OBSTRUCTIONS TO FIBERING A MANIFOLD

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Dedicated to Bruce Williams on the occasion of his 60th birthday

ABSTRACT. Given a map $f\colon M \to N$ of closed topological manifolds we define torsion obstructions whose vanishing is a necessary condition for f being homotopy equivalent to a projection of a locally trivial fiber bundle. If $N = S^1$, these torsion obstructions are identified with the ones due to Farrell [6].

INTRODUCTION

Given a map $f\colon M \to N$ of closed topological manifolds we define torsion obstructions whose vanishing is a necessary condition for f being homotopy equivalent to a projection of a locally trivial fiber bundle. If $N = S^1$, these torsion obstructions are identified with the ones due to Farrell [6].

The basic idea of the construction is as follows. A simple structure ξ on a space Y with the homotopy type of a finite CW-complex is the choice of an equivalence class of homotopy equivalences $X \to Y$ with a finite CW-complex as domain, where we call two such maps $f_i\colon X_i \to Y$ for $i = 1, 2$ equivalent, if $f_2^{-1} \circ f_1\colon X_1 \to X_2$ is a simple homotopy equivalence. The classical theory of Whitehead torsion for homotopy equivalences between finite CW-complexes extends to homotopy equivalences of space with simple structures.

Consider a map $f\colon M \to N$ of topological manifolds whose homotopy fiber hofib(f) has the homotopy type of a finite CW-complex. Suppose for simplicity that M and N are connected. By inspecting the fiber transport of the fibration $\mu_f\colon \mathrm{FIB}(f) \to N$ associated to f, one obtains a homomorphism from $\pi(N)$ to the group of homotopy classes of selfhomotopy equivalences of the homotopy fiber. If we pick a simple structure on the homotopy fiber and consider the image of the Whitehead torsion of self homotopy equivalences of the homotopy fiber under $\mathrm{Wh}(\pi(\mathrm{hofib}(f))) \to \mathrm{Wh}(\pi(M))$, we obtain a homomorphism $\pi_1(N) \to \mathrm{Wh}(\pi(M))$. It defines an element

$$\Theta(p) \in H^1(N; \mathrm{Wh}(M)).$$

The element $\Theta(f)$ depends only on the homotopy class of f. If f is the projection of a locally trivial fiber bundle, then the fiber transport is given by homeomorphisms and by the topological invariance of Whitehead torsion this implies $\Theta(f) = 0$.

From now on suppose $\Theta(f) = 0$. Assume for simplicity of the outline that the Euler characteristic $\chi(N)$ of N is zero. Then one can construct a simple structure $\xi(\mathrm{FIB}(f))$ on $\mathrm{FIB}(f)$. This is obvious if the fibration $\mathrm{FIB}(f)$ is trivial since the cross product of a homotopy equivalence of finite CW-complexes with a finite CW-complex of Euler characteristic zero is simple. The general case is done by induction over the cells of B using a construction of a pushout simple structure and the fact that a fibration over D^n is trivial. Let $\mu_f\colon \mathrm{FIB}(f) \to M$ be the canonical homotopy equivalence. Since M is a topological manifold, it carries a preferred

simple structure. Hence the Whitehead torsion $\tau(\mu_f)$ of μ_f makes sense, and we define a second invariant

$$\tau_{\mathrm{fib}}(f) := \tau(\mu_f) \in \mathrm{Wh}(\pi(M)).$$

The element $\tau_{\mathrm{fib}}(f)$ depends only on the homotopy class of f. If f is the projection of a locally trivial fiber bundle, then $\tau_{\mathrm{fib}}(f) = 0$.

As an illustration we study the case, where the base space is S^1 and identify our invariants with the obstruction of fibering M over S^1 due to Farrell [6]. Notice that in this case Farrell [6] shows that the vanishing of the obstructions imply that f homotopic to the projection of a locally trivial fiber bundle provided $\dim(M) \geq 5$. For an arbitrary closed manifold N as target of f the vanishing of these obstructions will be necessary but not sufficient for f being homotopic to the projection of a locally trivial fiber.

We give a composition formula for τ_{fib}.

We introduce Poincaré torsion which is the obstruction for a finite Poincaré complex to be homotopy equivalent to a simple Poincaré complex.

Finally we briefly give a connection to the parametrized A-theoretic characteristic due to Dwyer-Weiss-Williams [4] and discuss some open questions.

The paper is organized as follows:

1. Simple structures and Whitehead torsion

In this section we extend the definition of the Whitehead torsion of homotopy equivalences between finite CW-complexes to homotopy equivalences between more general spaces, namely, spaces with simple structures.

Let Y be a space of the homotopy type of a finite CW-complex. We call two maps $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$ with finite CW-complexes as source and Y as target *simply equivalent* if the Whitehead torsion $\tau(f_2^{-1} \circ f_1 \colon X_1 \to X_2) \in \mathrm{Wh}(\pi(X_2))$ vanishes. (For the notion of Whitehead torsion and Whitehead group we refer to [3].)

Definition 1.1. *A* simple structure ξ *on a space Y with the homotopy type of a finite CW-complex is a choice of a simple equivalence class of homotopy equivalences $u \colon X \to Y$ with a finite CW-complex as source and Y as target. If Y is a finite CW-complex, we refer to the simple structure represented by id_Y as* canonical simple structure $\xi_{\mathrm{can}}(Y)$ *on Y.*

Let

$$Y_0 \xrightarrow{i_1} Y_1$$
$$\downarrow{\scriptstyle i_2} \qquad \downarrow{\scriptstyle j_1}$$
$$Y_2 \xrightarrow{j_2} Y$$

be a pushout of spaces with $i_1 \colon Y_0 \to Y_1$ a cofibration. Suppose that Y_i has the homotopy type of a finite CW-complex and comes with a simple structure ξ_i for $i = 0, 1, 2$. Then Y has the homotopy type of a finite CW-complex and there is a preferred simple structure ξ on Y which we will call *the pushout simple structure* and which is constructed as follows. Choose a pushout of finite CW-complexes

(1.2)
$$X_0 \xrightarrow{a_1} X_1$$
$$\downarrow{\scriptstyle a_2} \qquad \downarrow{\scriptstyle b_1}$$
$$X_2 \xrightarrow{b_2} X$$

together with homotopy equivalences $u_i \colon X_i \to Y_i$ representing ξ_i for $i = 0, 1, 2$ such that the maps a_1 and b_1 are inclusions of CW-subcomplexes, the maps a_2 and b_2 are cellular and the n-skeleton X_n of X is the subspace $b_1\big((X_1)_n\big) \cup b_2\big((X_2)_n\big)$ for every $n \geq -1$. The pushout property yields a map $u \colon X \to Y$ which is a homotopy equivalence. Let the pushout simple structure ξ be the one represented by u.

The proof that such a diagram (1.2) together with maps u_i exists and that ξ only depends on ξ_i and not on the choice of (X_i, u_i) can be found in [15, page 74 ff.].

Given two spaces (X, ξ) and (Y, η) with simple structures, the *product simple structure* $\xi \times \eta$ on $X \times Y$ is represented by crossing some representative for ξ with some representative for η. This is well-defined since the product of two simple homotopy equivalences between finite CW-complexes is again a simple homotopy equivalence.

Given a homotopy equivalence $f \colon (X, \xi) \to (Y, \eta)$ of spaces with simple structures, we define its Whitehead torsion

(1.3)
$$\tau(f) \quad \in \quad \mathrm{Wh}(\pi(Y))$$

by $v_*(\tau(v^{-1} \circ f \circ u))$, where $u \colon X' \to X$ and $v \colon Y' \to Y$ are representatives of the simple structures, $\tau(v^{-1} \circ f \circ u) \in \mathrm{Wh}(Y')$ is the classical Whitehead torsion of a homotopy equivalence of finite CW-complexes and $v_* \colon \mathrm{Wh}(\pi(Y')) \to \mathrm{Wh}(\pi(Y))$ is the isomorphism induced by v. The standard properties of the Whitehead torsion of a homotopy equivalence of finite CW-complexes carry over to homotopy equivalences of spaces with simples structure. Namely, we get (see [3, (22.1), (23.1), (23.2)], [15, Theorem 4.33])

Lemma 1.4. *(i) Homotopy invariance*

Let $f, g \colon X \to Y$ be maps of spaces with simple structures. If f and g are homotopic, then

$$\tau(f) = \tau(g);$$

(ii) Composition formula

Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps of spaces with simple structures. Then

$$\tau(g \circ f) = \tau(g) + g_*(\tau(f));$$

(iii) Sum formula

Let

$$
\begin{array}{ccc}
X_0 \xrightarrow{a_1} X_1 & & Y_0 \xrightarrow{i_1} Y_1 \\
\downarrow{a_2} \quad \downarrow{b_1} & and & \downarrow{i_2} \quad \downarrow{j_1} \\
X_2 \xrightarrow{b_2} X & & Y_2 \xrightarrow{j_2} Y
\end{array}
$$

be pushouts of spaces with a_1 and i_1 cofibrations. Let $j_0 \colon Y_0 \to Y$ be $j_1 \circ i_1 = j_2 \circ i_2$. Suppose that all spaces come with simple structures such that X and Y carry the pushout simple structure. Let $f_i \colon X_i \to Y_i$ be homotopy equivalences for $i = 0, 1, 2$ such that $f_1 \circ a_1 = i_1 \circ f_0$ and $f_2 \circ a_2 = i_2 \circ f_0$. Let $f \colon X \to Y$ be the map induced by the pushout property.

Then f is a homotopy equivalence and

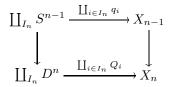$$\tau(f) = (j_1)_*(\tau(f_1)) + (j_2)_*(\tau(f_2)) - (j_0)_*(\tau(f_0));$$

(iv) Product formula

Let $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ be homotopy equivalences of path-connected spaces with simple structures. Equip $X_1 \times X_2$ and $Y_1 \times Y_2$ with the product simple structures. Define $i_1 \colon Y_1 \to Y_1 \times Y_2$ to be the inclusion of Y_1 into $Y_1 \times Y_2$ with respect to some base point $y_2 \in Y_2$ and analogously define $i_2 \colon Y_2 \to Y_1 \times Y_2$.

Then

$$\tau(f_1 \times f_2) = \chi(Y_1) \cdot (i_2)_*(\tau(f_2)) + \chi(Y_2) \cdot (i_1)_*(\tau(f_1)).$$

Remark 1.5. Let X be a finite CW-complex. Consider any pushout describing how X_n is obtained from X_{n-1} by attaching cells

$$
\begin{array}{ccc}
\coprod_{I_n} S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i} & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{I_n} D^n & \xrightarrow{\coprod_{i \in I_n} Q_i} & X_n
\end{array}
$$

Equip the two upper corners and the left lower corner with the canonical simple structure with respect to any CW-structure. Then the pushout simple structure on the right lower corner agrees with the canonical simple structure with respect to any CW-structure.

This is obvious if we equip each S^{n-1} with some finite CW-structure, D^n with the CW-structure which is obtained from the one on S^{n-1} by attaching one n-cell with the identity $S^{n-1} \to S^{n-1}$, we equip X_{n-1} and X_n with given CW-structures and each map q_i is cellular. The general case follows using the cellular approximation theorem, the fact that changing the attaching maps by a homotopy does not change the simple homotopy type (see [3, see (7.1) on page 23] and the topological invariance of Whitehead torsion (see [2]).

2. Fibrations

In this section we record some basis facts about fibrations.

Recall that a fibration $p \colon E \to B$ is a map which has the *homotopy lifting property*, i.e., for any homotopy $h \colon X \times [0, 1] \to B$ and map $f \colon X \to E$ with $p \circ f = h_0$ there is a homotopy $H \colon X \times [0, 1] \to E$ satisfying $p \circ H = h$ and $H_0 = f$, where here and in the sequel $h_t(x) := h(x, t)$ and $H_t(x) := H(x, t)$. For general information about fibrations we refer for instance to [24, page 342 ff.], [27, I.7]. We mention that we will work in the category of compactly generated spaces [22], [27,

I.4]. A map $(\overline{f}, f)\colon p \to p'$ of fibrations $p\colon E \to B$ to $p'\colon E' \to B'$ consists of a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ \overline{f}\ } & E' \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\
B & \xrightarrow{\ f\ } & B'
\end{array}
$$

A homotopy $h\colon X \times [0,1] \to E$ is called *fiber homotopy* if $p \circ h$ is stationary, i.e., $p \circ h(x,t) = p \circ h(x,0)$ for all $(x,t) \in X \times [0,1]$. Two maps $f_0, f_1\colon X \to E$ with $p \circ f_0 = p \circ f_1$ are called *fiber homotopic* $f_0 \simeq_p f_1$ if there is a fiber homotopy $h\colon X \times [0,1] \to E$ with $h_0 = f_0$ and $h_1 = f_1$. A *fiber homotopy equivalence* from the fibration $p\colon E \to B$ to the fibration $p'\colon E' \to B$ over the same base is a map of fibrations of the shape $(\overline{f}, \mathrm{id})\colon p \to p'$ such that there exists a map of fibrations $(\overline{g}, \mathrm{id})$ with $\overline{g} \circ \overline{f} \simeq_p \mathrm{id}$ and $\overline{f} \circ \overline{g} \simeq_p \mathrm{id}$. The pullback of a fibration $p\colon E \to B$ with a map $f\colon B' \to B$

$$
\begin{array}{ccc}
f^*E & \xrightarrow{\ \overline{f}\ } & E \\
\downarrow{\scriptstyle p_f} & & \downarrow{\scriptstyle p} \\
B' & \xrightarrow{\ f\ } & B
\end{array}
$$

is again a fibration $p_f\colon f^*E \to B$. The elementary proof of the next lemma can be found for instance in [24, page 342 ff.].

Lemma 2.1. *Let $p\colon E \to B$ be a fibration.*

(i) *Let $H\colon X \times [0,1] \to B$ be a homotopy $f_0 \simeq f_1\colon X \to B$. Let $H'\colon f_0^*E \times [0,1] \to E$ be a solution of the homotopy lifting problem for $H \circ (p_{f_0} \times \mathrm{id}_{[0,1]})\colon f_0^*E \times [0,1] \to B$ and $\overline{f_0}\colon f_0^*E \to E$. Define $g_H\colon f_0^*E \to f_1^*E$ by H_1' and p_{f_0} using the pull back property of f_1^*E.*
   Then $(g_H, \mathrm{id})\colon f_0^*E \to f_1^*E$ is a fiber homotopy equivalence and H' is a homotopy $\overline{f_1} \circ g_H \simeq \overline{f_0}$.

(ii) *Let $K\colon X \times [0,1] \to B$ be a second homotopy $f_0 \simeq f_1\colon X \to B$ and $M\colon X \times [0,1] \times [0,1] \to B$ be homotopy relative $X \times \{0,1\}$ between H and K. Then M induces a fiber homotopy from g_H to g_K.*

Let $p\colon E \to B$ be a fibration. Denote by F_b the fiber $p^{-1}(b)$ for $b \in B$. For any homotopy class $[w]$ of paths $w\colon [0,1] \to B$ we obtain by Lemma 2.1 a homotopy class $t([w])$ of maps $F_{w(0)} \to F_{w(1)}$ called the *fiber transport along w*. If v and w are paths with $v(1) = w(0)$, then $t([w]) \circ t([v]) = t([v * w])$. The constant path c_b induces the identity on F_b. We mention that in the situation of Lemma 2.1 (i) for each $x \in X$ the map $F_{f_0(x)} \to F_{f_1(x)}$ induced by g_H represents the fiber transport along the path $H(x, -)$.

