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A Caveat on the Isomorphism Conjecture in L-theory

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Abstract. The Isomorphism Conjecture of Farrell and Jones for L-theory [5] has only been formulated for $L^{-\infty}$ and in this formulation been proved for a large class of groups, for instance for discrete cocompact subgroups of a virtually connected Lie group. The question arises whether the corresponding conjecture is true for L^{ε} for the decorations $\varepsilon = p, h, s$. We give examples showing that it fails for any of the decorations $\varepsilon = p, h, s$. The groups involved are of the shape $\mathbb{Z}^2 \times F$ for finite F.

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We first summarize the Farrell-Jones-Conjecture for L-theory as far as needed here. We will follow the notation and setup of [6], which is different, but equivalent to the original setup in [5], [8], and slightly more convenient for our purposes. Let G be a group. A *family* $\mathscr{F}$ is a class of subgroups of G which is closed under conjugation and taking subgroups. Our main examples will be the families $\mathscr{VC}$ of virtually cyclic subgroups, i.e. subgroups which are either finite or contain $\mathbb{Z}$ as a normal subgroup of finite index, and the family $\mathscr{ALL}$ of all subgroups. Let L_n^{ε} for $\varepsilon \in \mathbb{Z}$, $\varepsilon \leq 2$ and $\varepsilon = -\infty$ be defined as in [9, §17]. Notice that with this convention $L_n^2 = L_n^s$, $L_n^1 = L_n^h$ and $L_n^0 = L_n^p$ holds. Denote by $\mathbf{L}^{\varepsilon}$ the corresponding $\mathrm{Or}(G)$-spectrum, i.e. covariant functor from the orbit category $\mathrm{Or}(G)$ to the category of spectra. It has the property that $\pi_n(\mathbf{L}^{\varepsilon}(G/H)) = L_n^{\varepsilon}(\mathbb{Z}H)$. An $\mathrm{Or}(G)$-space X is a contravariant functor X from $\mathrm{Or}(G)$ to the category of topological spaces. Associated to X and $\mathbf{L}^{\varepsilon}$ there are homology groups $H_n^G(X; \mathbf{L}^{\varepsilon})$ (see [4], [6, Section 1] for more details). Given a family $\mathscr{F}$, we denote by $\star_{G,\mathscr{F}}$ the $\mathrm{Or}(G)$-space, which sends G/H to the space consisting of one point, if $H \in \mathscr{F}$, and to the empty set, if $H \notin \mathscr{F}$. There is an obvious map $\star_{G,\mathscr{VC}} \to \star_{G,\mathscr{ALL}}$ and an obvious identification of $H_n^G(\star_{G,\mathscr{ALL}}; \mathbf{L}^{\varepsilon})$ with $\pi_n(\mathbf{L}^{\varepsilon}(G/G)) = L_n(\mathbb{Z}G)$. Thus we obtain a map, called *assembly map*

$$(1) \qquad H_n^G(\star_{G,\mathscr{VC}}; \mathbf{L}^{\varepsilon}) \to H_n^G(\star_{G,\mathscr{ALL}}; \mathbf{L}^{\varepsilon}) = L_n^{\varepsilon}(\mathbb{Z}G).$$

The *Farrell-Jones-Conjecture* says that (1) is an isomorphism for all $n \in \mathbb{Z}$, if $\varepsilon = -\infty$. We want to give examples which show that (1) is not an isomorphism for all $n \in \mathbb{Z}$, if

ε is p, h or s, or in other words, that the version of the Farrell-Jones-Conjecture for $\varepsilon = p, h, s$ is not true.

