similiar to the one in [148, Theorem 2.3 on page 27] using Lemma 9.5.5 instead of [148, Lemma 1.12 on page 25]. The case, where the fundamental group belongs to $\mathcal{G}$ follows from [229, Theorem 1.14].

■

We mention the following conjecture

Conjecture 9.10 *The map $\Phi_\Gamma : \mathrm{Wh}(\Gamma) \longrightarrow \mathbb{R}^{>0}$ defined in Theorem 9.9.1 is always trivial.*
■

Suppose that M is a closed Riemannian manifold. Then one defines its L^2-torsion analogously to the classical definition of $\rho(\widetilde{M}; V)$. If the L^2-Betti numbers are not all trivial, one has to invoke a correction term analogously to the classical definition which involves the L^2-Hodge-deRham isomorphism. Details can be found, for instance, in [163, section 5].

As for L^2-Betti numbers and for Novikov-Shubin invariants there are analytic versions of the L^2-torsion due to Lott [142] and Matthai [170].

Definition 9.11 *Let $\widetilde{M} \longrightarrow M$ be the universal covering of a closed Riemannian manifold M with $\Gamma = \pi_1(M)$. Suppose that $\widetilde{M}$ is of determinant class. Define*

$$\operatorname{tr}_{\mathcal{N}(\Gamma)}\left(e^{-t\Delta_p}\right) = \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}\left(e^{-t\Delta_p}(\widetilde{x},\widetilde{x})\right)\, d\widetilde{x}$$

using the notation of Remark 1.9. Then we define the analytic L^2-torsion of $\widetilde{M}$

$$\rho^{(2)}(\widetilde{M}) = \sum_{p\geq 0}(-1)^p \cdot p \cdot \left(\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{\epsilon} t^{s-1}\cdot\left(\operatorname{tr}_{\mathcal{N}(\Gamma)}\left(e^{-t\Delta_p}\right) - b_p^{(2)}(\widetilde{M})\right)\, dt\Big|_{s=0} + \int_{\epsilon}^{\infty} t^{-1}\cdot\left(\operatorname{tr}_{\mathcal{N}(\Gamma)}\left(e^{-t\Delta_p}\right) - b_p^{(2)}(\widetilde{M})\right)\, dt\right). \quad \blacksquare$$

The definition is independent of the choice of ϵ by the following calculation. The Γ-function

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\, dt$$

satisfies $\Gamma(s+1) = s\cdot\Gamma(s)$ and $\Gamma(1) = 1$. In the sequel we abbreviate

$$T(t) = \operatorname{tr}_{\mathcal{N}(\Gamma)}\left(e^{-t\Delta_p}\right) - b_p^{(2)}(\widetilde{M}).$$

We compute for $0 < \epsilon \leq \delta$

$$\begin{aligned}
&\frac{d}{ds}\frac{1}{\Gamma(s)}\int_{\epsilon}^{\delta} t^{s-1}\cdot T(t)\, dt\Big|_{s=0}\\
=\ &\frac{d}{ds}\, s\cdot\frac{1}{\Gamma(s+1)}\int_{\epsilon}^{\delta} t^{s-1}\cdot T(t)\, dt\Big|_{s=0}\\
=\ &\frac{d}{ds}s\Big|_{s=0}\cdot\frac{1}{\Gamma(s+1)}\int_{\epsilon}^{\delta} t^{s-1}\cdot T(t)\, dt\Big|_{s=0} + 0\cdot\frac{d}{ds}\frac{1}{\Gamma(s+1)}\int_{\epsilon}^{\delta} t^{s-1}\cdot T(t)\, dt\Big|_{s=0}\\
=\ &\int_{\epsilon}^{\delta} t^{-1}\cdot T(t)\, dt.
\end{aligned}$$

The following calculation shows the relation of the definition above with the classical Ray-Singer torsion. Namely, we get in the setting of the Ray-Singer torsion the following equation, where λ runs over the eigenvalues of the Laplace operator in dimension p listed with multiplicity:

$$
\begin{aligned}
\sum_{\lambda>0} \lambda^{-s} &= \sum_{\lambda>0} \frac{1}{\Gamma(s)} \cdot \lambda^{-s} \cdot \int_0^\infty t^{s-1} e^{-t}\, dt \\
&= \sum_{\lambda>0} \frac{1}{\Gamma(s)} \cdot \int_0^\infty (t\lambda^{-1})^{s-1} e^{-\lambda(t\lambda^{-1})} \lambda^{-1}\, dt \\
&= \sum_{\lambda>0} \frac{1}{\Gamma(s)} \cdot \int_0^\infty t^{s-1} e^{-\lambda t}\, dt \\
&= \frac{1}{\Gamma(s)} \cdot \int_0^\infty t^{s-1} \cdot \sum_{\lambda>0} e^{-\lambda t}\, dt \\
&= \frac{1}{\Gamma(s)} \cdot \int_0^\infty t^{s-1} \cdot \left(\mathrm{tr}_{\mathbb{C}}\left(e^{-t\Delta_p}\right) - \dim_{cc}(H_p(M;V))\right)\, dt.
\end{aligned}
$$

The integral from 0 to ϵ appearing in the Definition 9.11 exists by an argument analogous to the proof that the zeta-function is meromorphic without pole in 0 in the classical case. Given $p \geq 0$, the integral from ϵ to ∞ obviously converges if the Novikov-Shubin invariant $\alpha_p(\widetilde{M})$ is positive. It does converge if and only if for the analytic defined spectral density function F_p of the Laplace operator acting on p-forms on the universal covering 1.10 the integral $\int_{0+}^1 \ln(\lambda)\, dF$ does converge [43, Proposition 2.12]. Since the analytic and combinatorial spectral density functions are dilatationally equivalent by Theorem 8.7.2, $\int_{0+}^1 \ln(\lambda)\, dF$ converges if and only if $\widetilde{M}$ is of determinant class.

Next we mention the important result of Burghelea, Friedlander, Kappeler and McDonald which generalizes the Theorem of Cheeger and Müller to the L^2-case. The main technical tool is the generalization of the calculus of elliptic pseudo-differential operators, and of the Helffer-Sjöstrand analysis of the Witten deformation of the deRham complex for a closed Riemannian manifold with coefficients in a unitary finite-dimensional representation to a unitary representation on a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module.

Theorem 9.12 (Burghelea-Friedlander-Kappeler-McDonald [43]) *The analytic and the combinatorial L^2-torsion of the universal covering of a closed Riemannian manifold agree.* ∎

Theorem 9.12 above allows us to combine the results we have already mentioned for the combinatorial version with analytic results. The following result is taken from [53, Proposition 6.4 on page 149]

Lemma 9.13 *Let X be a simply-connected Riemannian manifold and $f : X \to \mathbb{R}$ be a function which is invariant under the isometries of X. Then there is a constant $C(f)$ with* *the property that for any cocompact free proper action of a discrete group Γ by isometries and any fundamental domain $\mathcal{F}$*

$$\int_{\mathcal{F}} f \, d\,\mathrm{vol}_X \quad = \quad C(f) \cdot \mathrm{vol}(X/\Gamma)$$

holds.

The next result is a consequence of the analytic definitions of L^2-Betti numbers, Novikov-Shubin invariants and L^2-torsion and Lemma 9.13.

Lemma 9.14 *Let X be a complete Riemannian manifold. Then there are constants $B_p^{(2)}(X)$ for $p \geq 0$, $A_p(X)$ for $p \geq 1$ and $T(X)$ such that for any closed Riemannian manifold M whose universal covering is isometrically diffeomorphic to X the following holds*

$$\begin{aligned} b_p^{(2)}(M) &= B_p^{(2)}(X) \cdot \mathrm{vol}(M); \\ \alpha_p^{(2)}(M) &= A_p^{(2)}(X); \\ \rho^{(2)}(M) &= T^{(2)}(M) \cdot \mathrm{vol}(M). \end{aligned}$$

The constant $T^{(2)}(X)$ appearing in Lemma 9.14 can be computed for $X = \mathbb{H}^d$ as follows. Consider the polynomial with integer coefficients for $j \in \{0, 1, 2, \ldots n-1\}$

$$P_j^n(\nu) \quad := \quad \frac{\prod_{i=0}^n (\nu^2 + i^2)}{\nu^2 + (n-j)^2} \quad = \quad \sum_{k \geq 0}^{2n} K_{k,j}^n \cdot \nu^{2k}.$$

Define

$$\begin{aligned} C_d \quad := \quad & \sum_{j=0}^{n-1} (-1)^{n+j+1} \frac{n!}{(2n)! \cdot \pi^n} \cdot \begin{pmatrix} 2n \\ j \end{pmatrix} \\ & \cdot \sum_{k=0}^{n} K_{k,j}^n \cdot \frac{(-1)^{k+1}}{2k+1} \cdot (n-j)^{2k+1}. \end{aligned} \tag{9.15}$$

The first values of C_d are computed in [118, Theorem 2]

$$\begin{aligned} C_3 &= \frac{1}{3\pi} &&\approx 0.106103; \\ C_5 &= \frac{62}{45\pi^2} &&\approx 0.139598; \\ C_7 &= \frac{221}{35\pi^3} &&\approx 0.203645; \\ C_{39} &\approx 4.80523 \cdot 10^7, \end{aligned}$$

and the constants C_d are positive, strictly increasing and grow very fast, namely they satisfy [118, Proposition 6]

$$\begin{aligned} C_{2n+1} &\geq \frac{n}{2\pi} \cdot C_{2n-1}; \\ C_{2n+1} &\geq \frac{2n!}{3(2\pi)^n}. \end{aligned}$$

The next result has been proven for 3-manifolds by Lott [142, Proposition 16] and Matthai [170, Corollary 6.7].

Theorem 9.16 (Hess-Schick [118], Theorem 2) *Let M be a closed hyperbolic d-dimensional manifold for odd $d = 2n + 1$. Let $C_d > 0$ be the constant introduced in 9.15. Then*

$$\rho^{(2)}(\widetilde{M}) \;=\; (-1)^n \cdot C_d \cdot \operatorname{vol}(M).$$

Recall that we have introduced the basic notions and results about 3-manifolds in Section 3.

