THE STRUCTURE OF CROSSED PRODUCTS OF IRRATIONAL ROTATION ALGEBRAS BY FINITE SUBGROUPS OF $\mathrm{SL}_2(\mathbb{Z})$

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ABSTRACT. Let $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite subgroup (necessarily isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, or $\mathbb{Z}_6$), and let F act on the irrational rotational algebra A_θ via the restriction of the canonical action of $\mathrm{SL}_2(\mathbb{Z})$. Then the crossed product $A_\theta \rtimes_\alpha F$ and the fixed point algebra A_θ^F are AF algebras. The same is true for the crossed product and fixed point algebra of the flip action of $\mathbb{Z}_2$ on any simple d-dimensional noncommutative torus A_Θ. Along the way, we prove a number of general results which should have useful applications in other situations.

0. Introduction and statement of the main results

For $\theta \in \mathbb{R}$ let A_θ be the rotation algebra, which is the universal C*-algebra generated by unitaries u and v satisfying $vu = \exp(2\pi i\theta)uv$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1}\theta) u^{n_{1,1}} v^{n_{2,1}} \text{ and } \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2}\theta) u^{n_{1,2}} v^{n_{2,2}}. \tag{0.1}$$

In this paper we are concerned with the structure of the crossed products $A_\theta \rtimes_\alpha F$ for the finite subgroups $F \subseteq \mathrm{SL}_2(\mathbb{Z})$, which are, up to conjugacy, the groups $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ (where $\mathbb{Z}_m$ stands for the cyclic group of order m) generated by the matrices

$$\begin{array}{llll} \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) & (\text{for } \mathbb{Z}_2), & \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) & (\text{for } \mathbb{Z}_3), \\ \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & (\text{for } \mathbb{Z}_4), \quad \text{and} & \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) & (\text{for } \mathbb{Z}_6). \end{array} \tag{0.2}$$

We refer to Proposition 21 of [32] for a proof that one obtains essentially all interesting actions of the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ on A_θ by restricting the action α of $\mathrm{SL}_2(\mathbb{Z})$ to these subgroups. The main results of this paper culminate in a proof of the following theorem:

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Theorem 0.1 (Theorems 4.9, 6.3, and 6.4)**.** *Let F be any of the finite subgroups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \subseteq \mathrm{SL}_2(\mathbb{Z})$ with generators given as above and let $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$. Then the crossed product $A_\theta \rtimes_\alpha F$ is an AF algebra. For all $\theta \in \mathbb{R}$ we have*

$$K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}^6, \quad K_0(A_\theta \rtimes_\alpha \mathbb{Z}_3) \cong \mathbb{Z}^8,$$

$$K_0(A_\theta \rtimes_\alpha \mathbb{Z}_4) \cong \mathbb{Z}^9, \quad \textit{and} \quad K_0(A_\theta \rtimes_\alpha \mathbb{Z}_6) \cong \mathbb{Z}^{10}.$$

For $F = \mathbb{Z}_k$ for $k = 2, 3, 4, 6$, the image of $K_0(A_\theta \rtimes_\alpha F)$ under the canonical tracial state on $A_\theta \rtimes_\alpha F$ (which is unique) is equal to $\frac{1}{k}(\mathbb{Z}+\theta\mathbb{Z})$. As a consequence, $A_\theta \rtimes_\alpha \mathbb{Z}_k$ is isomorphic to $A_{\theta'} \rtimes_\alpha \mathbb{Z}_l$ if and only if $k = l$ and $\theta' = \pm\theta$ mod $\mathbb{Z}$.

If $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$ then the fixed point algebra A_θ^F is Morita equivalent to $A_\theta \rtimes_\alpha F$. (This follows from the Proposition in [66].) Thus, as a direct corollary of Theorem 0.1 we get:

Corollary 0.2 (Corollary 6.5)**.** *Let $\alpha : F \to \mathrm{Aut}(A_\theta)$ be as above with $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. Then the fixed point algebras A_θ^F are AF for all $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$.*

The proof of Theorem 0.1 has three independent steps:

(1) Computation of the K-theory of the crossed product.
(2) Proof that the crossed product satisfies the Universal Coefficient Theorem.
(3) Proof that the action of the group has the tracial Rokhlin property of [59].

Given these steps, one uses Theorem 2.6 of [59] to prove that the crossed product has tracial rank zero in the sense of [39, 40], and then Huaxin Lin's classification theorem (Theorem 5.2 of [41]) to conclude that the crossed product is AF.

Our results have been motivated by previous studies of the algebras $A_\theta \rtimes_\alpha F$ and A_θ^F by several authors. See, for example, [7, 9, 26, 27, 28, 29, 30, 31, 32, 38, 62, 72, 73, 74]. The most was known about the crossed product by the flip: its (unordered) K-theory has been computed in [38], and the crossed product has been proved to be an AF algebra in [9]. The next best understood case is the case of $\mathbb{Z}_4$, which has been intensively studied in a series of papers culminating in [74]. It is proved there that for most irrational θ (that is, a dense G_δ-set of them) the crossed product $A_\theta \rtimes_\alpha \mathbb{Z}_4$ has tracial rank zero in the sense of [39], and that for most of those values of θ, it is an AF algebra. Step (1) is done for most θ in [73], Step (2) is done for all θ in [74], and, in place of Step (3), a direct proof is given in [74] that the crossed product has tracial rank zero for most θ. That $A_\theta \rtimes_\alpha \mathbb{Z}_4$ is an AF algebra for most θ then follows from Lin's classification theorem.

Using completely different methods, we prove in this paper several general results which will imply, as special cases, the three steps described above for *all* $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$ and for *all* choices for $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. For the computation of the K-theory (Step (1) above), we observe that the crossed products $A_\theta \rtimes_\alpha F$ can be realized as (reduced) twisted group algebras $C_r^*(\mathbb{Z}^2 \rtimes F, \widetilde{\omega}_\theta)$, where $\mathbb{Z}^2 \rtimes F$ is the semidirect product by the obvious action of F on $\mathbb{Z}^2$ and $\widetilde{\omega}_\theta \in Z^2(\mathbb{Z}^2 \rtimes F, \mathbb{T})$ is a suitable circle valued 2-cocycle on $\mathbb{Z}^2 \rtimes F$. Since $\widetilde{\omega}_\theta$ is homotopic to the trivial cocycle in a sense explained below, the following result will imply that the K-groups of $A_\theta \rtimes_\alpha F$ are isomorphic to the K-groups of the untwisted group algebra $C_r^*(\mathbb{Z}^2 \rtimes F) \cong C(\mathbb{T}^2) \rtimes F$, which we compute (together with an explicit basis) by using some general methods from [18].

Theorem 0.3 (Theorem 1.9)**.** *Suppose that $\omega_0, \omega_1 \in Z^2(G, \mathbb{T})$ are homotopic Borel 2-cocycles on the second countable locally compact group G, that is, there exists a*

Borel 2-cocycle $\Omega \in Z^2(G, C([0,1], \mathbb{T}))$ *such that* $\omega_j = \Omega(\,\cdot\,,\,\cdot\,)(j)$ *for* $j = 0, 1$. *Suppose further that* G *satisfies the Baum-Connes conjecture with coefficients. (If* G *is amenable or a-T-menable, this is automatic by Theorem 1.1 of* [33]*.) Then* $K_*(C^*_r(G, \omega_0)) \cong K_*(C^*_r(G, \omega_1))$.

Indeed, using the homotopy $\Omega \in Z^2(G, C([0,1], \mathbb{T}))$ one can construct a C*-algebra $C^*_r(G, \Omega)$ which is a continuous field of C*-algebras over $[0,1]$ with fibers isomorphic to the twisted group algebras $C^*_r(G, \omega_\theta)$, with $\omega_\theta = \Omega(\,\cdot\,,\,\cdot\,)(\theta)$ for $\theta \in [0,1]$. We show, under the conditions on G in Theorem 0.3 (in fact, slightly more generally), that the quotient maps $q_\theta \colon C^*_r(G, \Omega) \to C^*_r(G, \omega_\theta)$ induce isomorphisms on K-theory. This special structure of the isomorphism will be used for our description of an explicit basis of $A_\theta \rtimes_\alpha F$: we extend the basis of $K_0(C^*_r(\mathbb{Z}^2 \rtimes F))$ to a basis of $K_0(C^*_r(\mathbb{Z}^2 \rtimes F, \Omega))$ and then evaluate at the other fibers.

The K_0-groups of the crossed products of the irrational rotation algebras by the actions of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ have been computed independently in [62], using completely different methods. These methods give no information about the K_1-groups, and therefore do not suffice for the proof that the crossed products are AF algebras. We should also mention that Georges Skandalis pointed out to us an alternative way to compute the K-theory of the crossed products $A_\theta \rtimes_\alpha F$, also using the Baum-Connes conjecture. However, the method described above seems best suited for computing an explicit basis for $K_0(A_\theta \rtimes_\alpha F)$.

Step (2) (the Universal Coefficient Theorem) will be obtained from the Baum-Connes conjecture by slightly extending ideas first used in Section 10.2 of [71] and Lemma 5.4 of [15].

Step (3) (the tracial Rokhlin property) is obtained via Theorem 5.5, according to which an action of a finite group on a simple separable unital nuclear C*-algebra with unique tracial state has the tracial Rokhlin property if and only if the induced action is outer on the type II_1 factor obtained from the trace using the Gelfand-Naimark-Segal construction. Along the way, we give a characterization of the tracial Rokhlin property in terms of trace norms which does not require uniqueness of the tracial state.

The methods developed here can also be applied to the crossed products of all simple d-dimensional noncommutative tori A_Θ, for $d \geq 2$, by the flip action of $\mathbb{Z}_2$, which sends the unitary generators $u_1, u_2, \ldots, u_d$ to their adjoints $u_1^*, u_2^*, \ldots, u_d^*$. Using the general methods explained above together the main result of [60], which shows that every simple higher dimensional d-torus is an AT algebra, we prove:

Theorem 0.4 (Corollary 1.14, Theorem 6.6, and Corollary 6.7)**.** *Let* A_Θ *be the noncommutative* d*-torus corresponding to a real* $d \times d$ *skew symmetric matrix* Θ. *Let* $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(A_\Theta)$ *denote the flip action. Then*

$$K_0(A_\Theta \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}^{3 \cdot 2^{d-1}} \quad \textit{and} \quad K_1(A_\Theta \rtimes_\alpha \mathbb{Z}_2) = \{0\}.$$

If A_Θ *is simple, then* $A_\Theta \rtimes_\alpha \mathbb{Z}_2$ *and the fixed point algebra* $A_\Theta^{\mathbb{Z}_2}$ *are AF algebras.*

This result generalizes Theorem 3.1 of [6] and completely answers a question raised in the introduction of [28].

In Section 1, we give the necessary background on twisted group algebras and the proof of Theorem 0.3. The realization of the crossed products $A_\theta \rtimes_\alpha F$ as twisted group algebras of the group $\mathbb{Z}^2 \rtimes F$ is done in Section 2. The K-theory computations for $C^*_r(\mathbb{Z}^2 \rtimes F)$ are done in Sections 3 and 4. Section 5 contains the proof that the relevant actions have the tracial Rokhlin property. In Section 6, we prove the Universal Coefficient Theorem and put everything together. We also prove Theorem 0.4.

This paper contains the main result of Section 10 of the unpublished long preprint [58], Theorem 0.4 here. Although the three steps, as described at the beginning of the introduction, are the same, the proofs of all three of them differ substantially from the proofs given in [58]. This paper also supersedes Sections 8 and 9 of [58]. Again, the proofs of Steps (2) and (3) differ substantially from those in [58]. We improve on [58] by calculating the K-theory of the crossed products $A_\theta \rtimes_\alpha F$. No K-theory calculations were done in [58], so that the crossed products were merely shown to be simple AH algebras with real rank zero for $F = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and all irrational θ, and to be AF algebras for $F = \mathbb{Z}_4$ and a dense G_δ-set of values of θ. (The case $F = \mathbb{Z}_2$ was already known, and in any case it is covered by Theorem 0.4.)

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1. Twisted group algebras and actions on $\mathcal{K}$

We begin by recalling some basic facts about group C*-algebras twisted by cocycles. This is a special case of the theory of crossed product C*-algebras twisted by cocycles of [52]. Also see [77] for the case in which the group is discrete.

Let G be a second countable locally compact group, with modular function $\Delta_G \colon G \to (0, \infty)$. Let $\omega \colon G \times G \to \mathbb{T}$ be a Borel 2-cocycle on G. (Recall the algebraic conditions: $\omega(s,t)\omega(r,st) = \omega(r,s)\omega(rs,t)$ and $\omega(1,s) = \omega(s,1) = 1$ for $r,s,t \in G$.) Then the twisted convolution algebra $L^1(G,\omega)$ is defined to be the vector space of all integrable complex functions on G with convolution and involution given by

$$(f *_\omega g)(s) = \int_G f(t)g(t^{-1}s)\omega(t,\, t^{-1}s)\, dt \quad \text{and} \quad f^*(s) = \Delta_G(s^{-1})\overline{\omega(s,s^{-1})f(s^{-1})}.$$

An *ω-representation* of G on a Hilbert space $\mathcal{H}$ is a Borel map $V \colon G \to \mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$, with respect to the strong operator topology on $\mathcal{U}(\mathcal{H})$, satisfying

$$V(s)V(t) = \omega(s,t)V(st)$$

far all $s,t \in G$. The *regular ω-representation* of G is the representation $L_\omega \colon G \to \mathcal{U}(L^2(G))$ given by the formula

$$(L_\omega(s)\xi)(t) = \omega(s,\, s^{-1}t)\xi(s^{-1}t)$$

for $\xi \in L^2(G)$. Every ω-representation $V \colon G \to \mathcal{U}(\mathcal{H})$ determines a contractive $*$-homomorphism, also denoted V, from $L^1(G,\omega)$ to $B(\mathcal{H})$ via the formula

$$V(f) = \int_G f(s)V(s)\, ds$$

for $f \in L^1(G,\omega)$, and every nondegenerate representation of $L^1(G,\omega)$ appears in this way. The full twisted group algebra $C^*(G,\omega)$ is defined to be the enveloping C*-algebra of $L^1(G,\omega)$ and the reduced twisted group algebra $C^*_r(G,\omega)$ is defined to be the image of $C^*(G,\omega)$ under the regular ω-representation L_ω. If G is amenable, then both algebras coincide (Theorem 3.11 of [52]). In this case we simply write $C^*(G,\omega)$.

Recall that two cocycles $\omega, \omega' \in Z^2(G, \mathbb{T})$ are cohomologous if there exists a Borel function $u \colon G \to \mathbb{T}$ such that, with $\partial u(s,t) = u(s)u(t)\overline{u(st)}$, we have $\omega' = \partial u \cdot \omega$. (See Chapter 1 of [48]; the action of G on $\mathbb{T}$ is trivial.) It is then easy to verify that the map $L^1(G, \omega') \to L^1(G, \omega)$, given by $f \mapsto u \cdot f$, extends to isomorphisms of the full and reduced twisted group C*-algebras.

In this section, let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable infinite dimensional Hilbert space $\mathcal{H}$, let $\mathcal{U}$ be the group of unitary operators on $\mathcal{H}$, and let $\mathcal{PU} = \mathcal{U}/\mathbb{T}1$ denote the projective unitary group. If $V \colon G \to \mathcal{U}$ is an ω-representation, then V determines an action $\alpha \colon G \to \mathcal{PU} \cong \mathrm{Aut}(\mathcal{K})$ by $\alpha_s = \mathrm{Ad}(V(s))$ for $s \in G$. (Continuity follows from Proposition 5(a) of [50].) With $\overline{\omega}(s,t) = \overline{\omega(s,t)}$, the full and reduced crossed products $\mathcal{K} \rtimes_\alpha G$ and $\mathcal{K} \rtimes_{\alpha,r} G$ are then isomorphic to $C^*(G, \overline{\omega}) \otimes \mathcal{K}$ and $C^*_r(G, \overline{\omega}) \otimes \mathcal{K}$. On the level of L^1-algebras the isomorphisms are given by the map $\Phi \colon L^1(G, \overline{\omega}) \odot \mathcal{K} \to L^1(G, \mathcal{K})$ determined by

$$\Phi(f \otimes k)(s) = f(s)kV(s)^* \tag{1.1}$$

for $f \in L^1(G, \overline{\omega})$, $k \in \mathcal{K}$, and $s \in G$. (See for example Theorem 1.4.15 and Example 1.1.4 of [19], but this is easily proved directly.) Conversely, if $\alpha \colon G \to \mathrm{Aut}(\mathcal{K})$ is any action of G on $\mathcal{K}$, then by choosing a Borel section $c \colon \mathcal{PU} \to \mathcal{U}$ we obtain a Borel map $V^\alpha = c \circ \alpha \colon G \to \mathcal{U}$ such that $\alpha_s = \mathrm{Ad}(V^\alpha(s))$ for all $s \in G$. It is then easy to check that there exists a Borel cocycle $\omega_\alpha \in Z^2(G, \mathbb{T})$ such that $V^\alpha(s)V^\alpha(t) = \omega_\alpha(s,t)V^\alpha(st)$ for all $s, t \in G$. Thus, any action $\alpha \colon G \to \mathrm{Aut}(\mathcal{K})$ is given by an ω_α-representation for some cocycle $\omega_\alpha \in Z^2(G, \mathbb{T})$.

If A is any C*-algebra, then two actions $\alpha, \alpha' \colon G \to \mathrm{Aut}(A)$ are called *exterior equivalent* if there exists a strictly continuous map $v \colon G \to UM(A)$ (the unitary group in the multiplier algebra of A) such that $\alpha'_s = \mathrm{Ad}(v_s) \circ \alpha_s$ and $v_{st} = v_s \alpha_s(v_t)$ for all $s, t \in G$. (Compare with 8.11.3 of [54].) It is easily seen that exterior equivalent actions have isomorphic crossed products and isomorphic reduced crossed products, with isomorphisms given on $L^1(G, A)$ by $f \mapsto f \cdot v$.

Following [21], if X is any locally compact space we denote by $\mathcal{E}_G(X)$ the set of all exterior equivalence classes of $C_0(X)$-linear actions of G on $C_0(X, \mathcal{K})$, that is, actions $\alpha \colon G \to \mathrm{Aut}\big(C_0(X, \mathcal{K})\big)$ such that $\alpha_s(f \cdot g) = f \cdot \alpha_s(g)$ for all $f \in C_0(X)$ and $g \in C_0(X, \mathcal{K})$. Any such action is completely determined by its evaluations $\alpha^x \colon G \to \mathrm{Aut}(\mathcal{K})$ for $x \in X$, defined by $\alpha^x_s(g(x)) = \big(\alpha_s(g)\big)(x)$ for $g \in C_0(X, \mathcal{K})$ and $x \in X$. Thus, we should understand a $C_0(X)$-linear action as a continuous family of actions on $\mathcal{K}$. It is shown in [17] (see Lemma 3.1 and Theorem 3.6) and [21] that $\mathcal{E}_G(X)$ is a group with multiplication $[\alpha] \cdot [\beta] = [\alpha \otimes_X \beta]$, where $(\alpha \otimes_X \beta)^x = \alpha^x \otimes \beta^x$ for $x \in X$. The following result, basically due to Mackey, is well known; for example see 6.3 in [17]:

Proposition 1.1. *Let G be a second countable locally compact group. Two actions α and α' of G on $\mathcal{K}$ are exterior equivalent if and only if $[\omega_\alpha] = [\omega_{\alpha'}] \in H^2(G, \mathbb{T})$. Moreover, the mapping $[\alpha] \mapsto [\omega_\alpha]$ is a bijective homomorphism from $\mathcal{E}_G(\mathrm{pt})$ to $H^2(G, \mathbb{T})$.*

Proof. A calculation, which we omit, shows that if α and α' are two actions with corresponding cocycles ω and ω', then α is exterior equivalent to α' if and only if ω is cohomologous to ω'. This shows that $[\alpha] \mapsto [\omega_\alpha]$ is well defined and injective. For surjectivity, given ω, take α to be conjugation by the regular ω-representation of G on $L^2(G)$ (or $L^2(G) \otimes \mathcal{H}$ if G is finite). (Continuity of α follows from Proposition 5(a) of [50].) $\square$

We write $\mathcal{K}_\omega$ if we consider $\mathcal{K}$ equipped with an action corresponding to ω as above.

For an action $\alpha\colon G \to \mathrm{Aut}(A)$, we denote by $K_*^{\mathrm{top}}(G; A)$ the left hand side of the Baum-Connes Conjecture with coefficients A. See Section 9 of [3], where it is called $K_*^G(\underline{E}G; A)$. As there, we let $\mu\colon K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\alpha,r} G)$ be the assembly map. Versions of the following definition have appeared in [45, 11].

Definition 1.2. Let $[\omega] \in H^2(G, \mathbb{T})$. Then the twisted topological K-theory of G with respect to $[\omega]$ is defined to be the topological K-theory $K_*^{\mathrm{top}}(G; \omega) = K_*^{\mathrm{top}}(G; \mathcal{K}_{\overline{\omega}})$. The twisted assembly map for G with respect to ω is then defined to be

$$\mu_\omega\colon K_*^{\mathrm{top}}(G; \mathcal{K}_{\overline{\omega}}) \to K_*(\mathcal{K}_{\overline{\omega}} \rtimes_r G) \xrightarrow{\cong} K_*(C_r^*(G, \omega)).$$

Up to obvious isomorphisms, the definition does not depend on the choice of the representative ω of the class $[\omega] \in H^2(G, \mathbb{T})$ and the representative α of the class in $\mathcal{E}_G(\mathrm{pt})$ corresponding to $\overline{\omega}$.

