

Strategy of proof: From Farrell-Jones to flow spaces (Lecture III)

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- We first indicate how a proof of the Farrell-Jones Conjecture can be achieved if one has appropriate **flow spaces**.
- We give a idea of **controlled topology** and how to achieve control by flow spaces
- We briefly explain some basics concerning **coverings**.
- We state and explain an **axiomatic approach** to the proof which works for hyperbolic groups.
- We discuss that the existence of an appropriate flow space together with an appropriate **flow estimate** leads to a proof of the Farrell-Jones Conjecture for hyperbolic groups.

- The assembly map can be thought of an **approximation** of the algebraic K - or L -theory **by a homology theory**.
The basic feature between the left and right side of the assembly map is that for the left side one has **excision** which is not present on the right side.
- In general excision is available if one can make **representing cycles small**.
A best illustration for this is the proof of excision for simplicial or singular homology which is based on the **subdivision** whose effect is to make the support of cycles arbitrary small.
- The first big step in the proof of the Farrell-Jones Conjecture is to interpret the assembly map as a **forget control** map.
- Then the basic idea of proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class, i.e., **gain control**.

- The next two results are prototypes of this idea.

Theorem (Controlled h -Cobordism Theorem, Ferry (1977))

Let M be a compact Riemannian manifold of dimension ≥ 5 . Then there exists an $\epsilon = \epsilon_M > 0$, such that every ϵ -controlled h -cobordism over M is trivial.

Theorem (α -approximation theorem, Ferry (1979))

If M is a closed topological manifold of dimension ≥ 5 and α is an open cover of M , then there is an open cover β of M with the following property:

If N is a topological manifold of the same dimension and $f: N \rightarrow M$ is a proper β -homotopy equivalence, then f is α -close to a homeomorphism.

- One basic idea is to pass to **geometric modules** by remembering the position a basis.
- For instance, if we have a simplicial complex X , each basis element of the simplicial chain complex has a position in X , namely the barycenter of the simplex. Similar, one may assign to a handlebody a position in the underlying manifold.
- Given a metric space X , let $\mathcal{C}(X, R)$ be the following category:
 Objects are collections $\{M_x\} = \{M_x \mid x \in X\}$, where each M_x is a finitely generated free R -module and the support is required to be locally finite.
 Morphisms $\{f_{x,y}\}: \{M_x\} \rightarrow \{N_y\}$ are given by collection of R -morphisms $f_{x,y}: M_x \rightarrow N_y$ respecting certain finiteness conditions so that the composition can be defined by the usual formula for the multiplication of matrices.

- If X comes with a G -action, then G acts on $\mathcal{C}(X; R)$ and we can consider the G -fixed point set $\mathcal{C}(X, R)^G$. Denote by $\mathcal{T}(X; G)$ the full subcategory of $\mathcal{C}(X; R)^G$ where we additionally require that the support of a module is cocompact.
- Obviously $\mathcal{T}(G; R) = \mathcal{C}(G, R)^G$ is the category of finitely generated free RG -modules and hence

$$\pi_n(\mathbf{K}(\mathcal{T}(G; R))) = K_n(RG).$$

- If X a G -space, then the projection induces an equivalence of categories $\mathcal{T}(G \times X; R) \rightarrow \mathcal{T}(G; R)$. It induces for $n \in \mathbb{Z}$ a homotopy equivalence after taking K -theory

$$\pi_n(\mathbf{K}(\mathcal{T}(G \times X; R))) \xrightarrow{\cong} K_n(RG).$$

- Imposing appropriate **control conditions** on $\mathcal{T}(G \times X; R)$, leads to a subcategory $\mathcal{T}_c(G \times X; R)$ with the property that $X \mapsto \pi_*(\mathbf{K}(\mathcal{T}_c(G \times X; R)))$ yields a G -homology theory.

- The **forget control map**

$$\pi_n(\mathbf{K}(\mathcal{T}_c(G \times E_{\text{veyc}}(G); R))) \rightarrow \pi_n(\mathbf{K}(\mathcal{T}(G \times E_{\text{veyc}}(G); R)))$$

can be identified with the assembly map appearing in the K -theoretic Farrell-Jones Conjecture.

