

# Status and Methods of Proof (Lecture V)

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# Flashback

- We introduced the **Farrell-Jones Conjecture** and the **Baum-Connes Conjecture** in general

$$H_n^G(E_{\mathcal{VCyc}}(G), \mathbf{K}_R) \xrightarrow{\cong} H_n^G(\text{pt}, \mathbf{K}_R) = K_n(RG);$$

$$H_n^G(E_{\mathcal{VCyc}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} H_n^G(\text{pt}, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG);$$

$$K_n^G(\underline{EG}) = H_n^G(E_{\mathcal{Fin}}(G), \mathbf{K}^{\text{top}}) \xrightarrow{\cong} H_n^G(\text{pt}, \mathbf{K}^{\text{top}}) = K_n(C_r^*(G)).$$

- We discussed further applications of these conjectures.
- **Cliffhanger**

## Question (Status)

*For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true?*

## Question (Methods of proof)

*What are the methods of proof?*

- We give a **status report** of the Farrell-Jones Conjecture.
- We discuss open cases and the search for potential **counterexamples**.
- We discuss **methods of proof**.

# Status report of the Farrell-Jones Conjecture

- There are certain generalizations of the Farrell-Jones Conjectures.
- One can allow **coefficients in additive categories** or consider **fibred versions** or the **version with finite wreath products**.
- In what follows, the **Full Farrell-Jones Conjecture** will mean the most general form with coefficients in additive categories and with finite wreath products and require it for both  $K$  and  $L$ -theory.
- The strong version encompasses twisted group rings  $R_\phi G$ , or even crossed product rings  $R * G$ , and includes orientation characters  $w: G \rightarrow \{\pm 1\}$  in the  $L$ -theory setting.

## Theorem (Bartels, Echterhoff, Farrell, Lück, Reich, Roushon, Rüping, Wegner)

Let  $\mathcal{FJ}$  be the class of groups for which the Full Farrell-Jones Conjecture holds. Then  $\mathcal{FJ}$  contains the following groups:

- Hyperbolic groups belong to  $\mathcal{FJ}$ ;
- $\text{CAT}(0)$ -groups belong to  $\mathcal{FJ}$ ;
- Virtually poly-cyclic groups belong to  $\mathcal{FJ}$ ;
- Solvable groups belong to  $\mathcal{FJ}$ ;
- Cocompact lattices in almost connected Lie groups belong to  $\mathcal{FJ}$ ;
- All 3-manifold groups belong to  $\mathcal{FJ}$ ;
- If  $R$  is a ring whose underlying abelian group is finitely generated free, then  $SL_n(R)$  and  $GL_n(R)$  belong to  $\mathcal{FJ}$  for all  $n \geq 2$ ;
- All arithmetic groups belong to  $\mathcal{FJ}$ .

## Theorem (continued)

Moreover,  $\mathcal{FJ}$  has the following inheritance properties:

- If  $G_1$  and  $G_2$  belong to  $\mathcal{FJ}$ , then  $G_1 \times G_2$  and  $G_1 * G_2$  belong to  $\mathcal{FJ}$ ;
  - If  $H$  is a subgroup of  $G$  and  $G \in \mathcal{FJ}$ , then  $H \in \mathcal{FJ}$ ;
  - If  $H \subseteq G$  is a subgroup of  $G$  with  $[G : H] < \infty$  and  $H \in \mathcal{FJ}$ , then  $G \in \mathcal{FJ}$ ;
  - Let  $\{G_i \mid i \in I\}$  be a directed system of groups (with not necessarily injective structure maps) such that  $G_i \in \mathcal{FJ}$  for  $i \in I$ . Then  $\operatorname{colim}_{i \in I} G_i$  belongs to  $\mathcal{FJ}$ ;
- 
- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., **Bökstedt, Carlsson, Davis, Ferry, Hambleton, Hsiang, Jones, Linnell, Madsen, Pedersen, Quinn, Ranicki, Rognes, Rosenthal, Tessera, Varisco, Weinberger, Yu.**