Definition 2.2. *Let $p\colon E \to B$ be a fibration, $f\colon X \to B$ be a map from a contractible space X to B and $x \in X$ and $b \in B$. Let $w\colon [0,1] \to B$ be a path from b to $f(x)$. A fiber trivialization of f^*E with respect to (b, x, w) is a fiber homotopy equivalence $T\colon F_b \times X \to f^*E$ over X such that the map $F_b \to F_{f(x)}$ induced by T represents the fiber transport $t([w])$ for p along w.*

Lemma 2.1 implies

Lemma 2.3. *Consider the situation of Definition 2.2.*

(i) *The exists a fiber trivialization with respect to (b, x, w);*

(ii) *Two fiber trivializations with respect to (b, x, w) are fiber homotopic;*

(iii) *Let* $T_i \colon F_{b_i} \times X \to f^*E$ *be a fiber trivialization with respect to* (b_i, x_i, w_i) *for* $i = 0, 1$. *Choose a path* $v \colon [0, 1] \to X$ *from* x_0 *to* x_1. *Let* $t \colon F_{b_0} \to F_{b_1}$ *be a representative of the fiber transport of* p *along* $w_0 * f(v) * w_1^-$. *Then we get a fiber homotopy*

$$T_1 \circ (t \times \mathrm{id}_X) \simeq_{p_f} T_0;$$

(iv) *Let* $H \colon X \times [0, 1] \to B$ *be a homotopy* $f_0 \simeq f_1$. *Let* v *be the path in* B *from* $f_0(x)$ *to* $f_1(x)$ *given by* $h(x, -)$ *and let* w_0 *be a path from* b *to* $f_0(x)$. *Put* $w_1 = w_0 * v$. *Let* $T_i \colon F_b \times X \to f_i^*E$ *be the fiber trivialization for* f_i *with respect to* (b, x, w_i) *for* $i = 0, 1$ *and* $g_H \colon f_0^*E \to f_1^*E$ *be the fiber homotopy equivalence of Lemma 2.1 (i). Then we get a fiber homotopy over* X

$$g_H \circ T_0 \simeq_{p_{f_1}} T_1.$$

3. The simple structure on a total space of a fibration

In this section we explain how the total space of a fibration inherits a simple structure from the base space and the fiber.

Definition 3.1. *Let* B *be a connected* CW-*complex with base point* $b \in B$. *Denote by* $I(B)$ *the set of open cells of* B *and by* $\dim(c)$ *the dimension of a cell* $c \in I(B)$. *A spider at* b *for* B *is a collection of paths* w_c *indexed by* $c \in I(B)$ *such that* $w_c(0) = b$ *and* $w_c(1)$ *is a point in the open cell* c.

Let $p \colon E \to B$ be a fibration such that B is a path-connected finite CW-complex and the fiber has the homotopy type of a finite CW-complex. Given a base point $b \in B$, a spider s at b and a simple structure ζ on F_b, we want to construct a preferred simple structure

$$(3.2) \qquad\qquad \xi(b, s, \zeta) \text{ on } E$$

as follows. Let B_n be the n-skeleton of B and $E_n = p^{-1}(B_n)$. We construct the preferred simple structure on E_n inductively for $n = -1, 0, 1, \ldots$. The case $n = -1$ is trivial; the induction step from $(n-1)$ to n is done as follows. Choose a pushout

$$
\begin{array}{ccc}
\coprod_{I_n} S^{n-1} & \xrightarrow{\;\coprod_{i \in I_n} q_i\;} & B_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{I_n} D^n & \xrightarrow{\;\coprod_{i \in I_n} Q_i\;} & B_n
\end{array}
$$

Choose for $i \in I_n$, the n-cells of B, a point $x_i \in D^n - S^{n-1}$ such that $Q_i(x_i) = w_i(1)$, where w_i is the path with $w_i(0) = b$ associated by the spider s to the cell indexed by $i \in I_n$. We get from Lemma 2.3 (i) a fiber trivilization $T_i \colon F_b \times D^n \to Q_i^*E$. It yields a homotopy equivalence of pairs

$$T_i \colon F_b \times (D^n, S^{n-1}) \to (Q_i^*E, q_i^*E).$$

Equip Q_i^*E and q_i^*E with the simple structures induced by T_i and the product simple structure $(\zeta \times \xi_{\mathrm{can}}(D^n))$ on $F_b \times D^n$ and and $(\zeta \times \xi_{\mathrm{can}}(S^{n-1}))$ on $F_b \times S^{n-1}$.

By induction hypothesis we have already constructed a simple structure on E_{n-1}. Since there is a pushout with a cofibration as left vertical map (see [15, Lemma 1.26])

$$
\begin{array}{ccc}
\coprod_{I_n} q_i^*E & \xrightarrow{\;\coprod_{i \in I_n} \overline{q_i}\;} & E_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{I_n} Q_i^*E & \xrightarrow{\;\coprod_{i \in I_n} \overline{Q_i}\;} & E_n
\end{array}
$$

we can equip E_n with the pushout simple structure. Lemma 2.3 (ii) implies that the choice of T_i does not matter.

Notice that the choice of the characteristic maps (Q_i, q_i) does not belong to the structure of a CW-complex. Only the skeletal filtration $(X_n)_{n \geq -1}$ is part of the structure and the existence of a pushout as above is required but not specified. One can recover the open cells by the path-components of $X_n - X_{n-1}$ and the closed cells by the closure of the open cells but not the characteristic maps Q_i. Therefore one has to show that the simple structure on E_n is independent of the choice of these pushouts. This is done by thickening X_{n-1} into X_n. The details of the argument are similar to the one given in the proof of [15, Lemma 7.13] and can be found in [23, Subsection 3.2].

Remark 3.3. If p is trivial, i.e., $p \colon B \times F \to B$ is the projection map, and F is a finite CW-complex, then for any spider s, the simple structure $\xi(b, s, \xi_{\mathrm{can}}(F))$ on $B \times F$ agrees with the product simple structure.

The dependence of the simple structure on the choice of (b, s, ζ) is described in the next lemma. Therefore suppose that another choice (b', s', ζ') has been made, with $b' \in B$, s' a spider at b', and ζ' a simple structure on the fiber $F_{b'}$.

Lemma 3.4. *Suppose that B is path-connected. Given a cell $c \in I(B)$, let u_c be any path in the interior of c from $w_c(1)$ to $w'_c(1)$, where w_c and w'_c are given by the spiders s and s', and let v_c be the path $w_c * u_c * (w'_c)^-$. Then the homotopy class relative endpoints $[v_c]$ is independent of u_c. If we denote by $(i_{b'})_* \colon \mathrm{Wh}(\pi(F_{b'})) \to \mathrm{Wh}(\pi(E))$ the homomorphism induced by the inclusion $i_{b'} \colon F_{b'} \to E$, the following holds in $\mathrm{Wh}(\pi(E))$*

$$\tau\big((E, \xi(b, s, \zeta)) \xrightarrow{\mathrm{id}} (E, \xi(b', s', \zeta'))\big)$$
$$= \sum_{c \in I(B)} (-1)^{\dim(c)} \cdot (i_{b'})_* \tau\big((F_b, \zeta) \xrightarrow{t([v_c])} (F_{b'}, \zeta')\big)$$

Proof. This follows from Lemma 1.4 and Lemma 2.3. □

Let $p \colon E \to B$ be a fibration whose fiber has the homotopy type of a finite CW-complex. We can assign to it a class

(3.5) $$\Theta(p) \quad \in \quad H^1\big(B, \mathrm{Wh}(\pi(E))\big)$$

as follows. For simplicity we assume that B is path-connected. Given $b \in B$, a loop w at b in B and a simple structure ζ on F_b, we can compute the Whitehead torsion of the fiber transport along w

$$(i_b)_* \tau\big(t([w]) \colon (F_b, \zeta) \to (F_b, \zeta)\big) \quad \in \quad \mathrm{Wh}(\pi(E))$$

for $i_b \colon F_b \to E$ the inclusion. From Lemma 1.4 and Lemma 2.1 one concludes that this element in independent of the choice of ζ and that we obtain a group homomorphism $\pi_1(B, b) \to \mathrm{Wh}(\pi(E))$. It defines an element $\Theta(p) \in H^1\big(B; \mathrm{Wh}(\pi(E))\big)$ which is independent of the choice of $b \in B$.

Definition 3.6. *Let $p \colon E \to B$ be a fibration whose fiber has the homotopy type of a finite CW-complex. We call p simple if $\Theta(p|_C) = 0$ holds for any component $C \in \pi_0(B)$ with respect to the restriction $E|_C \to C$.*

Lemma 3.7. *Let $p \colon E \to B$ be a locally trivial fiber bundle with a finite CW-complex as typical fiber and paracompact base space. Then it is a simple fibration.*

Proof. It is a fibration by [27, page 33]. It is simple, since the fiber transport in such a bundle is given by homeomorphisms and the Whitehead torsion of a homeomorphism is trivial (see [2]). □

Corollary 3.8. *Consider the situation of Lemma 3.4. Assume that p is simple. Define*

$$\tau_0 := (i_{b'})_* \tau\big(t\colon (F_b, \zeta) \to (F_{b'}, \zeta')\big),$$

where $t\colon F_b \to F_{b'}$ is a homotopy equivalence representing the fiber transport $t([w])$ for some path w from b to b'.

Then τ_0 is independent of the choice of w and

$$\tau\big(\mathrm{id}\colon (E, \xi(b, s, \zeta)) \to (E, \xi(b', s', \zeta'))\big) \quad = \quad \chi(B) \cdot \tau_0.$$

Notation 3.9. Let $p\colon E \to B$ be a fibration with path-connected finite CW-complex as base space B such the homotopy fiber has the homotopy type of a finite CW-complex. Suppose that p is simple. Then the simple structure $\xi(b, s, \zeta)$ of (3.2) is independent of the spider s by Corollary 3.8 and will be denoted briefly by $\xi(b, \zeta)$.

Corollary 3.10. *Let $p\colon E \to B$ be a fibration such that B is a path-connected finite CW-complex with $\chi(B) = 0$ and the fiber has the homotopy type of a finite CW-complex. Suppose that p is simple. Then E carries a preferred simple structure.*

The next three lemmas describe the extent of compatibility of our construction with fiber homotopy equivalences, push-outs, and pull-backs by simple homotopy equivalences.

Lemma 3.11. *Let $p\colon E \to B$ and $p'\colon E' \to B$ be fibrations and $(\overline{f}, \mathrm{id})\colon p \to p'$ be a fiber homotopy equivalence. Let $b \in B$ be a base point and let s be a spider at b. Fix simple structures ζ and ζ' on the fibers F_b and F_b' of p and p' over b. Let $\overline{f}_b\colon F_b \to F_b'$ be the homotopy equivalence induced by $\overline{f}$. From $\overline{f}$ we obtain an isomorphism $\overline{f}_*\colon H^1\big(B, \mathrm{Wh}(\pi(E))\big) \xrightarrow{\cong} H^1\big(B; \mathrm{Wh}(\pi(E'))\big)$.*

Then we get

$$\tau\big(\overline{f}\colon (E, \xi(b, s, \zeta)) \to (E', \xi(b, s, \zeta'))\big) \quad = \quad \chi(B) \cdot (i_b)_* \tau\big(\overline{f}_b\colon (F_b, \zeta) \to (F_b', \zeta')\big);$$
$$\Theta(p') \quad = \quad \overline{f}_*(\Theta(p)).$$

Proof. This follows from Lemma 1.4 and Lemma 2.3. $\qquad\qquad\square$

Let

$$
\begin{array}{ccc}
B_0 & \xrightarrow{\ i_1\ } & B_1 \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\
B_2 & \xrightarrow{\ j_2\ } & B
\end{array}
$$

be a push-out of finite CW-complexes, such that (B_2, B_0) and (B, B_1) are CW-pairs, the maps i and j are inclusions, and B is obtained as a CW-complex from B_1 by attaching the relative cells of (B_2, B_0). Put $j_0 := j_2 \circ i_2 = j_1 \circ i_1$. Suppose that $p\colon E \to B$ is a simple fibration.

For every path component C of B choose a base point $x(C) \in C$. For every $i \in \{0, 1, 2\}$ and path component $C_i \in \pi_0(B_i)$ choose a base point $x(C_i) \in C_i$, a spider $s(C_i)$ for $(C_i, x(C_i))$ and a path $u(C_i)$ in B from $j_i(x(C_i))$ to the point $x(C) \in C$ for the component C of $\pi_0(B)$ containing $j_i(C_i)$.

For each $C \in \pi_0(B)$ choose a simple structure $\zeta(F_{x(C)})$ on fiber $F_{x(C)}$ of E over $x(C)$. For every $i \in \{0, 1, 2\}$ equip the fiber $F_{j_i(x(C_i))}$ of $j_i^* E$ over $x(C_i)$ with the simple structure $\zeta(F_{j_i(x(C_i))})$ for which the fiber transport in E along the path $u(C_i)$ yields a simple homotopy equivalence. With these choices we obtain simple structures ξ_i on $E|_{B_i}$ for every $i \in \{0, 1, 2\}$. Since p is by assumption simple, we obtain from the choices of base points $x(C) \in C$ and simple structures $\zeta(F_{x(C)})$ on $F_{x(C)}$ a simple structure ξ on E without specifying spiders (see Notation 3.9).

Lemma 3.12. *Under the assumptions above the square*

$$
\begin{array}{ccc}
\left(\phi^*(E|_{B_1}), \xi_0\right) & \xrightarrow{\overline{i_1}} & \left(E|_{B_1}, \xi_1\right) \\
\downarrow{\scriptstyle \overline{i_2}} & & \downarrow{\scriptstyle \overline{j_1}} \\
\left(\Phi^*E, \xi_2\right) & \xrightarrow{\overline{j_2}} & (E, \xi)
\end{array}
$$

is a simple pushout, i.e., the pushout simple structure on E agrees with the given one. (Here $\phi := i_1$ and $\Phi := j_2$.)

Proof. The diagram

$$
\begin{array}{ccc}
\phi^*(E|_{B_1}) & \xrightarrow{\overline{\phi}} & E|_{B_1} \\
\downarrow{\scriptstyle \overline{i}} & & \downarrow{\scriptstyle \overline{j}} \\
\Phi^*E & \xrightarrow{\overline{\Phi}} & E
\end{array}
$$

is a pushout and $\overline{i}$ is a cofibration (see [15, Lemma 1.26]). (Here $i := i_2$ and $j := j_1$.)

The pullback of $p\colon E \to B$ to B_i is not necessarily a simple fibration since the map induced from the Whitehead group of the total space of the pullback to E need not to be injective. Hence the simple structures on the pullbacks may depend on the choice of basepoints and spiders. Lemma 3.4 yields an explicit term which describes the dependence on the choice of base points and spiders. Since p is simple, this term vanishes when it is pushed forward to E. Hence the pushout simple structure on E does not depend on the choices of the base points and spiders on the path components of B_i for $i \in \{0, 1, 2\}$ and it suffices to prove the claim for one particular choice.

We proceed by induction over the dimension of the relative CW-complex (B_2, B_0). If its dimension is n, let $B_2^{(n-1)}$ be the $(n-1)$-skeleton and $i'\colon B_0 \to B_2^{(n-1)}$ and $i''\colon B_2^{(n-1)} \to B_2$ be the inclusions. Denote by $\Phi'\colon B_2^{(n-1)} \to B^{(n-1)}$ the map induced by Φ for $B^{(n-1)}$ the $(n-1)$-skeleton of (B, B_1). We obtain the following commutative diagram

$$
\begin{array}{ccc}
\phi^*(E|_{B_1}) & \xrightarrow{\overline{\phi}} & E|_{B_1} \\
\downarrow{\scriptstyle \overline{i'}} & & \downarrow \\
(\Phi')^*(E|_{B^{(n-1)}}) & \xrightarrow{\overline{\Phi'}} & E|_{B^{(n-1)}} \\
\downarrow{\scriptstyle \overline{i''}} & & \downarrow \\
\Phi^*E & \xrightarrow{\overline{\Phi}} & E
\end{array}
$$

Fix a base points in $b_0 \in B_0$ which will be used as a base points for all the spiders we construct next. Choose for any component of B_0 a spider. The choose for any component of B_1 a spider. Extend these collections of spiders B_0 to an analogous collection of spiders for $B_2^{(n-1)}$ and pushforward this extension to a collection of spiders on $B^{(n-1)}$ which extends the given collection of spiders on B_1. Then apply the same procedure to get spiders on B_2 and B. We equip all the six spaces appearing in the diagram above with the simple structure appearing in (3.2) with respect to the choice of spiders.