Let F be a finite group. Put $G = \mathbb{Z}^2 \times F$. Let $\mathcal{K}$ be the set of all subgroups $K \subset \mathbb{Z}^2$ such that $\mathbb{Z}^2/K \cong \mathbb{Z}$. In other words, an element in $\mathcal{K}$ is a subgroup $K \subset \mathbb{Z}^2$ which is neither trivial nor $\mathbb{Z}^2$ and has the property that for $x \in \mathbb{Z}^2$ and $n \in \mathbb{Z}$, $n \neq 0$ the implication $n \cdot x \in K \Rightarrow x \in K$ holds. Let $\mathcal{SUB}(F)$ and $\mathcal{SUB}(K \times F)$ be the family of all subgroups of F and $K \times F$. Consider the following diagram of $\mathrm{Or}(G)$-spaces

$$\begin{array}{ccc}
\coprod_{K \in \mathcal{K}} \star_{G, \mathcal{SUB}(F)} & \longrightarrow & \coprod_{K \in \mathcal{K}} \star_{G, \mathcal{SUB}(K \times F)} \\
\big\downarrow & & \big\downarrow \\
\star_{G, \mathcal{SUB}(F)} & \longrightarrow & \star_{G, \mathcal{VC}}
\end{array}$$

where the horizontal arrows are the obvious inclusions and the vertical arrows are the disjoint unions of the obvious inclusions. Any finite subgroup $H \subset G$ is already contained in F. Let $V \subset G$ be an infinite virtually cyclic subgroup. Define $\overline{V} \subset \mathbb{Z}^2$ to be the subgroup of elements $x \in \mathbb{Z}^2$ for which there exists $n \in \mathbb{Z}$, $n \neq 0$ with $n \cdot x \in V$. Then $\overline{V} \in \mathcal{K}$ and $V \in \mathcal{SUB}(\overline{V} \times F)$. If $K \in \mathcal{K}$ satisfies $V \subset K \times F$, then $\overline{V} = K$. Now one easily checks that the diagram of $\mathrm{Or}(G)$-spaces above evaluated at a subgroup $H \subset G$ looks as follows

$$\begin{array}{ccc}
\coprod_{K \in \mathcal{K}} * & \xrightarrow{\mathrm{Id}} & \coprod_{K \in \mathcal{K}} * \\
\big\downarrow & & \big\downarrow \\
* & \xrightarrow{\mathrm{Id}} & *
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \longrightarrow & * \\
{\scriptstyle \mathrm{Id}}\big\downarrow & & {\scriptstyle \mathrm{Id}}\big\downarrow \\
\emptyset & \longrightarrow & *,
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \xrightarrow{\mathrm{Id}} & \emptyset \\
{\scriptstyle \mathrm{Id}}\big\downarrow & & {\scriptstyle \mathrm{Id}}\big\downarrow \\
\emptyset & \xrightarrow{\mathrm{Id}} & \emptyset,
\end{array}$$

$$\text{if } |H| < \infty, \qquad \text{if } |H| = \infty,\ H \in \mathcal{VC}, \qquad \text{else,}$$

where $*$ denotes the space consisting of one point. In each case we get a pushout of spaces whose upper horizontal arrow is a cofibration. Hence by excision the inclusion induces an isomorphism [4, Lemma 4.4], [6, Lemma 1.8]

$$(2) \qquad \bigoplus_{K \in \mathcal{K}} H_n^G(\star_{G, \mathcal{SUB}(K \times F)}, \star_{G, \mathcal{SUB}(F)}; \mathbf{L}^\varepsilon) \xrightarrow{\cong} H_n^G(\star_{G, \mathcal{VC}}, \star_{G, \mathcal{SUB}(F)}; \mathbf{L}^\varepsilon).$$

We get from [6, Lemma 2.7] isomorphisms

$$H_n^G(\star_{G, \mathcal{SUB}(F)}; \mathbf{L}^\varepsilon) \cong H_n(B(G/F); \mathbf{L}^\varepsilon(\mathbb{Z}F));$$

$$H_n^G(\star_{G, \mathcal{SUB}(K \times F)}; \mathbf{L}^\varepsilon) \cong H_n(B(G/K \times F); \mathbf{L}^\varepsilon(\mathbb{Z}[K \times F])),$$

where $H_n(X; \mathbf{L}^\varepsilon(\mathbb{Z}F))$ denotes the homology of a space X associated to the L-theory spectrum $\mathbf{L}^\varepsilon(\mathbb{Z}F)$ and analogously for $H_n(X; \mathbf{L}^\varepsilon(\mathbb{Z}[K \times F]))$. Since $B(G/F) = T^2$ and $B(G/K \times F) = S^1$, we obtain isomorphisms

