K_1 and Whitehead torsion (Lecture II)

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Outline

- Introduce $K_1(R)$ and the Whitehead group $\mathrm{Wh}(G)$.
- We define the Whitehead torsion of a homotopy equivalence of finite connected CW-complexes
- Discuss its algebraic and topological significance (e.g., s-cobordism theorem).
- Introduce negative K-theory and the Bass-Heller-Swan decomposition.

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- Let $GL_n(R)$ be the group of invertible (n,n)-matrices.
- We get a sequence of inclusions

$$R^{\times} = GL_1(R) \subseteq GL_2(R) \subseteq \cdots \subseteq GL_n(R) \subseteq GL_{n+1}(R) \subseteq \cdots$$

  by sending an (n,n)-matrix A to the $(n+1,n+1)$-matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
- Let $GL(R) := \bigcup_{n \geq 1} GL_n(R)$.
- The obvious maps $GL_n(R) \to K_1(R)$ induce an epimorphism $GL(R) \to K_1(R))$.
- It induces an isomorphism

$$GL(R)/[GL(R), GL(R)] \xrightarrow{\cong} K_1(R).$$

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the reduced K_1-group

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

Exercise

Show for a commutative ring R that the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times}.$$

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable for finite groups G.

Exercise

Show that $t - 1 - t^{-1} \in \mathbb{Z}[\mathbb{Z}/5]$ for $t \in \mathbb{Z}/5$ the generator is a unit and hence defines an element η in $\mathrm{Wh}(\mathbb{Z}/5)$. Prove that we obtain a well-defined map

$$\mathrm{Wh}(\mathbb{Z}/5) \to \mathbb{R}$$

by sending the class represented by the $\mathbb{Z}[\mathbb{Z}/5]$-automorphism $f \colon \mathbb{Z}[\mathbb{Z}/5]^n \to \mathbb{Z}[\mathbb{Z}/5]^n$ to $\ln(|\det(\overline{f})|)$, where $\overline{f} \colon \mathbb{C}^n \to \mathbb{C}^n$ is the $\mathbb{C}$-linear map $f \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathrm{id}_{\mathbb{C}} \colon \mathbb{Z}[\mathbb{Z}/5]^n \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathbb{C} \to \mathbb{Z}[\mathbb{Z}/5]^n \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathbb{C}$ with respect to the $\mathbb{Z}/5$-action on $\mathbb{C}$ given by multiplication with $\exp(2\pi i/5)$. Finally show that η generates an infinite cyclic subgroup in $\mathrm{Wh}(\mathbb{Z}/5)$.

Whitehead torsion

- Let C_* be a contractible finite based free R-chain complex.
- Choose a chain contraction γ_*.
- Then $(c_* + \gamma_*)_{\text{odd}} : C_{\text{odd}} \to C_{\text{ev}}$ is an isomorphism of finitely generated based free R-modules and hence defines an element called Reidemeister torsion

$$\rho(C_*) := [(c_* + \gamma_*)_{\text{odd}}] \quad \in K_1(R).$$

- Next we show that it is independent of the choice of γ_*.

- Let δ_* be another chain contraction.
- Define R-homomorphisms

$$\begin{aligned}(e_* + \gamma_*)_{\text{odd}} : E_{\text{odd}} &\to E_{\text{ev}};\\ (e_* + \delta_*)_{\text{ev}} : E_{\text{ev}} &\to E_{\text{odd}}.\end{aligned}$$

- Put

$$\begin{aligned}\mu_n &:= (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n;\\ \nu_n &:= (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n.\end{aligned}$$

- One easily checks that

$$(\mathrm{id} + \mu_*)_{\mathrm{odd}},$$
$$(\mathrm{id} + \nu_*)_{\mathrm{ev}}$$

  and both compositions

$$(c_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (c_* + \delta_*)_{\mathrm{ev}}$$
$$(c_* + \delta_*)_{\mathrm{ev}} \circ (\mathrm{id} + \nu_*)_{\mathrm{ev}} \circ (c_* + \gamma_*)_{\mathrm{odd}}$$

  are given by upper triangular matrices whose diagonal entries are identity maps.
- In particular they represent zero in $K_1(R)$.
- This implies $[(c_* + \gamma_*)_{\mathrm{odd}}] = -[(c_* + \delta_*)_{\mathrm{ev}}]$ in $K_1(R)$ and hence that $[(c_* + \gamma_*)_{\mathrm{odd}}] \in K_1(R)$ does not depend on the choice of γ.