We have to show that for the outer square the simple structure on E agrees with the pushout simple structure. One easily checks that it suffices to show that the corresponding statement holds for the upper and the lower square. This is true for the upper square by induction hypothesis and follows for the lower square from the construction of the simple structure on E. □

Lemma 3.13. *Let $f\colon B' \to B$ be map of finite CW-complexes. Let $p\colon E \to B$ be a fibration whose homotopy fiber has the homotopy type of a finite CW-complex. Suppose that p is simple. Suppose that $f\colon B' \to B$ is a simple homotopy equivalence. Let*

$$
\begin{array}{ccc}
f^*E & \xrightarrow{\ \overline{f}\ } & E \\
\big\downarrow{\scriptstyle p_f} & & \big\downarrow{\scriptstyle p} \\
B' & \xrightarrow{\ f\ } & B
\end{array}
$$

be the pullback. For every component $C \in \pi_0(B')$ choose a base point $x(C)$. For every $C \in \pi_0(B')$ equip the fiber $(p_f)^{-1}(x(C))$ with a simple structure ζ'_C and the fiber $p^{-1}\big(f(x(C))\big)$ with a simple structure ζ_C such that

$$
\tau\bigg(\overline{f}|_{(p_f)^{-1}(x(C))}\colon \big((p_f)^{-1}(x(C)), \zeta'_C\big) \to \big(p^{-1}(f(x(C))), \zeta_C\big)\bigg) = 0.
$$

Equip f^*E and E with the simple structures ξ' and ξ associated to these choices in Notation 3.9. Then

$$
\tau\big(\overline{f}\colon (f^*E, \xi) \to (E, \xi)\big) = 0.
$$

Proof. Recall that a map is a simple homotopy equivalence if it is, up to homotopy, a composition of elementary collapses and expansions. Because of Lemma 1.4 we can assume without loss of generality that $f\colon B' \to B$ is an elementary expansion, i.e., f is the inclusion of a CW-subcomplex, where B is obtained from B' as a push-out $B = D^{n+1} \cup_{D^n} B'$, with an attaching map $D^n \to B'$ mapping into the n-skeleton and its restriction onto S^{n-1} mapping into the $(n-1)$-skeleton. The inclusion of D^n into D^{n+1} is given by identifying D^n with the upper hemisphere of S^n.

By Lemma 3.12, it is enough to show that the inclusion from $E|_{D^n}$ to $E|_{D^{n+1}}$ is simple. The base spaces in these fibrations are contractible; hence we can assume by Lemma 3.11 that the fibrations are products. In that case, by Remark 3.3, the simple structures on the total spaces are the product simple structures, and by the product formula the claim follows from the fact that the inclusion $D^n \to D^{n+1}$ is simple. $\qquad\square$

Remark 3.14. Let $p\colon E \to B$ be a simple fibration over a path-connected base space B carrying a simple structure η, and suppose we are given a simple structure ζ of the fiber over a point b. Notice that Lemma 3.13 gives us the possibility to define a simple structure on the total space E: Choose a finite CW-model $f\colon X \to B$ representing η and consider the pullback structure map $\overline{f}\colon f^*E \to E$. We can arrange by possibly changing f up to homotopy that $b = f(x)$ for some $x \in X$. Then f^*E carries the simple structure $\xi(x, \zeta)$; give E the simple structure for which the torsion of $\overline{f}$ vanishes. We are going to denote this simple structure by $\xi(\eta, b, \zeta)$.

Let M be a closed topological manifold. Then, by Kirby-Siebenmann [10, Essay III, Theorem 4.1 on page 118], there is a preferred simple structure

$$(3.15) \qquad\qquad\qquad \xi^{\mathrm{Top}}(M) \text{ on } M,$$

which is defined by considering any triangulated closed disc bundle over M: The simple structure on the disc bundle obtained from the triangulation induces the preferred simple structure on M via the retraction onto M. This simple structure agrees with the one obtained by any triangulation or by any handlebody decomposition (more generally what they call TOP s-decomposition) of M, whenever they exist (see [10, Essay III, Theorem 5.10 on page 131 and Theorem 5.11 on page 132]).

Lemma 3.16. *Let $F \to M \to B$ be a locally trivial bundle of closed topological manifolds with path connected B. Then we get:*

$$
\begin{aligned}
\Theta(p) &= 0; \\
\xi^{\mathrm{Top}}(M) &= \xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big),
\end{aligned}
$$

where $\xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)$ has been defined in Remark 3.14.

Proof. We have already proved $\Theta(p) = 0$ in Lemma 3.7. Moreover, if the bundle happens to be globally trivial, then the simple structure $\xi(M, b, \xi^{\mathrm{Top}}(F))$ agrees with $\xi^{\mathrm{Top}}(B \times F)$ by Remark 3.3.

Consider now the general case. We need not take care of the base point b, as the torsion of every fiber transport is zero (see Corollary 3.8). First suppose that $\dim(B) \geq 6$. Then there exists a handlebody decomposition

$$D^n = B_0 \subset B_1 \subset \cdots \subset B_k = B$$

[10, Essay III, §2], and proceed by induction over k. If $k = 0$, then the bundle is trivial, and the claim follows. For the induction step, consider the pushout which attaches to B_k a handle H to get B_{k+1}. By Lemma 3.12, the pushout

$$
\begin{array}{ccc}
\big(E|_{H \cap B_k}, \xi(\xi^{\mathrm{Top}}(H \cap B_k), b, \xi^{\mathrm{Top}}(F))\big) & \longrightarrow & \big(E|_{B_k}, \xi(\xi^{\mathrm{Top}}(B_k), b, \xi^{\mathrm{Top}}(F))\big) \\
\downarrow & & \downarrow \\
\big(E|_H, \xi(\xi^{\mathrm{Top}}(H), b, \xi^{\mathrm{Top}}(F))\big) & \longrightarrow & \big(E|_{B_{k+1}}, \xi(\xi^{\mathrm{Top}}(B_{k+1}), b, \xi^{\mathrm{Top}}(F))\big)
\end{array}
$$

is also simple. Here, the simple structures of the left column agree with the preferred structures as the bundles are trivial; by induction hypothesis, the same is true for the upper right space. Now the above pushout is one of the equivalent methods in [10, Essay III, §5] to give $E|_{B_{k+1}}$ its preferred simple structure $\xi^{\mathrm{Top}}(E|_{B_{k+1}})$. Hence the two structures on $E|_{B_{k+1}}$ agree.

We still have to treat the case where $\dim(B) \leq 5$. Take a 1-connected closed topological manifold N with $\dim(N) \geq 6$ and $\chi(N) = 1$, e.g., $(\mathbb{CP}^2 \times \mathbb{CP}^2) \sharp 4(S^3 \times S^5)$. Now apply what we have already proved to the fiber bundle $M \times N \to B \times N$ which is the product of the original bundle by the identity map on N. This leads to the equality

$$(3.17) \qquad \xi^{\mathrm{Top}}(M \times N) = \xi\big(\xi^{\mathrm{Top}}(B \times N), (b, n), \xi^{\mathrm{Top}}(F)\big)$$

for any $n \in N$. It is not hard to check that the right hand side of (3.17) coincides with $\xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big) \times \xi^{\mathrm{Top}}(N)$. Since $\dim(N \times B) \geq 0$, we get

$$
\begin{aligned}
& \tau\big(\mathrm{id} \colon (M, \xi^{\mathrm{Top}}(M)) \to (M, \xi(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)))\big) \\
=\ & \tau\big(\mathrm{id} \colon (M, \xi^{\mathrm{Top}}(M)) \to (M, \xi(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)))\big) \cdot \chi(N) \\
=\ & \tau\big(\mathrm{id} \colon (M \times N, \xi^{\mathrm{Top}}(M) \times \xi^{\mathrm{Top}}(N)) \to (M \times N, \xi(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)) \times \xi^{\mathrm{Top}}(N))\big) \\
=\ & \tau\big(\mathrm{id} \colon (M \times N, \xi^{\mathrm{Top}}(M \times N)) \to (M \times N, \xi(\xi^{\mathrm{Top}}(B \times N), (b, n), \xi^{\mathrm{Top}}(F)))\big) \\
=\ & 0
\end{aligned}
$$

by Lemma 1.4 (iv). $\qquad\qquad\square$

4. Turning a map into a fibration

Let $f \colon X \to B$ be a map. Let $\mathrm{FIB}(f)$ be the subspace of $X \times \mathrm{map}([0, 1], B)$ consisting of pairs (x, w) which satisfy $w(0) = f(x)$. Let $\widehat{f} \colon \mathrm{FIB}(f) \to B$ be the map sending (x, w) to $w(1)$. Let $\lambda_f \colon X \to \mathrm{FIB}(f)$ be the map which sends $x \in X$ to $(x, c_{f(x)})$ for $c_{f(x)}$ the constant path at $f(x)$ in B. Denote by $\mu_f \colon \mathrm{FIB}(f) \to X$ the map $(x, w) \mapsto x$. Then $\widehat{f} \colon \mathrm{FIB}(f) \to B$ is a fibration, λ_f is a homotopy equivalence

and $\widehat{f} \circ \lambda_f = f$, $\mu_f \circ \lambda_f = \mathrm{id}$, $f \circ \mu_f \simeq \widehat{f}$ and $\lambda_f \circ \mu_f \simeq \mathrm{id}$ [27, Theorem 7.30 on page 42]. The fiber of $\widehat{f} \colon \mathrm{FIB}(f) \to B$ over b is called the homotopy fiber of f over b and denoted by $\mathrm{hofib}(f)_b$.

Lemma 4.1.   *(i) If $f \colon E \to B$ is already a fibration, then $\lambda_f \colon E \to \mathrm{FIB}(f)$ is a fiber homotopy equivalence over B;*

(ii) If $H \colon X \times [0,1]$ is a homotopy, $f \simeq g \colon X \to B$, then it induces a natural fiber homotopy equivalence $\widehat{H} \colon \mathrm{FIB}(f) \to \mathrm{FIB}(g)$.

Proof. (i) see [27, Theorem 7.31 on page 43].

(ii) $\widehat{H}$ sends $(x, w) \in \mathrm{FIB}(f)$ to $(x, v) \in \mathrm{FIB}(g)$ for the following path $v \colon [0,1] \to B$

$$v(t) = \begin{cases} H(x, 1 - 2t) & 0 \le t \le 1/2; \\ w(2t - 1) & 1/2 \le t \le 1. \end{cases}$$

$\square$

5. Fiber torsion obstructions

Theorem 5.1. *Let $f \colon M \to B$ be a map of closed topological manifolds for path-connected B. Suppose that there is a locally trivial fiber bundle $F \to M \xrightarrow{g} B$ of closed topological manifolds such that $f \simeq g$. Then for one (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ has the homotopy type of a finite CW-complex and carries a simple structure ζ such that*

$$\Theta(\widehat{f} \colon \mathrm{FIB}(f) \to B) \quad = \quad 0;$$

$$\tau\big((M, \xi^{\mathrm{Top}}(M)) \xrightarrow{\lambda_f} (\mathrm{FIB}(f), \xi(b, \zeta))\big) \quad = \quad 0.$$

Proof. From Lemma 4.1 we obtain a fiber homotopy equivalence (λ_g, id) from $g \colon M \to B$ to $\widehat{g} \colon \mathrm{FIB}(g) \to B$ and a fiber homotopy equivalence $(\overline{k}, \mathrm{id})$ from $\widehat{g} \colon \mathrm{FIB}(g) \to B$ to $\widehat{f} \colon \mathrm{FIB}(f) \to B$ such that $\overline{k} \circ \lambda_g$ is homotopic to $\lambda_f \colon M \to \mathrm{FIB}(f)$. We get $\Theta(\widehat{f}) = 0$ from Lemma 3.11 and Lemma 3.16. Equip $\mathrm{hofib}(f)_b$ with the simple structure ζ for which the Whitehead torsion of the homotopy equivalence $(g^{-1}(b), \xi^{\mathrm{Top}}(g^{-1}(b))) \to (\mathrm{hofib}(f)_b, \zeta)$ induced by $\overline{k} \circ \lambda_g$ is trivial. Now the second claim follows from Lemma 3.11 and Lemma 3.16. $\square$

Definition 5.2 (Fiber torsion obstructions)**.** *Let $f \colon M \to B$ be a map of closed topological manifolds for path-connected B. Suppose that for some (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ has the homotopy type of a finite CW-complex.*

(i) Define the element

$$\Theta(f) \in H^1\big(B; \mathrm{Wh}(\pi(E))\big)$$

to be the image of $\Theta(\widehat{f})$ under the isomorphism $H^1\big(B; \mathrm{Wh}(\pi(\mathrm{FIB}(f)))\big) \to H^1\big(B; \mathrm{Wh}(\pi(M))\big)$ induced by the homotopy equivalence $\mu_f \colon \mathrm{FIB}(f) \to M$;

(ii) Suppose that $\Theta(f)$ vanishes. Let $(\mu_f \circ i_b)_* \colon \mathrm{Wh}\big(\pi(\mathrm{hofib}(f)_b)\big) \to \mathrm{Wh}(\pi(M))$ be the map induced by the composite $\mathrm{hofib}(f)_b \xrightarrow{i_b} \mathrm{FIB}(f) \xrightarrow{\mu_f} M$. Define the fiber torsion obstruction

$$\tau_{\mathrm{fib}}(f) \in \mathrm{cok}\big(\chi(B) \cdot (\mu_f \circ i_b)_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(f)_b)) \to \mathrm{Wh}(\pi(M))\big)$$

to be the image of the Whitehead torsion

$$\tau\big(\lambda_f \colon (M, \xi^{\mathrm{Top}}(M)) \to (\mathrm{FIB}(f), \xi(b, \zeta))\big)$$

under the isomorphism $(\mu_f)_* \colon \mathrm{Wh}\big(\pi(\mathrm{FIB}(f))\big) \to \mathrm{Wh}(\pi(M))$ for some choice of base point $b \in B$ and simple structure ζ on F_b.

Theorem 5.3. *Let $f\colon M \to B$ be a map of closed topological manifolds for path-connected B. Then*

(i) *The element $\Theta(f)$ depends only on the homotopy class of f and is independent of the choice of b and ζ. If $\Theta(f)$ vanishes, the same statement holds for the fiber torsion obstruction $\tau_{\mathrm{fib}}(f)$.*

(ii) *If f is homotopic to a map $g\colon M \to B$ which is the projection of a locally trivial fiber bundle with a closed topological manifold as fiber, then both $\Theta(f)$ and $\tau_{\mathrm{fib}}(f)$ vanish.*

Proof. This follows from Lemma 3.11 and Theorem 5.1. □

Remark 5.4. Let $f\colon M \to B$ be a map of closed topological manifolds for path-connected B. If $\chi(B)$ happens to be zero and $\Theta(f)$ vanishes, the invariant defined in Definition 5.2 lives in

$$(5.5) \qquad\qquad \tau_{\mathrm{fib}}(f) \quad\in\quad \mathrm{Wh}(\pi(M)).$$

In other words, if $\chi(B) = 0$, then $\mathrm{FIB}(f)$ carries a preferred simple structure ξ by Corollary 3.10 and the element $\tau_{\mathrm{fib}}(f)$ is the image of the Whitehead torsion of the map $\lambda\colon (M, \xi^{\mathrm{Top}}(M)) \to (\mathrm{FIB}(f), \xi)$ under the isomorphism $(\mu_f)_*\colon \mathrm{Wh}(\pi(\mathrm{FIB}(f))) \to \mathrm{Wh}(\pi(M))$.