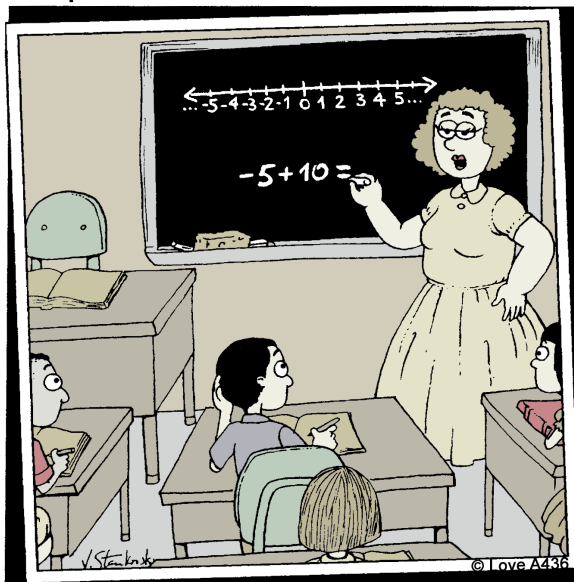
- **Limit groups** in the sense of **Zela** are CAT(0)-groups (**Alibegovic-Bestvina**).
- There are many **constructions of groups with exotic properties** which arise as colimits of hyperbolic groups.
- One example is the construction of **groups with expanders** due to **Gromov**, see **Arzhantseva-Delzant**. These yield **counterexamples** to the **Baum-Connes Conjecture with coefficients** due to **Higson-Lafforgue-Skandalis**.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
- Many groups of the region '**Hic abundant leones**' in the universe of groups in the sense of **Bridson** do satisfy the Full Farrell-Jones Conjecture.

- **Davis-Januszkiewicz** have constructed exotic closed aspherical manifolds using **hyperbolization techniques**. For instance there are examples which do **not admit a triangulation** or whose **universal covering is not homeomorphic to Euclidean space**.
- However, in all cases the universal coverings are CAT(0)-spaces and the fundamental groups are CAT(0)-groups. Hence they satisfy the Full Farrell-Jones Conjecture and in particular the Borel Conjecture in dimension  $\geq 5$ .
- The Baum-Connes Conjecture is open for CAT(0)-groups, cocompact lattices in almost connected Lie groups and  $SL_n(\mathbb{Z})$  for  $n \geq 3$ , but known, for instance, for all **a-T-menable groups** due to work of **Higson-Kasparov**.