- The control conditions say, very roughly speaking, that for morphisms $\{(f_{x,y})\}$ the set $\{d_X(x, y) \mid (f_{x,y} \neq 0)\}$ is small.
- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.
- The idea is to use the geodesic flow on the universal covering to gain the necessary control.
- We will briefly explain this in the case, where the universal covering is the two-dimensional hyperbolic space \mathbb{H}^2 .

- Consider two points with coordinates (x_1, y_1) and (x_2, y_2) in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the y -direction.
- There is a fundamental problem: if $x_1 = x_2$, then the distance of these points is unchanged. Therefore we make the following prearrangement. Suppose that $y_1 < y_2$. Then we first let the point (x_1, y_1) flow so that it reaches a position where $y_1 = y_2$. Inspecting the hyperbolic metric, one sees that the distance between the two points (x_1, τ) and (x_2, τ) goes to zero if τ goes to infinity. This is the basic idea to gain control in the negatively curved case.

- Why is the non-positively curved case harder?
- Again, consider the upper half plane, but this time equip it with the flat Riemannian metric coming from Euclidean space.
- Then the same construction makes sense, but the distance between two points (x_1, τ) and (x_2, τ) is unchanged if we change τ .
- The basic first idea is to choose a focal point far away, say $f := ((x_1 + x_2)/2, \tau + 169356991)$, and then let (x_1, τ) and (x_2, τ) flow along the rays emanating from them and passing through the focal point f .
- In the beginning the effect is indeed that the distance becomes smaller, but as soon as we have passed the focal point the distance grows again. Either one chooses the focal point very far away or uses the idea of moving the focal point towards infinity while the points flow.

- The problem with this idea is obvious, we must describe this process in a functorial way and carefully check all the estimates to guarantee the desired effects.
- The comments above are all rather vague. At least we want to give a precise **axiomatic approach**, which works at least in the case of hyperbolic groups. The axiomatic approach for $CAT(0)$ -groups is more complicated and will be omitted, some of the extra difficulties will be discussed when appropriate.

Definition (Open covering)

An **open covering** \mathcal{U} of a space X is a collection of open subsets $\{U_i \mid i \in I\}$ satisfying $X = \cup_{i \in I} U_i$.

- The **dimension** $\dim(\mathcal{U})$ of \mathcal{U} is the smallest natural number n for which every element in X is contained in at most $(n + 1)$ members of \mathcal{U} .
- An open covering \mathcal{V} is a refinement of the open covering \mathcal{U} if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ with $V \subseteq U$.

Definition (Covering dimension)

The (topological) dimension or covering dimension $\dim(X)$ of a space X is the smallest natural number n for which every open covering possesses an n -dimensional refinement. (If no such n exists, we write $\dim(X) = \infty$.)

- If $Y \subseteq X$ is closed, then $\dim(Y) \leq \dim(X)$.
- If $X = Y \cup Z$ for closed $Y, Z \subseteq X$, then

$$\dim(X) = \max\{\dim(Y), \dim(Z)\}.$$

- If M is a n -dimensional manifold, then $\dim(M) = n$.
- If X is a n -dimensional CW-complex, then $\dim(X) = n$.

Definition (Nerve)

Let \mathcal{U} be an open covering. The **realization of its nerve** $|\mathcal{U}|$ is the following simplicial complex: The set of vertices is \mathcal{U} itself. The vertices U_0, U_1, \dots, U_n span a n -simplex if and only if $\bigcap_{i=0}^n U_i \neq \emptyset$.

- Points $|\mathcal{U}|$ are formal sums $x = \sum_{U \in \mathcal{U}} x_U U$, with $x_U \in [0, 1]$ such that $\sum_{U \in \mathcal{U}} x_U = 1$ and the intersection of all the U with $x_U \neq 0$ is non-empty, i.e., $\{U \mid x_U \neq 0\}$.
- Every simplicial complex and in particular the realization of the nerve of an open cover can be equipped with the l^1 -metric $d_{|\mathcal{U}|}$, i.e., the metric where the distance between points $x = \sum_U x_U U$ and $y = \sum_U y_U U$ is given by $d^1(x, y) = \sum_U |x_U - y_U|$.

