The Isomorphism Conjectures in the torsion free case (Lecture III)

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Flashback

- We introduced $K_n(R)$ for $n \in \mathbb{Z}$.
- We discussed the topological relevance of $K_0(RG)$ and the Whitehead group $\operatorname{Wh}(G)$, e.g., the finiteness obstruction and the s-cobordism theorem.
- We stated the conjectures that $\widetilde{K}_0(\mathbb{Z}G)$ and $\operatorname{Wh}(G)$ vanish for torsion free G.
- We presented the Bass-Heller-Swan decomposition and indicated some similarities between $K_n(RG)$ and group homology.
- Cliffhanger

Question (K-theory of group rings and group homology)

Is there a relationship between $K_n(RG)$ and the group homology of G?

Outline

- We introduce spectra and how they yield homology theories.
- We state the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsion free groups.
- We discuss applications of these conjectures, such as the Kaplansky Conjecture and the Borel Conjecture.
- We explain that the formulations for torsion free groups cannot extend to arbitrary groups.

Definition (Spectrum)

A spectrum

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called structure maps

$$\sigma(n) \colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

- Given two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$, their one-point-union and their smash product are defined to be the pointed spaces

$$\begin{aligned} X \vee Y &:= \{(x, y_0) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\} \subseteq X \times Y; \\ X \wedge Y &:= (X \times Y)/(X \vee Y). \end{aligned}$$

- If X is a pointed space and $\mathbf{E}$ is a spectrum, then we obtain a new spectrum by $X \wedge \mathbf{E}$.

Exercise

Show $S^{n+1} \cong S^n \wedge S^1$.

Definition (Homotopy groups of a spectrum)

Given a spectrum **E**, define for $n \in \mathbb{Z}$ its *n-th homotopy group*

$$\pi_n(\mathbf{E}) := \operatorname*{colim}_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$

$$\cdots \xrightarrow{\sigma(k-1)_*} \pi_{k+n}(E(k)) \xrightarrow{\sigma(k)_*} \pi_{k+n+1}(E(k+1)) \xrightarrow{\sigma(k+1)_*} \cdots .$$

- Notice that a spectrum, in contrast to a space, can have non-trivial negative homotopy groups.

- Algebraic K-theory spectrum

  For a ring R, there is the algebraic K-theory spectrum $\mathbf{K}_R$ with the property

  $$\pi_n(\mathbf{K}_R) = K_n(R) \quad \text{for } n \in \mathbb{Z}.$$

- Algebraic L-theory spectrum

  For a ring with involution R, there is the algebraic L-theory spectrum $\mathbf{L}_R^{\langle -\infty \rangle}$ with the property

  $$\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R) \quad \text{for } n \in \mathbb{Z}.$$

Definition (Homology theory)

A homology theory $\mathcal{H}_*$ is a covariant functor from the category of *CW*-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Theorem (Homology theories and spectra)

Let $\mathbf{E}$ *be a spectrum.*
Then we obtain a homology theory $H_*(-;\mathbf{E})$ *by*

$$H_n(X,A;\mathbf{E}) := \pi_n\left((X \cup_A \operatorname{cone}(A)) \wedge \mathbf{E}\right).$$

It satisfies

$$H_n(pt;\mathbf{E}) = \pi_n(\mathbf{E}).$$

The Isomorphism Conjectures for torsion free groups

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsion free groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsion free group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of R;
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$ for $n \in \mathbb{Z}$.
- BG is the classifying space of the group G.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsion free groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsion free group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture for torsion free groups)

The Baum-Connes Conjecture for the torsion free group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for every $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G.

Exercise

Let G be the fundamental group of a closed orientable 2-manifold. Compute $K_n(BG)$.

Conclusions of the Isomorphism Conjectures for torsion free groups

- In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$, respectively $\mathcal{FJ}_L(R)$, be the class of groups that satisfy the K-theoretic, respectively L-theoretic, Farrell-Jones Conjecture for the coefficient ring R.
- Let $\mathcal{BC}$ be the class of groups that satisfy the Baum-Connes Conjecture.

Lemma

Let R be a regular ring. Suppose that G is torsion free and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

Lemma

Suppose that G is torsion free and $G \in \mathcal{FJ}_K(\mathbb{Z})$. *Then the Whitehead group* $\mathrm{Wh}(G)$ *is trivial.*

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

□

In particular, for a torsion free group $G \in \mathcal{FJ}_K(\mathbb{Z})$ we get:

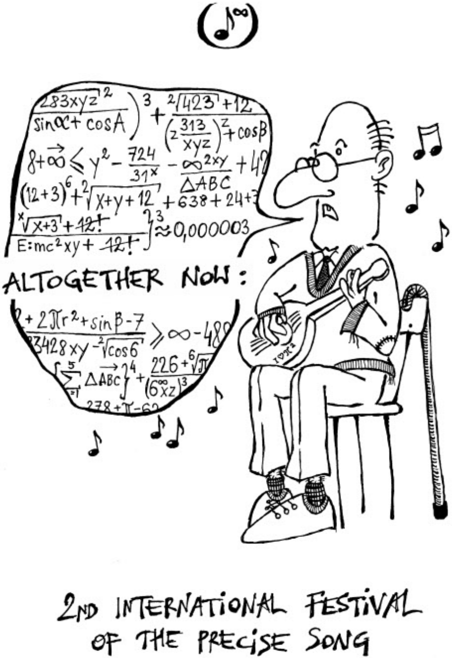
- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;

- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;

- Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;

- If G belongs to $\mathcal{FJ}_K(\mathbb{Z})$, then it is of type FF if and only if it is of type FP (Serre's problem).