Of course, if G is a group which satisfies the Baum-Connes conjecture with coefficients, then the twisted assembly map is an isomorphism for all $\omega \in Z^2(G, \mathbb{T})$. But it is sometimes possible to show that the Baum-Connes conjecture holds for $\mathcal{K}$ (with respect to any action of G on $\mathcal{K}$) without knowing that the conjecture holds for all coefficients. For example, it is shown in [14] that every almost connected group satisfies the conjecture for $\mathcal{K}$ but the Baum-Connes conjecture with arbitrary coefficients is not known in general for those groups.

We now consider homotopies between cocycles. The definition of the twisted assembly map suggests that it is actually useful to consider homotopies for actions on $\mathcal{K}$. The first part of the following definition is a special case, in different language, of Definition 3.1 of [57].

Definition 1.3. A *homotopy* of actions on $\mathcal{K}$ is a $C([0,1])$-linear action of G on $C([0,1], \mathcal{K})$. A homotopy between two classes $[\beta^0], [\beta^1] \in \mathcal{E}_G(\mathrm{pt})$ is a class $[\beta] \in \mathcal{E}_G([0,1])$ with evaluations $[\beta^0]$ and $[\beta^1]$ at the points 0 and 1.

Remark 1.4. The notion of homotopy in Definition 1.3 is, at least in principle, weaker than the notion of homotopy between classes in $H^2(G, \mathbb{T}) \cong \mathcal{E}_G(\mathrm{pt})$ used in Theorem 0.3. Indeed, let $\Omega\colon G \times G \to C([0,1], \mathbb{T})$ be a 2-cocycle with evaluations $\omega_x = \Omega(\,\cdot\,, \,\cdot\,)(x)$ for $x \in [0,1]$. By Proposition 3.1 of [34], and with L_{ω_x} being the regular ω_x-representation on $\mathcal{K}(L^2(G))$, the formula

$$\alpha_s(g)(x) = L_{\omega_x}(s) g(x) L_{\omega_x}(s)^*,$$

for $g \in C([0,1], \mathcal{K})$, defines a continuous action $\alpha\colon G \to \mathrm{Aut}\,\big(C([0,1], \mathcal{K})\big)$. Thus, if $[\alpha_0], [\alpha_1] \in \mathcal{E}_G(\mathrm{pt})$ correspond to $[\omega_0], [\omega_1] \in H^2(G, \mathbb{T})$ under the correspondence described in Proposition 1.1, we see that $[\alpha_0]$ and $[\alpha_1]$ are homotopic classes in $\mathcal{E}_G(\mathrm{pt})$.

Using the results of [21, 20] one can show that, conversely, homotopy of classes in $\mathcal{E}_G(\mathrm{pt})$ implies homotopy of the corresponding cocycles for a very large class of groups. This class includes all almost connected groups by Theorem 1.4 of [20] and all smooth groups in the sense of [49] by Theorem 5.4 of [21]; note that it is shown in [49] that all discrete groups and all compact groups are smooth. However, it is not clear to us whether this direction holds in general.

An even weaker notion of homotopy of classes in $\mathcal{E}_G(\mathrm{pt}) \cong H^2(G, \mathbb{T})$ is obtained by using the topology on $H^2(G, \mathbb{T})$ defined by Moore in [50] and defining two classes to be homotopic if they can be connected via a continuous path $\gamma\colon [0,1] \to H^2(G,\mathbb{T})$. Again, Theorem 1.4 of [20] and Theorem 5.4 of [21] imply that this notion of homotopy coincides with the previous notions whenever G is smooth or almost connected, but we do not know this in general. The problem is that the topology of $H^2(G,\mathbb{T})$, which is defined via pointwise almost everywhere convergence of cocycles, can have rather poor separation properties.

The structure of the group $\mathcal{E}_G(X)$ of all exterior equivalence classes of $C_0(X)$-linear actions on $C_0(X,\mathcal{K})$ was extensively studied in [21, 20]. As an easy corollary of the results obtained in [21] we get:

Proposition 1.5. *Suppose that G is compact and X is contractible. Then, for any $x \in X$, the evaluation map $\mathcal{E}_G(X) \to \mathcal{E}_G(\mathrm{pt})$, given by $[\alpha] \mapsto [\alpha^x]$, is an isomorphism. In particular, every $C_0(X)$-linear action $\alpha\colon G \to \mathrm{Aut}\big(C_0(X,\mathcal{K})\big)$ is exterior equivalent to the diagonal action $\mathrm{id}_X \otimes \alpha^x$ on $C_0(X) \otimes \mathcal{K} \cong C_0(X,\mathcal{K})$.*

Proof. Since compact groups are smooth in the sense of Moore (see the discussion preceding Proposition 3.1 in [49] and also Remark 4.2 of [21]), it follows from Theorem 5.4 of [21] and its proof that there is a short exact sequence

$$1 \to \check{H}^1(X;\widehat{G_{\mathrm{ab}}}) \to \mathcal{E}_G(X) \to C(X,H^2(G,\mathbb{T})) \to 1,$$

where $\check{H}^1(X;\widehat{G_{\mathrm{ab}}})$ denotes Čech cohomology with coefficients in the (discrete) dual group of the abelianization G_{ab} of G and $H^2(G,\mathbb{T})$ is topologized as in [50]. The quotient map in this sequence sends a class $[\alpha] \in \mathcal{E}_G(X)$ to the mapping $x \mapsto [\alpha^x] \in \mathcal{E}_G(\mathrm{pt}) \cong H^2(G,\mathbb{T})$. Since X is contractible, $\check{H}^1(X;\widehat{G_{\mathrm{ab}}}) = 0$. Since G is compact, $H^2(G,\mathbb{T})$ is countable by Corollary 1, on page 56, of [48]. Since it is locally compact Hausdorff (see Remark 4.2 of [21]), $H^2(G,\mathbb{T})$ is discrete. Since X is connected, it follows that $C(X,H^2(G,\mathbb{T})) \cong H^2(G,\mathbb{T})$. This completes the proof. □

The following result follows from Theorem 1.5 of [15].

Proposition 1.6. *Let G be a second countable locally compact group and let A and B be G-C^*-algebras. Let $z \in KK_0^G(A,B)$ have the property that for all compact subgroups L of G, the Kasparov product with the restriction $\mathrm{res}_L^G(z) \in KK_0^L(A,B)$ induces bijective homomorphisms $KK_*^L(\mathbb{C},A) \to KK_*^L(\mathbb{C},B)$. Then the Kasparov product with z induces an isomorphism $K_*^{\mathrm{top}}(G;A) \cong K_*^{\mathrm{top}}(G;B)$.*

Proof. For a closed subgroup H of G and a proper H-space Y we define $\mathcal{F}_H^n(C_0(Y)) = KK_n^G(C_0(Y),\, A)$ and $\mathcal{G}_H^n(C_0(Y)) = KK_n^G(C_0(Y),\, B)$. Let $\mathcal{S}(G)$ be the collection of subgroups of G defined in the introduction to Section 1 of [15]. Then, by Remark 1.2 of [15], the functors $\mathcal{F}^* = (\mathcal{F}_H^*)_{H\in\mathcal{S}(G)}$ and $\mathcal{G}^* = (\mathcal{G}_H^*)_{H\in\mathcal{S}(G)}$ are going-down functors in the sense of Definition 1.1 of [15]. For any $z \in KK_0^G(A,B)$, the family of transformations

$$\mathcal{F}_H^n(C_0(Y)) = KK_n^H(C_0(Y),\, A) \overset{\otimes_A \mathrm{res}_H^G(z)}{\longrightarrow} KK_n^H(C_0(Y),\, B) = \mathcal{G}_H^n(C_0(Y))$$

is a going-down transformation in the sense of Definition 1.4 of [15]. Indeed, the assumptions (1) and (2) of that definition follow from naturality of the Kasparov product. Thus, the proof of the proposition will follow from Theorem 1.5 of [15] if we know that

$$KK_n^L(C_0(V),\, A) \overset{\otimes_A \mathrm{res}_L^G(z)}{\longrightarrow} KK_n^L(C_0(V),\, B) \tag{1.2}$$

is an isomorphism for all compact subgroups L of G and for all real finite dimensional representation spaces V of L.

Let V be a real finite dimensional representation space of L. Lemma 7.7(2) and Lemma 7.7(1) of [12], in order, give isomorphisms

$$KK_n^L(C_0(V),\, D) \xrightarrow{\cong} KK_n^L(C_0(V)\otimes C_0(V),\, C_0(V)\otimes D) \xrightarrow{\cong} KK_n^L(\mathbb{C},\, C_0(V)\otimes D)$$

for any D, which respect Kasparov products. To prove that (1.2) is an isomorphism, it is therefore enough to show that Kasparov product with $1_{C_0(V)} \otimes_{\mathbb{C}} \mathrm{res}_L^G(z)$ induces an isomorphism

$$KK_n^L(\mathbb{C},\, C_0(V)\otimes A) \cong KK_n^L(\mathbb{C},\, C_0(V)\otimes B). \tag{1.3}$$

We prove this by a bootstrap process.

First, let $L' \subseteq L \subseteq G$ be compact subgroups of G. We claim that multiplication with $1_{C(L/L')} \otimes_{\mathbb{C}} \mathrm{res}_L^G(z)$ induces isomorphisms $KK_n^L(\mathbb{C},\, C(L/L')\otimes A) \cong KK_n^L(\mathbb{C}, C(L/L')\otimes B)$. For any G-C*-algebra D, let $q_D \colon C(L/L')\otimes D \to D$ be the quotient map given by evaluation at the identity coset $L' \in L/L'$. By Theorem 20.5.5 of [5], the chain of maps

$$KK_n^L(\mathbb{C},\, C(L/L')\otimes D) \xrightarrow{\mathrm{res}_{L'}^L} KK_n^{L'}(\mathbb{C},\, C(L/L')\otimes D) \xrightarrow{q_*} KK_n^{L'}(\mathbb{C}, D)$$

is an isomorphism. Moreover, when applied to A and B in place of D, it transforms Kasparov product with $1_{C(L/L')} \otimes_{\mathbb{C}} \mathrm{res}_L^G(z)$ to Kasparov product with $\mathrm{res}_{L'}^G(z)$. The claim thus follows from our assumption on z.

Now let V be any real finite dimensional representation space of L. Let

$$\varnothing = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l = V$$

be a stratification as in the proof of Lemma 2.8 of [15]. Thus, the U_j are L-invariant open subsets of V such that the difference sets $M_j = U_j \setminus U_{j-1}$ and their quotients M_j/L are smooth manifolds, and M_j is locally L-homeomorphic to spaces of the form $X \times L/L_j$ for suitable compact subgroups $L_j \subseteq L$. Now M_j/L is triangulable by Theorem 10.6 of [46], and refining the triangulation allows us to assume that the inverse image of each open simplex S is L-homeomorphic to $S \times L/L_j$ with the trivial action of L on S. By Bott periodicity and the previous paragraph, (1.3) is an isomorphism when $C_0(V)$ is replaced by $C_0(S \times L/L')$. Using six term exact sequences and the Five Lemma, we conclude that (1.3) is an isomorphism when $C_0(V)$ is replaced by $C_0(X)$ for the inverse image X in M_j of any finite complex in the triangulation of M_j/L, or the inverse image of the topological interior of one. Using continuity of equivariant K-theory for compact group actions (Theorem 2.8.3(6) of [56] or Proposition 11.9.2 of [5]), we find that (1.3) is an isomorphism when $C_0(V)$ is replaced by $C_0(M_j)$. Again using six term exact sequences and the Five Lemma, we conclude that (1.3) is an isomorphism for our given real representation space V. □

Theorem 1.7. *Let $\alpha \colon G \to \mathrm{Aut}\big(C([0,1], \mathcal{K})\big)$ be a homotopy of actions on $\mathcal{K}$. Write $\mathcal{K}_x$ for $\mathcal{K}$ when we consider $\mathcal{K}$ as the fiber at x equipped with the action $\alpha^x \colon G \to \mathrm{Aut}(\mathcal{K})$, and let $q_x \colon C\big([0,1], \mathcal{K}\big) \to \mathcal{K}_x$ denote evaluation at x. Then*

$$(q_x)_* \colon K_*^{\mathrm{top}}\big(G;\, C\big([0,1], \mathcal{K}\big)\big) \to K_*^{\mathrm{top}}(G; \mathcal{K}_x)$$

is an isomorphism for all $x \in [0,1]$.

Proof. By Proposition 1.6 it is enough to check that

$$(q_x)_* \colon KK_*^L\big(\mathbb{C},\, C\big([0,1],\, \mathcal{K}\big)\big) \to KK_*^L(\mathbb{C}, \mathcal{K}_x)$$

is an isomorphism for every compact subgroup $L \subseteq G$. By the Green-Julg theorem (Theorem 11.7.1 of [5] or Theorem 2.8.3(7) of [56]), this is equivalent to saying that

$$(q_x \rtimes L)_* \colon K_*\big(C\big([0,1],\, \mathcal{K}\big) \rtimes_\alpha L\big) \to K_*(\mathcal{K}_x \rtimes_{\alpha^x} L)$$

is an isomorphism for all such L. We know from Proposition 1.5 that the restriction $\mathrm{res}_L^G(\alpha)$ of α to L is exterior equivalent to $\mathrm{id}_{[0,1]} \otimes \mathrm{res}_L^G(\alpha^x)$. So $C\big([0,1],\, \mathcal{K}\big) \rtimes_\alpha L \cong C\big([0,1],\, \mathcal{K}_x \rtimes_{\alpha^x} L\big)$, and it is easy to check that $q_x \rtimes L$ corresponds to the evaluation map $\mathrm{ev}_x \colon C\big([0,1],\, \mathcal{K}_x \rtimes_{\alpha^x} L\big) \to \mathcal{K}_x \rtimes_{\alpha^x} L$. This map induces an isomorphism on K-theory because it is a homotopy equivalence. □

Corollary 1.8. *Let $\alpha \colon G \to \mathrm{Aut}\,\big(C\big([0,1],\, \mathcal{K}\big)\big)$ be a homotopy for actions on $\mathcal{K}$ and assume that G satisfies the Baum-Connes conjecture with coefficients $C\big([0,1],\, \mathcal{K}\big)$ (with respect to α) and with coefficients $\mathcal{K}_x$ for all $x \in [0,1]$. Then, for any $x \in [0,1]$, the map*

$$(q_x \rtimes G)_* \colon K_*(C\big([0,1],\, \mathcal{K}\big) \rtimes_{\alpha,r} G) \to K_*(\mathcal{K}_x \rtimes_{\alpha^x,r} G)$$

is an isomorphism.

Corollary 1.8 fails if $\mathcal{K}$ is replaced by a general AF algebra, or by a general commutative C*-algebra, even for $G = \mathbb{Z}_2$. See Examples 3.3 and 3.5 of [57]. Sometimes something can be said; see Theorem 4.3 of [57].

Theorem 1.9 (Theorem 0.3)**.** *Suppose that $\omega_0, \omega_1 \in Z^2(G, \mathbb{T})$ are homotopic Borel 2-cocycles on the second countable locally compact group G, that is, there exists a Borel 2-cocycle $\Omega \in Z^2(G,\, C([0,1],\, \mathbb{T}))$ such that $\omega_j = \Omega(\,\cdot\,,\,\cdot\,)(j)$ for $j = 0,1$. Suppose further that G satisfies the Baum-Connes conjecture with coefficients. Then $K_*(C_r^*(G, \omega_0)) \cong K_*(C_r^*(G, \omega_1))$.*

Proof. If G satisfies the Baum-Connes conjecture with arbitrary coefficients, then G certainly satisfies the hypotheses of Corollary 1.8. The proof follows by combining this corollary with the isomorphism (1.1) and Remark 1.4. □

Remark 1.10. Let us briefly discuss the assumption on the Baum-Connes conjecture as given in Corollary 1.8. If G satisfies the conjecture with coefficients $C\big([0,1],\, \mathcal{K}\big)$ for *all* homotopies $\alpha \colon G \to \mathrm{Aut}\,\big(C\big([0,1],\, \mathcal{K}\big)\big)$ (we then say that G satisfies the conjecture for homotopies), then G automatically satisfies the conjecture with coefficients $\mathcal{K}$ for any G-action on $\mathcal{K}$. This follows because for any action β of G on $\mathcal{K}$, the action $\mathrm{id}_{[0,1]} \otimes \beta$ of G on $C\big([0,1],\, \mathcal{K}\big)$ is equivariantly KK-equivalent to $\mathcal{K}_\beta$. Conversely, if we just know that G satisfies the conjecture for $\mathcal{K}$ with respect to any given G-action, then it follows from Proposition 3.1 of [14] that G satisfies the conjecture for homotopies as soon as G is exact and has a γ-element in the sense of Kasparov.

For later use it is convenient to restate Corollary 1.8 completely in terms of homotopies between cocycles. We will need the reduced twisted crossed product C*-algebra $C([0,1]) \rtimes_{\Omega,r} G$ for a cocycle $\Omega \in Z^2\big(G,\, C\big([0,1],\, \mathbb{T}\big)\big)$. It is a completion of the convolution algebra $L^1\big(G,\, C([0,1]),\, \Omega\big)$ of all $C([0,1])$-valued L^1-functions on G with convolution given by

$$f *_\Omega g(s,x) = \int_G f(t,x) g(t^{-1}s,\, x) \Omega(t,\, t^{-1}s)(x)\, dx.$$

See [52], and, for the case of a discrete group, [77].

Corollary 1.11. *Suppose that G satisfies the Baum-Connes conjecture with respect to any homotopy of actions on $\mathcal{K}$. Let G act trivially on $C\big([0,1],\mathbb{T}\big)$. Then for any $\Omega \in Z^2\big(G, C\big([0,1],\mathbb{T}\big)\big)$ and any $x \in [0,1]$, the canonical quotient map $p_x \colon C([0,1]) \rtimes_{\Omega,r} G \to C^*_r(G,\omega_x)$ induces an isomorphism on K-theory.*

Proof. As explained in Remark 1.4, the cocycle Ω determines a homotopy $\alpha \colon G \to \operatorname{Aut}\big(C\big([0,1],\mathcal{K}\big)\big)$. We claim that, if $\overline{\alpha} \colon G \to \operatorname{Aut}\big(C\big([0,1],\mathcal{K}\big)\big)$ is the homotopy corresponding to the inverse $\overline{\Omega}$ of Ω, then there is an isomorphism

$$\big(C([0,1]) \rtimes_{\Omega,r} G\big) \otimes \mathcal{K} \cong C\big([0,1],\mathcal{K}\big) \rtimes_{\overline{\alpha},r} G.$$

For the full twisted crossed products, this follows from Proposition 4.6 of [22] (which is a special case of Theorem 3.4 of [52]). Remark 3.12 and the proof of Theorem 3.11 of [52] imply that this is correct for the reduced crossed products as well.

With $\omega_x = \Omega(\,\cdot\,,\,\cdot\,)(x)$ for $x \in [0,1]$, there are obvious quotient maps $p_x \colon C([0,1]) \rtimes_{\Omega,r} G \to C^*_r(G,\omega_x)$. Moreover, with the right vertical arrow being the isomorphism of (1.1), the diagram

$$\begin{array}{ccc}
\big(C([0,1]) \rtimes_{\Omega,r} G\big) \otimes \mathcal{K} & \xrightarrow{p_x \otimes \mathrm{Id}_{\mathcal{K}}} & C^*_r(G,\omega_x) \otimes \mathcal{K} \\
\cong \big\downarrow & & \big\downarrow \cong \\
C\big([0,1],\mathcal{K}\big) \rtimes_{\overline{\alpha},r} G & \xrightarrow[q_x \rtimes G]{} & \mathcal{K} \rtimes_{\overline{\alpha}^x,r} G
\end{array}$$

commutes. Using the induced maps in K-theory for this diagram together with Corollary 1.8 and Remark 1.10, the result follows. □

Definition 1.12. We call a class $[\omega] \in H^2(G,\mathbb{T})$ *real* if there exists a cocycle $c \in Z^2(G,\mathbb{R})$ such that ω is equivalent to the cocycle $(s,t) \mapsto \exp\big(ic(s,t)\big)$.

In this case we obtain a homotopy $\Omega \in Z^2\big(G, C\big([0,1],\mathbb{T}\big)\big)$ between $[\omega]$ and the trivial cocycle $[1]$ by the formula

$$\Omega(s,t)(x) = e^{ix\cdot c(s,t)}$$

for $x \in [0,1]$ and $s,t \in G$. As a consequence we get the following corollary. Several special cases of it appear in [24, 53, 11, 43, 44]. (In [45], Mathai states a theorem saying that for any discrete G and any real 2-cocycle ω on G, one has $K_0(C^*_r(G,\omega)) \cong K_0(C^*_r(G))$. Unfortunately, the proof given in [45] has a substantial gap, and hence the result is not available so far.)

Corollary 1.13. *Let G be a group which satisfies the Baum-Connes conjecture for homotopies of actions on $\mathcal{K}$, and let $\omega \in Z^2(G,\mathbb{T})$ be a cocycle such that $[\omega]$ is real. Then $K_*(C^*_r(G,\omega)) \cong K_*(C^*_r(G))$. More generally, if G is any second countable locally compact group and $[\omega] \in H^2(G,\mathbb{T})$ is real, then $K^{\mathrm{top}}_*(G,\omega) \cong K^{\mathrm{top}}_*(G)$.*

Let Θ be any skew symmetric real $d \times d$ matrix. Then Θ determines a 2-cocycle on $\mathbb{Z}^d$ via

$$\omega_\Theta(m,n) = \exp(\pi i \langle \Theta m, n\rangle)$$

for $m,n \in \mathbb{Z}^d$. The corresponding d-dimensional noncommutative torus [65] is the twisted group C*-algebra $A_\Theta = C^*(\mathbb{Z}^d,\omega_\Theta)$. Equivalently, A_Θ is the universal C*-algebra generated by unitaries $u_1, u_2, \ldots, u_d$ subject to the relations

$$u_k u_j = \exp(2\pi i \theta_{j,k}) u_j u_k$$

for $1 \leq j, k \leq d$. (Of course, if all $\theta_{j,k}$ are integers, it is not really noncommutative.)