- Let (X, d_X) be a metric space and \mathcal{U} an open covering of finite dimension N .
- Suppose that $\beta \geq 1$ is a **Lebesgue number** for \mathcal{U} , i.e., for every $x \in X$ there exists $U \in \mathcal{U}$ with $B_\beta(x) \subseteq U$.
- There is a map

$$f = f^\mathcal{U}: Z \rightarrow |\mathcal{U}|, \quad x \mapsto \sum_{U \in \mathcal{U}} f_U(x)U,$$

where

$$f_U(x) = \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)};$$

$$a_U(x) = d(x, Z - U) = \inf\{d(x, u) \mid u \notin U\}.$$

Theorem (Contracting map)

If $x, y \in X$ satisfy $d_X(x, y) \leq \frac{\beta}{4(N+1)}$, then we get

$$d_{|U|}(f(x), f(y)) \leq \frac{12(N+1)^2}{\beta} \cdot d_X(x, y).$$

- The larger β is, the estimate applies more often and the stronger map f is contracting.
- The larger N is, the estimate applies less often and the weaker f is contracting. If $N = \infty$, there is no conclusion at all.

Proof:

- Put $b_V(x, y) := a_V(x) - a_V(y)$ for $V \in \mathcal{U}$.
- $|b_V(x, y)| \leq d_X(x, y)$ since d_X is a metric.
- Since there are at most $2(N + 1)$ elements $V \in \mathcal{U}$ with $b_V(x, y) \neq 0$, we get

$$\sum_V |b_V(x, y)| \leq 2(N + 1)d(x, y) \leq \frac{\beta}{2}.$$

- Since there is $U \in \mathcal{U}$ with $B_\beta(x) \subseteq U$, we get

$$\sum_V a_V(x) \geq a_U(x) \geq \beta.$$

We compute:

$$\begin{aligned} & f_U(y) - f_U(x) \\ &= \frac{a_U(y)}{\sum_{V \in \mathcal{U}} a_V(y)} - \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)} \\ &= \frac{a_U(y) \cdot (\sum_{V \in \mathcal{U}} a_V(x)) - a_U(x) \cdot (\sum_{V \in \mathcal{U}} a_V(y))}{(\sum_{V \in \mathcal{U}} a_V(x)) \cdot (\sum_{V \in \mathcal{U}} a_V(y))} \\ &= \frac{a_U(x) \cdot \sum_V b_V(x, y) - a_U(x) \cdot \sum_V a_V(x) + a_U(y) \cdot \sum_V a_V(x)}{(\sum_V a_V(x)) \cdot (\sum_V a_V(x) - b_V(x, y))} \\ &= \frac{a_U(x) \cdot \sum_V b_V(x, y) - b_U(x, y) \cdot \sum_V a_V(x)}{(\sum_V a_V(x)) \cdot (\sum_V a_V(x) - b_V(x, y))}. \end{aligned}$$

We estimate:

$$\begin{aligned} & \sum_U |f_U(x) - f_U(y)| \\ &= \sum_U \frac{a_U(x) \cdot \sum_V b_V(x, y) - b_U(x, y) \cdot \sum_V a_V(x)}{(\sum_V a_V(x)) \cdot (\sum_V a_V(x) - b_V(x, y))} \\ &\leq \sum_{U, a_U(x) \neq 0} \left| \frac{\sum_V b_V(x, y)}{\sum_V a_V(x) - b_V(x, y)} \right| + \\ &\quad \sum_{U, b_U(x, y) \neq 0} \left| \frac{b_U(x, y)}{\sum_V a_V(x) - b_V(x, y)} \right| \\ &\leq 3(N+1) \frac{\sum_V |b_V(x, y)|}{|\sum_V a_V(x) - b_V(x, y)|} \end{aligned}$$

$$\begin{aligned}
&= 3(N+1) \frac{\sum_V |b_V(x, y)|}{|\sum_V a_V(x) - b_V(x, y)|} \\
&\leq 3(N+1) \frac{2(N+1)d(x, y)}{|\sum_V a_V(x) - b_V(x, y)|} \\
&\leq \frac{6(N+1)^2 d(x, y)}{\sum_V a_V(x) - \sum |b_V(x, y)|} \\
&\leq \frac{6(N+1)^2 d(x, y)}{\beta - 2(N+1)d(x, y)} \\
&\leq \frac{6(N+1)^2 d(x, y)}{\beta - \frac{\beta}{2}} \\
&= \frac{12(N+1)^2 d(x, y)}{\beta}. \quad \square
\end{aligned}$$