Theorem 9.17 (Lück-Schick [164], Theorem 0.7) *Let M be a compact connected orientable prime 3-manifold with infinite fundamental group such that the boundary of M is empty or a disjoint union of incompressible tori. Suppose that M satisfies Thurston's Geometrization Conjecture which implies that there is a decomposition along disjoint incompressible 2-sided tori in M whose pieces are Seifert manifolds or hyperbolic manifolds. Let M_1, M_2, ..., M_r be the hyperbolic pieces. They all have finite volume [179, Theorem B on page 52]. Then M is admissible and*

$$\rho^{(2)}(\widetilde{M}) \;=\; -\frac{1}{3\pi} \cdot \sum_{i=1}^{r} \operatorname{vol}(M_i).$$

In particular, $\rho^{(2)}(\widetilde{M})$ is 0 if and and only if there are no hyperbolic pieces. ■

Examples of manifolds satisfying the assumptions of Theorem 9.17 are complements of knots in S^3.

We recall the definition of simplicial volume of an n-dimensional oriented closed manifold M [106, Section 0.2]. Let $C_*^{\text{sing}}(M,\mathbb{R})$ be the singular chain complex of M with coefficients in the real numbers $\mathbb{R}$. An element c in $C_p^{\text{sing}}(M,\mathbb{R})$ is given by a finite $\mathbb{R}$-linear combination $c = \sum_{i=1}^{s} r_i \cdot \sigma_i$ of singular p-simplices σ_i in M. Define the *l^1-norm* of c by setting

$$\|c\|_1 \;=\; \sum_{i=1}^{s} |r_i|.$$

For $\alpha \in H_m(M;\mathbb{R})$ define

$$\|\alpha\|_1 \;=\; \inf\left\{\|c\|_1 \;\mid c \in C_m^{\text{sing}}(M;\mathbb{R}) \text{ is a cycle representing } \alpha\right\}.$$

The *simplicial volume* of M is defined by

$$\|M\| \;=\; \|\,[M]\,\|_1,$$

where $[M]$ is the image of the fundamental class of M under the change of ring homomorphism on singular homology $H_n(M;\mathbb{Z}) \longrightarrow H_n(M;\mathbb{R})$. We mention the following extension of a question of Gromov [109, section 8A].

Conjecture 9.18 *Let M be a closed aspherical orientable manifold with vanishing simplicial volume. Then M is admissible and its L^2-torsion is trivial.* ∎

This conjecture is based on a variety of calculations and similarities of the properties of L^2-torsion and simplicial volume. The simplicial volume and the L^2-torsion are multiplicative under finite coverings (see [106, page 8] and Theorem 9.9.7). If the universal coverings of two closed Riemannian manifolds M and N are isometrically diffeomorphic, then [106, page 11]

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}. \tag{9.19}$$

The corresponding proportionality principle holds for the L^2-torsion by Lemma 9.14. There are constants T_n and V_n, depending only on the dimension n, such that for any closed orientable hyperbolic manifold of dimension n we have

$$\|M\| = V_n \cdot \mathrm{vol}(M); \tag{9.20}$$

$$\rho^{(2)}(\widetilde{M}) = T_n \cdot \mathrm{vol}(M), \tag{9.21}$$

where V_n^{-1} is the supremum of all n-dimensional geodesic simplices, i.e. the convex hull of $(n+1)$ points in general position, in the n-dimensional hyperbolic space $\mathbb{H}^n$, and T_n is zero for even n and $T_3 = -\frac{1}{3\pi}$ (see [106, section 2.2] and Theorem 9.16). Conjecture 9.18 is true for a closed aspherical orientable manifold with non-trivial smooth S^1-action. Namely, then its simplicial volume vanishes [106, Section 3.1], [253] and the map induced by evaluation $\pi_1(S^1) \longrightarrow \pi_1(M)$ is injective [66, Lemma 5.1 on page 242 and Corollary 5.3 on page 243] which implies that M is admissible and $\rho^{(2)}(\widetilde{M}) = 0$ by Theorem 9.9.8. Theorem 9.17 is true for $\|M\|$ instead of $\rho^{(2)}(\widetilde{M})$, if one substitutes V_3 for $-\frac{1}{3\pi}$, by [236],[240]. Hence Conjecture 9.18 is true for 3-manifolds satisfying the hypothesis of Theorem 9.17.

The considerations above may suggest that one should conjecture for a closed oriented n-dimensional aspherical manifold M of dimension n that

$$\rho^{(2)}(\widetilde{M}) = C_n \cdot \|M\| \tag{9.22}$$

holds for a dimension constant C_n. This is true with $C_n = 0$ for even n and possibly true for $n = 3$ with $C_3 = -\frac{1}{3\pi \cdot V_3}$. However, it is definitely false in all odd dimensions $n \geq 9$ by the following argument. Let M be a 3-dimensional oriented closed hyperbolic 3-manifold. Let N be $M \times M \times M$ and F be an oriented closed hyperbolic surface. Let F^d be the d-fold cartesian product of F with itself. We get from 9.20, 9.21, Theorem 9.9.5 and the product inequality for the simplicial volume in [106, page 10]

$$\begin{aligned} \rho^{(2)}(\widetilde{N \times F^d}) &= 0; \\ \rho^{(2)}(\widetilde{M \times F^{d+3}}) &\neq 0; \\ \|N \times F^d\| &\neq 0. \end{aligned}$$

Since all the manifolds appearing in the list above are orientable closed aspherical manifolds of dimension $9 + 2d$, equation 9.22 is wrong for odd $n \geq 9$.

The simplicial volume of an n-dimensional closed oriented manifold and its vanishing can be interpreted cohomologically as follows. Recall that a singular p-cochain on M with coefficients in $\mathbb{R}$ can be interpreted as a function $f : S_p(M) \longrightarrow \mathbb{R}$, where $S_p(M)$ is the set of all singular p-simplices on M. Define its norm

$$\|f\|_\infty \;=\; \sup\{|f(s)| \mid s \in S_p(M)\}.$$

The *bounded cochain complex* $\widehat{C}^*(M)$ is the subcochain complex of the singular cochain complex $C^*(M;\mathbb{R})$ with real coefficients which consists of bounded cochains f, i.e. cochains f with $\|f\|_\infty < \infty$. The *bounded cohomology* $\widehat{H}^*(M)$ is the cohomology of this chain complex.

For $\beta \in H^n(M;\mathbb{R})$ define

$$\|\beta\|_\infty \;=\; \inf\{\|f\|_\infty \mid f \in C^n(M;\mathbb{R}) \text{ is a cocycle representing } \beta\} \;\in [0,\infty].$$

Let $\beta(M) \in H^n(M;\mathbb{R})$ be the cohomological fundamental class of M. Then we get

$$\begin{array}{lcll} \|M\| & = & 0 & \text{if } \|\beta(M)\|_\infty = \infty; \\ \|M\| & = & \|\beta(M)\|_\infty^{-1} & \text{if } \|\beta(M)\|_\infty < \infty. \end{array}$$

Moreover, $\|M\|$ vanishes if and only if the canonical map $\widehat{H}^n(M) \longrightarrow H^n(M;\mathbb{R})$ is trivial [17, pages 278,279], [106, page 17].

If $f : M \longrightarrow B\pi_1(M)$ is the classifying map of M, then the induced map on bounded cohomology $f^* : \widehat{H}^p(B\pi_1(M)) \longrightarrow \widehat{H}^p(M)$ is an isometric isomorphism [106, page 40], [126, page 1105]. This implies that $\|M\|$ depends only on the image of the fundamental class under the classifying map $f_*([M]) \in H_n(B\pi_1(M);\mathbb{R})$. Namely, it is given by

$$\|M\| \;=\; \|f_*(\beta(M))\|_1. \tag{9.23}$$

Of course the L^2-torsion of a closed Riemannian manifold depends in general on more than $f_*([M]) \in H_n(B\pi_1(M);\mathbb{R})$, so that we see that Conjecture 9.18 only has a chance to be true for aspherical manifolds. Since the bounded cohomology $\widehat{H}^p(B\Gamma)$ vanishes for $p \geq 1$ for an amenable group, we get for any closed orientable manifold with amenable fundamental group that $\|M\| = 0$ [106, page 40], [126, Theorem 4.3 on page 1105]. Hence Conjecture 9.18 implies for an aspherical closed Riemannian manifold with amenable fundamental group that its L^2-Betti numbers vanish and its L^2-torsion is trivial. We have already proven the statement for the L^2-Betti numbers in Theorem 4.1. We conjecture

Conjecture 9.24 *If X is an aspherical connected finite CW-complex such that its fundamental group contains a non-trivial normal amenable subgroup, then its universal covering $\widetilde{X}$ is admissible and*

$$\rho^{(2)}(\widetilde{X}) \;=\; 0. \qquad \blacksquare$$

Conjecture 9.24 is true if the fundamental group contains $\mathbb{Z}^n$ as normal subgroup for $n \geq 1$, because then Theorem 9.9.4 applies to the fibration $T^n \longrightarrow X \longrightarrow B(\pi_1(X)/\mathbb{Z}^n)$.

Any closed smooth manifold satisfies

$$\|M\| \leq (n-1)^n n! \cdot \operatorname{minvol}(M) \tag{9.25}$$

where $\operatorname{minvol}(M)$ is the minimum over all volumes of M for all Riemannian metrics on M whose sectional curvature is pinched between -1 and 1 [106, page 12]. So the vanishing of the minimal volume implies the vanishing of the simplicial volume. In view of Conjecture 9.18, the question arises whether for a closed aspherical manifold the vanishing of the minimal volume implies that M is admissible and its L^2-torsion is trivial. Under certain conditions on the sectional curvature there are also estimates of the volume from above by the simplicial volume due to Thurston [106, page 10], [240].

Since the simplicial volume of a closed orientable manifold with non-trivial S^1-action vanishes [106, Section 3.1], [253], one could ask whether the vanishing of the simplicial volume is an obstruction to the existence of an S^1-foliation. Boileau, Druck and Vogt have dealt with this question in [27], [28]. A positive answer would rule out the existence of an S^1- foliation on a closed hyperbolic manifold because the simplicial volume in that case is (up to a non-zero factor) the volume. In view of Conjecture 9.18, one could ask whether the existence of an S^1-foliation on a closed aspherical manifold implies the vanishing of all the L^2-Betti numbers and of the L^2-torsion. Again a positive answer to this question would settle the problem of the existence of an S^1-foliation on a closed hyperbolic manifold.