Since ω_Θ is a real cocycle, Corollary 1.13 immediately gives the following well known computation of the K-theory of a higher dimensional noncommutative torus, as found, for example, in [24] and [64]. Of course, we do not immediately get the additional information given in [24] and [64].

Corollary 1.14. *For any $d \geq 1$ and any skew symmetric real $d \times d$ matrix Θ, we have $K_j\big(C^*(\mathbb{Z}^d, \omega_\Theta)\big) \cong \mathbb{Z}^{2^{d-1}}$ for $j = 0, 1$.*

It is shown in Theorem 3.2 of [1] (see Section 2 of [2] for terminology) that every class in $H^2(\mathbb{Z}^d, \mathbb{T})$ has a representative of the form $\omega_\Theta(n, m) = e^{i\langle \Theta n, m\rangle}$ with Θ a skew symmetric real $d \times d$ matrix.

2. Twisted group C*-algebras of semidirect products

As mentioned in the introduction, the rotation algebra A_θ can be realized as the twisted group algebra $C^*(\mathbb{Z}^2, \omega_\theta)$ with

$$\omega_\theta\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right)\right) = e^{\pi i \theta (n'm - nm')}.$$

Indeed, if we define $u_\theta = \delta_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}$ and $v_\theta = \delta_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}$, then u_θ and v_θ are unitary generators of $C^*(\mathbb{Z}^2, \omega_\theta)$ which satisfy the commutation relation $v_\theta u_\theta = e^{2\pi i \theta} u_\theta v_\theta$. This (and other computations below) follows because $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) \mapsto \delta_{\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)} \in C^*(\mathbb{Z}^2, \omega_\theta)$ is an ω_θ-representation of $\mathbb{Z}^2$.

As explained in the introduction, we want to study the crossed products $A_\theta \rtimes_\alpha F$ with F a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and the action α^θ of $\mathrm{SL}_2(\mathbb{Z})$ on on A_θ as given in (0.1). We will realize the crossed product $A_\theta \rtimes_{\alpha^\theta} H$, for *any* subgroup $H \subseteq \mathrm{SL}_2(\mathbb{Z})$, as a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta)$, where $H \subseteq \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ via matrix multiplication and $\widetilde{\omega}_\theta$ is a suitable extension of ω_θ to $\mathbb{Z}^2 \rtimes H$. A short computation shows that the cocycle $\omega_\theta \in Z^2(\mathbb{Z}^2, \mathbb{T})$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$, that is

$$\omega_\theta\left(N\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), N\left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right)\right) = \omega_\theta\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right)\right)$$

for all $N \in \mathrm{SL}_2(\mathbb{Z})$. This implies that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ defines an action $\beta^\theta \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}\left(C^*(\mathbb{Z}^2, \omega_\theta)\right)$ via

$$\beta_N^\theta(f)\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) = f\left(N^{-1}\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)\right)$$

for f in the dense subalgebra $l^1(\mathbb{Z}^2, \omega_\theta)$. It follows from this formula and the fact that $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) \mapsto \delta_{\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)} \in C^*(\mathbb{Z}^2, \omega_\theta)$ is an ω_θ-representation that

$$\begin{aligned} \beta_N^\theta(u_\theta) = \beta_N^\theta(\delta_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}) = \delta_{\left(\begin{smallmatrix} n_{1,1} \\ n_{2,1} \end{smallmatrix}\right)} &= \overline{\omega_\theta\left(\left(\begin{smallmatrix} n_{1,1} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ n_{21} \end{smallmatrix}\right)\right)}\delta_{\left(\begin{smallmatrix} n_{1,1} \\ 0 \end{smallmatrix}\right)}\delta_{\left(\begin{smallmatrix} 0 \\ n_{2,1} \end{smallmatrix}\right)} \\ &= e^{\pi i n_{11} n_{21} \theta}(\delta_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)})^{n_{1,1}}(\delta_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)})^{n_{2,2}} = e^{\pi i n_{1,1} n_{2,1} \theta} u_\theta^{n_{1,1}} v_\theta^{n_{2,1}} \end{aligned}$$

and, similarly, $\beta_N^\theta(v_\theta) = e^{\pi i n_{1,2} n_{2,2} \theta} u_\theta^{n_{1,2}} v_\theta^{n_{2,2}}$, so $\beta^\theta = \alpha^\theta$ for all $\theta \in [0, 1]$.

We recall some facts about semidirect products. Let M and H be locally compact groups; in view of our intended application ($M = \mathbb{Z}^2$ and $H \subseteq \mathrm{SL}_2(\mathbb{Z})$), we write the group operation on M additively and the operation on H multiplicatively. Assume further that H acts continuously on M by automorphisms; we write the action as $(h, m) \mapsto hm$ for $h \in H$ and $m \in M$. Then the semidirect product $M \rtimes H$ is the set $M \times H$, equipped with the product topology and the multiplication

$(m,h)\cdot(m',h') = (m+hm',\, hh')$. We let $\mu\colon H \to (0,\infty)$ be the function such that $\int_M f(m)\,dm = \mu(h)\int_M f(h^{-1}\cdot m)\,dm$ for all $f \in L^1(M)$, where dm and dh denote left Haar measures on M and H. Recall that left Haar measure on $M \rtimes H$ is given by the formula

$$\int_{M\rtimes H} f(m,h)\,d(m,h) = \int_H \int_M f(m,h)\mu(h)\,dm\,dh.$$

We will also need a continuously parametrized version of the following lemma. A general statement is awkward to formulate, but what we actually need (see Remark 2.3 below) has essentially the same proof, so we just refer to the proof.

Lemma 2.1. *Suppose that $M \rtimes H$ is a semidirect product of the second countable locally compact groups M and H and assume that $\omega \in Z^2(M,\mathbb{T})$ is invariant under the action of H on M, that is, $\omega(h\cdot n, h\cdot m) = \omega(n,m)$ for all $n,m \in M, h \in H$. Then:*

(1) *There is a cocycle $\widetilde{\omega} \in Z^2(M \rtimes H,\, \mathbb{T})$ defined by*
$$\widetilde{\omega}\big((m,h),(m',h')\big) = \omega(m,\, h\cdot m').$$
(2) *There is an action $\alpha\colon H \to \mathrm{Aut}(C^*(M,\omega))$ (and similarly for $C^*_r(M,\omega)$) determined by $\alpha_h(f)(m) = \mu(h)f(h^{-1}m)$ for $f \in L^1(M,\omega)$.*
(3) *There are isomorphisms*
$$C^*(M\rtimes H,\, \widetilde{\omega}) \cong C^*(M,\omega)\rtimes_\alpha H \quad \textit{and} \quad C^*_r(M\rtimes H,\, \widetilde{\omega}) \cong C^*_r(M,\omega)\rtimes_{\alpha,r} H$$
given on the level of L^1-functions by $f \mapsto \Phi(f) \in L^1\big(H,\, L^1(M,\omega)\big)$ with $\Phi(f)(h) = \mu(h)f(\,\cdot\,,h)$.

Proof. Part (1) follows from straightforward computations. For full twisted group algebras, Parts (2) and (3) are very special cases of Theorem 4.1 of [52]. Since the deduction from that result is quite tedious, we sketch the proof of (2) and (3) in the case of reduced twisted group algebras. Note first that we have a faithful representation $L^M_\omega\colon C^*_r(M,\omega) \to B(L^2(M))$ given by $L^M_\omega(f)\xi = f *_\omega \xi$ for $f \in L^1(M,\omega)$ and $\xi \in L^2(M)$. It is easy to check on L^1-functions that $\alpha_h\colon L^1(M,\omega) \to L^1(M,\omega)$ preserves all algebraic operations. Define a unitary operator $U_h\colon L^2(M) \to L^2(M)$ by $(U_h\xi)(m) = \sqrt{\mu(h)}\xi(h^{-1}\cdot m)$. We check, as a sample, that $U_h^* L^M_\omega(\alpha_h(f))U_h = L^M_\omega(f)$ for all $f \in L^1(M,\omega)$, which then implies that α_h is norm preserving. Let $\xi \in L^2(M)$. Then for almost all $n \in M$, we have, using H-invariance of ω at the fourth step and the definition of μ at the fifth step,

$$\begin{aligned}
\big(U_h^* L^M_\omega(\alpha_h(f))U_h\xi\big)(n) &= \sqrt{\mu(h^{-1})}\big(\alpha_h(f) *_\omega U_h(\xi)\big)(h\cdot n)\\
&= \sqrt{\mu(h^{-1})}\int_M \alpha_h(f)(m)U_h(\xi)\big(h\cdot n - m\big)\omega\big(m,\, h\cdot n - m\big)\,dm\\
&= \mu(h)\int_M f(h^{-1}\cdot m)\xi\big(n - h^{-1}\cdot m\big)\omega\big(m,\, h\cdot n - m\big)\,dm\\
&= \mu(h)\int_M f(h^{-1}\cdot m)\xi\big(n - h^{-1}\cdot m\big)\omega\big(h^{-1}\cdot m,\, n - h^{-1}\cdot m\big)\,dm\\
&= \int_M f(m)\xi(n-m)\omega(m,\, n-m)\,dm = \big(L^M_\omega(f)\xi\big)(n).
\end{aligned}$$

This proves (2). To check the isomorphism of (3) in the reduced case one first checks, using Fubini's theorem, that the map $\Phi\colon L^1(M\rtimes H,\, \widetilde{\omega}) \to L^1(H,\, L^1(M,\omega))$ is a bijection which preserves all algebraic operations. To see that it extends to an isomorphism of the reduced C*-algebras, consider the unitary $V\colon L^2(M \rtimes H) \to L^2(H, L^2(M))$ given by $(V\xi)(h)(n) = \xi(h \cdot n, h)$. Let $L_{\widetilde{\omega}}^{M \rtimes H}$ denote the $\widetilde{\omega}$-regular representation of $M \rtimes H$. Let $\Lambda\colon L^1(H, C_r^*(M,\omega)) \to B\big(L^2\big(H, L^2(M)\big)\big)$ denote the integrated from of the regular representation of $(C_r^*(M,\omega), H, \alpha)$ induced from the faithful representation $L_\omega^M\colon C_r^*(M,\omega) \to B(L^2(M))$. It is given by

$$\big(\Lambda(g)\eta\big)(h) = \int_H L_\omega^M(\alpha_{h^{-1}}(g(l))\eta(l^{-1}h)\, dl$$

for $g \in L^1(H, C_r^*(M,\omega))$ and $\eta \in L^2(H, L^2(M))$. Using this formula, one can check that

$$V L_{\widetilde{\omega}}^{M \rtimes H}(f)\xi = \Lambda(\Phi(f))V\xi$$

for all $f \in L^1(M \rtimes H, \widetilde{\omega})$ and $\xi \in L^2(M \rtimes H)$. This implies that Φ preserves the C*-norm. $\square$

Corollary 2.2. *Suppose that H is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Define a cocycle on $\mathbb{Z}^2 \rtimes H$ by*

$$\widetilde{\omega}_\theta\left(\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), N\right), \left(\left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right), N'\right)\right) = \omega_\theta\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), N \cdot \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right)\right)$$

for $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right) \in \mathbb{Z}^2$ and $N, N' \in H$. Then there is a canonical isomorphism

$$A_\theta \rtimes_{\alpha^\theta, r} H = C^*(\mathbb{Z}^2, \omega_\theta) \rtimes_{\alpha^\theta, r} H \cong C_r^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta)$$

(and similarly for the full crossed products). Moreover, for all $\theta \in [0,1]$, we have graded isomorphisms

$$K_*(A_\theta \rtimes_{\alpha^\theta, r} H) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta)) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes H)).$$

Proof. The isomorphism of C*-algebras follows directly from Lemma 2.1. For the isomorphisms on K-theory, first note that, because $\mathrm{SL}_2(\mathbb{Z})$ satisfies the Baum-Connes Conjecture with coefficients, Theorem 2.5 and Corollary 3.14 of [13] imply that $\mathbb{Z}^2 \rtimes H$ does too. Then use Corollary 1.13 and the fact that $[\widetilde{\omega}_\theta]$ is real in the sense of Definition 1.12. $\square$

Remark 2.3. In our application, to the classification of crossed products $A_\theta \rtimes_{\alpha,r} H$ for certain subgroups $H \subseteq \mathrm{SL}_2(\mathbb{Z})$, we will need the K-groups together with a basis for each. This problem can be simplified substantially, by working with continuous families (in a suitable sense) of projections or unitaries in matrix algebras over the twisted group algebras $C_r^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta)$. Thus, for any interval $[a,b]$, consider the cocycle $\widetilde{\Omega}\colon (\mathbb{Z}^2 \rtimes H) \times (\mathbb{Z}^2 \rtimes H) \to C\big([a,b], \mathbb{T}\big)$ given by $\widetilde{\Omega}(\,\cdot\,,\,\cdot\,)(\theta) = \widetilde{\omega}_\theta$ for $\theta \in [a,b]$. By a continuous family, we simply mean a projection (or unitary) in some matrix algebra $M_n\big(C\big([0,1]\big) \rtimes_{\widetilde{\Omega},r} (\mathbb{Z}^2 \rtimes H)\big)$. It follows from Corollary 1.11 that the evaluation map

$$\mathrm{ev}_\theta\colon C\big([0,1]\big) \rtimes_{\widetilde{\Omega},r} (\mathbb{Z}^2 \rtimes H) \to C_r^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta)$$

induces an isomorphism in K-theory, and hence it maps a basis to a basis. Lemma 2.1 provides an isomorphism $C_r^*(\mathbb{Z}^2 \rtimes H, \widetilde{\omega}_\theta) \cong A_\theta \rtimes_{\alpha^\theta, r} H$. An obvious generalization, proved by an easy modification of its proof, provides an isomorphism

$$C\big([a,b]\big) \rtimes_{\widetilde{\Omega},r} (\mathbb{Z}^2 \rtimes H) \cong \big(C\big([a,b]\big) \rtimes_{\Omega,r} \mathbb{Z}^2\big) \rtimes H$$

which moreover respects the evaluation maps ev_θ on both algebras. In particular, for projections $p_1, p_2, \ldots, p_n$ in matrix algebras over $\big(C\big([a,b]\big) \rtimes_{\Omega,r} \mathbb{Z}^2\big) \rtimes H$, the following are equivalent:

(1) $[p_1], [p_2], \ldots, [p_n]$ form a basis for $K_0\big(\big(C\big([a,b]\big) \rtimes_{\Omega,r} \mathbb{Z}^2\big) \rtimes H\big)$.
(2) For some $\theta \in [a,b]$, the evaluated classes $[\mathrm{ev}_\theta(p_1)]$, $[\mathrm{ev}_\theta(p_2)]$, $\ldots$, $[\mathrm{ev}_\theta(p_n)]$ form a basis for $K_0(A_\theta \rtimes_{\alpha^\theta,r} H)$.
(3) For all $\theta \in [a,b]$, the evaluated classes $[\mathrm{ev}_\theta(p_1)]$, $[\mathrm{ev}_\theta(p_2)]$, $\ldots$, $[\mathrm{ev}_\theta(p_n)]$ form a basis for $K_0(A_\theta \rtimes_{\alpha^\theta,r} H)$.

Remark 2.4. We should also remark at this point that twisted group algebras of semidirect products $\Gamma = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, with A a fixed element of $\mathrm{SL}_2(\mathbb{Z})$ and the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ given by powers of A, were extensively studied by Packer and Raeburn in [53]. They use the embedding of Γ into $\mathbb{R}^2 \rtimes_A \mathbb{R}$ together with Connes's Thom isomorphism to show that $K_*(C^*(\Gamma,\omega)) \cong K_*(C^*(\Gamma))$ for any cocycle $\omega \in H^2(\Gamma,\mathbb{T})$. (See Corollary 2.12 of [53].) Since they also show (Corollary 3.6 of [53]) that every cocycle of such a group Γ is the exponential of a real cocycle, this result also follows from Corollary 1.13 and the proof of Corollary 2.2.

3. The untwisted case

As mentioned above, we are interested in applying the results of the previous section to the case of finite groups $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ which are conjugate to one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators as given in (0.2). In this section we give an explicit computation of the K-theory of the untwisted group algebras $C^*(\mathbb{Z}^2 \rtimes F)$. We need some notation.

Notation 3.1. Let G be a *discrete* group.
(1) For a subgroup $H \subseteq G$, let $\mathrm{ind}_H^G \colon K_*(C_r^*(H)) \to K_*(C_r^*(G))$ be the induction homomorphisms induced by the obvious maps of the reduced group C*-algebras $C_r^*(H) \to C_r^*(G)$.
(2) For a subgroup $H \subseteq G$ such that the index $[G : H]$ is finite, the restriction homomorphism $\mathrm{res}_G^H \colon K_n(C_r^*(G)) \to K_n(C_r^*(H))$ (sometimes written res_H^G) is given by restriction of scalars: every finitely generated projective $C_r^*(G)$-module is also finitely generated and projective as a module over the subalgebra $C_r^*(H)$.
(3) We denote by $\widetilde{K}_n(C_r^*(G))$ the cokernel of the map $K_n(\mathbb{C}) = K_n(C^*(\{1\})) \to K_n(C_r^*(G))$ induced by the inclusion of the trivial subgroup $\{1\} \to G$, and we let $\mathrm{pr} \colon K_n(C_r^*(G)) \to \widetilde{K}_n(C_r^*(G))$ be the canonical projection.

The following lemma, known as the double coset formula, will be used several times. It actually holds in greater generality, but all we need here is the version below, which follows from the arguments in the proof of Proposition 5.6(b) in Chapter III of [10].

Lemma 3.2. *Suppose that G is a discrete group and H and C are subgroups of G such that C has finite index in G. For $g \in G$, let $\varphi_g \colon C_r^*(H) \to C_r^*(gHg^{-1})$ denote the isomorphism induced by conjugation. Then*

$$\mathrm{res}_G^C \circ \mathrm{ind}_H^G = \sum_{[g] \in C \backslash G/H} \Big(\mathrm{ind}_{gHg^{-1}\cap C}^C \circ \mathrm{res}_{gHg^{-1}}^{gHg^{-1}\cap C} \circ (\varphi_g)_*\Big)$$

as maps from $K_0(C_r^*(H))$ to $K_0(C_r^*(C))$.

Remark 3.3. Let H be a finite group. Then the homomorphism $\mathrm{ind}_{\{1\}}^H \colon K_0(\mathbb{C}) = K_0(C^*(\{1\})) \to K_0(C^*(H))$ has a left inverse $r \colon K_0(C^*(H)) \to K_0(\mathbb{C})$. Letting $\mathbb{C}$ carry the trivial representation of H, it sends the class $[V]$ of a finite-dimensional

H-representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{C^*(H)} V) \cdot [\mathbb{C}]$. (The number $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{C^*(H)} V)$ is just the multiplicity of the trivial representation in the decomposition of V into irreducibles.) It induces a splitting

$$s\colon \widetilde{K}_0(C^*(H)) \to K_0(C^*(H)) \tag{3.1}$$

of $\mathrm{pr}\colon K_0(C^*(H)) \to \widetilde{K}_0(C^*(H))$. Namely, s sends $x \in \widetilde{K}_0(C^*(H))$ to $y - \big(\mathrm{ind}_{\{1\}}^H \circ r\big)(y)$ for any choice of $y \in K_0(C_r^*(G))$ with $\mathrm{pr}(y) = x$.

Notation 3.4. Let G be a discrete group, and let $H \subseteq G$ be a finite subgroup. We denote by b_H the composition

$$b_H\colon \widetilde{K}_0(C^*(H)) \xrightarrow{s} K_0(C^*(H)) \xrightarrow{\mathrm{ind}_H^G} K_0(C_r^*(G)).$$

Theorem 3.5. *Suppose that $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ is one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators as in (0.2). Then there exist only finitely many conjugacy classes of maximal finite subgroups $M \subseteq \mathbb{Z}^2 \rtimes F$. Let $M_0, M_1, \ldots, M_l$ be a list of representatives for these conjugacy classes. Let $b_j\colon \widetilde{K}_0(C^*(M_j)) \to K_0(C^*(\mathbb{Z}^2 \rtimes F))$ for $j = 0, 1, \ldots, l$ be the map b_{M_j} of Notation 3.4. Then the following sequence is exact:*

$$0 \to K_0(C^*(\{1\})) \oplus \left(\bigoplus_{j=0}^{l} \widetilde{K}_0(C^*(M_j))\right) \xrightarrow{\mathrm{ind}_{\{1\}}^G \oplus \left(\bigoplus_{j=0}^{l} b_j\right)} K_0(C^*(\mathbb{Z}^2 \rtimes F))$$
$$\xrightarrow{\mathrm{pr} \circ \mathrm{res}_{\mathbb{Z}^2 \rtimes F}^{\mathbb{Z}^2}} \widetilde{K}_0(C^*(\mathbb{Z}^2)) \to 0.$$

Moreover we have $K_1(C^*(\mathbb{Z}^2 \rtimes F)) = \{0\}$ for all F as above.

Remark 3.6. The image of the homomorphism

$$K_0(C^*(\{1\})) \oplus \left(\bigoplus_{j=0}^{l} \widetilde{K}_0(C^*(M_j))\right) \xrightarrow{\mathrm{ind}_{\{1\}}^G \oplus \left(\bigoplus_{j=0}^{l} b_j\right)} K_0(C^*(\mathbb{Z}^2 \rtimes F))$$

does not depend on the special choice of the section $s_j\colon \widetilde{K}_0(C^*(M_j)) \to K_0(C^*(M_j))$ in (3.1) which was used in the definition of the maps $b_j\colon \widetilde{K}_0(C^*(M_j)) \to K_0(C^*(\mathbb{Z}^2 \rtimes F))$. So the theorem remains true if we replace it by any other section $\widetilde{K}_0(C^*(M_j)) \to K_0(C^*(M_j))$.