Definition (Open \mathcal{F} -covering)

Let \mathcal{F} be a family of subgroups of G and let Y be a G -space. An **open \mathcal{F} -covering** \mathcal{U} is an open covering of Y satisfying

- $U \in \mathcal{U}, g \in G \implies gU \in \mathcal{U}$;
- $U \in \mathcal{U}, g \in G, gU \cap U \neq \emptyset \implies gU = U$;
- For $U \in \mathcal{U}$ the subgroup $G_U := \{g \in G \mid gU = U\}$ belongs to \mathcal{F} .

Example

Put $X = \mathbb{R}$ and $G = \mathbb{Z}$. Then

$$\begin{aligned}\mathcal{U} &= \{(n, n+1) \mid n \in \mathbb{Z}\} \cup \{(n+1/2, n+3/2) \mid n \in \mathbb{Z}\}; \\ \mathcal{V} &= \{(n-1/2, n+3/2) \mid n \in \mathbb{Z}\},\end{aligned}$$

are open G -invariant coverings.

\mathcal{U} is a open \mathcal{TR} -covering, whereas there is no \mathcal{F} for which \mathcal{V} is an open \mathcal{F} -covering.

Lemma

Let \mathcal{U} be an open \mathcal{F} -covering. Then \mathcal{U} is a simplicial complex with simplicial G -action and also a G -CW-complex such that all isotropy groups belong to \mathcal{F} .

Definition (Weak Z -set condition)

A pair (\bar{X}, X) satisfies the **weak Z -set condition** if exists a homotopy $H: \bar{X} \times [0, 1] \rightarrow \bar{X}$, such that $H_0 = \text{id}_{\bar{X}}$ and $H_t(\bar{X}) \subset X$ for every $t > 0$.

- If M is a manifold with boundary, then $(M, \partial M)$ satisfies the weak Z -set condition because of the existence of a collar.

Theorem (Axiomatic Formulation)

Let G be a finitely generated group. Let \mathcal{F} be a family of subgroups of G . Suppose:

- There exists a G -space X such that the underlying space X is the realization of an abstract simplicial complex;
- There exists a G -space \bar{X} which contains X as an open G -subspace such that the underlying space of \bar{X} is compact, metrizable and contractible;
- The pair (\bar{X}, X) satisfies the weak Z -set condition.

Theorem (continued)

- There exists *wide open \mathcal{F} -coverings*, i.e.:

There is $N \in \mathbb{N}$, which only depends on the G -space \overline{X} , such that for every $\beta \geq 1$ there exists an open \mathcal{F} -cover $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:

- *For every $g \in G$ and $x \in \overline{X}$ there exists $U \in \mathcal{U}(\beta)$ such that*

$$B_\beta(g) \times \{x\} \subset U;$$

- *The dimension of the open cover $\mathcal{U}(\beta)$ is smaller than or equal to N .*

Then both the K - and L -theoretic Farrell-Jones Conjecture (with coefficients) holds for (G, \mathcal{F}) .

- An obvious choice for (\overline{X}, X) is $\overline{X} = X = \{\bullet\}$. But then the existence of wide open coverings implies $\mathcal{F} = \mathcal{ALL}$.
- Proof: We can choose β so large that $B_\beta(e)$ contains a (finite) set of generators S . Choose $U \in \mathcal{U}$ with $B_\beta(e) \in U$. Then we have $gU \cap U \neq \emptyset$ and hence $gU = U$ for all $g \in S$. This implies $G_U = G$ and hence $G \in \mathcal{F}$.
- In some sense we will need the space X to obtain some additional spaces to maneuver open sets around in order avoid too many intersections.
- It is crucial that \overline{X} is compact.
- In some sense N and β conflict another. The larger we take β , the higher is the chance that many members of \mathcal{U} intersect.

