# The relation between the Baum-Connes Conjecture and the Trace Conjecture

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**Conjecture 1 (Baum-Connes Conjecture for $G$)** *The assembly map*

$$\text{asmb} : K_p^G(\underline{E}G) \to K_p(C_r^*(G))$$

*which sends* $[M, P^*]$ *to* $\text{index}_{C_r^*(G)}(P^*)$ *is bijective.*

- $C_r^*(G)$ is the reduced $C^*$-algebra of $G$;

- $K_p(C_r^*(G))$ is the topological $K$-theory of $C_r^*(G)$. This is for $p = 0$ the same as the algebraic $K$-group. So elements in $K_0(C_r^*(G))$ are represented by finitely generated modules over the ring $C_r^*(G)$;

- $\underline{E}G$ is the classifying space for proper $G$-actions. It is characterized uniquely up to $G$-homotopy by the property that it is a $G$-$CW$-complex whose isotropy groups are all finite and whose $H$-fixed point sets for $H \subset G$ are contractible. If $G$ is torsionfree, this coincides with $EG$;

- $K_p^G(X)$ for a proper $G$-$CW$-complex $X$ is the equivariant $K$-homology of $X$ as defined for instance by Kasparov. If $G$ acts freely on $X$, there is a canonical isomorphism

$$K_0^G(X) \xrightarrow{\cong} K_0(G\backslash X)$$

to the $K$-homology of $G\backslash X$. For $H \subset G$ finite, $K_0^G(G/H)$ is $\mathrm{Rep}_{\mathbb{C}}(H)$.

An element in $K_p^G(\underline{E}G)$ is given by a pair $(M, P^*)$ which consists of a smooth manifold with proper cocompact $G$-action and an elliptic $G$-complex $P^*$ of differential operators of order 1;

- $\mathrm{index}_{C_r^*(G)}$ is the $C_r^*(G)$-valued index due to Mishchenko and Fomenko;

Next we explain the relevance of the Baum-Connes Conjecture.

- Since $K_p^G(-)$ is an equivariant homology theory for proper $G$-$CW$-complexes, it is much easier to compute $K_p^G(\underline{E}G)$ than to compute $K_p(C_r^*(G))$;

- Novikov-Conjecture for $G$

  The Hirzebruch signature formula says

$$\text{sign}(M) \ = \ \langle \mathcal{L}(M), [M] \rangle.$$

  Given a map $f : M \to BG$ and $x \in H^*(BG)$, define the higher signature by

$$\text{sign}_x(M, f) \ = \ \langle f^*(x) \cup \mathcal{L}(M), [M] \rangle.$$

  The Novikov Conjecture says that these are homotopy invariants, i.e. for $f : M \to BG$, $g : N \to BG$ and a homotopy equivalence $u : M \to N$ with $g \circ u \simeq f$ we have

$$\text{sign}_x(M, f) \ = \ \text{sign}_x(N, g).$$

  The Baum-Connes Conjecture for $G$ implies the Novikov Conjecture for $G$.

- Stable Gromov-Lawson-Rosenberg Conjecture for $G$

  Let $M$ be a closed Spin-manifold with fundamental group $G$ of dimension $\geq$ 5. Let $B$ be the Bott manifold. Then $M \times B^k$ carries a Riemannian metric of positive scalar curvature for some $k \geq 0$ if and only if

  $$\text{index}_{C_r^*(G)}(\widetilde{M}, \widetilde{D}) = 0.$$

  Here $D$ is the Dirac operator and $\widetilde{D}$ its lift to $\widetilde{M}$.

  Stolz has shown that the Baum-Connes Conjecture for $G$ implies the stable Gromov-Lawson-Rosenberg Conjecture for $G$. The unstable version of the Gromov-Lawson-Rosenberg Conjecture , i.e. $k = 0$, is false in general by a construction of Schick;

**Conjecture 2 (Trace Conjecture for $G$)** *The image of the composite*

$$K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*is the additive subgroup of $\mathbb{R}$ generated by $\{\frac{1}{|H|} \mid H \subset G, |H| < \infty\}$. Here $\mathcal{N}(G)$ is the group von Neumann algebra and $\mathrm{tr}_{\mathcal{N}(G)}$ the von Neumann trace.*

Notice that $\mathbb{C}[G] \subset C_r^*(G) \subset \mathcal{N}(G)$ and equality holds if and only if $G$ is finite.