We give two proofs of Theorem 3.5, one depending on the constructions of Sam Walters [72, 75] and some computations from Section 4 below, and the other more topological, although it has much in common with the first. Before doing so, however, we should describe what one can get from known results on the structure of $A_\theta \rtimes_\alpha F$ for θ rational. Suppose that $\theta = p/q$ in lowest terms, with $q > 0$. Then the fixed point algebras A_θ^F and the crossed products $A_\theta \rtimes_\alpha F$ have been computed explicitly, in a form from which it is easy to determine the K-theory. In particular, this covers the case $C^*(\mathbb{Z}^2 \rtimes F)$. For $F = \mathbb{Z}_2$, see Theorems 1.2 and 1.3 of [8]; for $F = \mathbb{Z}_3$, see Theorem 3.4.1 of [29] for the fixed point algebra and the Theorem on page 244 for the crossed product; for $F = \mathbb{Z}_4$, see Theorem 3.2.5 of [27] for the fixed point algebra and the Theorem 6.2.1 for the crossed product; and for $F = \mathbb{Z}_6$, see Theorem 3.2.8 of [30] for the fixed point algebra and the Theorem on page 2 for the crossed product. (The proofs for the crossed products are similar to the proofs for the fixed point algebras, and in the cases $F = \mathbb{Z}_3$ and $F = \mathbb{Z}_6$ they are not actually given in the papers.)

We describe briefly what happens for $F = \mathbb{Z}_6$; the other cases are similar, but sometimes simpler. Identify the algebras $(M_{3q})^2 = M_{3q} \oplus M_{3q}$, $(M_{2q})^3$, and $(M_q)^6$ with algebras of block diagonal matrices in M_{6q}. Choose three distinct points $x_0, x_1, x_2 \in S^2$. Then

(3.2)
$$A_{p/q} \rtimes_\alpha \mathbb{Z}_6 \cong \big\{ f \in C(S^2, M_{6q}) \colon f(x_0) \in (M_q)^6,\ f(x_1) \in (M_{2q})^3, \text{ and } f(x_2) \in (M_{3q})^2 \big\}.$$

The fixed point algebra has a similar description, in which the functions take values in M_q, and the subalgebras $(M_{3q})^2$, $(M_{2q})^3$, and $(M_q)^6$ are replaced by algebras with complicated formulas for the summands and, if $q \leq 6$, there are sometimes fewer summands. In particular, if $q = 1$, then the fixed point algebra is $C(S^2)$. Note that the isomorphism class of an algebra given by such a description does not depend on the choice of the points in S^2 or the particular identifications of the direct sums with subalgebras of M_{6q} or M_q.

It follows immediately that $\mathbb{T}^2/\mathbb{Z}_6 \cong S^2$ (and similarly in the other cases), a fact that will be proved by much less computational methods in the course of the proof of Theorem 3.5. It is also relatively easy to see that, for $\theta = p/q$, we have

$$K_0(A_\theta \rtimes_\alpha \mathbb{Z}_6) \cong \mathbb{Z}^{10} \quad \text{and} \quad K_1(A_\theta \rtimes_\alpha \mathbb{Z}_6) = 0, \tag{3.3}$$

and to write down nine projections in the algebra on the right hand side of (3.2) such that their classes, together with the Bott class, form a basis for $K_0(A_{p/q} \rtimes_\alpha \mathbb{Z}_6)$. (It was this computation for the fixed point algebra, and the similar results for $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$, that led to the conjecture that the fixed point algebras are AF.) It now follows from Remark 2.3, as in the proof of Theorem 4.9, that (3.3) holds for all values of θ. This is enough to make the proof of Theorem 6.3 go through, and conclude that $A_\theta \rtimes_\alpha \mathbb{Z}_6$ is an AF algebra. But it does not allow us to decide which AF algebra. Probably the most serious difficulty is that we do not know the trace of the last generator of $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_6)$ at irrational values of θ. To find it, we presumably must compute, for some rational value of θ (such as $\theta = 1$), the K_0-class of the module $\mathcal{E}_\theta^F$ of Notation 4.7 in terms of the algebra on the right hand side of (3.2).

First proof of Theorem 3.5. Put $G = \mathbb{Z}^2 \rtimes F$. The group G is amenable. Thus, $C^*(G) \to C^*_r(G)$ is an isomorphism, and we write $C^*(G)$ throughout. Being amenable, G satisfies the Baum-Connes Conjecture by Corollary 9.2 of [33], that is, the assembly map

$$\mu \colon K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C^*(G))$$

is an isomorphism for all $n \in \mathbb{Z}$. Let $p \colon K_0^G(\underline{E}G) \to K_0(G \backslash \underline{E}G))$ be the natural map. The sequence $1 \to \mathbb{Z}^2 \to G \to F \to 1$ has the property that the conjugation action of F on $\mathbb{Z}^2$ is free away from the origin $0 \in \mathbb{Z}^2$. We now claim that there is an isomorphism

$$K_1(C^*(G)) \cong K_1(G \backslash \underline{E}G) \tag{3.4}$$

and an exact sequence

$$(3.5)\ 0 \to \bigoplus_{j=1}^{l} \widetilde{K}_0(C^*(M_j)) \xrightarrow{\bigoplus_{j=0}^{l} b_j} K_0(C^*(G)) \xrightarrow{p\circ\mu^{-1}} K_0(G\backslash \underline{E}G) \to 0.$$

To get these, apply Theorem 5.1(a) and Remark 5.2 of [18] to the extension $1 \to \mathbb{Z}^2 \to G \to F \to 1$, or apply the more general Theorem 1.6 of [42], taking there $K = \{1\}$ and $G = Q$. (Theorem 1.6 of [42] actually gives a long exact sequence, which breaks up into short exact sequences after tensoring with $\mathbb{Q}$. However, every group appearing in the long exact sequence is torsion free.)

Clearly $G\backslash \underline{E}G$ is the same as $F\backslash \mathbb{T}^2$ for the obvious F-action on the two-dimensional torus $\mathbb{T}^2 = \mathbb{Z}^2\backslash \underline{E}G = \mathbb{Z}^2\backslash E\mathbb{Z}^2$. Since F is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, its action on $\mathbb{T}^2$ is orientation preserving. Also, there are only finitely many points of $\mathbb{T}^2$ at which the action is not free. Analyzing these, for example as on page 407 of [68], one can show that $F\backslash \mathbb{T}^2$ is a compact 2-dimensional manifold. The rational cohomology $H^*(G\backslash \underline{E}G; \mathbb{Q})$ agrees with $H^*(\mathbb{T}^2;\mathbb{Q})^F$. Since the action is orientation preserving, F acts trivially on $H^p(\mathbb{T}^2;\mathbb{Q})$ for $p = 0, 2$. Since F acts freely on $\mathbb{Z}^2 = H_1(\mathbb{T}^2;\mathbb{Z})$ away from $\{0\}$, we conclude $H^1(\mathbb{T}^2;\mathbb{Q})^F \cong \mathrm{Hom}_{\mathbb{Z}}\big(H_1(\mathbb{T}^2;\mathbb{Z})^F, \mathbb{Q}\big) = \{0\}$. Hence $G\backslash \underline{E}G = F\backslash \mathbb{T}^2$ has the rational cohomology of S^2 and hence is homeomorphic to S^2. This implies that $K_0(G\backslash \underline{E}G) \cong \mathbb{Z}^2$ and $K_1(G\backslash \underline{E}G) = 0$. We conclude from (3.4) that $K_1(C^*(G)) = 0$.

Using the fact that $G\backslash \underline{E}G$ is 2-dimensional, there is an edge homomorphism $\mathrm{edge}\colon K_0(G\backslash \underline{E}G) \to H_2(G\backslash \underline{E}G)$ which comes from the Atiyah-Hirzebruch spectral sequence converging to $K_{p+q}(G\backslash \underline{E}G)$ with E^2-term $E^2_{p,q} = H_p(G\backslash \underline{E}G, K_q(\mathrm{pt}))$. (See, for example, Section 13.6 of [76] for the Atiyah-Hirzebruch spectral sequence.) Let $f\colon G\backslash \underline{E}G \to \mathrm{pt}$ be the projection onto the one-point space pt. An easy spectral sequence argument shows that

$$K_0(f) \oplus \mathrm{edge}\colon K_0(G\backslash \underline{E}G) \xrightarrow{\cong} K_0(\mathrm{pt}) \oplus H_2(G\backslash \underline{E}G;\, \mathbb{Z})$$

is an isomorphism. The composition

$$K_0(C^*(\{1\})) \xrightarrow{\mathrm{ind}_1^G} K_0(C^*(G)) \xrightarrow{\mu^{-1}} K_0^G(\underline{E}G) \xrightarrow{p} K_0(G\backslash \underline{E}G) \xrightarrow{f} K_0(\mathrm{pt})$$

is an isomorphism, namely the inverse of the assembly map $K_0(\mathrm{pt}) \to K_0(\mathbb{C})$ of the assembly map for the trivial group. Denote by e_G the composition

$$(3.6)\qquad e_G\colon K_0(C^*(G)) \xrightarrow{\mu^{-1}} K_0^G(\underline{E}G) \xrightarrow{p} K_0(G\backslash \underline{E}G) \xrightarrow{\mathrm{edge}} H_2(G\backslash \underline{E}G;\, \mathbb{Z}).$$

Now the exact sequence (3.5) yields the exact sequence

$$(3.7)\quad 0 \to K_0(C^*(\{1\})) \oplus \left(\bigoplus_{j=1}^{l} \widetilde{K}_0(C^*(M_j))\right) \xrightarrow{\mathrm{ind}_{\{1\}}^G \oplus \left(\bigoplus_{j=0}^{l} b_j\right)} K_0(C^*(G)) \xrightarrow{e_G} H_2(G\backslash \underline{E}G;\, \mathbb{Z}) \to 0.$$

Let $\mathrm{res}_G^{\mathbb{Z}^2}\colon K_0(C^*(G)) \to K_0(C^*(\mathbb{Z}^2))$ be the homomorphism induced by restriction to the subgroup of finite index. By the double coset formula of Lemma 3.2 and because each conjugate of each finite group M_j has trivial intersection with $\mathbb{Z}^2$ in

G, the composition

$$K_0(C^*(\{1\}))\oplus\left(\bigoplus_{j=0}^{l}\widetilde{K}_0(C^*(M_j))\right)\xrightarrow{\operatorname{ind}_{\{1\}}^G\oplus\left(\bigoplus_{j=0}^{l}b_j\right)}K_0(C^*(G))\xrightarrow{\operatorname{res}_G^{\mathbb{Z}^2}}K_0(C^*(\mathbb{Z}^2))$$

factors through the map $\operatorname{ind}_{\{1\}}^{\mathbb{Z}^2}\colon K_0(C^*(\{1\}))\to K_0(C^*(\mathbb{Z}^2))$. Combining this with the exact sequence (3.7) yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K_0(C^*(\{1\}))\oplus\left(\bigoplus_{j=0}^{l}\widetilde{K}_0(C^*(M_j))\right) \arrow[r, "\mathrm{id}"] \arrow[d, "\operatorname{ind}_{\{1\}}^G\oplus\left(\bigoplus_{j=0}^{l}b_j\right)"'] & K_0(C^*(\{1\}))\oplus\left(\bigoplus_{j=0}^{l}\widetilde{K}_0(C^*(M_j))\right) \arrow[d, "\operatorname{ind}_{\{1\}}^G\oplus\left(\bigoplus_{j=0}^{l}b_j\right)"] \\
K_0(C^*(G)) \arrow[r, "\mathrm{id}"] \arrow[d, "e_G"'] & K_0(C^*(G)) \arrow[d, "\mathrm{pr}\circ\operatorname{res}_G^{\mathbb{Z}^2}"] \\
H_2(G\backslash\underline{E}G;\,\mathbb{Z}) \arrow[r, "\omega"] \arrow[d] & \widetilde{K}_0(C^*(\mathbb{Z}^2)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

for an appropriate map ω. Lemma 4.8 below (based on Sam Walters' construction of Fourier modules [72, 75]; the proof does not depend on Theorem 3.5) provides explicit elements in $K_0(C^*(G))$ which are mapped to a generator in $\widetilde{K}_0(C^*(\mathbb{Z}^2))$ under $\mathrm{pr}\circ\operatorname{res}_G^{\mathbb{Z}^2}$. This implies that $\mathrm{pr}\circ\operatorname{res}_G^{\mathbb{Z}^2}$ is surjective. Hence ω is a surjection of infinite cyclic groups and therefore an isomorphism. Since the left column in the diagram above is exact, the same is true for the right column. □

We now give a second proof, not relying on Lemma 4.8. Parts of the argument are the same as in the first proof, and we refer freely to it.

Second proof of Theorem 3.5. Analogously to the definition of e_G in (3.6), we define a surjection

$$e_{\mathbb{Z}^2}\colon K_0(C^*(\mathbb{Z}^2))\to H_2(\mathbb{Z}^2\backslash E\mathbb{Z}^2)$$

for the subgroup $\mathbb{Z}^2\subseteq G$. Let $q\colon \mathbb{T}^2=\mathbb{Z}^2\backslash\underline{E}\mathbb{Z}^2\to F\backslash\mathbb{T}^2=G\backslash\underline{E}G$ be the projection. Let $i\colon\mathbb{Z}^2\to G$ be the inclusion. Then the following diagram commutes:

$$\begin{tikzcd}
K_0(C^*(\mathbb{Z}^2)) \arrow[r, "e_{\mathbb{Z}^2}"] \arrow[d, "K_0(i)"'] & H_2(\mathbb{T}^2;\mathbb{Z}) \arrow[d, "H_2(q)"] \\
K_0(C^*(G)) \arrow[r, "e_G"'] & H_2(F\backslash\mathbb{T}^2;\,\mathbb{Z}).
\end{tikzcd} \tag{3.8}$$

The F-action on $\mathbb{T}^2$ is orientation preserving and has at least one free orbit. The quotient $F\backslash\mathbb{T}^2$ is a two dimensional manifold as in the first proof, so $q\colon\mathbb{T}^2\to F\backslash\mathbb{T}^2$ is a map of closed oriented surfaces of degree $\mathrm{card}(F)$ and thus induces an injective map $H_2(q)$ of infinite cyclic groups whose image has index $\mathrm{card}(F)$.

An easy spectral sequence argument shows that the image of

$$\operatorname{ind}_{\{1\}}^{\mathbb{Z}^2}\colon K_0(C^*(\{1\}))\to K_0(C^*(\mathbb{Z}^2))$$

is the kernel of the surjection $e_{\mathbb{Z}^2}\colon K_0(C^*(\mathbb{Z}^2)) \to H_2(\mathbb{T}^2;\mathbb{Z})$. Hence we obtain from (3.7) a map $\mu\colon H_2(F\backslash\mathbb{T}^2;\,\mathbb{Z}) \to H_2(\mathbb{T}^2;\mathbb{Z})$ such that the following square commutes:

$$\begin{array}{ccc} K_0(C^*(G)) & \xrightarrow{e_G} & H_2(F\backslash\mathbb{T}^2;\,\mathbb{Z}) \\ {\scriptstyle \mathrm{res}_G^{\mathbb{Z}^2}}\downarrow & & \downarrow{\scriptstyle \mu} \\ K_0(C^*(\mathbb{Z}^2)) & \xrightarrow[e_{\mathbb{Z}^2}]{} & H_2(\mathbb{T}^2;\mathbb{Z}), \end{array} \tag{3.9}$$

and an isomorphism

$$\nu\colon H_2(\mathbb{T}^2;\mathbb{Z}) \xrightarrow{\cong} \widetilde{K}_0(C^*(\mathbb{Z}^2))$$

such that the composition $\nu \circ e_{\mathbb{Z}^2}$ agrees with the projection $\mathrm{pr}\colon K_0(C^*(\mathbb{Z}^2)) \to \widetilde{K}_0(C^*(\mathbb{Z}^2))$. Hence the following diagram commutes:

$$\begin{array}{ccc} K_0(C^*(G)) & \xrightarrow{e_G} & H_2(F\backslash\mathbb{T}^2;\,\mathbb{Z}) \\ {\scriptstyle \mathrm{res}_G^{\mathbb{Z}^2}}\downarrow & & \downarrow{\scriptstyle \nu\circ\mu} \\ K_0(C^*(\mathbb{Z}^2)) & \xrightarrow[\mathrm{pr}]{} & \widetilde{K}_0(C^*(\mathbb{Z}^2)). \end{array} \tag{3.10}$$

In view of the exact sequence (3.7) it remains to show that μ is bijective. Again by the double coset formula (Lemma 3.2), the composition

$$K_0(C^*(\mathbb{Z}^2)) \xrightarrow{K_0(i)} K_0(C^*(G)) \xrightarrow{\mathrm{res}_G^{\mathbb{Z}^2}} K_0(C^*(\mathbb{Z}^2))$$

is $\sum_{j=1}^{\mathrm{card}(F)} K_0(\sigma)^j$, where $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by multiplication by the generator of the finite cyclic group F. The map σ induces an orientation preserving automorphism of $\mathbb{T}^2$ and hence the identity $H_2(\mathbb{T}^2)$. Therefore the following diagram commutes:

$$\begin{array}{ccc} K_0(C^*(\mathbb{Z}^2)) & \xrightarrow{e_{\mathbb{Z}^2}} & H_2(\mathbb{T}^2;\mathbb{Z}) \\ {\scriptstyle \mathrm{res}_G^{\mathbb{Z}^2}\circ K_0(i)}\downarrow & & \downarrow{\scriptstyle \mathrm{card}(F)\cdot\mathrm{id}} \\ K_0(C^*(\mathbb{Z}^2)) & \xrightarrow[e_{\mathbb{Z}^2}]{} & H_2(\mathbb{T}^2;\mathbb{Z}). \end{array} \tag{3.11}$$

Fix a generator x of the infinite cyclic group $H_2(\mathbb{T}^2;\mathbb{Z})$. Choose $y \in K_0(C^*(\mathbb{Z}^2))$ with $e_{\mathbb{Z}^2}(y) = x$. Let $z \in K_0(C^*(G))$ be the image of y under the homomorphism $K_0(i)\colon K_0(C^*(\mathbb{Z}^2)) \to K_0(C^*(G))$. We conclude from (3.8) and the fact that $H_2(\mathrm{pr})$ is injective and has cokernel of order $\mathrm{card}(F)$ that $e_G(z)$ is $\mathrm{card}(F)\cdot a$ for a suitable generator a of the infinite cyclic group $H_2(F\backslash\mathbb{T}^2;\,\mathbb{Z})$. We get from (3.9) and (3.11) that

$$\mathrm{card}(F)\cdot\mu(a) = \mu\circ e_G(z) = \mathrm{card}(F)\cdot x.$$

Since a and x are generators of infinite cyclic groups, μ is bijective. (Notice that the map ω agrees with the map $\nu\circ\mu$ in the first proof.) This finishes the second proof of Theorem 3.5. □

It is clear how to use Theorem 3.5 to give an explicit basis for $K_0(\mathbb{Z}^2 \rtimes F)$, and we do this in all four cases. Note first that $K_0(C^*(\mathbb{Z}_k))$ is the free abelian group of rank k. If t is a generator for $\mathbb{Z}_k$, identified with its canonical image in $C^*(\mathbb{Z}_k)$, and $\zeta = e^{2\pi i/k}$, then an explicit basis is given by the classes of the projections

$$p_j = \frac{1}{k} \sum_{l=0}^{k-1} \left(\zeta^j t\right)^l$$

for $j = 0, \dots, k-1$. One easily checks that $\sum_{j=0}^{k-1} p_j = 1 \in C^*(\mathbb{Z}_k)$, and therefore we may replace any one of the above projections by [1] to obtain a new basis. In particular, it follows that $\widetilde{K}_0(C^*(\mathbb{Z}_k))$, as defined in Notation 3.1(3), is a free abelian group of rank $k-1$ with basis

$$\big([p_0],\, [p_1],\, \dots,\, [p_{k-2}]\big).$$

In the following examples we give a complete set of representatives of the conjugacy classes of maximal finite subgroups. The elementary calculation that this is indeed such a system is carried out for $F = \mathbb{Z}_4$ in Lemma 2.2 of [42]. The analogous calculations in the other cases are left to the reader.

Example 3.7. (a) The case $F = \mathbb{Z}_2$. The group $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ has the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_2 = \big\langle u, v, t \colon t^2 = 1,\ uv = vu,\ tut = u^{-1},\ tvt = v^{-1} \big\rangle.$$

The maximal finite subgroups are represented, up to conjugacy, by the four groups

$$M_0 = \langle t \rangle, \quad M_1 = \langle ut \rangle, \quad M_2 = \langle vt \rangle, \quad M_3 = \langle uvt \rangle,$$

which are all isomorphic to $\mathbb{Z}_2$. Hence it follows from Theorem 3.5 that, for any $S \in K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_2))$ which maps onto a generator of $\widetilde{K}_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}$ via $\mathrm{pr} \circ \mathrm{res}^{\mathbb{Z}^2}_{\mathbb{Z}^2 \rtimes \mathbb{Z}_2}$,

$$\big([1],\ \big[\tfrac{1}{2}(1+t)\big],\ \big[\tfrac{1}{2}(1+ut)\big],\ \big[\tfrac{1}{2}(1+vt)\big],\ \big[\tfrac{1}{2}(1+uvt)\big],\ S\big)$$

is a basis for $K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_2)) \cong \mathbb{Z}^6$.