Example 5.6. Let $f\colon M \to B$ be a map of closed topological manifolds for path-connected B and M. Suppose that for some (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ has the homotopy type of a finite CW-complex. Suppose that the Whitehead group of the kernel of $\pi_1(f)\colon \pi_1(M) \to \pi_1(B)$ is trivial. This is the case if $\pi_1(f)$ is bijective. Then $\Theta(f)$ vanishes.

This follows from the long exact homotopy sequence of $\mathrm{FIB}(f) \to B$ which implies that under the conditions above the map $\mathrm{Wh}\big(\pi(\mathrm{hofib}(f)_b)\big) \to \mathrm{Wh}(\pi(M))$ is trivial.

6. Base space S^1

In this section we consider the case, where the base space is the one-dimensional sphere S^1, i.e., we consider a map

$$f\colon M \to S^1$$

from a connected closed manifold M to S^1 whose homotopy fiber has the homotopy type of a finite CW-complex. In this special situation we can find a single obstruction $\tau'_{\mathrm{fib}}(f)$ which carries the same information as our previous invariants $\Theta(f)$ and $\tau_{\mathrm{fib}}(f)$ and has a nice description in terms of mapping tori. $\tau'_{\mathrm{fib}}(f)$ agrees with the obstruction $\tau(f)$ defined in [5].

We begin with the definition of $\tau'_{\mathrm{fib}}(f)$. Let $e\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi it)$ be the universal covering of S^1. We abbreviate the homotopy fiber over $e(0)$ by $F :=\mathrm{hofib}(f)_{e(0)} = \mathrm{FIB}(f)_{e(0)}$.

Equip S^1 with the CW-structure whose 0-skeleton is $e(0)$ and whose 1-skeleton is S^1. Let s be the spider based at $e(0)$ which is given by the constant path at $e(0)$ for the 0-cell and by the path $w\colon [0, 1] \to S^1$ sending t to $\exp(\pi it)$ for the 1-cell. Equip $\mathrm{FIB}(f)$ with the simple structure $\xi\big(e(0), s, \zeta\big)$ defined in (3.2) for any choice of simple structure ζ on F. Because of Lemma 3.4 the simple structure $\xi\big(e(0), s, \zeta\big)$ is independent of the choice of ζ and we will write $\xi\big(e(0), s\big)$. Then

$$(6.1) \qquad\qquad \tau'_{\mathrm{fib}}(f) \quad\in\quad \mathrm{Wh}(\pi(M))$$

is defined to be the Whitehead torsion of the canonical homotopy equivalence $\mu_f\colon \mathrm{FIB}(f) \to M$ with respect to the simple structure $\xi\big(e(0), s\big)$ on $\mathrm{FIB}(f)$ and

the simple structure associated to the structure $\xi^{\mathrm{Top}}(M)$ of a closed topological manifold on M.

In the sequel we identify $H^1(S^1; \mathrm{Wh}(\pi(M)) = \mathrm{Wh}(\pi(M))$ using the standard generator of $\pi_1(S^1) \cong H_1(S^1) \cong \mathbb{Z}$ represented by the identity map $\mathrm{id}\colon S^1 \to S^1$. In particular $\Theta(f)$ becomes an element in $\mathrm{Wh}(\pi(M))$.

Complex conjugation defines an orientation reversing selfdiffeomorphism

$$\mathrm{con}\colon S^1 \to S^1, \quad z \mapsto \overline{z}.$$

Lemma 6.2.

   (i) We have
$$\Theta(f) = \tau'_{\mathrm{fib}}(f) - \tau'_{\mathrm{fib}}(\mathrm{con} \circ f);$$

   (ii) If $\Theta(f) = 0$, then
$$\tau_{\mathrm{fib}}(f) = \tau'_{\mathrm{fib}}(f);$$

   (iii) We have $\tau'_{\mathrm{fib}}(f) = 0$ if $\Theta(f) = 0$ and $\tau_{\mathrm{fib}}(f) = 0$ hold.

Proof. (i) Let $\overline{s}$ be the spider on S^1 with base point $e(0)$ which is given by the constant path at $e(0)$ for the 0-cell and by the path $\overline{w}\colon [0,1] \to S^1$ sending t to $\overline{\exp(\pi it)} = \exp(-\pi it)$ for the 1-cell. Obviously $\tau'_{\mathrm{fib}}(\mathrm{con} \circ f)$ is the Whitehead torsion of the canonical homotopy equivalence $\mu_f\colon \mathrm{FIB}(f) \to M$ with respect to the simple structure $\xi\big(e(0), \overline{s}\big)$ on $\mathrm{FIB}(f)$ and the simple structure associated to the structure of a closed manifold on M. Hence we conclude from Lemma 1.4 (ii)

$$\tau'_{\mathrm{fib}}(\mathrm{con} \circ f) - \tau'_{\mathrm{fib}}(f) = \tau\big(\mathrm{id}\colon (\mathrm{FIB}(f), \xi(e(0), \overline{s})) \to (\mathrm{FIB}(f), \xi(e(0), s))\big).$$

Now the claim follows from Lemma 3.4 and the definition of $\Theta(f)$.

(ii) This follows from the definitions.

(iii) This follows from assertions (i) and (ii). $\square$

Remark 6.3 (Mapping tori). Given a selfmap $v\colon Y \to Y$, define its *mapping torus* T_v by the pushout

$$
\begin{array}{ccc}
Y \times \{0\} \coprod Y \times \{1\} & \xrightarrow{\mathrm{id} \coprod v} & Y \\
\downarrow & & \downarrow \\
Y \times [0,1] & \longrightarrow & T_v
\end{array}
$$

where the left horizontal arrow is the obvious inclusion. This corresponds to identifying in $Y \times [0,1]$ the point $(y,0)$ with $(v(y),1)$ for all $y \in Y$.

If Y has the homotopy type of a finite CW-complex, we can choose a simple structure on Y and equip the mapping torus T_v with the pushout simple structure (see Section 1). This simple structure is independent of the choice of the simple structure on Y by Lemma 1.4. Hence a mapping torus of a selfmap of a space with the homotopy type of a finite CW-complex has a preferred simple structure which we will use in the sequel without any further notice.

Consider selfhomotopy equivalences $v_i\colon Y_i \to Y_i$ of spaces of the homotopy type of a finite CW-complexes for $i = 0, 1$ such that $v_1 \circ u \simeq u \circ v_0$. Choose a homotopy $h\colon v_1 \circ u \simeq u \circ v_0$, where $u\colon Y_1 \to Y_2$ is a homotopy equivalence. Then the map $Y_1 \times [0,1] \to Y_2 \times [0,1]$ sending x, t to $(h(x,t), t)$ induces a map $T_{u,h}\colon T_{v_0} \to T_{v_1}$. Its homotopy class depends on the choice of u and h, but not its Whitehead torsion. Namely, Lemma 1.4 implies

$$\tau\big(T_{u,h}\colon T_{v_0} \to T_{v_1}\big) = 0.$$

Consider the pullback

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ \overline{e}\ } & M \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow {\scriptstyle f} \\ \mathbb{R} & \xrightarrow{\ e\ } & S^1 \end{array}$$

of f with the universal covering e. Consider the map $l_1 \colon \overline{M} \to \overline{M}$, induced by the action of $1 \in \mathbb{Z} \cong \pi_1(S^1)$ by deck transformations. Since $\overline{e} \circ l_1 = \overline{e}$, the map $\overline{M} \times [0,1] \to M$ sending (x,t) to $\overline{e}(x)$ induces a homotopy equivalence

$$\widehat{e} \colon T_{l_1} \to M.$$

Lemma 6.4. *We get*

$$\begin{aligned} \Theta(f) &= \overline{e}_*\big(\tau(l_1 \colon \overline{M} \to \overline{M})\big); \\ \tau'_{\mathrm{fib}}(f) &= \tau\big(\widehat{e} \colon T_{l_1} \to M\big), \end{aligned}$$

where we use the preferred simple structures on the mapping torus T_{l_1} and on the closed manifold M, and any simple structure on $\overline{M}$.

Proof. There is an explicit homotopy equivalence

$$h \colon \overline{M} \xrightarrow{\ \simeq\ } F,$$

which sends $x \in \overline{M}$ to $(\overline{e}(x), w) \in F$ for the path $w(t) = \exp\big(2\pi i \overline{f}(x)(1-t)\big)$. Let $t \colon F \to F$ be given by the canonical fiber transport along the standard generator of S^1. It sends a pair $(x, w) \in F$ to the pair $(x, w * v)$ for the path $v = e|_{[0,1]}$. We have by definition

$$\Theta(f) = (\mu_f \circ i)_*\big(\tau(t \colon F \to F)\big)$$

for any choice of simple structure on F, where $i \colon F \to \mathrm{FIB}(f)$ is the inclusion and $\mu_F \colon \mathrm{FIB}(f) \to M$ is the canonical map. We have $h \circ l_1 = t \circ h$. Lemma 1.4 implies that for any choice of simple structure on $\overline{M}$

$$\Theta(f) = (\mu_f \circ i \circ h)_*\big(\tau(l_1 \colon \overline{M} \to \overline{M})\big).$$

Since $\overline{e} = \mu_f \circ i \circ h$, we conclude

$$\Theta(f) = \overline{e}_*\big(\tau(l_1 \colon \overline{M} \to \overline{M})\big).$$

Define

$$\alpha' \colon F \times [0,1] \to \mathrm{FIB}(f), \quad \big((x,w), s\big) \mapsto (x, w_s),$$

where w_s is the path sending $s' \in [0,1]$ to

$$w_s(s') := \begin{cases} w\big((s+1)s'\big) & 0 \le s' \le \frac{1}{s+1}; \\ \exp\big(2\pi i(s'(s+1) - 1)\big) & \frac{1}{s+1} \le s' \le 1. \end{cases}$$

The following diagram commutes

$$\begin{array}{ccc} F \times [0,1] & \xrightarrow{\ \alpha'\ } & \mathrm{FIB}(f) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow {\scriptstyle f} \\ [0,1] & \xrightarrow{\ e\ } & S^1 \end{array}$$

The map α' induces over 0 the identity $F \to F$ and over 1 the map $t \colon F \to F$. Hence the map α' induces an explicit homotopy equivalence

$$\alpha \colon T_t \xrightarrow{\ \simeq\ } \mathrm{FIB}(f).$$

By definition

$$\tau'_{\mathrm{fib}}(f) = \tau\big(\mu_f \circ \alpha \colon T_t \to M\big).$$

Since $h \circ l_1 = t \circ h$, the map h induces an explicit homotopy equivalence

$$\beta \colon T_{l_1} \to T_t.$$

We conclude from Remark 6.3

$$\tau'_{\mathrm{fib}}(f) = \tau\big(\mu_f \circ \alpha \circ \beta \colon T_{l_1} \to M\big).$$

Since $\widehat{e} = \mu_f \circ \alpha \circ \beta$, we conclude

$$\tau'_{\mathrm{fib}}(f) = \tau\big(\widehat{e} \colon T_{l_1} \to M\big). \qquad \square$$

7. GLUING h-COBORDISMS

In this section we consider the illuminating example of a map $M \to S^1$ which is obtained from gluing the two ends of an h-cobordism together.

Let $(W, \partial_0 W, \partial_1 W)$ be a topological h-cobordism, i.e., a closed manifold W whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$ such that both inclusions $i_k \colon \partial_k W \to W$ are homotopy equivalences. Its Whitehead torsion

(7.1) $$\tau(W) \quad \in \quad \mathrm{Wh}(\partial_0 W)$$

is defined by the image of the Whitehead torsion $\tau(i_0 \colon \partial_0 W \to W)$ under the isomorphism $\mathrm{Wh}(\pi(W)) \xrightarrow{\cong} \mathrm{Wh}(\pi(\partial_0 W))$ induced by i_0^{-1}. Let $g \colon \partial_1 W \to \partial_0 W$ be a homeomorphism. Let W_g be the closed topological manifold obtained from W by gluing $\partial_1 W$ to $\partial_0 W$ by g. Choose any continuous map $f' \colon W \to [0,1]$ with $f'(\partial_0 W) = \{0\}$ and $f'(\partial_1 W) = \{1\}$. Let

$$f_g \colon W_g \to S^1$$

be the map induced by f'. Since $[0,1]$ is convex, f_g is unique up to homotopy. Let $l \colon \partial_0 W \to W_g$ be the obvious inclusion. Let $w_1 \colon \pi_1(\partial_0 W) \to \{\pm 1\}$ be the orientation homomorphism of $\partial_0 W$. The w_1-twisted anti-involution on the group ring $\mathbb{Z}\big[\pi_1(\partial_0 W)\big]$ is given by

$$\overline{\sum_{g \in \pi_1(\partial_0 W)} \lambda_g \cdot g} \quad = \quad \sum_{g \in \pi_1(\partial_0 W)} w_1(g) \cdot \lambda_g \cdot g^{-1}.$$

Let

$$\ast \colon \mathrm{Wh}(\pi(\partial_0 W)) \to \mathrm{Wh}(\pi(\partial_0 W))$$

be the induced involution. It corresponds geometrically to turning an h-cobordism upside down [17, §10]. Namely, if W^* is the h-cobordism with the roles of $\partial_0 W$ and $\partial_1 W$ interchanged, we get (see [17, Duality Theorem on page 394]

$$\tau(W^*) = (-1)^{\dim(\partial_0 W)} \cdot (i_1)_*^{-1} \circ (i_0)_* \circ \ast(\tau(W)).$$

Lemma 7.2. *Let*

$$\ast \colon \mathrm{Wh}(\pi(W_g)) \to \mathrm{Wh}(\pi(W_g))$$

be the $w_1(W_g)$-twisted involution, where $w_1(W_g) \colon \pi_1(W_g) \to \{\pm 1\}$ is the orientation homomorphism of W_g. Then:

(i) *We have*

$$\Theta(f_g) = l_*\big(\tau(g \circ i_1^{-1} \circ i_0)\big) = \big((-1)^{\dim(W)} \cdot \ast + \mathrm{id}\big) \circ l_*(\tau(W));$$

(ii) *We have*

$$\tau'_{\mathrm{fib}}(f_g) = (-1)^{\dim(W)} \cdot \ast \circ l_*(\tau(W)) = l_*(\tau(W)) - \Theta(f_g);$$

(iii) *If $\Theta(f_g) = 0$, then*

$$\tau_{\mathrm{fib}}(f_g) = -l_*(\tau(W)).$$

(iv) *The following assertions are equivalent:*
    (a) *$l_*(\tau(W)) = 0$;*

(b) $\tau'_{\mathrm{fib}}(f_g) = 0$;

(c) $\Theta(f_g) = 0$ and $\tau_{\mathrm{fib}}(f_g) = 0$.

Proof. (i) Since $\partial_0 W$ is part of the boundary of W, we get $w_1(W_g) \circ \pi_1(l) = w(\partial_0 W)$. Hence $l_* \colon \mathrm{Wh}(\pi(\partial_0 W)) \to \mathrm{Wh}(\pi(W_g))$ is compatible with the involutions.