**Conjecture 3 (Kadison Conjecture for $G$)** *Let $G$ be torsionfree. Let $p \in C_r^*(G)$ be an idempotent, i.e. $p^2 = p$. Then $p = 0, 1$.*

**Lemma 4** *The Trace Conjecture for $G$ implies the Kadison Conjecture for torsion-free $G$.*

Proof:

$$\begin{aligned} 0 \le p \le 1 \ &\Rightarrow\ 0 = \mathrm{tr}(0) \le \mathrm{tr}(p) \le \mathrm{tr}(1) = 1 \\ &\Rightarrow\ \mathrm{tr}(p) \in \mathbb{Z} \cap [0,1] \ \Rightarrow\ \mathrm{tr}(p) = 0, 1 \\ &\Rightarrow\ \mathrm{tr}(p) = \mathrm{tr}(0), \mathrm{tr}(1) \ \Rightarrow\ p = 0, 1. \end{aligned}$$

**Lemma 5** *Let $G$ be torsionfree. Then the Baum-Connes Conjecture for $G$ implies the Trace Conjecture for $G$.*

Proof: The following diagram commutes

$$\begin{array}{ccccccc} K_0^G(EG) & \longrightarrow & K_0(C_*^r(G)) & \longrightarrow & K_0(\mathcal{N}(G)) & \longrightarrow & \mathbb{R} \\ \downarrow \cong & & & & & & \uparrow \\ K_0(BG) & \longrightarrow & K_0(*) & & \xrightarrow{\quad\cong\quad} & & \mathbb{Z} \end{array}$$

This follows from the Atiyah index theorem. Namely, the upper horizontal composite sends $[M, P^*] \in K_0^G(EG)$ to the $L^2$-index in the sense of Atiyah

$$L^2-\text{index}(M, P^*) \in \mathbb{R},$$

the right vertical arrow sends $[M, P^*]$ to $[G\backslash M, G\backslash P^*]$ and the lower horizontal composite sends $[G\backslash M, G\backslash P^*]$ to the ordinary index

$$\text{index}(G\backslash M, G\backslash P^*) \in \mathbb{Z}.$$

The $L^2$-index theorem of Atiyah says

$$L^2-\text{index}(M, P^*) \;=\; \text{index}(G\backslash M, G\backslash P^*).$$

**Theorem 6 (Roy 99)** *The Trace Conjecture is false in general.*

Proof: Define an algebraic smooth variety

$$M = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^{15} + z_1^{15} + z_2^{15} + z_3^{15} = 0\}.$$

The group $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ acts on it by

$$[z_0, z_1, z_2, z_3] \mapsto [\exp(2\pi i/3) \cdot z_0, z_1, z_2, z_3]$$
$$[z_0, z_1, z_2, z_3] \mapsto [z_0, z_3, z_1, z_2]$$

One obtains

$$M^G = \emptyset;$$
$$\text{sign}(M) = -1105;$$
$$\pi_1(M) = \{1\}.$$

An equivariant version of a construction due to Davis and Januszkiewicz yields

- A closed oriented aspherical manifold $N$ with $G$-action;
- A $G$-map $f : N \to M$ of degree one;
- An isomorphism $f^*TM \cong TN$.

There is an extension of groups

$$1 \to \pi = \pi_1(N) \to \Gamma \xrightarrow{p} G \to 1$$

and a $\Gamma$-action on $\widetilde{N}$ extending the $\pi$-action on $\widetilde{N}$ and covering the $G$-action on $N$. We compute using the Hirzebruch signature formula

$$\begin{aligned} &\operatorname{sign}(N) = \langle \mathcal{L}(N), [N] \rangle = \langle f^*\mathcal{L}(M), [N] \rangle \\ &= \langle \mathcal{L}(M), f_*([N]) \rangle = \langle \mathcal{L}(M), [M]) \rangle = \operatorname{sign}(M). \end{aligned}$$

Next we prove that any finite subgroup $H \subset \Gamma$ satisfies

$$|H| \in \{1, 3\}.$$

Since $\widetilde{N}$ turns out to be a CAT(0)-space, any finite subgroup $H \subset \Gamma$ has a fixed point by a result of Bruhat and Tits. This implies

$$\widetilde{N}^H \neq \emptyset \Rightarrow N^{p(H)} \neq \emptyset \Rightarrow M^{p(H)} \neq \emptyset \Rightarrow p(H) \neq G.$$

Since $\pi_1(N)$ is torsionfree, $p|_H : H \to p(H)$ is bijective.

