Introduction to Algebraic K-theory (Lecture I)

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Outline

- Introduce the projective class group $K_0(R)$.
- Discuss its algebraic and topological significance (e.g., finiteness obstruction).
- Introduce $K_1(R)$ and the Whitehead group $\mathrm{Wh}(G)$.
- Discuss its algebraic and topological significance (e.g., s-cobordism theorem).
- Introduce negative K-theory and the Bass-Heller-Swan decomposition.

The projective class group

Definition (Projective R-module)

An R-module P is called *projective* if it satisfies one of the following equivalent conditions:

- P is a direct summand in a free R-module;
- The following lifting problem has always a solution

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & {\scriptstyle \overline{f}}\nwarrow & \uparrow{\scriptstyle f} & & \\ & & P & & \end{array}$$

- If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

- Over a field or, more generally, over a principal ideal domain every projective module is free.
- If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree. For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.
- Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Definition (Group ring)

Let G be a group and R be a ring. The group ring RG is the R-algebra whose underlying R-module is the free R-module with G as basis and whose multiplicative structure is induced by the group structure

- The cellular chain complex $C_*(\widetilde{X})$ of the universal covering $\widetilde{X}$ of a CW-complex X with fundamental group π is a free $\mathbb{Z}\pi$-chain complex.
- Let F be a field of characteristic p for p a prime number or 0. Let G be a finite group. A finite-dimensional G-representation of G with coefficients in F is the same as a finitely generated FG-module.
- F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G, and is a free FG-module if and only if G is trivial.

Definition (Projective class group $K_0(R)$)

Let R be an (associative) ring (with unit). Define its *projective class group*

$$K_0(R)$$

to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.
- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.
- Induction

  Let $f \colon R \to S$ be a ring homomorphism. Given an R-module M, let f_*M be the S-module $S \otimes_R M$. We obtain a homomorphism of abelian groups

$$f_* \colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P].$$

- Compatibility with products

  The two projections from $R \times S$ to R and S induce isomorphisms

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

- Morita equivalence

  Let R be a ring and $M_n(R)$ be the ring of (n,n)-matrices over R. We can consider R^n as a $M_n(R)$-R-bimodule and as a R-$M_n(R)$-bimodule.
  Tensoring with these yields mutually inverse isomorphisms

$$\begin{array}{lclcl} K_0(R) & \xrightarrow{\cong} & K_0(M_n(R)), & [P] & \mapsto & [{}_{M_n(R)}R^n{}_R \otimes_R P]; \\ K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R), & [Q] & \mapsto & [{}_R R^n{}_{M_n(R)} \otimes_{M_n(R)} Q]. \end{array}$$

Example (Principal ideal domains)

If R is a principal ideal domain. Let F be its quotient field. Then we obtain mutually inverse isomorphisms

$$\begin{array}{lclclcl} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & & n & \mapsto & [R^n]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example (Representation ring)

Let G be a finite group and let F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$. Taking the character of a representation yields an isomorphism

$$R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} = K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \text{class}(G, \mathbb{C}),$$

where $\text{class}(G; \mathbb{C})$ is the complex vector space of class functions $G \to \mathbb{C}$, i.e., functions, which are constant on conjugacy classes.

Example (Dedekind domains)

- Let R be a Dedekind domain, for instance the ring of integers in an algebraic number field.
- The ideal class group C(R) is the abelian group of equivalence classes ideals under multiplication of ideals.
- Then we obtain an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

- The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p.

Theorem (Swan (1960))

If G is finite, then $\widetilde{K}_0(\mathbb{Z}G)$ is finite.

- Topological K-theory

  Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X.
  This is the zero-th term of a generalized cohomology theory $K^*(X)$ called topological K-theory.
  It is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = \{0\}$.
- Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Theorem (Swan (1962))

There is an isomorphism

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Wall's finiteness obstruction

Definition (Finitely dominated)

A CW-complex X is called *finitely dominated* if there exists a finite (= compact) CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

- A finite CW-complex is finitely dominated.
- A closed manifold is a finite CW-complex.

Problem

Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

Definition (Wall's finiteness obstruction)

A finitely dominated *CW*-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its *finiteness obstruction* as follows.

- Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$.
- Let $C_*(\widetilde{X})$ be the cellular chain complex. It is a free $\mathbb{Z}\pi$-chain complex.
- Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$.
- Define

$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

Theorem (Wall (1965))

A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.

- A finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex since $\widetilde{K}_0(\mathbb{Z}) = \{0\}$.
- Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Theorem (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$)

The following statements are equivalent for a finitely presented group G:

- *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*
- $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$*.*

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the *K_1-group of a ring R*

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- This is the same as $GL(R)/[GL(R), GL(R)]$.
- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

  which in general is not bijective.
- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of the universal determinant for R.

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ can be computed for finite groups.
- For instance we ge for an odd prime number p

$$\mathrm{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{(p-1)/2}.$$

If $p = 5$, a generator of $\mathrm{Wh}(\mathbb{Z}/5) \cong \mathbb{Z}$ is given by the unit $1 - t - t^{-1}$.

Whitehead torsion

Definition (*h*-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorpic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

□

- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$, since the Alexander trick does not work smoothly.
- Indeed, there exists so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- Given a finitely presented group G, an element $\xi \in \mathrm{Wh}(G)$ and a closed manifold M of dimension $n \geq 5$ with $G \cong \pi_1(M)$, there exists an h-cobordism W over M with $\tau(W, M) = \xi$.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Negative K-theory

Definition (Bass-Nil-groups)

Define for $n = 0, 1$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for K_1 (1964))

There is an isomorphism, natural in R,

$$K_0(R) \oplus K_1(R) \oplus \mathrm{NK}_1(R) \oplus \mathrm{NK}_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]) = K_1(R[\mathbb{Z}]).$$

Definition (Negative K-theory)

Define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t, t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for negative K-theory)

For $n \leq 1$ there is an isomorphism, natural in R,

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition (Regular ring)

A ring R is called *regular* if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

- Principal ideal domains are regular. In particular $\mathbb{Z}$ and any field are regular.
- If R is regular, then $R[t]$ and $R[t, t^{-1}] = R[\mathbb{Z}]$ are regular.
- If R is regular, then RG in general is not Noetherian or regular.

Theorem (Bass-Heller-Swan decomposition for regular rings)

Suppose that R is regular. Then

$$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \leq -1; \\ \mathrm{NK}_n(R) &= 0 \quad \text{for } n \leq 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition reduces for $n \leq 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

- There are also higher algebraic K-groups $K_n(R)$ for $n \geq 2$ constructed by Quillen (1973) as homotopy groups of certain spaces or spectra.
- Most of the well known features of $K_0(R)$ and $K_1(R)$ extend to both negative and higher algebraic K-theory.

- Notice the following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$, which look similar:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

- If G and K are groups, then we have the following formulas, which look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G * K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G * K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

Question (*K-theory of group rings and group homology*)

Is there a relation between $K_n(RG)$ and the homology of BG?