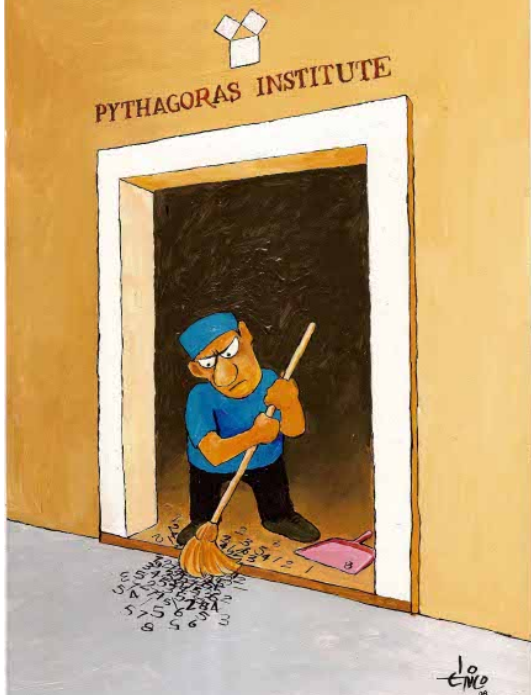


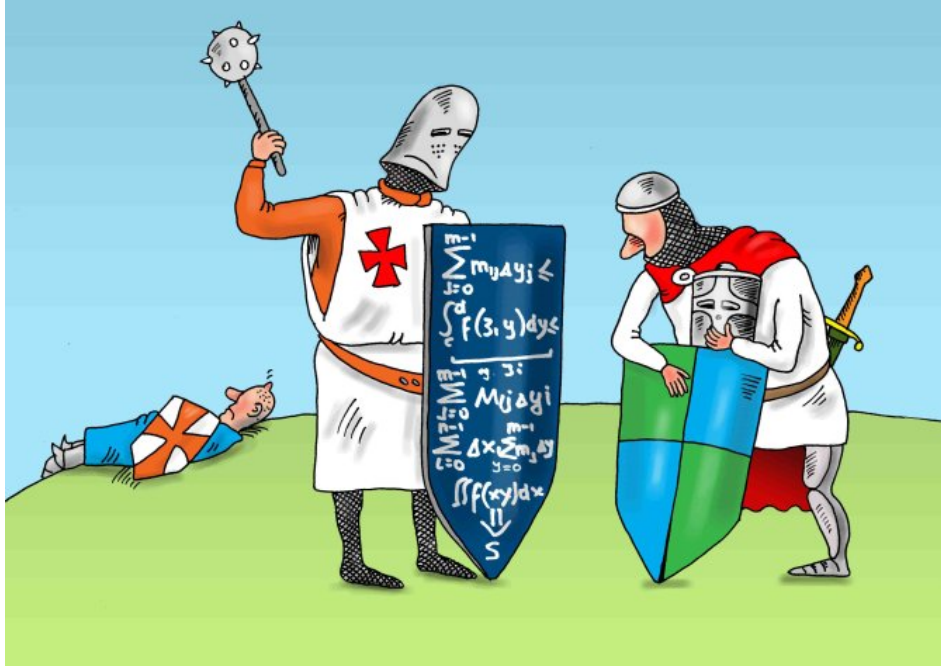
- What are candidates for groups or closed aspherical manifolds for which the conjectures due to Farrell-Jones, Novikov or Borel may be false?
- There are still many interesting groups for which the Farrell-Jones Conjecture is open. Examples are:
  - Amenable groups;
  - Mapping class groups;
  - $\text{Out}(F_n)$ ;
  - Thompson groups.
- We have no good candidate for a group (or for a property of groups) for which the Farrell-Jones Conjecture may fail.

How do you feel about  
mathematics?



"If you go on a diet and lose five pounds, only to gain back ten the following month, how many infuriating, godforsaken pounds do you weigh?"







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- The assembly map can be thought of an **approximation** of the algebraic  $K$ - or  $L$ -theory **by a homology theory**.
- The basic feature between the left and right side of the assembly map is that on the left side one has **excision** which is not present on the right side.
- In general excision is available if one can make **representing cycles small**.
- A best illustration for this is the proof of excision for simplicial or singular homology based on **subdivision** whose effect is to make the support of cycles arbitrary small.

- The first big step in the proof of the Farrell-Jones Conjecture is to interpret the assembly map as a **forget control** map.
- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class, i.e., **gain control**.
- The following result is a prototype of this idea.

### Theorem (Controlled $h$ -Cobordism Theorem, Ferry)

*Let  $M$  be a compact Riemannian manifold of dimension  $\geq 5$ . Then there exists an  $\epsilon = \epsilon_M > 0$ , such that every  $\epsilon$ -controlled  $h$ -cobordism over  $M$  is trivial.*



- One basic idea is to pass to **geometric modules** by remembering the position of a basis.
- For instance, if we have a simplicial complex  $X$ , each basis element of the simplicial chain complex has a position in  $X$ , namely the barycenter of the simplex.
- Similarly, one may assign to a handlebody a position in the underlying manifold.

- Given a metric space  $X$ , let  $\mathcal{C}(X, R)$  be the following category:
- Objects are collections  $\{M_x\} = \{M_x \mid x \in X\}$ , where each  $M_x$  is a finitely generated free  $R$ -module and the support is required to be locally finite.
- Morphisms  $\{f_{x,y}\}: \{M_x\} \rightarrow \{N_y\}$  are given by collection of  $R$ -morphisms  $f_{x,y}: M_x \rightarrow N_y$  respecting certain finiteness conditions so that the composition can be defined by the usual formula for the multiplication of matrices.

- If  $X$  comes with a  $G$ -action, then  $G$  acts on  $\mathcal{C}(X; R)$  and we can consider the  $G$ -fixed point set  $\mathcal{C}(X, R)^G$ . Denote by  $\mathcal{T}(X; G)$  the full subcategory of  $\mathcal{C}(X; R)^G$  where we additionally require that the support of a module is cocompact.
- Obviously  $\mathcal{T}(G; R) = \mathcal{C}(G, R)^G$  is the category of finitely generated free  $RG$ -modules and hence

$$\pi_n(\mathbf{K}(\mathcal{T}(G; R))) = K_n(RG).$$

- If  $X$  is a  $G$ -space, then projection induces an equivalence of categories  $\mathcal{T}(G \times X; R) \rightarrow \mathcal{T}(G; R)$ . It induces for  $n \in \mathbb{Z}$  a homotopy equivalence after taking  $K$ -theory