(b) The case $F = \mathbb{Z}_3$. Here we have the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_3 = \big\langle u, v, t \colon t^3 = 1,\ uv = vu,\ tut^{-1} = u^{-1}v,\ tvt^{-1} = u^{-1} \big\rangle.$$

The conjugacy classes of the maximal finite subgroups are represented by

$$M_0 = \langle t \rangle, \quad M_1 = \langle ut \rangle, \quad M_2 = \langle u^2 t \rangle,$$

which are all isomorphic to $\mathbb{Z}_3$. Set $\zeta = \frac{1}{2}\big(-1 + i\sqrt{3}\big)$, and define

$$p_0 = \tfrac{1}{3}(1 + t + t^2), \quad q_0 = \tfrac{1}{3}\big(1 + ut + (ut)^2\big), \quad r_0 = \tfrac{1}{3}\big(1 + u^2 t + (u^2 t)^2\big),$$

$$p_1 = \tfrac{1}{3}\big(1 + \zeta t + (\zeta t)^2\big), \quad q_1 = \tfrac{1}{3}\big(1 + \zeta ut + (\zeta ut)^2\big), \quad r_1 = \tfrac{1}{3}\big(1 + \zeta u^2 t + (\zeta u^2 t)^2\big).$$

Then, if $S \in K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_3))$ maps to a generator of $\widetilde{K}_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}$, we obtain from Theorem 3.5 that

$$([1],\ [p_0],\ [p_1],\ [q_0],\ [q_1],\ [r_0],\ [r_1],\ S)$$

is a basis for $K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_3)) \cong \mathbb{Z}^8$.

(c) The case $F = \mathbb{Z}_4$. In this case we have the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_4 = \big\langle u, v, t \colon t^4 = 1,\ uv = vu,\ tut^{-1} = v,\ tvt^{-1} = u^{-1} \big\rangle.$$

The maximal finite subgroups are given up to conjugacy by

$$M_0 = \langle t \rangle, \quad M_1 = \langle ut \rangle, \quad M_2 = \langle ut^2 \rangle.$$

The groups M_0 and M_1 are isomorphic to $\mathbb{Z}_4$, and $M_2 \cong \mathbb{Z}_2$. Define

$$p_0 = \tfrac{1}{4}\big(1 + t + t^2 + t^3\big), \quad p_1 = \tfrac{1}{4}\big(1 + it - t^2 - it^3\big), \quad p_2 = \tfrac{1}{4}\big(1 - t + t^2 - t^3\big),$$

$$q_0 = \tfrac{1}{4}\big(1 + ut + (ut)^2 + (ut)^3\big), \quad q_1 = \tfrac{1}{4}\big(1 + iut - (ut)^2 - i(ut)^3\big),$$

$$q_2 = \tfrac{1}{4}\big(1 - ut + (ut)^2 - (ut)^3\big), \quad \text{and} \quad r = \tfrac{1}{2}(1 + ut^2).$$

Then, if $S \in K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_4))$ maps to a generator of $\widetilde{K}_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}$, we obtain from Theorem 3.5 the basis

$$\big([1],\ [p_0],\ [p_1],\ [p_2],\ [q_0],\ [q_1],\ [q_2],\ [r],\ S\big)$$

of $K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_4)) \cong \mathbb{Z}^9$.

(d) The case $F = \mathbb{Z}_6$. Here we have the presentation

$$\mathbb{Z}^2 \rtimes \mathbb{Z}_6 = \big\langle u, v, t \colon t^6 = 1,\ uv = vu,\ tut^{-1} = v,\ tvt^{-1} = u^{-1}v\big\rangle.$$

The maximal finite subgroups are given up to conjugacy by

$$M_0 = \langle t\rangle, \quad M_1 = \langle ut^2\rangle, \quad M_2 = \langle ut^3\rangle,$$

with $M_0 \cong \mathbb{Z}_6$, $M_1 \cong \mathbb{Z}_3$, and $M_2 \cong \mathbb{Z}_2$. Set $\zeta = \frac{1}{2}\big(1 + i\sqrt{3}\big)$, define

$$p_j = \tfrac{1}{6}\big(1 + (\zeta^j t) + (\zeta^j t)^2 + (\zeta^j t)^3 + (\zeta^j t)^4 + (\zeta^j t)^5\big)$$

for $j = 0, 1, 2, 3, 4$, and define

$$q_0 = \tfrac{1}{3}\big(1 + ut^2 + (ut^2)^2\big), \quad q_1 = \tfrac{1}{3}\big(1 + \zeta^2 ut^2 + (\zeta^2 ut^2)^2\big), \quad \text{and} \quad r = \tfrac{1}{2}(1 + ut^3).$$

Then, if $S \in K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_6))$ maps to a generator of $\widetilde{K}_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}$, we obtain from Theorem 3.5 that

$$\big([1],\ [p_0],\ [p_1],\ [p_2],\ [p_3],\ [p_4],\ [q_0],\ [q_1],\ [r],\ S\big)$$

is a basis of $K_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_6)) \cong \mathbb{Z}^{10}$.

Using the construction of the Fourier modules due to Sam Walters (see [72, 75]) we later explicitly construct candidates for S in all four cases considered above.

As a direct consequence of Example 3.7 and Corollary 2.2 we now get:

Corollary 3.8. *Let $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators as described above and let F act on the rotation algebra A_θ for $\theta \in [0,1]$ via the restriction of the action of $\mathrm{SL}_2(\mathbb{Z})$ as described in (0.1) and (0.2). Then $K_1(A_\theta \rtimes_\alpha F) = 0$ for all F and θ, and for all $\theta \in [0,1]$ we have*

$$K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}^6, \quad K_0(A_\theta \rtimes_\alpha \mathbb{Z}_3) \cong \mathbb{Z}^8,$$

$$K_0(A_\theta \rtimes_\alpha \mathbb{Z}_4) \cong \mathbb{Z}^9, \quad \textit{and} \quad K_0(A_\theta \rtimes_\alpha \mathbb{Z}_6) \cong \mathbb{Z}^{10}.$$

4. A basis for $K_0(A_\theta \rtimes_\alpha F)$

Let $\theta \in [0,1]$. In this section we present an explicit basis for $K_0(A_\theta \rtimes_\alpha F)$ for $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

Notation 4.1. To simplify notation we write from now on $e(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$.

Recall from Corollary 2.2 that $A_\theta \rtimes_\alpha F \cong C^*(\mathbb{Z}^2 \rtimes F, \widetilde{\omega}_\theta)$.

Lemma 4.2. *Let $\theta \in [0,1]$, and let F be one of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. Then $A_\theta \rtimes_\alpha F$ is the universal C^*-algebra generated by three unitaries*

$$u_\theta, \quad v_\theta, \quad t_\theta,$$

subject to the relations $v_\theta u_\theta = e(\theta) u_\theta v_\theta$ and

$$\begin{array}{llll}
\text{for } F = \mathbb{Z}_2: & t_\theta^2 = t_\theta, & t_\theta u_\theta t_\theta^{-1} = u_\theta^{-1}, & t_\theta v_\theta t_\theta^{-1} = v_\theta^{-1} \\
\text{for } F = \mathbb{Z}_3: & t_\theta^3 = t_\theta, & t_\theta u_\theta t_\theta^{-1} = e\big(-\tfrac{1}{2}\theta\big) u_\theta^{-1} v_\theta, & t_\theta v_\theta t_\theta^{-1} = u_\theta^{-1} \\
\text{for } F = \mathbb{Z}_4: & t_\theta^4 = t_\theta, & t_\theta u_\theta t_\theta^{-1} = v_\theta, & t_\theta v_\theta t_\theta^{-1} = u_\theta^{-1} \\
\text{for } F = \mathbb{Z}_6: & t_\theta^6 = t_\theta, & t_\theta u_\theta t_\theta^{-1} = v_\theta, & t_\theta v_\theta t_\theta^{-1} = e\big(-\tfrac{1}{2}\theta\big) u_\theta^{-1} v_\theta.
\end{array}$$

The C^*-algebra generated by u_θ and v_θ is A_θ. Moreover, the homomorphism given on the generators by

$$u_0 \mapsto u, \quad v_0 \mapsto v \quad \text{and} \quad t_0 \mapsto t$$

is an isomorphism $A_0 \rtimes_\alpha F \to C^*(\mathbb{Z}^2 \rtimes F)$, and the homomorphism given on the generators by

$$u_1 \mapsto -u, \quad v_1 \mapsto -v \quad \text{and} \quad t_1 \mapsto t$$

is an isomorphism $A_1 \rtimes_\alpha F \to C^*(\mathbb{Z}^2 \rtimes F)$.

Proof. The first part follows from the description of the action of $\mathrm{SL}_2(\mathbb{Z})$ on A_θ as given in Corollary 2.2 and the realization of the groups $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \subseteq \mathrm{SL}_2(\mathbb{Z})$ with generators as in (0.2). The last part is trivial for $\theta = 0$, and follows from $e\big(-\tfrac{1}{2}\theta\big) = -1$ for $\theta = 1$. $\square$

Although the cocycle $\widetilde{\omega}_1$ is in the class of the trivial cocycle, it is not trivial itself, so it has to be taken into account in the computations below!

We now fix some $a \in (0,1]$. Consider the 2-cocycle

$$\widetilde{\Omega}_a \colon (\mathbb{Z}^2 \rtimes F) \times (\mathbb{Z}^2 \rtimes F) \to C\big([a,1], \mathbb{T}\big)$$

given by $\widetilde{\Omega}_a(\,\cdot\,,\cdot, \theta) = \widetilde{\omega}_\theta(\,\cdot\,,\,\cdot\,)$ for $\theta \in [a,1]$. It is clear that it restricts at θ to the cocycle ω_θ on $\mathbb{Z}^2$. For convenience, we write $C^*\big(\mathbb{Z}^2 \rtimes F, \widetilde{\Omega}_a\big)$ for the twisted crossed product $C\big([a,1]\big) \rtimes_{\widetilde{\Omega}_a} (\mathbb{Z}^2 \rtimes F)$ and $C^*(\mathbb{Z}^2, \Omega_a)$ for the twisted crossed product $C\big([a,1]\big) \rtimes_{\Omega_a} \mathbb{Z}^2$. As in Remark 2.3, we may write

$$C^*\big(\mathbb{Z}^2 \rtimes F, \widetilde{\Omega}_a\big) \cong C^*(\mathbb{Z}^2, \Omega_a) \rtimes F \tag{4.1}$$

where the action of F on $C^*(\mathbb{Z}^2, \Omega_a)$ is given fiberwise by the actions on the fibers $A_\theta = C^*(\mathbb{Z}^2, \omega_\theta)$ for $\theta \in [a,1]$.

We want to construct suitable elements of $K_0\big(C^*\big(\mathbb{Z}^2 \rtimes F, \widetilde{\Omega}_a\big)\big)$ which, with respect to the identification $C^*(\mathbb{Z}^2 \rtimes F, \widetilde{\omega}_1) \cong C^*(\mathbb{Z}^2 \rtimes F)$ of Lemma 4.2, project to classes in the fiber at 1 as given in Example 3.7.

Notation 4.3. Let $a \in (0,1]$. Since $G = \mathbb{Z}^2 \rtimes F$ is discrete, there is a canonical map from $\mathbb{Z}^2 \rtimes F$ into the group of unitaries of $C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$ which sends a group element to the corresponding Dirac function. Let $U_a, V_a, T_a \in C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$ be the images of the generators $u, v, t \in \mathbb{Z}^2 \rtimes F$ under this map. We use the same notation for these unitaries regarded as elements of suitable dense subalgebras, and, for U_a and V_a, with $C^*\big(\mathbb{Z}^2, \Omega_a\big)$ in place of $C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$.

These unitaries clearly project to the generators $u_\theta, v_\theta, t_\theta \in C^*(\mathbb{Z}^2 \rtimes F,\, \widetilde{\omega}_\theta)$ for $\theta \in [a,1]$. Moreover, every function $\varphi \in C\big([a,1]\big)$ can be considered canonically as a central element of $C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$ by identifying φ with $\delta_1 \otimes \varphi \in C_c\big(\mathbb{Z}^2 \rtimes F,\, C([a,1])\big) \subseteq C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$ (where 1 denotes the identity of $\mathbb{Z}^2 \rtimes F$). In the lemma below, we identify the generators u, v, t of $C^*(\mathbb{Z}^2 \rtimes F)$ with the generators $-u_1$, $-v_1$, t_1 of $C^*(\mathbb{Z}^2 \rtimes F,\, \widetilde{\omega}_1)$ as in the last part of Lemma 4.2.

Lemma 4.4. *Recalling Notation 4.1, define functions $\varphi_1, \varphi_2, \varphi_3 \in C\big([a,1]\big)$ by*

$$\varphi_1(\theta) = -e\big(\tfrac{1}{2}\theta\big), \quad \varphi_2(\theta) = e\big(\tfrac{1}{6}(2+\theta)\big), \quad \text{and} \quad \varphi_3(\theta) = ie\big(\tfrac{1}{4}\theta\big).$$

Then:

(1) *If $F = \mathbb{Z}_2$, the elements T_a, $-U_aT_a$, $-V_aT_a$, and $\varphi_1 U_a V_a T_a$ are unitaries of order two which project onto the elements t, ut, vt, and uvt in the fiber at* 1.
(2) *If $F = \mathbb{Z}_3$, the elements T_a, $\varphi_2 U_a T_a$, and $U_a^2 T_a$ are unitaries of order three which project onto the elements t, ut, and u^2t in the fiber at* 1.
(3) *If $F = \mathbb{Z}_4$, the elements T_a, and $\varphi_3 U_a T_a$ are unitaries of order four and $-U_a T_a^2$ is a unitary of order two which project onto the elements t, ut, and ut^2 in the fiber at* 1.
(4) *If $F = \mathbb{Z}_6$, then T_a is a unitary of order six, $\varphi_2 U_a T_a^2$ is a unitary of order three, and $-U_a T_a^3$ is a unitary of order two, and they project onto the elements t, ut^2, and ut^3 in the fiber at* 1.

Proof. The proof is a straight-forward (but tedious) computation using the relations for the unitaries $u_\theta, v_\theta, t_\theta$ in each fiber $C^*(\mathbb{Z}^2 \rtimes F,\, \widetilde{\omega}_\theta)$. We leave the details for the reader. □

As a consequence of Lemma 4.4, we can now use exactly the same formulas as for the explicit projections in $C^*(\mathbb{Z}^2 \rtimes F)$ presented in Example 3.7, replacing the combinations of u, v, t in the example by appropriate combinations of U_a, V_a, T_a, to obtain projections in $C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$ which project onto the ones in $C^*(\mathbb{Z}^2 \rtimes F) \cong C^*(\mathbb{Z}^2 \rtimes F,\, \widetilde{\omega}_1)$ via evaluation at 1. Therefore, all we have to do is to find a class $S_a \in K_0\big(C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)\big)$ which projects onto a class $S \in K_0(C^*(\mathbb{Z}^2 \rtimes F))$ via evaluation at 1 with the properties as in Example 3.7. Before we construct this element, it is convenient to recall some basic properties about equivariant K-theory of C*-algebras with respect to finite group actions.

Proposition 4.5. *Suppose that F is a finite group and $\alpha\colon F \to \mathrm{Aut}(A)$ is an action of F on the C*-algebra A. Suppose further that $\mathcal{E}$ is a finitely generated projective (right) A-module together with a right action $W\colon F \to \mathrm{Aut}(\mathcal{E})$, written $(\xi, g) \mapsto \xi W_g$, such that $(\xi W_g)a = (\xi \alpha_g(a))W_g$ for all $\xi \in \mathcal{E}$, $a \in A$, and $g \in F$. Then $\mathcal{E}$ becomes a finitely generated projective $A \rtimes_\alpha F$-module with action defined* *by*

$$\xi \cdot \left(\sum_{g \in F} a_g \delta_g \right) = \sum_{g \in F} (\xi a_g) W_g.$$

Moreover, the restriction to A of this module structure is just the original A-module $\mathcal{E}$, with the action of F forgotten.

Proof. This is the definition of the map in the Green-Julg theorem (Theorem 11.7.1 of [5] or Theorem 2.6.1 of [56]). The last statement is obvious. □

We will use Proposition 4.5 and the isomorphism (4.1) to construct finitely generated projective $C^*\big(\mathbb{Z}^2 \rtimes F,\, \widetilde{\Omega}_a\big)$-modules which represent a suitable class S_a. To do this, we exploit below some basic constructions and ideas of Alain Connes and Marc Rieffel in [16, 63] and Sam Walters in [72, 75]. Define a new cocycle $\Omega_{1/a} \in Z^2\big(\mathbb{Z}^2,\, C\big([a,1],\, \mathbb{T}\big)\big)$ by setting $\Omega_{1/a}(\,\cdot\,,\,\cdot\,)(\theta) = \omega_{1/\theta}$ for all $\theta \in [a,1]$. Set $A = C^*(\mathbb{Z}^2, \Omega_a)$ and $B = C^*\big(\mathbb{Z}^2,\, \Omega_{1/a}\big)$. Then the fiber B_θ of B at $\theta \in [a,1]$ is the rotation algebra $A_{1/\theta} = C^*(\mathbb{Z}^2,\, \omega_{1/\theta})$. Consider the space $\mathcal{S}[a,1]$ consisting of all complex functions on $\mathbb{R} \times [a,1]$ which are smooth and rapidly decreasing in the first variable and continuous in the second variable in each derivative of the first variable. Denote by $\mathcal{S}(\mathbb{Z}^2, \Omega_a)$ the set of rapidly decreasing $C\big([a,1]\big)$-valued functions on $\mathbb{Z}^2$, viewed as a (dense) subalgebra of $C^*(\mathbb{Z}^2, \Omega_a)$, and denote by $\mathcal{S}(\mathbb{Z}^2, \Omega_{1/a})$ the same set, viewed as a (dense) subalgebra of $C^*(\mathbb{Z}^2,\, \Omega_{1/a})$. With U_a and V_a as in Notation 4.3, we may represent each element $\varphi \in \mathcal{S}(\mathbb{Z}^2, \Omega_a)$ as an infinite sum

$$\varphi = \sum_{n,m \in \mathbb{Z}} \beta_{n,m}\, U_a^n V_a^m \tag{4.2}$$

with $\beta_{n,m} \in C\big([a,1]\big) \subseteq Z(C^*(\mathbb{Z}^2, \Omega_a))$ and $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) \to \|\beta_{n,m}\|_\infty$ rapidly decreasing on $\mathbb{Z}^2$. Similarly, we may write each element $\psi \in \mathcal{S}(\mathbb{Z}^2, \Omega_{1/a})$ as a sum

$$\psi = \sum_{n,m \in \mathbb{Z}} \alpha_{n,m}\, U_{1/a}^n V_{1/a}^m \tag{4.3}$$

with $\alpha_{n,m} \in C\big([a,1]\big)$. We define a right action of $\mathcal{S}(\mathbb{Z}^2, \Omega_a)$ on $\mathcal{S}[a,1]$ via the actions of U_a, V_a, and $f \in C\big([a,1]\big)$ given by

$$(\xi \cdot U_a)(s,\theta) = \xi(s+\theta,\, \theta), \quad (\xi \cdot V_a)(s,\theta) = e(s)\xi(s,\theta), \quad \text{and} \quad (\xi \cdot f)(s,\theta) = f(\theta)\xi(s,\theta). \tag{4.4}$$

Moreover, we define an $\mathcal{S}(\mathbb{Z}^2, \Omega_a)$-valued inner product on $\mathcal{S}[a,1]$ as follows. For $n, m \in \mathbb{Z}$, set

$$\langle \xi, \eta \rangle_{n,m}^a(\theta) = \theta \int_{-\infty}^{\infty} \overline{\xi(x + n\theta,\, \theta)}\, \eta(x,\theta) e(-mx)\, dx. \tag{4.5}$$

Then set

$$\langle \xi, \eta \rangle_{\mathcal{S}(\mathbb{Z}^2, \Omega_a)} = \sum_{n,m \in \mathbb{Z}} \langle \xi, \eta \rangle_{n,m}^a U_a^n V_a^m. \tag{4.6}$$

There is also a left action of $\mathcal{S}(\mathbb{Z}^2, \Omega_{1/a})$ on $\mathcal{S}[a,1]$ given by

$$U_{1/a} \cdot \xi(s,\theta) = \xi(s+1,\, \theta), \quad V_{1/a} \cdot \xi(s,\theta) = e(-s/\theta)\xi(s,\theta), \quad \alpha \cdot \xi(s,\theta) = \alpha(\theta)\xi(s,\theta), \tag{4.7}$$

for $\alpha \in C\big([a,1]\big)$, and a left inner product

$$\mathcal{S}(\mathbb{Z}^2,\,\Omega_{1/a})\langle \xi, \eta \rangle = \sum_{n,m\in\mathbb{Z}} \langle \xi, \eta \rangle_{n,m}^{1/a} U_{1/a}^n V_{1/a}^m \tag{4.8}$$

with

$$\langle \xi, \eta \rangle_{n,m}^{1/a}(\theta) = \int_{-\infty}^{\infty} \xi(x-n,\,\theta)\overline{\eta(x,\theta)}e(mx/\theta)\,dx.$$

Evaluated at each fixed $\theta \in [a,1]$, these actions and inner products determine $B_\theta - A_\theta$ imprimitivity bimodules $\mathcal{E}_\theta = \overline{\mathcal{S}(\mathbb{R})}$ as in [63] (Theorem 1.1 and the proof of Lemma 1.5). The formulas given here are from the beginning of Section 3 of [72]. (The paper [63] uses a dense subalgebra of A_θ consisting of functions on $\mathbb{Z} \times \mathbb{T}$.) It follows then easily from the fact that all operations are $C\big([a,1]\big)$-linear that $\mathcal{S}[a,1]$ completes to give a $B-A$ imprimitivity bimodule $\mathcal{E}$. Since both B and A are unital, it follows from the arguments given after Notation 1.3 of [63] that $\mathcal{E}$ is a finitely generated projective A-module with respect to the given action of A on $\mathcal{E}$. In order to construct a module over $C^*\big(\mathbb{Z}^2 \rtimes F,\,\widetilde{\Omega}_a\big) = A \rtimes F$, it thus suffices to construct a suitable action of F on $\mathcal{E}$. The operator W_t in the following proposition is really the action of the unitary T_a on the module.