$$(3) \qquad H_n^G(\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon) \cong L_n^\varepsilon(\mathbb{Z}F) \oplus L_{n-1}^\varepsilon(\mathbb{Z}F) \oplus L_{n-1}^\varepsilon(\mathbb{Z}F) \oplus L_{n-2}^\varepsilon(\mathbb{Z}F);$$

$$(4) \qquad H_n^G(\star_{G,\mathcal{SUB}(K\times F)};\mathbf{L}^\varepsilon) \cong L_n^\varepsilon(\mathbb{Z}[K\times F]) \oplus L_{n-1}^\varepsilon(\mathbb{Z}[K\times F]).$$

We claim that the two maps induced by the inclusions

$$(5) \qquad H_n^G(\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon)[1/2] \xrightarrow{\cong} H_n^G(\star_{G,\mathcal{VC}};\mathbf{L}^\varepsilon)[1/2] \xrightarrow{\cong} H_n^G(\star_{G,\mathcal{ALL}};\mathbf{L}^\varepsilon)[1/2]$$

are isomorphism. The first one is an isomorphism by (2), (3) and (4), the target of the second one can be computed directly from the splitting due to Shaneson [10] and Wall [11], which yields the claim. Notice that we have inverted 2 so that we do not have to worry about decorations by the Rothenberg sequence. One can also deduce the bijectivity of the two maps in (5) from more general results, namely, from [6, Theorem 2.4] and [13, Theorem 4.11]. There are isomorphisms coming from the splitting due to Shaneson [10] and Wall [11] for L^s and for other decoration from [9, §16]

$$(6) \qquad L_n^\varepsilon(\mathbb{Z}G) \cong L_n^\varepsilon(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}F) \oplus L_{n-2}^{\varepsilon-2}(\mathbb{Z}F);$$

$$(7) \qquad L_n^\varepsilon(\mathbb{Z}[\mathbb{Z}\times F]) \cong L_n^\varepsilon(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}F).$$

Since the L-groups of integral group rings of finite groups are finitely generated abelian groups [12], we conclude from (3) and (6) that $H_n^G(\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon)$ and $H_n^G(\star_{G,\mathcal{ALL}};\mathbf{L}^\varepsilon) = L_n^\varepsilon(\mathbb{Z}G)$ are finitely generated abelian groups. We conclude from (5) that the inclusion induces a 2-*isomorphism*, i.e. a map whose kernel and cokernel are finite 2-groups.

$$(8) \qquad H_n^G(\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon) \xrightarrow{\cong_2} H_n^G(\star_{G,\mathcal{ALL}};\mathbf{L}^\varepsilon).$$

This implies together with the long exact sequence of a pair of $\mathrm{Or}(G)$-spaces

Lemma 9. *Suppose that the assembly map* (1) *for* $\mathbf{L}^\varepsilon$ *is bijective for all* $n \in \mathbb{Z}$. *Then, for all* $n \in \mathbb{Z}$, *the inclusion induces a* 2-*isomorphism*

$$H_n^G(\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon) \xrightarrow{\cong_2} H_n^G(\star_{G,\mathcal{VC}};\mathbf{L}^\varepsilon)$$

and the group $H_n^G(\star_{G,\mathcal{VC}},\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon)$ *is a finite* 2-*group.*

Given $K \subset \mathcal{K}$, there is an isomorphism $G \xrightarrow{\cong} G$ which sends K bijectively to $0\times\mathbb{Z}$ and leaves F fixed. It induces an isomorphism

$$H_n^G(\star_{G,\mathcal{SUB}(K\times F)},\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon) \xrightarrow{\cong} H_n^G(\star_{G,\mathcal{SUB}(0\times\mathbb{Z}\times F)},\star_{G,\mathcal{SUB}(F)};\mathbf{L}^\varepsilon).$$

Thus we obtain from (2) an isomorphism

$$(10)\qquad \bigoplus_{K\in\mathscr{K}} H_n^G(\star_{G,\mathscr{SUB}(0\times\mathbb{Z}\times F)},\star_{G,\mathscr{SUB}(F)};\mathbf{L}^\varepsilon) \xrightarrow{\cong} H_n^G(\star_{G,\mathscr{VC}},\star_{G,\mathscr{SUB}(F)};\mathbf{L}^\varepsilon).$$