Consider the pullback

$$
\begin{array}{ccc}
\overline{W_g} & \xrightarrow{\ \overline{e}\ } & W_g \\
\ \downarrow{\scriptstyle \overline{f_g}} & & \ \downarrow{\scriptstyle f_g} \\
\mathbb{R} & \xrightarrow{\ e\ } & S^1
\end{array}
$$

of f_g with the universal covering e. Notice that $\overline{W_g}$ is obtained from $W \times \mathbb{Z}$ by identifying $(g(x), n)$ and $(x, n+1)$ for $x \in \partial_1 W$ and $n \in \mathbb{Z}$ and the map $l_1 \colon \overline{W_g} \to \overline{W_g}$ is induced by the map $(x, n) \mapsto (x, n+1)$. The inclusion $l \colon \partial_0 W \to W_g$ lifts uniquely to an inclusion $\overline{l} \colon \partial_0 W \to \overline{W_g}$ which satisfies $\overline{e} \circ \overline{l}(W_g) = \{0\}$ and is a homotopy equivalence. In the model above this corresponds to sending x to $(x, 0)$ for $x \in \partial_0 W$. Now

$$
\Theta(f_g) = l_* \big(\tau(g \circ i_1^{-1} \circ i_0) \big)
$$

with respect to the simple structure on $\partial_0 W$ coming from the structure of a topological manifold follows Remark 6.3 and Lemma 6.4 since $(g \circ i_1^{-1} \circ i_0) \simeq \overline{l}^{-1} \circ l_1 \circ \overline{l}$. Now the assertion (i) follows from

$$
\begin{aligned}
l_* \big(\tau(g \circ i_1^{-1} \circ i_0) \big) &= l_*(\tau(g)) + (l \circ g)_*(\tau(i_1^{-1})) + (l \circ g \circ i_1^{-1})_*(\tau(i_0)) \\
&= 0 - l_* \circ (i_0)_*^{-1} \circ (i_1)_* \circ (i_1)_*^{-1}(\tau(i_1)) + l_* \circ (i_0)_*^{-1}(\tau(i_0)) \\
&= l_* \big(-(i_0)_*^{-1} \circ (i_1)_* \circ (i_1)_*^{-1}(\tau(i_1)) + (i_0)_*^{-1}(\tau(i_0)) \big) \\
&= l_* \big(-(-1)^{\dim(\partial_0 W)} \cdot *(\tau(W)) + \tau(W) \big) \\
&= l_* \circ \big((-1)^{\dim(W)} \cdot * + \mathrm{id} \big)(\tau(W)) \\
&= \big((-1)^{\dim(W)} \cdot * + \mathrm{id} \big) \circ l_*(\tau(W)).
\end{aligned}
$$

(ii) Consider the commutative diagram

$$
\begin{array}{ccccc}
\partial_0 W & \xleftarrow{\ \mathrm{id} \amalg g \circ i_1^{-1} \circ i_0\ } & \partial_0 W \amalg \partial_0 W & \xrightarrow{\ j\ } & \partial_0 W \times [0,1] \\
\ \downarrow{\scriptstyle \mathrm{id}} & & \ \downarrow{\scriptstyle \mathrm{id} \amalg i_1^{-1} \circ i_0} & & \ \downarrow{\scriptstyle h \circ (i_0 \times \mathrm{id}_{[0,1]})} \\
\partial_0 W & \xleftarrow{\ \mathrm{id} \amalg g\ } & \partial_0 W \amalg \partial_1 W & \xrightarrow{\ i_0 \amalg i_1\ } & W
\end{array}
$$

where $j \colon \partial_0 W \amalg \partial_0 W = \partial_0 W \times \{0,1\} \to \partial_0 W \times [0,1]$ is the inclusion, $i_1^{-1} \colon W \to \partial_1 W$ is a homotopy inverse of i_1 and $h \colon \mathrm{id}_W \simeq i_1 \circ i_1^{-1}$ is some homotopy. The pushout of the upper row is the mapping torus $T_{g \circ i_1^{-1} \circ i_0}$. The pushout of the lower row is W_g and the structure $\xi^{\mathrm{Top}}(W_g)$ on the closed manifold W_g is just the pushout of the simple structures. All vertical arrows are homotopy equivalences. We obtain a homotopy equivalence

$$
\lambda \colon T_{g \circ i_1^{-1} \circ i_0} \to W_g.
$$

We conclude from Remark 6.3 and Lemma 6.4 since $\overline{l} \circ (g \circ i_1^{-1} \circ i_0) \simeq l_1 \circ \overline{l}$.

$$
\tau'_{\mathrm{fib}}(f_g) = \tau(\lambda).
$$

Let $\mathrm{pr}\colon W \to W_g$ be the canonical projection. We conclude from Lemma 1.4, assertion (i) and the diagram above

$$
\begin{aligned}
\tau'_{\mathrm{fib}}(f_g) &= \tau(\lambda) \\
&= -(\mathrm{pr}\circ i_1)_*(\tau(i_1^{-1} \circ i_0)) + \mathrm{pr}_*(\tau(h \circ (i_0 \times \mathrm{id}_{[0,1]}))) \\
&= -(\mathrm{pr}\circ i_1)_*(\tau(i_1^{-1})) - (\mathrm{pr}\circ i_1 \circ i_1^{-1})_*(\tau(i_0)) + \mathrm{pr}_*(\tau(h)) \\
&\qquad\qquad\qquad\qquad\qquad +(\mathrm{pr}\circ h)_*(\tau(i_0 \times \mathrm{id}_{[0,1]})) \\
&= (\mathrm{pr}\circ i_1 \circ i_1^{-1})_*(\tau(i_1)) - \mathrm{pr}_*(\tau(i_0)) + \mathrm{pr}_*(\tau(\mathrm{id}_W)) + \mathrm{pr}_*(\tau(i_0)) \\
&= (\mathrm{pr}\circ i_0)_* \circ (i_0^{-1} \circ i_1 \circ i_1^{-1})_*(\tau(i_1)) \\
&= l_* \circ (i_0^{-1} \circ i_1)_*\big(\tau(W^*)\big) \\
&= (-1)^{\dim(\partial_0 W)} \cdot l_* \circ *(\tau(W)) \\
&= (-1)^{\dim(\partial_0 W)} \cdot * \circ l_*(\tau(W)) \\
&= l_*(\tau(W)) - \Theta(f_g).
\end{aligned}
$$

(iii) This follows from Lemma 6.2 (ii) and assertion (ii).

(iv) This follows from assertions (i),(ii) and (iii). $\square$

Hence $\Theta(f)$ and $\tau_{\mathrm{fib}}(f)$ are given in terms of $\tau(W)$. The map induced by l on the fundamental groups can be identified with the inclusion of $\pi_1(\partial_0 W)$ into the semi-direct product $\pi_1(\partial_0 W) \rtimes_\phi \mathbb{Z} = \pi_1(W_g)$, where ϕ is the automorphism of $\pi_1(\partial_0 W)$ induced by $g \circ i_1^{-1} \circ i_0$. The map $l_*\colon \mathrm{Wh}(\pi_1(W)) \to \mathrm{Wh}(\pi(W_g))$ is injective if $\phi = \mathrm{id}$ but not in general. So it can happen that the h-cobordism W is non-trivial but both elements $\Theta(f_g)$ and $\tau_{\mathrm{fib}}(f_g)$ vanish. Moreover, for a fixed h-cobordism W the answer to the question, whether $\Theta(f_g)$ or $\tau_{\mathrm{fib}}(f_g)$ vanishes, does in general depend on $\pi_1(g)$.

8. Comparison with Farrell's obstruction over S^1

In this section we show in the case of S^1 as base space that the torsion obstructions defined in this article are equivalent to the ones defined by Farrell [6]. As in the paper [6] we will work in the smooth category in this section.

Throughout this section we consider a map

$$f\colon M \to S^1$$

from a connected closed smooth manifold M to S^1 such that its homotopy fiber has the homotopy type of a finite CW-complex, the homomorphism $\pi_1(f)\colon \pi_1(M) \to \pi_1(S^1)$ is surjective and the dimension of M is at least five. Let G be the kernel of $\pi_1(f)$. Choose an element $t \in \pi_1(M)$ which is sent under $\pi_1(f)$ to the standard generator of the infinite cyclic group $\pi_1(S^1)$. Conjugation with t induces an automorphism $\alpha\colon G \xrightarrow{\cong} G$. We obtain an isomorphism $G \rtimes_\alpha \mathbb{Z} \xrightarrow{\cong} \pi_1(M)$ which is the identity on G and sends $1 \in \mathbb{Z}$ to $t \in \pi_1(M)$. We will use it to identify $G \rtimes_\alpha \mathbb{Z} = \pi_1(M)$.

A *splitting* of M with respect to f is a pair (N, ν) such that N is a closed submanifold of M of codimension one together with a framing ν of the normal bundle such that under the Pontrjagin Thom construction ν corresponds to f. Such a splitting can be obtained by changing f in its homotopy class to a smooth map which is transversal to $\{e(0)\} \in S^1$ and taking N to be the preimage of $e(0)$. If we take out a tubular neighborhood of N in M, we obtain a cobordism M_N with two boundary components $\partial_0 M_N = N$ to $\partial_1 M_N = N$. A splitting is called a *pseudo fibering* if $\big(M_N, \partial_0 M_N, \partial_1 M_N\big)$ is an h-cobordism. We use the convention that going from $\partial_0 M_N$ to $\partial_1 M_N$ corresponds to going in the circle in the anticlockwise sense.

Farrell [6, Chapter III] introduces an element $c(f) \in C(\mathbb{Z}G; \alpha)$ which depends only on the homotopy class of f. It vanishes if and only if there exists a pseudo fibering (see [6, Chapter V]). Farrell [6, Chapter IV] constructs a duality isomorphism $C(\mathbb{Z}G; \alpha) \xrightarrow{\cong} C(\mathbb{Z}G; \alpha^{-1})$ which sends $c(f)$ to $c(\mathrm{con} \circ f)$. Hence the vanishing of $c(f)$ is equivalent to the vanishing of $c(\mathrm{con} \circ f)$.

Farrell [6, Chapter I] defines a map $p \colon \mathrm{Wh}(G \times_\alpha \mathbb{Z}) \to C(\mathbb{Z}G, \alpha)$. By inspecting a highly connected splitting one sees that it sends $\tau'_{\mathrm{fib}}(f)$ to $c(f)$ (see [6, Lemma 3.7]). In particular the vanishing of $\tau'_{\mathrm{fib}}(f)$ implies the vanishing of $c(f)$.

Now suppose that $c(f)$ is trivial. Then we can find a pseudo fibering (N, ν) for f. Recall that associated to it is an h-cobordism M_N obtained from M by deleting a tubular neighborhood of N. Its Whitehead torsion $\tau(M_N)$ lives in $\mathrm{Wh}(\pi(\partial_0 M_N))$. Let $i \colon \partial_0 M_N \to M$ be the inclusion and $\mathrm{pr} \colon \mathrm{Wh}(G) \to \mathrm{Wh}(G) \otimes_\alpha \mathbb{Z}$ be the canonical projection. Then Farrell [6, Chapter VI] defines

$$\tau(f) \in \mathrm{Wh}_\alpha(G) := \mathrm{Wh}(G) \otimes_\alpha \mathbb{Z}$$

to be the image of $\tau(M_N)$ under the map $\mathrm{pr} \circ i_* \colon \mathrm{Wh}(\partial_0 M_N) \to \mathrm{Wh}(G) \otimes_\alpha \mathbb{Z}$. The inclusion $G \to G \rtimes_\alpha \mathbb{Z}$ induces a map

$$j \colon \mathrm{Wh}(G) \otimes_\alpha \mathbb{Z} \to \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z})$$

which is injective by [7].

The identity on N yields a diffeomorphism $g \colon \partial_0 M_N \xrightarrow{\cong} \partial_1 M_N$ and we can consider in the notation of Section 7 the manifold $(M_N)_g$ together with a up to homotopy well-defined map $f' \colon M_N \to S^1$. We can construct a diffeomorphism $\psi \colon (M_N)_g \to M$ such that up to homotopy $f \circ \psi = f'$. Now we conclude from Lemma 7.2 (ii) that the injective map $j \colon \mathrm{Wh}(G) \otimes_\alpha \mathbb{Z} \to \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z})$ sends $\tau(f)$ to $(-1)^{\dim(W)} \cdot *\big(\tau'_{\mathrm{fib}}(f)\big)$. Hence $\tau(f)$ vanishes if and only if $\tau'_{\mathrm{fib}}(f)$ vanishes.

Thus we have shown that the vanishing of $\tau'_{\mathrm{fib}}(f)$ implies the vanishing of the obstructions $c(f)$ and $\tau(f)$ of Farrell. Exploiting the main theorem of Farrell [6] that $c(f)$ and $\tau(f)$ vanish if and only if f is homotopic to a smooth fiber bundle, we conclude from Theorem 5.3 (ii), Lemma 7.2 (iv):

Theorem 8.1. *Let $f \colon M \to S^1$ be a map from a connected closed smooth manifold M to S^1 such that its homotopy fiber has the homotopy type of a finite CW-complex, the homomorphism $\pi_1(f) \colon \pi_1(M) \to \pi_1(S^1)$ is surjective and the dimension of M is at least five. Then the following assertions are equivalent:*

(i) $\tau'_{\mathrm{fib}}(f)$ vanishes;
(ii) $\Theta(f)$ and $\tau_{\mathrm{fib}}(f)$ vanish;
(iii) $c(f)$ and $\tau(f)$ vanish;
(iv) f is homotopic to a smooth fiber bundle.

Remark 8.2. Siebenmann [21, Section 1] says that the main theorem of Farrell [6] and hence Theorem 8.1 hold also in the topological category.

9. A composition formula

In this section we want to to express $\tau_{\mathrm{fib}}(g \circ f)$ in terms of $\tau_{\mathrm{fib}}(f)$ and $\tau_{\mathrm{fib}}(g)$.

Let $f \colon M \to N$ and $g \colon N \to B$ be maps of closed path-connected manifolds. Assume that the homotopy fibers of both f and g have the homotopy type of a finite CW-complex. Then the same is true for the composite $g \circ f$ since there is a fibration $\mathrm{hofib}(f) \to \mathrm{hofib}(g \circ f) \to \mathrm{hofib}(g)$. So the elements $\Theta(f) \in H^1\big(N, \mathrm{Wh}(\pi(M))\big)$, $\Theta(g) \in H^1\big(N, \mathrm{Wh}(\pi(N))\big)$ and $\Theta(g \circ f) \in H^1\big(B; \mathrm{Wh}(\pi(M))\big)$ are defined. Assume that $\Theta(f)$, $\Theta(g)$ and $\Theta(g \circ f)$ vanish. We obtain fiber torsion obstructions
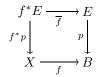
(see Definition 5.2)

$$\tau_{\mathrm{fib}}(f) \quad \in \quad \mathrm{cok}\big(\chi(N) \cdot (\mu_f \circ i_f)_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(f))) \to \mathrm{Wh}(\pi(M)))\big);$$

$$\tau_{\mathrm{fib}}(g) \quad \in \quad \mathrm{cok}\big(\chi(B) \cdot (\mu_g \circ i_g)_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g))) \to \mathrm{Wh}(\pi(N)))\big);$$

$$\tau_{\mathrm{fib}}(g \circ f) \quad \in \quad \mathrm{cok}\big(\chi(B) \cdot (\mu_{g\circ f} \circ i_{g\circ f})_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f))) \to \mathrm{Wh}(\pi(M)))\big),$$

where i_f, i_g and $i_{g\circ f}$ denote the inclusions of the homotopy fibers over $z \in N$ or $f(z) \in B$ respectively for a fixed base point $z \in N$.