Proposition 4.6. *Let F be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with fixed generator t. We define an action of F on the dense subspace $\mathcal{S}[a,1]$ (described above) of the $B-A$ imprimitivity bimodule $\mathcal{E}$ via the action of the generator t given by:*

$(\xi W_t)(s,\theta) = \xi(-s,\,\theta)$ *(for $F = \mathbb{Z}_2$)*

$(\xi W_t)(s,\theta) = e^{-\pi i/12}\theta^{-1/2}e(s^2/(2\theta))\int_{-\infty}^{\infty}\xi(x,\theta)e(sx/\theta)\,dx$ *(for $F = \mathbb{Z}_3$)*

$(\xi W_t)(s,\theta) = \theta^{-1/2}\int_{-\infty}^{\infty}\xi(x,\theta)e(sx/\theta)\,dx$ *(for $F = \mathbb{Z}_4$)*

$(\xi W_t)(s,\theta) = e^{\pi i/12}\theta^{-1/2}\int_{-\infty}^{\infty}\xi(x,\theta)e\big((2sx-x^2)/(2\theta)\big)\,dx$ *(for $F = \mathbb{Z}_6$).*

Then these actions extend to actions on $\mathcal{E}$ such that $\mathcal{E}$ becomes an F-equivariant finitely generated projective A-module as in Proposition 4.5.

Proof. The operator which appears in the case $F = \mathbb{Z}_6$ is, in the fiber over $\theta \in [a,1]$, the hexic transform of Walters [75] with respect to the parameter $\mu = \frac{1}{2\theta}$. It follows from Theorem 1 of [75] that it has period six and that its square is the operator which appears in the case $F = \mathbb{Z}_3$, which therefore has period three. The operator which appears in case $F = \mathbb{Z}_4$ is, in the fiber over θ, the one introduced by Walters in Section 3 of [72], where it is shown that it has period four with square equal to the operator which appears in case $F = \mathbb{Z}_2$. We have to show that, in all four cases, there exists a unique extension of W_t to $\mathcal{E}$. This will be clear as soon as we verify the equation

$$\langle \xi W_t,\, \eta W_t \rangle_A = \alpha_{t^{-1}}(\langle \xi, \eta \rangle_A)$$

for all $\xi, \eta \in \mathcal{S}[a,1]$ and in all cases $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. The cases $F = \mathbb{Z}_2$ and $F = \mathbb{Z}_3$ will follow directly from the cases $F = \mathbb{Z}_4$ and $F = \mathbb{Z}_6$. As a sample, we do the details for the case $F = \mathbb{Z}_6$ here and leave the computation for the case $F = \mathbb{Z}_4$ to the reader. (Note that these relations in the case $F = \mathbb{Z}_4$ have been used implicitly in Section 3 of [72].)

Replacing ξ by ξW_t^{-1} and applying α_t on both sides, we reduce to checking the equation

$$\langle \xi W_t^{-1},\, \eta \rangle_A = \alpha_t(\langle \xi,\, \eta W_t \rangle_A). \tag{4.9}$$

The formula for the transformation $\xi \mapsto \xi W_t^{-1}$ is given in [75] (see the remark immediately after Theorem 1 of [75], and continue to take $\mu = \frac{1}{2\theta}$), and is

$$(\xi W_t^{-1})(s,\theta) = \frac{e^{-\pi i/12}}{\sqrt{\theta}} e\big(\tfrac{1}{2}\theta^{-1}s^2\big) \int_{-\infty}^{\infty} \xi(x,\theta) e(-sx/\theta)\, dx. \tag{4.10}$$

Let $\varphi \in \mathcal{S}(\mathbb{Z}^2, \Omega_a)$ be given $\varphi = \sum_{n,m\in\mathbb{Z}} \beta_{n,m} U_a^n V_a^m$ as in (4.2). We use the definition of the action α_t (for example, see the commutation relations in Lemma 4.2 for the case $F = \mathbb{Z}_6$) and, at the third step, the commutation relation $v_\theta u_\theta^{-1} = e(-\theta) u_\theta^{-1} v_\theta$, to get, for $\theta \in [a,1]$,

$$\begin{aligned}
\big(\alpha_t(\varphi)\big)(\theta) &= \sum_{n,m\in\mathbb{Z}} \beta_{n,m}(\theta) \alpha_t^\theta(u_\theta)^n \alpha_t^\theta(v_\theta)^m \\
&= \sum_{n,m} \beta_{n,m}(\theta) e\big(-\tfrac{1}{2}m\theta\big) v_\theta^n \big(u_\theta^{-1}v_\theta\big)^m \\
&= \sum_{n,m} \beta_{n,m}(\theta) e\big(-\tfrac{1}{2}\theta(m^2+2nm)\big) u_\theta^{-m} v_\theta^{n+m} \\
&= \sum_{k,l} \beta_{l+k,\,-k}(\theta) e\big(\tfrac{1}{2}\theta(2k(l+k)-k^2)\big) u_\theta^k v_\theta^l.
\end{aligned}$$

Using this formula together with (4.6), we see that we have to check the identity

$$\langle \xi W_t^{-1}, \eta\rangle_{k,l}^a(\theta) = \langle \xi, \eta W_t\rangle_{k+l,\,-k}^a(\theta) e\big(\tfrac{1}{2}\theta[2k(k+l)-k^2]\big) \tag{4.11}$$

for all $k,l \in \mathbb{Z}$. Combining (4.5) with (4.10) gives for the left hand side:

(4.12)

$$\begin{aligned}
&\langle \xi W_t^{-1}, \eta\rangle_{k,l}^a(\theta) \\
&\quad= e^{\pi i/12}\sqrt{\theta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\xi(r,\theta)}\eta(x,\theta) e\big(\tfrac{1}{2}\theta^{-1}\big[2(x+k\theta)r-(x+k\theta)^2-2\theta lx\big]\big)\, dr\, dx.
\end{aligned}$$

For the right hand side of (4.11), first define

$$\begin{aligned}
&\psi(r,x,\theta,k,l) \\
&\quad= e\left(\frac{1}{2\theta}\big[2\big(r-(l+k)\theta\big)x - x^2 + 2\theta k\big(r-(l+k)\theta\big) + \theta^2\big(2k(k+l)-k^2\big)\big]\right).
\end{aligned}$$

Calculate, using at the first step (4.5) and the formula for W_t as given in the proposition, and using at the second step the change of variables r to $r-(k+l)\theta$, to get:

$$\begin{aligned}
&\langle \xi, \eta W_t\rangle_{k+l,\,-k}^a(\theta) e\big(\tfrac{1}{2}\theta[2k(k+l)-k^2]\big) \\
&\qquad= e^{\pi i/12}\sqrt{\theta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\xi(r+(k+l)\theta,\,\theta)}\eta(x,\theta) \\
&\qquad\qquad\qquad e\big(\tfrac{1}{2}\theta^{-1}(2rx-x^2+2\theta kr)\big) e\big(\tfrac{1}{2}\theta[2k(l+k)-k^2]\big)\, dr\, dx \\
&\qquad= e^{\pi i/12}\sqrt{\theta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\xi(r,\theta)}\eta(x,\theta)\psi(r,x,\theta,k,l)\, dr\, dx.
\end{aligned}$$

So the result follows from the identity

$$\begin{aligned}
&2(x+k\theta)r-(x+k\theta)^2-2\theta lx \\
&\qquad= 2\big(r-(l+k)\theta\big)x - x^2 + 2\theta k\big(r-(l+k)\theta\big) + \theta^2\big(2k(k+l)-k^2\big).
\end{aligned}$$

We still have to check the equation

$$(\xi W_t)\varphi = (\xi \alpha_t(\varphi))W_t$$

for all $\xi \in \mathcal{S}[a,1]$ and $\varphi \in \mathcal{S}(\mathbb{Z}^2, \Omega_a)$, which will then imply that $\mathcal{E}$ is F-equivariant in the sense of Proposition 4.5. Of course it is enough to check this identity on the generating set $C\big([a,1]\big) \cup \{U_a, V_a\}$. Since F acts trivially on $C\big([a,1]\big)$, this amounts to verifying the identities

$$(\xi W_t)U_a = (\xi \alpha_t(U_a))W_t, \quad (\xi W_t)V_a = (\xi \alpha_t(V_a))W_t, \quad \text{and} \quad (\xi W_t)f = (\xi f)W_t,$$

for all $f \in C\big([a,1]\big)$. The last equation is easy, so we only have to consider the first two equations. For $F = \mathbb{Z}_4$, these equations were used (but not explicitly checked) in Section 3 of [72]. The case $F = \mathbb{Z}_2$ follows directly from the case $F = \mathbb{Z}_4$, since the operator in the quadratic case is the square of the operator in the quartic case. We do the computations for the case $F = \mathbb{Z}_6$ below. The case $F = \mathbb{Z}_3$ will then follow from this in the same way as the case $\mathbb{Z}_2$ follows from the case $\mathbb{Z}_4$.

The computations in the case that t is the generator of $\mathbb{Z}_6$ are basically the same as the ones given by Walters in Section 3 of [75]. By analogy with Lemma 4.2, and with $g \in C\big([a,1]\big)$ defined by $g(\theta) = e(-\theta/2)$, we have

$$\alpha_t(U_a) = V_a \quad \text{and} \quad \alpha_t(V_a) = g U_a^{-1} V_a.$$

Using this, we have to check the identities

$$(\xi W_t)U_a = (\xi V_a)W_t \quad \text{and} \quad (\xi W_t)V_a = (\xi g U_a^{-1} V_a)W_t.$$

The first of these identities is very easy, and we omit the proof. For the second identity, we compute

(4.13)
$$\big((\xi W_t)V_a\big)(s,\theta) = e(s)(\xi W_t)(s,\theta) = e(s)\frac{e^{\pi i/12}}{\sqrt{\theta}} \int_{-\infty}^{\infty} \xi(x,\theta) e\big(\tfrac{1}{2}\theta^{-1}(2sx - x^2)\big)\, dx.$$

On the other side, changing the variable x to $x+\theta$ at the last step, we have

$$\begin{aligned}
\big((\xi g U_a^{-1} V_a)W_t\big)(s,\theta) &= e\big(-\tfrac{1}{2}\theta\big)\frac{e^{\pi i/12}}{\sqrt{\theta}} \int_{-\infty}^{\infty} \big(\xi U_a^{-1} V_a\big)(x,\theta) e\big(\tfrac{1}{2}\theta^{-1}(2sx - x^2)\big)\, dx \\
&= e\big(-\tfrac{1}{2}\theta\big)\frac{e^{\pi i/12}}{\sqrt{\theta}} \int_{-\infty}^{\infty} e(x)\xi(x-\theta,\, \theta) e\big(\tfrac{1}{2}\theta^{-1}(2sx - x^2)\big)\, dx \\
&= e\big(-\tfrac{1}{2}\theta\big)\frac{e^{\pi i/12}}{\sqrt{\theta}} \int_{-\infty}^{\infty} e(x+\theta)\xi(x,\theta) e\big(\tfrac{1}{2}\theta^{-1}[2s(x+\theta) - (x+\theta)^2]\big)\, dx.
\end{aligned}$$

The last expression is equal to $\big((\xi W_t)V_a\big)(s,\theta)$ by (4.13) and the easy to verify identity

$$e\big(-\tfrac{1}{2}\theta\big)e(x+\theta)e\big(\tfrac{1}{2}\theta^{-1}[2s(x+\theta) - (x+\theta)^2]\big) = e(s)e\big(\tfrac{1}{2}\theta^{-1}(2sx - x^2)\big),$$

which holds for all s, x and θ. □

Notation 4.7. Let F be one of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, and let $\mathcal{E}$ be the F equivariant A-module of Proposition 4.6. We denote by $\mathcal{E}^F$ the finitely generated $A \rtimes F$ module constructed from $\mathcal{E}$ by the procedure described in Proposition 4.5. We write $\mathcal{E}_\theta^F$ and $\mathcal{E}_\theta$ for the evaluations of $\mathcal{E}^F$ and $\mathcal{E}$ at $\theta \in [a,1]$.

Thus, with ev_θ denoting evaluation at θ (on both $A = C^*(\mathbb{Z}^2, \Omega_a)$ and $A \rtimes F$), in K-theory we have

$$(\mathrm{ev}_\theta)_*([\mathcal{E}]) = [\mathcal{E}_\theta] \in K_0(A_\theta) \quad \text{and} \quad (\mathrm{ev}_\theta)_*\big([\mathcal{E}^F]\big) = \big[\mathcal{E}_\theta^F\big] \in K_0(A_\theta \rtimes_\alpha F).$$

We now look at the special value $\theta = 1$, for which we have $A_1 \rtimes_\alpha F \cong C^*(\mathbb{Z}^2 \rtimes F)$.

Lemma 4.8. *Let F be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and let $[\mathcal{E}_1^F] \in K_0(C^*(\mathbb{Z}^2 \rtimes F))$ be as in Notation 4.7 for $\theta = 1$. Then $\big[\mathcal{E}_1^F\big]$ is mapped onto a generator of the group $\widetilde{K}_0(C^*(\mathbb{Z}^2)) \cong \mathbb{Z}$ under the projection*

$$\mathrm{pr} \circ \mathrm{res}_{\mathbb{Z}^2 \rtimes F}^{\mathbb{Z}^2} \colon K_0(C^*(\mathbb{Z}^2 \rtimes F)) \to \widetilde{K}_0(C^*(\mathbb{Z}^2))$$

which appears in the sequence of Theorem 3.5.

Proof. Since $\mathrm{res}_{\mathbb{Z}^2 \rtimes F}^{\mathbb{Z}^2}([\mathcal{E}_1^F]) = [\mathcal{E}_1]$, it suffices to show that $([1], [\mathcal{E}_1])$ is a basis for $K_0(C^*(\mathbb{Z}^2))$. By Remark 2.3 (applied to the trivial group $H = \{1\}$) it suffices to check that there exists one $\theta \in [a, 1]$ such that $([1], [\mathcal{E}_\theta])$ is a basis for $K_0(A_\theta)$. Choose $\theta \in [a, 1] \smallsetminus \mathbb{Q}$ and let $\tau \colon A_\theta \to \mathbb{R}$ be the canonical tracial state on A_θ. In Theorem 1.4 of [63] it is shown that $\tau_*([\mathcal{E}_\theta]) = \theta$. It follows from the Appendix in [55] that τ_* is an isomorphism from $K_0(A_\theta)$ to $\mathbb{Z} + \theta\mathbb{Z}$, so the result follows. $\square$

We are now ready to give an explicit basis for $K_0(A_\theta \rtimes F)$ for all $\theta \in (0, 1]$ and for $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

Theorem 4.9. *Let $\theta \in (0, 1]$. Let u_θ, v_θ, and t_θ be the generators of $A_\theta \rtimes_\alpha F = C^*(\mathbb{Z}^2 \rtimes F, \widetilde{\omega}_\theta)$ of Lemma 4.2. Choose a with $0 < a \leq \theta$, and let $\big[\mathcal{E}_\theta^F\big] \in K_0(A_\theta \rtimes_\alpha F)$ be as in Notation 4.7. For $n = 2, 3, 4, 6$ let $\tau_{n,\theta}$ denote the canonical tracial state on $A_\theta \rtimes_\alpha \mathbb{Z}_n$. Then:*

(1) *If $F = \mathbb{Z}_2$, define*

$$p^\theta = \tfrac{1}{2}(1 + t_\theta), \qquad r^\theta = \tfrac{1}{2}\big(1 - e\big(\tfrac{1}{2}\theta\big) u_\theta v_\theta t_\theta\big),$$

$$q_0^\theta = \tfrac{1}{2}(1 - u_\theta t_\theta), \qquad \text{and} \qquad q_1^\theta = \tfrac{1}{2}(1 - v_\theta t_\theta).$$

Then the classes

$$[1], \ \big[p^\theta\big], \ \big[q_0^\theta\big], \ \big[q_1^\theta\big], \ \big[r^\theta\big], \ \big[\mathcal{E}_\theta^{\mathbb{Z}_2}\big]$$

form a basis for $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}^6$. Moreover, $(\tau_{2,\theta})_*$ takes the following values (in this order) on these classes:

$$1, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \ \frac{\theta}{2}.$$

(2) *If $F = \mathbb{Z}_3$, set $\zeta = \frac{1}{2}\big(-1 + i\sqrt{3}\big)$, and define*

$$p_0^\theta = \tfrac{1}{3}\big(1 + t_\theta + t_\theta^2\big), \qquad p_1^\theta = \tfrac{1}{3}\big(1 + \zeta t_\theta + (\zeta t_\theta)^2\big),$$

$$q_0^\theta = \tfrac{1}{3}\big(1 + e\big(\tfrac{1}{6}[2+\theta]\big) u_\theta t_\theta + e\big(\tfrac{1}{3}[2+\theta]\big)(u_\theta t_\theta)^2\big),$$

$$q_1^\theta = \tfrac{1}{3}\big(1 + e\big(\tfrac{1}{6}[2+\theta]\big) \zeta u_\theta t_\theta + e\big(\tfrac{1}{3}[2+\theta]\big)(\zeta u_\theta t_\theta)^2\big),$$

$$r_0^\theta = \tfrac{1}{3}\big(1 + u_\theta^2 t_\theta + (u_\theta^2 t_\theta)^2\big), \qquad \text{and} \qquad r_1^\theta = \tfrac{1}{3}\big(1 + \zeta u_\theta^2 t_\theta + (\zeta u_\theta^2 t_\theta)^2\big).$$

Then the classes

$$[1], \ \big[p_0^\theta\big], \ \big[p_1^\theta\big], \ \big[q_0^\theta\big], \ \big[q_1^\theta\big], \ \big[r_0^\theta\big], \ \big[r_1^\theta\big], \ \big[\mathcal{E}_\theta^{\mathbb{Z}_3}\big]$$

form a basis for $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_3) \cong \mathbb{Z}^8$. *Moreover,* $(\tau_{3,\theta})_*$ *takes the following values on these classes:*

$$1,\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{\theta}{3}.$$

(3) *If* $F = \mathbb{Z}^4$, *define*

$$p_0^\theta = \tfrac{1}{4}\big(1 + t_\theta + t_\theta^2 + t_\theta^3\big), \qquad p_1^\theta = \tfrac{1}{4}\big(1 + it_\theta - t_\theta^2 - it_\theta^3\big),$$

$$p_2^\theta = \tfrac{1}{4}\big(1 - t_\theta + t_\theta^2 - t_\theta^3\big), \qquad r^\theta = \tfrac{1}{2}\big(1 - u_\theta t_\theta^2\big),$$

$$q_0^\theta = \tfrac{1}{4}\big(1 + ie\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta - \big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta\big]^2 - i\big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta\big]^3\big),$$

$$q_1^\theta = \tfrac{1}{4}\big(1 - e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta + \big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta\big]^2 - \big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta\big]^3\big),$$

and

$$q_2^\theta = \tfrac{1}{4}(1 - ie\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta - \big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta)^2 + i\big[e\big(\tfrac{1}{4}\theta\big)u_\theta t_\theta\big]^3\big).$$

Then the classes

$$[1],\ \big[p_0^\theta\big],\ \big[p_1^\theta\big],\ \big[p_2^\theta\big],\ \big[q_0^\theta\big],\ \big[q_1^\theta\big],\ \big[q_2^\theta\big],\ \big[r^\theta\big],\ \big[\mathcal{E}_\theta^{\mathbb{Z}_4}\big]$$

form a basis for $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_4) \cong \mathbb{Z}^9$. *Moreover,* $(\tau_{4,\theta})_*$ *takes the following values on these classes:*

$$1,\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{\theta}{4}.$$

(4) *If* $F = \mathbb{Z}_6$, *set* $\zeta = \frac{1}{2}\big(1 + i\sqrt{3}\big)$, *define*

$$p_j^\theta = \tfrac{1}{6}\big(1 + (\zeta^j t_\theta) + (\zeta^j t_\theta)^2 + (\zeta^j t_\theta)^3 + (\zeta^j t_\theta)^4 + (\zeta^j t_\theta)^5\big)$$

for $0 \leq j \leq 4$, *and define*

$$q_0^\theta = \tfrac{1}{3}\big(1 + e\big(\tfrac{1}{6}(2+\theta)\big)u_\theta t_\theta^2 + \big[e\big(\tfrac{1}{6}(2+\theta)\big)u_\theta t_\theta^2\big]^2\big),$$

$$q_1^\theta = \tfrac{1}{3}\big(1 + \zeta^2 e\big(\tfrac{1}{6}(2+\theta)\big)u_\theta t_\theta^2 + \big[\zeta^2 e\big(\tfrac{1}{6}(2+\theta)\big)u_\theta t_\theta^2\big]^2\big),$$

and

$$r^\theta = \tfrac{1}{2}\big(1 - u_\theta t_\theta^3\big).$$

Then the classes

$$[1],\ \big[p_0^\theta\big],\ \big[p_1^\theta\big],\ \big[p_2^\theta\big],\ \big[p_3^\theta\big],\ \big[p_4^\theta\big],\ \big[q_0^\theta\big],\ \big[q_1^\theta\big],\ \big[r^\theta\big],\ \big[\mathcal{E}_\theta^{\mathbb{Z}_6}\big]$$

form a basis for $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_6) \cong \mathbb{Z}^{10}$. *Moreover,* $(\tau_{6,\theta})_*$ *takes the following values on these classes:*

$$1,\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{2},\ \frac{\theta}{6}.$$

Proof. In each case, we define projections in $C^*\big(\mathbb{Z}^2 \rtimes_\alpha F, \widetilde{\Omega}_a\big)$ as follows. In the formulas in Example 3.7 (excluding S, which we treat below), substitute for t, ut, etc. the unitaries given in Lemma 4.4 whose images in the fiber at 1 are t, ut, etc. Thus, for example, for $F = \mathbb{Z}_6$ we replace ut^2 by $\varphi_2 U_a T_a^2$, and in place of the projection q_0 we obtain $Q_0 = \frac{1}{3}\big(1 + \varphi_2 U_a T_a^2 + (\varphi_2 U_a T_a^2)^2\big)$. In each case, by Lemma 4.4, the evaluation at $\theta \in [0,1]$ of the projection obtained this way is the corresponding projection in the statement of the theorem. For example, for $F = \mathbb{Z}_6$ we get $\mathrm{ev}_\theta(Q_0) = q_0^\theta$. To get our replacement for S, choose a projection in some matrix algebra over $C^*\big(\mathbb{Z}^2 \rtimes F, \widetilde{\Omega}_a\big)$ corresponding to the module $\mathcal{E}^F$ as in Notation 4.7. Thus, for each choice of F, we have a collection $\mathcal{P}$ of projections in $C^*\big(\mathbb{Z}^2 \rtimes F, \widetilde{\Omega}_a\big)$ or a matrix algebra over this algebra.