Suppose that the closed oriented manifold M admits a selfmap of degree different from 0 and ± 1. Then the simplicial volume is trivial [106, page 8]. Assume additionally that M is aspherical. Then Conjecture 9.18 implies that $\rho^{(2)}(\widetilde{M})$ is trivial. This would be obvious if M would cover itself non-trivially. This raises the question of when a map $f : M \longrightarrow M$ for M a closed aspherical manifold of positive degree d is homotopic to a covering of degree d.

Obviously the L^2-torsion is hard to compute, even in the combinatorial version where one does not have to deal with the regularization process. The problem is that it is very difficult to compute the spectral density function. Next we state a more algorithmic approach which was developed in [154]. Here we again exploit the fact that the combinatorial Laplace operator already lives over the integral group ring of the fundamental group.

Let $A \in M(n, m, \mathbb{C}\Gamma)$ be an (n, m)-matrix over $\mathbb{C}\Gamma$. It induces, by right multiplication, a $\mathbb{C}\Gamma$-homomorphism of left $\mathbb{C}\Gamma$-modules

$$R_A : \oplus_{i=1}^n \mathbb{C}\Gamma \longrightarrow \oplus_{i=1}^m \mathbb{C}\Gamma \qquad\qquad x \mapsto xA$$

and by completion a bounded Γ-equivariant operator

$$R_A^{(2)} : \oplus_{i=1}^n l^2(\Gamma) \longrightarrow \oplus_{i=1}^m l^2(\Gamma).$$

We define an involution of rings on $\mathbb{C}\Gamma$ by

$$\overline{\sum_{w\in\Gamma}\lambda_w\cdot w}=\sum_{w\in\Gamma}\overline{\lambda_w}\cdot w^{-1}.$$

Denote by A^* the (m,n)-matrix obtained from A by transposing and applying the involution above to each entry. As the notation suggests, the bounded Γ-equivariant operator $R^{(2)}_{A^*}$ is the adjoint of the bounded Γ-equivariant operator $R^{(2)}_A$. Define the $\mathbb{C}\Gamma$-*trace* of an element $u=\sum_{w\in\Gamma}\lambda_w\cdot w\in\mathbb{C}\Gamma$ by

$$tr_{\mathbb{C}\Gamma}(u)=\lambda_e\ \in\mathbb{C}$$

for e the unit element in Γ. This extends to a square (n,n)-matrix A over $\mathbb{C}\Gamma$ by

$$tr_{\mathbb{C}\Gamma}(A)\ =\ \sum_{i=1}^{n}\mathrm{tr}_{\mathbb{C}\Gamma}(a_{i,i})\ \in\mathbb{C}. \tag{9.26}$$

It follows directly from the definitions that the $\mathbb{C}\Gamma$-trace $tr_{\mathbb{C}\Gamma}(A)$ of 9.26 agrees with the von Neumann trace $tr_{\mathcal{N}(\Gamma)}(R^{(2)}_A)$ defined in 1.2.

Let $A\in M(n,m,\mathbb{C}\Gamma)$ be an (n,m)-matrix over $\mathbb{C}\Gamma$. In the sequel let K be any positive real number satisfying

$$K\geq\ \|R^{(2)}_A\|$$

where $\|R^{(2)}_A\|$ is the operator norm of $R^{(2)}_A$. For $u=\sum_{w\in\Gamma}\lambda_w\cdot w\in\mathbb{C}\Gamma$, define $|u|_1=\sum_{w\in\Gamma}|\lambda_w|$. Then a possible choice for K is given by

$$K=\sqrt{m}\cdot\max\left\{|a_{i,j}|_1\mid 1\leq i\leq n,1\leq j\leq m\right\}.$$

The bounded Γ-equivariant operator $1-K^{-2}\cdot R^*_AR_A:\oplus_{i=1}^n l^2(\Gamma)\longrightarrow\oplus_{i=1}^n l^2(\Gamma)$ is positive. Let $(1-K^{-2}\cdot A^*A)^p$ be the p-fold product of matrices and $(1-K^{-2}\cdot R^*_AR_A)^p$ be the p-fold composition of operators.

Definition 9.27 *The* characteristic sequence *of a matrix $A\in M(n,m,\mathbb{C}\Gamma)$ and a non-negative real number K satisfying $K\geq\ \|R^{(2)}_A\|$ is the sequence of real numbers*

$$c(A,K)_p\ =\mathrm{tr}_{\mathbb{C}\Gamma}\left((1-K^{-2}\cdot AA^*)^p\right)\ =\ tr_{\mathcal{N}(\Gamma)}\left(\left(1-K^{-2}\cdot(R^{(2)}_A)^*R^{(2)}_A\right)^p\right). \qquad \blacksquare$$

Theorem 9.28 (Lück [154], Theorem 4.4 on page 100) *Let $A\in M(n,m,\mathbb{C}\Gamma)$ be an (n,m)-matrix over $\mathbb{C}\Gamma$. Denote by F the spectral density function of $R^{(2)}_A$. Let K be a positive real number satisfying $K\geq\ \|R^{(2)}_A\|$. Then*

1. *The characteristic sequence $c(A,K)_p$ is a monotone decreasing sequence of non-negative real numbers;*

2. *We have*

$$\dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)})) \;=\; F(0) \;=\; \lim_{p\to\infty} c(A,K)_p;$$

3. *Define* $\beta(A) \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ *by*

$$\beta(A) = \sup\left\{\beta \in \mathbb{R}^{\geq 0} \mid \lim_{p\to\infty} p^{\beta} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right) = 0\right\}.$$

Then we have

$$\begin{array}{lclr} \beta(A) & \geq & \alpha(R_A^{(2)}) & \text{if } \alpha(R_A^{(2)}) \text{ is a real number;} \\ \beta(A) & = & \infty & \text{otherwise;} \end{array}$$

where $\alpha(R_A^{(2)})$ *is the Novikov-Shubin invariant of Definition 8.4;*

4. *Let* K *be any positive real number satisfying* $K \geq \|R_A^{(2)}\|$. *Then the sum of positive real numbers*

$$\sum_{p=1}^{\infty} \frac{1}{p} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right)$$

converges if and only if $R_A^{(2)}$ *is of determinant class, i.e. the integral* $\int_{0+}^{\infty} \ln(\lambda)\; dF$ *converges. If* $R_A^{(2)}$ *is of determinant class, then*

$$\begin{array}{rcl} 2 \cdot \ln(\det_{\mathcal{N}(\Gamma)}(R_A^{(2)})) & = & 2 \cdot \left(n - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right) \cdot \ln(K) \\ & & - \displaystyle\sum_{p=1}^{\infty} \frac{1}{p} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right); \end{array}$$

5. *Suppose* $\alpha(R_A^{(2)}) > 0$. *Then* $\det_{\mathcal{N}(\Gamma)}(R_A^{(2)})$ *is a positive real number. Given a real number* α *satisfying* $0 < \alpha < \alpha(R_A^{(2)})$, *there is a real number* C *such that we have for all* $L \geq 1$

$$0 \leq c(A,K)_L - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)})) \leq \frac{C}{L^{\alpha}}$$

and

$$\begin{array}{rcl} 0 & \leq & -2 \cdot \ln(\det_{\mathcal{N}(\Gamma)}(R_A^{(2)})) + 2 \cdot \left(n - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right) \cdot \ln(K) \\ & & - \displaystyle\sum_{p=1}^{L} \frac{1}{p} \cdot \left(c(A,K)_p - \dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))\right) \;\leq\; \frac{C}{L^{\alpha}} \qquad \blacksquare \end{array}$$

Theorem 9.28 gives the possibility of computing $\dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))$ and $\ln(\det_{\mathcal{N}(\Gamma)}(R_A^{(2)}))$ by a sequence whose individual terms can be computed by an algorithm, provided a concrete presentation of Γ is given and the word problem can be solved for Γ. The speed of convergence can be predicted by the Novikov-Shubin invariants. However, we do not have a concrete value for the constant C appearing in Theorem 9.28.5. At any rate one gets upper bounds for $\dim_{\mathcal{N}(\Gamma)}(\ker(R_A^{(2)}))$ and $\ln(\det_{\mathcal{N}(\Gamma)}(R_A^{(2)}))$ since the characteristic sequence is monotone decreasing and positive. In this context Conjecture 2.1 is interesting. If, for instance, Γ is torsionfree and one of the elements of the characteristic sequence is smaller than 1, then Conjecture 2.1 implies that $\ker(R_A^{(2)})$ is trivial.

In particular, 3-manifolds are interesting since the cellular $\mathbb{Z}\Gamma$-chain complex of the universal covering can be computed from an appropriate presentation of the fundamental group. Theorem 9.28 implies [154, Theorem 2.4 on page 84]:

Theorem 9.29 *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group Γ. Let*

$$\Gamma = \langle s_1, s_2, \dots s_g \mid R_1, R_2, \dots R_r \rangle$$

be a presentation of Γ. Let the (r, g)-matrix

$$F = \begin{pmatrix} \frac{\partial R_1}{\partial s_1} & \cdots & \frac{\partial R_1}{\partial s_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial R_r}{\partial s_1} & \cdots & \frac{\partial R_r}{\partial s_g} \end{pmatrix}$$

be the Fox matrix of the presentation. Denote by $\alpha_2(M)$ the second Novikov-Shubin invariant of M. Now there are two cases:

1. *Suppose ∂M is non-empty. We make the assumption that ∂M is a union of incompressible tori and that $g = r - 1$. Then M is admissible. Define A to be the $(g-1, g-1)$-matrix with entries in $\mathbb{Z}\Gamma$ obtained from the Fox matrix by deleting one of the columns. Let α be any real number satisfying $0 < \alpha < \frac{2 \cdot \alpha_2(M)}{\alpha_2(M)+2}$;*
2. *Suppose ∂M is empty. We make the assumption that a finite covering of M is homotopy equivalent to a hyperbolic, Seifert or Haken 3-manifold and that the given presentation comes from a Heegaard decomposition. Then M is admissible and $g = r$. Define A to be the $(g-1, g-1)$-matrix with entries in $\mathbb{Z}\Gamma$ obtained from the Fox matrix by deleting one of the columns and one of the rows. Let α be any real number satisfying $0 < \alpha < \frac{2 \cdot \alpha_2(M)}{\alpha_2(M)+1}$;*

Let K be any positive real number satisfying $K \geq \|R_A^{(2)}\|$. A possible choice for K is the product of $(g-1)^2$ and the maximum over the word length of those relations R_i whose Fox derivatives appear in A.