- If M is a closed manifold with non-positive sectional curvature and $G = \pi_1(M)$, then the canonical choice for X is \tilde{M} and for \bar{X} its standard compactification $\bar{M} = \tilde{M} \cup \partial\tilde{M}$.
- If G is a hyperbolic group, one uses for X the **Rips complex** and for $\bar{X} = X \cup \partial G$, where ∂G is the **boundary** of a hyperbolic group. In the sequel we consider this case.
- The main technical point is then the construction of the wide \mathcal{VCC} -covering $\mathcal{U}(\beta)$.
- This will be achieved with the help of a flow space $\text{FS}(X)$. We will use a variant which is closely related to the construction of **Mineyev(2005)**.
- Our main contribution to the flow space in the case of a hyperbolic group is the following flow estimate.

Theorem (Flow space estimate)

There exists a continuous G -equivariant map

$$j: G \times \bar{X} \rightarrow \text{FS}(X)$$

such that for every $\alpha > 0$ there exists a number $\beta = \beta(\alpha)$ such that the following holds:

If $g, h \in G$ with $d_G(g, h) \leq \alpha$ and $x \in \bar{X}$ then there is $\tau_0 \in [-\beta, \beta]$ such that for all $\tau \in \mathbb{R}$

$$d_{\text{FS}}(\phi_\tau j(g, x), \phi_{\tau+\tau_0} j(h, x)) \leq f_\alpha(\tau).$$

Here $f_\alpha: \mathbb{R} \rightarrow [0, \infty)$ is a function that depends only on α and has the property that $\lim_{\tau \rightarrow \infty} f_\alpha(\tau) = 0$.

- Then the next big step is to construct an appropriate open $\mathcal{V}\mathcal{C}\mathcal{V}\mathcal{C}$ -covering on the flow space $\text{FS}(X)$ such that the desired covering on $G \times X$ is obtained by pulling back this open covering on $\text{FS}(X)$ with $\Phi_\tau \circ j$ for appropriate τ .

Theorem (Long thin coverings)

There exists a natural number N such that for every $\beta > 0$ there is an $\mathcal{V}\mathcal{C}\mathcal{V}\mathcal{C}$ -cover \mathcal{U} of $\text{FS}(X)$ with the following properties:

- $\dim \mathcal{U} \leq N$;
- *For every $x \in X$ there exists $U \in \mathcal{U}$ such that*

$$\Phi_{[-\beta, \beta]}(x) := \{\Phi_\tau(x) \mid \tau \in [-\beta, \beta]\} \subseteq U;$$

- $G \setminus \mathcal{U}$ is finite.

- One ingredient in the proof that the existence of long thin coverings implies the existence of wide open coverings is the conclusion by a compactness argument, that there exists $\delta > 0$ such that for every $c \in \text{FS}(X)$ there exists $U_c \in \mathcal{U}$ with $B_\delta(\Phi_{[-\beta, \beta]}(x)) \subseteq U_c$, if \mathcal{U} is the long thin covering for β .
- Next we explain why our strategy will not work for a smaller family than $\mathcal{V}\mathcal{C}\mathcal{V}\mathcal{C}_I$.
- Consider a subgroup $H \subseteq G$ which can be written as an extension $1 \rightarrow F \rightarrow H \rightarrow \mathbb{Z} \rightarrow 1$ for a finite group H . Choose $g \in H$ which maps to a generator of \mathbb{Z} . Then there are $x \in X$ and $t \in (0, \infty)$ such that $\phi_t(x) = gx$ and $hx = x$ holds for all $h \in F$. If α satisfies $t < \alpha$, then $\Phi_{[-\alpha, \alpha]}(x) \subseteq U$ implies $gx \in U$ and $hx \in U$ for all $h \in F$. Hence $gU \cap U \neq \emptyset$ and $hU \cap U \neq \emptyset$ for all $h \in F$. This implies $g \in G_U$ and $h \in G_U$ for all $h \in F$. Hence G_U contains H .

Question

Does there exist an appropriate flow space for CAT(0)-spaces and CAT(0)-groups?