$$\pi_n(\mathbf{K}(\mathcal{T}(G \times X; R))) \xrightarrow{\cong} K_n(RG).$$

- Imposing appropriate **control conditions** on  $\mathcal{T}(G \times X; R)$ , leads to a subcategory  $\mathcal{T}_c(G \times X; R)$  with the property that  $X \mapsto \pi_*(\mathbf{K}(\mathcal{T}_c(G \times X; R)))$  yields a  $G$ -homology theory.
- The **forget control map**

$$\pi_n(\mathbf{K}(\mathcal{T}_c(G \times E_{\mathcal{V}Cyc}(G); R))) \rightarrow \pi_n(\mathbf{K}(\mathcal{T}(G \times E_{\mathcal{V}Cyc}(G); R)))$$

can be identified with the assembly map appearing in the  $K$ -theoretic Farrell-Jones Conjecture.

- Suppose that  $G = \pi_1(M)$  for a closed Riemannian manifold with negative sectional curvature.
- The idea is to use the **geodesic flow** on the universal covering to gain the necessary control.
- We will briefly explain this in the case, where the universal covering is the two-dimensional hyperbolic space  $\mathbb{H}^2$ .

- Consider two points with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the  $y$ -direction.
- There is a fundamental problem: if  $x_1 = x_2$ , then the distance between these points is unchanged. Therefore we make the following prearrangement. Suppose that  $y_1 < y_2$ . Then we first let the point  $(x_1, y_1)$  flow so that it reaches a position where  $y_1 = y_2$ . Inspecting the hyperbolic metric, one sees that the distance between the two points  $(x_1, \tau)$  and  $(x_2, \tau)$  goes to zero if  $\tau$  goes to infinity. This is the basic idea to gain control in the negatively curved case.

- Why is the non-positively curved case harder?
- Again, consider the upper half plane, but this time equip it with the flat Riemannian metric coming from Euclidean space.
- Then the same construction makes sense, but the distance between  $(x_1, \tau)$  and  $(x_2, \tau)$  is unchanged if we change  $\tau$ .
- The basic first idea is to choose a **focal point** far away, say  $f := ((x_1 + x_2)/2, \tau + 169356991)$ , and then let  $(x_1, \tau)$  and  $(x_2, \tau)$  flow along the rays emanating from them and passing through the focal point  $f$ .
- In the beginning the effect is indeed that the distance becomes smaller, but as soon as we have passed the focal point the distance grows again. Either one chooses the focal point very far away or uses the idea of moving the focal point towards infinity while the points flow. All this has to be carried out **functorially**.

# Coverings and Contracting Maps

- Let  $(X, d_X)$  be a metric space and  $\mathcal{U}$  an open covering of finite (topological) dimension  $N$ . Let  $|\mathcal{U}|$  be its nerve.
- There is a canonical map

$$f = f^{\mathcal{U}}: X \rightarrow |\mathcal{U}|, \quad x \mapsto \sum_{U \in \mathcal{U}} f_U(x)U,$$

where

$$\begin{aligned} f_U(x) &= \frac{a_U(x)}{\sum_{V \in \mathcal{U}} a_V(x)}; \\ a_U(x) &= d(x, Z - U) = \inf\{d(x, u) \mid u \notin U\}. \end{aligned}$$

- Suppose that  $\beta \geq 1$  is a **Lebesgue number** for  $\mathcal{U}$ .



## Theorem (Contracting map)

If  $x, y \in X$  satisfy  $d_X(x, y) \leq \frac{\beta}{4(N+1)}$ , then we get

$$d_{|U|}(f(x), f(y)) \leq \frac{12(N+1)^2}{\beta} \cdot d_X(x, y).$$

- The larger  $\beta$  is, the estimate applies more often and the map  $f$  is stronger contracting.
- The larger  $N$  is, the estimate applies less often and the weaker  $f$  is contracting. If  $N = \infty$ , there is no conclusion at all.

## Definition (Open $\mathcal{F}$ -covering)

Let  $\mathcal{F}$  be a family of subgroups of  $G$  and let  $Y$  be a  $G$ -space. An **open  $\mathcal{F}$ -covering**  $\mathcal{U}$  is an open covering of  $Y$  satisfying

- $U \in \mathcal{U}, g \in G \implies gU \in \mathcal{U}$ ;
- $U \in \mathcal{U}, g \in G, gU \cap U \neq \emptyset \implies gU = U$ ;
- For  $U \in \mathcal{U}$  the subgroup  $G_U := \{g \in G \mid gU = U\}$  belongs to  $\mathcal{F}$ .