Then the sum of non-negative rational numbers $\sum_{p=1}^{L} \frac{1}{p} \cdot \operatorname{tr}_{\mathbb{Z}\pi}\left((1 - K^{-2} \cdot AA^*)^p\right)$ *converges to the real number* $\rho^{(2)}(\widetilde{M}) + 2(g-1) \cdot \ln(K)$. *More precisely, there is a constant* $C > 0$ *such that we get for all* $L \geq 1$

$$0 \leq \rho^{(2)}(\widetilde{M}) + 2(g-1) \cdot \ln(K) - \sum_{p=1}^{L} \frac{1}{p} \cdot \operatorname{tr}_{\mathbb{Z}\pi}\left((1 - K^{-2} \cdot AA^*)^p\right) \leq \frac{C}{L^{\alpha}}. \qquad \blacksquare$$

Remark 9.30 Let M be a closed n-dimensional hyperbolic manifold. Then Mostow's Rigidity Theorem says that the isometric diffeomorphism type, and in particular the volume of M, depends only on its fundamental group [17, Theorem C.0 on page 83], [182]. We get from Theorem 9.12, Theorem 9.16 and Theorem 9.29 a way of computing the volume purely in terms of a presentation of the fundamental group without using information about M itself. If for a group Γ the classifying space $B\Gamma$ is a finite CW-complex and $E\Gamma$ is admissible, then its L^2-torsion is defined and is a generalization of the volume in the case where Γ is the fundamental group of an odd-dimensional hyperbolic closed manifold. $\blacksquare$

Example 9.31 In [154, Example 2.7] the complement M of the figure eight knot is computed. For the presentation of $\Gamma = \pi_1(M)$

$$\Gamma = \langle s_1, s_2, t \mid t s_1 t^{-1} s_2^{-1} = t s_2 t^{-1} s_1 s_2^{-3} = 1 \rangle$$

and the $(2,2)$-matrix

$$B = \begin{pmatrix} 13 + s_2 + s_2^{-1} & -1 + s_2 + s_1 s_2^3 - s_2 s_1 s_2^{-3} - t s_1 s_2^{-3} \\ -1 + s_2^{-1} + s_2^3 s_1^{-1} - s_2^3 s_1^{-1} s_2^{-1} - s_2^3 s_1^{-1} t^{-1} & 13 + s_2^3 s_1^{-1} + s_1 s_2^{-3} \end{pmatrix}$$

we obtain

$$\rho^{(2)}(\widetilde{M}) = -8\ln(2) + \sum_{p=1}^{\infty} \frac{1}{p \cdot 16^p} \cdot \operatorname{tr}_{\mathbb{Z}\Gamma}(B^p). \qquad \blacksquare$$

We have already mentioned in Conjecture 2.1 and Conjecture 8.9 what possible values we expect for the L^2-Betti numbers and Novikov-Shubin invariants. We do not have a good guess in the case of L^2-torsion for spaces of acyclic determinant class. This question only makes sense if the space is L^2-acyclic, otherwise one could vary the Riemannian metric to get any real number as the L^2-torsion. Recall that for an (m,n)-matrix A, we denote by $R_A : l^2(\Gamma)^m \longrightarrow l^2(\Gamma)^n$ the bounded Γ-equivariant operator induced by right multiplication with A.

Definition 9.32 *For a group* Γ, *define a multiplicative subgroup of the positive real numbers*

$$\begin{aligned} R(\Gamma) = & \left\{\det\nolimits_{\mathcal{N}(\Gamma)}(R_A) \mid A \in M(n,n,\mathbb{Z}\Gamma), \det\nolimits_{\mathcal{N}(\Gamma)}(R_A) \neq 0, \ker(R_A) = 0\right\} \\ & \cup \left\{\left(\det\nolimits_{\mathcal{N}(\Gamma)}(R_A)\right)^{-1} \mid A \in M(n,n,\mathbb{Z}\Gamma), \det\nolimits_{\mathcal{N}(\Gamma)}(R_A) \neq 0, \ker(R_A) = 0\right\}. \qquad \blacksquare \end{aligned}$$

Lemma 9.33 *Let Γ be a finitely presented group Γ. Suppose that there is at least one connected finite CW-complex Y with $\Gamma = \pi_1(Y)$ such that $\widetilde{Y}$ is of acyclic determinant class. Then*

$$\begin{aligned} 2 \cdot \ln\left(R(\Gamma)\right) &\subset \Big\{\rho^{(2)}(\widetilde{X}) \mid X \textit{ a connected finite } CW\textit{-complex with } \pi_1(X) = \Gamma \\ &\qquad \textit{and } \widetilde{X} \textit{ of acyclic determinant class}\Big\} \\ &\subset \ln\left(R(\Gamma)\right) \end{aligned}$$

and

$$\begin{aligned} 4 \cdot \ln\left(R(\Gamma)\right) &\subset \Big\{\rho^{(2)}(\widetilde{M}) \mid M \textit{ a closed manifold with } \pi_1(M) = \Gamma \\ &\qquad \textit{and } \widetilde{M} \textit{ of acyclic determinant class}\Big\} \\ &\subset \ln\left(R(\Gamma)\right). \end{aligned}$$

Proof : The non-trivial inclusion in the first assertion is

$$\begin{aligned} 2 \cdot \ln\left(R(\Gamma)\right) &\subset \Big\{\rho^{(2)}(\widetilde{X}) \mid X \text{ a connected finite } CW\text{-complex with } \pi_1(X) = \Gamma \\ &\qquad \text{and } \widetilde{X} \text{ of acyclic determinant class}\Big\}. \end{aligned}$$

Let $Z = Y \times S^3$. Then $\pi_1(Z) = \Gamma$ and $\widetilde{Z}$ is of acyclic determinant class with $\rho^{(2)}(\widetilde{Z}) = 0$ by Theorem 9.9.5. Let A be an (n,n)-matrix over $\mathbb{Z}\Gamma$ such that $\ker(R_A) = 0$ and $\det_{\mathcal{N}(\Gamma)}(R_A) \neq 0$. Let n be an integer such that $2n$ is greater than or equal to the dimension of Z. By attaching cells to Z in dimensions $2n+2$ and $2n+3$ we obtain a connected finite CW-complex X such that $\Gamma = \pi_1(X)$ and the cellular $\mathbb{Z}\Gamma$-chain complex of $\widetilde{X}$ is the direct sum of the one of $\widetilde{Z}$ and the chain complex concentrated in dimensions $2n+2$ and $2n+3$ whose only non-trivial differential is given by R_A. Then Lemma 9.5 implies that $\widetilde{X}$ is of determinant class and

$$\begin{aligned} \rho^{(2)}(\widetilde{X}) &= -(-1)^{2n+3} \cdot (n+3) \cdot \ln\left(\det_{\mathcal{N}(\Gamma)}(R_A^* R_A)\right) \\ &\quad -(-1)^{2n+2} \cdot (n+2) \cdot \ln\left(\det_{\mathcal{N}(\Gamma)}(R_A R_A^*)\right) \\ &= 2 \cdot \ln\left(\det_{\mathcal{N}(\Gamma)}(R_A)\right). \end{aligned}$$

This shows the first assertion.

Let X be a connected finite CW-complex such that $\Gamma = \pi_1(X)$ and $\widetilde{X}$ is of acyclic determinant class. Let n be an integer such that $2n$ is greater or equal to the dimension of X. We embed X into $\mathbb{R}^{2n+2}$. Let M be the boundary of a regular neighborhood N of X. Now L^2-torsion and the notions of acyclic determinant class can also be defined for pairs and simple homotopy invariance, the sum formula and Poincaré duality extend to this case [154]. Since the inclusion of X into N is a simple homotopy equivalence and X is of acyclic determinant class, $\widetilde{N}$ by homotopy invariance, $(\widetilde{N}, \widetilde{M})$ by Poincaré duality, and hence $\widetilde{M}$ by additivity (= formula for pairs) are of acyclic determinant class, and we get

$$\begin{aligned}\rho^{(2)}(\widetilde{M}) &= \rho^{(2)}(\widetilde{N}) - \rho^{(2)}(\widetilde{N}, \widetilde{M}) \\ &= \rho^{(2)}(\widetilde{N}) + \rho^{(2)}(\widetilde{N}) \\ &= 2 \cdot \rho^{(2)}(\widetilde{N}) \\ &= 2 \cdot \rho^{(2)}(\widetilde{X}).\end{aligned}$$

This finishes the proof of Lemma 9.33. ■

If Γ is countable, then $R(\Gamma)$ is countable because then $\mathbb{Z}\Gamma$, and hence $M(n, n, \mathbb{Z}\Gamma)$, is countable for all n and $R(\Gamma)$ is a countable union of countably sets.

If $\Gamma = \mathbb{Z}$, then each element in $R(\Gamma)$ is an algebraic number, i.e. the root of a non-trivial polynomial with rational coefficients, by the following argument.