On $\widetilde{N}$ we have the signature operator $\widetilde{S}$. We claim that the composite

$$K_0^\Gamma(\underline{E}\Gamma) \xrightarrow{\text{asmb}} K_0(C_r^*(\Gamma)) \to K_0(\mathcal{N}(\Gamma)) \xrightarrow{\text{tr}_{\mathcal{N}(\Gamma)}} \mathbb{R}$$

sends $[\widetilde{N}, \widetilde{S}]$ to

$$\frac{1}{[\Gamma : \pi]} \cdot \text{sign}(N) \;=\; \frac{-1105}{9}.$$

The Trace Conjecture for $\Gamma$ says

$$\frac{-1105}{9} \in \{r \in \mathbb{R} \mid 3 \cdot r \in \mathbb{Z}\}.$$

This is not true (by some very deep number theoretic considerations).

**Conjecture 7 (Modified Trace Conjecture)** *Let $\Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting the orders of finite subgroups of $G$. Then the image of composite*

$$K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*is contained in $\Lambda^G$.*

**Theorem 8 (L 01)** *The image of the composite*

$$K_0^G(\underline{E}G) \xrightarrow{\text{asmb}} K_0(C_r^*(G)) \\ \to K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

*is contained in* $\Lambda^G$.

*In particular the Baum-Connes Conjecture for* $G$ *implies the Modified Trace Conjecture for* $G$.

**Theorem 9 (Generalized $L^2$-index theorem (L 01))** *The following diagram commutes*

$$\begin{array}{ccccccc} K_0^G(EG) & \longrightarrow & K_0^G(\underline{E}G) & \longrightarrow & K_0(C_r^*(G)) & \longrightarrow & K_0(\mathcal{N}(G)) \\ \downarrow \cong & & & & & & \uparrow \\ K_0(BG) & \longrightarrow & K_0(*) & & \xrightarrow{\cong} & & K_0(\mathcal{N}(1)) \end{array}$$

*or, equivalently, we get for a free cocompact* $G$*-manifold* $M$ *with elliptic* $G$*- complex* $P^*$ *of differential operators of order* 1 *in* $K_0(\mathcal{N}(G))$

$$\text{index}_{\mathcal{N}(G)}(M, P^*) = \text{index}(G\backslash M, G\backslash P^*)\cdot[\mathcal{N}(G)].$$

**Example 10** Let $M$ be a closed oriented $4k$-dimensional manifold. Suppose that the finite group $G$ acts on $M$ freely and orientation preserving. Define the *equivariant signature*

$$\mathrm{sign}^G(M) \in \mathrm{Rep}_{\mathbb{C}}(G)$$

by

$$\mathrm{sign}^G(M) = \left[H_{2k}(M;\mathbb{C})^+\right] - \left[H_{2k}(M;\mathbb{C})^-\right].$$

Then the theorem above implies the well-known statement that for a free $G$-action we get

$$\begin{aligned} \mathrm{sign}^G(M) &= \mathrm{sign}(G\backslash M) \cdot [\mathbb{C}G]; \\ \mathrm{sign}(M) &= |G| \cdot \mathrm{sign}(G\backslash M). \end{aligned}$$

**Theorem 11 (Artin's Theorem)** *Let $G$ be finite. Then the map*

$$\bigoplus_{C\subset G} \mathrm{ind}_C^G : \bigoplus_{C\subset G} \mathrm{Rep}_{\mathbb{C}}(C) \to \mathrm{Rep}_{\mathbb{C}}(G)$$

*is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of $G$.*

Let $C$ be a finite cyclic group. The *Artin defect* is the cokernel of the map

$$\bigoplus_{D\subset C, D\neq C} \mathrm{ind}_D^C : \bigoplus_{D\subset C, D\neq C} \mathrm{Rep}_{\mathbb{C}}(D) \to \mathrm{Rep}_{\mathbb{C}}(C).$$

For an appropriate idempotent

$$\theta_C \in \mathrm{Rep}_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

the Artin defect becomes after inverting the order of $|C|$ canonically isomorphic to

$$\theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

**Theorem 12 (L 01)** *Let $X$ be a proper $G$-$CW$-complex. For a finite cyclic subgroup $C \subset G$ let $(C)$ be its conjugacy class, $N_GC$ its normalizer, $C_GC$ its centralizer and $W_GC = N_GC/C_GC$. Then there is a natural isomorphism called* equivariant Chern character

$$\begin{array}{c} \bigoplus_{(C)} K_p(C_GC\backslash X^C) \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\ \Big\downarrow {\scriptstyle \mathrm{ch}^G}\ \cong \\ K_p^G(X) \otimes_{\mathbb{Z}} \Lambda^G \end{array}$$