- Consider a R-chain homotopy equivalence $f_*\colon C_* \to D_*$ of finite based free R-chain complexes.
- Then its mapping cone $\operatorname{cone}(f_*)$ is a contractible finite based free R-chain complexes.
- Define the Whitehead torsion of f_* to be

$$\tau(f_*) := \rho(\operatorname{cone}(f_*)) \quad \in K_1(R)$$

Exercise

Let $r \in \mathbb{Q}$ be a rational number. Show that the following finite based free $\mathbb{Q}$-chain complex concentrated in dimensions 2, 1 and 0 is contractible and compute its Reidemeister torsion

$$\mathbb{Q} \xrightarrow{\begin{pmatrix} 0 \\ r \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q} \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathbb{Q}$$

Definition (Whitehead torsion of maps)

Given a homotopy equivalence of connected finite CW-complexes $f\colon X \to Y$, define its Whitehead torsion

$$\tau(f) \in \mathrm{Wh}(\pi_1(Y))$$

as follows.

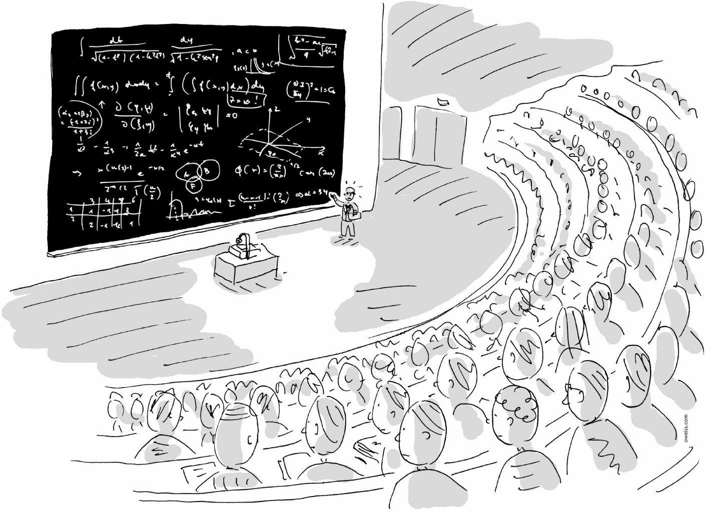
- Let $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ be a lift of f to the universal coverings.
- It is a $\pi_1(Y)$-equivariant homotopy equivalence and hence induces a $\mathbb{Z}[\pi_1(Y)]$-chain homotopy equivalence $C_*(\widetilde{f})\colon C_*(\widetilde{X}) \to C_*(\widetilde{Y})$.
- The CW-structures induce $\mathbb{Z}[\pi_1(Y)]$-basis on $C_*(\widetilde{X})$ and $C_*(\widetilde{Y})$ which are unique up to multiplying a basis element with some element $\pm w$ for $w \in \pi_1(Y)$ and up to permutation of the basis elements.
- Define $\tau(f)$ to be the Whitehead torsion of $C_*(\widetilde{f})$ considered in $\mathrm{Wh}(\pi_1(Y))$.

Definition (simple homotopy equivalence)

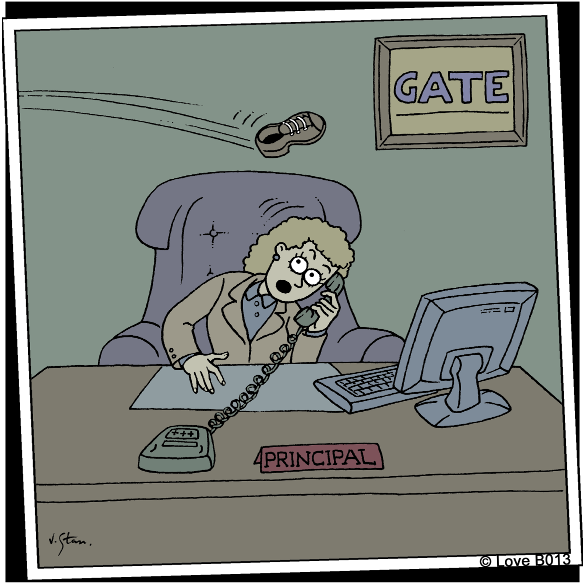
A homotopy equivalence of connected finite CW-complex is called simple if its Whitehead torsion is trivial.

- A homotopy equivalence is a simple homotopy equivalence if and only if it is homotopic to a composition of so called expansions and collapses.
- Any element in $\mathrm{Wh}(\pi_1(Y))$ can be realized as $\tau(f)$ for some homotopy equivalence $f\colon X \to Y$.
- Since there exist connected finite CW-complexes Y with $\mathrm{Wh}(\pi_1(Y)) \neq 0$, there exists homotopy equivalences of connected finite CW-complexes which are not simple.