Given a simple fibration $p \colon E \to B$ over a finite CW-complex one can define a *transfer map*

$$(9.1) \qquad\qquad p^* \colon \mathrm{Wh}(B) \quad \to \quad \mathrm{Wh}(E)$$

as follows, provided that the fiber is homotopy equivalent to a finite CW-complex. For simplicity assume that B is path connected, the general case is then done componentwise. Given an element $\omega \in \mathrm{Wh}(\pi(B))$, choose a finite CW-complex X together with a homotopy equivalence $f \colon X \to B$ such $\tau(f) = \omega$. Consider the pullback diagram

$$
\begin{array}{ccc}
f^*E & \xrightarrow{\ \overline{f}\ } & E \\
{\scriptstyle f^*p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
X & \xrightarrow{\ f\ } & B
\end{array}
$$

Choose a point $x \in X$. Let $f(x) \in B$ be its image under f. Choose a simple structure on the fiber of E over $f(x)$. Use the same simple structure ζ on the fiber of p^*E over x. We obtain well-defined simple structures $\xi(x, \zeta)$ on f^*E and $\xi(f(x), \zeta)$ on E (see Notation 3.9). Now define

$$p^*(\omega) = \tau\big(\overline{f} \colon (p^*E, (\xi(x, \zeta)) \to (E, (\xi(f(x), \zeta))\big).$$

This is a well-defined homomorphism because of Lemma 1.4, Lemma 3.12 and Lemma 3.13. By construction the transfer is compatible with pullbacks and by Lemma 3.11 with fiber homotopy equivalences. More information about this transfer map including its algebraic description and computational tools can be found for instance in [12], [13] and [14].

We obtain a transfer map

$$f^* \colon \mathrm{Wh}(\pi(N)) \to \mathrm{Wh}(\pi(M))$$

from the transfer associated in (9.1) to the fibration $\widehat{f} \colon \mathrm{FIB}(f) \to N$ and the isomorphism $(\mu_f)_* \colon \mathrm{Wh}\big(\pi(\mathrm{FIB}(f))\big) \xrightarrow{\cong} \mathrm{Wh}(\pi(M))$ induced by the homotopy equivalence $\mu_f \colon \mathrm{FIB}(f) \to M$. Since the transfer is compatible with pullbacks and fiber homotopy equivalences, the transfer induces a map, also denoted by f^*,

$$(9.2) \quad f^* \colon \mathrm{cok}\big(\chi(N) \cdot (\mu_g \circ i_g)_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g))) \to \mathrm{Wh}(\pi(N)))$$
$$\to \mathrm{cok}\big(\chi(B) \cdot (\mu_{g\circ f} \circ i_{g\circ f})_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f))) \to \mathrm{Wh}(\pi(M)))\big).$$

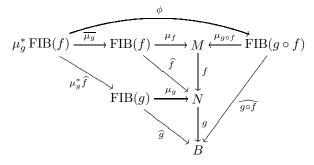Since $\chi(N) = \chi(M) \cdot \chi(B)$, the element $\tau_{\mathrm{fib}}(f)$ defines an element $\overline{\tau}_{\mathrm{fib}}(f) \in \mathrm{cok}\big(\chi(B) \cdot (\mu_{g\circ f} \circ i_{g\circ f})_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f))) \to \mathrm{Wh}(\pi(M)))\big).$

Theorem 9.3. *Under the conditions above we get*

$$\tau_{\mathrm{fib}}(g \circ f) = \overline{\tau}_{\mathrm{fib}}(f) + f^*(\tau_{\mathrm{fib}}(g))$$

in $\mathrm{cok}\big(\chi(B) \cdot (\mu_{g\circ f} \circ i_{g\circ f})_* \colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f))) \to \mathrm{Wh}(\pi(M)))\big).$
If we additionally assume $\chi(B) = 0$, *we get*

$$\tau_{\mathrm{fib}}(g \circ f) = \tau_{\mathrm{fib}}(f) + f^*(\tau_{\mathrm{fib}}(g))$$

in $\mathrm{Wh}(\pi(M))$.

Proof. Consider the following commutative diagram

Here $\mu_g^* \widehat{f}\colon \mu_g^* \mathrm{FIB}(f) \to \mathrm{FIB}(g)$ is the pullback of the fibration $\widehat{f}\colon \mathrm{FIB}(f) \to N$ with $\mu_g\colon \mathrm{FIB}(g) \to N$ and ϕ is an appropriate fiber homotopy equivalence of fibrations over B from $\widehat{g} \circ \mu_g^* \widehat{f}$ to $\widehat{g \circ f}$ such that $\mu_{g \circ f} \circ \phi$ is homotopic to $\widehat{\mu_f} \circ \overline{\mu_g}$. Since $\widehat{g \circ f}$ is simple, the fibration $\widehat{g} \circ \mu_g^* \widehat{f}$ is also simple

We equip M, N and B with canonical simple structure coming from the manifold structure.

Choose simple structures on $\mathrm{hofib}(f)$, $\mathrm{hofib}(g)$, $\mathrm{hofib}(g \circ f)$ and the fiber of the fibration given by the composite $\widehat{g} \circ \mu_g^* \widehat{f}$. We equip the total spaces of the simple fibrations over finite CW-complexes $\widehat{g}\colon \mathrm{FIB}(g) \to B$, $\widehat{f}\colon \mathrm{FIB}(f) \to N$, $\widehat{g \circ f}\colon \mathrm{FIB}(g \circ f) \to B$ and $\widehat{g} \circ \mu_g^* \widehat{f}\colon \mu_g^* \mathrm{FIB}(f) \to B$ with the simple structure coming from Notation 3.9.

Since ϕ is a fiber homotopy equivalence, we conclude from Lemma 3.12

$$\tau(\phi) = 0$$

in $\mathrm{cok}\big(\chi(B) \cdot (\mu_{g \circ f} \circ i_{g \circ f})_*\colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f)) \to \mathrm{Wh}(\pi(M)))\big)$.

There is a second simple structure on $\mu_g^* \mathrm{FIB}(f)$ which comes from Remark 3.14 applied to the fibration $\mu_g^* \widehat{f}\colon \mu_g^* \mathrm{FIB}(f) \to \mathrm{FIB}(g)$ and the simple structure defined on $\mathrm{FIB}(g)$ above. With respect to this simple structure we conclude from the definition of the transfer maps

$$f^*(\tau(\mu_g)) = (\mu_f)_* \big(\tau(\overline{\mu_g})\big).$$

These two simple structures on $\mu_g^* \mathrm{FIB}(f)$ are not necessarily the same. But a modification of the proof of Lemma 3.11 and Lemma 3.12 show that the image of the Whitehead torsion of the identity map on $\mu_g^* \mathrm{FIB}(f)$ under the isomorphism

$$(\mu_f \circ \overline{\mu_g})_*\colon \mathrm{Wh}\big(\pi(\mu_g^* \mathrm{FIB}(f))\big) \xrightarrow{\cong} \mathrm{Wh}(M)$$

with respect to these two different simple structures becomes zero when regarded in $\mathrm{cok}\big(\chi(B) \cdot (\mu_{g \circ f} \circ i_{g \circ f})_*\colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f)) \to \mathrm{Wh}(\pi(M)))\big)$. Hence it does not matter which simple structure we use.

¿From the composition formula for Whitehead torsion and the equalities above we conclude in $\mathrm{cok}\big(\chi(B) \cdot (\mu_{g \circ f} \circ i_{g \circ f})_*\colon \mathrm{Wh}(\pi(\mathrm{hofib}(g \circ f))) \to \mathrm{Wh}(\pi(M)))\big)$

$$
\begin{aligned}
\tau_{\mathrm{fib}}(g \circ f) &= \tau(\mu_{g \circ f}) \\
&= \tau(\mu_{g \circ f}) + (\mu_{g \circ f})_*(\tau(\phi)) \\
&= \tau\big(\mu_{g \circ f} \circ \phi\big) \\
&= \tau\big(\mu_f \circ \overline{\mu_g}\big) \\
&= \tau(\mu_f) + (\mu_f)_* \big(\tau(\overline{\mu_g})\big) \\
&= \tau(\mu_f) + f^*\big(\tau(\mu_g)\big) \\
&= \bar{\tau}_{\mathrm{fib}}(f) + f^*(\tau_{\mathrm{fib}}(g)).
\end{aligned}
$$

$\square$

10. POINCARÉ TORSION

The definitions and probably most of the properties of the Poincaré torsion are known to the experts but since we could not find a good reference in the literature, we elaborate on them in this section. Some information can be found for instance in [11, Proposition 26].

Let X be a finite CW-complex. Suppose that X is connected. Denote by π the fundamental group $\pi_1(X)$. Let $p\colon \widetilde{X} \to X$ be the universal covering. Denote by $C_*(\widetilde{X})$ the cellular $\mathbb{Z}\pi$-chain complex. Let $C^{n-*}(\widetilde{X})$ denote the dual $\mathbb{Z}\pi$-chain complexes, where we always use the involution on $\mathbb{Z}\pi$ sending g to $w(g) \cdot g^{-1}$ for a given homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$. We call X an *n-dimensional Poincaré complex*, if there exist an element called *fundamental class* $[X] \in H_n(X;\mathbb{Z})$ and a so called *orientation homomorphism* $w_1(X)\colon \pi_1(X) \to \{\pm 1\}$ such that the up to $\mathbb{Z}\pi$-chain homotopy uniquely defined $\mathbb{Z}\pi$-chain map

$$(10.1) \qquad\qquad - \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence. If a finite CW-complex carries some structure of a Poincaré complex, then $H_n(X;\mathbb{Z})$ is infinite cyclic and the fundamental class $[X]$ is a generator and hence unique up to sign, and one can rediscover the orientation homomorphism w from X as a CW-complex (see [16, paragraph before 1.3]).

If X is not connected, we require that each component $C \in \pi_0(X)$ is an n-dimensional Poincaré pair in the sense above.

Let $(X, \partial X)$ be a finite CW-pair such that X is n-dimensional and ∂X is $(n-1)$-dimensional. Suppose that X is connected. Denote by π the fundamental group $\pi_1(X)$. Let $p\colon \widetilde{X} \to X$ be the universal covering and put $\widetilde{\partial X} = p^{-1}(\partial X)$. Denote by $C_*(\widetilde{X})$ and $C_*(\widetilde{X}, \widetilde{\partial X})$ the cellular $\mathbb{Z}\pi$-chain complexes. Let $C^{n-*}(\widetilde{X})$ and $C^{n-*}(\widetilde{X}, \widetilde{\partial X})$ denote the dual $\mathbb{Z}\pi$-chain complexes, where we always use the involution on $\mathbb{Z}\pi$ sending g to $w(g) \cdot g^{-1}$ for a given homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$. We call $(X, \partial X)$ a *n-dimensional Poincaré pair*, if there exists an element called *fundamental class* $[X, \partial X] \in H_n(X, \partial X;\mathbb{Z})$ and a so called *orientation homomorphism* $w_1(X)\colon \pi_1(X) \to \{\pm 1\}$ such that the up to $\mathbb{Z}\pi$-chain homotopy uniquely defined $\mathbb{Z}\pi$-chain map

$$(10.2) \qquad\qquad - \cap [X, \partial X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{\partial X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence and ∂X is a Poincaré complex with respect to the fundamental classes of each component of ∂X coming from the image of the fundamental class of X under the boundary homomorphism $H_n(X, \partial X;\mathbb{Z}) \to H_{n-1}(\partial X;\mathbb{Z})$.

If X is not connected, we require that for each component $C \in \pi_0(X)$ the pair $(C, C \cap \partial X)$ is an n-dimensional Poincaré pair in the sense above.

The chain complexes $C^{n-*}(\widetilde{X})$ and $C_*(\widetilde{X}, \widetilde{\partial X})$ inherit from the CW-structure a cellular $\mathbb{Z}\pi$-basis which is unique up to permuting the elements of the basis or multiplying with elements of the form $\pm g$ for $g \in \pi$. Hence one can associate to the $\mathbb{Z}\pi$-chain homotopy equivalence defined in (10.2) its Whitehead torsion $\tau\big(\cap[X, \partial X]\big) \in \mathrm{Wh}(\pi)$. Since X is connected and hence $H^0(X) = \mathbb{Z}$, we get from Poincaré duality that $H_n(X, \partial X;\mathbb{Z}) = \mathbb{Z}$ and $[X, \partial X]$ must be a generator. If we replace $[X, \partial X]$ by $-[X, \partial X]$, we get $\tau\big(- \cap (-[X, \partial X])\big) = \tau\big(- \cap [X, \partial X]\big)$ since the Whitehead torsion satisfies the composition formula $\tau(g \circ f) = \tau(f) + \tau(g)$ and $\tau(-\,\mathrm{id}) = 0$.

Definition 10.3. *Let $(X, \partial X)$ be an n-dimensional Poincaré pair. If X is connected, define its Poincaré torsion*

$$\rho(X, \partial X) \in \mathrm{Wh}(\pi(X))$$

by the Whitehead torsion $\tau\left(- \cap [X, \partial X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{\partial X})\right)$ for any choice of fundamental class $[X, \partial X] \in H_n(X, \partial X; \mathbb{Z})$.

If X is not connected, define

$$\rho(X, \partial X) \in \mathrm{Wh}(\pi(X)) = \bigoplus_{C \in \pi_0(X)} \mathrm{Wh}(\pi(C))$$

by the various elements $\rho(C, C \cap \partial X) \in \mathrm{Wh}(\pi(C))$.

We call an n-dimensional Poincaré pair $(X, \partial X)$ simple if $\rho(X, \partial X) = 0$.

Next we collect the basic properties of this invariant (see also [26, Proposition 2.7]). Notice that because of Theorem 10.4 (ii) we can extend the Definition 10.3 of $\rho(X, A)$ to pairs of spaces (X, A) with simple structures for which there exists a simple homotopy equivalence $(X, A) \to (Y, \partial Y)$ with a finite Poincaré pair as target. This applies in particular to $(X, A) = (M, \partial M)$ for a compact topological manifold M with boundary ∂M.