We get from [163, section 4]

$$R(\mathbb{Z}) = \left\{\det_{\mathcal{N}(\mathbb{Z})}(R_p) \mid p \in \mathbb{Z}[\mathbb{Z}], p \neq 0\right\}.$$

We can write p in $\mathbb{C}[\mathbb{Z}]$ as

$$p(z) = C \cdot z^n \cdot \prod_{k=1}^{l} (z - a_i)$$

for complex numbers $C, a_0, a_1, \ldots, a_l$ and integers n and $l \geq 0$. Since p is a non-zero polynomial with integer coefficients, C must be a non-zero integer and each a_i is algebraic. We get from Lemma 9.5.4

$$\det_{\mathcal{N}(\mathbb{Z})}(R_p) = \det_{\mathcal{N}(\mathbb{Z})}(R_C) \cdot \det_{\mathcal{N}(\mathbb{Z})}(R_{z^n}) \cdot \prod_{k=1}^{l} \det_{\mathcal{N}(\mathbb{Z})}(R_{(z-a_i)}) = \prod_{k=1}^{l} \det_{\mathcal{N}(\mathbb{Z})}(R_{(z-a_i)}).$$

Hence the claim follows from the following equation

$$\det_{\mathcal{N}(\mathbb{Z})}(R_{(z-a)}) = \begin{cases} |a| & \text{for } |a| \geq 1 \\ 1 & \text{for } |a| \leq 1 \end{cases} \tag{9.34}$$

which we prove next. We have to show that, for $a \in \mathbb{C}$, if we equip S^1 with the obvious measure satisfying $\mathrm{vol}(S^1) = 1$

$$\int_{S^1} \ln((z - a)(z^{-1} - \overline{a}))\, d\mathrm{vol} = \begin{cases} 2 \cdot \ln(|a|) & \text{for } |a| \geq 1 \\ 0 & \text{for } |a| \leq 1 \end{cases}. \tag{9.35}$$

We have

$$\int_{S^1} \ln((z - a)(z^{-1} - \overline{a}))\, d\mathrm{vol} = \int_{S^1} \ln((z - |a|)(z^{-1} - |a|))\, d\mathrm{vol}.$$

Hence we may suppose in the sequel $a \in \mathbb{R}^{\geq 0}$.

We compute for $a \neq 1$ and the path $\gamma : [0,1] \longrightarrow S^1 \quad t \mapsto \exp(2\pi i t)$, using the Residue Theorem

$$\begin{aligned}
& \int_{S^1} \frac{d}{da} \ln((z-a)(z^{-1}-a)) \, d\mathrm{vol} \\
= & \int_{S^1} \frac{1}{a-z} + \frac{1}{a-z^{-1}} \, d\mathrm{vol} \\
= & \ 2 \cdot \int_{S^1} \frac{1}{a-z} \, d\mathrm{vol} \\
= & \ 2 \cdot \int_{S^1} \frac{1}{(a-z)\cdot 2\pi i z} \cdot 2\pi i z \cdot \ d\mathrm{vol} \\
= & \ \frac{2}{2\pi i} \cdot \int_{\gamma} \frac{1}{(a-z)\cdot z} \, dz \\
= & \begin{cases} \frac{2}{a} & \text{for } a > 1 \\ 0 & \text{for } a < 1 \end{cases}.
\end{aligned}$$

This implies for $a \in \mathbb{R}^{\geq 0}, a \neq 1$

$$\frac{d}{da} \int_{S^1} \ln((z-a)(z^{-1}-a)) \, d\mathrm{vol} = \begin{cases} \frac{2}{a} & \text{for } a > 1 \\ 0 & \text{for } a < 1 \end{cases}.$$

We conclude for an appropriate number C

$$\begin{array}{lcll}
\int_{S^1} \ln((z-a)(z^{-1}-a)) \, d\mathrm{vol} & = & 2 \cdot \ln(a) + C & \text{for } a > 1 \\
\int_{S^1} \ln((z-a)(z^{-1}-a)) \, d\mathrm{vol} & = & 0 & \text{for } a < 1
\end{array}.$$

We get from Levi's Theorem of Monotone Convergence

$$\int_{S^1} \ln((z-1)(z^{-1}-1)) \, d\mathrm{vol} = C.$$

We get from Lebesgue's Theorem of Majorized Convergence

$$\int_{S^1} \ln((z-1)(z^{-1}-1)) \, d\mathrm{vol} = 0.$$

This proves 9.35, and hence 9.34.

More information about L^2-torsion can be found in [42], [44], [47], [48] [70], [71], [146], [172].

10. Algebraic dimension theory of finite von Neumann algebras

In this section we give a purely algebraic approach to L^2-Betti numbers and the von Neumann dimension and extend all these notions for a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module which is essentially the same as a finitely generated projective $\mathcal{N}(\Gamma)$-module to arbitrary $\mathcal{N}(\Gamma)$-modules.

We mention the following observations and facts which first were made by Farber [89], [90] and then independently by the author [157]. Farber's approach is different in that he works with a more abstract category than the category of finitely presented modules. Both approaches are compared and identified in [157, Theorem 0.9].

There is an equivalence of categories

$$\nu : \{\text{fin. gen. proj. } \mathcal{N}(\Gamma) - \text{mod.}\} \longrightarrow \{\text{fin. gen. Hilb. } \mathcal{N}(\Gamma) - \text{mod.}\} \tag{10.1}$$

where $\{\text{fin. gen. proj. } \mathcal{N}(\Gamma)-\text{mod.}\}$ is the category of finitely generated projective modules over the ring $\mathcal{N}(\Gamma)$ with $\mathcal{N}(\Gamma)$- linear maps as morphisms, and $\{\text{fin. gen. Hilb. } \mathcal{N}(\Gamma)-\text{mod.}\}$ is the category of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules with bounded $\mathcal{N}(\Gamma)$-equivariant operators as morphisms [157, Theorem 2.2]. It sends $\mathcal{N}(\Gamma)$ to $l^2(\Gamma)$. It is compatible with finite direct sums, with the complex vector space structures on the set of morphisms and with the involutions given by taking dual $\mathcal{N}(\Gamma)$-modules and dual homomorphisms in $\{\text{fin. gen. proj. } \mathcal{N}(\Gamma)-\text{mod.}\}$ and adjoint operators in $\{\text{fin. gen. Hilb. } \mathcal{N}(\Gamma)-\text{mod.}\}$. The category of finitely generated projective $\mathcal{N}(\Gamma)$-modules is a subcategory of the category of finitely presented $\mathcal{N}(\Gamma)$-modules

$$\{\text{fin. gen. proj. } \mathcal{N}(\Gamma) - \text{mod.}\} \subset \{\text{fin. pres. } \mathcal{N}(\Gamma) - \text{mod.}\}. \tag{10.2}$$

The point is that $\mathcal{N}(\Gamma)$ is a semi-hereditary ring, i.e. finitely generated $\mathcal{N}(\Gamma)$-submodules of projective $\mathcal{N}(\Gamma)$-modules are projective and the category $\{\text{fin. pres. } \mathcal{N}(\Gamma)-\text{mod.}\}$ is abelian, i.e. the kernel, the image and the cokernel of an $\mathcal{N}(\Gamma)$-linear map of finitely presented $\mathcal{N}(\Gamma)$-modules are again finitely presented [90, §2], [157, Theorem 1.2 and Corollary 2.4]. Let M be an $\mathcal{N}(\Gamma)$-submodule of N. Define the *closure of M in N* to be the $\mathcal{N}(\Gamma)$-submodule of N

$$\overline{M} \quad = \quad \{x \in N \mid f(x) = 0 \text{ for all } f \in \hom_{\mathcal{N}(\Gamma)}(N, \mathcal{N}(\Gamma)) \text{ with } M \subset \ker(f)\}. \tag{10.3}$$

The functor ν of 10.1 respects exact and weakly exact sequences [157, Lemma 2.3] where weakly exact for $\{\text{fin. gen. Hilb. } \mathcal{N}(\Gamma)-\text{mod.}\}$ was defined in section 1 and translates to $\{\text{fin. gen. proj. } \mathcal{N}(\Gamma)-\text{mod.}\}$ using 10.3. For an $\mathcal{N}(\Gamma)$-module M define the $\mathcal{N}(\Gamma)$-submodule $\mathbf{T}M$ and the $\mathcal{N}(\Gamma)$-quotient module $\mathbf{P}M$ by (see also [90, §3])

$$\mathbf{T}M \quad = \quad \{x \in M \mid f(x) = 0 \text{ for all } f \in \hom_{\mathcal{N}(\Gamma)}(M, \mathcal{N}(\Gamma))\}; \tag{10.4}$$

$$\mathbf{P}M \quad = \quad M/\mathbf{T}M. \tag{10.5}$$

Notice that $\mathbf{T}M$ is the closure of the trivial module in M. If M is finitely generated, then $\mathbf{P}M$ is finitely generated projective [157, Theorem 1.2]. These notions now allow one to read off the L^2-Betti numbers and the Novikov-Shubin invariants of a regular covering

$\overline{X} \longrightarrow X$ with Γ as group of deck transformations from the finitely presented $\mathcal{N}(\Gamma)$-module $H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C(\overline{X}))$ [90], [157]. Moreover, one can generalize all these invariants using the universal center-valued trace of $\mathcal{N}(\Gamma)$ as carried out in [157].

Because of 10.1, one can define for a finitely generated projective $\mathcal{N}(\Gamma)$-module

$$\dim_{\mathcal{N}(\Gamma)}(P) = \dim_{\mathcal{N}(\Gamma)}(\nu(P)) \qquad \in [0, \infty) \tag{10.6}$$

where $\dim_{\mathcal{N}(\Gamma)}(\nu(P))$ is defined in 1.3.

Theorem 10.7 (Lück, [159], Theorem 0.6) *There is a dimension function*

$$\dim_{\mathcal{N}(\Gamma)} : \{\mathcal{N}(\Gamma) - \textit{modules}\} \longrightarrow [0, \infty]$$

which has the following properties:

1. *Extension property*

   If M is finitely generated projective, then $\dim_{\mathcal{N}(\Gamma)}(M)$ agrees with the number given in 10.6;

2. *Invariance under closure*

   If $K \subset M$ is a submodule of the finitely generated $\mathcal{N}(\Gamma)$-module M, then

   $$\dim_{\mathcal{N}(\Gamma)}(K) = \dim_{\mathcal{N}(\Gamma)}(\overline{K});$$

3. *Cofinality*

   Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \cup_{i\in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then

   $$\dim_{\mathcal{N}(\Gamma)}(M) = \sup\{\dim_{\mathcal{N}(\Gamma)}(M_i) \mid i \in I\};$$

4. *Additivity*

   If $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$ is an exact sequence of $\mathcal{N}(\Gamma)$-modules, then

   $$\dim_{\mathcal{N}(\Gamma)}(M_1) = \dim_{\mathcal{N}(\Gamma)}(M_0) + \dim_{\mathcal{N}(\Gamma)}(M_2);$$

5. *If M is a finitely generated $\mathcal{N}(\Gamma)$-module, then*

   $$\begin{aligned} \dim_{\mathcal{N}(\Gamma)}(M) &= \dim_{\mathcal{N}(\Gamma)}(\mathbf{P}M); \\ \dim_{\mathcal{N}(\Gamma)}(\mathbf{T}M) &= 0; \end{aligned}$$

6. *Uniqueness*

   This dimension function is uniquely determined by the extension property, invariance under closure, cofinality and additivity. ■

Meanwhile this dimension function (and its properties) has been extended from the von Neumann algebra $\mathcal{N}(\Gamma)$ to the associated algebra of affiliated operators by Reich [214].