Evaluate at $\theta = 1$. The classes of the projections $\mathrm{ev}_1(p)$ for $p \in \mathcal{P}$ are exactly the classes listed in Example 3.7, except with S replaced by $\left[\mathcal{E}_\theta^F\right]$. By Example 3.7 and Lemma 4.8, these classes form a basis for $K_0(C^*(\mathbb{Z}^2 \rtimes F))$. By Remark 2.3, for any $\theta \in [a, 1]$, the classes of the projections $\mathrm{ev}_\theta(p)$ for $p \in \mathcal{P}$ form a basis for $K_0(A_\theta \rtimes_\alpha F)$.

The computation of the trace is clear from the description for all classes except the classes $\left[\mathcal{E}_\theta^F\right]$. The values of the traces of the classes $\left[\mathcal{E}_\theta^F\right]$ are computed in Proposition 3.3 of [72] for $F = \mathbb{Z}_4$ and in Theorem 2 of [75] for $F = \mathbb{Z}_3, \mathbb{Z}_6$. The case $F = \mathbb{Z}_2$ is done similarly. $\square$

5. The tracial Rokhlin property

In this section, we prove that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the actions of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ on A_θ, given by (0.1) for the groups generated as in (0.2), have the tracial Rokhlin property, and that for every nondegenerate skew-symmetric matrix Θ, the flip action of $\mathbb{Z}_2$ on A_Θ has the tracial Rokhlin property. We do this by showing that, under suitable conditions, the tracial Rokhlin property is equivalent to outerness of the corresponding action on the type II_1 factor obtained as the weak operator closure of A in the Gelfand-Naimark-Segal representation associated with the tracial state. A preliminary result, stated entirely in terms of C*-algebras, gives a criterion for the tracial Rokhlin property in terms of trace norms. For use elsewhere, we state and prove this result for C*-algebras with an arbitrary simplex of tracial states.

We will make extensive use of the L^2-norm (or seminorm) associated with a tracial state τ of a C*-algebra A, given by $\|a\|_{2,\tau} = \tau(a^*a)^{1/2}$. See the discussion before Lemma V.2.20 of [70] for more on this seminorm in the von Neumann algebra context. All the properties we need are immediate from its identification with the seminorm in which one completes A to obtain the Hilbert space H_τ for the Gelfand-Naimark-Segal representation associated with τ, and from the relation $\tau(ba) = \tau(ab)$. In particular, we always have $\|abc\|_{2,\tau} \leq \|a\| \cdot \|b\|_{2,\tau} \cdot \|c\|$.

Since the C*-algebras in this section will almost all be finite, we recall the tracial Rokhlin property for actions of finite groups on finite infinite dimensional simple unital C*-algebras. The following result is Lemma 1.16 of [59].

Proposition 5.1. *Let A be a finite infinite dimensional simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the tracial Rokhlin property if and only if for every finite set $S \subseteq A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in S$.
(3) *With* $e = \sum_{g \in G} e_g$, *the projection* $1 - e$ *is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of* A *generated by* x.

We begin with a simple reformulation of the tracial Rokhlin property which is valid in the presence of good comparison properties for projections. Recall that, if A is a unital C*-algebra, then we say that the order on projections over A is determined by traces if Blackadar's Second Fundamental Comparability Question (1.3.1 in [4]) holds for all matrix algebras over A. That is, whenever $n \in \mathbb{N}$ and $p, q \in M_n(A)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states τ on A, then $p \precsim q$.

Lemma 5.2. *Let A be a finite infinite dimensional simple separable unital C*-algebra with Property (SP) (every nonzero hereditary subalgebra contains a nonzero projection) and such that the order on projections over A is determined by traces. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the tracial Rokhlin property if and only if for every finite set $S \subseteq A$ and every $\varepsilon > 0$, there exist orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) *$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.*
(2) *$\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in S$.*
(3) *With $e = \sum_{g \in G} e_g$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in T(A)$.*

Proof. Suppose that the condition of Proposition 5.1 holds, and let S and ε be given. Choose $n \in \mathbb{N}$ with $n > 1/\varepsilon$. By Lemma 1.10 of [59], there exist n nonzero mutually orthogonal projections in A. Let x be one of them. Then $\tau(x) < \varepsilon$ for all $\tau \in T(A)$. Apply Proposition 5.1 with this x and with S and ε as given. Conversely, assume the condition of the lemma, and let S, ε, and x be given. Using Property (SP), choose a nonzero projection $q \in \overline{xAx}$, and apply the condition of the lemma with ε replaced by $\min\left(\varepsilon, \, \inf_{\tau \in T(A)} \tau(q)\right)$. The assumption that the order on projections over A is determined by traces implies that $1 - e \precsim q$, giving (3) of Proposition 5.1. ∎

Theorem 5.3. *Let A be an infinite dimensional simple separable unital C*-algebra with tracial rank zero. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the tracial Rokhlin property if and only if for every finite set $S \subseteq A$ and every $\varepsilon > 0$, there exist orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) *$\|\alpha_g(e_h) - e_{gh}\|_{2,\tau} < \varepsilon$ for all $g, h \in G$ and all $\tau \in T(A)$.*
(2) *$\|[e_g, a]\|_{2,\tau} < \varepsilon$ for all $g \in G$, all $a \in S$, and all $\tau \in T(A)$.*
(3) *$\sum_{g \in G} e_g = 1$.*

The equivalence is also valid if one substitutes for Condition (3) the following condition:

(3′) *$\tau\left(1 - \sum_{g \in G} e_g\right) < \varepsilon$ for all $\tau \in T(A)$.*

Proof. We first prove that if α has the tracial Rokhlin property, then the version of the condition of the theorem using Conditions (1), (2), and (3′) holds. Let $S \subseteq A$ be finite and let $\varepsilon > 0$. Since A has tracial rank zero, Corollary 5.7 and Theorems 5.8 and 6.8 of [40] imply that the order on projections over A is determined by traces. Therefore we may apply Lemma 5.2 with S and ε as given. Since $\|a\|_{2,\tau} \leq \|a\|$ for all $\tau \in T(A)$, it is easy to show that the resulting projections satisfy (1), (2), and (3′).

Next, assume that the version with Conditions (1), (2), and (3′) holds. Let $S \subseteq A$ be finite and let $\varepsilon > 0$. We may assume that $\|a\| \leq 1$ for all $a \in S$. Apply the hypothesis with S as given and with $\min\left(\frac{1}{3}\varepsilon, \frac{1}{9}\varepsilon^2\right)$ in place of ε. Call the resulting projections p_g for $g \in G$. Set $p = \sum_{g \in G} p_g$, set $e_1 = p_1 + 1 - p$, and set $e_g = p_g$ for $g \in G \setminus \{1\}$. Since $\|e_1 - p_1\|_{2,\tau} = \tau(1 - p)^{1/2} < \frac{1}{3}\varepsilon$ for all $\tau \in T(A)$, it is easy to show that the e_g satisfy (1), (2), and (3).

Now we prove that if (1), (2), and (3) hold, then α has the tracial Rokhlin property. We follow the proof of Theorem 2.14 of [51] (but note that the finite set there is called F). We describe the choices and constructions carefully, but omit many of the details in the verification of the estimates. We verify the condition of

Lemma 5.2. So let $S \subseteq A$ be a finite set, and let $\varepsilon > 0$. Without loss of generality $\varepsilon < 1$.

Choose $\varepsilon_0 > 0$ with

$$\varepsilon_0 < \frac{\varepsilon}{3\mathrm{card}(G)}$$

and so small that whenever p_g, for $g \in G$, are projections in a C*-algebra B which satisfy $\|p_g p_h\| < 4\varepsilon_0$ for $g \neq h$, then there are orthogonal projections $e_g \in B$, for $g \in G$, such that $e_1 = p_1$ and $\|e_g - p_g\| < \frac{1}{3}\varepsilon$ for $g \in G$.

Apply Lemma 2.8 of [51] with ε_0 in place of ε and with $n = \mathrm{card}(G) - 1$, and let $\varepsilon_1 > 0$ be the resulting value of δ. We also require $\varepsilon_1 \leq \varepsilon_0$. Then set

$$\varepsilon_2 = \min\left(1,\, \frac{\varepsilon_1^2}{16},\, \frac{\varepsilon_1}{8},\, \frac{\varepsilon}{18}\right).$$

Set $T = \bigcup_{g \in G} \alpha_g(S)$. Apply Lemma 2.13 of [51] with ε_2 in place of ε, with T in place of F, and with G in place of S. We obtain projections $q, q_0 \in A$, unital finite dimensional subalgebras $E \subseteq qAq$ and $E_0 \subseteq q_0 A q_0$, and automorphisms $\varphi_g \in \mathrm{Aut}(A)$ for $g \in G$, such that:

(1) $\varphi_1 = \mathrm{id}_A$ and $\|\varphi_g - \alpha_g\| < \varepsilon_2$ for all $g \in G$.
(2) For every $g \in G$ and $x \in E$, we have $q_0 \varphi_g(x) = \varphi_g(x) q_0$ and $q_0 \varphi_g(x) q_0 \in E_0$.
(3) For every $a \in T$, we have $\|qa - aq\| < \varepsilon_2$ and $\mathrm{dist}(qaq,\, E) < \varepsilon_2$.
(4) $\tau(1-q)$, $\tau(1-q_0) < \varepsilon_2$ for all $\tau \in T(A)$.

Apply Lemma 2.12 of [51] with ε_2 in place of ε and $E_0 + \mathbb{C}(1-q_0)$ in place of E, obtaining $\delta > 0$. Also require $\delta \leq \varepsilon_2$.

Apply the hypothesis with δ in place of ε, and with a system of matrix units for $E_0 + \mathbb{C}(1-q_0)$ in place of S, getting projections p_g for $g \in G$. Let $B_0 = A \cap [E_0 + \mathbb{C}(1-q_0)]'$, the subalgebra of A consisting of all elements which commute with everything in $E_0 + \mathbb{C}(1-q_0)$. Apply the choice of δ using Lemma 2.12 of [51] to p_1, obtaining a projection $f \in B_0$ which satisfies $\|f - p_1\|_{2,\tau} < \varepsilon_2$ for all $\tau \in T(A)$. Since q_0 is in the center of $E_0 + \mathbb{C}(1-q_0)$, the element $f_1 = q_0 f$ is also a projection in B_0. For $g \in G$, since $q_0 \varphi_{g^{-1}}(E) q_0 \subseteq E_0$, it follows that f_1 commutes with all elements of $q_0 \varphi_{g^{-1}}(E) q_0$ and hence with all elements of $\varphi_{g^{-1}}(E)$. Therefore $f_g = \varphi_g(f_1)$ commutes with all elements of E, including q. So f_g also commutes with $1-q$.

Following the analogous estimates in the proof of Theorem 2.14 of [51], we get $\|f_1 - p_1\|_{2,\tau} < \frac{1}{4}\varepsilon_1 + \varepsilon_2$ for $\tau \in T(A)$ and $\|f_g - \alpha_g(f_1)\| < \varepsilon_2$ for $g \in G$. We then get (with a slight difference: the sum in the estimate in [51] is not needed here),

$$\|f_g - p_g\|_{2,\tau} < \varepsilon_2 + \left(\tfrac{1}{4}\varepsilon_1 + \varepsilon_2\right) + \varepsilon_2$$

for $\tau \in T(A)$. Continuing as there, for $g \in G \setminus \{1\}$ we get

$$\|f_1 f_g\|_{2,\tau} = \|f_1 f_g - p_1 p_g\|_{2,\tau} < \tfrac{1}{2}\varepsilon_1 + 4\varepsilon_2 \leq \varepsilon_1.$$

We saw that $f_g \in B = A \cap [E + \mathbb{C}(1-q)]'$ for $g \in G$. This algebra has real rank zero because $E + \mathbb{C}(1-q)$ is finite dimensional. Therefore Lemma 2.8 of [51], applied to B with $\{\tau|_B \colon \tau \in T(A)\}$ in place of T, and the choice of ε_1, provide a projection $r \in A \cap [E + \mathbb{C}(1-q)]'$ such that $r \leq f_1$, such that $\|r f_g\| < \varepsilon_0$ for $g \in G$, and such that $\tau(r) > \tau(f_1) - \varepsilon_0$ for all $\tau \in T(A)$.

Now use Corollary 2.4 of [51] to find a projection $e_1 \in A \cap [E + \mathbb{C}(1-q)]'$ such that $e_1 \leq q$, such that $\|r e_1 - e_1\| < \varepsilon_0$, and such that $[e_1] \geq [1] - ([1]-[q]) - ([1]-[r])$

in $K_0(A \cap [E + \mathbb{C}(1-q)]')$. The last inequality implies that

$$\tau(e_1) \geq \tau(r) - \tau(1-q) > \tau(f_1) - \varepsilon_0 - \varepsilon_2 \geq \tau(f_1) - 2\varepsilon_0$$

for all $\tau \in T(A)$.

We now show that the $\alpha_g(e_1)$, for $g \in G$, are approximately orthogonal. It suffices to estimate $\|e_1 \alpha_g(e_1)\|$ for $g \in G$. We first use $\varphi_g(r) \leq \varphi_g(f_1) = f_g$ to get $\|r\varphi_g(r)\| \leq \|r f_g\| < \varepsilon_0$. Then, following the proof of Theorem 2.14 of [51], for $g \in G$ we get $\|e_1 \varphi_g(e_1)\| < 3\varepsilon_0$ and

$$\|e_1 \alpha_g(e_1)\| \leq \|\alpha_g - \varphi_g\| + \|e_1 \varphi_g(e_1)\| < 4\varepsilon_0.$$

Now use the choice of ε_0 to find orthogonal projections $e_g \in A \cap [E + \mathbb{C}(1-q)]'$ for $g \in G \setminus \{1\}$, all orthogonal to e_1, such that $\|e_g - \alpha_g(e_1)\| < \frac{1}{3}\varepsilon$ for $g \in G \setminus \{1\}$. Then the e_g, for $g \in G$, satisfy Conditions (1) through (3) of Lemma 5.2, by estimates essentially the same as in the last part of the proof of Theorem 2.14 of [51]. $\square$

Notation 5.4. If A is a simple C*-algebra with unique tracial state τ, and $\alpha \in \mathrm{Aut}(A)$, we write α'' for the automorphism of $\pi_\tau(A)''$ determined by α.

Theorem 5.5. *Let A be a simple separable unital C*-algebra with tracial rank zero, and suppose that A has a unique tracial state τ. Let $\pi_\tau \colon A \to B(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. Then α has the tracial Rokhlin property if and only if α''_g is an outer automorphism of $\pi_\tau(A)''$ for every $g \in G \setminus \{1\}$.*

Proof. Assume that α''_g is outer for every $g \in G \setminus \{1\}$. We verify the hypotheses of Theorem 5.3. Thus let $\varepsilon > 0$ and let $S \subseteq A$ be a finite subset. Without loss of generality $\|a\| \leq 1$ for all $a \in S$. Set $n = \mathrm{card}(G)$, and set $\varepsilon_0 = (4n+1)^{-1}\varepsilon$. Choose $\delta > 0$ as in Lemma 2.9 of [51] with n as given and with ε_0 in place of ε.

We regard A as a subalgebra of $N = \pi_\tau(A)''$, and we let τ also denote the extension of τ to N. The algebra N is hyperfinite by Lemma 2.16 of [51].

We apply Lemma 5.2.1 of [35]. The term "equivariant s.m.u." [system of matrix units] is defined in Section 1.5 of [35], and $N(\psi)$ is defined in Section 1.2 of [35]. (Also note that pages 44 and 45 of [35] are switched.) We take the set X of Lemma 5.2.1 of [35] to be G with the left translation action. The lemma provides, ignoring the off diagonal matrix units, projections $p_g \in \pi_\tau(A)''$ such that $\alpha''_g(p_h) = p_{gh}$ for $g, h \in G$, and such that $\|[p_g, a]\|_{2,\tau} < \delta$ for $g \in G$ and $a \in S$.

For $g \in G$ use Lemma 2.15 of [51] to find a projection $q_g \in A$ such that $\|q_g - p_g\|_{2,\tau} < \min\left(\frac{1}{2}\delta, \varepsilon_0\right)$. Then $\|q_g q_h\|_{2,\tau} < \delta$ for $g \neq h$, so the choice of δ using Lemma 2.9 of [51] provides mutually orthogonal projections $e_g \in A$ for $g \in G \setminus \{1\}$ such that $\|e_g - q_g\|_{2,\tau} < \varepsilon_0$, and therefore $\|e_g - p_g\|_{2,\tau} < 2\varepsilon_0$. Set $e_1 = 1 - \sum_{g \in G \setminus \{1\}} e_g$. Since $\sum_{g \in G} p_g = 1$, we get

$$\|e_1 - p_1\|_{2,\tau} \leq \sum_{g \in G \setminus \{1\}} \|e_g - p_g\|_{2,\tau} < 2(n-1)\varepsilon_0.$$

So $\|e_g - p_g\|_{2,\tau} < 2n\varepsilon_0$ for all $g \in G$.

For $g, h \in G$, using $\alpha''_g(p_h) = p_{gh}$ we now get

$$\|\alpha_g(e_h) - e_{gh}\|_{2,\tau} \leq \|\alpha''_g(e_h - p_h)\|_{2,\tau} + \|p_{gh} - e_{gh}\|_{2,\tau} < 2n\varepsilon_0 + 2n\varepsilon_0 < \varepsilon.$$

Also, if $a \in S$ and $g \in G$ then

$$\|[a, e_g]\|_{2,\tau} \leq 2\|e_g - p_g\|_{2,\tau} \cdot \|a\| + \|[a, p_g]\|_{2,\tau} < 2 \cdot 2n\varepsilon_0 + \varepsilon_0 \leq \varepsilon.$$

This completes the proof that outerness in the trace representation implies the tracial Rokhlin property.

The proof of the converse is similar to the proof of Lemma 1.5 of [59]. Set $n = \mathrm{card}(G)$ as before, and set

$$\varepsilon = \min\left(\frac{1}{2}, \, \frac{1}{4\sqrt{2n}}\right).$$

Let $g_0 \in G \smallsetminus \{1\}$, and let $u \in \pi_\tau(A)''$ be a unitary; we show that $\alpha''_{g_0} \neq \mathrm{Ad}(u)$. Let ξ_τ be the standard cyclic vector for π_τ, so that $\|x\|_{2,\tau} = \|x\xi_\tau\|$ for all $x \in \pi_\tau(A)''$. The Kaplansky Density Theorem implies that the unit ball of $\pi_\tau(A)$ is strong operator dense in the unit ball of $\pi_\tau(A)''$. Therefore there is $a \in A$ with $\|a\| \leq 1$ such that $\|u - a\|_{2,\tau} < \varepsilon$. Choose a nonzero projection $f \in A$ with $\tau(f) < \varepsilon$, and choose mutually orthogonal projections $e_g \in A$ for $g \in G$ such that $\|e_g a - a e_g\| < \varepsilon$ for $g \in G$, such that $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for $g, h \in G$, and such that $1 - \sum_{g \in G} e_g \precsim f$. Since τ is the unique tracial state, it is G-invariant, so that

$$\tau(e_1) = \tfrac{1}{n} \textstyle\sum_{g \in G} \tau(e_g) \geq \tfrac{1}{n}(1 - \tau(f)) \geq \tfrac{1}{2n}.$$

Now

$$\begin{aligned} \|e_1 - e_{g_0}\|_{2,\tau} &\leq 2\|u - a\|_{2,\tau} + \|a e_1 - e_1 a\| + \|u e_1 u^* - \alpha_{g_0}(e_1)\| + \|\alpha_{g_0}(e_1) - e_{g_0}\| \\ &< 2\varepsilon + \varepsilon + \|u e_1 u^* - \alpha_{g_0}(e_1)\| + \varepsilon. \end{aligned}$$

On the other hand, using orthogonality,

$$\|e_1 - e_{g_0}\|_{2,\tau} \geq \|e_1\|_{2,\tau} \geq \frac{1}{\sqrt{2n}}.$$

Therefore

$$\|u e_1 u^* - \alpha_{g_0}(e_1)\| > \frac{1}{\sqrt{2n}} - 4\varepsilon \geq 0.$$

Thus $u e_1 u^* \neq \alpha_{g_0}(e_1)$, and $\alpha''_{g_0} \neq \mathrm{Ad}(u)$. □

In case the group is $\mathbb{Z}$, Theorem 2.18 of [51] gives two more equivalent conditions for an action $\alpha \colon G \to \mathrm{Aut}(A)$ on a simple unital C*-algebra with tracial rank zero and unique tracial state to have the tracial Rokhlin property, namely that the crossed product $A \rtimes_\alpha G$ have real rank zero, and that it have a unique tracial state. For finite groups, the tracial Rokhlin property implies these conditions, but neither condition implies the tracial Rokhlin property.

The tracial Rokhlin property implies that the crossed product has real rank zero under these hypotheses, because it implies that the crossed product has tracial rank zero (Theorem 2.6 of [59]). The converse is easily seen to be false, by considering the trivial action of any finite group on any simple unital C*-algebra with tracial rank zero and a unique tracial state. Example 2.9 of [61] shows that it does not help to require that the action be outer.

We now consider the condition that $A \rtimes_\alpha G$ have a unique tracial state. One can give a direct argument that this follows from the tracial Rokhlin property. The line of reasoning we give here uses some machinery and an additional useful lemma. For this lemma, we do not require the algebra to be finite, so the tracial Rokhlin property is as in Definition 1.2 of [59].