Theorem 10.4.   *(i) If X is an n-dimensional Poincaré complex, then*

$$\rho(X) = (-1)^n \cdot *\left(\rho(X)\right),$$

where $* \colon \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(X))$ is the $w_1(X)$-twisted involution.
More generally, we get for an n-dimensional Poincaré pair $(X, \partial X)$

$$(j_{\partial X})_*\left(\rho(\partial X)\right) = (-1)^n \cdot *\left(\rho(X, \partial X)\right) - \rho(X, \partial X),$$

where $j_{\partial X} \colon \partial X \to X$ is the inclusion;
(ii) If $(f, \partial f) \colon (X, \partial X) \to (Y, \partial Y)$ is a homotopy equivalence of n-dimensional Poincaré pairs, then

$$\rho(Y, \partial Y) - f_*\left(\rho(X, \partial X)\right) \ = \ \tau(f) + (-1)^n \cdot *\left(\tau(f)\right) - (j_{\partial M})_*\left(\tau(\partial f)\right),$$

where $f_* \colon \mathrm{Wh}(\pi(X)) \to \mathrm{Wh}(\pi(Y))$ and $(j_{\partial M})_* \colon \mathrm{Wh}(\pi(\partial Y)) \to \mathrm{Wh}(\pi(Y))$ are the homomorphisms induced by $f \colon X \to Y$ and the inclusion $j_{\partial M} \colon \partial Y \to Y$, $\tau(f) \in \mathrm{Wh}(\pi(Y))$ and $\tau(\partial f) \in \mathrm{Wh}(\pi(\partial Y))$ denote the Whitehead torsion of the homotopy equivalences of finite CW-complexes f and ∂f, and $* \colon \mathrm{Wh}(\pi(Y)) \to \mathrm{Wh}(\pi(Y))$ is the $w_1(Y)$-twisted involution;
(iii) Let $(X, \partial X)$ and $(Y, \partial Y)$ be n-dimensional Poincaré pairs such that X and Y are connected. Let $f \colon \partial X \to \partial Y$ be a homotopy equivalence. Let $X \cup_f Y$ be the space obtained by gluing X to Y along f. Denote by $j_X \colon X \to X \cup_f Y$, $j_Y \colon Y \to X \cup_f Y$ and $j_{\partial Y} \colon \partial Y \to X \cup_f Y$ the canonical inclusions. Then $X \cup_f Y$ is a connected n-dimensional Poincaré complex and

$$\rho(X \cup_f Y) \ = \ (-1)^n \cdot * \circ (j_X)_*\left(\rho(X, \partial X)\right) + (j_Y)_*\left(\rho(Y, \partial Y)\right)$$
$$+ (j_{\partial Y})_*\left(\tau(f)\right),$$

where $* \colon \mathrm{Wh}\left(\pi(Z \cup_f Y)\right) \to \mathrm{Wh}\left(\pi(Z \cup_f Y)\right)$ is the $w_1(Z \cup_f Y)$-twisted involution.
(iv) Let $(X, \partial X)$ resp. $(Y, \partial Y)$ be a m- resp. n-dimensional Poincaré pair such that X and Y are connected. Then

$$(X, \partial X) \times (Y, \partial Y) = (X \times Y, X \times \partial Y \cup \partial X \times Y)$$

is an $(n+m)$-dimensional Poincaré pair with

$$\rho\big((X,\partial X)\times(Y,\partial Y)\big) = \chi(X,\partial X)\cdot(k_Y)_*\big(\rho(Y,\partial Y)\big)$$
$$+ \chi(Y,\partial Y)\cdot(k_X)_*\big(\rho(X,\partial X)\big),$$

where $k_X\colon X \to X\times Y$ and $k_Y\colon Y \to X\times Y$ are the inclusions and χ denotes the Euler characteristic;

(v) Let M be a compact topological manifold (possibly with boundary ∂M). Then

$$\rho(M,\partial M) = 0.$$

Proof. (i) The $\mathbb{Z}\pi$-chain map $-\cap[X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$ is self dual in the sense that it is $\mathbb{Z}\pi$-chain homotopic to the one obtained from $-\cap[X]$ by applying the functor C^{n-*} and the obvious identification $(C^{n-*}(\widetilde{X}))^{n-*} = C_*(\widetilde{X})$. This follows from the fact the chain map $-\cap[X]$ is the zeroth part of a cocycle in the Q^n-group associated to $C_*(\widetilde{X})$ (see [20, Section 1 and Proposition 2.1]). This implies

$$\rho(X) = \tau(-\cap[X]) = \tau\big(C^{n-*}(-\cap[X])\big) = (-1)^n\cdot *\big(\tau(-\cap[X])\big) = (-1)^n\cdot *\big(\rho(X)\big).$$

The case of a pair is more complicated. For the definition of the mapping cylinder and cone of a chain map and the basic properties of these we refer for instance to [15, page 213 ff.]. We have the short based exact sequences of finite based free $\mathbb{Z}\pi$-chain complexes

$$0 \to C_*(\widetilde{\partial X}) \xrightarrow{i_*} C_*(\widetilde{X}) \xrightarrow{p_*} C_*(\widetilde{X},\widetilde{\partial X}) \to 0.$$

and

$$0 \to C^{n-*}(\widetilde{X},\widetilde{\partial X}) \xrightarrow{p^{n-*}} C^{n-*}(\widetilde{X}) \xrightarrow{i^{n-*}} C^{n-*}(\widetilde{\partial X}) \to 0.$$

We obtain short based exact sequences of finite based free $\mathbb{Z}\pi$-chain complexes

$$0 \to C_*(\widetilde{X}) \xrightarrow{j_*} \mathrm{cyl}(p_*) \xrightarrow{q_*} \mathrm{cone}(p_*) \to 0$$

and

$$0 \to \Sigma C_*(\widetilde{\partial X}) \xrightarrow{k_*} \mathrm{cone}(p_*) \xrightarrow{r_*} \mathrm{cone}\big(C_*(\widetilde{X},\widetilde{\partial X})\big) \to 0.$$

Since the chain map $0_* \to \mathrm{cone}\big(C_*(\widetilde{X},\widetilde{\partial X})\big)$ is a simple $\mathbb{Z}\pi$-chain homotopy equivalence, the chain map

$$k_*\colon \Sigma C_*(\widetilde{\partial X}) \xrightarrow{\;\simeq_s\;} \mathrm{cone}(p_*)$$

is a simple chain $\mathbb{Z}\pi$-chain homotopy equivalence.

Analogously we get short based exact sequences of finite based free $\mathbb{Z}\pi$-chain complexes

$$0 \to C^{n-*}(\widetilde{X},\widetilde{\partial X}) \xrightarrow{l_*} \mathrm{cyl}(p^{n-*}) \xrightarrow{r_*} \mathrm{cone}(p^{n-*}) \to 0$$

and

$$0 \to \mathrm{cone}\big(C^{n-*}(\widetilde{X},\widetilde{\partial X})\big) \xrightarrow{m_*} \mathrm{cone}(p^{n-*}) \xrightarrow{s_*} C^{n-*}(\widetilde{\partial X}) \to 0$$

and

$$s_*\colon \mathrm{cone}(p^{n-*}) \xrightarrow{\;\simeq_s\;} C^{n-*}(\widetilde{\partial X})$$

is a simple chain $\mathbb{Z}\pi$-chain homotopy equivalence.

We obtain an up to chain homotopy commutative diagram of finite based free $\mathbb{Z}\pi$-chain complexes

$$
\begin{array}{ccc}
C^{n-*}(\widetilde{X},\widetilde{\partial X}) & \xrightarrow{\;p^{n-*}\;} & C^{n-*}(\widetilde{X}) \\
{\scriptstyle -\cap[X,\partial X]}\big\downarrow & & \big\downarrow{\scriptstyle -\cap[X,\partial X]} \\
C_*(\widetilde{X}) & \xrightarrow{\;p_*\;} & C_*(\widetilde{X},\widetilde{\partial X})
\end{array}
$$

Actually there is a preferred chain homotopy which is unique up to higher homotopies coming from the cocycle in the Q^n-group associated to $C_*(\widetilde{X})$ (see [20, Section 1 and Proposition 2.1]). Hence we obtain up to chain homotopy unique chain maps α_* and β_* making the following diagram with based exact rows commutative

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n-*}(\widetilde{X}, \widetilde{\partial X}) & \xrightarrow{\ l_*\ } & \mathrm{cyl}(p^{n-*}) & \xrightarrow{\ r_*\ } & \mathrm{cone}(p^{n-*}) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle -\cap[X,\partial X]} & & \Big\downarrow{\scriptstyle \alpha_*} & & \Big\downarrow{\scriptstyle \beta_*} & & \\
0 & \longrightarrow & C_*(\widetilde{X}) & \xrightarrow{\ j_*\ } & \mathrm{cyl}(p_*) & \xrightarrow{\ q_*\ } & \mathrm{cone}(p_*) & \longrightarrow & 0
\end{array}
$$

Additivity of the Whitehead torsion implies

$$
\begin{aligned}
\rho(X, \partial X) &= \tau\big(-\cap[X,\partial X]\colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{\partial X})\big) \\
&= (-1)^n \cdot * \Big(\tau\big(-\cap[X,\partial X]\colon C^{n-*}(\widetilde{X}, \widetilde{\partial X}) \to C_*(\widetilde{X})\big)\Big) \\
&= (-1)^n \cdot * \big(\tau(\alpha_*) - \tau(\beta_*)\big).
\end{aligned}
$$

Moreover, we obtain a commutative diagram

$$
\begin{array}{ccc}
C^{n-*}(\widetilde{X}) & \xrightarrow{\ u_*\ } & \mathrm{cyl}(p^{n-*}) \\
\Big\downarrow{\scriptstyle -\cap[X,\partial X]} & & \Big\downarrow{\scriptstyle \alpha_*} \\
C_*(\widetilde{X}, \widetilde{\partial X}) & \xrightarrow{\ v_*\ } & \mathrm{cyl}(p_*)
\end{array}
$$

where u_* and v_* the canonical inclusions and simple chain homotopy equivalences. From the composition formula for Whitehead torsion we conclude

$$
\rho(X, \partial X) = \tau(\alpha_*).
$$

Finally we obtain an up to chain homotopy commutative diagram with simple homotopy equivalences as rows

$$
\begin{array}{ccc}
\mathrm{cone}(p^{n-*}) & \xrightarrow{\ s_*\ } & \Sigma C^{n-1-*}(\widetilde{\partial X}) \\
\Big\downarrow{\scriptstyle \beta_*} & & \Big\downarrow{\scriptstyle \Sigma(-\cap[\partial X])} \\
\mathrm{cone}(p_*) & \xleftarrow{\ k_*\ } & \Sigma C_*(\widetilde{\partial X})
\end{array}
$$

From the composition formula for Whitehead torsion we conclude

$$
(j_{\partial X})_*\big(\rho(\partial X)\big) = \tau\big(-\cap[\partial X]\big) = -\tau\big(\Sigma(-\cap[\partial X])\big) = -\tau(\beta_*).
$$

This implies

$$
\begin{aligned}
(j_{\partial X})_*\big(\rho(\partial X)\big) &= -\tau(\beta_*) \\
&= (-1)^n \cdot * \big(\rho(X, \partial X)\big) - \tau(\alpha_*) \\
&= (-1)^n \cdot * \big(\rho(X, \partial X)\big) - \rho(X, \partial X).
\end{aligned}
$$

This finishes the proof of assertion (i).

(ii) Obviously it suffices to treat the case, where X and Y are connected, the general case follows componentwise. Choose the fundamental classes such that $H_n(f, \partial f)$

maps $[X, \partial X]$ to $[Y, \partial Y]$. Then the following diagram of $\mathbb{Z}\pi$-chain complexes commutes where we identify $\pi = \pi_1(X) = \pi_1(Y)$ by $\pi_1(f)$

$$
\begin{array}{ccc}
C^{n-*}(\widetilde{X}) & \xleftarrow{\ C^{n-*}(\widetilde{f})\ } & C^{n-*}(\widetilde{Y}) \\
{\scriptstyle \cap[X,\partial X]}\downarrow & & \downarrow{\scriptstyle \cap[Y,\partial Y]} \\
C_*(\widetilde{X}, \widetilde{\partial X}) & \xrightarrow{\ C_*(\widetilde{f},\widetilde{\partial f})\ } & C_*(\widetilde{Y}, \widetilde{\partial Y})
\end{array}
$$

The composition formula for Whitehead torsion implies

$$
\rho(Y, \partial Y) = \tau\big(C_*(\widetilde{f}, \widetilde{\partial f})\big) + \rho(X, \partial X) + \tau\big(C^{n-*}(\widetilde{f})\big).
$$

We get from additivity and the definitions

$$
\begin{aligned}
\tau\big(C_*(\widetilde{f}, \widetilde{\partial f})\big) &= \tau(f) - (j_{\partial Y})_*(\tau(\partial f)); \\
C^{n-*}(\widetilde{f}) &= (-1)^n \cdot *\big(\tau(f)\big).
\end{aligned}
$$

Now assertion (ii) follows.

(iii) Define Z by the pushout

$$
\begin{array}{ccc}
\partial X & \xrightarrow{\ f\ } & \partial Y \\
{\scriptstyle i_X}\downarrow & & \downarrow \\
X & \xrightarrow[\ \overline{f}\ ]{} & Z
\end{array}
$$

Then $(\overline{f}, f) \colon (X, \partial X) \to (Z, \partial Y)$ is a homotopy equivalence of pairs of spaces. In the sequel we treat only the case, where X and hence also Z are connected, the general case follows by inspecting the individual components. The pair $(\overline{f}, f)$ induces a base preserving isomorphism of cellular $\mathbb{Z}[\pi]$-chain complexes

$$
C_*(\widetilde{\overline{f}}, \widetilde{f}) \colon C_*(\widetilde{X}, \widetilde{\partial X}) \xrightarrow{\cong} C_*(\widetilde{Z}, \widetilde{\partial Y}),
$$

where we identify $\pi = \pi_1(X) = \pi_1(Z)$, $\widetilde{X} \to X$ and $\widetilde{Z} \to Z$ are the universal coverings and $\widetilde{\partial X}$ and $\widetilde{\partial Y}$ are obtained by restriction to ∂X and ∂Y. In particular $\tau\big(C_*(\widetilde{\overline{f}}, \widetilde{f})\big) = 0$. We conclude from assertion (ii) and the composition formula and additivity of Whitehead torsion.

$$
\begin{aligned}
& \rho(Z, \partial Y) - \rho(X, \partial X) \\
&= \tau(\overline{f}) + (-1)^n \cdot *\big(\tau(\overline{f})\big) - (j_{\partial Y})_*\tau(f) \\
&= \big(\tau(\overline{f}) - (j_{\partial Y})_*(\tau(f))\big) + (-1)^n \cdot *\big(\tau(\overline{f}) - (j_{\partial Y})_*(\tau(f)) + (j_{\partial Y})_*(\tau(f))\big) \\
&= \tau\big(C_*(\widetilde{f}, \widetilde{\overline{f}})\big) + (-1)^n \cdot *\big(\tau(C_*(\widetilde{f}, \widetilde{\overline{f}}) + (j_{\partial Y})_*(\tau(f))\big) \\
&= (-1)^n \cdot * \circ (j_{\partial Y})_*\big(\tau(f)\big).
\end{aligned}
$$

There is an obvious homeomorphism $X \cup_f Y \xrightarrow{\cong} Z \cup_{\partial Y} Y$. Hence it remains to show

$$
(10.5) \quad \rho(Z \cup_{\partial Y} Y) = (-1)^n \cdot * \circ (j_Z)_*\big(\rho(Z, \partial Y)\big) + (j_Y)_*\big(\rho(Y, \partial Y)\big),
$$

where $j_Z \colon Z \to Z \cup_{\partial Y} Y$ and $j_Y \colon Y \to Z \cup_{\partial Y} Y$ are the canonical inclusions. In the following let $\widetilde{Z \cup_{\partial Y} Y} \to Z \cup_{\partial Y} Y$ be the universal covering. Denote by $\widetilde{X} \to X$, $\widetilde{Y} \to Y$, $\widetilde{\partial Y} \to \partial Y$ the restriction of it to Y, X and ∂Y. Notice that the these are not necessarily the universal coverings. By excision we obtain an isomorphism

$$
H_n(Z, \partial Y; \mathbb{Z}) \oplus H_n(Y, \partial Y; \mathbb{Z}) \xrightarrow{\cong} H_n(Z \cup_{\partial Y} Y, \partial Y; \mathbb{Z}).
$$

The boundary homomorphisms $H_n(Y, \partial Y; \mathbb{Z}) \to H_{n-1}(\partial Y; \mathbb{Z})$ and $H_n(Z, \partial Y; \mathbb{Z}) \to H_{n-1}(\partial Y; \mathbb{Z})$ are injective and we can arrange such that $[Y, \partial Y]$ and $[Z, \partial Y]$ are mapped to $[\partial Y]$. The Mayer-Vietoris sequence yields an exact sequence

$$0 \to H_n(Z \cup_{\partial Y} Y; \mathbb{Z}) \to H_n(Y; \partial Y; \mathbb{Z}) \oplus H_n(Z, \partial Y; \mathbb{Z}) \to H_{n-1}(\partial Y; \mathbb{Z}).$$

Let $[Z \cup_{\partial Y} Y]$ be the unique element in $H_n(Z \cup_{\partial Y} Y; \mathbb{Z})$ which is mapped to $([Z, \partial Y], [Y, \partial Y])$. Then $[Z \cup_{\partial Y} Y]$ generates the infinite cyclic group $H_n(Z \cup_{\partial Y} Y; \mathbb{Z})$ and we obtain a commutative diagram of based free $\mathbb{Z}\pi$-chain complexes whose vertical maps are the Poincaré duality chain homotopy equivalences and whose rows are based exact sequences of finite based free $\mathbb{Z}\pi$-chain complexes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n-*}(\widetilde{Z}, \widetilde{\partial Y}) & \longrightarrow & C^{n-*}(\widetilde{Z \cup_{\partial Y} Y}) & \longrightarrow & C^{n-*}(\widetilde{Y}) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle -\cap[Z,\partial Y]} & & \Big\downarrow{\scriptstyle -\cap[Z\cup_{\partial Y}Y]} & & \Big\downarrow{\scriptstyle -\cap[Y,\partial Y]} & & \\
0 & \longrightarrow & C_*(\widetilde{Z}) & \longrightarrow & C_*(\widetilde{Z \cup_{\partial Y} Y}) & \longrightarrow & C_*(\widetilde{Y}, \widetilde{\partial Y}) & \longrightarrow & 0
\end{array}
$$

Additivity of the Whitehead torsion implies that Whitehead torsion of the middle vertical arrow is the sum of the Whitehead torsions of the left and of the right vertical arrow. The Whitehead torsion of the right vertical arrow is by definition $(j_Y)_*\big(\rho(Y, \partial Y)\big)$ and the Whitehead torsion of the middle arrow is $\rho(Z \cup_{\partial Y} Y)$. If we apply C^{n-*} to the left arrow, we obtain

$$- \cap [Z, \partial Y] \colon C^{n-*}(\widetilde{Z}) \to C_*(\widetilde{Z}, \widetilde{\partial Y})$$

Hence Whitehead torsion of the left arrow is $(-1)^n \cdot * \circ (j_Z)_*\big(\tau(Z, \partial Y)\big)$. This proves (10.5) and hence assertion (iii).