Let $i : \Delta \longrightarrow \Gamma$ be an injective group homomorphism. We claim that associated to i there is a ring homomorphism of the group von Neumann algebras, also denoted by

$$i : \mathcal{N}(\Delta) \quad \longrightarrow \quad \mathcal{N}(\Gamma).$$

Recall from 1.1 that $\mathcal{N}(\Delta)$ is the same as the ring $\mathcal{B}(l^2(\Delta), l^2(\Delta))^\Delta$ of bounded Δ-equivariant operators $f : l^2(\Delta) \longrightarrow l^2(\Delta)$. Notice that $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ can be viewed as a dense subspace of $l^2(\Gamma)$ and that f defines a $\mathbb{C}\Gamma$-homomorphism $\mathrm{id} \otimes_{\mathbb{C}\Delta} f : \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta) \longrightarrow \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ which is bounded with respect to the pre-Hilbert structure induced on $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ from $l^2(\Gamma)$. Hence $\mathrm{id} \otimes_{\mathbb{C}\Delta} f$ extends to a Γ-equivariant bounded operator $i(f) : l^2(\Gamma) \longrightarrow l^2(\Gamma)$.

Given an $\mathcal{N}(\Delta)$-module M, define *induction with* i to be the $\mathcal{N}(\Gamma)$-module

$$i_*(M) \quad = \quad \mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} M.$$

Obviously i_* is a covariant functor from the category of $\mathcal{N}(\Delta)$-modules to the category of $\mathcal{N}(\Gamma)$-modules, preserves direct sums and the properties finitely generated and projective and sends $\mathcal{N}(\Delta)$ to $\mathcal{N}(\Gamma)$. We get from [159, Theorem 3.3]

Theorem 10.8 *Let $i : \Delta \longrightarrow \Gamma$ be an injective group homomorphism. Then*

1. *i_* is an exact functor, i.e. for any exact sequence of $\mathcal{N}(\Delta)$-modules $M_0 \longrightarrow M_1 \longrightarrow M_2$ the induced sequence of $\mathcal{N}(\Gamma)$-modules $i_*M_0 \longrightarrow i_*M_1 \longrightarrow i_*M_2$ is exact;*
2. *For any $\mathcal{N}(\Delta)$-module M we have*

$$\dim_{\mathcal{N}(\Delta)}(M) \quad = \quad \dim_{\mathcal{N}(\Gamma)}(i_*M). \qquad ■$$

This allows us to extend the definition of L^2-Betti numbers to arbitrary topological spaces with Γ-action and to arbitrary groups.

Definition 10.9 *Let Γ be a group acting on the topological space Z. Let $C_*^{\mathrm{sing}}(Z)$ be the singular chain complex, which becomes a $\mathbb{Z}\Gamma$-chain complex by the Γ-action. Define*

$$\begin{aligned} H_p^\Gamma(Z; \mathcal{N}(\Gamma)) \quad &= \quad H_p\left(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(Z)\right); \\ b_p^{(2)}(Z) \quad &= \quad \dim_{\mathcal{N}(\Gamma)}\left(H_p^\Gamma(Z; \mathcal{N}(\Gamma))\right) \qquad \in [0, \infty]. \end{aligned}$$

Given a group Γ, *define*

$$b_p^{(2)}(\Gamma) \;=\; b_p^{(2)}(E\Gamma) \qquad \in [0,\infty]. \qquad \blacksquare$$

These L^2-Betti numbers are investigated in [159], [160] and [214], where also their relation with the generalized L^2-Betti numbers defined in [54] is explained. Obviously they depend only on the Γ-homotopy type of Z and they agree with the L^2-Betti numbers of Definition 1.5 in the special case where Z is the total space $\overline{X}$ of a regular covering $\overline{X} \longrightarrow X$ of a CW-complex X of finite type with Γ as deck transformation group. The point of this extension is that it is useful to have the notion of L^2-Betti numbers for arbitrary Γ-spaces, even if one wants to compute them only for a regular covering of a CW-complex of finite type. For instance in Theorem 4.1 one wants to compute the L^2-Betti numbers of $E\Gamma$ in the case where $B\Gamma$ is assumed to be of finite type, using the information that Γ contains a non-trivial amenable subgroup $\Delta \subset \Gamma$, but no information on $B\Delta$ is given. Next we sketch how Theorem 4.1 follows from Lemma 4.4. Details of this proof and a comparison with the original proof in [54] are given in [159, section 5]. We will show the more general result

Theorem 10.10 *1. Let* Γ *be an infinite amenable group. Then*

$$b_p^{(2)}(\Gamma) \;=\; 0 \qquad \textit{for } p \geq 0;$$

2. Let Δ *be a normal subgroup of* Γ *with* $b_p^{(2)}(\Delta) = 0$ *for* $p \geq 0$*. Then*

$$b_p^{(2)}(\Gamma) \;=\; 0 \qquad \textit{for } p \geq 0.$$

Proof : 1.) In the sequel colimits are taken over the directed system of finite subcomplexes Y of $B\Gamma$ and $\overline{Y}$ is the restriction of the universal Γ-principal bundle $E\Gamma \longrightarrow B\Gamma$ to Y. Notice that a colimit over a directed system is an exact functor and compatible with tensor products. Hence the following diagram commutes and has isomorphisms as horizontal maps

$$\begin{array}{ccc}
\operatorname{colim} \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} H_p(\overline{Y}) & \xrightarrow{\cong} & \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} H_p(E\Gamma) \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\
\operatorname{colim} H_p^{\Gamma}(\overline{Y};\mathcal{N}(\Gamma)) & \xrightarrow[\cong]{} & H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))
\end{array}$$

where the horizontal arrows are given by the inclusions of $\overline{Y}$ into $E\Gamma$ and the vertical arrows are the canonical maps. It is not hard to deduce from Lemma 4.4 the in (some sense dual) statement that the dimension $\dim_{\mathcal{N}(\Gamma)}$ of the cokernel of each of the maps

$$\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} H_p(\overline{Y}) \longrightarrow H_p^{\Gamma}(\overline{Y};\mathcal{N}(\Gamma))$$

is zero because Γ is amenable. Since colimit over a directed system is an exact functor, we conclude from cofinality and additivity of $\dim_{\mathcal{N}(\Gamma)}$ of Theorem 10.7 that the dimension $\dim_{\mathcal{N}(\Gamma)}$ of the cokernel of the left vertical arrow in the diagram above is zero. From additivity of the dimension $\dim_{\mathcal{N}(\Gamma)}$ (see Theorem 10.7) we conclude

$$\dim_{\mathcal{N}(\Gamma)}\left(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} H_p(E\Gamma)\right) \;\geq\; \dim_{\mathcal{N}(\Gamma)}\left(H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))\right).$$

Since $H_p(E\Gamma)$ vanishes for $p\geq 1$, we get $b_p^{(2)}(\Gamma)=0$ for $p\geq 1$. Since Γ is infinite, a direct calculation shows $b_0^{(2)}(\Gamma)=0$.

2.) We have the fibration $B\Delta \longrightarrow B\Gamma \longrightarrow B\pi$ for $\pi=\Gamma/\Delta$. Since $B\pi$ is a CW-complex and we are dealing with homology, the associated Leray-Serre spectral sequence with coefficients in $\mathcal{N}(\Gamma)$ converges to $H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$. Its E^2-term is given by

$$E^2_{p,q} \;=\; H_p^{\pi}\left(E\pi; H_q^{\Gamma}(\Gamma\times_{\Delta} E\Delta;\mathcal{N}(\Gamma))\right)$$

for a certain action of π on the $\mathcal{N}(\Gamma)$-module $H_q^{\Gamma}(\Gamma\times_{\Delta} E\Delta;\mathcal{N}(\Gamma))$ which comes from the fiber transport [150, section 1], [151, section 4]. We get from Theorem 10.8 and the assumption

$$\begin{aligned}\dim_{\mathcal{N}(\Gamma)}\left(H_q^{\Gamma}(\Gamma\times_{\Delta} E\Delta;\mathcal{N}(\Gamma))\right) &= \dim_{\mathcal{N}(\Delta)}\left(\mathcal{N}(\Gamma)\otimes_{\mathcal{N}(\Delta)} H_q^{\Delta}(E\Delta;\mathcal{N}(\Delta))\right)\\ &= \dim_{\mathcal{N}(\Delta)}\left(H_q^{\Delta}(E\Delta;\mathcal{N}(\Delta))\right)\\ &= 0.\end{aligned}$$

Now we conclude from additivity of the dimension $\dim_{\mathcal{N}(\Gamma)}$ (see Theorem 10.7)

$$\begin{aligned}\dim_{\mathcal{N}(\Gamma)}\left(E^2_{p,q}\right) &= 0 \qquad \text{for } p,q\geq 0;\\ \dim_{\mathcal{N}(\Gamma)}\left(H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))\right) &= 0 \qquad \text{for } p\geq 0.\end{aligned}$$

This finishes the proof of Theorem 10.10. ■

More information about extending the definition of L^2-cohomology can be found in [91], [235]. The same program has been carried out for Novikov-Shubin invariants in [162].

11. The zero-in-the-spectrum Conjecture

In this section we deal with the zero-in-the-spectrum Conjecture. To our knowledge this conjecture is due to John Lott. We recommend to the reader the survey article [144] where more information can be found. Gromov deals with this problem in the aspherical case in [109].