Since $\mathscr{K}$ is an infinite set, we conclude from Lemma 9 and (10) together with the long exact sequence of a pair of $\mathrm{Or}(G)$-spaces

Lemma 11. *Suppose that the assembly map* (1) *for* $\mathbf{L}^\varepsilon$ *is bijective for all* $n\in\mathbb{Z}$. *Then, for all* $n\in\mathbb{Z}$,

$$H_n^G(\star_{G,\mathscr{SUB}(0\times\mathbb{Z}\times F)},\star_{G,\mathscr{SUB}(F)};\mathbf{L}^\varepsilon) = 0$$

and the inclusion induces an isomorphism

$$(12)\qquad H_n^G(\star_{G,\mathscr{SUB}(F)};\mathbf{L}^\varepsilon) \xrightarrow{\cong} H_n^G(\star_{G,\mathscr{SUB}(0\times\mathbb{Z}\times F)};\mathbf{L}^\varepsilon).$$

Notice that we have identified already the source and the target of the homomorphism (12) in (3) and (4). Using furthermore the isomorphism (7) the homomorphism (12) can be identified with a homomorphism

$$(13)\qquad L_n^\varepsilon(\mathbb{Z}F)\oplus L_{n-1}^\varepsilon(\mathbb{Z}F)\oplus L_{n-1}^\varepsilon(\mathbb{Z}F)\oplus L_{n-2}^\varepsilon(\mathbb{Z}F)$$
$$\to L_n^\varepsilon(\mathbb{Z}F)\oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}F)\oplus L_{n-1}^\varepsilon(\mathbb{Z}F)\oplus L_{n-2}^{\varepsilon-1}(\mathbb{Z}F).$$

Example 14. Consider the cyclic group $G=\mathbb{Z}/29$ of order 29. Then we get from [1, Theorem 1, 3 and 5] and [2, Corollary 4.3 on page 58].

$$L_n^\varepsilon(\mathbb{Z}[\mathbb{Z}/29]) = 0 \qquad \varepsilon = h,s,p, n \text{ odd};$$

$$L_0^\varepsilon(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}^{15} \qquad \varepsilon = s,p;$$

$$L_2^\varepsilon(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}/2\oplus\mathbb{Z}^{14} \qquad \varepsilon = s,p;$$

$$L_0^h(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}^{15}\oplus\hat{H}^2(\mathbb{Z}/2;\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29]));$$

$$L_2^h(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}/2\oplus\mathbb{Z}^{14}\oplus\hat{H}^2(\mathbb{Z}/2;\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29])).$$

We have $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ [7, page 30]. Hence $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29])$ is $\mathbb{Z}[\mathbb{Z}/2]$-isomorphic either to $\mathbb{Z}/2[\mathbb{Z}/2]\oplus\mathbb{Z}/2$ or to $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$, where $\mathbb{Z}/2$ carries the trivial $\mathbb{Z}[\mathbb{Z}/2]$-structure. In both cases $\hat{H}^i(\mathbb{Z}/2;\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29]))$ is non-trivial for all $i\in\mathbb{Z}$. This implies that the homomorphism (13) is for no $n\in\mathbb{Z}$ an isomorphism, if ε is s or h. We conclude from Lemma 11 that the assembly map (1) is not an isomorphism for all $n\in\mathbb{Z}$, if ε is s or h and $G=\mathbb{Z}^2\times\mathbb{Z}/29$.