(iv) follows from the product formula of Whitehead torsion and the Künneth isomorphism

$$
\begin{array}{rcl}
C_*(\widetilde{X}, \widetilde{\partial X}) \otimes C_*(\widetilde{Y}, \widetilde{\partial Y}) & \xrightarrow{\cong} & C_*((\widetilde{X}, \widetilde{\partial X}) \times (\widetilde{Y}, \widetilde{\partial Y})); \\
C^{m-*}(\widetilde{X}) \otimes C^{n-*}(\widetilde{Y}) & \xrightarrow{\cong} & C^{n+m-*}(\widetilde{X \times Y}),
\end{array}
$$

which are compatible with the various Poincaré chain duality maps.

(v) Kirby-Siebenmann [10, Essay III, Theorem 5.13 on page 136]) prove that there is a simple homotopy equivalence of pairs $(M, \partial M) \to (X, \partial X)$ for a simple finite Poincaré pair $(X, \partial X)$. This finishes the proof of Theorem 10.4. $\qquad\square$

Denote by $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}(\pi))$ the Tate homology of $\mathbb{Z}/2$ with coefficients in $\mathrm{Wh}(\pi)$ with respect to the involution $*$ introduced above. Explicitly

$$\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}(\pi)) = \{x \in \mathrm{Wh}(\pi) \mid *x = (-1)^n \cdot x\}/\{y + (-1)^n \cdot *y \mid y \in \mathrm{Wh}(\pi))\}.$$

Let X be a space which has the homotopy type of a finite n-dimensional Poincaré complex. Let $f \colon X \to Y$ be any homotopy equivalence to a finite n-dimensional Poincaré complex Y. The Poincaré torsion $\rho(Y) \in \mathrm{Wh}(\pi(Y))$ satisfies $\rho(Y) = (-1)^n \cdot *\rho(Y)$ by Theorem 10.4 (i) and hence defines a class in $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}(\pi(Y)))$. Denote by

$$\widehat{\rho}(X) \in \widehat{H}^n\big(\mathbb{Z}/2; \mathrm{Wh}(\pi(X))\big)$$

the image of $\rho(Y)$ under the bijection $\widehat{H}^n\big(\mathbb{Z}/2; \mathrm{Wh}(\pi(Y))\big) \xrightarrow{\cong} \widehat{H}^n\big(\mathbb{Z}/2; \mathrm{Wh}(\pi(X))\big)$ induced by f^{-1}. This is independent of the choice of $f \colon X \to Y$ by Theorem 10.4 (ii).

Definition 10.6. *Given a space X of the homotopy type of a finite n-dimensional Poincaré CW-complex we call*

$$\widehat{\rho}(X) \in \widehat{H}^n\big(\mathbb{Z}/2; \mathrm{Wh}(\pi(X))\big)$$

the Tate-Poincaré *torsion of* X.

We conclude from Theorem 10.4 and the definitions.

Theorem 10.7.   *(i) If* $f\colon X \to Y$ *is a homotopy equivalence of spaces of the homotopy type of finite n-dimensional Poincaré complexes, then the induced isomorphism* $\widehat{H}^n\big(\mathbb{Z}/2, \mathrm{Wh}(\pi(X))\big) \xrightarrow{\cong} \widehat{H}^n\big(\mathbb{Z}/2, \mathrm{Wh}(\pi(Y))\big)$ *maps* $\widehat{\rho}(X)$ *to* $\widehat{\rho}(Y)$.

   (ii) Let $(X, \partial X)$ *be a Poincaré pair. Let* $j_{\partial M}\colon \partial X \to X$ *be the inclusion. Then the map*

$$(j_{\partial X})_*\colon \widehat{H}^n\big(\mathbb{Z}/2, \mathrm{Wh}(\pi(\partial X))\big) \to \widehat{H}^n\big(\mathbb{Z}/2, \mathrm{Wh}(\pi(X))\big)$$

   induced by $j_{\partial M}$ *sends* $\widehat{\rho}(\partial X)$ *to zero.*

   (iii) Let X *resp.* Y *be a space of the homotopy type of a connected m- resp. n-dimensional finite Poincaré complex. Then* $X \times Y$ *has the homotopy type of a connected finite* $(m+n)$*-dimensional Poincaré CW-complex and*

$$\widehat{\rho}(X \times Y) = \chi(X) \cdot (k_Y)_*\big(\widehat{\rho}(Y)\big) + \chi(Y) \cdot (k_X)_*\big(\widehat{\rho}(X)\big),$$

   where $k_X\colon X \to X \times Y$ *and* $k_Y\colon Y \to X \times Y$ *are the inclusions.*

   (iv) If X *has the homotopy type of an n-dimensional closed topological manifold, then* X *is homotopy equivalent to a simple n-dimensional Poincaré complex and in particular*

$$\widehat{\rho}(X) = 0.$$

   (v) Let X *be an n-dimensional Poincaré complex. Then it is homotopy equivalent to a simple n-dimensional Poincaré complex if and only if* $\widehat{\rho}(X) = 0$ *holds in* $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}(X))$.

Proof. (i) This follows from Theorem 10.4 (ii).

(ii) This follows from Theorem 10.4 (i).

(iii) This follows from Theorem 10.4 (iv).

(iv) This follows from Theorem 10.4 (v) and assertion (i).

(v) Since $\widehat{\rho}(X) = 0$ in $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}(X))$, we can find $y \in \mathrm{Wh}(X)$ with $-\rho(X) = y + (-1)^n * y$. Choose a finite CW-complex Y together with a homotopy equivalence $f\colon Y \to X$ satisfying $\tau(f) = y \in \mathrm{Wh}(X)$. Then we conclude $\rho(Y) = 0$ from Theorem 10.4 (ii). $\qquad\square$

Remark 10.8. Let $f\colon M \to N$ be a map of closed manifolds. Suppose that M and N are connected. Suppose that the homotopy fiber $\mathrm{hofib}(f)$ has the homotopy type of a finite CW-complex. Then we have the fibration of spaces of the homotopy type of finite CW-complexes

$$\mathrm{hofib}(f) \to \mathrm{FIB}(f) \to N$$

such that the total space has the homotopy type of a finite Poincaré complex and the base space is a finite Poincaré complex. We conclude from [9] that also the homotopy fiber has the homotopy type of a finite Poincaré complex. Hence we can define

(10.9) $\widehat{\rho(f)} \ = \ \widehat{\rho}\big(\mathrm{hofib}(f)\big) \in \widehat{H}^n\big(\mathbb{Z}/2, \mathrm{Wh}(\pi(\mathrm{hofib}(f)))\big).$

Suppose that f is homotopic to a map $p\colon M \to N$ which is the projection of a locally trivial fiber bundle with a closed manifold F as fiber. Then the homotopy fiber $\mathrm{hofib}(f)$ of f is homotopy equivalent to F. Theorem 10.7 implies

$$\widehat{\rho}(\mathrm{hofib}(f)) = 0.$$

Hence we have besides the obstructions appearing in Definition 5.2 another torsion obstruction for f to be homotopic to a bundle projection.

11. CONNECTION TO THE PARAMETRIZED A-THEORY CHARACTERISTIC

The element $\Theta(f)$ defined here is a shadow of the parametrized A-theory characteristic, as defined by Dwyer-Weiss-Williams [4]. In this section we give a sketch of the relationship.

The parametrized A-theory characteristic $\chi(p)$ is defined for any fibration $p \colon E \to B$ with homotopy finitely dominated fibers and can be understood as a section of the fibration obtained from p by applying the (connective) A-theory functor fiberwise. We are going to write

$$\chi(p) \in \pi_0 \Gamma \begin{pmatrix} A_B(E) \\ \downarrow \\ B \end{pmatrix} =: H^0(B; A(F_b)),$$

thinking of this as the zeroth cohomology of B with twisted coefficients in the spectrum $A(F_b)$ (where F_b denotes the fiber of p over b).

The natural transformation $A(X) \to \mathrm{Wh}^{PL}(X)$ induces a map $H^0(B; A(F_b)) \to H^0(B; \mathrm{Wh}^{PL}(F_b))$; denote the image of $\chi(p)$ by $\mathrm{Wall}(p)$, the parametrized Wall obstruction. Dwyer-Weiss-Williams show the following:

Theorem 11.1. *The fibration p is fiber homotopy equivalent to a bundle of compact topological manifolds (possibly with boundary) if and only if $\mathrm{Wall}(p) = 0$.*

Therefore, if a map $f \colon M \to B$ between manifolds is homotopic to a fiber bundle, then $\mathrm{Wall}(p)$ is defined and vanishes, with $p \colon E \to B$ the fibration associated to f. There is an Atiyah-Hirzebruch type spectral sequence

$$E_2^{pq} = H^p(B; \pi_{-q} \mathrm{Wh}^{PL}(F_b)) \implies H^{p+q}(B; \mathrm{Wh}^{PL}(F_b)),$$

where the cohomology on the left hand side is ordinary cohomology with twisted coefficients in the system $\{b \mapsto \pi_{-q} \mathrm{Wh}^{PL}(F_b)\}$. Denote by $P_1 \mathrm{Wh}^{PL}(F_b)$ the first Postnikov approximation of $\mathrm{Wh}^{PL}(F_b)$, such that $\pi_n P_1 \mathrm{Wh}^{PL}(F_b) = 0$ for $n \geq 2$. The Atiyah-Hirzebruch spectral sequence reduces to the exact sequence

$$0 \to H^1(B; \pi_1 \mathrm{Wh}^{PL}(F_b)) \to H^0(B; P_1 \mathrm{Wh}^{PL}(F_b)) \to H^0(B; \pi_0 \mathrm{Wh}^{PL}(F_b)) \to 0.$$

Proposition 11.2.   *(i) The image of $\mathrm{Wall}(p)$ in*

$$H^0(B; \pi_0 \mathrm{Wh}^{PL}(F_b)) \cong \bigoplus_{[b] \in \pi_0 B} \tilde{K}_0(\mathbb{Z}[\pi_1(F_b, b)])^{\pi_1(B,b)}$$

     consists of the Wall obstructions of the fibers over every path component of B.

     (ii) Suppose that all the fibers have the homotopy type of a finite CW-complex. The lift of $\mathrm{Wall}(p)$ to $H^1(B; \pi_1 \mathrm{Wh}^{PL}(F_b))$ maps to $\Theta(p)$ under the map

$$H^1(B; \pi_1 \mathrm{Wh}^{PL}(F_b)) \to H^1(B; \pi_1 \mathrm{Wh}^{PL}(E)) \cong H^1(B; \mathrm{Wh}(\pi(E)))$$

     induced by the inclusion.

Proof. (i) This assertion mainly depends on the fact that the path component of the unparametrized A-theory characteristic gives the unreduced Wall obstruction (which follows rather easily from the linearization map to K-theory).

(ii) One needs to show that a simple structure on a space X is the same thing as a (homotopy class of a) lift of $\chi(X)$ to an "excisive characteristic" $\chi^\%(X) \in A^\%(X)$, and that the naturality of the A-theory characteristic for homotopy equivalences allows to describe the Whitehead torsion with respect to these lifts.

Then observe that Waldhausen's description of the A-theory assembly map [25] defines a canonical excisive characteristic for finite CW complexes. The equivalence of the algebraic and the geometric definition of the Whitehead group [3, §21] implies that the corresponding simple structure is just the canonical one. Once one has identified Waldhausen's excisive characteristic with the excisive characteristic $\chi^{\%}(X)$ defined by Dwyer-Weiss-Williams for compact ENRs X (in the case where both are defined), assertion (ii) follows by the construction of the short exact sequence. $\qquad\square$

12. SOME QUESTIONS

Let $f\colon M \to N$ and $g\colon N \to B$ be maps of closed path-connected manifolds. Assume that the homotopy fiber of both f and g has the homotopy type of a finite CW-complex. Then the same is true for the composite $g \circ f$ since there is a fibration $\mathrm{hofib}(f) \to \mathrm{hofib}(g \circ f) \to \mathrm{hofib}(g)$. So the elements $\Theta(f) \in H^1\big(N, \mathrm{Wh}(\pi(M))\big)$, $\Theta(g) \in H^1\big(N, \mathrm{Wh}(\pi(N))\big)$ and $\Theta(g \circ f) \in H^1\big(B; \mathrm{Wh}(\pi(M))\big)$ are defined.

Question 12.1. What is the relation between $\Theta(g \circ f)$, $\Theta(f)$ and $\Theta(g)$?

Question 12.2. If N is aspherical, what are the other obstructions besides the torsion obstructions presented in this paper for f to be homotopic to a bundle projection?

Notice that in the case $B = S^1$ there are no other obstructions because of Theorem 8.1.

Question 12.3. Suppose that M and N are aspherical. Is then the homotopy fiber a closed manifold?

The question may have a positive answer in favorite circumstances because of the following remarks. Suppose that the Farrell-Jones Conjecture for algebraic K- and L-theory with arbitrary coefficients is true for the fundamental group of E. (This is known to be true for a large class of groups.) Assume that the difference of the dimensions of E and B is at least six. Moreover, assume that the resolution obstruction of Quinn (see [19]) vanishes for all aspherical closed ANR-homology manifolds. (There is no counterexample to this assumption known to the authors.) Then one can deduce that the homotopy fiber is homotopy equivalent to a closed topological manifold and this closed topological manifold is unique up to homeomorphism (see [1]).

Question 12.4. Suppose that M and N are aspherical. Are there any obstructions for p being homotopic to a bundle projection of a locally trivial bundle, or for weaker notions, such as block bundles?

Quinn developed a technique addressing the block bundle case of this question in [18, Section 1]. Using Quinn's technique a partial result on the block bundle question was obtained in [8, Theorem 10.7].

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