Conjecture 11.1 *(zero-in-the-spectrum Conjecture) There is no connected closed Riemannian manifold M such that for all $p\geq 0$ zero is not in the spectrum of the Laplace operator Δ_p acting on smooth p-forms of the universal covering $\widetilde{M}$.* ■

Lott [144] gives five versions of this conjecture, stated as a question, namely, that for some $p \geq 0$ zero is in the spectrum of the Laplace operator Δ_p acting on smooth p-forms of $\widetilde{M}$ if

1. $\widetilde{M}$ is a complete Riemannian manifold;
2. $\widetilde{M}$ is a complete Riemannian manifold with bounded geometry, i.e. the injectivity radius is positive and the sectional curvature is pinched between -1 and 1;
3. $\widetilde{M}$ is a uniformly contractible Riemannian manifold, i.e. for all $r > 0$ there is an $R(r) > 0$ such that for all $m \in M$ the metric ball $B_r(m)$ is contractible within $B_{R(r)}(m)$;
4. $\widetilde{M}$ is the universal covering of a closed Riemannian manifold;
5. $\widetilde{M}$ is the universal covering of a closed aspherical Riemannian manifold.

We emphasize that the next definition makes sense for arbitrary groups and spaces.

Definition 11.2 *Let Z be a topological space with an action of the group Γ. Let $H_p^\Gamma(Z;\mathcal{N}(\Gamma))$ be the $\mathcal{N}(\Gamma)$-module given by the singular homology with coefficients in $\mathcal{N}(\Gamma)$ as defined in 10.9. We say that Z is* $\mathcal{N}(\Gamma)$-acyclic. *(resp. n-*$\mathcal{N}(\Gamma)$-connected*) if $H_p^\Gamma(Z;\mathcal{N}(\Gamma))$ vanishes for $p \geq 0$ (resp. $0 \leq p \leq n$).*

A group Γ is called $\mathcal{N}(\Gamma)$-acyclic. *resp. n-*$\mathcal{N}(\Gamma)$-connected *if the universal Γ-space $E\Gamma$ has this property.* ■

Lemma 11.3 *The following statements are equivalent for a finitely presented group Γ:*

1. *There is no connected closed Riemannian manifold M with fundamental group Γ such that, for all $p \geq 0$, zero is not in the spectrum of the Laplace operator Δ_p acting on smooth p-forms of the universal covering $\widetilde{M}$;*
2. *There is no connected closed Riemannian manifold M with fundamental group Γ such that*
$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \qquad p \geq 0; \\ \alpha_p(\widetilde{M}) &= \infty^+ \qquad p \geq 0; \end{aligned}$$
3. *There is no connected finite CW-complex X with fundamental group Γ such that*
$$\begin{aligned} b_p^{(2)}(\widetilde{X}) &= 0 \qquad p \geq 0; \\ \alpha_p(\widetilde{X}) &= \infty^+ \qquad p \geq 0; \end{aligned}$$

4. *There is no connected finite CW-complex X with fundamental group Γ such that for all $p \geq 0$ the combinatorial Laplace operator Δ_p acting on the p-th chain module of the L^2-chain complex of the universal covering $\widetilde{X}$ is invertible;*

5. *There is no connected finite CW-complex X with fundamental group Γ such that $\widetilde{X}$ is $\mathcal{N}(\Gamma)$-acyclic.*

Proof : The equivalence 1.) $\Longleftrightarrow$ 2.) follows from the analytic definition of L^2-Betti numbers and Novikov-Shubin invariants.

2.) $\Longleftrightarrow$ 3.) The implication 3.) $\Longrightarrow$ 2.) is obvious since a closed manifold is a finite CW-complex. To prove 2.) $\Longrightarrow$ 3.), let X be a connected finite CW-complex with fundamental group Γ such that

$$\begin{array}{rcll} b_p^{(2)}(\widetilde{X}) & = & 0 & p \geq 0; \\ \alpha_p(\widetilde{X}) & = & \infty^+ & p \geq 0; \end{array}$$

Let n be the dimension of X. Since X cannot be S^1 because of Example 8.5, and cannot be $\vee_{i=1}^r S^1$ for $r \geq 2$ because the Euler characteristic of X must be trivial (see Theorem 1.7), we have $n \geq 2$. Let M be the boundary of a regular neighbourhood N of an embedding of X into $\mathbb{R}^{2n+1}$ [221, chapter 3]. Then M is $2n$-dimensional and there is an n-connected map from M to X. We conclude from Theorem 1.7 and Theorem 8.7

$$\begin{array}{rclcll} b_p^{(2)}(\widetilde{M}) & = & b_p^{(2)}(\widetilde{X}) & = & 0 & \text{for } p \leq n-1; \\ b_p^{(2)}(\widetilde{M}) & = & b_{2n-p}^{(2)}(\widetilde{M}) & = & 0 & \text{for } p \geq n+1; \\ b_n^{(2)}(\widetilde{M}) & = & (-1)^n \cdot \chi(M); & & & \\ \alpha_p(\widetilde{M}) & = & \alpha_p(\widetilde{X}) & = & \infty^+ & \text{for } p \leq n; \\ \alpha_p(\widetilde{M}) & = & \alpha_{2n+1-p}(\widetilde{X}) & = & \infty^+ & \text{for } p \geq n+1. \end{array}$$

Poincaré duality applied to (M, N), the fact that N is homotopy equivalent to X, and Theorem 1.7.2 imply

$$\chi(M) \;=\; 2 \cdot \chi(N) \;=\; 2 \cdot \chi(X) \;=\; 0.$$

Hence M is a closed Riemannian manifold satisfying

$$\begin{array}{rcll} b_p^{(2)}(\widetilde{M}) & = & 0 & p \geq 0; \\ \alpha_p(\widetilde{M}) & = & \infty^+ & p \geq 0; \end{array}$$

3.) $\Longleftrightarrow$ 4.) follows from [148, Lemma 2.5 on page 31].

4.) $\Longleftrightarrow$ 5.) follows from [157, remark after Definition 3.11, Theorem 6.1]. ■

Conjecture 11.4 *(zero-in-the-spectrum Conjecture for a group) For any finitely presented group Γ the five equivalent assertions of Lemma 11.3 are true.* ■

Remark 11.5 It makes sense to formulate a version of Conjecture 11.4 for arbitrary groups Γ, where one has to substitute the universal coverings $\widetilde{M}$ (resp. $\widetilde{X}$) by regular coverings of the closed manifold M (resp. the finite CW-complex X) with Γ as group of deck transformations. Of course, one then has to drop the condition that Γ is the fundamental group of M (resp. X). Also a lot of the following results can be reformulated for arbitrary groups Γ. It is likely that this more general version is true if Conjecture 11.4 holds, because the decisive condition seems to be the finiteness of X, not that $\widetilde{X}$ is simply-connected. Moreover, one may weaken the condition that X is finite to the condition that X is of finite type. ■

Lemma 11.6 *Let X be a connected finite CW-complex with fundamental group Γ which is a counterexample to the zero-in-the-spectrum Conjecture 11.4 for the group Γ. Then*

1. *Γ is 2-$\mathcal{N}(\Gamma)$-connected;*
2. *$\chi(X) = 0$;*
3. *If X is a closed manifold, its signature is trivial;*
4. *If X is a closed Riemannian manifold, then $\widetilde{X}$ is not hyperEuclidean, where hyper-Euclidean means that there is a proper distance non-increasing map from $\widetilde{X}$ to $\mathbb{R}^{\dim(X)}$ of nonzero degree. In particular, $\widetilde{X}$ and hence X, do not admit Riemannian metrics with non-positive sectional curvature;*
5. *If X is an oriented closed manifold and $f : X \longrightarrow M$ a map to an oriented closed manifold of the same dimension as X which has non-zero degree and induces an isomorphism on the fundamental groups, then M is also a counterexample to the zero-in-the-spectrum Conjecture 11.4 for the group Γ;*
6. *If $X \longrightarrow E \longrightarrow B$ is a fibration of connected finite CW-complexes and the inclusion of X into E induces an injection on the fundamental groups, then E is a counterexample to the zero-in-the-spectrum Conjecture 11.4 for $\pi_1(E)$.*

Proof : 1.) The classifying map $f : \widetilde{X} \longrightarrow E\Gamma$ is a Γ-equivariant 2-connected map and induces an isomorphism $H_p^{\Gamma}(\widetilde{X};\mathcal{N}(\Gamma)) \longrightarrow H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$ for $p = 0, 1$ and an epimorphism for $p = 2$.

2.) This follows from Theorem 1.7.

3.) This is shown as in the proof of Lemma 6.5.

4.) This is proven in Gromov [109, section 8] and in [144, Proposition 7].

5.) If d is the degree of f and n the dimension of X and M, then the following diagram commutes and has isomorphisms as vertical maps by Poincaré duality

$$\begin{array}{ccc} H_p^{\Gamma}(\widetilde{X};\mathcal{N}(\Gamma)) & \xrightarrow{\widetilde{f}_*} & H_p^{\Gamma}(\widetilde{M};\mathcal{N}(\Gamma)) \\ {\scriptstyle \cap[X]}\uparrow\cong & & {\scriptstyle d\cdot\cap[M]}\uparrow\cong \\ H_{\Gamma}^{n-p}(\widetilde{X};\mathcal{N}(\Gamma)) & \xleftarrow{\widetilde{f}^*} & H_{\Gamma}^{n-p}(\widetilde{M};\mathcal{N}(\Gamma)) \end{array} .$$

Hence the upper horizontal map is split-surjective and the claim follows.

6.) This is proven by a spectral sequence argument analogous to the proof of Theorem 10.10. ■

Hence the zero-in-the-spectrum Conjecture 11.4 is true for any finitely presented group Γ which is not 2-$\mathcal{N}(\Gamma)$-connected. The next result collects some information about such groups.