**Example 13** Suppose that $G$ is torsion-free. Then the trivial subgroup $\{1\}$ is the only finite cyclic subgroup of $C$. We have $C_G\{1\} = N_G\{1\} = G$ and $W_G\{1\} = \{1\}$. We get an isomorphism

$$\begin{array}{c} \bigoplus_{(C)} K_p(C_GC\backslash X^C) \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\ \downarrow \cong \\ K_p(G\backslash X) \otimes_{\mathbb{Z}} \mathbb{Z} \\ \downarrow \cong \\ K_p(G\backslash X) \end{array}$$

Under this identification the inverse of $\mathrm{ch}^G$ becomes the canonical isomorphism

$$K_p^G(X) \xrightarrow{\cong} K_p(G\backslash X).$$

**Example 14** Let $G$ be finite and $X = \{*\}$. Then we get an improvement of Artin's theorem, namely, the equivariant Chern character induces an isomorphism

$$\begin{array}{c} \bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right] \\ \mathrm{ch}^G \downarrow \cong \\ \mathrm{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right] \end{array}$$

**Example 15** Take $G$ to be any (discrete) group and $X = \underline{E}G$. There is a natural isomorphism

$$K_p(BC_GC) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_p(C_GC \backslash (\underline{E}G)^C) \otimes_{\mathbb{Z}} \Lambda^G.$$

The equivariant Chern character induces an isomorphism

$$\begin{array}{c} \bigoplus_{(C)} K_p(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\ \mathrm{ch}^G \Big\downarrow \cong \\ K_p^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda^G \end{array}$$

**Corollary 16** *The ordinary Chern character induces for a $CW$-complex $Y$ an isomorphism*

$$\oplus_k H_{2k+p}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_p(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*If the Baum-Connes Conjecture holds for $G$, then we obtain an isomorphism*

$$\begin{array}{c} \bigoplus_{(C)} \bigoplus_k H_{p+2k}(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathrm{ch}^G \Big\downarrow \cong \\ K_p^G(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

Let $X$ be a proper $G$-$CW$-complex. Define two homomorphisms

$$\begin{array}{c} \bigoplus_{(C)} K_0(C_GC\backslash X^C) \otimes_{\mathbb{Z}[W_GC]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\ \downarrow \xi_i \\ K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G \end{array}$$

as follows. The first one is the composition of the equivariant Chern character with the assembly map

$$\mathrm{asmb}^G \otimes \mathrm{id} : K_0^G(X) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G$$

and the change of rings homomorphism

$$K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G.$$

This is the homomorphism which we want to understand. In particular we are interested in its image. We want to identify it with the easier to compute homomorphism $\xi_2$.

The homomorphism $\xi_2$ is induced by the composition

$$\begin{array}{c}
\bigoplus_{(C)} K_0(C_G C\backslash X^C) \otimes_{\mathbb{Z}} \theta_c \cdot \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\
\Big\downarrow {\scriptstyle \bigoplus_{(C)} K_0(\mathrm{pr}) \otimes_{\mathbb{Z}} \mathrm{incl}} \\
\bigoplus_{(C)} K_0(*) \otimes_{\mathbb{Z}} \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\
\Big\downarrow \cong \\
\bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}} \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\
\Big\downarrow \cong \\
\bigoplus_{(C)} \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \\
\Big\downarrow {\scriptstyle \bigoplus_{(C)} \mathrm{ind}_C^G} \\
K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G
\end{array}$$

The proof of the next result uses the generalized $L^2$-Atiyah index theorem.

**Theorem 17** *Let $X$ be a proper $G$-$CW$-complex. Then the maps $\xi_1$ and $\xi_2$ agree.*

**Theorem 18** *The image of the composite*

$$K_0(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \\ \to K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G$$

*is contained in the image of*

$$\bigoplus_{(C)} \mathrm{ind}_C^G : \bigoplus_{(C)} \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \to K_0(\mathcal{N}(G) \otimes_{\mathbb{Z}} \Lambda^G.$$

**Remark 19** If we compose the second map above with

$$\mathrm{tr}_{\mathcal{N}(G)} : K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G \to \mathbb{R}$$

it is easy to see that its image is contained in $\Lambda^G$. Hence the following composition has $\Lambda^G$ as image

$$K_0^G(\underline{E}G) \xrightarrow{\mathrm{asmb}} K_0(C_r^*(G)) \\ \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}.$$