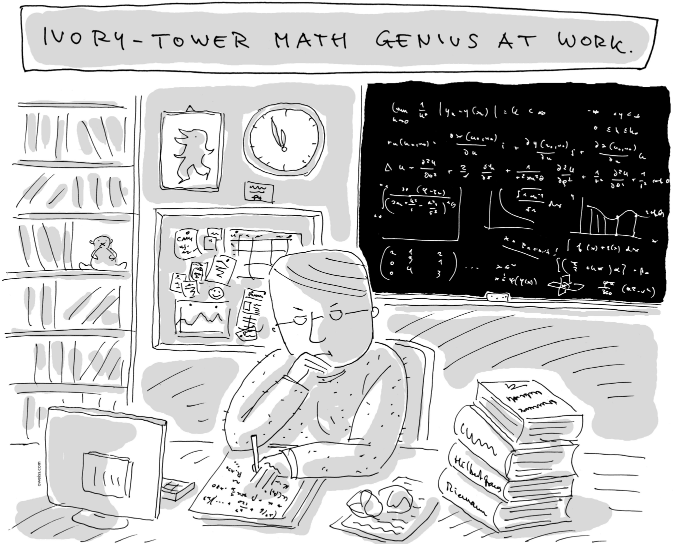
Carrying out mathematics



„*Umfragen haben gezeigt, dass 17 von 23 Betroffenen diese Lösung als richtig empfinden.*“



"Ms. Thomas, you'll have to come pick up your son. He's swinging from a chandelier claiming that he can and will divide by zero."



"Now, let's see... if I took the second derivative of α, this would yield the new natural constant β_0 which would in turn revolutionize the way we see the universe today... oops, time for lunch!"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Definition (*h*-cobordism)

An *h*-cobordism over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorphic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_0^n and D_1^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_0^n \times [0,1]$ that is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{\partial D_1^n}\colon \partial D_1^n \xrightarrow{\cong} \partial D_0^n$ to a homeomorphism $g\colon D_1^n \to D_0^n$.
- The three homeomorphisms $id_{D_0^n}$, f and g fit together to a homeomorphism $h\colon M \to D_0^n \cup_{\partial D_0^n \times \{0\}} \partial D_0^n \times [0,1] \cup_{\partial D_0^n \times \{1\}} D_0^n$. The target is obviously homeomorphic to S^n.

□

- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$, since the Alexander trick does not work smoothly.
- Indeed, there exist so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- Given a finitely presented group G, an element $\xi \in \mathrm{Wh}(G)$ and a closed manifold M of dimension $n \geq 5$ with $G \cong \pi_1(M)$, there exists an h-cobordism W over M with $\tau(W, M) = \xi$.

- The s-Cobordism Theorem is one step in the surgery program due to Browder, Novikov, Sullivan and Wall to decide whether two closed manifolds M and N are diffeomorphic what is in general a very hard question. It consists of the following steps.

1. Construct a simple homotopy equivalence $f\colon M \to N$;
2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id})\colon (W; M, N) \to (N \times [0,1], N \times \{0\}, N \times \{1\})$;
3. Modify W and F relative boundary by surgery such that F becomes a simple homotopy equivalence and thus W becomes an h-cobordism whose Whitehead torsion is trivial.
4. Apply the s-Cobordism Theorem.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsion free G)

If G is torsion free, then

$$\mathrm{Wh}(G) = \{0\}.$$

Negative and higher K-theory

- There exist K-groups $K_n(R)$ for every $n \in \mathbb{Z}$. The negative K-groups were introduced by Bass, the higher algebraic K-groups by Quillen.

Theorem (Bass-Heller-Swan decomposition)

For $n \in \mathbb{Z}$ there is an isomorphism, natural in R,

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition (Regular ring)

A ring R is called regular if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

Theorem (Bass-Heller-Swan decomposition for regular rings)

Suppose that R is regular. Then

- $$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \leq -1; \\ \mathrm{NK}_n(R) &= 0 \quad \text{for } n \in \mathbb{Z}; \end{aligned}$$

- *The Bass-Heller-Swan decomposition reduces for $n \in \mathbb{Z}$ to the natural isomorphism*

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Example (Eilenberg swindle)

- Consider a ring R. Let $\mathcal{P}(R)$ be the additive category of finitely generated projective R-modules.
- Suppose that there exists a functor $S\colon \mathcal{P}(R) \to \mathcal{P}(R)$ of additive categories together with a natural equivalence $S \oplus \mathrm{id}_{\mathcal{P}(R)} \xrightarrow{\cong} S$.
- Then $K_n(R) = 0$ for $n \in \mathbb{Z}$ since $K_n(S) + \mathrm{id}_{K_n(R)} = K_n(S \oplus \mathrm{id}_{\mathcal{P}(R)}) = K_n(S)$ holds.

Exercise

Let R be a ring. Consider the ring E of R-endomorphisms of $\bigoplus_{i\in\mathbb{N}} R$. Show that $K_n(E) = 0$ for $n \in \mathbb{Z}$.

- Notice the similarity between following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

- If G and K are groups, then we have the following formulas, which also look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G * K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G * K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

Cliffhanger

Question (K-theory of group rings and group homology)

Is there a relationship between $K_n(RG)$ and the group homology of G?

To be continued

Stay tuned

Next talk: Thursday 14:30