Lemma 5.6. *Let A be an infinite dimensional simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Let H be a subgroup of G. Then $\alpha|_H$ has the tracial Rokhlin property.*

Proof. Set $m = \mathrm{card}(G)/\mathrm{card}(H)$. Choose a set S of right coset representatives for H in G, so that $\mathrm{card}(S) = m$ and every element of G can be written uniquely as hs with $h \in H$ and $s \in S$.

Let $S \subseteq A$ be finite, let $\varepsilon > 0$, and let $x \in A$ be a positive element with $\|x\| = 1$. Set $\varepsilon_0 = \varepsilon/m$. The tracial Rokhlin property for α provides mutually orthogonal projections $p_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon_0$ for all $g, h \in G$.
(2) $\|p_g a - a p_g\| < \varepsilon_0$ for all $g \in G$ and all $a \in S$.
(3) With $p = \sum_{g \in G} p_g$, the projection $1-p$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) With p as in (3), we have $\|pxp\| > 1 - \varepsilon$.

For $h \in H$ define $e_h = \sum_{s \in S} p_{hs}$. These are clearly mutually orthogonal projections. We verify the analogs of Conditions (1) through (4). For $h, k \in H$, we have

$$\|\alpha_k(e_h) - e_{kh}\| \leq \sum_{s \in S} \|\alpha_k(p_{hs}) - p_{khs}\| < m\varepsilon_0 = \varepsilon.$$

For $h \in H$ and $a \in S$, we have

$$\|e_h a - a e_h\| \leq \sum_{s \in S} \|p_{hs} a - a p_{hs}\| < m\varepsilon_0 = \varepsilon.$$

The projection $e = \sum_{h \in H} e_h$ is equal to p, so that $1 - e = 1 - p$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x. Finally, $\|exe\| = \|pxp\| > 1 - \varepsilon$. $\square$

Proposition 5.7. *Let A be an infinite dimensional simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Let $r \colon T(A \rtimes_\alpha G) \to T(A)$ be the restriction map between the tracial state spaces. Then r is an affine homeomorphism from $T(A \rtimes_\alpha G)$ to the subspace $T(A)^G$ of G-invariant tracial states on A.*

Proof. It is easy to see that the image of r is exactly $T(A)^G$, and that r is continuous and affine. It therefore suffices to show that r is injective, that is, that a tracial state τ on $A \rtimes_\alpha G$ is uniquely determined by its values on A. We prove this by proving that if $a \in A$, if $g \in G \setminus \{1\}$, and if $u_g \in A \rtimes_\alpha G$ is the corresponding unitary, then $\tau(a u_g) = 0$.

Let H be the subgroup of G generated by g, and let $\beta = \alpha|_H$. Set $\sigma = \tau|_{A \rtimes_\beta H}$. Then β has the tracial Rokhlin property by Lemma 5.6. Therefore $\widehat{\beta}$ is tracially approximately representable, by Theorem 3.12(1) of [59]. Choose $\mu \in \widehat{H}$ such that $\mu(g) \neq 1$. It follows from Proposition 6.1 of [59] that $\sigma \circ \widehat{\beta}_\mu = \sigma$. Since $\widehat{\beta}_\mu(a u_g) = \overline{\mu(g)} a u_g$, this implies that $\sigma(a u_g) = 0$. So $\tau(a u_g) = 0$. $\square$

In particular, if A has a unique tracial state and $\alpha \colon G \to \mathrm{Aut}(A)$ has the tracial Rokhlin property, then $A \rtimes_\alpha G$ has a unique tracial state.

Again, the converse is false.

Example 5.8. First take $A = M_2$ and $G = (\mathbb{Z}_2)^2$. Let the two generators act by conjugation by the unitaries $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Because the matrices commute up to a scalar, one gets a well defined action $\alpha \colon G \to \mathrm{Aut}(M_2)$, such that α_g is inner for every $g \in G$. It is well known (see, for example, Example 4.2.3 of [56]) that $A \rtimes_\alpha G \cong M_4$, which has a unique tracial state.

This example is finite dimensional, and the tracial Rokhlin property was only defined for infinite dimensional C*-algebras. To get an infinite dimensional example, tensor the above example with, say, the trivial action of G on the 2^∞ UHF algebra.

The following lemma is well known, and is given in this form as Lemma 3.1 of [51].

Lemma 5.9. *Let A be a separable unital C*-algebra with a faithful tracial state τ. Let $(y_i)_{i\in I}$ be a family of unitaries such that $\tau(y_i^* y_j) = 0$ for $i \neq j$ and whose linear span is dense in A. Let $\pi\colon A \to B(H)$ be the Gelfand-Naimark-Segal representation associated with τ. We identify A with its image in $\pi(A)''$. Then every $a \in \pi(A)''$ has a unique representation as $a = \sum_{i\in I} \lambda_i y_i$, with convergence in $\|\cdot\|_{2,\tau}$ and whose coefficients satisfy $\sum_{i\in I} |\lambda_i|^2 = \|a\|_{2,\tau}^2$. If a is unitary then $\sum_{i\in I} |\lambda_i|^2 = 1$.*

Recall that a real skew symmetric $d \times d$ matrix Θ is *nondegenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i\langle x, \Theta y\rangle) = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. This definition is essentially from Section 1.1 of [69]. It is well known that the noncommutative torus A_Θ (defined after Corollary 1.13) is simple if and only if Θ is nondegenerate; the main part is Theorem 3.7 of [69], and the complete statement is Theorem 1.9 of [60].

Lemma 5.10. *Let Θ be a nondegenerate real skew symmetric $d \times d$ matrix. Let $1 \leq k \leq d$, and let $\alpha \in \operatorname{Aut}(A_\Theta)$ be an automorphism such that $\alpha(u_k) = \rho u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ for some $\rho \in \mathbb{C}$ with $|\rho| = 1$, and for some $m \in \mathbb{Z}^d$ not equal to the kth standard basis vector δ_k. Then α'' (as in Notation 5.4) is outer.*

Proof. Without loss of generality $k = 1$.

Let τ be the unique tracial state on A_Θ, and identify A_Θ with its image in $\pi_\tau(A_\Theta)''$.

Suppose α'' is inner, so $\alpha'' = \operatorname{Ad}(w)$ for some unitary $w \in \pi_\tau(A_\Theta)''$. We apply Lemma 5.9 with $I = \mathbb{Z}^d$ and $y_n = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$ for $n \in \mathbb{Z}^d$, and write

$$w^* = \sum_{n\in\mathbb{Z}^d} \lambda_n u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$$

with convergence in $\|\cdot\|_{2,\tau}$. Using the commutation relations, there are numbers ζ_n for $n \in \mathbb{Z}^d$, with $|\zeta_n| = 1$, such that

$$\big(u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}\big)\big(u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}\big) = \zeta_n u_1^{n_1+m_1} u_2^{n_2+m_2} \cdots u_d^{n_d+m_d}.$$

Then, also with convergence in $\|\cdot\|_{2,\tau}$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} \lambda_n u_1^{n_1+1} u_2^{n_2} \cdots u_d^{n_d} &= u_1 w^* = w^* \alpha(u_1)\\ &= \sum_{n\in\mathbb{Z}^d} \rho\zeta_n \lambda_n u_1^{n_1+m_1} u_2^{n_2+m_2} \cdots u_d^{n_d+m_d}\\ &= \sum_{n\in\mathbb{Z}^d} \rho\zeta_{n-m+\delta_1} \lambda_{n-m+\delta_1} u_1^{n_1+1} u_2^{n_2} \cdots u_d^{n_d}.\end{aligned}$$

We have $\lambda_l \neq 0$ for some $l \in \mathbb{Z}^d$. Since ρ and the ζ_n have absolute value 1, uniqueness of the series representation implies

$$|\lambda_l| = |\lambda_{l-m+\delta_1}| = |\lambda_{l-2m+2\delta_1}| = \cdots.$$

Since $m \neq \delta_1$, this contradicts $\sum_{n\in\mathbb{Z}^d} |\lambda_n|^2 < \infty$. $\square$

Corollary 5.11. *Let $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, and let $\alpha\colon F \to \mathrm{Aut}(A_\theta)$ be given by (0.1) for the groups generated as in (0.2). Suppose $\theta \notin \mathbb{Q}$. Then α has the tracial Rokhlin property.*

Proof. In each case, Lemma 5.10 implies that α''_g is outer for $g \in G \smallsetminus \{1\}$. Apply Theorem 5.5. ☐

Let $\mathbb{Z}_2$ act on $\mathbb{Z}^d$ via the flip $n \mapsto -n$. It is immediately clear that for any $d \times d$ skew symmetric real matrix Θ, the cocycle ω_Θ is invariant under this action. It thus induces an action, also called the flip, of $\mathbb{Z}_2$ on the higher dimensional noncommutative torus A_Θ. On the standard generators, this action sends u_k to u_k^* for $1 \leq k \leq d$.

Corollary 5.12. *Let Θ be a nondegenerate real $d \times d$ skew symmetric matrix. Then the flip action on A_Θ has the tracial Rokhlin property.*

Proof. Lemma 5.10 implies that the flip automorphism is outer. Theorem 3.5 of [60] implies that A_Θ is a simple separable unital C*-algebra with tracial rank zero. Apply Theorem 5.5. ☐

6. The structure of the crossed products

In this section, we prove that for $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$ the crossed products $A_\theta \rtimes_\alpha \mathbb{Z}_2$, $A_\theta \rtimes_\alpha \mathbb{Z}_3$, $A_\theta \rtimes_\alpha \mathbb{Z}_4$, and $A_\theta \rtimes_\alpha \mathbb{Z}_6$, by the action (0.1) for the groups generated as in (0.2), are AF algebras, and we determine when two of them are isomorphic. For every nondegenerate Θ, we also prove that the crossed product $A_\Theta \rtimes \mathbb{Z}_2$ by the flip is an AF algebra. We further prove that all the fixed point algebras are AF. As mentioned in the introduction, this was already known for $A_\theta \rtimes_\alpha \mathbb{Z}_2$ for all irrational θ [9], for $A_\theta \rtimes_\alpha \mathbb{Z}_4$ for "most" irrational θ [74], and for $A_\Theta \rtimes \mathbb{Z}_2$ for "most" skew symmetric real matrices Θ [6].

We need to show that the crossed products satisfy the Universal Coefficient Theorem, as in Theorem 1.17 of [67]. A much stronger result than the one formulated in the proposition below has been obtained by Meyer and Nest in Proposition 8.5 of [47]. Since the arguments used in the special case we consider here are much easier, we include the short proof.

Proposition 6.1. *Let G be an amenable (or, more generally, an a-T-menable) group which can be embedded as a closed subgroup of some almost connected group L. Then the full and reduced crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$ of any type I algebra A by G are KK-equivalent to commutative C*-algebras.*

Proof. Let C denote the maximal compact subgroup of L and let $T^*(L/C)$ denote the cotangent space of the quotient L/C (which is a manifold). Then it follows from the work of Kasparov (see Theorem 2 of [36]; Definitions 4.1 and 5.1 and Theorems 5.2 and 5.7 of [37]) that there exist Dirac and dual-Dirac elements

$$D \in KK_0^L\big(C_0(T^*(L/C)), \mathbb{C}\big) \quad \text{and} \quad \beta \in KK_0^L\big(\mathbb{C}, C_0(T^*(L/C))\big)$$

such that

$$D \otimes_{\mathbb{C}} \beta = 1 \in KK_0^L\big(C_0(T^*(L/C)), C_0(T^*(L/C))\big)$$

and such that $\gamma = \beta \otimes_{C_0(T^*(L/C))} D \in KK_0^L(\mathbb{C}, \mathbb{C})$ is the so-called γ-element of L. The restriction γ_G of γ from L to G is the γ-element of G. (For example, see Remark 6.4 of [12].) Since G is a-T-menable, we know from the work of Higson and

Kasparov (Theorem 1.1 of [33]) that $\gamma_G = 1 \in KK_0^G(\mathbb{C}, \mathbb{C})$. But this implies that $\mathbb{C}$ is KK^G-equivalent to the commutative algebra $C_0(T^*(L/C))$, on which G acts properly. Now proceed as in the proof of Lemma 5.4 of [15]. □

The algebras in the previous proposition thus satisfy the Universal Coefficient Theorem. In particular, we have:

Corollary 6.2. *Let G be an amenable group which can be embedded as a closed subgroup of some almost connected group L. Let $\omega \colon G \times G \to \mathbb{T}$ be a Borel 2-cocycle on G. Then $C^*(G, \omega)$ satisfies the Universal Coefficient Theorem (Theorem 1.17 of* [67]*).*

Proof. As in the discussion at the beginning of Section 1, there is action $\alpha \colon G \to \mathrm{Aut}(\mathcal{K})$ such that $C^*(G, \omega) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_\alpha G$. Thus Proposition 6.1 implies that $C^*(G, \omega)$ is KK-equivalent to a commutative C*-algebra. It therefore satisfies the Universal Coefficient Theorem, by Proposition 7.1 of [67]. □

Theorem 6.3. *Let $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, and let $\alpha \colon F \to A_\theta$ be given by (0.1) for the groups generated as in (0.2). Suppose $\theta \notin \mathbb{Q}$. Then $A_\theta \rtimes_\alpha F$ is an AF algebra.*

Proof. We know that A_θ is a simple separable unital C*-algebra with tracial rank zero, say by the Elliott-Evans Theorem [25]. (See Theorem 3.5 of [60].) By Corollary 5.11, we may apply Corollary 1.6 and Theorem 2.6 of [59] to conclude that $A_\theta \rtimes_\alpha F$ is a simple separable unital C*-algebra with tracial rank zero. Corollary 6.2 implies that $A_\theta \rtimes_\alpha F$ satisfies the Universal Coefficient Theorem. Therefore Huaxin Lin's classification theorem (Theorem 5.2 of [41]), in the form presented in Proposition 3.7 of [60], together with the K-theory computation of Corollary 3.8, implies the conclusion. □

The case $F = \mathbb{Z}_2$ is not new [9].

Using the results of Section 4, one can determine exactly which AF algebra the crossed product is. As an example, we show that, for fixed F, the crossed product algebras are isomorphic if and only if the original irrational rotation algebras are isomorphic.

Theorem 6.4. *Let $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ be one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, and let $\alpha \colon F \to A_\theta$ be given by (0.1) for the groups generated as in (0.2). Let $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$. Then $A_{\theta_1} \rtimes_\alpha F \cong A_{\theta_2} \rtimes_\alpha F$ if and only if $\theta_1 = \pm\theta_2$ mod $\mathbb{Z}$.*

Proof. We give the proof for the case $F = \mathbb{Z}_2$, using Theorem 4.9(1). The proofs for the other parts are the same, using instead Parts (2), (3), and (4) of Theorem 4.9, and choosing in place of p^θ projections with traces $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$.

We clearly have $A_\theta \rtimes_\alpha \mathbb{Z}_2 \cong A_{\theta+n} \rtimes_\alpha \mathbb{Z}_2$ for $n \in \mathbb{Z}$, so without loss of generality $\theta_1, \theta_2 \in (0, 1)$. From Theorem 4.9(1), it is clear that for $\theta \in (0, 1) \setminus \mathbb{Q}$ the canonical tracial state $\tau_{2,\theta}$ on $A_\theta \rtimes_\alpha \mathbb{Z}_2$ satisfies

$$(\tau_{2,\theta})_*(K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2)) = \tfrac{1}{2}\mathbb{Z} + \tfrac{1}{2}\theta\mathbb{Z}.$$

It is well known (and also follows directly from Proposition 5.7 and Corollary 5.11) that $\tau_{2,\theta}$ is the unique tracial state on $A_\theta \rtimes_\alpha \mathbb{Z}_2$. Accordingly, if $\theta_1 \neq \pm\theta_2$ mod $\mathbb{Z}$, then

$$2(\tau_{2,\theta_1})_*(K_0(A_{\theta_1} \rtimes_\alpha \mathbb{Z}_2)) = \mathbb{Z} + \theta_1\mathbb{Z} \neq \mathbb{Z} + \theta_2\mathbb{Z} = 2(\tau_{2,\theta_2})_*(K_0(A_{\theta_2} \rtimes_\alpha \mathbb{Z}_2))$$

whence $A_{\theta_1} \rtimes_\alpha \mathbb{Z}_2 \not\cong A_{\theta_2} \rtimes_\alpha \mathbb{Z}_2$.

If $\theta_1, \theta_2 \in (0,1)$ with $\theta_1 = \pm\theta_2 \mod \mathbb{Z}$, then $\theta_2 = \theta_1$ or $\theta_2 = 1 - \theta_1$. Thus, it remains to prove that $A_\theta \rtimes_\alpha \mathbb{Z}_2 \cong A_{1-\theta} \rtimes_\alpha \mathbb{Z}_2$ for $\theta \in (0,1) \setminus \mathbb{Q}$. By Theorem 6.3 and George Elliott's classification theorem for AF algebras (Theorem 4.3 of [23]), it suffices to exhibit an isomorphism $f \colon K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2) \to K_0(A_{1-\theta} \rtimes_\alpha \mathbb{Z}_2)$ of scaled ordered K_0-groups. Since each algebra has a unique tracial state, and since the order on the K_0-group of a simple unital AF algebra is determined by its tracial state (this is true much more generally: see Corollary 5.7 and Theorems 5.8 and 6.8 of [40]), it suffices to find a group isomorphism f as above such that $(\tau_{2,\,1-\theta})_* \circ f = (\tau_{2,\theta})_*$ and $f([1]) = [1]$.

Define f on the basis elements listed in Theorem 4.9(1) by $f\big(\big[\mathcal{E}_{\theta_1}^{\mathbb{Z}_2}\big]\big) = [p^\theta] - \big[\mathcal{E}_{\theta_2}^{\mathbb{Z}_2}\big]$, and by sending each of the other basis elements for $K_0(A_\theta \rtimes_\alpha \mathbb{Z}_2)$ to the corresponding basis element for $K_0(A_{1-\theta} \rtimes_\alpha \mathbb{Z}_2)$, that is, $f([1]) = [1]$, $f([p^\theta]) = [p^{1-\theta}]$, etc. The computations of the trace on the generators given in Theorem 4.9(1) show that $(\tau_{2,\,1-\theta})_* \circ f = (\tau_{2,\theta})_*$, as desired. □

In the cases $F = \mathbb{Z}_2$ and $F = \mathbb{Z}_4$, it also is easy to use the isomorphism $\varphi \colon A_\theta \to A_{1-\theta}$, given by $u_\theta \mapsto v_{1-\theta}$ and $v_\theta \mapsto u_{1-\theta}$, to directly exhibit an isomorphism $A_\theta \rtimes_\alpha F \cong A_{1-\theta} \rtimes_\alpha F$. Presumably this can be done in the other two cases as well.

Theorem 6.3 also has the following corollary.

Corollary 6.5. *Under the hypotheses of Theorem 6.3, the fixed point algebra A_θ^F is AF.*

Proof. By the Proposition in [66], the fixed point algebra A_θ^F is isomorphic to a corner of $A_\theta \rtimes_\alpha F$. □

We now turn to the crossed product of a simple higher dimensional noncommutative torus by the flip, defined before Corollary 5.12. The following theorem generalizes Theorem 3.1 of [6], and completely answers a question raised in the introduction to [28].

Theorem 6.6. *Let Θ be a nondegenerate real $d \times d$ skew symmetric matrix. Let $\varphi \colon \mathbb{Z}_2 \to \mathrm{Aut}(A_\Theta)$ be the flip action. Then $A_\Theta \rtimes_\varphi \mathbb{Z}_2$ is an AF algebra.*

Proof. By Theorem 3.5 of [60], we know that A_Θ is a simple separable unital C*-algebra with tracial rank zero. By Corollary 5.12, we may apply Corollary 1.6 and Theorem 2.6 of [59] to conclude that $A_\Theta \rtimes_\varphi \mathbb{Z}_2$ is a simple separable unital C*-algebra with tracial rank zero.

Lemma 2.1 implies that the cocycle ω_Θ extends to a cocycle $\widetilde{\omega}_\Theta$ on the semidirect product $\mathbb{Z}^d \rtimes \mathbb{Z}_2$ such that $C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_2, \widetilde{\omega}_\Theta) \cong A_\Theta \rtimes_\varphi \mathbb{Z}_2$. Since $\mathbb{Z}^d \rtimes \mathbb{Z}_2$ can be embedded into $\mathbb{R}^d \rtimes \mathbb{Z}_2$, Corollary 6.2 therefore implies that $A_\Theta \rtimes_\varphi \mathbb{Z}_2$ satisfies the Universal Coefficient Theorem.

Since ω_Θ and $\widetilde{\omega}_\Theta$ are real cocycles (Definition 1.12), it follows from Corollary 1.13 that

$$K_0(A_\Theta \rtimes_\varphi \mathbb{Z}_2) = K_0(C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_2, \widetilde{\omega}_\Theta)) \cong K_0(C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_2)),$$

and $K_1(A_\Theta \rtimes_\varphi \mathbb{Z}_2) \cong K_1(C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_2))$ for all Θ. But in [28] it was shown that $K_0(A_\Theta \rtimes_\varphi \mathbb{Z}_2) \cong \mathbb{Z}^{3 \cdot 2^{d-1}}$ and $K_1(A_\Theta \rtimes_\varphi \mathbb{Z}_2) = 0$ for some special values for Θ. This must then be true for *all* Θ. Therefore Lin's classification theorem (Theorem 5.2 of [41]), in the form presented in Proposition 3.7 of [60], implies the conclusion. □

Corollary 6.7. *Let Θ be a nondegenerate real $d \times d$ skew symmetric matrix. Let $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(A_\Theta)$ be the flip action. Then the fixed point algebra $A_\Theta^{\mathbb{Z}_2}$ is AF.*

Proof. The proof is the same as for Corollary 6.5. □

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