Remark 15. With some further effort the homomorphism (13) and hence the homomorphism (12) can be identified with

$$(16)\qquad (\mathrm{Id}, f, \mathrm{Id}, f) : L_n^{\varepsilon}(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon}(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon}(\mathbb{Z}F) \oplus L_{n-2}^{\varepsilon}(\mathbb{Z}F)$$
$$\to L_n^{\varepsilon}(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}F) \oplus L_{n-1}^{\varepsilon}(\mathbb{Z}F) \oplus L_{n-2}^{\varepsilon-1}(\mathbb{Z}F),$$

where f denotes the forgetful map given by lowering the decoration ε to $\varepsilon - 1$. Because of the Rothenberg sequence we see that (12) is bijective for all $n \in \mathbb{Z}$ if and only if $\hat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_{\varepsilon-1}(F)) = 0$ for all $n \in \mathbb{Z}$. We conclude from Lemma 11 that, given $\varepsilon \leq 2$, the assembly map (1) for $\mathbf{L}^{\varepsilon}$ is bijective for all $n \in \mathbb{Z}$, only if $\hat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_{\varepsilon-1}(F)) = 0$ for all $n \in \mathbb{Z}$. Since $\mathrm{Wh}_{-1}(\mathbb{Z}[\mathbb{Z}/6])$ is isomorphic to $\mathbb{Z}$ [3], $\hat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_{-1}(\mathbb{Z}/6)) = 0$ cannot be zero for all $n \in \mathbb{Z}$. Hence the assembly map (1) for $\mathbf{L}^p$ cannot be an isomorphism for all $n \in \mathbb{Z}$, if $G = \mathbb{Z}^2 \times \mathbb{Z}/6$.

Remark 17. One may speculate which family of subgroups one has to choose so that the Isomorphism Conjecture in L-theory has a chance to be true for all decorations ε. Of course the family $\mathcal{ALL}$ of all subgroups works, but the point is to find an efficient family. For instance we do not know whether the family of virtually polycyclic is enough. The smallest possible candidate is the family $\mathcal{B}$ of subgroups H of G for which there is a virtually cyclic group V and an integer $k \geq 0$ with $H \subset V \times \mathbb{Z}^k$. With this choice all the counterexamples above are taken care of, but for the trivial reason that for them $\mathcal{B} = \mathcal{ALL}$. The family $\mathcal{B}$ at least passes the following test. Namely, it can be shown that the splitting formula due to Shaneson and Wall is also true for the source of the assembly map

$$H_n^G(\star_{G,\mathcal{B}}; \mathbf{L}^{\varepsilon}) \to H_n^G(\star_{G,\mathcal{ALL}}; \mathbf{L}^{\varepsilon}) = L_n^{\varepsilon}(\mathbb{Z}G).$$

Namely, one has a canonical splitting compatible under the assembly map with the splitting formula due to Shaneson and Wall on the target

$$H_n^{G\times\mathbb{Z}}(\star_{G\times\mathbb{Z},\mathcal{B}}; \mathbf{L}^{\varepsilon}) = H_n^G(\star_{G,\mathcal{B}}; \mathbf{L}^{\varepsilon}) \oplus H_{n-1}^G(\star_{G,\mathcal{B}}; \mathbf{L}^{\varepsilon-1});$$
$$L_n^{\varepsilon}(\mathbb{Z}[G \times \mathbb{Z}]) = L_n^{\varepsilon}(\mathbb{Z}G) \oplus L_{n-1}^{\varepsilon-1}(\mathbb{Z}G).$$

Suppose that G is an extension $1 \to \mathbb{Z}^n \to G \to F \to 1$ such that F is finite and the conjugation action of F on $\mathbb{Z}^n$ is free outside the origin or that G is a cocompact Fuchsian group. Then the Farrell-Jones-Conjecture with respect to the family $\mathcal{VC}$ holds for $\mathbf{L}^{\varepsilon}$ for all decorations ε if it holds for $\varepsilon = -\infty$, as explained in [6, Remark 4.32, Remark 6.5]. This fits together with the fact that in the first case $\mathcal{B} = \mathcal{VC} \cup \mathcal{SUB}(\mathbb{Z}^n)$ and in the second case $\mathcal{B} = \mathcal{VC}$. This implies in both cases that the obvious map $H_n^G(\star_{G,\mathcal{VC}}; \mathbf{L}^{\varepsilon}) \to H_n^G(\star_{G,\mathcal{B}}; \mathbf{L}^{\varepsilon})$ is bijective for all $n \in \mathbb{Z}$ and $\varepsilon \leq 2$.

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