## Definition (Weak Z-set condition)

A pair  $(\bar{X}, X)$  satisfies the **weak Z-set condition** if there exists a homotopy  $H: \bar{X} \times [0, 1] \rightarrow \bar{X}$ , such that  $H_0 = \text{id}_{\bar{X}}$  and  $H_t(\bar{X}) \subset X$  for every  $t > 0$ .

- If  $M$  is a manifold with boundary, then  $(M, \partial M)$  satisfies the weak Z-set condition because of the existence of a collar.

## Theorem (Axiomatic Formulation)

Let  $G$  be a finitely generated group. Let  $\mathcal{F}$  be a family of subgroups of  $G$ . Suppose:

- There exists a  $G$ -space  $X$  such that the underlying space  $X$  is the realization of an abstract simplicial complex;
- There exists a  $G$ -space  $\bar{X}$  which contains  $X$  as an open  $G$ -subspace such that the underlying space of  $\bar{X}$  is compact, metrizable and contractible;
- The pair  $(\bar{X}, X)$  satisfies the weak  $Z$ -set condition.

## Theorem (continued)

- There exists *wide open  $\mathcal{F}$ -coverings*, i.e.:

There is  $N \in \mathbb{N}$ , which only depends on the  $G$ -space  $\overline{X}$ , such that for every  $\beta \geq 1$  there exists an open  $\mathcal{F}$ -cover  $\mathcal{U}(\beta)$  of  $G \times \overline{X}$  with the following two properties:

- For every  $g \in G$  and  $x \in \overline{X}$  there exists  $U \in \mathcal{U}(\beta)$  such that

$$B_\beta(g) \times \{x\} \subset U;$$

- The dimension of the open cover  $\mathcal{U}(\beta)$  is smaller than or equal to  $N$ .

Then both the  $K$ - and  $L$ -theoretic Farrell-Jones Conjecture (with coefficients) hold for  $(G, \mathcal{F})$ .

- An obvious choice for  $(\overline{X}, X)$  is  $\overline{X} = X = \text{pt.}$  But then the existence of wide open coverings implies  $\mathcal{F} = \mathcal{ALL}$ .
- Proof: We can choose  $\beta$  so large that  $B_\beta(e)$  contains a (finite) set of generators  $S$ . Choose  $U \in \mathcal{U}$  with  $B_\beta(e) \in U$ . Then we have  $gU \cap U \neq \emptyset$  and hence  $gU = U$  for all  $g \in S$ . This implies  $G_U = G$  and hence  $G \in \mathcal{F}$ .
- We will need the space  $X$  to obtain some additional spaces to maneuver open sets around in order avoid too many intersections.
- The numbers  $N$  and  $\beta$  conflict with each another. The larger we take  $\beta$ , the higher the chance is that many members of  $\mathcal{U}$  intersect.

- If  $M$  is a closed manifold with negative sectional curvature and  $G = \pi_1(M)$ , then the canonical choice for  $X$  is  $\tilde{M}$  and for  $\bar{X}$  its standard compactification  $\bar{M} = \tilde{M} \cup \partial\tilde{M}$ .
- If  $G$  is a hyperbolic group, one uses for  $X$  the **Rips complex** and for  $\bar{X} = X \cup \partial G$ , where  $\partial G$  is the **boundary** of a hyperbolic group. We consider this case in what follows.
- The main technical point then is the construction of the wide open  $\mathcal{VCyc}$ -covering  $\mathcal{U}(\beta)$ .
- This will be achieved with the help of a flow space  $\text{FS}(X)$ . We will use a variant that is closely related to the construction of **Mineyev(2005)**.
- Our main contribution to the flow space in the case of a hyperbolic group is the following flow estimate.