Mathematicians!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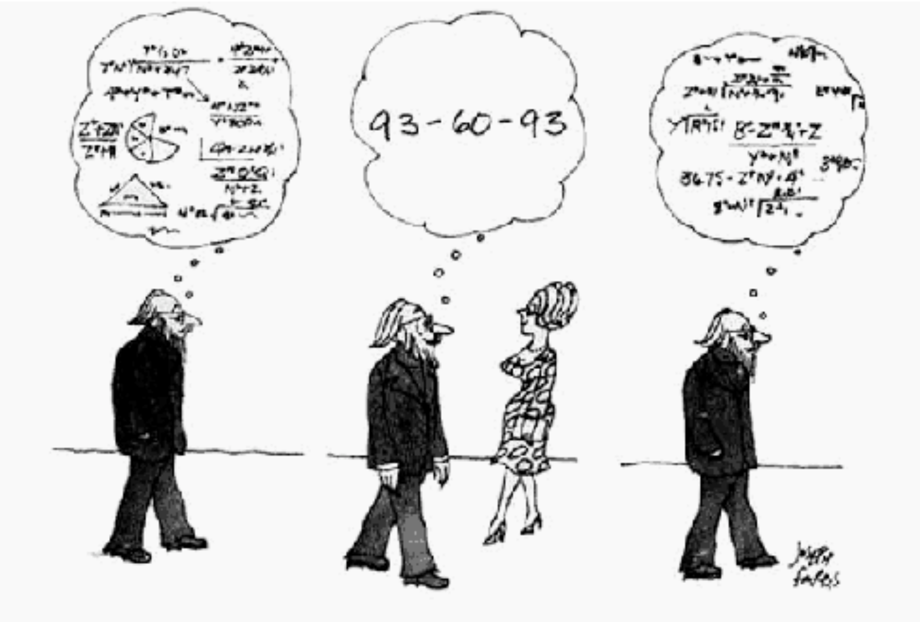
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... und was kostet eine Kurvendiskussion?
KITTIHAWK

What my parents think I do

What my friends think I do

What my students think I do

What my spouse thinks I do

What my colleagues think I do

What I actually do

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says that for a torsion free group G and an integral domain R the elements 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture)

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero, and G is torsion free;*
- *G is torsion free and sofic;*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic;*

Then 0 and 1 are the only idempotents in FG.

Proof.

- We only treat the case of fields of characteristic zero.
- Let p be an idempotent in FG. We want to show $p \in \{0, 1\}$.
- Denote by $\epsilon \colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. It suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p, which is a finitely generated projective FG-module.
- Since $G \in \mathcal{FJ}_K(F)$, we can conclude that

$$i_* \colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is surjective.
- Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

$$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

Proof (continued).

- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$, then we obtain

$$FG^m \cong i_*(P) \oplus FG^n.$$

- Inserting this in the first equation yields

$$(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.$$

- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$.

□

Exercise

Let p be a prime. Find all idempotents in $R[\mathbb{Z}/p]$ for $R = \mathbb{Z}$, $R = \mathbb{C}$ and $R = \mathbb{F}_p$.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- In particular the Borel Conjecture predicts that two closed aspherical manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.
  A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones.
- There are also non-aspherical manifolds that are topologically rigid in the sense of the Borel Conjecture (see Kreck-L.).

Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.

Exercise

Prove the Borel Conjecture in dimensions 1 and 2.

Definition (Structure set)

The structure set $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

Theorem (Ranicki)

There is an exact sequence of abelian groups, called the algebraic surgery exact sequence, for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; \mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M; \mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M; \mathbf{L}\langle 1\rangle) \to H_k(M; \mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$.

What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above are false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict that for a finite group G

$$K_0(BG) \cong K_0(C_r^*(G)) \cong R_{\mathbb{C}}(G).$$

However, $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} K_0(\text{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ and $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ holds if and only if G is trivial.

- If G is torsion free, then the version of the K-theoretic Farrell-Jones Conjecture predicts

$$\begin{aligned} H_n(B\mathbb{Z};\mathbf{K}_R) = H_n(S^1;\mathbf{K}_R) = H_n(\mathrm{pt};\mathbf{K}_R) \oplus H_{n-1}(\mathrm{pt};\mathbf{K}_R) \\ = K_n(R) \oplus K_{n-1}(R) \cong K_n(R\mathbb{Z}). \end{aligned}$$

  In view of the Bass-Heller-Swan decomposition this is only possible if $\mathrm{NK}_n(R)$ vanishes which is true for regular rings R but not for general rings R.

Cliffhanger

Question (Arbitrary groups and rings)

Are there versions of the Farrell-Jones Conjecture for arbitrary groups and rings and of the Baum-Connes Conjecture for arbitrary groups?

To be continued

Stay tuned

Next talk: Friday 9:15