Definition 11.7 (Lott [144], Definition 8 in section 5.1) *A finitely presented group Γ is called* big , *if Γ is non-amenable, $b_1^{(2)}(E\Gamma) = 0$ and $\alpha_2(E\Gamma) = \infty^+$. It is called* small *if it is not big.* ■

Lemma 11.8 *1. Let Γ be a group and n an integer such that $B\Gamma$ has finite $(n+1)$-skeleton. Then Γ is n-$\mathcal{N}(\Gamma)$-connected if and only if*

$$\begin{array}{rcll} b_p^{(2)}(E\Gamma) & = & 0 & 0 \le p \le n; \\ \alpha_p(E\Gamma) & = & \infty^+ & 0 \le p \le n+1; \end{array}$$

2. *A finitely presented group Γ is big in the sense of Lott's Definition 11.7 if and only if Γ is 1-$\mathcal{N}(\Gamma)$-connected. A finitely presented group Γ is non-amenable if and ony if it is 0-$\mathcal{N}(\Gamma)$-acyclic;*
3. *The fundamental group of a compact 2-manifold is small;*
4. *The fundamental group of a compact connected orientable 3-manifold, which satisfies the assumptions of Theorem 3.3 that none of its prime factors is exceptional, is small;*
5. *Let Δ be a normal subgroup of Γ. If Δ is p-$\mathcal{N}(\Delta)$-connected (resp. $\mathcal{N}(\Delta)$-acyclic), then Γ is p-$\mathcal{N}(\Gamma)$-connected (resp. $\mathcal{N}(\Gamma)$-acyclic). In particular, a finitely presented normal subgroup of a small finitely presented group is small;*
6. *Let Γ_0 and Γ_1 be non-trivial groups. Then $\Gamma_0 * \Gamma_1$ is not 1-$\mathcal{N}(\Gamma_0 * \Gamma_1)$- connected;*

7. *Let Γ be finitely presented. If Γ is big, then we get for the integer (see [115])*

$$q(\Gamma) \;=\; \min\{\chi(M) \mid M \textit{ connected closed oriented } 4\textit{-manifold with } \pi_1(M)=\Gamma\}$$

and the deficiency $\operatorname{def}(\Gamma)$

$$\begin{aligned} \operatorname{def}(\Gamma) &\leq 1; \\ q(\Gamma) &\geq 0. \end{aligned}$$

If Γ satisfies the zero-in-the-spectrum Conjecture 11.4, then

$$\begin{aligned} \operatorname{def}(\Gamma) &\leq 0; \\ q(\Gamma) &\geq 1. \end{aligned}$$

Proof : 1.) This follows from [157, remark after Definition 3.11, Theorem 6.1].

2.) This follows from 1.), Theorem 1.7.9 and Theorem 8.7.6.

3.) This follows from [144, Proposition 12].

4.) This follows from [148] as carried out in [144, Proposition 13].

5.) This is proven by a spectral sequence argument analogously to the proof of Theorem 10.10.

6.) We abbreviate $\Gamma = \Gamma_0 * \Gamma_1$. Assume that Γ_0 and Γ_1 are non-trivial and Γ is 1-$\mathcal{N}(\Gamma)$-connected. A model for $B\Gamma$ is the wedge of $B\Gamma_1$ and $B\Gamma_2$. Hence one obtains a pushout of Γ-spaces

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma \times_{\Gamma_1} E\Gamma_1 \\ \big\downarrow & & \big\downarrow \\ \Gamma \times_{\Gamma_0} E\Gamma_0 & \longrightarrow & E\Gamma\ . \end{array}$$

The low-dimensional part of the associated Mayer-Vietoris sequence looks like

$$\ldots \longrightarrow H_1^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma)) \longrightarrow l^2(\Gamma) \longrightarrow H_0^{\Gamma}(\Gamma\times_{\Gamma_0} E\Gamma_0;\mathcal{N}(\Gamma)) \oplus H_0^{\Gamma}(\Gamma\times_{\Gamma_1} E\Gamma_1;\mathcal{N}(\Gamma)) \longrightarrow H_0^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma)).$$

As Γ is by assumption 1-$\mathcal{N}(\Gamma)$-connected, we get an isomorphism

$$l^2(\Gamma) \xrightarrow{\cong} H_0^{\Gamma}(\Gamma\times_{\Gamma_0} E\Gamma_0;\mathcal{N}(\Gamma)) \oplus H_0^{\Gamma}(\Gamma\times_{\Gamma_1} E\Gamma_1;\mathcal{N}(\Gamma)).$$

If one applies $\dim_{\mathcal{N}(\Gamma)}$, then Theorem 10.8 implies

$$1 \;=\; b_0^{(2)}(\Gamma_0) + b_0^{(2)}(\Gamma_1).$$

Theorem 1.7.9 extends to arbitrary groups [160, section 3]. Hence both Γ_0 and Γ_1 are of order 2 and Γ is $\mathbb{Z}/2 * \mathbb{Z}/2$. This group contains $\mathbb{Z}$ as a normal subgroup of index 2. We conclude from Example 8.5 and Theorem 8.7.5 that $\alpha_1(\Gamma) = 1$, a contradiction.

7.) Theorem 6.6 gives the first two inequalities. The improved ones in the case that Γ satisfies the zero-in-the-spectrum Conjecture are proven in [144, section 5.2 and 5.3]. ∎

Remark 11.9 In view of Lemma 11.8.7, Lott has conjectured that $\operatorname{def}(\Gamma) \leq 0$ and $q(\Gamma) \geq 1$ holds for any finitely presented group Γ which is big [144, Conjecture 1 in 5.2 and Conjecture 2 in 5.3]. We remark that it suffices to prove this conjecture for finitely presented 2-$\mathcal{N}(\Gamma)$-connected groups. Namely, suppose that the finitely presented group Γ satisfies $b_1^{(2)}(\Gamma) = 0$ and $H_2^{\Gamma}(E\Gamma; \mathcal{N}(\Gamma)) \neq 0$. Then $\operatorname{def}(\Gamma) \leq 0$ and $q(\Gamma) \geq 1$ follow from [157, section 6.6]. An example of a finitely presented group Γ which is big, but not 2-$\mathcal{N}(\Gamma)$-acyclic, is $(\mathbb{Z} * \mathbb{Z}) \times (\mathbb{Z} * \mathbb{Z})$. ∎

Example 11.10 If Γ_k is an n_k-$\mathcal{N}(\Gamma_k)$-connected group for $k = 0, 1$, then $\Gamma_0 \times \Gamma_1$ is $(n_0{+}n_1)$-$\mathcal{N}(\Gamma_0 \times \Gamma_1)$-connected. This follows from the fact that the canonical $\mathbb{Z}[\Gamma_0 \times \Gamma_1]$-chain map

$$C_*(E\Gamma_0) \otimes_{\mathbb{Z}} C_*(E\Gamma_1) \longrightarrow C_*(E(\Gamma_0 \times \Gamma_1))$$

is an isomorphism. The (not finitely generated) group $\Gamma = \prod_{i=1}^{\infty}(\mathbb{Z} * \mathbb{Z})$ is $\mathcal{N}(\Gamma)$-acyclic because of Lemma 11.8.5, because it contains for each p the normal subgroup $\Gamma_n = \prod_{i=1}^{n}(\mathbb{Z} * \mathbb{Z})$ which is n-$\mathcal{N}(\Gamma_n)$-acyclic. This shows that it is crucial in the formulation of the zero-in-the-spectrum Conjecture 11.4 that in Lemma 11.3.3 to 11.3.5, the CW-complex X in question satsifies some finiteness conditions such as being finite. It may be possible that being of finite type suffices. ∎

For residually finite Γ the results of section 5 can be extended to the question whether zero is in the spectrum of the Laplace operator on the universal covering $\widetilde{X}$ of a connected finite CW-complex with Γ as fundamental group. Namely, the answer to this question can be read off from the low eigenvalue distributions of the Laplace operators of the various finite coverings of X given by a tower of coverings [156, Theorem 0.2 on page 456].

12. Miscellaneous

In this section we briefly mention some further aspects of L^2-invariants and give references for the reader who wants to know more about them.

The L^2-version of the index theorem was proven by Atiyah [3]. Let P be an elliptic differential operator on a closed Riemannian manifold M and let $\overline{M} \longrightarrow M$ be a regular covering of M with Γ as group of deck transformations. Then we can lift the Riemannian metric and the operator P to $\overline{M}$. The operator P is Fredholm and its *index* is defined by

$$\operatorname{ind}(P) \quad = \quad \dim_{\mathbb{C}}(\ker(P)) - \dim_{\mathbb{C}}(\ker(P^*)).$$

Using the von Neumann trace one can define the *L^2-index* of the lifted operator $\overline{P}$ analogously

$$\operatorname{ind}_{\mathcal{N}(\Gamma)}(\overline{P}) \quad = \quad \dim_{\mathcal{N}(\Gamma)}(\ker(\overline{P})) - \dim_{\mathcal{N}(\Gamma)}(\ker(\overline{P}^*)).$$

Then the L^2-index theorem says

$$\operatorname{ind}_{\mathcal{N}(\Gamma)}(\overline{P}) \quad = \quad \operatorname{ind}(P).$$

If one puts elliptic boundary conditions on the operator, this result was generalized to the case where M is compact and has a boundary by Schick [228]. This generalization is the L^2-version of the index theorem in [4]. These boundary conditions are local. There are also versions of the index theorem for manifolds with boundary using global boundary conditions which apply in contrast to the local conditions, to important geometrically defined operators such as the signature operator due to Atiyah, Patodi and Singer [5], [6], [7]. This index theorem involves, as a correction term, the eta-invariant. The L^2-version of the eta-invariant is defined and studied by Cheeger and Gromov [52], [53]. The L^2-version of this index theorem for manifolds with boundary and global boundary conditions is proven by Ramachandran [209] for Dirac type operators. Further references on L^2-index theory are [8], [10], [37], [38], [57], [58], [63], [64], [65], [78], [86], [133], [180], [185], [217], [218], [219], [233], [237], [238].

Of course L^2-cohomology is not only of interest for regular coverings of closed manifolds or CW-complexes of finite type. See for instance [30], [49], [145], [174]. In particular, the Cheeger-Goresky-MacPherson Conjecture [51] and the Zucker Conjecture [256] have created a lot of activity. They link the L^2-cohomology of the regular part with the intersection homology of an algebraic variety. References on this topic are [29], [31], [123], [141], [192], [193], [194], [198], [199], [204], [205], [222], [223], [224], [225], [226], [255], [257], [258], [259].

Connections of L^2-cohomology and discrete series of representations of Lie groups are investigated, for instance, in [8], [65], [102], [103], [230].

One can also define and investigate L^p-cohomology, as done for instance by Gromov [109, section 8] and Pansu [201],[202].

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