## Theorem (Flow space estimate)

*There exists a continuous  $G$ -equivariant map*

$$j: G \times \bar{X} \rightarrow \text{FS}(X)$$

*such that for every  $\alpha > 0$  there exists a number  $\beta = \beta(\alpha)$  such that the following holds:*

*If  $g, h \in G$  with  $d_G(g, h) \leq \alpha$  and  $x \in \bar{X}$  then there is  $\tau_0 \in [-\beta, \beta]$  such that for all  $\tau \in \mathbb{R}$*

$$d_{\text{FS}}(\phi_\tau j(g, x), \phi_{\tau+\tau_0} j(h, x)) \leq f_\alpha(\tau).$$

*Here  $f_\alpha: \mathbb{R} \rightarrow [0, \infty)$  is a function that depends only on  $\alpha$  and has the property that  $\lim_{\tau \rightarrow \infty} f_\alpha(\tau) = 0$ .*



- Then the next big step is to construct an appropriate open  $\mathcal{V}Cyc$ -covering on the flow space  $FS(X)$  such that the desired covering on  $G \times X$  is obtained by pulling back this open covering on  $FS(X)$  with  $\Phi_\tau \circ j$  for appropriate  $\tau$ .

### Theorem (Long thin coverings)

*There exists a natural number  $N$  such that for every  $\beta > 0$  there is a  $\mathcal{V}Cyc$ -cover  $\mathcal{U}$  of  $FS(X)$  with the following properties:*

- $\dim \mathcal{U} \leq N$ ;
- *For every  $x \in X$  there exists  $U \in \mathcal{U}$  such that*

$$\Phi_{[-\beta, \beta]}(x) := \{\Phi_\tau(x) \mid \tau \in [-\beta, \beta]\} \subseteq U;$$

- $G \setminus \mathcal{U}$  is finite.

- Next we explain why our strategy will not work for a smaller family than  $\mathcal{VCyc}_I$ .
- Consider a subgroup  $H \subseteq G$  which can be written as an extension  $1 \rightarrow F \rightarrow H \rightarrow \mathbb{Z} \rightarrow 1$  for a finite group  $H$ . Choose  $g \in H$  which maps to a generator of  $\mathbb{Z}$ .
- Then there are  $x \in X$  and  $t \in (0, \infty)$  such that  $\phi_t(x) = gx$  and  $hx = x$  holds for all  $h \in F$ .
- If  $\alpha$  satisfies  $t < \alpha$ , then  $\Phi_{[-\alpha, \alpha]}(x) \subseteq U$  implies  $gx \in U$  and  $hx \in U$  for all  $h \in F$ . Hence  $gU \cap U \neq \emptyset$  and  $hU \cap U \neq \emptyset$  for all  $h \in F$ . This implies  $g \in G_U$  and  $h \in G_U$  for all  $h \in F$ .
- Hence  $G_U$  contains  $H$ .

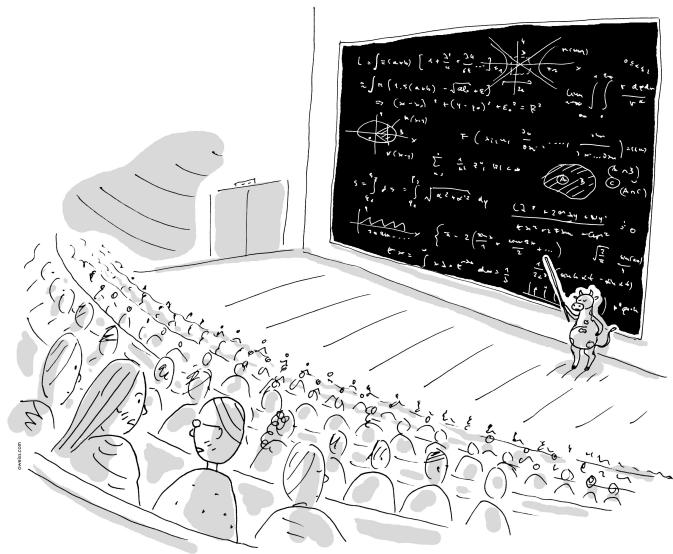
## Question

*Is the Farrell-Jones Conjecture true for all groups?*

- **Female student:** “So you don’t think this is weird at all, Anatol?”
- **Anatol:** “Absolutely, There should be an  $\epsilon_0$  in the second integral as opposed to an  $\epsilon$ ”.

- Main actor





- “So you don’t think this is weird at all, Anatol?”
- “Absolutely. There should be an  $\varepsilon_0$  in the second integral as opposed to an  $\varepsilon$ .”

Thank you for your attention!